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# W. Stenger's and M.A. Nudelman's results and resolvent formulas involving compressions

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## Abstract

In the first part of this note we give a rather short proof of a generalization of Stenger's lemma about the compression $A_0$ to $\mathfrak{H}_0$ of a self-adjoint operator $A$ in some Hilbert space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. In this situation, $S := A \cap A_0$ is a symmetry in $\mathfrak{H}_0$ with the canonical self-adjoint extension $A_0$ and the self-adjoint extension $A$ with exit into $\mathfrak{H}$. In the second part we consider relations between the resolvents of $A$ and $A_0$ like M.G. Krein's resolvent formula, and corresponding operator models.

**Keywords** Hilbert space · Dissipative operator · Symmetric operator · Self-adjoint operator · Dilation · Compression · Extension · Generalized resolvent · Nevanlinna function · Krein's resolvent formula

**Mathematics Subject Classification** 47B25 · 47A20 · 47A56

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Dedicated to Franciszek Hugon Szafraniec on the occasion of his 80-th birthday.

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## 1 Introduction

Let $A$ be a closed densely defined operator with nonempty resolvent set $\rho(A)$ in a Hilbert space $\mathfrak{H}$ which is the orthogonal sum of the two Hilbert spaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$ : $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$; $P_{\mathfrak{H}_0}$ denotes the orthogonal projection in $\mathfrak{H}$ onto $\mathfrak{H}_0$. We study the compression $A_0 := P_{\mathfrak{H}_0}A\big|_{\mathfrak{H}_0 \cap \operatorname{dom} A}$ of $A$ to $\mathfrak{H}_0$. Our starting point is the block matrix representation of the resolvent of $A$:

$$(A-z)^{-1} = \begin{pmatrix} S(z) & L(z) \\ M(z) & D(z) \end{pmatrix}, \quad z \in \rho(A).$$

Under the assumption that $D(z)$ in $\mathfrak{H}_1$ is boundedly invertible (meaning that $D(z)^{-1}$ exists and is defined on all of $\mathfrak{H}_1$ and bounded) we show (see Theorem 1) that the compression $A_0$ is also a closed densely defined operator with nonempty resolvent set. Since $D(z)$ for $z \in \rho(A) \setminus \sigma_p(A_0)$ is injective (see Lemma 1), it is boundedly invertible e.g. if $\dim \mathfrak{H}_1 < \infty$. Hence Theorem 1 implies the well-known results of Stenger [11] and Nudelman [10] about self-adjointness or maximal dissipativity of finite-codimensional compressions of a self-adjoint or maximal dissipative operator as well as corresponding results for maximal symmetric operators and dilations.

If $A$ and also its compression $A_0$ are self-adjoint, then $S := A \cap A_0$ is a symmetric operator in $\mathfrak{H}_0$ with equal defect numbers. Clearly, $A_0$ in $\mathfrak{H}_0$ is a canonical self-adjoint extension of $S$, and $A$ in $\mathfrak{H}$ is a self-adjoint extension of $S$ with exit from $\mathfrak{H}_0$ into the larger Hilbert space $\mathfrak{H}$. So if we choose for $S$ and its canonical self-adjoint extension $A_0$ a corresponding $\gamma$-field and $Q$-function, M.G. Krein's resolvent formula connects the compressed resolvent of $A$ with the resolvent of the compression $A_0$ through a parameter which is a (matrix or operator) Nevanlinna function (see the Appendix). If the $\gamma$-field and the Q-function are chosen properly and $\ker L(z) = \{0\}$, this parameter is the function $T(z) = zI$. In the general case this parameter is considered in Theorem 3. Finally, in Theorem 4 we extend Krein's resolvent formula for $A$ and $A_0$ to a model for the resolvent of $A$.

This note is a continuation of our studies in Refs. [3–5], but it can be read independently.

About notation: sometimes also for (single valued) operators $T$ we use the relation or subspace notation, that is the operator is described by its graph in the product space: instead of $y = Tx$ we write $\{x, y\} \in T$. Let $T$ be a densely defined operator on a Hilbert space $\mathfrak{H}$ with inner product $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$. $T$ is called *dissipative* if $\operatorname{Im} \langle Tf, f \rangle_{\mathfrak{H}} \geq 0$ for all $f \in \operatorname{dom} T$, the domain of $T$, and *maximal dissipative* if it is dissipative and not properly contained in another dissipative operator in $\mathfrak{H}$. If $T$ is dissipative, then it is maximal dissipative if and only if $\mathbb{C}_- \cap \rho(T) \neq \emptyset$, and then $\mathbb{C}_- \subset \rho(T)$. The operator $T$ is called *symmetric* if $T \subset T^*$, the adjoint of $T$ in $\mathfrak{H}$, and then the *upper/lower defect number* $n_\pm(T)$ is

$$n_\pm(T) := \dim(\operatorname{ran}(T-z))^\perp = \dim(\ker(T^* - z^*)), \quad z \in \mathbb{C}_\pm.$$

$T$ is called *maximal symmetric* if it is symmetric and not properly contained in another symmetric operator in $\mathfrak{H}$. If $T$ is symmetric, then it is maximal symmetric if and only if at least one of its defect numbers equals zero. Finally, $T$ is called *self-adjoint* if $T = T^*$ and this holds if and only if $T$ is symmetric and its defect numbers are zero. We assume that the reader is familiar with the spectral properties of such operators. We denote by $\rho(T)$ the resolvent set, by $\sigma(T)$ the spectrum, and by $\sigma_p(T)$ the point spectrum of $T$. An operator $A$ in a Hilbert space $\mathfrak{K}$ is called a *dilation* of $T$, if $\mathfrak{H}$ is a subspace of $\mathfrak{K}$, $\rho(A) \cap \rho(T) \neq \emptyset$ and $P_{\mathfrak{H}}(A-z)^{-1}|_{\mathfrak{H}} = (T-z)^{-1}$ for $z \in \rho(A) \cap \rho(T)$ (see [8]); here $P_{\mathfrak{H}}$ is the projection in $\mathfrak{K}$ onto $\mathfrak{H}$. The dilation $A$ is called *minimal* if for some $w \in \rho(A)$

$$\overline{\operatorname{span}}\Big\{ \big(I + (z-w)(A-z)^{-1}\big)h \,:\, z \in \rho(A),\ h \in \mathfrak{H} \Big\} = \mathfrak{K}.$$

Finally we recall the *Schur factorization* of a $2 \times 2$ block operator matrix of a bounded operator on a Hilbert space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ in which $D$ is boundedly invertible:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_0 \\ \mathfrak{H}_1 \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_0 \\ \mathfrak{H}_1 \end{pmatrix}.$$

The entry $A - BD^{-1}C$ is called the *first Schur complement* of the matrix on the left.

## 2 A general Stenger–Nudelman result

Let $A$ be a closed densely defined operator in a Hilbert space $\mathfrak{H}$ with a nonempty resolvent set $\rho(A)$ and resolvent operator $R(z) := (A-z)^{-1}$, $z \in \rho(A)$. We decompose $\mathfrak{H}$ into two orthogonal subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_1$: $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. Then the resolvent $R(z)$ can be decomposed as a $2 \times 2$ block operator matrix:

$$R(z) = \begin{pmatrix} S(z) & L(z) \\ M(z) & D(z) \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_0 \\ \mathfrak{H}_1 \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_0 \\ \mathfrak{H}_1 \end{pmatrix}, \quad z \in \rho(A). \tag{1}$$

It follows that, written as a relation,

$$A = \left\{ \left\{ \begin{pmatrix} S(z)f_0 + L(z)f_1 \\ M(z)f_0 + D(z)f_1 \end{pmatrix}, \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} + z \begin{pmatrix} S(z)f_0 + L(z)f_1 \\ M(z)f_0 + (z)f_1 \end{pmatrix} \right\} : f_0 \in \mathfrak{H}_0,\ f_1 \in \mathfrak{H}_1 \right\}. \tag{2}$$

Recall that the *compression* $A_0$ of $A$ to the space $\mathfrak{H}_0$ is the operator defined by

$$\begin{aligned} A_0 &:= P_{\mathfrak{H}_0} A\big|_{\mathfrak{H}_0 \cap \operatorname{dom} A_0} \\ &= \Big\{ \big\{ S(z)f_0 + L(z)f_1, f_0 + z(S(z)f_0 + L(z)f_1) \big\} : \\ &\qquad M(z)f_0 + D(z)f_1 = 0,\ f_0 \in \mathfrak{H}_0,\ f_1 \in \mathfrak{H}_1 \Big\}. \end{aligned} \tag{3}$$

**Lemma 1** *If $z \in \rho(A) \setminus \sigma_p(A_0)$, then $D(z)$ is injective.*

***Proof*** Assume $D(z)f_1 = 0$ for some $f_1 \in \mathfrak{H}_1$. Then with $f_0 = 0$ from the relation (3) we obtain $\{L(z)f_1, zL(z)f_1\} \in A_0$. The assumption $z \notin \sigma_p(A_0)$ implies $L(z)f_1 = 0$ and hence

$$R(z)\begin{pmatrix} 0 \\ f_1 \end{pmatrix} = \begin{pmatrix} L(z)f_1 \\ D(z)f_1 \end{pmatrix} = 0.$$

Apply $A - z$ to both sides of this equality to obtain $f_1 = 0$. □

**Theorem 1** *If $z \in \rho(A)$ and $D(z)$ in (1) is boundedly invertible, then $A_0$ is a closed densely defined operator in $\mathfrak{H}_0$ given by*

$$A_0 = \Big\{\{(S(z) - L(z)D(z)^{-1}M(z))f_0, \\ f_0 + z(S(z) - L(z)D(z)^{-1}M(z))f_0\} : f_0 \in \mathfrak{H}_0\Big\}. \tag{4}$$

*Moreover, $z \in \rho(A_0)$ and*

$$R_0(z) := (A_0 - z)^{-1} = S(z) - L(z)D(z)^{-1}M(z). \tag{5}$$

The relation (5) means that the resolvent of the compression $A_0$ of $A$ is the first Schur complement of the block operator matrix of the resolvent $R(z)$ of $A$ in (1).

***Proof of Theorem 1*** The relation (4) follows from (3). It implies that $A_0$ is closed and the equalities

$$\operatorname{dom} A_0 = \operatorname{ran}\left(S(z) - L(z)D(z)^{-1}M(z)\right) \tag{6}$$

and (5). The latter relation implies that $(A_0 - z)^{-1}$ is a bounded operator on $\mathfrak{H}_0$ and hence $z \in \rho(A_0)$. The Schur factorization of $R(z)$ takes the form

$$R(z) = U(z)\begin{pmatrix} S(z) - L(z)D(z)^{-1}M(z) & 0 \\ 0 & D(z) \end{pmatrix} V(z)$$

with

$$U(z) = \begin{pmatrix} I & L(z)D(z)^{-1} \\ 0 & I \end{pmatrix}, \quad V(z) = \begin{pmatrix} I & 0 \\ D(z)^{-1}M(z) & I \end{pmatrix}.$$

To show that $\operatorname{dom} A_0$ is dense in $\mathfrak{H}_0$ we assume that an element $g_0 \in \mathfrak{H}_0$ is orthogonal to $\operatorname{ran}\left(S(z) - L(z)D(z)^{-1}M(z)\right)$. Then we have in the inner product of $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ that for all $f_0 \in \mathfrak{H}_0$ and $f_1 \in \mathfrak{H}_1$

$$\begin{aligned}
&\left\langle R(z)\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, U(z)^{-*}\begin{pmatrix} g_0 \\ 0 \end{pmatrix}\right\rangle_{\mathfrak{H}} \\
&\quad = \left\langle U(z)\begin{pmatrix} S(z)-L(z)D(z)^{-1}M(z) & 0 \\ 0 & D(z) \end{pmatrix} V(z)\begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, U(z)^{-*}\begin{pmatrix} g_0 \\ 0 \end{pmatrix}\right\rangle_{\mathfrak{H}} \\
&\quad = \left\langle \begin{pmatrix} \big(S(z)-L(z)D(z)^{-1}M(z)\big)f_0 \\ M(z)f_0 + D(z)f_1 \end{pmatrix}, \begin{pmatrix} g_0 \\ 0 \end{pmatrix}\right\rangle_{\mathfrak{H}} \\
&\quad = 0.
\end{aligned}$$

Since $\operatorname{ran} R(z)$ is dense in $\mathfrak{H}$, $U(z)^{-*}\begin{pmatrix} g_0 \\ 0 \end{pmatrix} = 0$ and hence $g_0 = 0$. By (6), this proves that $\operatorname{dom} A_0$ is dense in $\mathfrak{H}_0$. □

The first and the third of the following corollaries of Theorem 1 contain the results of Nudelman [10] and Stenger [11] (see also [1, Section 3], [2, Sections 3 and 4] and [6, Theorem 3.3]), and the fourth corollary contains the operator case of [2, Theorem 5.3]. These references concern the case $\dim \mathfrak{H}_1 < \infty$. Under this assumption Lemma 1 assures the invertibility of $D(z)$.

**Corollary 1** *Assume that $T$ is a densely defined maximal dissipative operator in the space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ with the block matrix representation (1) of its resolvent. If, for some $z \in \mathbb{C}_-$, $D(z)$ is boundedly invertible, then the compression $T_0$ of $T$ to $\mathfrak{H}_0$ is densely defined and maximal dissipative in $\mathfrak{H}_0$.*

**Corollary 2** *Assume that $S$ is a densely defined maximal symmetric operator with lower defect number $n_-(S) = 0$ (upper defect number $n_+(S) = 0$) in $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. Suppose that the block matrix representation of the resolvent of $S$ is given by the right-hand side of (1). If $D(z)$ is boundedly invertible for some $z \in \mathbb{C}_-$ $(z \in \mathbb{C}_+)$, then the compression $S_0$ of $S$ to $\mathfrak{H}_0$ is a densely defined maximal symmetric operator with $n_-(S_0) = 0$ $(n_+(S_0) = 0)$ in $\mathfrak{H}_0$.*

**Corollary 3** *Assume that $A$ is a densely defined self-adjoint operator in $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ with the block matrix representation (1) of the resolvent. If $D(z)$ is boundedly invertible for some $z \in \rho(A)$, then the compression $A_0$ of $A$ to $\mathfrak{H}_0$ is a densely defined self-adjoint operator in $\mathfrak{H}_0$ and $z \in \rho(A_0)$.*

As to the proof of Corollary 3, by the observation preceeding the relation (11) below, $D(z)$ is boundedly invertible on an open subset of $\rho(A)$ around $z$ and $z^*$. By Theorem 1, this set is also contained in $\rho(A_0)$. Hence the symmetric operator $A_0$ is in fact self-adjoint.

**Corollary 4** *Assume that $T$ is a densely defined maximal dissipative operator in the space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ with the block matrix representation (1) of its resolvent in which $D(z)$ is boundedly invertible for some $z \in \mathbb{C}_-$. If the operator $A$ in the Hilbert space $\mathfrak{K}$ is a minimal self-adjoint dilation of $T$, then its compression $A_0$ to the space $\mathfrak{K} \ominus \mathfrak{H}_1$ is a minimal self-adjoint dilation of the compression $T_0$ of $T$ to the space $\mathfrak{H}_0$.*

# 3 Resolvent formulas based on the compression of a self-adjoint operator

## 3.1 A first decomposition

In this subsection let $A$ be a self-adjoint operator in the Hilbert space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$. With respect to this decomposition of the space $\mathfrak{H}$ we write again

$$R(z) := (A - zI)^{-1} = \begin{pmatrix} S(z) & L(z) \\ M(z) & D(z) \end{pmatrix}, \quad z \in \rho(A). \tag{7}$$

The relation $R(z)^* = R(z^*)$ implies

$$S(z)^* = S(z^*), \quad D(z)^* = D(z^*), \quad L(z)^* = M(z^*), \quad z \in \rho(A). \tag{8}$$

Moreover, the resolvent equation

$$\frac{R(z) - R(w)^*}{z - w^*} = R(w)^* R(z), \quad z,\ w \in \rho(A),$$

is equivalent to the relations

$$\begin{aligned} \frac{S(z) - S(w)^*}{z - w^*} &= S(w)^* S(z) + L(w^*) M(z), \quad z,\ w \in \rho(A), \\ \frac{L(z) - L(w^*)}{z - w^*} &= S(w)^* L(z) + L(w^*) D(z), \quad z,\ w \in \rho(A), \end{aligned} \tag{9}$$

$$\frac{D(z) - D(w)^*}{z - w^*} = D(w)^* D(z) + L(w)^* L(z), \quad z,\ w \in \rho(A). \tag{10}$$

Now we assume that $D(z)$ is boundedly invertible for some $z \in \rho(A)$. As an analytic function of $z$ it is also boundedly invertible in a neighborhood of $z$ and because of (8) also for $z^*$. For those points $z$, $w$ the relation (10) implies

$$\frac{D(w)^{-*} - D(z)^{-1}}{z - w^*} = I + \left(L(w) D(w)^{-1}\right)^* L(z) D(z)^{-1}. \tag{11}$$

We introduce the operator functions

$$Q(z) := -D(z)^{-1} - z, \quad \Gamma(z) := L(z) D(z)^{-1}. \tag{12}$$

Then (5) and (8) imply that $R_0(z) = (A_0 - z)^{-1}$ is given by

$$R_0(z) = S(z) + L(z)(Q(z) + z) L(z^*)^* = S(z) + \Gamma(z)(Q(z) + z)^{-1} \Gamma(z^*)^*. \tag{13}$$

**Theorem 2** *Let $A$ be a self-adjoint operator in the Hilbert space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ with the block matrix representation (1) of the resolvent. Suppose that for some $z \in \rho(A)$ the operator $D(z)$ is boundedly invertible. Then, with the operator functions $Q(z)$*

*and* $\Gamma(z)$ *from* (12) *and the compression* $A_0 = P_{\mathfrak{H}_0} A|_{\mathfrak{H}_0 \cap \operatorname{dom} A}$, *the matrix representation* (7) *takes the form*

$$(A-z)^{-1} = \begin{pmatrix} (A_0 - z)^{-1} - \Gamma(z)(Q(z)+z)^{-1}\Gamma(z^*)^* & -\Gamma(z)(Q(z)+z)^{-1} \\ -(Q(z)+z)^{-1}\Gamma(z^*)^* & -(Q(z)+z)^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} (A_0-z)^{-1} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \Gamma(z) \\ I \end{pmatrix} (Q(z)+z)^{-1} \begin{pmatrix} \Gamma(z^*)^* & I \end{pmatrix}. \tag{14}$$

*The functions* $Q(z)$ *and* $\Gamma(z)$ *satisfy the relations*

$$\frac{Q(z)-Q(w)^*}{z-w^*} = \Gamma(w)^*\Gamma(z), \quad (A_0 - w)^{-1}\Gamma(z) = \frac{\Gamma(z)-\Gamma(w)}{z-w}, \quad z,\ w \in \rho(A_0). \tag{15}$$

***Proof*** The equality (14) follows from (7), (12) and (13). It remains to prove the relations (15). The first relation follows from (11) and (12). To prove the second one, we use (9), (13) and (11) to obtain

$$\begin{aligned} \frac{L(z)-L(w^*)}{z-w^*} &= R_0(w^*)L(z) - L(w^*)(Q(w)+w)L(w)^*L(z) + L(w^*)D(z) \\ &= R_0(w^*)L(z) + L(w^*)D(w)^{-*}L(w)^*L(z) + L(w^*)D(z) \\ &= R_0(w^*)L(z) + L(w^*)\big(D(w)^{-*}L(w)^*L(z)D(z)^{-1}\big)D(z) \\ &\quad + L(w^*)D(z) \\ &= R_0(w^*)L(z) + L(w^*)\frac{D(w)^{-*} - D(z)^{-1}}{z-w^*}D(z) \\ &= R_0(w^*)L(z) + \frac{L(w^*)D(w)^{-*}D(z) - L(w^*)}{z-w^*}. \end{aligned}$$

This implies

$$\frac{L(z)}{z-w^*} = R_0(w^*)L(z) + \frac{L(w^*)D(w)^{-*}D(z)}{z-w^*},$$

or

$$R_0(w^*)L(z)D(z)^{-1} = \frac{L(z)D(z)^{-1} - L(w^*)D(w^*)^{-1}}{z-w^*},$$

which yields the second equality in (15). □

The left upper corner in the first matrix in (14) is in general not yet the right-hand side of a Krein resolvent formula (see the Appendix) since $\Gamma(z)$ may have a nontrivial kernel. In the next subsection we replace $\Gamma(z)$ by $\Gamma_z$ being injective.

### 3.2 Krein's resolvent formula

In the following we establish a connection between (14) with Krein's resolvent formula (see the Appendix). Assume that the conditions of Theorem 2 are satisfied. The second equality in (15) implies that the kernel $\ker \Gamma(z)$ of $\Gamma(z)$ is independent of $z$. We decompose $\mathfrak{H}_1 = \mathfrak{H}_{1,1} \oplus \mathfrak{H}_{1,2}$ with $\mathfrak{H}_{1,2} := \ker \Gamma(z)$. Then $\Gamma(z)$ and $Q(z)$ have the block matrix representation

$$\Gamma(z) = \begin{pmatrix} \Gamma_z & 0 \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_{1,1} \\ \mathfrak{H}_{1,2} \end{pmatrix} \to \mathfrak{H}_0 \tag{16}$$

with $\ker \Gamma_z = \{0\}$, and

$$Q(z) = \begin{pmatrix} Q_{11}(z) & Q_{12} \\ Q_{12}^* & Q_{22} \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_{1,1} \\ \mathfrak{H}_{1,2} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_{1,1} \\ \mathfrak{H}_{1,2} \end{pmatrix}. \tag{17}$$

By the first equality in (15), the entry $Q_{11}(z)$ in the representation of $Q(z)$ is a bounded operator function satisfying

$$\frac{Q_{11}(z) - Q_{11}(w)^*}{z - w^*} = \Gamma_w^* \Gamma_z, \tag{18}$$

the other two entries $Q_{12}$ and $Q_{22}$ are bounded operators independent of $z$, and $Q_{22} = Q_{22}^*$.

**Theorem 3** *In the situation of Theorem 2, the operator $S := A_0 \cap A = A \cap \mathfrak{H}_0^2$ is symmetric in $\mathfrak{H}_0$ with equal defect numbers* $\dim \mathfrak{H}_{1,1}$. *For the canonical self-adjoint extension $A_0$ of $S$ in $\mathfrak{H}_0$ and the self-adjoint extension $A$ of $S$ in $\mathfrak{H}$ the following formula holds*:

$$P_{\mathfrak{H}_0}(A - z)^{-1}|_{\mathfrak{H}_0} = (A_0 - z)^{-1} - \Gamma_z(Q_{11}(z) + T(z))^{-1}\Gamma_{z^*}^* \tag{19}$$

*with the Nevanlinna function*

$$T(z) := z - Q_{12}(Q_{22} + z)^{-1}Q_{12}^*. \tag{20}$$

*Here $\Gamma_z$ is a $\gamma$-field and $Q_{11}(z)$ is a corresponding $Q$-function for the symmetric operator $S$ and its canonical self-adjoint extension $A_0$.*

Clearly, (19) is a Krein resolvent formula, where the function $T(z)$ plays the role of the parameter. In the particular case $\ker \Gamma(z) = \{0\}$, that is $\ker L(z) = \{0\}$, this parameter becomes $T(z) = zI$. Formally, in Krein's resolvent formula, on the left-hand side $A$ is often replaced by the minimal self-adjoint operator in $\mathfrak{H}$ which contains the restriction of $A$ to $\mathfrak{H}_0 \cap \operatorname{dom} A$.

***Proof of Theorem 3*** Since $A$ is a self-adjoint operator, $S$ is a closed symmetric operator in $\mathfrak{H}_0$. From (2) we obtain that

$$S = A \cap \mathfrak{H}_0^2 = \left\{\{S(z)f_0, f_0 + zS(z)f_0\} : M(z)f_0 = 0,\ f_0 \in \mathfrak{H}_0\right\} = A \cap A_0.$$

From $S \subset A_0 = A_0^*$ it follows that $S$ has equal defect numbers.

By Theorem 2, $\operatorname{ran}(S-z) = \ker M(z) = \ker \Gamma(z^*)^*$. The decomposition (16) implies that the defect numbers are equal to the dimension of the space $\mathfrak{H}_{1,1}$:

$$\ker(S^* - z) = \left(\operatorname{ran}(S-z^*)\right)^\perp = \left(\ker \Gamma(z)^*\right)^\perp = \overline{\operatorname{ran}}\,\Gamma(z) = \overline{\operatorname{ran}}\,\Gamma_z = \mathfrak{H}_{1,1}. \tag{21}$$

The relations $T(z)^* = T(z^*)$ and

$$\frac{\operatorname{Im} T(z)}{\operatorname{Im} z} = I + Q_{12}^*(Q_{22}+z^*)^{-1}(Q_{22}+z)^{-1}Q_{12} \geq 0, \quad z \in \mathbb{C}\backslash\mathbb{R},$$

show that $T(z)$ in (20) is an operator Nevanlinna function. The equality (19) is obtained from Theorem 2, the relation (16) and the relation

$$\begin{aligned}\Gamma(z)\left(Q(z)+z\right)^{-1}\Gamma(z^*)^* &= \Gamma_z\left(Q_{11}(z)+z-Q_{12}(Q_{22}+z)^{-1}Q_{12}^*\right)^{-1}\Gamma_{z^*}^* \\ &= \Gamma_z\left(Q_{11}(z)+T(z)\right)^{-1}\Gamma_{z^*}^*,\end{aligned}$$

which follows from the form of the inverse of the $2\times 2$ block matrix for $Q(z)+z$.

To prove the last statement we only need to show (see the Appendix), that $\Gamma_z$ maps $\mathfrak{H}_{1,1}$ into $\ker(S^*-z)$ and has zero kernel, $\Gamma_z = (I+(z-w)(A_0-z)^{-1})\Gamma_w$ and $Q_{11}(z) - Q_{11}(w)^* = (z-w^*)\Gamma_w^*\Gamma_z$, $z, w \in \mathbb{C}\backslash\mathbb{R}$. But this follows from (21), the second equality in (15) and (18). □

We end this subsection with a simple example. Let the self-adjoint operator $A$ in $\mathbb{C}^3$ be given by the symmetric matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_0 \\ \mathfrak{H}_{1,1} \\ \mathfrak{H}_{1,2} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_0 \\ \mathfrak{H}_{1,1} \\ \mathfrak{H}_{1,2} \end{pmatrix} \quad \text{with} \quad \mathfrak{H}_0 = \mathfrak{H}_{1,1} = \mathfrak{H}_{1,2} = \mathbb{C}.$$

Then, with $d(z) := (z-1)(-z^2+z+2)$,

$$L(z) = M(z^*)^* = \frac{1}{d(z)}\begin{pmatrix} z-1 & 1 \end{pmatrix}, \quad D(z) = \frac{1}{d(z)}\begin{pmatrix} (z-1)^2 & z-1 \\ z-1 & z^2-z-1 \end{pmatrix}.$$

Hence

$$\begin{aligned} D(z)^{-1} &= -(Q(z)+z) \quad \text{with} \quad Q(z) = \begin{pmatrix} \dfrac{1}{1-z} & -1 \\ -1 & -1 \end{pmatrix}, \\ \Gamma(z) &= L(z)D(z)^{-1} = \begin{pmatrix} \dfrac{1}{z-1} & 0 \end{pmatrix}, \quad \Gamma_z = \frac{1}{z-1}. \end{aligned}$$

and

$$\frac{1}{1-z} = (A_0 - z)^{-1} = S(z) - \frac{1}{d(z)}.$$

### 3.3 A refined decomposition

In analogy to [4, Theorem 2.4] and [5, Proposition 3.3], the formulas in Theorem 2 and Theorem 3 can be given a more symmetric form, which is at the same time a refinement with respect to the self-adjoint parts of the operator $A$ in $\mathfrak{H}_1$. To this end with the function $T(z)$ in (20):

$$T(z) = z - Q_{12}(Q_{22} + z)^{-1} Q_{12}^* \tag{22}$$

we associate the following operator model:

(i) $\mathfrak{H}_T$ is the Hilbert space $\mathfrak{H}_T = \mathfrak{H}_{1,1} \oplus \widehat{\mathfrak{H}}_{1,2} \subset \mathfrak{H}_1$ where

$$\widehat{\mathfrak{H}}_{1,2} = \overline{\operatorname{span}} \left\{ (Q_{22} + z)^{-1} Q_{12}^* f_{11} \ : f_{11} \in \mathfrak{H}_{1,1},\ z \in \mathbb{C} \backslash \mathbb{R} \right\} \subset \mathfrak{H}_{1,2},$$

(ii) $B_T$ is the self-adjoint relation in $\mathfrak{H}_T$ with resolvent

$$R_T(z) := (B_T - z)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & -(Q_{22} + z)^{-1} \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_{1,1} \\ \widehat{\mathfrak{H}}_{1,2} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_{1,1} \\ \widehat{\mathfrak{H}}_{1,2} \end{pmatrix}, \quad z \in \mathbb{C} \backslash \mathbb{R},$$

(iii) $\delta_z$ is the operator function

$$\delta_z = \begin{pmatrix} I \\ -(Q_{22} + z)^{-1} Q_{12}^* \end{pmatrix} : \mathfrak{H}_{1,1} \to \begin{pmatrix} \mathfrak{H}_{1,1} \\ \widehat{\mathfrak{H}}_{1,2} \end{pmatrix}, \quad z \in \mathbb{C} \backslash \mathbb{R}.$$

Note that $\widehat{\mathfrak{H}}_{1,2}$ contains $\operatorname{ran} Q_{12}^*$ and that $Q_{22}$ maps $\widehat{\mathfrak{H}}_{1,2}$ to $\widehat{\mathfrak{H}}_{1,2}$ and is bounded.

The proof of the following proposition is straightforward and therefore omitted.

**Proposition 1** *The operator Nevanlinna function $T(z)$ from (22) in the space $\mathfrak{H}_{1,1}$ has the representation*

$$T(z) = T(w)^* + (z - w^*)\delta_w^* \left( I + (z - w^*)(B_T - z)^{-1} \right) \delta_{w^*}, \quad z,\ w \in \mathbb{C} \backslash \mathbb{R},$$

*which is minimal in the sense that*

$$\mathfrak{H}_T = \overline{\operatorname{span}} \left\{ \delta_z h_{11} \ : \ h_{11} \in \mathfrak{H}_{1,1},\ z \in \mathbb{C} \backslash \mathbb{R} \right\}. \tag{23}$$

*Moreover, for $z, w \in \mathbb{C} \backslash \mathbb{R}$*

$$\frac{T(z) - T(w)^*}{z - w^*} = \delta_w^* \delta_z, \quad \delta_z = \left( I + (z - w)(B_T - z)^{-1} \right) \delta_w.$$

In the following we set $\mathfrak{H}'_{1,2} := \mathfrak{H}_{1,2} \ominus \widehat{\mathfrak{H}}_{1,2}$. From the inclusion $\operatorname{ran} Q^*_{1,2} \subset \widehat{\mathfrak{H}}_{1,2}$ it follows that $Q_{1,2}\mathfrak{H}'_{1,2} = \{0\}$. Since $Q_{22}$ maps $\widehat{\mathfrak{H}}_{1,2}$ to $\widehat{\mathfrak{H}}_{1,2}$ and is self-adjoint on $\mathfrak{H}_{1,2}$, $Q_{22}$ has a diagonal form with respect to the decomposition $\mathfrak{H}_{1,2} = \widehat{\mathfrak{H}}_{1,2} \oplus \mathfrak{H}'_{1,2}$:

$$Q_{22} = \begin{pmatrix} \widehat{Q}_{22} & 0 \\ 0 & Q'_{22} \end{pmatrix} : \begin{pmatrix} \widehat{\mathfrak{H}}_{1,2} \\ \mathfrak{H}'_{1,2} \end{pmatrix} \to \begin{pmatrix} \widehat{\mathfrak{H}}_{1,2} \\ \mathfrak{H}'_{1,2} \end{pmatrix}.$$

This implies that the resolvent $R_T(z)$ can be written as

$$R_T(z) := \begin{pmatrix} 0 & 0 \\ 0 & (-\widehat{Q}_{22} - z)^{-1} \end{pmatrix} : \begin{pmatrix} \mathfrak{H}_{1,1} \\ \widehat{\mathfrak{H}}_{1,2} \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}_{1,1} \\ \widehat{\mathfrak{H}}_{1.2} \end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

The theorem below shows that in general $A$ need not be $\mathfrak{H}_0$-minimal with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ in the sense that for some $w \in \mathbb{C}\backslash\mathbb{R}$

$$\mathfrak{H} = \overline{\operatorname{span}}\left\{ \left(I + (z-w)(A-z)^{-1}\right)\begin{pmatrix} h_0 \\ 0 \end{pmatrix} \,:\, h_0 \in \mathfrak{H}_0,\ z \in \mathbb{C}\backslash\mathbb{R} \right\}.$$

In fact the theorem implies that the gap $\mathfrak{H} \ominus (\mathfrak{H}_0 \oplus \mathfrak{H}_T) = \mathfrak{H}'_{1,2}$ between the space on the right-hand side and $\mathfrak{H}$ is an invariant subspace for $A$ on which $A$ coincides with the self-adjoint operator $-Q'_{22}$.

**Theorem 4** *Under the conditions of Theorem 2 and with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_T \oplus \mathfrak{H}'_{1,2}$ the resolvent $(A-z)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, has the $3 \times 3$ block matrix representation*

$$\begin{aligned}(A-z)^{-1} &= \begin{pmatrix} R_0(z) - \Gamma_z\,\Delta(z)^{-1}\Gamma^*_{z^*} & -\Gamma_z\,\Delta(z)^{-1}\delta^*_{z^*} & 0 \\ -\delta_z\,\Delta(z)^{-1}\Gamma^*_{z^*} & R_T(z) - \delta_z\,\Delta(z)^{-1}\delta^*_{z^*} & 0 \\ 0 & 0 & (-Q'_{22}-z)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} R_0(z) & 0 & 0 \\ 0 & R_T(z) & 0 \\ 0 & 0 & (-Q'_{22}-z)^{-1} \end{pmatrix} - \begin{pmatrix} \Gamma_z \\ \delta_z \\ 0 \end{pmatrix} \Delta(z)^{-1} \begin{pmatrix} \Gamma^*_{z^*} & \delta^*_{z^*} & 0 \end{pmatrix},\end{aligned} \tag{24}$$

*where $\Delta(z) := Q_{11}(z) + T(z)$. Moreover, for each $w \in \mathbb{C}\backslash\mathbb{R}$*

$$\overline{\operatorname{span}}\left\{ \left(I + (z-w)(A-z)^{-1}\right)\begin{pmatrix} h_0 \\ 0 \end{pmatrix} \,:\, h_0 \in \mathfrak{H}_0,\ z \in \mathbb{C}\backslash\mathbb{R} \right\} = \begin{pmatrix} \mathfrak{H}_0 \\ \mathfrak{H}_T \end{pmatrix}, \tag{25}$$

*and under the identification of $\mathfrak{H} \ominus (\mathfrak{H}_0 \oplus \mathfrak{H}_T)$ with $\mathfrak{H}'_{1,2}$ the restriction of the operator $A$ to $\mathfrak{H} \ominus (\mathfrak{H}_0 \oplus \mathfrak{H}_T)$ coincides with the self-adjoint operator $-Q'_{22}$ on $\mathfrak{H}'_{1,2}$.*

***Proof*** The first equality in (24) follows from Theorem 2, the decompositions (16) and (17) and the inverse of the Schur factorization of $Q(z)+z$. We find relative to the decomposition $\mathfrak{H}_1=\mathfrak{H}_{1,1}\oplus\mathfrak{H}_{1,2}$ and with $X(z):=Q_{12}(Q_{22}+z)^{-1}$ the relation

$$\begin{aligned}-D(z)&=\begin{pmatrix}Q_{11}(z)+z & Q_{12}\\ Q_{12}^* & Q_{22}+z\end{pmatrix}^{-1}\\&=\begin{pmatrix}\Delta(z)^{-1} & \Delta(z)^{-1}X(z)\\ X(z^*)^*\Delta(z)^{-1} & X(z^*)^*\Delta(z)^{-1}X(z)+(Q_{22}+z)^{-1}\end{pmatrix}.\end{aligned}$$

Now we write $\mathfrak{H}_{1,2}=\widehat{\mathfrak{H}}_{1,2}\oplus\mathfrak{H}'_{1,2}$ and use that, since the operator $X(z^*)^*=(Q_{22}+z)^{-1}Q_{12}^*$ maps $\mathfrak{H}_{1,1}$ to $\widehat{\mathfrak{H}}_{1,2}\subset\mathfrak{H}_{1,2}$,

$$X(z)\mathfrak{H}'_{1,2}=Q_{12}(Q_{22}+z)^{-1}\mathfrak{H}'_{1,2}=\{0\},$$

to obtain with respect to the decomposition $\mathfrak{H}_1=\mathfrak{H}_{1,1}\oplus\widehat{\mathfrak{H}}_{1,2}\oplus\mathfrak{H}'_{1,2}$

$$-D(z)=\begin{pmatrix}\Delta(z)^{-1} & \Delta(z)^{-1}X(z) & 0\\ X(z^*)^*\Delta(z)^{-1} & X(z^*)^*\Delta(z)^{-1}X(z)+(\widehat{Q}_{22}+z)^{-1} & 0\\ 0 & 0 & (Q'_{22}+z)^{-1}\end{pmatrix}.$$

A straightforward calculation shows that the left upper $2\times 2$ block matrix in this $3\times 3$ matrix is the block matrix representation of the operator

$$\delta_z\,\Delta(z)^{-1}\delta_{z^*}^*-R_T(z):\mathfrak{H}_T\to\mathfrak{H}_T$$

relative to the decomposition $\mathfrak{H}_T=\mathfrak{H}_{1,1}\oplus\widehat{\mathfrak{H}}_{1,2}$. Hence relative to this decomposition of $\mathfrak{H}_T$ we have

$$D(z)=\begin{pmatrix}R_T(z)-\delta_z\,\Delta(z)^{-1}\delta_{z^*}^* & 0\\ 0 & -(Q'_{22}+z)^{-1}\end{pmatrix}.$$

In a similar way we find that

$$L(z)=M(z^*)^*=-\Gamma_z\,\Delta(z)^{-1}\begin{pmatrix}\delta_{z^*}^* & 0\end{pmatrix},$$

and that $S(z)=P_{\mathfrak{H}_0}(A-z)^{-1}|_{\mathfrak{H}_0}$ is as in (19). The second equality in (24) follows from the first one. As to the equality (25), it holds if and only if

$$\overline{\operatorname{span}}\left\{\delta_z\Delta(z)^{-1}\Gamma_{z^*}^*\begin{pmatrix}h_0\\0\end{pmatrix}\,:\,h_0\in\mathfrak{H}_0,\ z\in\mathbb{C}\backslash\mathbb{R}\right\}=\mathfrak{H}_T.$$

Denote the space on the left-hand side by $\mathfrak{G}$. Then, by (23), $\mathfrak{G}\subset\mathfrak{H}_T$. We prove the reverse inclusion. For fixed $z\in\mathbb{C}\backslash\mathbb{R}$ the range of $\Gamma_{z^*}^*$ is dense in $\mathfrak{H}_{1,1}$, and therefore $\Delta(z)^{-1}\Gamma_{z^*}^*\mathfrak{H}_0$ is also dense in $\mathfrak{H}_{1,1}$. Since $\delta_z$ is continuous, we see that $\delta_z\mathfrak{H}_{1,1}$ belongs to $\mathfrak{G}$ for every $z\in\mathbb{C}\backslash\mathbb{R}$. By (23), $\mathfrak{H}_T\subset\mathfrak{G}$. This proves the third equality.

We prove the last statement using the identification of the space $\mathfrak{H} \ominus (\mathfrak{H}_0 \oplus \mathfrak{H}_T)$ with the space $\mathfrak{H}'_{1,2}$. Let $h \in \mathfrak{H}'_{1,2}$ and set $g = (-Q'_{22} - z)h$. Then $g \in \mathfrak{H}'_{1,2}$. If we apply both sides of the equality (24) to $g$ then we obtain

$$(A - z)^{-1} g = (-Q'_{22} - z)^{-1} g = h.$$

Hence $h \in \operatorname{dom} A$ and $Ah = -Q'_{22} h$. □

## Compliance with ethical standards

**Conflict of interest** The authors declare that they have no conflict of interest.



## Appendix

In the following we recall Krein's resolvent formula from Refs. [7] and [9] as needed in this paper. Let $S$ be a closed densely defined symmetric operator in a Hilbert space $\mathfrak{H}_0$ with equal defect numbers $n = n_-(S) = n_+(S) \leq \infty$. Let $A_0$ be a self-adjoint extension of $S$ in $\mathfrak{H}_0$. Let $\mathfrak{G}$ be a Hilbert space with $\dim \mathfrak{G} = n$. Fix a point $z_0 \in \rho(A_0)$, a bijection $\Gamma_{z_0} : \mathfrak{G} \to \ker(S^* - z_0)$ and define the so called *$\gamma$-field*

$$\Gamma_z := (I + (z - z_0)(A_0 - z)^{-1})\Gamma_{z_0}, \quad z \in \rho(A_0).$$

Then $\Gamma_z$ is a bounded bijection from $\mathfrak{G}$ onto $\ker(S^* - z)$ and satisfies the relation

$$\Gamma_z = (I + (z - w)(A_0 - z)^{-1})\Gamma_w, \quad z,\ w \in \rho(A_0).$$

Associate with $\Gamma_z$ a so called *Q-function* $Q(z)$. It is a bounded operator on $\mathfrak{G}$, defined for $z \in \rho(A_0)$ and it satisfies the relation

$$\frac{Q(z) - Q(w)^*}{z - w^*} = \Gamma_w^* \Gamma_z, \quad z,\ w \in \rho(A_0).$$

This relation uniquely defines $Q(z)$ up to an additive bounded self-adjoint operator on $\mathfrak{G}$. Let $A$ be a self-adjoint extension of $S$ in a Hilbert space $\mathfrak{H} \supset \mathfrak{H}_0$. The function

$$P_{\mathfrak{H}_0}(A - z)^{-1}|_{\mathfrak{H}_0},$$

where $P_{\mathfrak{H}_0}$ is the projection in $\mathfrak{H}$ onto $\mathfrak{H}_0$, is defined for $z \in \rho(A)$ and is a bounded operator on $\mathfrak{H}_0$. It is called a *generalized resolvent* of $S$. Krein's resolvent formula

$$P_{\mathfrak{H}_0}(A-z)^{-1}|_{\mathfrak{H}_0} = (A_0 - z)^{-1} - \Gamma_z(Q(z)+T(z))^{-1}\Gamma_{z^*}^*$$

establishes a one-to-one correspondence between the generalized resolvents of $S$ corresponding to self-adjoint extensions $A$ of $S$ satisfying $A \cap A_0 = S$ and the *operator Nevanlinna functions* $T(z)$ on $\mathfrak{G}$. The latter are bounded operators on $\mathfrak{G}$, defined for and holomorphic in $z \in \mathbb{C}\backslash\mathbb{R}$ and satisfy the relations

$$T(z^*) = T(z)^*, \quad \frac{T(z)-T(z)^*}{z-z^*} \geq 0, \quad z \in \mathbb{C}\backslash\mathbb{R}.$$

For example $Q(z)$ is a Nevanlinna function with the property

$$\frac{Q(z)-Q(z)^*}{z-z^*} > 0, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

If in Krein's formula the assumption $A \cap A_0 \supset S$ holds, then the operator Nevanlinna functions $T(z)$ have to be replaced by relation Nevanlinna functions, see Ref. [9].

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