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Course Portfolio for Math 309: Introduction to Mathematical Proofs - A Peer Review of Teaching Project Benchmark Portfolio

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Course Portfolio for Math 309: Introduction to Mathematical Proofs

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Abstract

In this course portfolio, I examine dimensions of an active learning curriculum developed for a sophomore-level undergraduate course serving as an introduction to mathematical proofs. Prominent course goals include developing effective practices for communicating mathematics using formal language, learning to read, comprehend, and evaluate the validity of mathematical proofs, and practicing to write rigorous and concise mathematical proofs. I explore a new piece of collaborative annotation software called Perusall to help students read and understand mathematics together, and I analyze mastery level grading scales across exams throughout the semester. The portfolio also contains information about the structure and syllabus for the course, the exams given, and samples of student work and collaborative annotations.

Keywords. Undergraduate education, mathematics, active learning, collaborative annotation, lecture based tutoring

Contents

Chapter 1

Introduction

With this being my first time teaching Introduction to Mathematical Proofs, and with the course being so new to the department (having been taught for just the first time in Fall 2019), the primary purpose of my course portfolio will be twofold:

- Critically evaluate and formalize the course objectives and purpose of the course
- Carefully organize the assessment strategies and evaluate their effectiveness in providing feedback about students' understanding

The evaluation of course objectives was done leading up to the spring semester, based on a survey of potential textbook options, looking over the presentation of material and standard proofs presented in subsequent math courses, and feedback from the sole faculty member who taught the first iteration of the course in Fall 2019. The assessment strategies were determined based on these course objectives and were numerous and varied, which was beneficial during the Spring 2020 semester as the effectiveness of the assessment strategies was evaluated.

On a personal note, the culture of the math department is one where faculty are open about sharing materials when people are teaching a course for the first time. In creating a course portfolio, the opportunity to hone in on specific course objectives and effective assessment strategies has the potential to benefit many students and other instructors as the materials are adapted in subsequent semesters.

Chapter 2

Benchmark Memo 1: Description of Course

Math 309, Introduction to Mathematical Proofs, is a course offered to sophomore and junior students intending to go on to take upper level mathematics courses requiring a sophisticated understanding of logic and proofs, and will be my focus in this course portfolio.

2.1 Course Description

This course is brand new in the department - Fall 2019 was the first time that the course was offered. The course was created as a result of the department realizing that a course was needed to serve as a bridge between the more "procedural" mathematics that students experience in first year math courses (College Algebra, Trigonometry, Calculus I/II/III, Differential Equations) and proof-heavy, conceptually-driven upper level math courses (Introduction to Modern Algebra, Linear Algebra, Elementary Analysis, etc.). The art of logic and proof is a tricky concept for students upon first exposure, and before the existence of Introduction to Mathematical Proofs, students were learning the intricacies of mathematical proof in the context of courses where an understanding of proof techniques was necessary, but where other material was the primary focus - the names of the aforementioned courses suggest that they focus on the algebra or analysis material that must be learned.

With Introduction to Mathematical Proofs being such a new course, there are a lot of components to the course that are not clearly defined. As I thought about planning for the course, I could not explicitly answer what students generally think about the course, or who is taking the course, or even how many students I would enroll, so I needed to allow flexibility in the planning of the course without answers to these types of questions in mind. Nonetheless, Introduction to Mathematical Proofs is serving an incredibly important and new role for our program, and teaching it in Spring 2020 was a great opportunity to have a thoughtful hand in shaping the purpose and reputation of this course.

2.2 Course Goals

Stated simply, the overarching goal of this course is to develop students' abilities and confidence with heuristic techniques in the context of mathematics, preparing them for the problem solving and mathematical proof reading and constructing that will be expected of them in subsequent math courses. I believe that students who meet the following learning objectives will be able to achieve this level of mastery with mathematical proofs:

G.1 Communicate effectively using the formal language of mathematics

G.2 Evaluate the validity of complex logic statements

G.3 Identify useful definitions and use them to make valid deductions

G.4 Recognize various mathematical proof techniques and know when to apply them

G.5 Read, comprehend, and evaluate the validity of mathematical proofs, and criticize invalid proofs

G.6 Write rigorous and concise mathematical proofs

G.7 Present and explain the reasoning behind mathematical proofs

Many different types of activities will be required of students as they work towards these learning objectives, where each activity will allow for a different form of assessment varying in formality and nature (summative and formative). The following list of major topics will be focused on throughout the course:

- Basic set theory
- Elements of logic
- Types of proofs
- Mathematical induction
- Study of relations and functions
- Cardinality of sets

For more information about the content of the course, see the course syllabus in Appendix 6.1.

Chapter 3

Benchmark Memo 2: Teaching Methods and Activities

3.1 Teaching Methods

A typical day of class consisted of a mixture of lecture, group work, and in-class presentations by students, all of which are outlined further below. The time spent in the classroom focused on student thinking and collaboration, and the activities chosen for class contributed to all of the Course Goals which do not require the extra time and focus afforded by working on homework sets.

3.1.1 Lecture

I used a method of lecturing known as "lecture based tutoring", which was developed by Todd Easton, a faculty member in the Industrial and Manufacturing Systems Engineering Department at Kansas State University.

With this style of lecture, the instructor calls on students throughout the lecture and asks them a specific question to contribute to moving the next step of a problem or proof forward. In my implementation, when I reach points in my lecture where I identify a good question to pose, I would call on a student by shuffling a stack of cards with names and randomly selecting one, helping to prevent implicit bias or re-scaling of problem difficulty based on any perceptions about how well the student understands the material. Once called upon, if the student gives a correct answer, then the lecture moves forward with that knowledge. If the student gives an incorrect answer, the instructor works with the student from the board, akin to a one-on-one tutoring scenario, to help guide them towards a helpful intellectual contribution.

Since my focus is on students collaborating during class, I keep my lectures relatively short, typically no more than 20 minutes in a 50-minute class period. Having used lecture based tutoring in other courses, I know that it is paramount to create a classroom environment wherein students feel comfortable making intellectual contributions in front of the class and where it is alright to make mistakes or not know the answer. I've learned that, in order to make lecture based tutoring successful, *the instructor must make sure that every single contribution by a student when called upon is meaningful to move the conversation forward*. This can be challenging sometimes when a student does not know the correct answer, but I am always prepared to think creatively and help students discover that even if they do not know the correct answer, they have some insight into the question to move us towards a correct answer.

Successful implementation of lecture based tutoring helps students to stay engaged during lecture, gives everyone in the class a voice while boosting students' confidence as they contribute to the discussion, and contributes to the active learning environment that will be so important in the course.

3.1.2 Group Work

A majority of time in class was spent on students discussing and working through problems. I taught in a classroom ideally structured for this type of active learning - circular tables seating up to 6 students and whiteboards all the way around the room. I encouraged groups to work live at the boards while discussing problems, giving them extra space and mobility to engage in creative problem solving.

The tasks that were chosen for students to discuss in groups proved to be incredibly important. In order to facilitate productive discussion among all groups and knowledge bases, I used the following guiding principles when selecting and creating tasks:

- Most tasks should be low-floor, high-ceiling, allowing students that are less comfortable with the material to have an entry point into the conversation, but giving room for more advanced students to still be challenged.
- All tasks should have limited focus on computations, allowing students to focus on discussing problem solving techniques and logic, and hopefully not getting bogged down by comparing steps of algebra.
- Some tasks should synthesize concepts from the current and previous lessons in the course. The curriculum of the course encompasses a large selection of somewhat distinct topics, all of which serve as different and important settings for learning some aspect of communicating mathematics. Having tasks which bring together these distinct topics will give purpose to the curriculum, exemplify communicating about different types of mathematics simultaneously, and show the interconnections of mathematics.

During group work, my role was to float among groups, helping to support their collaborations and making decisions about how to structure the in-class presentations and wrap-up discussions based on groups' progress. I supported their collaborations by asking probing questions to understand groups' thinking, and if needed, by asking leading questions to focus the groups' thinking to make progress on the task. As groups made progress on the tasks, I would encourage them to try to communicate using formal mathematical language so as to be clear about their ideas.

3.1.3 In-Class Presentations

Time was set aside most day during class for students to present their group's discussions and progress to the class as a whole. Students were encouraged to use formal mathematical language when making these presentations and preparing their thoughts on a whiteboard. Although students were not graded explicitly on these presentations, the presentations served an important role as formative assessment for me to gauge groups' knowledge of the material and maturity with mathematical communication. I encouraged others in the class to write down or take pictures of the board work, and I asked students presenting to not simply "read their solutions off the board", but to explain their reasoning for important or tricky steps within a task so as to highlight the importance of the problems solving, not just the final product.

3.2 Course Materials and Assessment

Students engaged with material outside of class in two phases: prior to class, students familiarized themselves with definitions and theorems that were to be explored more deeply in class, and following class students were assessed on their knowledge and ability to apply the definitions, theorems, and problems solving techniques from class.

3.2.1 Materials and assessment used prior to class

The focus of in-class time was on group work and problem solving, not on lecture. Therefore, it was necessary for students to engage seriously with the material prior to class, structured in two components:

- Collaborative Textbook Annotations.
- Pre-Class Notes.

During the semester, I tried out a new piece of software called Perusall with the textbook for the course. Students rented or purchased a digital copy of the textbook within Perusall, which they then accessed via Perusall's website. I also encouraged students who preferred a physical copy of the book to purchase an old edition at a much cheaper cost. The benefit of Perusall is that students are able to collaboratively annotate a single digital copy of the textbook. Anyone can highlight a portion of the text and make a comment or ask a question, then students are able to discuss the initial annotation (possibly even answering one another's questions) via a series of posts styled similar to a discussion board. I encouraged them to comment with points to clarify the text or extend the text, with specific examples connecting to general statements, with links to external resources that helped them better understand a concept, or with proposed solutions to exercises given in the text. Students received credit for the quality and quantity of their annotations, and Perusall came equipped with artificial intelligence algorithms to assign an initial score for this. Prior to class, I would skim over all of the annotations, using discussions and questions to gauge students' understanding of topics in the text so as to inform what to focus on during class time. I also made manual adjustments to students' scores if the AI grading in Perusall assigned a lower score than I felt was warranted. Samples of annotation threads by students can be found in Appendix 6.4.

While reading the textbook and making annotations, students also filled in Pre-Class Notes for each chapter. These Pre-Class Notes are worksheets that have space for every definition, proposition, or theorem that would be used during class time, along with space for accompanying proofs that would be presented in class and a few simple fill-in-the-blank practice problems. To account for the textbook being digital, this gives students a hard copy of all results that they may wish to reference during class. Note that students were welcome to open the digital textbook while working in groups, but I did not want to assume that students would bring an internet-connected device to class. Another important purpose of these notes was to minimize the amount of time spent on lecture during class. I did exhibit the process of mathematical proof frequently, but I was able to efficiently go straight into the proof without writing out any statements of definitions or results that would be proved since students had that in their notes. Students submitted these notes via Canvas upload and they were graded on completion.

3.2.2 Materials and assessment used following class

Associated with each chapter of the textbook, students were assigned a few associated problems for homework. On homework, students were expected to compile their thoughts much more formally than in class, using careful mathematical language, justifying each portion of their arguments, and clearly referencing any results used. Students were encouraged to collaborate on these problems, aligning with the emphasis on group work in class, but they were required to submit final formal solutions in their own words.

There were two midterm exams and a cumulative final exam in the course. All exams were split into two portions – around 50% to 60% of the exam was timed as an "in-class" portion, and the rest was distributed as a "take-home" portion.

For the in-class portion of the first midterm exam, students had a single 50-minute class period, and students were unable to use any additional resources.

On the second in-class midterm and in-class final exam, students took a remote exam which was available over a 24-hour period, but which was timed once they began the first problem, and students were allowed to use resources like the textbook and annotations, their Pre-Class Notes, and other notes from the course. Questions on all in-class portions of exams were simpler, focusing on having students craft and communicate simple proofs from given results or determine the validity of various statements based on big-picture concepts in the course.

For the take-home portions, students had a few days and were allowed to use notes and the course textbook to carefully construct clean, formal proofs to more challenging problems similar to homework assignments. This portion was graded more strictly on technical details since students had more time and resources to complete these more difficult problems.

3.3 Links to Broader Curriculum and Course Goals

Recall the course goals, which are the driving factors determining components of the course structure, materials, and assessment:

G.1 Communicate effectively using the formal language of mathematics

G.2 Evaluate the validity of complex logic statements

G.3 Identify useful definitions and use them to make valid deductions

G.4 Recognize various mathematical proof techniques and know when to apply them

G.5 Read, comprehend, and evaluate the validity of mathematical proofs, and criticize invalid proofs

G.6 Write rigorous and concise mathematical proofs

G.7 Present and explain the reasoning behind mathematical proofs

Components of the course are linked to the course goals as follows:

- Lecture - Although many of the course goals were exhibited by the various proofs that I presented during lecture time, my goal was for this to be a relatively small portion of the course, so my focus was to exhibit how to string together definitions or results to prove larger statements (G.3).
- Group Work - The focus of this task was on giving students lots of time to struggle through crafting proofs together, practicing communicating well with their groups (G.1) and clearly explaining their reasoning to each other (G.7).

- In-Class Presentations - This task was an important component to the structure of the class. Many different conversations and great ideas were discussed in small groups during class, and these presentations served to collate ideas and progress across groups. While giving the presentations, students were expected to be efficient and clear with their explanations (G.1, G.7).
- Textbook Annotations - As students were introduced to mathematical proofs in this course, many were learning to achieve a different type of reading comprehension while reading a mathematical text since mathematical proofs often have more nuances than computational examples (G.5). Having a common platform for annotations helped students achieve this, being able to think and understand together, much like the group work that was emphasized in class.
- Homework - Given that students had ample time to work on homework assignments, these assessments were a chance to rigorously gauge students' ability to bring together the big picture components of the course, including carefully using definitions and results from the textbook (G.3) to craft polished mathematical proofs (G.7). Some homework problems asked students to critique complex arguments and possibly correct flaws in logic (G.2).
- Exams - Nearly all course goals were connected to students' work on exams, as outlined more explicitly in Section 3.3.1 below.

3.3.1 Explicit Connection Between Exams and Course Goals

The exams given throughout the semester can be found in Appendix 6.2. Some connections may be highlighted to the course goals listed above:

- G.1 was assessed in a unique way on Problem 3 on the Take-Home Midterm 2, and anecdotal feedback from students indicated that they particularly enjoyed the problem.
- G.2 was assessed frequently by having students determine whether complex statements were true or false before being asked to prove or provide a counterexample (see In-Class Midterm 1 Problem 3, In-Class Midterm 2 Problems 3 and 5, and In-Class Final Exam Problems 2 and 3).
- On a few exam problems, students were explicitly given a definition to carefully apply in a certain context, relating to G.3 (see In-Class Midterm 2 Problem 4 and In-Class Final Exam Problem 6).
- In relation to G.4, some exam problems assessed students' abilities to identify and use proof techniques in a somewhat open-ended format (see In-Class Midterm 2 Problem 5 and In-Class Final Exam Problem 2).
- Many exam problems required students to write out clear mathematical proofs, relating to G.6 (see, in particular, problems on Take-Home portions

of exams).

Chapter 4

Benchmark Memo 3: Analysis of Student Learning

In this chapter, we analyze two dimensions of student work from throughout the semester. We consider student engagement via the collaborative annotation platform used to access the textbook and complete pre-class reading assignments as well as "mastery level" scores on exam problems throughout the semester.

4.1 Analysis of Annotation Statistics

Recall a few of the course goals:

- G.1 Communicate effectively using the formal language of mathematics
- G.5 Read, comprehend, and evaluate the validity of mathematical proofs, and criticize invalid proofs

Something can be said about students' progress towards these course goals based on statistics gathered from Perusall, the collaborative annotation software that students used to access the textbook this semester, See Appendix 6.4 for a visual of the software and a sample of student annotations threads.

To give some background, students are able to highlight sections of text in the textbook and type an annotation to go along with the text. Anyone in the course is able to "upvote" one another's annotations, or to contribute to an ongoing comment thread stemming from an initial annotation. Perusall uses artificial intelligence to grade student annotations, factoring in quality, length, and relevance of the annotation to the highlighted portion of the text. While

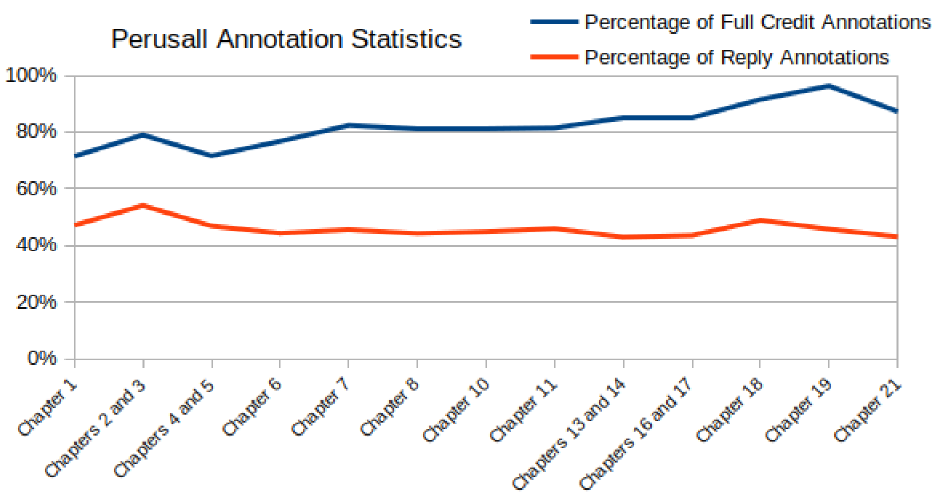


Figure 4.1: Graphs representing the percentage of responses which received full credit and the percentage of responses which were "replies" for each assignment across all 13 annotation assignments throughout the semester.

reviewing annotations, I make manual adjustments to scores when I feel that a good annotation was made, but the AI algorithm did not recognize this, although this was a relatively uncommon occurrence.

Figure 4.1 shows the percentage of student annotations on each assignment which received "full credit" (where annotations were scored as a 0, 1 or 2). This is shown by the blue trendline. This trendline followed a general upward trajectory throughout the semester, supporting the idea that students were making higher quality contributions to the conversation about the text. This indicates progress towards G.5, as students became more proficient with reading mathematical proofs.

Also depicted in Figure 4.1 is a trendline tracking the percentage of annotations on each assignment which were "replies" – that is, the percentage of annotations which were threaded beneath another student's initial annotation. This is shown by the red trendline. This trendline fluctuated between 43% and 54% throughout the semester. This seems to indicate that, on average, nearly every annotation in the textbook was the start to a conversation between students about the content – if every initial annotation had exactly one reply, this would indicate a 50% rate for reply annotations. Thus, students were having frequent conversations about mathematics even outside of class while they were preparing for the group work that would take place in class, contributing to G.1.

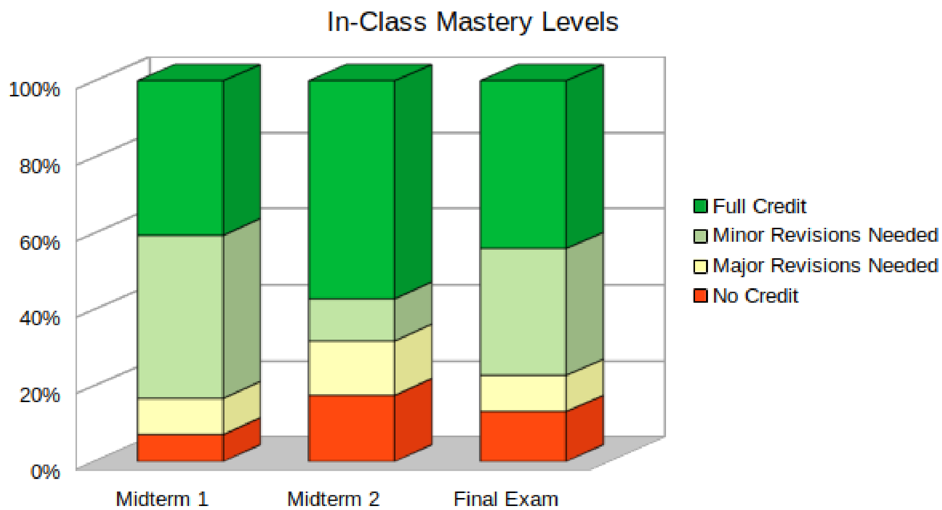


Figure 4.2: Percentage of students achieving different levels of mastery across in-class portions of semester exams.

4.2 Analysis of Exam Mastery Levels

Students were tasked with writing proofs on many exam questions throughout the semester. These problems were graded relatively uniformly throughout the semester, marked with a "mastery" level representing students' overall accuracy, structure, and presentation of a proof. Appendix 6.3 shows samples of student work on in-class and take-home portions of exams scored at the various mastery levels. An overall picture of the progression of proofs by the class as a whole across the semester can be understood by looking at this data.

4.2.1 In-Class Mastery Levels

The proof problems that were graded on a mastery level scale on the in-class portion of exams are as follows:

- Midterm 1 - Problem 2, Problem 4
- Midterm 2 - Problem 2 Part (c), Problem 4 Parts (a) and (b)
- Final Exam - Problem 2 Part (b), Problem 3 Part (a), Problem 6 Parts (a) and (b)

Figure 4.2 shows that on the in-class portion of the final exam, around 80% of submissions were in the "Full Credit" or "Minor Revisions Needed" mastery levels. These levels represent the quality of proofs that students should be able to produce in order to move on and communicate effectively in a subsequent math course.

The percentage of responses achieving the “Full Credit” or “Minor Revisions Needed” mastery levels on Midterm 1 was rather large. Of the in-class portions of exams, this one had the fewest proof problems graded at mastery levels, and the problems were more straightforward than other exams. This midterm also had the largest band of “Minor Revisions Needed” responses, indicating that although students were writing very good proofs, there was room for improvement.

The band of submissions achieving the “Full Credit” or “Minor Revisions Needed” mastery levels on Midterm 2 shrunk substantially. This was the first major assessment given after transitioning to remote learning in response to the COVID-19 pandemic, which affected students’ ability to engage with the material leading up to the exam. Interestingly, Midterm 2 had the largest percentage of responses across all exams in the ”Full Credit” category. It seems the transition was somewhat divisive. In fairness to all students in the class, limited resources were allowed on the in-class portions of the exam. Students typically in the “Minor Revisions Needed” group were likely pushed away from that group, either up into the “Full Credit” band if the transition to remote learning went smoothly and they were able to effectively use their additional resources, or into a lower band if the transition was more difficult.

4.2.2 Take-Home Mastery Levels

The proof problems that were graded on a mastery level scale on the in-class portion of exams are as follows:

- Midterm 1 - Problem 2, Problem 3
- Midterm 2 - Problem 1, Problem 2 Parts (a) and (b)
- Final Exam - Problem 1, Problem 2 Parts (a) and (b), Problem 4

Figure 4.3 indicates that around 70% of submissions on the take-home portion of the Final Exam were in the “Full Credit” or “Minor Revisions Needed” band. This portion of the exam had some of the most challenging questions that students were asked all semester, and problems were graded more strictly than problems on in-class portions of exams since students had additional time and resources. Therefore, having 70% of students achieving mastery or near-mastery is noteworthy on this final assessment.

The “Medium Revisions Needed” group on Midterm 2 was the only assessment that this mastery level was used along with the standard mastery levels. Formally, I would categorize this level as more aligned with the “Major Revisions Needed”, and Midterm 2 again saw the lowest percentage of responses

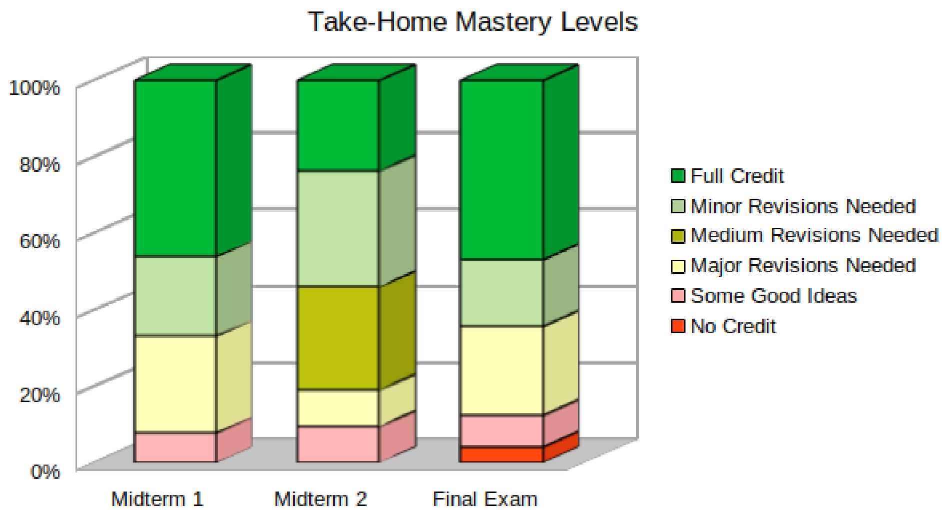


Figure 4.3: Percentage of students achieving different levels of mastery across in-class portions of semester exams.

in the highest two mastery level bands. This also means that the "Major Revisions Needed" band shrunk substantively between Midterm 2 and the Final Exam.

Around 10% of responses on the take-home portion of the final exam fell into the lowest two bands, "Some Good Ideas" and "No Credit". There were a few students who submitted partially-complete final exams, and most responses in these bands were assigned to those students.

Chapter 5

Summary

5.1 Successful Aspects of Course

Overall, the course ran very smoothly, and the time invested in intentional design of course components ensured that all of the most important aspects of the course could be truly emphasized. In particular, the careful planning for the course contributed to the successful implementation of (new-to-me) annotation software, focused time for in-class group work, and a strong foundation for a curriculum for the course for future semesters.

5.1.1 Collaborative Annotation Software

Giving students ample quality time to discuss mathematics actively in class was a driving force when designing the course. To make this possible, the amount of time needed for lecture had to somehow be reduced. This was made possible, in large part, by giving students a reason to engage seriously with the content prior to class, allowing time to be refocused towards active learning. Perusall, the collaborative annotation software used in the course, gave students deadlines and accountability to read prior to class, helped them become familiar with the material so as to apply the concepts more deeply during class, and gave them a space to extend the notion of collaboration from class while annotating and studying math together.

5.1.2 In-Class Group Work

The course started with an ice breaker wherein students received a few pages of logic puzzles – some quick and easy, others more involved and complicated – and they floated around the room, forming small groups to discuss problem solving strategies for the puzzles, and even to present solutions to some of them. I told students that this is what they would be doing all semester, working together to solve puzzles, just with a little more mathematics involved. This early precedent, which was set as a part of the careful planning that had been done for the course, along with the assignments that students completed prior to class, meant that group work continued to work well throughout the semester as students got more comfortable with the expectations and structure. Incorporating this type of active active learning successfully, which education literature supports as an effective teaching strategy, was a great accomplishment for the course.

5.1.3 Well-Developed Curriculum

At the end of the semester, I was very pleased with the curriculum that I developed, and I documented class activities and assignments in such a way to provide a versatile base structure for the next time that I teach the course, or the next time another faculty member in the department teaches the course and is interested in a flipped classroom style approach. The Department of Mathematics has a centralized Box course repository for instructors to upload any course materials, and I have made available my syllabus, all of my Pre-Class Notes and solutions, all of my homework assignments and solutions, and all of my exams (both in-class and take-home portions) and solutions. If an instructor were interested, I would be happy to share my lesson plans as well to give some inspiration on types of tasks which engaged students well throughout the semester. With this being only the second time ever that this course was offered in the department, I'm very happy to contribute to the pool of high quality materials that instructors throughout the department share around.

5.2 Limitations of Course

Some students prefer a more traditional lecture-style course. Being intentional at the start of the semester to get buy-in from students from the active learning structure is incredibly important, and although I tried to impress upon students the (research-backed) benefits to active learning, there were still students who expressed a preference for more lecture.

Additionally, the semester was particularly challenging for instructors around the nation due to the COVID-19 pandemic that arose partway through the semester. Having participated in the Peer Review of Teaching Project, it was clear to me what my objectives were for student learning and assessment, and the restructuring that had to take place for the course was made easier by having in mind the guiding principles that I had used to structure the course in the first place. The collaborative annotations and in-class group work proved to be absolutely vital in the transition to remote learning across campus - students had been practicing all semester collaborating in a digital space outside class, and had bought into the group work in-class. Without going on a tangent into the details of how the course was restructured for remote learning, I was incredibly impressed by the students' continued rich conversations about mathematics in the discussion boards which replaced in-class group work.

5.3 Future Plans for Course

Teaching a course for the first time is always a challenge, especially when little precedent has been set for the course. I have appreciated the opportunity to help define what this course looks like in the department, and I hope that other instructors are able to use some of the materials and structure developed this semester to incorporate active learning in their curriculum for the course.

At times in my planning, I thought about more alternative ways of assessing student learning, such as pre/post assessments on students' attitudes towards mathematics and peer-review assignments with proofs written by students. A formal way of assessing student presentations would also be beneficial to their learning. With so much other content to build from the outset of the course, I did not have time to prepare these components of the course, and this will be a priority for the next iteration of the course.

5.4 Final Thoughts

This course was my favorite that I have taught while at UNL. The intentional design that I engaged with in preparing for the course meant that I was more invested in the curriculum, and the frequent opportunities as part of the Peer Review of Teach Project to meet with colleagues across campus who were engaging in the same types of careful thought toward teaching helped me continue to reflect and adapt throughout the course of the semester.

I received the following email from a student shortly after the conclusion of the course:

> I am emailing you to talk about some critiques of the class that didn't really fit into a course evaluation, teacher evaluation, etc. I think this class would benefit greatly from a recitation section. Or to make this class an hour and 15 minutes or something like that. In class we would touch on a ton of topics and working in groups was beneficial but I felt like due to the time constraint of the class we were not able to thoroughly discuss some topics. Given how the class itself is structured I feel like 50 minutes with peers in class was not enough.

What the student does not realize is that this is one of the most genuine compliments that I have ever received about a course I taught. This comment says to me that the student was completely invested in the group work and active learning of the course and was willing to attend class longer to engage more deeply with the material in this way, a stark contrast to critiques of active learning courses where "the instructor didn't teach us anything, we had to do all of the work." The student went on to say:

> I also just want to mention some praise for this course. Even though I did not do as well in this class as I did other classes this semester,

> I do feel like this course stretched my rational thinking skills and problem solving abilities farther than other classes, not just this semester, have. I came into this class thinking I had a pretty general idea of mathematics and our discussions have just opened me up to an entire new world and new perspectives. I think the most humbling class I have ever taken due to the fact that it showed me that I really do not know math and no one really does until they can prove an idea. Thank you for facilitating this class and for bringing your personality. I hope to learn from you again.

The student has identified exactly the reason why I feel that Math 309 is such a special course - it can be a part of a mathematician's education at an incredibly important time as they transition from procedural and computationally-heavy courses like Calculus to proof-based courses at the 300-level and above, introducing them to a wider world of mathematics. I truly hope to work with this student again, along with the rest of the class from this past semester.

Chapter 6

Appendix

The appendix is organized as follows:

- Appendix 6.1 - Syllabus
- Appendix 6.2 - Midterms and Final Exam
- Appendix 6.3 - Samples of Work Representing Levels of Mastery
- Appendix 6.4 - Samples of Perusall Annotations

Spring 2020 **University of Nebraska–Lincoln**

MATH 309 – INTRODUCTION TO MATHEMATICAL PROOFS
COURSE SYLLABUS

Instructor: Professor Josh Brummer

Office:	Avery Hall 244
Email:	`jbrummer@unl.edu`
Office Hours:	Tuesday 11am-12pm and Friday 12:30pm-1:20pm, or by appointment

Course Objective: The goal of the course is to provide a foundation in understanding and writing rigorous mathematical proofs. This will be achieved by an introduction to formal mathematical language which includes mathematical notation and quantifiers, working with different mathematical objects (sets, functions, relations), and learning about different types of proofs. Specifically, a successful student in this course will be able to:

1. Communicate effectively using the formal language of mathematics.
2. Evaluate the validity of complex logic statements.
3. Identify useful definitions and use them to make valid deductions.
4. Recognize various mathematical proof techniques and know when to apply them.
5. Read, comprehend, and evaluate the validity of mathematical proofs, and criticize invalid proofs.
6. Write rigorous and concise mathematical proofs.
7. Present and explain the reasoning behind mathematical proofs.

The following major topics will be covered in this course: basic set theory, elements of logic, types of proofs, induction, relations and functions, and carinality of sets.

Course Prerequisites: A grade of P, C, or better in MATH 107 (Calculus II) or MATH 107H (Calculus II Honors).

Attendance Policy: Attendance is required in this class. Throughout the course, subtle aspects of mathematical logic will be discussed; the flow of ideas will have to be followed, and often the notes will not be sufficient to relay the content of a lecture. It is important that students are present to take notes that suit their own style and enhance their own individual learning, and to participate in group work and discussions about challenging problems. Please email the instructor about any missed lecture; it is the student's responsibility to make up the material that was missed.

Textbook: The textbook for this course is *A Concise Introduction to Pure Mathematics, Fourth Edition (Chapman & Hall/Crc Mathematics)*, by Martin Liebeck. We will be using Perusall, a collaborative and social ebook platform, to access the textbook in this course. To get access to the book:

1. Go to `https://app.perusall.com/` and either log in with a social media account, or create an account using your email address.
2. Select "I am a student" and enter the course access code: BRUMMER-CL7JA
3. The textbook must be purchased through Perusall to access the reading assignments. The first time you click on the book or a reading assignment, you will be prompted to purchase the book; you can choose either perpetual access (no time limit) or a 180-day rental.

Assessment: Grades for the course will be computed based on the following components, weighted correspondingly. Each component is explained in further detail below:

In-Class Participation	6%
Pre-Class Notes	6%
Textbook Annotations	8%
Written Homework	25%
Quizzes	15%
Midterms	20%
Final	20%

In-Class Participation (6%): Regular attendance and active participation are important to students' success in this course. A part of students' grades will be based on an assessment of participation in class. Note that this is an assessment of engagement in class, not a judgement about whether a student's work is correct. Active participation includes attending class on time, participating actively in group work, and in general, contributing positively to the academic environment of the classroom. The in-class participation score will be updated periodically throughout the semester.

Pre-Class Reading: For each chapter of the textbook, students will be required to carefully read over the chapter prior to the week of class when the material will be explored. This allows for more time to be spent in class going over examples and working in groups to gain deeper insight about the material. There are two components associated with these pre-class reading assignments, due weekly typically on Sundays at 10pm (which gives the instructor time to adjust the upcoming in-class material according to students' questions and feedback):

- **Pre-Class Notes (6%):** Students will be given a set of scaffolded notes that they will fill out while completing the pre-class reading. Students will turn this assignment in by uploading an image or scan of these notes to Canvas. Student are expected to bring these completed notes to class and use them as a resource while working through problems.
- **Textbook Annotations (8%):** Students will engage with the textbook together via Perusall, a collaborative annotation platform. These annotations are required and graded. The system will automatically grade each annotation. It keeps the 5 highest annotations, so to ensure the best possible score, students should aim for 7-10 good annotations. Annotations could be a summary of a piece of text for the class, asking an insightful question, answering other student's questions, linking to external resources you find helpful, etc. Annotations should be spread out through the whole assigned reading (and not all in one small area).

Written Homework (25%): There will be regular written homework assignments consisting of a selection of problems relating to a few chapters of the textbook. Written homework is usually due on Tuesdays. No late homework will be accepted, however, the lowest homework score will be dropped. Collaboration is encouraged on these assignments (unless otherwise specified), but each student needs to write their own solutions.
Note. Homework must be turned in to mailbox number 87. From the main math department office entrance (Avery 203), mailboxes are located down the hall (past the display cases with the pictures of faculty and graduate students) and just around the corner. It is the student's responsibility to turn in their homework at a time when Avery is accessible.

Quizzes (15%): There will be occasional in-class quizzes, some completed individually and some where collaboration will be allowed. They will give students targeted practice with the most important concepts studied. No make-up quizzes will be given, however, the lowest quiz score will be dropped.

Exams: There will be two midterms and a final exam given during the course. No references will be permitted during the exams. Additional details are as follows:

- **Midterms (10% each):** The (in-class) midterms are tentatively planned for Friday, February 28 and Wednesday, April 15.
- **Final Exam (20%):** The date, time, and location for the final exam are Friday, May 8, 7:30-9:30 am, in Brace 310.

Final Grades: The following table represents a "worst-case" scenario. This means that, for example, if a student earns an 84%, they are guaranteed at least a "B+", but the cut-off for a "B+" may be lower in the end.

Letter Grade	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-
Pct. Needed	96	90	87	84	80	77	74	70	67	64	60	57

Note. The minimum grade needed for a "P" if taking this class Pass-No Pass is a "C".

Course Evaluations: The Department of Mathematics course evaluation form will be available through Canvas during the last two weeks of class. Evaluations are anonymous and instructors do not see the responses until after final grades have been submitted. Evaluations are important – the department uses them to improve instruction. Please complete the evaluation and take the time to do so thoughtfully.

Academic Honesty Policy: Cheating, plagiarism, impermissible collaboration, and misrepresentation to avoid academic work are serious violations of the Student Code of Conduct (`https://studentconduct.unl.edu/student-code-conduct`) and may be subject to both academic and disciplinary sanctions as severe as giving a failing grade for the course. Further, your instructor may recommend the institution of disciplinary proceedings for the violation of the Student Code.

Grading Appeals Policy: The Department of Mathematics does not tolerate discrimination or harassment on the basis of race, gender, religion, or sexual orientation. If you believe you have been subject to such discrimination or harassment, in this or any other math course, please contact the department. If, for this or any other reason, you believe your grade was assigned incorrectly or capriciously, then appeals may be made to (in order) the instructor, the vice chair, the Department grading appeals committee, the College of Arts and Sciences grading appeals committee, and the University grading appeals committee.

Disability Accommodation: The University of Nebraska-Lincoln is committed to providing flexible and individualized accommodation to students with documented disabilities that may affect their ability to fully participate in course activities or to meet course requirements. Students with disabilities are encouraged to contact the instructor for a confidential discussion of their individual needs for academic accommodation. To receive accommodation services, students must be registered with the Services for Students with Disabilities (SSD) office, 132 Canfield Administration, 472-3787 voice or TTY.

Week	Chapter	Topic(s)
Week 1 (1/13–1/17)		
	1	Sets and Proofs
Week 2 (1/20–1/24)		
	2	Number Systems
	3	Decimals
	No class on Monday, January 20 - Martin Luther King Jr. Day	
	Tuesday, January 21 is last day to add a course or to drop with full refund	
Week 3 (1/27–1/31)		
	4	n^{th} Roots and Rational Powers
	5	Inequalities
Week 4 (2/3–2/7)		
	6	Complex Numbers
Week 5 (2/10–2/14)		
	7	Polynomial Equations
Week 6 (2/17–2/21)		
	8	Induction
Week 7 (2/24–2/28)		
	10	The Integers
	Midterm 1 tentatively scheduled for Friday, February 28.	
Week 8 (3/2–3/6)		
	11	Prime Factorization
	Friday, March 6 is last day to switch to or from "Pass/No Pass"	
Week 9 (3/9–3/13)		
	13	Congruence of Integers
	14	More on Congruence
Week 10 (3/16–3/20)		
	16	Counting and Choosing
	17	More on Sets
Week 11 (3/23–3/27)		
	No class, Spring Break	
Week 12 (3/30–4/3)		
	18	Equivalence Relations
	Friday, April 3 is last day to withdraw (grade of W)	
Week 13 (4/6–4/10)		
	19	Functions
Week 14 (4/13–4/17)		
	21	Infinity
	Midterm 2 tentatively scheduled for Wednesday, April 15.	
Week 15 (4/20–4/24)		
	22	Introduction to Analysis: Bounds
	23	More on Analysis: Limits
	24	Yet More on Analysis: Continuity
Week 16 (4/27–5/1)		
	Review for Final Exam	
	Final Exam is 7:30am-9:30am on Friday, May 8	

Math 309 - Introduction to Proof Theory
Friday, February 28, 2020

Name: ______________________

Midterm 1 In-Class Portion

This is the in-class portion of the midterm exam. No notes, books, or electronic devices may be used on this portion of the exam. Solutions and proofs do not need to be completely formal, but **write everything as cleanly as possible in the allotted time, clearly justifying all steps of proofs.**

Question	Points	Score
1	10	
2	10	
3	15	
4	10	
5	15	
Total:	60	

1. (10 points) Negate the following statements:

   (a) $\forall x, y \in \mathbb{R},\ \exists z \in \mathbb{R},\quad x + y = z^2$

   (b) The sum of every set of 3 integers is a positive number.

2. (10 points) Prove the following proposition:

   Proposition. For all integers n with $n \geq 1$,

$$4 + 8 + \ldots + 4n = 2n(n + 1)$$

3. (15 points) Mark all of the following statements as **True** or **False**. If the statement is false, briefly explain your reasoning (but you do not need to formally prove the statement is false).

______ $\mathbb{Z} \subseteq \mathbb{Q}$

______ $0.24224222422224222224\ldots$ is a rational number.

______ The notation $\sum_{r=1}^{3} 10^r$ represents the value 1110.

______ If $|z| = 1$ for some $z \in \mathbb{C}$, then it possible for z to be a solution to the equation $z^6 = 6i$.

______ A fourth degree polynomial could have exactly 1 real root along with 3 distinct complex roots.

4. (10 points) Determine all $x \in \mathbb{R}$ which satisfy the following statement:

$$4x < |x| + 5$$

5. (15 points) Consider the following three techniques associated with proving or disproving results:

   - Direct proof
   - Proof by contradiction
   - Disproof by counterexample

   Three statements are given below, two of which are true statements and one of which is a false statement. Use each of the techniques listed above *exactly once* to prove or disprove the three statements given below, and clearly indicate which technique you are using for each.

   (a) $x - (x + 5) < -3$ for any real number $x \in \mathbb{R}$.

   (b) The square of any odd number is odd.

   (c) The sum of any two irrational numbers is irrational.

Midterm 1 Take-Home Portion

Due at start of class on Wednesday, March 4, 2020

This is the take-home portion of the midterm exam. You may reference your notes, the textbook, and its annotations, but no other resources are allowed while working on this portion of the exam. You may **not** collaborate with anyone on this portion of the exam. Write complete, **formal** proofs for each question below. **Do not use any results from after Chapter 8 in the textbook**, and clearly reference anytime you use results from the textbook.

1. (12 points) Prove that $\sqrt[3]{3}$, the cube root of 3, is irrational.

2. (14 points) Find all solutions $z \in \mathbb{C}$ to the following equation:

$$z^8 + 5 = i5\sqrt{3}.$$

Clearly graph and label all solutions in an Argand diagram.

3. (14 points) Prove the following proposition using induction. Note that you will not receive any credit if you prove the proposition using any method other than induction.

Proposition. If $x \in \mathbb{R}$ with $x \neq 1$, then for all $n \in \mathbb{Z}$ with $n \geq 1$,

$$x + x^2 + \ldots + x^n = \frac{x(1 - x^n)}{1 - x}.$$

Math 309 - Introduction to Proof Theory
Wednesday, April 15, 2020

Name: ______________________________

Midterm 2 In-Class Portion

This is the in-class portion of the midterm exam. No notes, books, or electronic devices may be used on this portion of the exam. Solutions and proofs do not need to be completely formal, but **write everything as cleanly as possible in the allotted time, clearly justifying all steps of proofs.**

Question	Points	Score
1	0	
2	15	
3	15	
4	10	
5	10	
Total:	50	

1. (0 points) Write out the following statement and sign your name after it to verify that you will only use approved resources on this midterm.

   I will only use the course textbook and classmate annotations, Pre-Class notes, and content within the course Canvas page (discussion boards and wrap-up posts) to complete this portion of the midterm.

   (sign you name)

2. (15 points) This question involves prime factorizations, highest common factors, and least common multiples.

   (a) Write the prime factorizations of 84 and 1960.

   (b) Find $\text{hcf}(84, 1960)$ and $\text{lcm}(84, 1960)$.

   (c) Let $a, b \in \mathbb{Z}$ be arbitrary positive integers. Prove that $\text{hcf}(a, b)$ divides $\text{lcm}(a, b)$.

3. (15 points) Mark all of the following statements as **True** or **False**. If the statement is false, briefly explain your reasoning (but you do not need to formally prove the statement is false).

______ $8^2 \equiv -5 \pmod{23}$

______ It is possible for a positive integer of the form p^2, where p is a prime number, to also be a perfect cube. That is, p^2 could be equal to a^3 for some $a \in \mathbb{Z}$.

______ There exist distinct sets A and B such that both $A \cup B$ and $A \cap B$ are empty.

______ The number of 5-digit numbers greater than or equal to 50000 is equal to $5 \cdot 10^4$.

______ There exists an equivalence relation on $\{1, 2, 3, 4, 5\}$ which has 6 equivalence classes.

4. (10 points) Recall the following principle for sets:

Inclusion-Exclusion Principle. If A and B are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

(a) Use the Inclusion-Exclusion Principle to prove the following statement:

Statement A. Let S and T be sets. If $S \subseteq T$, then $|S \cup T| = |T|$.

You must use the Inclusion-Exclusion Principle to receive credit on this question.

(b) Use the Inclusion-Exclusion Principle to prove the following statement:

Statement B. Let M and N be sets. Then $|M \cup N| \leq |M| + |N|$.

You must use the Inclusion-Exclusion Principle to receive credit on this question.

5. (10 points) This problem involves relations defined on $\mathbb{Z}$. Only one of them is an equivalence relation. Prove that one relation is an equivalence relation, and prove that the other is not an equivalence relation.

   (a) $a \sim b \iff (a - b)$ is divisible by 5

   (b) $x \sim y \iff xy$ is even

Midterm 2 Take-Home Portion

Due on Gradescope at 11:59pm on Sunday, April 19, 2020

This is the take-home portion of the midterm exam. You may reference your notes, the textbook, and its annotations, but no other resources are allowed while working on this portion of the exam. You may **not** collaborate with anyone on this portion of the exam. Write complete, **formal** proofs for each question below. **Do not use any results from after Chapter 18 in the textbook**, and clearly reference anytime you use results from the textbook.

1. (10 points) Give a general formula (which may involve specific numbers and/or variables) for all integers n such that $28n$ is a perfect cube. Use results from the textbook to support your claim, and to prove why your formula represents all possible values of n.

2. (20 points) Let a be an "integer" in base 6. That is, if a has $n+1$ digits given by $a_n, \ldots, a_1, a_0$ (with $0 \leq a_i \leq 5$ for each $0 \leq i \leq n$), then a has the form

$$a = a_n a_{n-1} \ldots a_1 a_0 = a_n \cdot 6^n + +a_{n-1} \cdot 6^{n-1} + \ldots + a_1 \cdot 6^1 + a_0.$$

Prove the following divisibility rules in base 6. You must use the language of integer congruence, examples of which can be found in the textbook.

(a) "Rule of 4": If the last 2 digits of a are divisible by 4, then a is divisible by 4.

(b) "Rule of 5": If the sum of the digits of a is divisible by 5, then a is divisible by 5.

3. (20 points) Describe a common scenario where you might make a decision that has more than ten thousand possible outcomes. You do not have to account for every single aspect of the scenario, but you must **clearly justify that it has <u>at least</u> ten thousand outcomes**. Your scenario must incorporate all of the following components:

- At least 4 separate stages
- The Multiplication Principle
- A "$k!$" component for some integer k
- An "n choose r" component for some integers n and r
- Detailed answer laid out in paragraph form with full sentences

You will be graded on satisfying each component listed above correctly, and on how clearly you communicate these ideas. *It is fine to make assumptions in a problem like this, just make sure you clearly lay them out.*

An example of a decision and some relevant factors is given below, but **you must come up with your own scenario** other than this example. This example may not include all of the components indicated above, but is only to be used as a general idea of how you should be thinking about your solution.

Example. You decide to go to a movie with a friend, and you count the number of ways that you can experience this outing. For example, some factors that affect this include:

1. Which of your friends you go with (among your 10 movie-loving friends)
2. Which movie you see (among 8 possible movies)
3. Which two snacks you choose (among 30 possible snacks at the snack bar)

Math 309 - Introduction to Proof Theory
Friday, May 8, 2020

Name: ______________________

Final Exam In-Class Portion

This is the in-class portion of the final exam. Solutions and proofs do not need to be completely formal, but **write everything as cleanly as possible in the allotted time, clearly justifying all steps of proofs.**

Question	Points	Score
1	0	
2	10	
3	10	
4	10	
5	8	
6	12	
Total:	50	

1. (0 points) I certify that the work on this exam will be mine and mine alone. I understand that discussing this exam with anyone else in the class or with anyone outside the class, or consulting any website where people submit problems and other people provide solutions is considered cheating and will be referred to the Office of Student Affairs for investigation as a possible violation of the Student Code of Conduct.

   ○ I so certify (this is equivalent to your signature).

   ○ I do not so certify.

2. (10 points) Consider the following two statements:

   - Statement 1 - There exist integers $x, y \in \mathbb{Z}$ such that $x + y$ is not divisible by $\text{hcf}(x, y)$.
   - Statement 2 - For all positive integers $j, k \in \mathbb{Z}$, $jk < 10 \cdot \text{lcm}(j, k)$.

   Recall that the hcf, or highest common factor, of two numbers is the largest divisor that the numbers have in common. The lcm, or least common multiple, of two numbers is the smallest number which is divisible by both numbers.

   (a) Give the negation of **both** statements in the bullet points above.

   (b) Prove the negation of **one** of the statements from above. Explain what this means about the corresponding original statement listed in the bullet point given above.

3. (10 points) For each statement below, indicate whether it is true or false. If it is true, prove the statement. If it is false, provide a counterexample.

______ The average value of any even and odd integer is not itself an integer. That is, for any $a, b \in \mathbb{Z}$ with a even and b odd, it must be that $\frac{a+b}{2} \notin \mathbb{Z}$.

______ $p = 2$ is the smallest prime number. For any integer $n \in \mathbb{Z}$ with $n \geq 2^{10}$, n must have at least 10 prime numbers in its prime factorization.

4. (10 points) Each part of this problem gives a true statement. Give a brief explanation of the reasoning behind why each statement is true. **You do not need to fully prove any of the statements, just give the intuition.**

   (a) There exists a mapping $f : A \to B$ between two sets A and B which is neither one-to-one nor onto. Nonetheless, sets A and B are equivalent as sets (recall the definition for equivalent sets from Chapter 21).

   (b) If $t \equiv 0 \pmod{m}$ and $w \equiv 0 \pmod{n}$, then $tw \equiv 0 \pmod{mn}$. (Carefully note the mn piece in the last equivalence.)

5. (8 points) The two parts of this question are unrelated. For each part, either give an example of a pair of real numbers $a, c \in \mathbb{R}$ that satisfy the statement, or indicate why no such pair exists.

   (a) $\frac{a}{c}$ is irrational and $\frac{c}{a}$ is rational (with $a \neq 0$ and $c \neq 0$)

   (b) $\frac{a}{c}$ is irrational and $\frac{a^2}{c^2}$ is rational

6. (12 points) Recall the following proposition from the textbook about roots of polynomials:

Proposition 7.1

Let the roots of the equation

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$$

be $\alpha_1, \alpha_2, \ldots, \alpha_n$. If s_1 denotes the sum of the roots, s_2 denotes the sum of all products of pairs of roots, s_3 denotes the sum of all products of triples of roots, and so on, then

$$\begin{aligned} s_1 = \alpha_1 + \ldots + \alpha_n &= -a_{n-1}, \\ s_2 &= a_{n-2}, \\ s_3 &= -a_{n-3}, \\ \ldots \quad & \quad \ldots \\ s_n = \alpha_1\alpha_2\ldots\alpha_n &= (-1)^n a_0. \end{aligned}$$

Use this proposition to answer the following questions. You should be very explicit about how you are applying the proposition.

(a) Consider the polynomial $p(x) = 10x^5 + 20x^4 - 150x^3 + 70x^2 + 200x - 40$. What is the product of all roots of the equation $p(x) = 0$?

(b) Consider a generic polynomial of the form

$$q(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

but suppose that all coefficients of the polynomial $a_{n-1}, \ldots, a_1, a_0 \in \mathbb{Z}$ are integers. Prove that it is not possible to have exactly one non-integer root to the equation $q(x) = 0$.

Final Exam Take-Home Portion

Due on Gradescope at 11:59pm on Friday, May 8, 2020

This is the take-home portion of the midterm exam. You may reference your notes, the textbook and its annotations, and past homework assignments, quizzes, and exams, but no other resources are allowed while working on this portion of the exam. You may **not** collaborate with anyone on this portion of the exam. Write complete, **formal** proofs for each question below. **Do not use any results from sources other than the approved materials listed above**, and clearly reference anytime you use results from any of these materials.

1. (16 points) Let $k \in \mathbb{Z}$ be a fixed integer, and let $P_k(n)$ be the following statement associated to some natural number $n \in \mathbb{N}$:

$$P_k(n)\text{: } n^5 + 4n + k \text{ is divisible by } 5$$

   (a) Prove an "inductive step" for this statement. That is, prove $P_k(n) \implies P_k(n+1)$ for all natural numbers $n \in \mathbb{N}$.

   (b) Determine all values of $k \in \mathbb{Z}$ for which the statement $P_k(n)$ is true for all natural numbers n. Summarize your results in a concise statement, and indicate what proof techniques imply your statement is true, based on your results in this problem.

2. (14 points) Consider a positive natural number of the form $n = p_1^{a_1} \cdot \ldots \cdot p_m^{a_m}$, where $p_1, \ldots, p_m \in \mathbb{N}$ are prime numbers and $a_1, \ldots, a_m \in \mathbb{N}$. For each part, be sure to show and fully justify your computations, and clearly indicate why the solutions you have counted are unique.

   (a) How many unique divisors does n have (not counting 1)?

   (b) Assuming $m \geq 3$, how many unique divisors does n have of the form $(x \cdot y \cdot z)$, where $x, y, z \in \mathbb{N}$ are unique prime numbers.

3. (8 points) Suppose that $\sim$ is a relation defined on $\mathbb{R} \times \mathbb{R}$ via $(x_1, y_1) \sim (x_2, y_2) \iff x_1 = x_2$. Prove that $\sim$ is an equivalence relation, and explain geometrically all equivalence classes of $\sim$.

4. (12 points) Prove that the sets $X = [0, 10)$ and $Y = (0, 40]$ are equivalent (as sets) by defining an explicit bijective function between. You must prove that your function is a bijection.

6.3 Samples of Work Representing Levels of Mastery

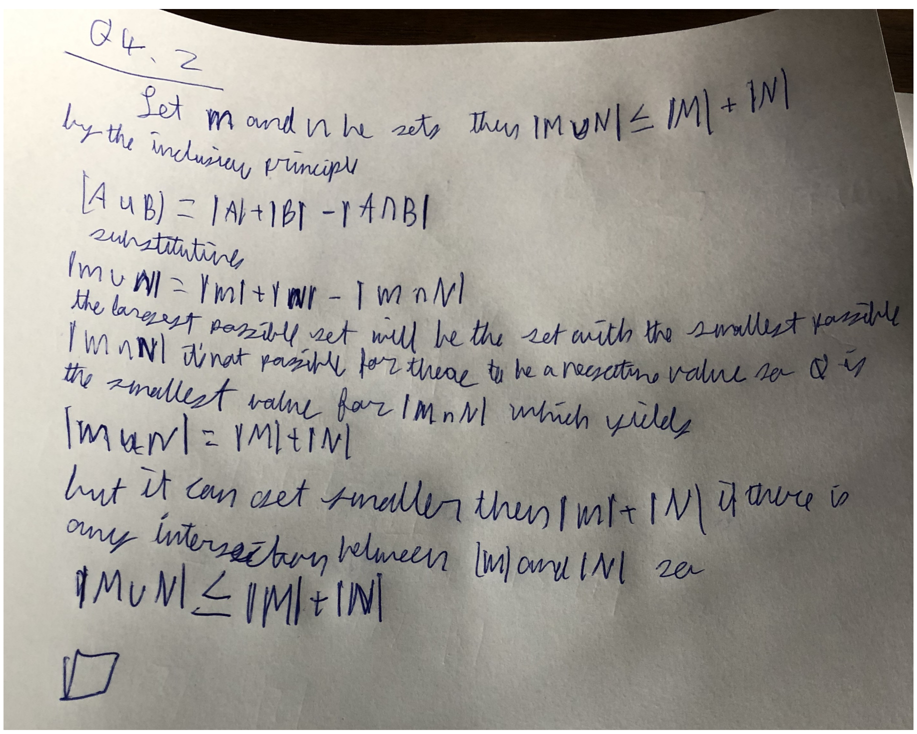
This section provides samples of student work that represent various levels of mastery on the grading scale used to assign points on exam problems requiring students to write proofs.

6.3.1 In-Class Exam Mastery Levels

Rubric items were assigned using the following scale for proof problems on the in-class portions of exams:

- Full Credit
- Minor Revisions Needed
- Major Revisions Needed
- No Credit

The following selections of student work exhibit the different levels of mastery for student responses to Problem 4 Part (b) on the In-Class portion of Midterm 2 (see Appendix 6.2.3).



Q4.2

Let M and N be sets then $|M \cup N| \leq |M| + |N|$
by the inclusion principle

$|A \cup B| = |A| + |B| - |A \cap B|$

substituting

$|M \cup N| = |M| + |N| - |M \cap N|$

the largest possible set will be the set with the smallest possible $|M \cap N|$ it's not possible for there to be a negative value so 0 is the smallest value for $|M \cap N|$ which yields

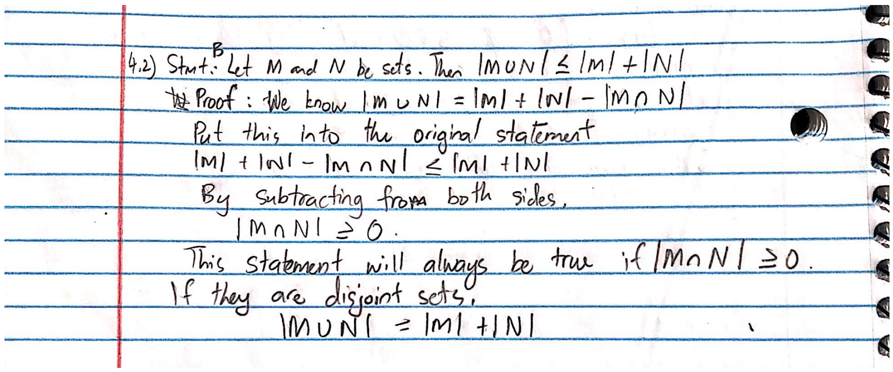
$|M \cup N| = |M| + |N|$

but it can get smaller then $|M| + |N|$ if there is any intersection between $|M|$ and $|N|$ so

$|M \cup N| \leq |M| + |N|$

□

In-Class Mastery, Full Credit



4.2) Stmt:^{B} Let M and N be sets. Then $|M \cup N| \leq |M| + |N|$

Proof: We know $|M \cup N| = |M| + |N| - |M \cap N|$

Put this into the original statement

$|M| + |N| - |M \cap N| \leq |M| + |N|$

By subtracting from both sides,

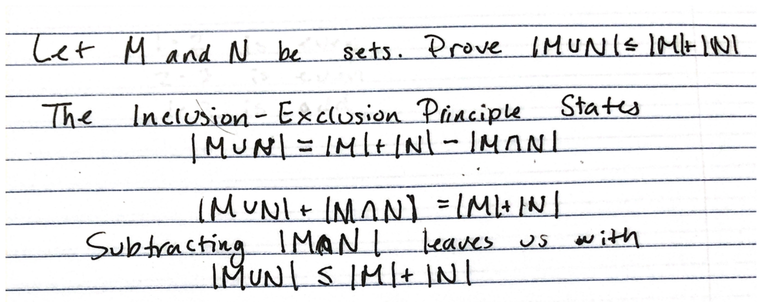
$|M \cap N| \geq 0$.

This statement will always be true if $|M \cap N| \geq 0$.

If they are disjoint sets,

$|M \cup N| = |M| + |N|$

In-Class Mastery, Minor Revisions Needed - Instructor comment: "You should argue why you can reverse the direction of your reasoning. You show that assuming the result implies a true statement. Instead, you should show that a true statement implies the desired result."



Let M and N be sets. Prove $|M \cup N| \leq |M| + |N|$

The Inclusion-Exclusion Principle States

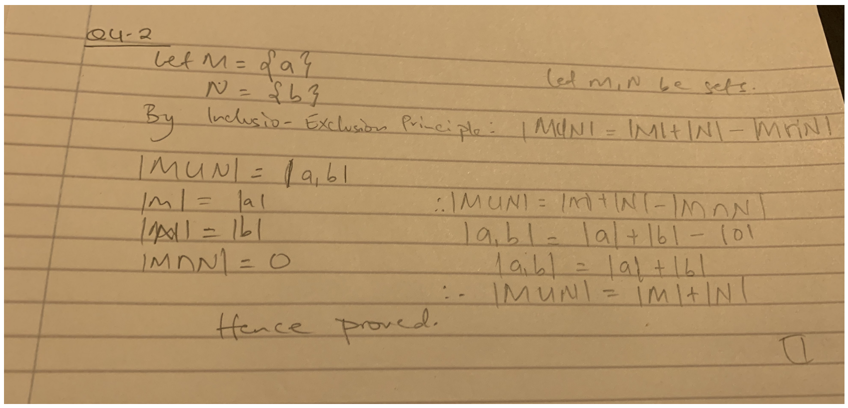
$|M \cup N| = |M| + |N| - |M \cap N|$

$|M \cup N| + |M \cap N| = |M| + |N|$

Subtracting $|M \cap N|$ leaves us with

$|M \cup N| \leq |M| + |N|$

In-Class Mastery, Major Revisions Needed - Instructor comment: "You need to argue that since $|M \cap N|$ is positive, the left hand side gets smaller by subtracting the quantity."



In-Class Mastery, No Credit - Instructor comment: "This result does not require that M and N are disjoint. Their intersection could be non-empty, and the result should still hold. This proof is for a single specific case. You need to prove the statement for arbitrary sets M and N."

6.3.2 Take-Home Exam Mastery Levels

Rubric items were assigned using the following scale for proof problems on the take-home portions of exams:

- Full Credit
- Minor Revisions Needed
- Major Revisions Needed
- Some Good Ideas
- No Credit

The following selections of student work exhibit the different levels of mastery for student responses to Problem 2 Part (b) on the Take-Home portion of the Final Exam (see Appendix 6.2.6).

b) Assuming $m \geq 3$, how many unique divisors does n have of the form $(x \cdot y \cdot z)$ where $x, y, z \in \mathbb{N}$ are prime numbers.

From the given form of n, we know that n has m distinct prime factors. We want to find how many different ways there are to choose 3 of these m prime factors.

This can be calculated using $\binom{m}{3}$ ("m choose 3")
Using Proposition 16.2 we know that the number of unique divisors of n of the form $(x \cdot y \cdot z)$ where $x, y, z \in \mathbb{N}$ are unique primes is

$$\frac{m!}{3!\,(m-3)!} = \frac{m!}{6\,(m-3)!}$$

2 b (cont)

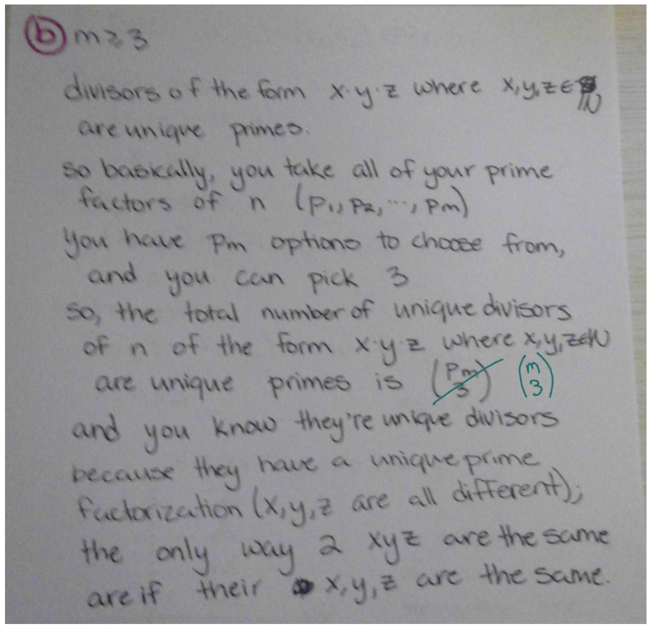
With $m \leq 3$ we know that $\binom{m}{3}$ will be defined for all m that fit this criteria because we won't have factorials of negative values or division by 0.

It doesn't matter if any of the prime factors have exponents that are greater than 1 because we defined the desired form $(x \cdot y \cdot z)$ to be a product of 3 unique primes. In other words, once one of factors is chosen for x, it can't be selected for y or z.

By definition of "n choose r" and assuming there were no repeated primes in the factorization of n

$1 \leq i, j \leq m$ → ($p_i = p_j$ only when $i = j$) we know that each possible factor of the desired form is unique.

Take-Home Mastery, Full Credit



(b) $m \geq 3$

divisors of the form $x \cdot y \cdot z$ where $x, y, z \in \mathbb{N}$ are unique primes.

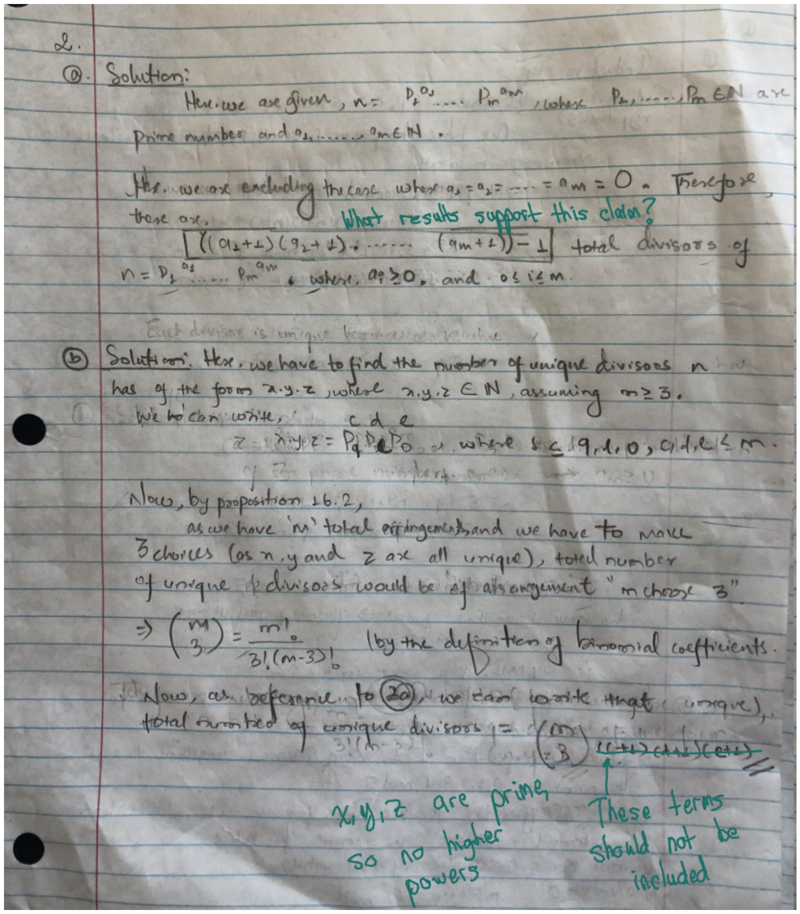
so basically, you take all of your prime factors of n $(p_1, p_2, \cdots, p_m)$

you have p_m options to choose from, and you can pick 3

so, the total number of unique divisors of n of the form $x \cdot y \cdot z$ where $x, y, z \in \mathbb{N}$ are unique primes is ~~$\binom{p_m}{3}$~~ $\binom{m}{3}$

and you know they're unique divisors because they have a unique prime factorization (x, y, z are all different); the only way 2 xyz are the same are if their x, y, z are the same.

Take-Home Mastery, Minor Revisions Needed



2.

(a). Solution:

Here, we are given, $n = p_1^{a_1} \cdots p_m^{a_m}$, where $p_1, \ldots, p_m \in \mathbb{N}$ are prime number and $a_1, \ldots, a_m \in \mathbb{N}$.

Here, we are excluding the case where $a_1 = a_2 = \cdots = a_m = 0$. Therefore, there are,

What results support this claim?

$\boxed{((a_1+1)(a_2+1) \cdots (a_m+1)) - 1}$ total divisors of $n = p_1^{a_1} \cdots p_m^{a_m}$, where, $a_i \geq 0$, and $0 \leq i \leq m$.

(b) Solution: Here, we have to find the number of unique divisors n has of the form $x \cdot y \cdot z$, where $x, y, z \in \mathbb{N}$, assuming $m \geq 3$.

We can write,

$x \cdot y \cdot z = p_q^{c} p_l^{d} p_o^{e}$, where $1 \leq q, l, o, c, d, e \leq m$.

Now, by proposition 16.2,

as we have 'm' total arrangements, and we have to make 3 choices (as x, y and z are all unique), total number of unique divisors would be of arrangement "m choose 3".

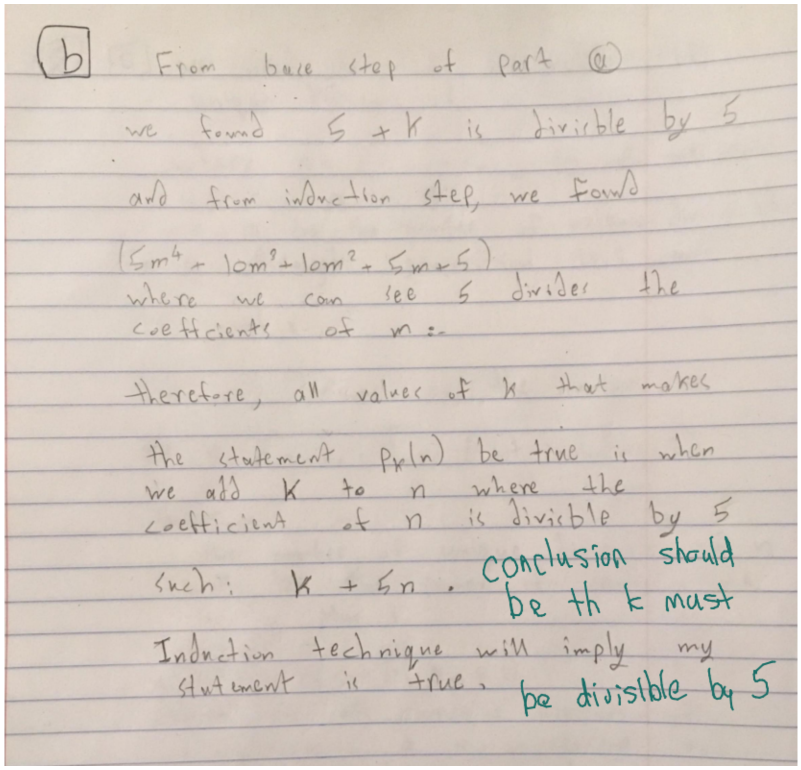
$\Rightarrow \binom{m}{3} = \frac{m!}{3!(m-3)!}$ | by the definition of binomial coefficients.

Now, as reference to (2a), we can write that (unique), total number of unique divisors $= \binom{m}{3}$ ~~$(c+1)(d+1)(e+1)$~~ //

x, y, z are prime, so no higher powers

These terms should not be included

Take-Home Mastery, Major Revisions Needed


b

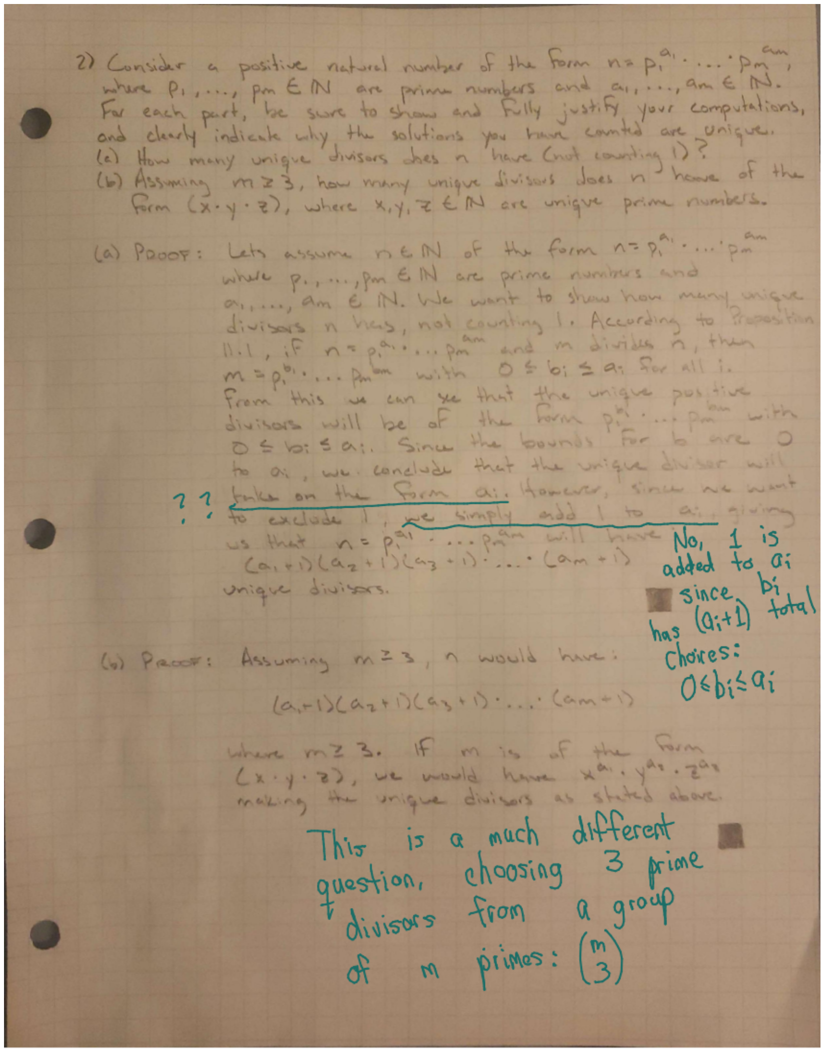
From base step of part (a)
we found $5 + k$ is divisible by 5
and from induction step, we found
$(5m^4 + 10m^3 + 10m^2 + 5m + 5)$
where we can see 5 divides the
coefficients of m.

therefore, all values of k that makes
the statement $P_k(n)$ be true is when
we add k to n where the
coefficient of n is divisible by 5
such: $k + 5n$.

Induction technique will imply my
statement is true.

conclusion should
be th k must
be divisible by 5

Take-Home Mastery, Some Good Ideas



2) Consider a positive natural number of the form $n = p_1^{a_1} \cdot \ldots \cdot p_m^{a_m}$, where $p_1, \ldots, p_m \in \mathbb{N}$ are prime numbers and $a_1, \ldots, a_m \in \mathbb{N}$. For each part, be sure to show and fully justify your computations, and clearly indicate why the solutions you have counted are unique.
(a) How many unique divisors does n have (not counting 1)?
(b) Assuming $m \geq 3$, how many unique divisors does n have of the form $(x \cdot y \cdot z)$, where $x, y, z \in \mathbb{N}$ are unique prime numbers.

(a) PROOF: Lets assume $n \in \mathbb{N}$ of the form $n = p_1^{a_1} \cdot \ldots \cdot p_m^{a_m}$ where $p_1, \ldots, p_m \in \mathbb{N}$ are prime numbers and $a_1, \ldots, a_m \in \mathbb{N}$. We want to show how many unique divisors n has, not counting 1. According to Proposition 11.1, if $n = p_1^{a_1} \cdots p_m^{a_m}$ and m divides n, then $m = p_1^{b_1} \cdots p_m^{b_m}$ with $0 \leq b_i \leq a_i$ for all i. From this we can see that the unique positive divisors will be of the form $p_1^{b_1} \cdots p_m^{b_m}$ with $0 \leq b_i \leq a_i$. Since the bounds for b are 0 to a_i, we conclude that the unique divisor will take on the form a_i. However, since we want to exclude 1, we simply add 1 to a_i, giving us that $n = p_1^{a_1} \cdots p_m^{a_m}$ will have $(a_1+1)(a_2+1)(a_3+1) \cdot \ldots \cdot (a_m+1)$ unique divisors.

?? No, 1 is added to a_i since b_i has (a_i+1) total choices: $0 \leq b_i \leq a_i$

(b) PROOF: Assuming $m \geq 3$, n would have:

$$(a_1+1)(a_2+1)(a_3+1) \cdot \ldots \cdot (a_m+1)$$

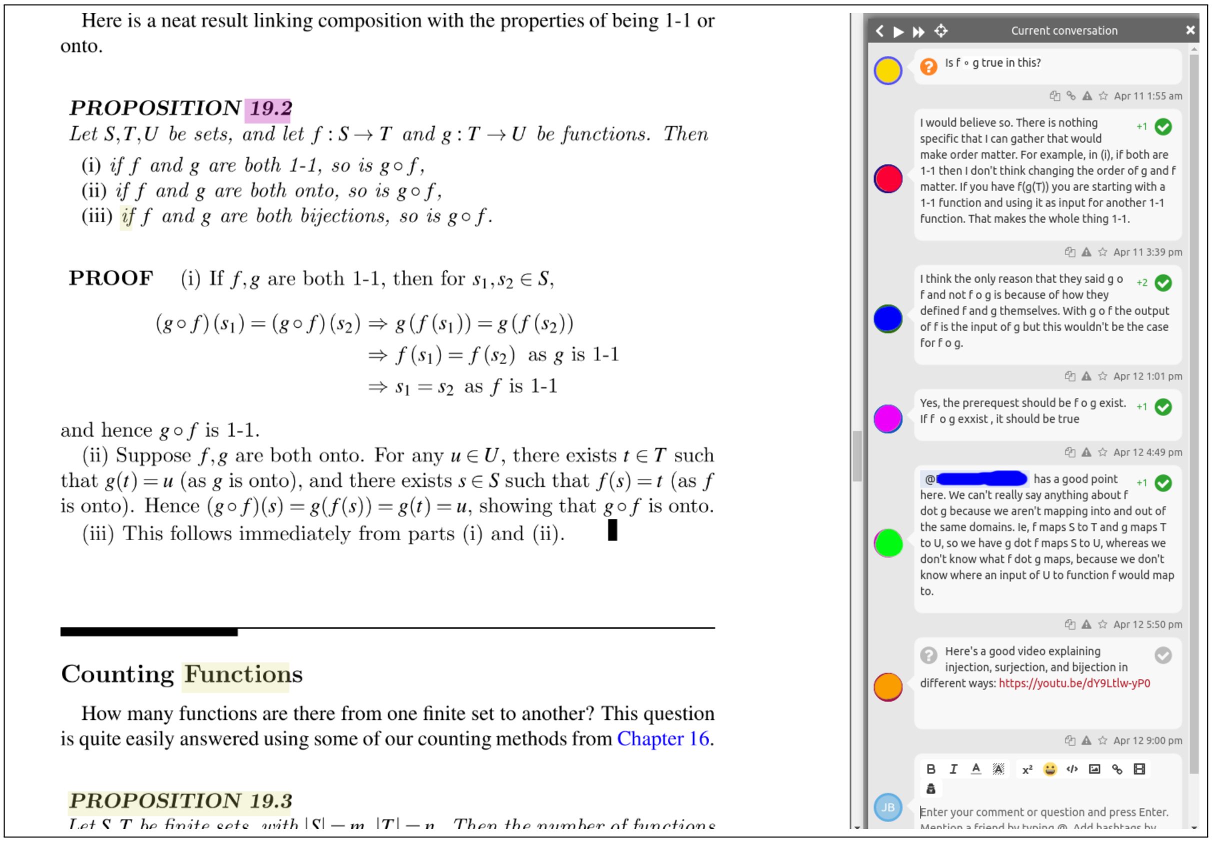
where $m \geq 3$. If m is of the form $(x \cdot y \cdot z)$, we would have $x^{a_1} \cdot y^{a_2} \cdot z^{a_3}$ making the unique divisors as stated above.

This is a much different question, choosing 3 prime divisors from a group of m primes: $\binom{m}{3}$

Take-Home Mastery, No Credit

6.4 Samples of Perusall Annotations

See below a few examples of the Perusall interface and annotation threads by students. The first image is a conversation that students had among themselves about some tricky subtleties of function compositions. The second image is a conversation about math terminology which I weighed in on. The third image are some fun memes posted by students as we introduced the chapter on prime factorization.



Here is a neat result linking composition with the properties of being 1-1 or onto.

PROPOSITION 19.2
Let S,T,U be sets, and let $f : S \to T$ and $g : T \to U$ be functions. Then
(i) *if f and g are both 1-1, so is $g \circ f$,*
(ii) *if f and g are both onto, so is $g \circ f$,*
(iii) *if f and g are both bijections, so is $g \circ f$.*

PROOF (i) If f,g are both 1-1, then for $s_1, s_2 \in S$,

$$\begin{aligned}(g\circ f)(s_1) = (g\circ f)(s_2) &\Rightarrow g(f(s_1)) = g(f(s_2))\\ &\Rightarrow f(s_1) = f(s_2) \quad \text{as } g \text{ is 1-1}\\ &\Rightarrow s_1 = s_2 \quad \text{as } f \text{ is 1-1}\end{aligned}$$

and hence $g \circ f$ is 1-1.
(ii) Suppose f,g are both onto. For any $u \in U$, there exists $t \in T$ such that $g(t) = u$ (as g is onto), and there exists $s \in S$ such that $f(s) = t$ (as f is onto). Hence $(g \circ f)(s) = g(f(s)) = g(t) = u$, showing that $g \circ f$ is onto.
(iii) This follows immediately from parts (i) and (ii). ∎

Counting Functions

How many functions are there from one finite set to another? This question is quite easily answered using some of our counting methods from Chapter 16.

PROPOSITION 19.3
Let S, T be finite sets, with $|S| = m$, $|T| = n$. Then the number of functions

necessarily true if we choose only n numbers — for example, we could choose the n numbers $1,3,5,\ldots,2n-1$.)

Answer This becomes easy when we make the following cunning choice of what the pigeonholes are. Define the pigeonholes to be the n sets

$$\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}.$$

Since we are choosing $n+1$ numbers, each of which belongs to one of these pigeonholes, the Pigeonhole Principle tells us that two of them must lie in the same pigeonhole. These two will then differ by 1. Pretty neat, eh?

More examples of the use of the Pigeonhole Principle can be found in Exercise 5 at the end of the chapter.

Inverse Functions

Given a function $f: S \to T$, under what circumstances can we define an "inverse function" from T to S, sending everything back to where it came from? (In other words, if f sends $s \to t$, the "inverse" function should send $t \to s$.) To define such a function from T to S, we need:

(a) f to be onto (otherwise some elements of T will not be sent anywhere

Current conversation

This sounds really weird to ask, but in a formal proof, can we refer to things as pigeonholes/pigeons or is there a formal term for those things?

Apr 10 11:26 am

@[redacted] I think you can refer to them as pigeonholes/pigeons. It seems that the principle is popular enough to be able to use it. You can also use part 2 of proposition 19.1.

Apr 10 7:33 pm

i think we can mention terms from the books as long as we state its from the book.

Apr 12 9:38 pm

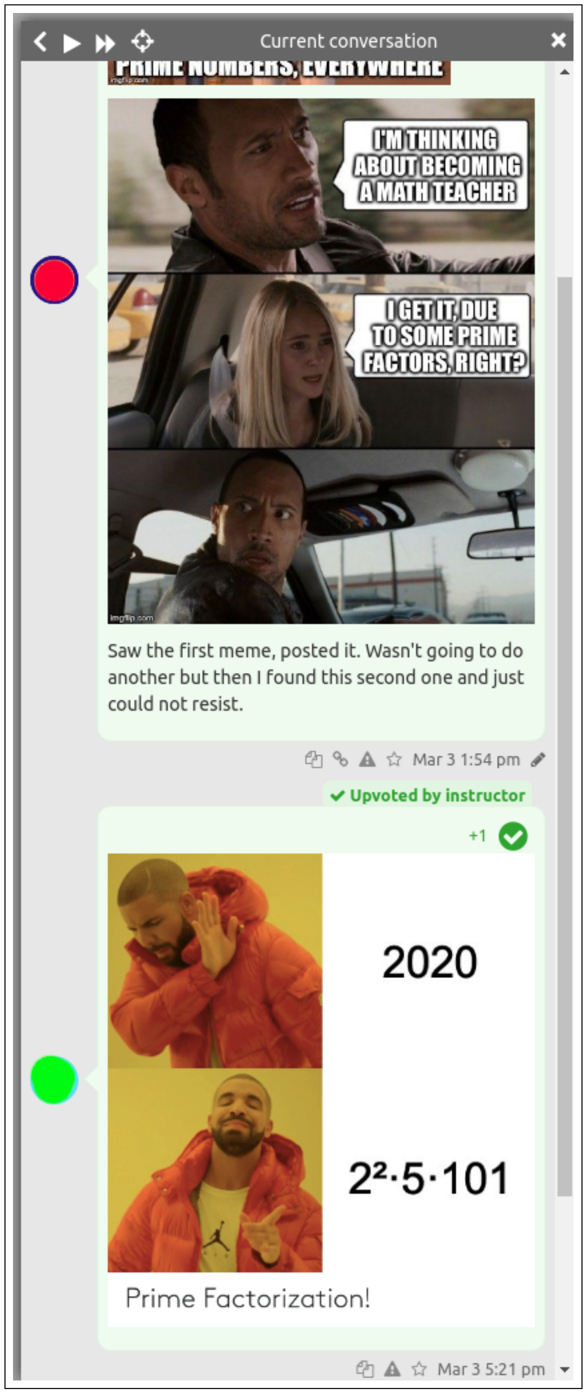
@[redacted] I'm happy to have you refer to things as pigeons and pigeonholes if using the Pigeonhole Principle! If you're ever writing a math paper for a prestigious journal, you'll want to check with the editors though...

JB

Apr 27 4:37 pm

JB

Enter your comment or question and press Enter. Mention a friend by typing @. Add hashtags by typing #.


Current conversation
I'M THINKING ABOUT BECOMING A MATH TEACHER
I GET IT, DUE TO SOME PRIME FACTORS, RIGHT?
Saw the first meme, posted it. Wasn't going to do another but then I found this second one and just could not resist.
Mar 3 1:54 pm
Upvoted by instructor
+1
2020
$2^2 \cdot 5 \cdot 101$
Prime Factorization!
Mar 3 5:21 pm