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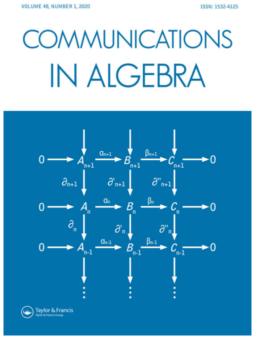

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The level of pairs of polynomials

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ABSTRACT
Given a polynomial f with coefficients in a field of prime characteristic p, it is known that there exists a differential operator that raises $1/f$ to its p^{th} power. We first discuss a relation between the "level" of this differential operator and the notion of "stratification" in the case of hyperelliptic curves. Next, we extend the notion of level to that of a pair of polynomials. We prove some basic properties and we compute this level in certain special cases. In particular, we present examples of polynomials g and f such that there is no differential operator raising g/f to its p^{th} power.

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1. Introduction

Let k be any perfect field and $R = k[x_1, \ldots, x_d]$ its polynomial ring in d variables. In this case, it is known [12, IV, Théorème 16.11.2] that the ring $\mathcal{D}_R$ of k-linear differential operators on R is the R-algebra (which we take here as a definition)

$$\mathcal{D}_R := R\langle D_{x_i,t} \mid i = 1, \ldots, d \text{ and } t \geq 1\rangle \subseteq \operatorname{End}_k(R),$$

generated by the operators $D_{x_i,t}$, defined as

$$D_{x_i,t}(x_j^s) = \begin{cases} \binom{s}{t} x_i^{s-t}, & \text{if } i = j \text{ and } s \geq t, \\ 0, & \text{otherwise.} \end{cases}$$

For a non-zero $f \in R$, let R_f be the localization of R at f; the natural action of $\mathcal{D}_R$ on R extends uniquely to R_f and it is known that $m \geq 1$ exists such that $R_f = \mathcal{D}_R \frac{1}{f^m}$. In characteristic 0, there are examples where the minimal such m is strictly larger than 1 (e.g. [14, Example 23.13]). On the other hand, if $\operatorname{char}(k) = p > 0$, one may always take $m = 1$ ([1, Theorem 3.7 and Corollary 3.8]). This is shown by proving the existence of a differential operator $\delta \in \mathcal{D}_R$ such that $\delta(1/f) = 1/f^p$, that is, δ acts as Frobenius on $1/f$. We want to mention here that the existence of this

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differential operator was used as key ingredient in [3] to prove that local cohomology modules over smooth $\mathbb{Z}$-algebras have finitely many associated primes. On the other hand, the fact that R_f is generated by $1/f$ as $\mathcal{D}_R$-module remains valid for more general classes of rings R: the interested reader may consult [1, Theorems 4.1 and 5.1], [13, Theorem 3.1], [26, Corollary 2.10 and Remark 2.11], and [2, Theorem 4.4] for details.

We will suppose that k is a perfect field of positive characteristic p, and we fix an algebraic closure $\bar{k}$ of k. For an integer $e \geq 0$, let $R^{p^e} \subseteq R$ be the subring of all the p^e powers of all the elements of R and set $\mathcal{D}_R^{(e)} := \mathrm{End}_{R^{p^e}}(R)$, the ring of -linear ring-endomorphism of R. Since R is a finitely generated R^p-module, by [29, 1.4.8 and 1.4.9], it is

$$\mathcal{D}_R = \bigcup_{e \geq 0} \mathcal{D}_R^{(e)}.$$

Therefore, for $\delta \in \mathcal{D}_R$, there exists $e \geq 0$ such that $\delta \in \mathcal{D}_R^{(e)}$ but $\delta \notin \mathcal{D}_R^{(e')}$ for any $e' < e$. This number e is called the level of δ. For a polynomial f, the level is defined as the lowest level of an operator δ such that $\delta(1/f) = 1/f^p$.

The level of a polynomial has been studied in [1, 6, 4]. In particular, results were established relating the level of a polynomial defining a (hyper)elliptic curve to p-torsion of the Jacobian; see also Section 2.

By [7, §4.4 and 4.5], the level of a polynomial f is closely related to the so-called Hartshorne–Speiser–Lyubeznik–Gabber number of the pair (R, f), and the latter number can be explicitly calculated using Macaulay2. On the other hand, one can also calculate the level of f in terms of F-jumping numbers [11, Proposition 6].

One of the goals of this article is to introduce and study the level of a pair of polynomials. Given f, g polynomials defined over $\mathbb{F}_p$, one may ask whether there is a differential operator $\delta \in \mathcal{D}_R$ mapping g/f to $(g/f)^p$. Such an operator exists when $g = 1$ by [1, Theorem 3.7 and Corollary 3.8], and more generally, when f itself has level one, as pointed out in [6]. Keeping in mind all of this, it seems natural to define the level of g and f as

$$\mathrm{level}(g, f) := \inf\left\{ e \geq 0 : \exists \delta \in \mathcal{D}^{(e)} \text{ such that } \delta(g/f) = (g/f)^p \right\}.$$

As we already mentioned, our goal in this article is to study this notion, and to calculate it in several interesting examples.

Part of our motivation for introducing it comes from [25], where the author gave a conceptual proof of a polynomial identity obtained in [24, Lemma 3.1] using hypergeometric series algorithms. This polynomial identity, and the corresponding results obtained by Singh concerning associated primes of local cohomology modules [24] were the basis of [20], where the authors proved, among other remarkable results, that local cohomology modules $H^m_{I_t(X)}(\mathbb{Z}[X])$ are rational vector spaces for any $m > \mathrm{height}(I_t(X))$, where X is a matrix of indeterminates, and $I_t(X)$ is the ideal of size t minors of this matrix [20, Theorem 1.2]. The proof presented in [25] used as key ingredient certain differential operators defined over the integers that, modulo a prime p, act as the Frobenius endomorphism on quotients of polynomials [25, page 244].

Another motivation comes from [9], where the authors use higher order differential operators to measure various kind of singularities in all characteristics. These higher order operators also play a key role in recent developments in the study of symbolic powers of ideals (see [10] and [9, Section 10] for details). We hope that the calculation of the level of a pair of polynomials might help in the understanding of these differential operators. The interplay between differential operators over the integers and their reduction modulo a prime p (which is a delicate issue, see [15, Section 6] for details) was a key technical ingredient to prove in [3, Theorem 3.1] that local cohomology modules over $\mathbb{Z}$ can have p-torsion for at most finitely many primes p.

Now, we provide a more detailed overview of the contents of this manuscript for the convenience of the reader; first of all, in Section 2, we give some connection between being stratified for a nonlinear differential equation and the level of a polynomial in the case of hyperelliptic curves. Second, in Section 3, we formally define the level of a pair of polynomials, listing some of the properties it satisfies. In Section 4, we focus on specific calculations when f and g are both homogeneous polynomials; in particular, we will show, among other things, that level(g, f) is, in general, not finite (see Proposition 4.9). We end this paper by raising some open questions to stimulate further research on this subject.

2. Stratified differential equations and hyperelliptic curves

The notion of stratification for nonlinear differential equations was introduced in [23]; we briefly recall it here. Let $C \supseteq \mathbb{F}_p$ be an algebraically closed field, let $C(z)$ be the one variable differential field extension of C with derivation $\frac{d}{dz}$ and let K be a finite separable extension of $C(z)$. Consider the differential equation $f(y', y) = 0$, where $f \in K[S, T]$ is an absolutely irreducible polynomial such that the image d of df/dS in $K[S, T]/(f)$ is nonzero; the differential algebra $A := K[y', y, 1/d]$ is given by the derivation D with $D(z) = 1$ and $D(y) = y'$. One says that $f(y', y) = 0$ is **stratified** if and only if $D^p = 0$ [23, Theorem 1.1]; it was also proved in [23, Proposition 2.3] that, if $p \geq 3$ and f is the defining equation of an elliptic curve E, then $f(y', y) = 0$ is stratified if and only if E is supersingular. By [6, Theorem 1.1], this is equivalent to the homogeneous polynomial corresponding to f having level two.

Keeping in mind these characterizations, one may ask what is the connection between being stratified and the level of a polynomial. For this, we recall the following terminology. Let X be a curve of genus g defined over an algebraically closed field k of characteristic $p > 0$. The p-rank f_X of X is defined as the $\mathbb{F}_p$-dimension of the p-torsion of the k-points of the Jacobian of X. The a-number a_X is defined as the dimension of the kernel of the Cartier–Manin matrix associated to X. Many properties of these numbers are discussed in the textbook [18]; the p-rank f_X and the a-number a_X satisfy $f_X + a_X \leq g$. Here equality does not hold in general, but $a_X = 0 \iff f_X = g \iff X$ is ordinary, and $a_X = g \iff X$ is superspecial (see [22, Theorem 2] and [21, Theorem 4.1] for the latter).

Proposition 2.1. *Given an algebraically closed field k of prime characteristic $p \geq 3$, consider the hyperelliptic curve $\mathcal{H}$ of genus $g \geq 1$ defined by the equation $y^2 = h(x)$, where $h(x) \in k[x]$ is squarefree and has degree $2g + 1$. The following statements are equivalent.*

(i) *$\mathcal{H}$ is not ordinary.*

(ii) *There exist $a_0, a_1, \ldots, a_{g-1} \in k$ with $a_j \neq 0$ for at least one j, such that the differential equation*

$$(x')^2 = \frac{h(x)}{(a_{g-1}x^{g-1} + \ldots + a_1x + a_0)^2}$$

is stratified.

(iii) *The a-number of the Jacobian of $\mathcal{H}$ is not zero.*

Proof. Let $\mathcal{C}'$ be the modified Cartier operator defined in [30, Definition 2.1.]; by the argument pointed out in [23, page 312], the differential equation is stratified if and only the differential form $\omega := ((a_{g-1}x^{g-1} + \ldots + a_1x + a_0)/y)dx$ is exact, which is equivalent to the condition $\mathcal{C}'(\omega) = 0$. Our goal now is to write down this condition in terms of the basis of differentials $\omega_i := (x^{i-1}/y)dx$ $(1 \leq i \leq g)$; it is easy to see that $\mathcal{C}'(\omega) = 0$ if and only if

$$\sum_{i=1}^{g} a_{i-1}^{1/p} \mathcal{C}'(\omega_i) = 0.$$

Now, if one writes $h(x)^{(p-1)/2} = \sum_{j=0}^{N} c_j x^j$, (where $N = ((p-1)/2)(2g+1)$) then one has [30, page 381] that

$$C'(\omega_i) = \sum_{j=1}^{g} c_{jp-i}\omega_j,$$

and therefore one ends up with the following equality:

$$\sum_{j=1}^{g}\left(\sum_{i=1}^{g} a_{i-1}^{1/p} c_{jp-i}\right)\omega_j = 0.$$

Equivalently, since the ω_j's are k-linearly independent, for any $1 \le j \le g$,

$$\sum_{i=1}^{g} a_{i-1}^{1/p} c_{jp-i} = 0.$$

Summing up, if one denotes by v the column vector $(a_0^{1/p}, \ldots, a_{g-1}^{1/p})$ and by C the Cartier–Manin matrix of the hyperelliptic curve $y^2 = h(x)$ [30, Definition 2.2], one has that our differential equation is stratified if and only if $C \cdot v = 0$, which, by [30, Theorem 3.1], is equivalent to the statement that the hyperelliptic curve $y^2 = h(x)$ is not ordinary. This proves the equivalence between (i) and (ii); finally, the equivalence between (i) and (iii) follows immediately from the fact that the a-number of $\mathrm{Jac}(\mathcal{H})$ equals the corank of the Cartier–Manin matrix of $\mathcal{H}$ [18, 5.2.8]. □

Combining Proposition 2.1 with [4, Theorems 1.3, 3.5 and 3.9], we obtain the following result.

Corollary 2.2. *Preserving the assumptions and notations of Proposition 2.1, let $g \ge 2, p > 2g^2 - 1$, and let $f = y^2 z^{2g-1} - z^{2g+1} h(x/z)$. If $\mathrm{level}(f) \ge 3$, then there are $a_0, a_1, \ldots, a_{g-1} \in k$ with $a_j \ne 0$ for at least one j such that the equation*

$$(x')^2 = \frac{h(x)}{(a_{g-1}x^{g-1} + \ldots + a_1 x + a_0)^2}$$

is stratified.

The next examples illustrate some of the results obtained above.

Example 2.3. Given $0 \ne b \in \mathbb{F}_p$, and $p > 7$, consider the equation

$$(x')^2 = \frac{x^5 + b}{(a_1 x + a_0)^2}, \tag{1}$$

and assume that $p \equiv 3 \pmod 5$ (e.g. $p = 13$). The hyperelliptic curve of genus two $\mathcal{H}$ defined by $y^2 = x^5 + b$ has the following Cartier–Manin matrix:

$$\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \text{ where } c := \binom{(p-1)/2}{(2p-1)/5} b^{(p-3)/10}.$$

In particular, $\mathcal{H}$ is not ordinary. In this case, $\mathcal{H}$ is supersingular (but not superspecial) and therefore $\mathrm{level}(y^2z^3 - x^5 - bz^5) \ge 3$ by [4, Corollary 3.10]. The equation (1) is stratified, if and only if $a_1 = 0$, as follows from the fact that the differential form dx/y is in the kernel of the Cartier operator, whereas for $a_1 \ne 0$ the form $a_0 dx/y + a_1 x dx/y$ is not in the kernel.

Assume that $p \equiv 4 \pmod 5$ (e.g. $p = 19$). In this case, by either [27, Theorem 2] or [28, Corollary of page 12], $\mathcal{H}$ is superspecial and therefore (1) is stratified for any value of a_1, a_0. In this case, $\mathrm{level}(y^2z^3 - x^5 - bz^5) \ge 3$ by [4, Example 4.4]. In contrast, where $p \equiv 1 \pmod 5$ (e.g. $p = 11$), one can easily check that $\mathcal{H}$ is ordinary (this also follows from [28, Theorem 3]) and therefore (1) is not stratified for any choice of a_1, a_0. In this case, using [4, Theorems 1.3, 3.5 and 3.9] one concludes that $\operatorname{level}(y^2z^3 - x^5 - bz^5) = 2$.

Example 2.4. Given $p > 17$, consider the equation

$$(x')^2 = \frac{(x-1)^8 - x^8}{(a_2x^2 + a_1x + a_0)^2}. \tag{2}$$

One can check that, under a Möbius transformation of the form

$$(x, y) \mapsto \left(\frac{1}{x+1}, \frac{y}{(x+1)^4}\right),$$

the hyperelliptic curve $\mathcal{H}$ defined by $y^2 = (x-1)^8 - x^8$ corresponds to $y^2 = x^8 - 1$, and therefore both have the same p-rank. As shown in [17, Section 2], $\mathcal{H}$ is ordinary if and only if $p \equiv 1 \pmod 8$, and supersingular (that is, its p-rank is 0) if and only if $p \equiv 7 \pmod 8$. In the ordinary case, we know that the level is 2, and at least three in the supersingular (not superspecial) case. However, in the remaining cases (where $p \equiv 3, 5 \pmod 8$) the curve has p-ranks 1 and 2 respectively, and in these two cases, while we can ensure that there are non-zero choices of a_2, a_1, a_0 such that (2) can be either stratified or not, we cannot predict in general what is the level.

3. The level of a pair of polynomials

Hereafter, let k be a perfect field of prime characteristic p, and let R be the polynomial ring $k[x_1, ..., x_d]$. The aim of this section is to study the following concept.

Definition 3.1. Given polynomials f, g with coefficients in k and $f \neq 0$, one defines the level of (g, f) as

$$\operatorname{level}(g,f) := \inf\left\{e \geq 0 : \exists \delta \in \mathcal{D}^{(e)} \text{ such that } \delta(g/f) = (g/f)^p\right\} \in \mathbb{N}_0 \cup \{\infty\}.$$

When $g=1$, one denotes $\operatorname{level}(f)$ instead of $\operatorname{level}(1,f)$; this is the notion of level of a polynomial introduced in [6, Definition 2.6].

Remark 3.2. Note that $\operatorname{level}(g,f)$ only depends on the quotient g/f, so one could also reasonably denote this notion by $\operatorname{level}(\frac{g}{f})$ instead. But this alternative notation is inconsistent with the one in [6] in the case $f=1$, so we stick with the notation $\operatorname{level}(g,f)$. In any case, one can usually assume that g and f are coprime, since common factors do not change the level of the pair.

Note also that $\operatorname{level}(g,f) = 0$ if and only if $g/f \in R$. If g and f are coprime, this only happens if f is a constant.

In Proposition 4.9, we give an example of polynomials f and g such that $\operatorname{level}(g,f) = \infty$.

Before going on studying this notion, we review the so-called ideals of p^eth roots; the interested reader can find a more detailed treatment in [1, page 465], [5, Definition 2.2], and [16, Definition 5.1]. For an ideal $I \subset R$ we denote by $I^{[p^e]}$ the ideal generated by the p^e-th powers of elements of I.

Definition 3.3. Given $g \in R$ and an integer $e \geq 0$, we define the *ideal of p^eth roots* $I_e(g)$ to be the smallest ideal $J \subseteq R$ such that $g \in J^{[p^e]}$.

Remark 3.4. Under our assumptions, R is a free R^{p^e}-module with basis given by the monomials $\left\{\mathbf{x}^{\alpha} \mid \|\alpha\| \leq p^e - 1\right\}$. A polynomial $g \in R$ can therefore be written as

$$g = \sum_{0 \le ||\alpha|| \le p^e - 1} g_\alpha^{p^e} \mathbf{x}^\alpha,$$

for unique $g_\alpha \in R$. Then $I_e(g)$ is the ideal of R generated by elements g_α [5, Proposition 2.5].

The main relation between these ideals and differential operators is the following equality, valid for any polynomial $g \in R$ and any integer $e \ge 0$ (see [1, Lemma 3.1]):

$$\mathcal{D}^{(e)} \cdot g = I_e(g)^{[p^e]}. \tag{3}$$

Using this, one can relate the level of a pair of polynomials to ideals of p^eth roots as follows.

Lemma 3.5. *Let $f, g \in R$ and $e \ge 0$ be given. Then the following are equivalent:*

(i) $\operatorname{level}(g,f) \le e$;
(ii) $I_e(g^p f^{p^e - p}) \subseteq I_e(gf^{p^e-1})$;
(iii) $I_e(g^p f^{p^e - p})^{[p^e]} \subseteq I_e(gf^{p^e-1})^{[p^e]}$.

In particular, $\operatorname{level}(g,f) = \inf\{e \ge 0 : I_e(g^p f^{p^e-p}) \subseteq I_e(gf^{p^e-1})\}$.

Proof. The equivalence of (ii) and (iii) is proved in the last paragraph of the proof of [1, Proposition 3.5]. We prove that (i) and (iii) are equivalent. Suppose that there is $\delta \in \mathcal{D}^{(e)}$ such that $\delta(g/f) = (g/f)^p$. Since δ is linear over p^e-powers, this implies that $\delta(gf^{p^e-1}) = g^p f^{p^e-p}$. By (3), this implies $g^p f^{p^e-p} \in I_e(g^p f^{p^e-p})^{[p^e]}$, so that $I_e(g^p f^{p^e-p})^{[p^e]} \subseteq I_e(gf^{p^e-1})^{[p^e]}$.

Conversely, suppose now that $I_e(g^p f^{p^e-p})^{[p^e]} \subseteq I_e(gf^{p^e-1})^{[p^e]}$. Again using (3), one has that $\mathcal{D}^{(e)}(g^p f^{p^e-p}) \subseteq \mathcal{D}^{(e)}(gf^{p^e-1})$. In particular $g^p f^{p^e-p} \in \mathcal{D}^{(e)}(gf^{p^e-1})$, hence there is $\delta \in D^{(e)}$ such that $\delta(gf^{p^e-1}) = g^p f^{p^e-p}$. Multiplying this equality by $1/f^{p^e}$ and using that δ is linear over p^eth powers, we get $\delta(g/f) = (g/f)^p$. □

Observe that the equality $\mathcal{D}^{(e)} \cdot g = I_e(g)^{[p^e]}$ is made explicit in, e.g., the proof of [6, Claim 3.4]. Using these techniques one can in case $e = \operatorname{level}(g,f) < \infty$, algorithmically construct an explicit operator $\delta \in \mathcal{D}_R^{(e)}$ with $\delta(g/f) = g^p/f^p$. However we do not know how to decide whether the level of a given pair is finite.

Remark 3.6. By the same argument as in [4, § 2.4], the level of a pair is invariant under linear coordinate changes.

In the next statement, our aim is to collect some properties that the level of a pair of polynomials satisfies.

Proposition 3.7. *Let $f, g \in R$ be non-zero polynomials such that $\frac{g}{f} \notin R$. Then the following statements hold.*

(i) $\operatorname{level}(g,f) = 1$ *if and only if* $g \in I_1(gf^{p-1})$.
(ii) *If* $\operatorname{level}(f) = 1$, *then* $\operatorname{level}(g,f) = 1$.
(iii) *If either* $I_e(g^p f^{p^e-p}) \not\subseteq I_e(f^{p^e-1})$ *or* $I_e(g^p f^{p^e-p}) \not\subseteq I_e(g)$, *then* $\operatorname{level}(g,f) > e$.
(iv) *If f and g are homogeneous, and $e \ge 1$ is an integer such that $p^e > \deg g - \deg f$, then $I_e(gf^{p^e-1})$ is generated by polynomials of degree at most $\deg f$.*

Proof. The assumption that f does not divide g in R implies that $\operatorname{level}(g,f) > 0$. Then (i) follows from Lemma 3.5 together with the easy observation that $I_1(g^p) = (g)$. Part (ii) was already proved in [6, page 248]; we repeat the proof for the sake of completeness. Let $\delta' \in \mathcal{D}^{(1)}$ such that $\delta'(1/f) = 1/f^p$. Then define $\delta := \delta' \circ (\cdot g^{p-1})$. We find that $\delta(g/f) = \delta'(g^p/f) = g^p\delta'(1/f) = (g/f)^p$, as required.

Part (iii) follows immediately combining Lemma 3.5 with the fact that $I_e(gf^{p^e-1}) \subseteq I_e(g)I_e(f^{p^e-1})$ [1, Lemma 3.3]. Finally, to prove part (iv) fix $e \geq 1$ an integer and write

$$gf^{p^e-1} = \sum_{0 \leq ||\alpha|| \leq p^e-1} c_\alpha^{p^e}\mathbf{x}^\alpha,$$

for some $c_\alpha \in R$. Since both f and g are homogeneous it follows that

$$\deg(g) + (p^e-1)\deg(f) = p^e\deg(c_\alpha) + \deg(\mathbf{x}^\alpha),$$

which implies that

$$\deg(c_\alpha) \leq \frac{(p^e-1)\deg(f) + \deg(g)}{p^e} = \deg(f) + \frac{\deg\, g - \deg\, f}{p^e}.$$

The second term on the right hand side is smaller than 1 by assumption, and since both sides are integers, we get $\deg\, c_\alpha \leq \deg\, f$. The result follows. □

4. Some examples

The goal of this section is to calculate the level of a pair of polynomials (g, f) for several particular choices of g and f; we will quickly see that, even for low degrees, most of the calculations are highly non-trivial. In particular, we show that $\text{level}(g,f)$ is, in general, not always finite (see Example 4.9).

We want to start with the case considered by Singh, see for example [25].

Lemma 4.1. *Let p be a prime number, $X = \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix}$ be a matrix of indeterminates defined over $R = k[u, v, w, x, y, z]$, and set $\Delta_1 := vz - wy, \Delta_2 := wx - uz$, and $\Delta_3 := uy - vx$. Then, $\text{level}(g,f) = 1$ for each pair $(g,f) \in \{(w, \Delta_1\Delta_2),\ (v, \Delta_1\Delta_3),\ (u, \Delta_2\Delta_3)\}$.*

Proof. By symmetry, it is enough to show that $\text{level}(g,f) = 1$ when $(g,f) = (w, \Delta_1\Delta_2)$. Set $f := \Delta_1\Delta_2$, and notice that $f = 1 \cdot (xzvw) + (-z^2) \cdot (uv) + (-w^2) \cdot (xy) + 1 \cdot (yzuw)$. This shows that, if $p = 2$, then $I_1(f) = R$ so $\text{level}(f) = 1$ and therefore $\text{level}(g,f) = 1$. Now, assume that $p \geq 3$, one can check that in the support of f^{p-1} appears the monomial $(xyuv)^{(p-1)/2}(zw)^{p-1}$ with coefficient $\begin{pmatrix} p-1 \\ (p-1)/2 \end{pmatrix}$; this shows again that $\text{level}(f) = 1$ and therefore $\text{level}(g,f) = 1$. □

Remark 4.2. Notice that, in the setting considered in Lemma 4.1, Singh shows in [25] that the differential operator $\delta := D_{u,p-1}D_{y,p-1}D_{z,p-1}$ (which is clearly of level one) is such that $\delta(g/f) = (g/f)^p$, for g/f any of the three fractions considered in Lemma 4.1.

Lemma 4.3. *Let k be a field of characteristic p, let $f = x^d$, assume that $p \geq d$, and let $g \in R = k[x, y]$ be a homogeneous polynomial of degree d which is not a multiple of f. Then, $\text{level}(g,f) = 2$ unless $g \in (x^{d-1})$, in which case $\text{level}(g,f) = 1$.*

Proof. Write $g = \sum_{i=0}^{d} a_i x^i y^{d-i}$; now, notice that

$$gf^{p-1} = \sum_{i=0}^{d} a_i x^{i+d(p-1)} y^{d-i}.$$

Given $0 \le i \le d$ write $i + d(p-1) = (d-1)p + (p+i-d)$, and notice that, unless $i = d$, $0 \le p+i-d \le p-1$ (here, we are also using that $d \le p$). This shows that $I_1(gf^{p-1}) = (a_d^{1/p} x^d, a_i^{1/p} x^{d-1} : 1 \le i \le d-1) = (x^{d-1})$, so $\text{level}(g,f) \ne 1$ unless $g \in (x^{d-1})$, in which case $\text{level}(g,f) = 1$. So, from now on, assume that $g \notin (x^{d-1})$.

We have $I_2(g^p f^{p^2-p}) = I_1(gf^{p-1}) = (x^{d-1})$. Now, write

$$gf^{p^2-1} = \sum_{i=0}^{d} a_i x^{i+d(p^2-1)} y^{d-i}.$$

Again, the equality $i + d(p^2-1) = (d-1)p^2 + (p^2+i-d)$ and the fact unless $i = d$, $p^2+i-d \le p^2-1$, shows that $I_2(gf^{p^2-1}) = (a_d^{1/p^2} x^d, a_i^{1/p^2} x^{d-1} : 1 \le i \le d-1) = (x^{d-1}) = I_2(g^p f^{p^2-p})$, and therefore $\text{level}(g,f) = 2$, as claimed. □

Lemma 4.3 has the following interesting consequence.

Lemma 4.4. *Let k be a field of prime characteristic p, and let $f, g \in k[x,y]$ be quadratic forms. If $\sqrt{(f)}$ denotes the radical of (f), then*

$$\text{level}(g,f) = \begin{cases} 0, & \text{if } g \text{ is a multiple of } f, \\ 1, & \text{if either } f \text{ is not the square of a linear form, or if } g \in \sqrt{(f)} \setminus (f), \\ 2, & \text{otherwise.} \end{cases}$$

Proof. First of all, if f is not the square of a linear form, then by [6, Proposition 5.7] $\text{level}(f) = 1$ and therefore part (ii) of Proposition 3.7 implies that $\text{level}(g,f) = 1$. So, hereafter we assume that f is the square of a linear form; By Remark 3.6 we can assume that $f = x^2$ and that g is again a quadratic form. Then, in this case, Lemma 4.3 says exactly that $\text{level}(g,f) = 2$ unless $g \in (x)$, in which case $\text{level}(g,f) = 1$; the proof is therefore completed. □

As a more elaborate example we now consider $\text{level}(g,f)$ with $f = x^3 + y^3 + z^3$ and g any homogeneous cubic in 3 variables which is not a scalar multiple of f. Since $\text{level}(f) = 1$ in case the characteristic $p \equiv 1 \pmod 3$, Proposition 3.7 (ii) shows $\text{level}(g,f) = 1$ for $p \equiv 1 \pmod 3$ and any such g.

We expect that the same holds for all characteristics $p \ge 5$. The next two special cases show that this is correct for most g. By Example 4.8, the same does not hold in characteristics $p = 2, 3$.

Claim 4.5. *Let $p \ge 5$ with $p \equiv 2 \pmod 3$, let $f = x^3 + y^3 + z^3$, and let $g \in R = k[x,y,z]$ be a homogeneous polynomial of degree 3 such that, if one writes $g = \sum_{a+b+c=3} g_{a,b,c} x^a y^b z^c$, and set $B := \binom{p-1}{(p-2)/3,\,(p-2)/3,\,(p+1)/3}$, $C := \binom{p-1}{(p-2)/3}$, $D := \binom{p-1}{1,\,2(p-2)/3,\,(p-2)/3}$, $E := \binom{p-1}{1,\,(2p-1)/3,\,(p-5)/3}$, and $F := \binom{p-1}{(p+4)/3,\,(p-2)/3,\,(p-5)/3}$, then the rank of*

$$A := \begin{pmatrix} Bg_{1,1,1} & Cg_{2,0,1} & Cg_{2,1,0} \\ Cg_{0,2,1} & Bg_{1,1,1} & Cg_{1,2,0} \\ Cg_{0,1,2} & Cg_{1,0,2} & Bg_{1,1,1} \\ Bg_{2,0,1} & Dg_{0,2,1} & Cg_{3,0,0}+Eg_{0,3,0}+Dg_{0,0,3} \\ Bg_{2,1,0} & Cg_{3,0,0}+Eg_{0,3,0}+Dg_{0,0,3} & Dg_{0,1,2} \\ Bg_{3,0,0}+Fg_{0,3,0}+Fg_{0,0,3} & Dg_{1,2,0} & Dg_{1,0,2} \\ Dg_{2,0,1} & Bg_{0,2,1} & Eg_{3,0,0}+Cg_{0,3,0}+Dg_{0,0,3} \\ Eg_{3,0,0}+Cg_{0,3,0}+Dg_{0,0,3} & Bg_{1,2,0} & Dg_{1,0,2} \\ Dg_{2,1,0} & Fg_{3,0,0}+Bg_{0,3,0}+Fg_{0,0,3} & Dg_{0,1,2} \\ Dg_{2,1,0} & Eg_{3,0,0}+Dg_{0,3,0}+Cg_{0,0,3} & Bg_{0,1,2} \\ Eg_{3,0,0}+Dg_{0,3,0}+Cg_{0,0,3} & Dg_{1,2,0} & Bg_{1,0,2} \\ Dg_{2,0,1} & Dg_{0,2,1} & Fg_{3,0,0}+Fg_{0,3,0}+Bg_{0,0,3} \end{pmatrix}$$

is three. Then level$(g,f) \leq 1$, *with equality exactly if g is not a multiple of f.*

Proof. Write $g = \sum_{a+b+c=3} g_{a,b,c} x^a y^b z^c$, and

$$gf^{p-1} = \sum_{a+b+c=3} \sum_{i+j+k=p-1} g_{a,b,c} \binom{p-1}{i,j,k} x^{3i+a} y^{3j+b} z^{3k+c}.$$

Then, if one picks $i = j = (p-2)/3$ and $k = (p+1)/3$, then the corresponding term of gf^{p-1} is

$$\sum_{a+b+c=3} g_{a,b,c} \binom{p-1}{i,j,k} z^p \cdot \left(x^{p-2+a} y^{p-2+b} z^{c+1}\right).$$

Again, if $i = k = (p-2)/3$ and $j = (p+1)/3$, then the corresponding term of gf^{p-1} is

$$\sum_{a+b+c=3} g_{a,b,c} \binom{p-1}{i,j,k} y^p \cdot \left(x^{p-2+a} y^{b+1} z^{p-2+c}\right).$$

By the same reason, if $j = k = (p-2)/3$ and $i = (p+1)/3$, then the corresponding term of gf^{p-1} is

$$\sum_{a+b+c=3} g_{a,b,c} \binom{p-1}{i,j,k} x^p \cdot \left(x^{a+1} y^{p-2+b} z^{p-2+c}\right).$$

The above expansions show that the basis elements $x^{p-1}y^{p-1}z^2$, $x^{p-1}y^2z^{p-1}$ and $x^2y^{p-1}z^{p-1}$ contain respectively in their coefficient the below term, where $B := \binom{p-1}{(p-2)/3,\,(p-2)/3,\,(p+1)/3}$:

$$g_{1,1,1}Bz^p, g_{1,1,1}By^p, g_{1,1,1}Bx^p.$$

Hereafter, we only plan to prove that the coefficient of $x^{p-1}y^{p-1}z^2$ is exactly $Cg_{0,1,2}x^p + Cg_{1,0,2}y^p + Bg_{1,1,1}z^p$ and one can show using the same arguments that the coefficient of $x^{p-1}y^2z^{p-1}$ (resp. $x^2y^{p-1}z^{p-1}$) is exactly $Cg_{0,2,1}x^p + Bg_{1,1,1}y^p + Cg_{1,2,0}z^p$ resp. $Bg_{1,1,1}x^p + Cg_{2,0,1}y^p + Cg_{2,1,0}z^p$.

Indeed, we want to calculate the coefficient of $x^{p-1}y^{p-1}z^2$, so suppose that there are non-negative integers λ, μ, γ such that $3i + a = \lambda p + p - 1$, $3j + b = \mu p + p - 1$, $3k + c = \gamma p + 2$. Since $\deg\left(gf^{p-1}\right) = 3p$, it follows that $3p = 3i + a + 3j + b + 3k + c = (\lambda + \mu + \gamma + 2)p$, which implies that $\lambda + \mu + \gamma = 1$, so we only have three possibilities for these integers; namely, (1, 0, 0), (0, 1,

0) and (0, 0, 1). For (1, 0, 0), we get $i=(2p-1-a)/3,\ j=(p-1-b)/3,\ k=(2-c)/3$. Since $p\equiv 2 \pmod 3$, this forces $a=0$, $b=1$ and $c=2$. By the same argument, for (0, 1, 0) one gets $a=1$, $b=0$ and $c=2$, and finally, for (0, 0, 1) one ends up with $a=b=c=1$. This shows that the coefficient of $x^{p-1}y^{p-1}z^2$ is exactly $B(g_{0,1,2}x^p+g_{1,0,2}y^p+g_{1,1,1}z^p)$, as claimed.

One might ask from where the other rows of matrix A appearing in our assumption comes from; following the same arguments, these rows corresponds to the calculation of the coefficients of the below basis elements:

$$\begin{gathered}x^3y^{p-2}z^{p-1},\ x^3y^{p-1}z^{p-2},\ x^4y^{p-2}z^{p-2},\\ x^{p-2}y^3z^{p-1},\ x^{p-1}y^3z^{p-2},\ x^{p-2}y^4z^{p-2},\\ x^{p-2}y^{p-1}z^3,\ x^{p-1}y^{p-2}z^3,\ x^{p-2}y^{p-2}z^4.\end{gathered}$$

Summing up, the foregoing implies, since by assumption the rank of A is 3, that $(x,y,z)=I_1(gf^{p-1})$, hence $g\in I_1(gf^{p-1})$ and this shows that $\operatorname{level}(g,f)=1$ by using part (i) of Proposition 3.7. □

Claim 4.6. *Let $p\geq 5$, let $f=x^3+y^3+z^3$, and let $g\in R=k[x,y,z]$ be a non-zero monomial of degree 3. Then,* $\operatorname{level}(g,f)=1$.

Proof. If $p\equiv 1 \pmod 3$, then $\operatorname{level}(f)=1$ and therefore $\operatorname{level}(g,f)=1$ by part (ii) of Proposition 3.7, so hereafter we will assume that $p\equiv 2 \pmod 3$. By symmetry, it is enough to consider the monomials $g=x^3$, $g=x^2y$ and $g=xyz$. In each of these cases, we will simply construct an explicit differential operator of level 1 that does what is needed. For $g=x^3$, consider first

$$\delta=D_{x,p-1}\circ D_{y,p-2}\circ D_{z,3}$$

(see the Introduction for the notation $D_{x,n}$). Clearly δ is of level 1, since $p>3$. We have that

$$gf^{p-1}=\sum_{i+j+k=p-1}\binom{p-1}{i,j,k}x^{3i+3}y^{3j}z^{3k}.$$

Applying δ gives us

$$\delta(gf^{p-1})=\sum_{i+j+k=p-1}\binom{p-1}{i,j,k}\binom{3i+3}{p-1}\binom{3j}{p-2}\binom{3k}{3}x^{3i+4-p}y^{3j+2-p}z^{3k-3},$$

where we use the convention that $\binom{n}{k}=0$ for $k>n$. We investigate for which indices i, j, k the coefficient in this term is zero. The first factor is never zero, since $p-1$, i, j and k are all between 0 and $p-1$. The second factor is zero unless $3i+3\equiv -1 \pmod p$, as can be seen by writing out the product. Since i lies between 0 and $p-1$, and since $p\equiv 2 \pmod 3$, the only integer value for i such that $3i+3\equiv -1 \bmod p$ is $i=(2p-4)/3$. This means that j is at most $(p+1)/3$. The third factor $\binom{3j}{p-2}$ is zero unless $3j$ is either -1 or -2 modulo p. In the allowed range for j, the only integer possibility is $j=(p-2)/3$. This leaves $k=1$, and for this value of k we have $\binom{3k}{3}=1\neq 0$. So we see that the only non-zero term in $\delta(gf^{p-1})$ is the one for indices $(i,j,k)=\left(\frac{2p-4}{3},\frac{p-2}{3},1\right)$. This gives

$$\delta(gf^{p-1})=\binom{p-1}{\frac{2p-4}{3},\frac{p-2}{3},1}\binom{2p-1}{p-1}\binom{p-2}{p-2}\binom{3}{3}x^p=\binom{p-1}{\frac{2p-4}{3},\frac{p-2}{3},1}x^p$$

Define now

$$\Delta = \binom{p-1}{\frac{2p-4}{3}, \frac{p-2}{3}, 1}^{-1} \cdot x^{2p} \cdot \delta,$$

then Δ is also a differential operator of level 1, and by construction we have $\Delta(gf^{p-1}) = x^{3p} = g^p$. Using that Δ is R^p-linear, we may divide both sides by f^p and get $\Delta(g/f) = g^p/f^p$, as needed.

For the other cases $g = x^2y$ and $g = xyz$, a similar analysis shows that the operators

$$C \cdot x^p y^p D_{x,p-2} D_{y,p-1} D_{z,3}, \quad \text{resp.} \quad C' \cdot y^p z^p, D_{x,p-3} D_{y,p-1} D_{z,4}$$

for suitably chosen non-zero constants $C, C' \in \mathbb{F}_p$, have the required property. □

Proposition 3.7(ii) shows that if level(f) = 1 then level(g,f) $\leq$ 1. In the example considered in Lemma 4.5 one has level(f) = 2 > level(g,f) = 1. One might ask whether in general level(g,f) $\leq$ level(f). This is not the case, as the following example shows.

Example 4.7. Let $R = k[x, y, z, w]$, g = y and $f = xy^{p+1} + yz^{p+1} + zw^{p+1}$. Using Magma [8] we computed for the cases $p \in \{2, 3, 5\}$ that level(g) = 1, level(f) = 2, but level(g,f) = 4.

For any prime p, what is easy to show in this example is that level(g,f) ≥ 2; indeed, notice that

$$gf^{p-1} = \sum_{\substack{0 \leq i,j,k \leq p-1 \\ i+j+k = p-1}} \binom{p-1}{i,j,k} (y^i z^j w^k)^p \cdot (x^i y^{p-k} z^{p-1-i} w^k).$$

We claim that, whereas $y^p \in I_1(gf^{p-1})$, $g = y \notin I_1(gf^{p-1})$. Indeed, if in the above expansion we pick $j = k = 0$ and $i = p-1$, then one gets that $gf^{p-1} = (y^p)^p(x^{p-1}) + \ldots$, and this choice is the only one that makes the basis element x^{p-1} appearing in this expansion. This shows that $y^p \in I_1(gf^{p-1})$; moreover, notice that, if one choices a i, j, k as above where $i < p-1$, then the coefficient of the corresponding basis element is made up by monomials that are divisible by either z or w. This shows that $\mathrm{y^p}$ is the smallest possible power of y that belongs to $I_1(gf^{p-1})$, hence $g = y \notin I_1(gf^{p-1})$ and therefore level(g,f) ≥ 2, as claimed.

Moreover, again about Lemma 4.6, we want to single out that the assumption $p \neq 2, 3$ can not be removed, as the following examples show.

Example 4.8. Let $p = 2$, let $R = k[x, y, z]$, $f = x^3 + y^3 + z^3$ and $g = xyz$; we claim level(g,f) = 2. Indeed, on the one hand, $gf^{p-1} = (x^2)^2 \cdot (yz) + (y^2)^2 \cdot (xz) + (z^2)^2 \cdot (xy)$, so $g = xyz \notin I_1(gf^{p-1}) = (x^2, y^2, z^2)$; this shows, by part (i) of Proposition 3.7, that level(g,f) ≥ 2. On the other hand,

$$gf^{p^2-1} = (x^2)^4 \cdot (x^2yz) + (y^2)^4 \cdot (xy^2z) + (z^2)^4 \cdot (xyz^2) + (xy)^4 \cdot (x^3z) + (xy)^4 \cdot (y^3z) + (xz)^4 \cdot (x^3y) + (xz)^4 \cdot (yz^3) + (yz)^4 \cdot (xyz^3),$$

and $g^pf^{p^2-p} = x^8(yz)^2 + y^8(xz)^2 + z^8(xy)^2$; these last two computations show that

$$g^pf^{p^2-p} \in (x^2, y^2, z^2, xy, xz, yz)^{[p^2]} = I_2\left(gf^{p^2-1}\right)^{[p^2]},$$

and therefore Lemma 3.5 ensures level(g,f) = 2, as claimed.

Now, assume that $p=3$ (g and f are the same); in this case, one can check that $J := I_1(gf^2) = (x^2+2xy+y^2+2xz+2yz+z^2)$ and $g = xyz \notin J$. One way to check it is the following; denote by $V(J)$ the hypersurface defined by J. This hypersurface contains the point $(1,1,1)$, which is a point which does not belong to $V(xyz)$. This shows that $xyz \notin J$.

The above argument shows that $\text{level}(g,f) \geq 2$ and, actually, one can check either by hand or by computer that $\text{level}(g,f) = 2$.

We conclude this section with an example showing that the level of a pair of polynomials is, in general, not finite. This in fact answers a question raised in [6, Section 5].

Proposition 4.9. *Let $R = k[x,y]$ with $\text{char}\, k = p$, and let $f = x^{p+1} + y^{p+1}$ and $g = x$. Then $\text{level}(g,f) = \infty$. In particular, no $\delta \in \mathcal{D}_R$ exists with $\delta(g/f) = g^p/f^p$.*

Proof. Let $e \geq 2$ be an arbitrary even integer. We will show that $\text{level}(g,f) > e$. By Lemma 3.5, this is equivalent to showing that $I_e(g^p f^{p^e-p})^{[p^e]} \not\subseteq I_e(gf^{p^e-1})^{[p^e]}$.

First, we show that $I_e(gf^{p^e-1})$ is a monomial ideal. Indeed, we have

$$gf^{p^e-1} = \sum_{i=0}^{p^e-1} \binom{p^e-1}{i} x^{i(p+1)+1} y^{(p^e-1-i)(p+1)}. \tag{4}$$

By the description of I_e in Remark 3.4, to find generators of $I_e(gf^{p^e-1})$, express gf^{p^e-1} as an R^{p^e}-linear combination of monomials with exponents below p^e, and take p^e-th roots of the coefficients. If for two indices i and j the corresponding terms in (4) differ by a p^e-th power, then they both contribute to the same generator. But this happens only if the exponents for x and y are congruent modulo p^e. From $i(p+1)+1 \equiv j(p+1)+1 \pmod{p^e}$ we obtain $i \equiv j \pmod{p^e}$ since $p+1$ is a unit modulo p^e. But if $0 \leq i,j \leq p^e-1$ and $i \equiv j \pmod{p^e}$, then $i=j$. So we see that the terms occurring in gf^{p^e-1} are independent over $\text{Frac}(R^{p^e})$. Hence the generators for $I_e(gf^{p^e-1})$ that we get from Remark 3.4 are monomials, and so $I_e(gf^{p^e-1})$ is a monomial ideal. It follows that also $I_e(gf^{p^e-1})^{[p^e]}$ is a monomial ideal.

Now we show that $g^p f^{p^e-p} \notin I_e(gf^{p^e-1})^{[p^e]}$. Since the latter is a monomial ideal, it is sufficient to find a monomial that occurs in $g^p f^{p^e-p}$ with non-zero coefficient which is not in this ideal. For this, set $m := x^{p^e-p^2+p} y^{p^{e+1}-p}$. We claim that this monomial occurs in $g^p f^{p^e-p}$ with non-zero coefficient. We have

$$g^p f^{p^e-p} = \sum_{i=0}^{p^e-p} \binom{p^e-p}{i} x^{i(p+1)+p} y^{(p^e-p-i)(p+1)}.$$

We see that our monomial m occurs for index $i = (p^e-p^2)/(p+1)$, which is an integer because e is even. To evaluate the binomial coefficient for this value of i, we can look at the p-adic digits of the numbers involved. We have $p^e - p = (p-1)p^{e-1} + (p-1)p^{e-2} + \ldots + (p-1)p$, and we have $i = (p-1)p^{e-2} + (p-1)p^{e-4} + \ldots + (p-1)p^2$. Using Lucas's theorem [19, pp. 51–52], we find that the binomial coefficient evaluates to 1, so in particular it is non-zero.

Now we need to show that $m \notin I_e(gf^{p^e-1})^{[p^e]}$. This ideal is generated by monomials which are also p^e-th powers, and m is an element of this ideal if and only if at least one of these monomials divides m. The largest p^e-th power dividing m is $y^{(p-1)p^e}$. Hence, it is enough to show that $y^{(p-1)p^e} \notin I_e(gf^{p^e-1})^{[p^e]}$, or equivalently, that $y^{p-1} \notin I_e(gf^{p^e-1})$. In view of Remark 3.4, we look at terms in the product gf^{p^e-1} that contribute something of the form y^n to $I_e(gf^{p^e-1})$. A term does this if and only if the exponent for x is strictly lower than p^e. In Equation (4) above, this happens for terms with index i for which $i(p+1)+1 \leq p^e-1$, which is equivalent to

$$i \leq \left\lfloor \frac{p^e - 2}{p+1} \right\rfloor = \frac{p^e - p - 2}{p+1},$$

where we used again that e is even. But for such indices i, the exponent for y is given by

$$(p^e - 1 - i)(p+1) \leq p^{e+1} + p^e - p - 1 - p^e + p + 2 = p^{e+1} + 1.$$

So the contribution of these terms to $I_e\left(gf^{p^e-1}\right)$ is at least y^p. Thus the lowest exponent n such that $y^n \in I_e\left(gf^{p^e-1}\right)$ is $n = p$, and in particular $y^{p-1} \notin I_e\left(gf^{p^e-1}\right)$. □

4.1. Some open questions

Question 4.10. The following questions are open, to the best of our knowledge.

(i) Does an algorithm exist which, on input polynomials f and g, decides whether $\text{level}(g, f) < \infty$?
(ii) Under which conditions one can ensure that $\text{level}(g, f) \leq \text{level}(f)$?
(iii) In [11, Proposition 6], it is shown that, if R is an F-finite ring of characteristic $p \geq 3$, $f \in R$, and e is the largest F-jumping number of f that lies inside (0, 1), then $\text{level}(f) = \lceil 1 - \log_p(1-e) \rceil$. Is it possible to obtain a similar result for $\text{level}(g, f)$?

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