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The study of reaction-diffusion equations involving nonlocal diffusion operators has recently flourished. The fractional Laplacian is an example of a nonlocal diffusion operator which allows long-range interactions in space, and it is therefore important from the application point of view.

The fractional Laplacian operator plays a similar role in the study of nonlocal diffusion operators as the Laplacian operator does in the local case. Therefore, the goal of this dissertation is a systematic treatment of steady state reaction-diffusion problems involving the fractional Laplacian as the diffusion operator on a bounded domain and to investigate existence (and nonexistence) results with respect to a bifurcation parameter. In particular, we establish existence results for positive solutions depending on the behavior of a nonlinear reaction term near the origin and at infinity. We use topological degree theory as well as the method of sub- and supersolutions to prove our existence results. In addition, using a moving plane argument, we establish that, for a class of steady state reaction-diffusion problems involving the fractional Laplacian, any nonnegative nontrivial solution in a ball must be positive, and hence radially symmetric and radially decreasing.

Finally, we provide numerical bifurcation diagrams and the profiles of numerical positive solutions, corresponding to theoretical results, using finite element methods in one and two dimensions.

NONNEGATIVE SOLUTIONS OF NONLINEAR FRACTIONAL LAPLACIAN EQUATIONS

by

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Approved by

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To all of my teachers.

APPROVAL PAGE

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CHAPTER I

INTRODUCTION

In 2005, *Physics World* published a featured article with the headline *An increasing number of natural phenomena do not fit into the relatively simple description of diffusion developed by Einstein a century ago*, [KS05]. This sentiment has been echoed in numerous articles that investigated models involving so called *anomalous diffusion* - modeled by the fractional Laplacian defined below - a type of diffusion to be investigated in this dissertation.

Definition 1.1 (Fractional Laplacian)**.** *([Lan72, pp.45],[Poz16]) For $s \in (0,1)$ and for a function $u : \mathbb{R}^N \to \mathbb{R}$, the fractional Laplacian is a linear operator defined pointwise for $x \in \mathbb{R}^N$ by the singular integral*

$$(-\triangle)^s u(x) := C_{N,s}\ P.V. \int\limits_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \mathrm{d}y\,. \tag{1.1}$$

Here P.V. stands for the Cauchy principal value of the singular integral, defined as

$$P.V. \int\limits_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \mathrm{d}y := \lim_{\epsilon \to 0^+} \int\limits_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \mathrm{d}y$$

and

$$C_{N,s} := \frac{s 2^{2s} \Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}} \Gamma(1-s)}$$

is a positive normalizing constant, with Γ the usual gamma function. Recently there has been widespread interest in the study of problems involving the fractional Laplacian operator as a diffusion operator especially since the seminal paper of Caffarelli and Silvestre [CS07]. They showed that the fractional Laplacian operator as defined in (1.1) can be interpreted as a Dirichlet to Neumann map, effectively relating the nonlocal operator in (1.1) to a local operator. This characterization allowed them to prove several regularity results by using local techniques and provides a framework for interested researchers to further the study of the still emerging field of fractional Laplacian problems. Since then, the progress has been swift and there are already several excellent survey papers and monographs available, see [AV19,Buc17,BV16,Gar19,LPG+20,MBRS16,Poz16,RO16,V́17] and references therein.

We contribute to these efforts in this dissertation with the theoretical and numerical investigation of nonnegative solutions of reaction-diffusion problems, involving the *fractional Laplacian* as the diffusion operator, of the form

$$\begin{cases} (-\Delta)^s u = \lambda f(x, u(x)) & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.2}$$

with respect to the bifurcation parameter $\lambda > 0$. We will assume $\Omega \subset \mathbb{R}^N$ to be a bounded domain and $f : \Omega \times [0, +\infty) \to \mathbb{R}$, satisfies additional assumptions, to be specified in later chapters. We discuss the organization of this dissertation at the end of this chapter. The rest of this chapter is devoted to understanding the basic properties of fractional Laplacian operator and discussing its importance in applications.

There are several equivalent definitions of the fractional Laplacian in addition to the one we consider in this dissertation. The article [Kwa17] serves as an excellent resource on various definitions of the fractional Laplacian and their equivalence.

1.1. Comparison of the Fractional Laplacian and Laplacian

In this section, we build some intuition about the fractional Laplacian operator, discuss the notation used for the fractional Laplacian, and compare with the classical Laplacian operator.

First, we show $(-\Delta)^s$ can be thought of as a second order difference operator weighted over all of $\mathbb{R}^N$ by showing that (1.1) can be expressed as

$$(-\triangle)^s u(x) = \frac{C_{N,s}}{2}\ P.V. \int\limits_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2s}} \mathrm{d}y\,. \tag{1.3}$$

Indeed, rewriting the integral in (1.1) as

$$\int\limits_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy = \frac{1}{2}\left(\int\limits_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy + \int\limits_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy \right) \tag{1.4}$$

and letting $z = y - x$ and $z' = x - y$ in the first and second integrals on the right hand side of (1.4), respectively, we get

$$\frac{1}{2}\left(\int\limits_{\mathbb{R}^N} \frac{u(x)-u(x+z)}{|z|^{N+2s}} dz + \int\limits_{\mathbb{R}^N} \frac{u(x)-u(x-z')}{|z'|^{N+2s}} dz' \right).$$

Then, combining the above two integrals (using $y = z$ and $y = z'$ in first and second integral, respectively), we arrive at the equivalent definition of the fractional Laplacian given by (1.3).

Next, we show that the fractional Laplacian operator as defined in (1.3), and hence in (1.1), is well defined for $u \in C_0^2(\mathbb{R}^N)$ (see [DNPV12, Remark 3.1, Lemma 3.2]).

For the purpose of illustration, we show the result only in one dimension. For $x \in \operatorname{supp}(u)$,

$$
\begin{aligned}
&(-\Delta)^s u(x) = \frac{C_{1,s}}{2} \int\limits_{\mathbb{R}} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{1+2s}} \mathrm{d}y \\
&= \frac{C_{1,s}}{2} \int\limits_{|y| \leqslant 1} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{1+2s}} \mathrm{d}y + \frac{C_{1,s}}{2} \int\limits_{|y|>1} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{1+2s}} \mathrm{d}y \\
&= I_1 + I_2 \,.
\end{aligned}
$$

It suffices to show that the integrals I_1 and I_2 are finite. For I_1, since $u \in C_0^2(\mathbb{R})$, using a second order Taylor expansion, the triangle inequality, and combining yields

$$
\begin{aligned}
&\frac{|2u(x) - u(x+y) - u(x-y)|}{|y|^{1+2s}} \\
&\leqslant \frac{|2u(x) - u(x) - u'(x)y - \frac{1}{2}u''(x)y^2 - u(x) + u'(x)y - \frac{1}{2}u''(x)y^2|}{|y|^{1+2s}} \\
&\leqslant \frac{\|u''\|_{L^\infty(\mathbb{R})}|y|^2}{|y|^{1+2s}} = \|u''\|_{L^\infty(\mathbb{R})}|y|^{1-2s}
\end{aligned}
\tag{1.5}
$$

Therefore, using (1.5), it follows that I_1 is finite since $s \in (0,1)$,

$$
|I_1| \leqslant \|u''\|_{L^\infty(\mathbb{R})} \int\limits_{|y| \leqslant 1} |y|^{1-2s} \mathrm{d}y < \infty \,.
$$

Similarly, for I_2, since $s > 0$ we get

$$
|I_2| \leqslant 4\|u\|_{L^\infty(\mathbb{R})} \int\limits_{|y|>1} |y|^{-(1+2s)} \mathrm{d}y < \infty \,.
$$

Therefore, since I_1 and I_2 are both finite, we get that $(-\Delta)^s u$ is well defined for $u \in C_0^2(\mathbb{R})$ and $s \in (0,1)$.

Next, we motivate the choice of the symbol $(-\Delta)^s$ for the fractional Laplacian and make a connection with the classical Laplacian. For u from a suitable class of functions, one has (see [BHS18], [DNPV12, ST10])

$$\lim_{s\to 1^-} (-\Delta)^s\, u = -\Delta\, u \text{ and } \lim_{s\to 0^+} (-\Delta)^s\, u = I\,, \tag{1.6}$$

where I is the identity operator. This connection can be seen by considering the Fourier transform of the operator (see [Poz16, Val09] for more details). Recall that the Fourier transform and the inverse Fourier transform in $\mathbb{R}^N$ are given by

$$\mathscr{F}[u](y) := \int\limits_{\mathbb{R}^N} u(x) e^{2\pi i\langle x,k\rangle} dx\,, \quad \mathscr{F}^{-1}[w](x) := \int\limits_{\mathbb{R}^N} w(x) e^{-i2\pi\langle x,k\rangle} \mathrm{d}k\,, \tag{1.7}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^N$. Consider the Schwartz space of rapidly decaying $C^\infty(\mathbb{R}^N)$ functions defined as

$$\mathcal{S} := \{u \in C^\infty(\mathbb{R}^N) \mid \sup_{x\in\mathbb{R}^N} |x|^\alpha D^\beta u(x) < +\infty \text{ for all } \alpha, \beta \in \mathbb{N}^N\}\,.$$

For a Schwartz function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, $(-\Delta)^s$ acts as a Fourier multiplier in the sense that

$$\mathscr{F}[(-\Delta)^s u](k) = C_{N,s}|k|^{2s}\mathscr{F}[u](k)\,. \tag{1.8}$$

Hence, with respect to the Fourier transform, the fractional Laplacian acts as the multiplication by the symbol (*multiplier*) $|k|^{2s}$.

Since $|k|^{2s} \to |k|^2$ as $s \to 1^-$, the Fourier multiplier of $(-\Delta)^s$, approaches the Fourier multiplier of $(-\Delta)$ as $s \to 1^-$ in equation (1.8).

Remark 1.1. The normalizing constant $C_{N,s}$ is chosen to satisfy (1.8). We drop the positive constant $C_{N,s}$ from the definition of fractional Laplacian in (1.1) in Chapters II - VI for theoretical investigation. However, the constant is important for numerical experiments, and we utilize the specific values depending on $N = 1, 2$ and $s \in (0, 1)$ in Chapter VII.

Despite the limiting relationship between $(-\Delta)$ and $(-\Delta)^s$, there are some important differences. A list of differences can be found in [AV19]. We mention two below.

(Local vs Nonlocal): The first difference is in noting that the classical Laplacian $(-\Delta)$ is a local operator and the fractional Laplacian $(-\Delta)^s$ is nonlocal. The classical Laplacian of u at a point x depends only on values of u in a neighborhood of x making it local. Whereas, it is evident from (1.1) that the fractional Laplacian is a *nonlocal* operator due to the integration over all of $\mathbb{R}^N$.

To demonstrate this difference, we find a function for which $(-\Delta)$ vanishes but $(-\Delta)^s$ does not. Let $B_i := \{x \in \mathbb{R}^N : |x| < i\}$ for $i = 1, 2$. Suppose $u \in C_0^2(B_1)$, $u > 0$ in B_1, and $u = 0$ in $\mathbb{R}^N \backslash B_1$. Then, $\Delta u(x) = 0$ for $x \in \mathbb{R}^N \backslash B_2$. But using the fact $u > 0$ in B_1, for $x \in \mathbb{R}^N \backslash B_2$, we get

$$(-\Delta)^s u(x) = P.V. \int\limits_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \mathrm{d}y = -P.V. \int\limits_{B_1} \frac{u(y)}{|x - y|^{N+2s}} \mathrm{d}y < 0\,.$$

Due to the nonlocal nature of $(-\Delta)^s$, an analogue of a boundary value problem for the Laplacian problem in a bounded domain $\Omega \subset \mathbb{R}^N$ is an exterior value problem, where values must be prescribed on the unbounded set $\mathbb{R}^N \backslash \Omega$.

(Growth near boundary): The second important difference is in the behavior of solutions near the boundary on a bounded domain. To illustrate, let $u : B_1 \to \mathbb{R}$ be a function in $C^2(B_1) \cap C^1(\overline{B_1})$ satisfying

$$\begin{cases} -\Delta u = f(x) & \text{in } B_1; \\ u = 0 & \text{in } \partial B_1\,, \end{cases} \tag{1.9}$$

where f is continuous. Then, it is well known (see e.g. [Dip19, Appendix C]) that

$$\frac{|u(x)|}{\delta(x)} \leqslant \text{const.} \sup_{B_1} |f|\,, \tag{1.10}$$

where $\delta(x) := \operatorname{dist}(x, \partial\Omega)$. In the fractional Laplacian case, we show an example where the estimate (1.10) does not hold. Consider the function defined by

$$e(x) := (1 - |x|^2)^s \text{ in } B_1 \text{ and } e(x) = 0 \text{ in } \mathbb{R}^N \backslash B_1\,,$$

where e satisfies $(-\Delta)^s e = \text{const.}$ in B_1 (see [ROS14a, eqn. (1.4)]). Therefore e does not satisfy the estimate (1.10) since $s \in (0,1)$ implies

$$\lim_{|x| \nearrow 1} \frac{(1-|x|^2)^s}{1-|x|} = \lim_{|x| \nearrow 1} \frac{(1-|x|)^s (1+|x|)^s}{1-|x|} = \lim_{|x| \nearrow 1} \frac{(1+|x|)^s}{(1-|x|)^{1-s}} = +\infty\,.$$

1.2. Lévy Flights and the Fractional Laplacian

The fractional Laplacian is an example of a nonlocal operator which allows for long-range interactions in space. Long jump random walks, often referred to as *Lévy Flights*, are characterized by a probability distribution which selects the length of jumps for the diffusing medium or particles. The singular integral (1.1) can be derived as the continuous limit of discrete, long jump random walks (see [Buc17, Mar16, Val09] for a detailed discussion of more general cases). Below we describe how the singular integral (1.1) arises as the continuous limit of discrete random walks with jumps and derive the fractional heat equation.

We assume that randomly jumping particles are equally likely to jump in any direction and pick a distribution or *kernel* which determines the jump lengths. The kernel is chosen so that jumps are forced to happen and long jumps occur with a small probability. We first define the symmetric kernel. For $s \in (0,1)$, let $K : \mathbb{R}^N \to [0, +\infty)$ be defined by (up to a normalization constant)

$$K(y) = \frac{1}{|y|^{N+2s}} \text{ for } y \neq 0 \text{ and } K(0) = 0 \text{ with } \sum_{y \in \mathbb{Z}^N} K(y) = 1\,. \tag{1.11}$$

The probability that a particle jumps from the point x to the point y is taken to be $K(x-y) = K(y-x)$. For a given step size $h > 0$, consider a particle randomly jumping on the lattice $h\mathbb{Z}^N$ with jump lengths defined by (1.11). Assume that at any unit of time τ, a particle may jump from any point in $h\mathbb{Z}^N$ to any other. We further assume for convenience that $\tau = h^{2s}$.

Let $u(x,t)$ be the probability that a particle lies at $x \in h\mathbb{Z}^N$ at time $t \in \tau\mathbb{Z}^+$. Then, since a particle may jump from any point in $h\mathbb{Z}^N$ to any other, we get

$$u(x,t+\tau) = \sum_{k\in\mathbb{Z}^N} |k|^{-(N+2s)} u(x+hk,t)\,.$$

Using the normalization of K, we see

$$u(x,t+\tau) - u(x,t) = \sum_{k\in\mathbb{Z}^N} |k|^{-(N+2s)} \left[u(x+hk,t) - u(x,t)\right]\,.$$

Dividing by $\tau = h^{2s}$ yields

$$\begin{aligned}
\frac{u(x,t+\tau) - u(x,t)}{\tau} &= \sum_{k\in\mathbb{Z}^N} \frac{1}{h^{2s}} |k|^{-(N+2s)} \left[u(x+hk,t) - u(x,t)\right] \\
&= \sum_{k\in\mathbb{Z}^N} \frac{h^N}{h^{N+2s}} |k|^{-(N+2s)} \left[u(x+hk,t) - u(x,t)\right] \\
&= h^N \sum_{k\in\mathbb{Z}^N} |hk|^{-(N+2s)} \left[u(x+hk,t) - u(x,t)\right]\,. \qquad (1.12)
\end{aligned}$$

Observe that the right hand side of (1.12) is the approximating Riemann sum of

$$(-\Delta)^s u(x,t) = \int_{\mathbb{R}^N} \frac{u(y,t) - u(x,t)}{|x-y|^{N+2s}} \mathrm{d}y$$

with $y = x + hk$. Then, letting $h \to 0^+$, we arrive at the fractional heat equation

$$\frac{\partial u(x,t)}{\partial t} + (-\Delta)^s u(x,t) = 0\,.$$

1.3. The Fractional Laplacian and Random Movement

In this section, we begin with the fractional heat equation and derive a probability distribution which will be used in generating random walks related to the fractional Laplacian with for $N = 2$. The derivation for $s = \frac{1}{3}$ and $N = 2$ was carried out in [Mon55] without the use of the modern notation $(-\Delta)^s$, see [Uch13] for more details. Consider the initial value problem

$$\frac{\partial u}{\partial t} + (-\Delta)^s u = 0 \text{ in } \mathbb{R}^2 \times [0, \infty) \quad \text{with} \quad u(x, 0) = \delta_x \,, \tag{1.13}$$

where δ_x is the Dirac-delta function. By computing the Fourier transform of (1.13), we get

$$\frac{\partial \widehat{u}}{\partial t} = -(2\pi|k|)^{2s}\widehat{u}(k, t) \quad \text{with} \quad \widehat{u}(k, 0) = \widehat{\delta_x} = e^{i\langle k,0\rangle} = 1 \,. \tag{1.14}$$

Then, (1.14) defines an ordinary differential equation in the t variable with solution

$$\widehat{u}(k, t) = e^{-(2\pi|k|)^{2s}t} \,.$$

Using the inverse Fourier transform, we get the solution of (1.13)

$$u(x, t) = \int\limits_{\mathbb{R}^2} e^{-i2\pi\langle k,x\rangle} e^{-(2\pi|k|)^{2s}t} \mathrm{d}k \,, \tag{1.15}$$

which defines a Lévy distribution with stretching factor t. To simplify the expression (1.15), we note that $f(k) = e^{-(2\pi|k|)^{2s}}$ is a radial function.

The inverse Fourier transform of the radial function f is given by (see [Gar19, Thm. 4.4])

$$\mathscr{F}^{-1}[f](x) = 2\pi \int\limits_0^\infty f(\rho) J_0(2\pi|x|\rho)\mathrm{d}\rho\,,$$

where J_0 is the Bessel function of order zero. Then, setting the stretching factor $t = 1$, (1.15) has the form

$$u(x,1) = \int\limits_{\mathbb{R}^2} e^{-i2\pi\langle k,x\rangle} e^{-(2\pi|k|)^{2s}}\mathrm{d}k = 2\pi \int\limits_{\mathbb{R}^2} e^{-(2\pi\rho)^{2s}} J_0(2\pi|x|\rho)\rho\mathrm{d}\rho\,. \tag{1.16}$$

Using the polar coordinates $x = (r\cos(\theta'), r\sin(\theta'))$, (1.16) becomes

$$u(r,1) = 2\pi \int\limits_0^\infty e^{-(2\pi\rho)^{2s}} J_0(2\pi\rho r)\rho\mathrm{d}\rho\,. \tag{1.17}$$

The expression (1.17) defines a Lévy distribution which is a type of heavy-tailed probability distribution.

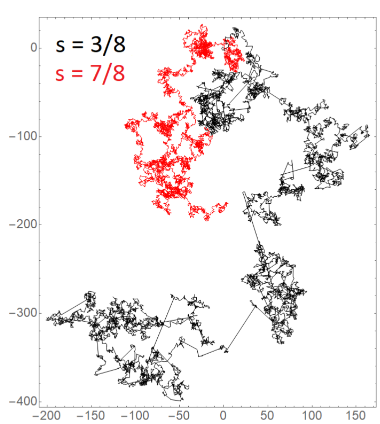


Figure 1.1. Comparison of Random Walks With $s = \frac{3}{8}$ and $\frac{7}{8}$

Heavy-tailed distributions are probability distributions whose tails are not bounded by an exponential distribution. We use the Lévy distribution (1.17) to generate random walks for $s = \frac{3}{8}$ and $s = \frac{7}{8}$ associated with the fractional Laplacian in Figure 1.1. We observe that the random movement for $s = \frac{7}{8} \approx 1$ resembles Brownian movement compared to $s = \frac{3}{8}$ which has occasional long jumps, a characteristic associated with the fractional Laplacian. For more on Lévy distributions and their connection to the fractional Laplacian see [LPG$^+$20].

Remark 1.2. The derivation of (1.17) above gives a probability distribution only when $s \in (0, 1)$. Moreover, the fractional Laplacian operator defined by (1.1) is the operator satisfying (1.8) only when $s \in (0, 1)$, see [SZ97]. However, when $s > 1$, the operator satisfying (1.8) is a hypersingular integral, see [Sam02].

1.4. Applications

The nonlocal nature of the fractional Laplacian appears to be advantageous, often suggested by empirical evidence, in modeling applications when the local counterpart seems inadequate. Next, we first mention a study of turbulent diffusion and its connection to fractional Laplacian. Then, we give a brief list of other applications involving the fractional Laplacian.

The fractional Laplacian operator describes the long-range correlations in particle displacements inherent to turbulent motion, see [DSU08, Uch13, US18] and the references therein. In turbulent diffusion, particles *disperse faster* than predicted by classical Brownian motion and is thus called *superdiffusion*. Indeed, it was observed by Richardson [Ric26] in 1926 that the field data for the diffusion of particles in the atmosphere does not obey Fick's law related to Brownian motion. Monin [Mon55] in 1955 and Monin–Yaglom [MY07, Sec. 24.4] in 1965 proposed an integro–differential equation modeling turbulent dispersal in two and three dimensions, respectively, based on Richardson's observations. The operator involved in these integro-differential equations turns out to be the fractional Laplacian operator, $(-\Delta)^s$, for $s = 1/3$. As mentioned in Section 1.3, the analysis was done by Monin [Mon55] for the case $N = 2$ and $s = \frac{1}{3}$. More recent field observations, as compiled in the survey paper by Gifford [Gif95] in 1994, indicate that the realistic values of s can be expected in the range $[1/3, 1]$ for long-range cloud spreading governed by atmospheric turbulence processes (where $s = (2\mathrm{p})^{-1}$, with p taken from [Gif95, p. 1729]). The case $s = 1/3$ corresponds to dispersal of particles driven by locally isotropic turbulent motion of fluid at sufficiently high Reynolds number, and the case $s = 1$ corresponds to the diffusion driven by Brownian motion (classical Laplacian). For more on turbulent dispersal and the fractional Laplacian, see [CR08, DSU08, EC18, US18, Uch13, YSF15] and the references therein.

In addition to turbulent diffusion, there are numerous equations and models involving the fractional Laplacian that have been proposed and studied. We list a few here: mathematical models of superdiffusion of living organism while foraging [CDV17,MV17,VdLRS11,AH90,LWS97], flame propagation and planar crack expansion [CRS10], phase transitions [dMG09, SV09], quasi-geostrophic flow [BKM10, CV10, CW99], mathematical finance [Sil07], water wave models [BV16], dislocation dynamics in crystals [DPV15], acoustic wave propagation in heterogeneous attenuating media [ZH14], modelling gases in porous media [Váz12], nonlinear porous medium equations [CV11a, CV11b, CSV13, dPQRV12, dPQRV11, Váz12] absorption and dispersion in viscoelastic solids [TC14], absorption and dispersion for acoustic propagation [TC10], spatial epidemic spreading [BS09], compressional and shear wave equations [HS10], linear and nonlinear lossy media [CH04], models of anomalous diffusion [Han02], generalized Fujita equation [BLMW05], and geophysical flows in the atmosphere [CMT94]. This list is not exhaustive, and more applications can be found in [BH04, MK00, MBRS16, Poz16, Uch13].

It was pointed out in [VNN13, p. 2] that in mathematical models involving superdiffusive systems, where the diffusion is characterized by spatial non-locality with no time memory effects (such as the fractional Laplacian), reaction terms can be incorporated in the model as with the classical Laplacian case. Therefore, it is of theoretical as well as practical interest to study reaction-diffusion equations involving the fractional Laplacian operator for $s \in (0, 1)$ such as (1.2).

In Chapter II, we define relevant terminology and introduce several preliminary results used to establish existence results. In Chapter III, we state and prove a sub- and supersolution result Theorem 3.1. In Chapter IV, we state and prove an existence result Theorem 4.1 for a class of superlinear problems. In Chapter V, we state Theorems 5.1 - 5.6 and prove the existence results Theorems 5.3 - 5.6 for sublinear, asymptotically linear, and logistic reaction terms, using Theorem 3.1. The proof of Theorems 5.1 and 5.2 are given in Appendix B. In Chapter V, we also state and prove the nonexistence result Theorem 5.7. In Chapter VI, we state and prove Theorem 6.1 showing radial symmetry and monotonicity of nonnegative solutions to fractional Laplacian problems. In Chapter VII, using the finite element method, we give numerical bifurcation diagrams and the profile of solutions corresponding existence results. In Chapter VIII, we discuss the conclusion of this dissertation and future directions to pursue. In Appendix 8.2, we give proofs of some auxiliary results for completeness. In Appendix B, we give proofs of some theorems from Chapter V.

The results in Chapter III, part of Chapter V (Theorem 5.1, Theorem 5.2, Theorem 5.5, Theorem 5.6) and the corresponding numerical experiments in Chapter VII are joint work with M. Chhetri and P. Girg that resulted in the manuscript [CGHb] (under review). Chapter IV is also a joint work with M. Chhetri and P. Girg that resulted in the manuscript [CGHa] (under review).

CHAPTER II

PRELIMINARIES

In this chapter, we first define the function spaces and types of solutions considered. Then, we discuss linear problems to define a solution operator, the maximum principles for the fractional Laplacian, and the degree theory as used in this dissertation.

2.1. Function Spaces and Solutions

Let $\Omega \subset \mathbb{R}^N (N \geqslant 1)$ be a bounded domain with smooth boundary $\partial\Omega$ if $N \geqslant 2$ and a bounded open interval if $N = 1$. We will specify the necessary boundary smoothness in later chapters.

First, we define the space of Hölder continuous functions. For $0 < \alpha \leqslant 1$, define

$$C^{0,\alpha}(\overline{\Omega}) := \{w : \overline{\Omega} \to \mathbb{R} \;\big|\; \|w\|_{C^{0,\alpha}(\overline{\Omega})} < \infty\}$$

where $\|w\|_{C^{0,\alpha}(\overline{\Omega})} := \|w\|_{C^0(\overline{\Omega})} + [w]_{C^{0,\alpha}(\overline{\Omega})}$ with $\|w\|_{C^0(\overline{\Omega})} := \sup\limits_{x \in \overline{\Omega}} |w(x)|$ and

$$[w]_{C^{0,\alpha}(\overline{\Omega})} := \sup_{\substack{x,y \in \overline{\Omega} \\ x \neq y}} \frac{|w(x) - w(y)|}{|x - y|^\alpha}\,.$$

For $D \subseteq \mathbb{R}^N$ and $1 \leqslant p \leqslant +\infty$, $L^p(D)$ denotes the usual Lebesgue space with norms denoted by $\|\cdot\|_{L^p(D)}$ for $1 \leqslant p < +\infty$ and $\|\cdot\|_\infty$ for $p = +\infty$.

Next, we define fractional Sobolev spaces (see [MBRS16, DNPV12] for more on these spaces). For a fixed $s \in (0,1)$, let

$$H^s(\mathbb{R}^N) := \left\{ w \in L^2(\mathbb{R}^N) \;\middle|\; \|w\|_{H^s(\mathbb{R}^N)} < +\infty \right\} ,$$

where $\|w\|_{H^s(\mathbb{R}^N)} := \left(\|w\|^2_{L^2(\mathbb{R}^N)} + [w]^2_{H^s(\mathbb{R}^N)} \right)^{\frac{1}{2}}$ and

$$[w]_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x) - w(y)|^2}{|x - y|^{N+2s}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{2}}$$

is the *Gagliardo seminorm* of w. Then, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is a Hilbert space with respect to the inner product

$$\langle v, w \rangle_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} vw \mathrm{d}x + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[v(x) - v(y)][w(x) - w(y)]}{|x - y|^{N+2s}} \mathrm{d}x \mathrm{d}y \,. \qquad (2.1)$$

Further, the fractional Sobolev space $H^s_0(\Omega) := \left\{ w \in H^s(\mathbb{R}^N) : w \equiv 0 \text{ a.e. } \mathbb{R}^N \backslash \Omega \right\}$ is also a Hilbert space with respect to the inner product

$$\langle v, w \rangle_{H^s_0(\Omega)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[v(x) - v(y)]\,[w(x) - w(y)]}{|x - y|^{N+2s}} \mathrm{d}x \mathrm{d}y \,. \qquad (2.2)$$

We will use the following equivalence of the inner product $\langle \cdot, \cdot \rangle_{H^s_0(\Omega)}$ and the fractional Laplacian defined by (1.1).

Proposition 2.1. *([Kwa17, Sec. 2.5]) Let $\psi, \phi \in H^s(\mathbb{R}^N)$. Then, the following equiavalence holds*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[\psi(x)-\psi(y)][\phi(x)-\phi(y)]}{|x-y|^{N+2s}} \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^{\mathbb{N}}} (-\Delta)^s \psi(x)\phi(x)\mathrm{d}x\,.$$

We also use the fact that the norms generated by $\langle \cdot\,,\cdot \rangle_{H_0^s(\Omega)}$ and $\langle \cdot\,,\cdot \rangle_{H^s(\mathbb{R}^N)}$ are equivalent in $H_0^s(\Omega)$, see [MBRS16, Lemma 1.28 & Lemma 1.29] for $N > 2s$. This equivalence holds also in dimension $N = 1$, say for $\Omega = (0,1) \subset \mathbb{R}$. The assumption $N > 2s$ leads to the restriction $s \in (0, 1/2)$. However, by carefully examining their proofs and taking advantage of computations in one dimension, we can show that the above equivalence of norms holds for all $s \in (0,1)$ when $N = 1$, in the lemma below. The proof is given in the Appendix 8.2.

Lemma 2.1. *The norms generated by $\langle \cdot\,,\cdot \rangle_{H_0^s(0,1)}$ and $\langle \cdot\,,\cdot \rangle_{H^s(\mathbb{R})}$ are equivalent in $H_0^s(0,1)$ for $s \in (0,1)$.*

Next, we consider the following linear fractional Laplacian problem

$$\begin{cases} (-\Delta)^s w &= \ell(x) & \text{in} \quad \Omega; \\ \qquad w &= 0 & \text{in} \quad \mathbb{R}^N \backslash \Omega\,. \end{cases} \tag{2.3}$$

Here we discuss the types of solutions to problem (2.3) considered in this dissertation. These definitions will be applied to all linear and nonlinear problems considered.

Definition 2.1. *We say that a function $u \in H_0^s(\Omega)$ is a* weak solution *of* (2.3) *if for all $\phi \in H_0^s(\Omega)$, it satisfies the integral identity*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(x)-u(y)]\,[\phi(x)-\phi(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y = \int_\Omega \ell(x)\phi(x)\mathrm{d}x\,. \tag{2.4}$$

Letting

$$\mathcal{E}(u,\phi) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(x)-u(y)]\,[\phi(x)-\phi(y)]}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\,,$$

(2.4) *can be simply expressed as*

$$\mathcal{E}(u,\phi) = \int_\Omega \ell(x)\phi(x)\mathrm{d}x\,. \tag{2.5}$$

Definition 2.2. *We say that a function $u : \mathbb{R}^N \to \mathbb{R}$ is a* classical solution *to* (2.3) *if the fractional Laplacian of u is defined at all points in Ω, according to definition* (1.1), *and if u satisfies equation* (2.3) *and the external condition in a pointwise sense.*

2.2. Linear Problems and Solution Operators

Let Ω be a bounded $C^{1,1}$ domain and consider the following linear fractional Laplacian problem

$$\begin{cases} (-\Delta)^s w = \ell(x) & \text{in } \Omega; \\ w = 0 & \text{in } \mathbb{R}^N\backslash\Omega\,. \end{cases} \tag{2.6}$$

For each $\ell \in H^{-s}(\Omega)$ (the dual of $H_0^s(\Omega)$), there exists a unique weak solution $w \in H_0^s(\Omega)$ of (2.6), see [LPPS15, Thm. 12] for $N \geqslant 2$ and [BHS18, Prop. 2.1] for $N = 1$. Moreover, if $\ell \in L^\infty(\Omega)$, then there exists $C > 0$ such that

$$\|w\|_{C^{0,s}(\overline{\Omega})} \leqslant C\|\ell\|_\infty\,, \tag{2.7}$$

see [RO16, Prop. 7.2] and [ROS14a, Prop. 1.1]. Then, the solution operator $(-\Delta)^{-s} \colon L^\infty(\Omega) \to L^\infty(\Omega)$ given by $\ell \mapsto w$ is well defined, continuous, and compact since the following holds for some $s' \in (0, s)$

$$L^\infty(\Omega) \xrightarrow{(-\Delta)^{-s}} C^{0,s}(\overline{\Omega}) \hookrightarrow\hookrightarrow C^{0,s'}(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)\,. \tag{2.8}$$

Finally, consider the following fractional linear problem

$$\begin{cases} (-\Delta)^s e &= 1 \quad \text{in} \quad \Omega; \\ e &= 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Omega\,. \end{cases} \tag{2.9}$$

Then, there exists a unique weak solution $e \in H_0^s(\Omega)$ of (2.9) such that $e > 0$ a.e. in Ω, see [LPPS15, Thm. 12] for $N \geqslant 2$, and for $N = 1$ the explicit formula of the solution is given in [ROS14a, eqn. (1.4)]. Moreover, it follows from [RO16, Lem. 7.3] and [ROS14a, Thm. 1.2] that there exist $c_1\,, c_2 > 0$ such that

$$c_1 \delta^s(x) \leqslant e(x) \leqslant c_2 \delta^s(x) \quad \text{a.e. in } \Omega, \tag{2.10}$$

where $\delta(x)$ is the distance function to the boundary $\partial\Omega$. Solutions of (2.6) can have at most $C^s(\overline{\Omega})$ regularity in the bounded domain. Indeed, in the unit ball, the explicit solution of (2.9) is given by a positive constant multiple of $w_0 := (1 - |x|^2)^s$, see [ROS14a]. Hence $w_0 \notin C^{s+\epsilon}(\overline{B_1})$ for any $\epsilon > 0$, see [RO16]. On the other hand, according to [RO16], if $\ell \in C^{0,\alpha}$, then solutions to (2.6) are $C^{0,2s+\alpha}$ inside Ω whenever α and $2s + \alpha$ are not integers.

2.3. Eigenvalue Problems

Let Ω be a bounded $C^{1,1}$ domain and consider the fractional Laplacian eigenvalue problem

$$\begin{cases} (-\Delta)^s \varphi = \lambda \varphi & \text{in } \Omega; \\ \varphi = 0 & \text{in } \mathbb{R}^N \backslash \Omega\,. \end{cases} \tag{2.11}$$

It is known that (2.11) has a simple eigenvalue $\lambda_1 > 0$ and a corresponding positive eigenfunction $\varphi_1 \in H_0^s(\Omega)$, see [MBRS16, Prop 3.1 & Cor. 4.8]. Moreover, it follows from [RO16, Lem. 7.3] and [ROS14a, Thm. 1.2] that there exist $d_1\,, d_2 > 0$ such that

$$d_1 \delta^s(x) \leqslant \varphi_1(x) \leqslant d_2 \delta^s(x) \quad \text{a.e. in } \Omega\,. \tag{2.12}$$

Next, we will consider the following weighted fractional Laplacian eigenvalue problems of the form

$$\begin{cases} (-\Delta)^s \varphi = \lambda q(x) \varphi & \text{in } \Omega; \\ \varphi = 0 & \text{in } \mathbb{R}^N \backslash \Omega\,, \end{cases} \tag{2.13}$$

where $q \in L^\infty(\Omega)$ is such that $q \geqslant 0$ a.e. in Ω and positive on a set of positive measure. Using arguments similar to the case $q \equiv 1$, cf. [MBRS16, Prop 3.1 & Cor. 4.8], we obtain the following result. We outline the proof in Appendix 8.2 for completeness.

Proposition 2.2. *Let $s \in (0,1)$ be fixed and $\Omega \subset \mathbb{R}^N (N \geqslant 1)$ be an open, bounded set. Then the following holds:*

(a) (2.13) *has an eigenvalue $\lambda_{1,q} > 0$ that can be characterized as*

$$\lambda_{1,q} = \inf_{\phi \in H_0^s(\Omega) \backslash \{0\}} \frac{\mathcal{E}(\phi, \phi)}{\displaystyle\int_\Omega q(x) |\phi(x)|^2 \mathrm{d}x}\,. \tag{2.14}$$

(b) There exists a nonnegative function $\varphi_{1,q} \in H_0^s(\Omega)$ that is an eigenfunction corresponding to $\lambda_{1,q}$, attaining the minimum in (2.14)*; that is,*

$$\lambda_{1,q} = \frac{\mathcal{E}(\varphi_{1,q}, \varphi_{1,q})}{\displaystyle\int_\Omega q(x)|\varphi_{1,q}(x)|^2 \mathrm{d}x} \,. \tag{2.15}$$

Moreover, it follows that $\varphi_{1,q}$ satisfies (2.13) *according to definition* (2.4)*; that is, for every $\phi \in H_0^s(\Omega)$*

$$\mathcal{E}(\varphi_{1,q}, \phi) = \lambda_{1,q} \int_\Omega q(x)\varphi_{1,q}(x)\phi(x)\mathrm{d}x \,.$$

(c) $\lambda_{1,q}$ is simple; that is, if $\psi \in H_0^s(\Omega)$ is a solution of the equation

$$\mathcal{E}(\psi, \phi) = \lambda_{1,q} \int_\Omega q(x)\psi(x)\phi(x)\mathrm{d}x$$

for every $\phi \in H_0^s(\Omega)$, then $\psi = k\varphi_{1,q}$ for some $k \in \mathbb{R}$.

(d) If Ω is $C^{1,1}$ for $N \geqslant 2$ (or bounded open interval if $N = 1$), then there exist positive constants $\tilde{c}_1(q), \tilde{c}_2(q)$ such that

$$0 < \tilde{c}_1(q)\delta^s(x) \leqslant \varphi_{1,q}(x) \leqslant \tilde{c}_2(q)\delta^s(x) \quad \textit{a.e. in } \Omega \,. \tag{2.16}$$

(e) If Ω is $C^{1,1}$ for $N \geqslant 2$ (or bounded open interval if $N = 1$), then

$$\lambda_{1,q} = \inf_{\substack{\phi \in H_0^s(\Omega) \\ \phi \geqslant \delta^s \text{ a.e. in } \Omega}} \frac{\mathcal{E}(\phi, \phi)}{\displaystyle\int_\Omega q(x)|\phi(x)|^2 \mathrm{d}x} \,. \tag{2.17}$$

Finally, we consider another class of weighted eigenvalue problems related to (2.13) and state and prove a result needed in Chapter V. For each $k = 2, 3, \ldots$, consider the weighted fractional eigenvalue problem

$$\begin{cases} (-\Delta)^s\varphi &= \lambda\, \gamma_k(x)\varphi \qquad \text{in} \quad \Omega; \\ \qquad \varphi &= 0 \qquad\qquad\quad \text{in} \quad \mathbb{R}^N\backslash\Omega\,, \end{cases} \tag{2.18}$$

where

$$\gamma_k(x) := \begin{cases} 0 & \text{if } 0 \leqslant q(x) < 1/k, \\ \\ q(x) & \text{if } q(x) \geqslant 1/k\,, \end{cases} \tag{2.19}$$

for $q \in L^\infty(\Omega)$ with $0 \leqslant q \leqslant 1$ a.e. in Ω and $q(x) > 1/2$ on a set of positive measure. Then the weighted fractional eigenvalue problem (2.18) has a principal eigenvalue λ_{1,γ_k} and a corresponding eigenfunction φ_{1,γ_k} satisfying (a)-(d) of Proposition 2.2.

Then we establish the following useful relationship between $\lambda_{1,q}$ and λ_{1,γ_k} that will be utilized in the investigation of the weighted logistic problem.

Proposition 2.3. *Let $q \in L^\infty(\Omega)$ with $0 \leqslant q \leqslant 1$ a.e. in Ω, $q(x) > 1/2$ on a set of positive measure, and γ_k be as given in (2.19). Then, $\lambda_{1,\gamma_k} \searrow \lambda_{1,q}$ as $k \to +\infty$.*

Proof. The properties of q and γ_k imply that the inequalities

$$\int_\Omega q(x)|\phi(x)|^2 \mathrm{d}x \geqslant \int_\Omega \gamma_{k+1}(x)|\phi(x)|^2 \mathrm{d}x \geqslant \int_\Omega \gamma_k(x)|\phi(x)|^2 \mathrm{d}x > 0 \tag{2.20}$$

hold for every $k \geqslant 2$ for all $\phi \in H_0^s(\Omega)$, $\phi \geqslant \delta^s$ a.e. in Ω. First we show $\lambda_{1,q} \leqslant \lambda_{1,\gamma_{k+1}} \leqslant \lambda_{1,\gamma_k}$ for each $k \geqslant 2$. Indeed, it follows from (2.20) that the inequalities

$$\begin{aligned} \frac{\mathcal{E}(\phi,\phi)}{\displaystyle\int_\Omega q(x)|\phi(x)|^2 \mathrm{d}x} &\leqslant \frac{\mathcal{E}(\phi,\phi)}{\displaystyle\int_\Omega \gamma_{k+1}(x)|\phi(x)|^2 \mathrm{d}x} \\ &\leqslant \frac{\mathcal{E}(\phi,\phi)}{\displaystyle\int_\Omega \gamma_k(x)|\phi(x)|^2 \mathrm{d}x} \end{aligned} \tag{2.21}$$

holds for all $\phi \in H_0^s(\Omega)$, $\phi \geqslant \delta^s$ a.e. in Ω. By taking the infimum over all such ϕ, inequalities (2.21) imply $\lambda_{1,q} \leqslant \lambda_{1,\gamma_{k+1}} \leqslant \lambda_{1,\gamma_k}$, using (2.17), as desired. Now we show $\lambda_{1,\gamma_k} \to \lambda_{1,q}$ as $k \to +\infty$. By (2.17) with $k \geqslant 2$, we see

$$\lambda_{1,\gamma_k} = \inf_{\substack{\phi \in H_0^s(\Omega) \\ \phi \geqslant \delta^s \text{ a.e. in } \Omega}} \frac{\mathcal{E}(\phi,\phi)}{\displaystyle\int_\Omega \gamma_k(x)|\phi(x)|^2 \mathrm{d}x}.$$

Let $\varphi_{1,q}$ be the principal eigenfunction scaled such that $\varphi_{1,q} \geqslant \delta^s$ a.e. in Ω. Then using the same argument as in the proof of Proposition 2.2 (e), we get

$$\lambda_{1,q} = \inf \left\{ \frac{\mathcal{E}(\phi,\phi)}{\displaystyle\int_\Omega q(x)|\phi(x)|^2 \mathrm{d}x} \,\middle|\, \phi \in H_0^s(\Omega)\,, \phi \geqslant \delta^s \text{ a.e. } \Omega\,, \|\phi\|_{H_0^s(\Omega)} \leqslant \|\varphi_{1,q}\|_{H_0^s(\Omega)} \right\}.$$

By the definition of the infimum, for each $k \in \mathbb{N}$, we can find $\phi_k \in H_0^s(\Omega)$, $\phi_k \geqslant \delta^s$ a.e. in Ω and $\|\phi_k\|_{H_0^s(\Omega)} \leqslant \|\varphi_{1,q}\|_{H_0^s(\Omega)}$ such that

$$\lambda_{1,q} \geqslant \frac{\mathcal{E}(\phi_k, \phi_k)}{\displaystyle\int_\Omega q(x)|\phi_k(x)|^2 \mathrm{d}x} - 2^{-k}\,.$$

Thus, for $k \geqslant 2$, we have

$$\begin{aligned}
\lambda_{1,q} &\geqslant \frac{\mathcal{E}(\phi_k, \phi_k)}{\displaystyle\int_\Omega q(x)|\phi_k(x)|^2 \mathrm{d}x} - 2^{-k} \\
&= \frac{\mathcal{E}(\phi_k, \phi_k)}{\displaystyle\int_\Omega \gamma_k(x)|\phi_k(x)|^2 \mathrm{d}x} \cdot \frac{\displaystyle\int_\Omega \gamma_k(x)|\phi_k(x)|^2 \mathrm{d}x}{\displaystyle\int_\Omega q(x)|\phi_k(x)|^2 \mathrm{d}x} - 2^{-k} \\
&\geqslant \left(\inf_{\substack{\phi \in H_0^s(\Omega) \\ \phi \geqslant \delta^s \text{ a.e. in } \Omega}} \frac{\mathcal{E}(\phi, \phi)}{\displaystyle\int_\Omega \gamma_k(x)|\phi(x)|^2 \mathrm{d}x} \right) \frac{\displaystyle\int_\Omega \gamma_k(x)|\phi_k(x)|^2 \mathrm{d}x}{\displaystyle\int_\Omega q(x)|\phi_k(x)|^2 \mathrm{d}x} - 2^{-k} \\
&\geqslant \lambda_{1,\gamma_k} \frac{\displaystyle\int_\Omega \gamma_k(x)|\phi_k(x)|^2 \mathrm{d}x}{\displaystyle\int_\Omega q(x)|\phi_k(x)|^2 \mathrm{d}x} - 2^{-k}\,.
\end{aligned}$$

This yields

$$\lambda_{1,q} \leqslant \lambda_{1,\gamma_k} \leqslant \left(\lambda_{1,q} + 2^{-k}\right) \frac{\displaystyle\int_\Omega q(x)|\phi_k(x)|^2 \mathrm{d}x}{\displaystyle\int_\Omega \gamma_k(x)|\phi_k(x)|^2 \mathrm{d}x}\,. \tag{2.22}$$

By the compact embedding of $H_0^s(\Omega)$ into $L^2(\Omega)$ and $\|\phi_k\|_{H_0^s(\Omega)} \leqslant \|\varphi_{1,q}\|_{H_0^s(\Omega)}$, we can find a subsequence $\phi_{k_j} \to \psi$ in $L^2(\Omega)$, where ψ is some element of $L^2(\Omega)$. Then $\phi_{k_j}^2 \to \psi^2$ in $L^1(\Omega)$.

Since $q \in L^\infty(\Omega)$, it follows that $\int_\Omega q(x)|\phi_{k_j}(x)|^2 \mathrm{d}x \to \int_\Omega q(x)|\psi(x)|^2 \mathrm{d}x$. Next, we will show that

$$\int_\Omega \gamma_{k_j}(x)|\phi_{k_j}(x)|^2 \mathrm{d}x \to \int_\Omega q(x)|\psi(x)|^2 \mathrm{d}x$$

as well. Indeed,

$$\int_\Omega \gamma_{k_j}(x)|\phi_{k_j}(x)|^2 \mathrm{d}x = \int_\Omega (\gamma_{k_j}(x) - q(x))|\phi_{k_j}(x)|^2 \mathrm{d}x + \int_\Omega q(x)|\phi_{k_j}(x)|^2 \mathrm{d}x\,.$$

By (2.19), $q(x) - \gamma_{k_j}(x) \leqslant 1/k_j$, thus

$$\left|\int_\Omega (\gamma_{k_j}(x) - q(x))|\phi_{k_j}(x)|^2 \mathrm{d}x\right| \leqslant 1/k_j \int_\Omega |\phi_{k_j}|^2 \mathrm{d}x \leqslant \frac{C}{k_j}\|\varphi_{1,q}\|^2_{H_0^s(\Omega)} \to 0$$

as $k_j \to +\infty$, where C is the constant of the embedding of $H_0^s(\Omega)$ into $L^2(\Omega)$. Observe that $\psi \geqslant \delta^s > 0$ a.e. in Ω since $\phi_{k_j} \geqslant \delta^s$ a.e. in Ω, and hence

$$\frac{\displaystyle\int_\Omega q(x)|\phi_k(x)|^2 \mathrm{d}x}{\displaystyle\int_\Omega \gamma_k(x)|\phi_k(x)|^2 \mathrm{d}x} \longrightarrow \frac{\displaystyle\int_\Omega q(x)|\psi(x)|^2 \mathrm{d}x}{\displaystyle\int_\Omega q(x)|\psi(x)|^2 \mathrm{d}x} = 1$$

as $k_j \to +\infty$. Thus, we established from (2.22) that $\lambda_{1,\gamma_{k_j}} \to \lambda_{1,q}$. Since λ_{1,γ_k} is a monotone sequence, it must hold for the entire sequence $\lambda_{1,\gamma_k} \searrow \lambda_{1,q}$. ■

2.4. Maximum Principles

Here we state maximum principles for the fractional Laplacian.

Proposition 2.4. *([Sil07, Prop. 2.2.8.]) Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain and* u *a lower semi-continuous function in* $\overline{\Omega}$ *such that* $(-\Delta)^s u \geqslant 0$ *in* Ω *and* $u \geqslant 0$ *in* $\mathbb{R}^N \backslash \Omega$*. Then,* $u \geqslant 0$ *in* $\mathbb{R}^N$*. Moreover, if* $u(x) = 0$ *for some point* x *in* Ω*, then* $u \equiv 0$ *in* $\mathbb{R}^N$*.*

Proposition 2.5. *([RO16, Lem. 7.3]) Let* Ω *be a bounded domain with* $C^{1,1}$ *boundary and* u *be any weak solution to* (2.6)*, with* $0 \leqslant \ell \in L^\infty(\Omega)$*. Then, either*

$$u \geqslant c\delta^s \text{ in } \Omega \text{ for some } c > 0$$

or $u \equiv 0$ *in* Ω*.*

We will utilize the following strong maximum principle in a unit ball $B_1 \subset \mathbb{R}^N$.

Proposition 2.6. *([Buc17, Thm. 2.1.8.]) If* $(-\Delta)^s u \geqslant 0$ *in* B_1 *and* $u \geqslant 0$ *in* $\mathbb{R}^N \backslash B_1$*, then* $u > 0$ *in* B_1*, unless* u *vanishes identically.*

We will also use the following maximum principle for thin domains.

Proposition 2.7. *([FW14, Prop. 2.2]) Let* Ω *be a bounded, connected open subset of* $\mathbb{R}^N$*. Suppose that* $\varphi : \Omega \to \mathbb{R}$ *is in* $L^\infty(\mathbb{R}^N)$ *and* w *is a classical solution of*

$$\begin{cases} (-\Delta)^s w(x) \geqslant \varphi(x) w(x) & \text{in } \Omega; \\ w(x) \geqslant 0 & \text{in } \mathbb{R}^N \backslash \Omega. \end{cases}$$

Then, there exists $\xi > 0$ *such that* $w \geqslant 0$ *in* Ω *whenever* $|\Omega^-| < \xi$*, where* $\Omega^- := \{x \in \Omega \mid w(x) < 0\}$*.*

2.5. Degree Theory

Here we briefly discuss topological degree which we use to prove an existence result in Chapter IV. First, we discuss topological degree and its properties in finite dimensional spaces. Then, we discuss topological degree in infinite dimensional spaces (Leray-Schauder degree). For more on topological degree theory, see [Llo78].

Degree Theory in Finite Dimensional Spaces: Let $D \subset \mathbb{R}^N$ be a bounded open domain and $T \in C^1(\overline{D})$. Define $J_T(x)$ as the as the determinant of the Jacobian of T at x. We say that x is a critical point of T if $J_T(x) = 0$. Define the set of critical points by $Z_T := \{x \in \overline{D} \mid J_T(x) = 0\}$ and the set of critical values by $T(Z_T)$. It follows that if $T \in C^1(\overline{D})$ and $p \notin T(Z_T)$, then $T^{-1}(p)$ is finite.

Now we can define the degree of T at p when T is a C^1 function and $p \notin T(Z_T)$.

Definition 2.3. *Suppose $T \in C^1(\overline{D})$, $p \notin T(Z_T)$, and $p \notin T(\partial D)$. The degree of T at p with respect to D is defined as*

$$\deg(T, D, p) := \sum_{x \in T^{-1}(p)} \operatorname{sign} J_T(x)\,.$$

For $p \notin T(\partial D)$ but $p \in T(Z_T)$, the degree of T at p with respect to Ω is defined to be $deg(T, D, p) := deg(T, D, q)$, where q is any point such that $q \notin T(\partial D)$, $q \notin T(Z_T)$, and $|p - q| < \delta(p, T(\partial D))$. Here, δ is the distance function from p to $T(\partial D)$.

For a continuous function G and $p \notin G(D)$, the degree of G at p with respect to D is defined to be

$$\deg(G, p, D) := \deg(T, p, D)\,,$$

where T is a C^1 function such that $|G(x) - T(x)| < \delta(p, G(\partial D))$ for all $x \in \overline{D}$. Then the following properties are satisfied:

1. **Normalization**: If I is the identity operator and $p \in D$, then $\deg(I, D, p) = 1$. If $p \notin \overline{D}$, then $\deg(I, D, p) = 0$.

2. **Solution**: If $\deg(T, D, p)$ is defined and non-zero, then there exists $x \in D$ such that $T(x) = p$.

3. **Excision**: If $D_1 \subset D$ and $D_2 \subset D$ are disjoint open subsets such that $p \notin T(\overline{D} \backslash (D_1 \cup D_2))$, then $\deg(T, D, p) = \deg(T, D_1, p) + \deg(T, D_2, p)$.

4. **Homotopy Invariance**: A homotopy between elements T, G of $C(\overline{D})$ is a function $h : [0,1] \times \overline{D} \to \mathbb{R}^N$ such that, if $h_t = H(t, x)$, then $h_0 = G$, $h_1 = T$, $h_t \in C(\overline{D})$ for $0 \leqslant t \leqslant 1$, and $h_s \to h_t$ in $C(\overline{D})$ as $s \to t$. If $p \notin h_t(\partial D)$ for $0 \leqslant t \leqslant 1$, then $\deg(h_t, D, p)$ is independent of $t \in [0,1]$ and

$$\deg(G, D, p) := \deg(T, D, p)\,.$$

Degree Theory in Infinite Dimensional Spaces: Here we consider maps of the form $I - T$ where I is the identity operator and T is compact. Let $(X, \|\cdot\|)$ be a normed linear space and $D \subset X$ be open and bounded. The map $T : X \to X$ is compact if T is continuous and $\overline{T(A)}$ is compact for every bounded subset $A \subset X$. Let $p \in X \backslash T(\partial D)$. Since T is compact, there is a continuous map $\widehat{T} : \overline{D} \to X$ whose range $\widehat{T}(D)$ is finite dimensional and $\|T(x) - \widehat{T}(x)\| < \delta(p, T(\partial D))$ for $x \in \overline{D}$. Let $\widehat{D} := D \cap \text{span}\{\widehat{T}(\overline{D}), p\}$.

Now we can define the degree of $I - T$ at p when T is compact and $p \notin T(\partial D)$.

Definition 2.4. *Suppose $T : \overline{D} \to X$ is compact and $p \notin T(\partial\Omega)$. The degree of T at p with respect to D is defined as*

$$\deg(I - T, D, p) := \deg(I - \widehat{T}, \widehat{D}, p)\,.$$

Topological degree in infinite dimensional spaces satisfies all the above properties $1 - 4$.

CHAPTER III

SUB- AND SUPERSOLUTION THEOREM

3.1. Introduction and Statement of Result

Here we state and prove a sub- and supersolution theorem, without monotone iteration, which we use to prove the existence of a positive weak solution for classes of sublinear, asymptotically linear, and logistic type nonlinearities.

Sub- and supersolution methods for the fractional Laplacian were discussed in [Aba15, Bah18, FT18]. However, in [Aba15] and [FT18], the authors consider L^1-very weak solutions thereby requiring a rather complicated structure of the space of test functions. In [Bah18], the author considers weak solutions for a fractional $p(x)$-Laplacian which requires a complicated functional framework necessary for the fractional $p(x)$-Laplacian operator. Therefore, we present a sub- and supersolution result, Theorem 3.1, with functional framework analogous with the weak formulation that is standard for the Laplacian case. The distinct advantage of our approach is in the possibility of employing the principal eigenfunction corresponding to the variational principal eigenvalue of $(-\Delta)^s$ in the construction of positive sub- and supersolutions. We consider the following problem

$$\begin{cases} (-\Delta)^s u = g(x,u) & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded domain with $C^{1,1}$ boundary and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

We say $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a *Carathéodory function* if $g(\cdot, \sigma) : \Omega \to \mathbb{R}$ is measurable for all $\sigma \in \mathbb{R}$, and $g(x, \cdot)$ is continuous for a.e $x \in \Omega$.

Definition 3.1. A function $\overline{u} \in H^s(\mathbb{R}^N)$ is called a *weak supersolution* of (3.1) if, for all $\phi \in H_0^s(\Omega)$ with $0 \leqslant \phi$ in Ω, the following inequalities hold:

$$\mathcal{E}(\overline{u}, \phi) \geqslant \int_\Omega g(x, \overline{u}(x))\phi(x)\mathrm{d}x \tag{3.2}$$

and

$$\overline{u} \geqslant 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\,. \tag{3.3}$$

A function $\underline{u} \in H^s(\mathbb{R}^N)$ is called a *weak subsolution* of (3.1) if the inequalities are reversed in (3.2) and (3.3).

Then, we prove:

Theorem 3.1. *Suppose*

(H1) *for all $r > 0$, there is $a_r \in L^\infty(\Omega)$ such that $|g(x, \sigma)| \leqslant a_r(x)$ for all $|\sigma| \leqslant r$ a.e. $x \in \Omega$;*

(H2) *for all $r > 0$, there is a continuous nondecreasing function b_r with $b_r(0) = 0$ such that $|g(x, \sigma_1) - g(x, \sigma_2)| \leqslant b_r(|\sigma_1 - \sigma_2|)$ for all $|\sigma_1|, |\sigma_2| \leqslant r$ a.e. $x \in \Omega$.*

Let $\underline{u}$ and $\overline{u} \in H^s(\mathbb{R}^N) \cap L^\infty(\Omega)$ be a weak subsolution and weak a supersolution, respectively, of (3.1) *satisfying $\underline{u} \leqslant \overline{u}$ a.e. in Ω. Then, there exists a weak solution u to* (3.1) *satisfying $\underline{u} \leqslant u \leqslant \overline{u}$ a.e. in Ω.*

Remark 3.1. The hypotheses of Theorem 3.1 are satisfied by a function of the form $g(x, \sigma) = k(x)\tilde{g}(\sigma)$, where $k \in L^\infty(\Omega)$ and $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is Hölder continuous. Indeed, clearly $g(x,\sigma)$ is a Carathéodory function. For any $r > 0$ and for all $|\sigma| \leqslant r$, we have $|g(x,\sigma)| \leqslant \|k\|_\infty \max_{|\sigma| \leqslant r} |\tilde{g}(\sigma)|$ and hence (H1) is satisfied. By the Hölder continuity of $\tilde{g}$, $|g(x,\sigma_1) - g(x,\sigma_2)| \leqslant A\|k\|_\infty |\sigma_1 - \sigma_2|^\eta$ for all $|\sigma_1|, |\sigma_2| \leqslant r$ for some $\eta \in (0,1)$ and $A > 0$. Then, (H2) is satisfied with $b_r(|\sigma_1 - \sigma_2|) := A\|k\|_\infty |\sigma_1 - \sigma_2|^\eta$.

3.2. Proof of Theorem 3.1

We follow the idea of the proof from Clement-Sweers [CS87], where a similar result was proven for the Laplacian case ($s = 1$) using the Schauder fixed point theorem.

Consider the modified function $g^* \colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$g^*(x,\sigma) := \begin{cases} g(x,\underline{u}(x)) & \text{if} \quad \sigma < \underline{u}(x)\,, \\ g(x,\sigma) & \text{if} \quad \underline{u}(x) \leqslant \sigma \leqslant \overline{u}(x)\,, \\ g(x,\overline{u}(x)) & \text{if} \quad \sigma > \overline{u}(x)\,, \end{cases}$$

and note that since g is clearly a Carathéodory function so is g^*. We observe that any weak solution u of (3.1) satisfying $\underline{u} \leqslant u \leqslant \overline{u}$ a.e. in Ω is also a weak solution of the modified problem

$$\begin{cases} (-\Delta)^s u = g^*(x,u) & \text{in} \quad \Omega; \\ u = 0 & \text{in} \quad \mathbb{R}^N \backslash \Omega\,. \end{cases} \tag{3.4}$$

Moreover, using the definition of g^*, it follows from the claim below that a weak solution of (3.4) is a weak solution of (3.1). Next, we claim that If u is a weak solution of (3.4), then $\underline{u} \leqslant u \leqslant \overline{u}$ a.e. in Ω.

First we establish $u \leqslant \overline{u}$ a.e. in Ω by showing that $\text{meas}(A) = 0$, where $A := \left\{x \in \mathbb{R}^N \;\middle|\; \overline{u}(x) < u(x)\right\}$. Clearly A is measurable (in the sense of Lebesgue) since $u \in H_0^s(\Omega)$ and $\overline{u} \in H^s(\mathbb{R}^N)$. Assume to the contrary that $\text{meas}(A) > 0$. We note that meas $\big(A \cap (\mathbb{R}^N \backslash \Omega)\big) = 0$ since $\overline{u} \geqslant 0 = u$ a.e. in $\mathbb{R}^N \backslash \Omega$. Hence, meas$(A \cap \Omega) > 0$. Setting $z^+ := \max\{0, z\} \geqslant 0$, we see that $[u - \overline{u}]^+ \in H^s_0(\Omega)$ since $[u - \overline{u}]^+ \in H^s(\mathbb{R}^N)$ and it vanishes almost everywhere outside $A \subset \Omega$. Taking $\phi := [u - \overline{u}]^+$ as a test function in (2.5) and (3.2), and using the definitions of g^* and A, we obtain

$$\begin{aligned} \mathcal{E}(u, [u - \overline{u}]^+) = & \int_\Omega g^*(x, u(x))\, [u - \overline{u}]^+ (x)\, \mathrm{d}x \\ = & \int_A g^*(x, u(x))\, [u - \overline{u}]^+ (x)\, \mathrm{d}x \\ = & \int_A g(x, \overline{u}(x))\, [u - \overline{u}]^+ (x)\, \mathrm{d}x \\ = & \int_\Omega g(x, \overline{u}(x))\, [u - \overline{u}]^+ (x)\, \mathrm{d}x \\ \leqslant & \ \mathcal{E}(\overline{u}, [u - \overline{u}]^+)\,. \end{aligned} \tag{3.5}$$

On one hand, subtracting the right-hand side from the left-hand side in (3.5) and rearranging the terms yields the following inequality

$$\mathcal{E}(u - \overline{u}, [u - \overline{u}]^+) \leqslant 0\,. \tag{3.6}$$

On the other hand, by taking $v = u - \overline{u}$, it follows from [MBRS16, Lem. 3.3] that

$$[v(x) - v(y)][v^+(x) - v^+(y)] \geqslant [v^+(x) - v^+(y)]^2 \text{ for a.e. } x, y \in \mathbb{R}^N\,.$$

Since the measure of A is positive, $v > 0$ in A, and $\|\cdot\|_{H^s_0(\Omega)}$ is a norm on $H^s_0(\Omega)$, it follows that

$$\mathcal{E}(u - \overline{u}, [u - \overline{u}]^+) \geqslant \|[u - \overline{u}]^+\|^2_{H^s_0(\Omega)} > 0\,,$$

a contradiction to (3.6). Hence meas$(A) = 0$, that is, $u(x) \leqslant \overline{u}(x)$ for a.e. $x \in \Omega$. Similarly, by letting $\phi := [\underline{u} - u]^+$ as a test function, and repeating the argument above we can show $\text{meas}(B) = 0$, where $B := \{x \in \mathbb{R}^N : \underline{u}(x) > u(x)\}$. Hence $u(x) \geqslant \underline{u}(x)$ a.e. $x \in \Omega$. This proves the claim. Therefore, it suffices to show the existence of a solution of (3.4) using the Schauder fixed point theorem.

The Nemytskii operator $\hat{g} \colon L^\infty(\Omega) \to L^\infty(\Omega)$ defined by $\hat{g}(u)(x) := g^*(x, u(x))$ is continuous (see [AZ90, Thm. 3.17, p. 110]) since g^* satisfies (H1) and (H2). Since the solution operator $(-\Delta)^{-s} \colon L^\infty(\Omega) \to L^\infty(\Omega)$, as defined in (2.8), is continuous and compact, it follows that $(-\Delta)^{-s} \circ \hat{g} \colon L^\infty(\Omega) \to L^\infty(\Omega)$ is continuous and compact. Clearly fixed points of $(-\Delta)^s \circ \hat{g}$ are solutions of (3.4).

Next, we find a nonempty, closed, convex subset of $L^\infty(\Omega)$ to apply the Schauder fixed point theorem. Since $\underline{u}, \overline{u} \in L^\infty(\Omega)$, $r^* := \max\{\|\underline{u}\|_\infty, \|\overline{u}\|_\infty\} > 0$. Then, it follows from (H1), applied to g^*, that there exists $a_{r*} \in L^\infty(\Omega)$ such that $|g^*(x,t)| \leqslant a_{r*}(x)$ for all $|t| \leqslant r^*$. Therefore, for any $u \in L^\infty(\Omega)$, we have

$$\|(-\Delta)^{-s} \circ \hat{g}(u)\|_\infty \leqslant \|(-\Delta)^{-s}\| \|\hat{g}(u)\|_\infty \leqslant \|(-\Delta)^{-s}\| \|a_{r*}\|_\infty ,$$

and hence the operator $(-\Delta)^{-s} \circ \hat{g}$ maps $\overline{B}_R(0)$ to itself where $R := \|(-\Delta)^{-s}\| \|a_{r*}\|_\infty$ and $\|\cdot\|$ is the operator norm. Thus, by the Schauder fixed point theorem, $(-\Delta)^{-s} \circ \hat{g}$ has a fixed point $u \in \overline{B}_R(0) \subset L^\infty(\Omega)$. This implies that the modified problem (3.4) and hence the original problem (3.1) has a weak solution $u \in L^\infty(\Omega)$. By the definition of $(-\Delta)^{-s}$, it follows that $u \in H^s_0(\Omega)$ as well. Hence the proof is complete. ∎

CHAPTER IV

SUPERLINEAR PROBLEMS

4.1. Introduction and Statement of Result

In this chapter, we study a nonlocal problem of the form

$$\begin{cases} (-\Delta)^s u &= \lambda f(u) \quad \text{in } \Omega\,; \\ u &= 0 \quad \text{in } \mathbb{R}^N \backslash \Omega\,, \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N (N \geqslant 2)$ is a bounded domain with C^2 boundary $\partial\Omega$, $s \in (0,1)$ is fixed, and $\lambda > 0$ is a bifurcation parameter. The nonlinearity $f : [0, \infty) \to \mathbb{R}$ is continuous and subcritical and superlinear at infinity, that is; there exists a constant $b > 0$ such that

$$(G1) \qquad \lim_{\sigma \to +\infty} \frac{f(\sigma)}{\sigma^p} = b \quad \text{with} \quad 1 < p < 2_s^* - 1 = \frac{N+2s}{N-2s}\,,$$

where $2_s^* := \frac{2N}{N-2s}$ is the fractional critical exponent (see e.g. [DNPV12]). The assumption $(G1)$ implies that f is positive for σ large. The goal of this chapter is to discuss the existence of a positive weak solution of (4.1) with respect to λ without assuming any additional sign condition on the reaction term f including near the origin.

The authors in [ROS14b, Prop. 1.2] established the existence of a minimal positive solution of (4.1) for λ small and discussed the existence and regularity of an extremal (positive) solution when $f(\sigma) > 0$ for $\sigma \geqslant 0$ and f superlinear. Theorem 4.1 complements the results of [ROS14b, Prop. 1.2] by capturing the branch of positive solutions bifurcating from infinity at $\lambda = 0$ for $1 < p < 2_s^* - 1$. See also [BJK19, Thm. 2.4], where existence of solutions for λ small is established. When $\lambda = 1$ and $f(0) = 0$, existence of a positive viscosity solution of (4.1) was obtained in [BDPGMQ18, Thm. 1.1] for continuous f satisfying $(G1)$. For the existence of nonnegative and positive solutions for superlinear problems using variational methods, see [ADM19, DI18, MBMS17, SV12, SV13, WZ19, ZF15]. For existence results for superlinear problems concerning the spectral fractional Laplacian operator, see [Amb17, CT10, Cap11] and the references therein.

In order to define some terminologies necessary to state our result, we first state the following lemma that establishes the L^∞ regularity of weak solutions of general superlinear, subcritical problems. We give the proof in Section 4.3.

Lemma 4.1. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying*

$$(G2) \qquad |g(x, \sigma)| \leqslant A(1 + |\sigma|^p) \quad \text{for a.e.} \quad x \in \Omega \text{ and all } \sigma \in \mathbb{R}$$

for some $p \in (1, 2_s^* - 1)$ and for some constant $A > 0$. If u is a weak solution of

$$\begin{cases} (-\Delta)^s u &= g(x, u) \quad \text{in} \quad \Omega\,; \\ \qquad u &= 0 \qquad\quad \text{in} \quad \mathbb{R}^N \backslash \Omega\,, \end{cases} \tag{4.2}$$

then $u \in L^\infty(\Omega)$.

Since f satisfies $(G1)$, it also satisfies $(G2)$. Then Lemma 4.1 implies that any weak solution of (4.1) belongs to $L^\infty(\Omega)$ making it possible to take $L^\infty(\Omega)$ as our underlying space. Therefore, we can define

$$\Sigma := \left\{(\lambda, u) \in [0, +\infty) \times L^\infty(\Omega) \;\middle|\; (\lambda, u) \text{ is a weak solution of (4.1)}\right\}.$$

We say that $\lambda_\infty \in \mathbb{R}$ is a bifurcation point from infinity for (4.1) if there exists a sequence $(\lambda_n, u_n) \in \Sigma$ such that $\lambda_n \to \lambda_\infty$ and $\|u_n\|_\infty \to +\infty$ as $n \to +\infty$. By a continuum of weak solutions of (4.1), we mean a subset $\mathscr{C} \subset \Sigma$ which is closed and connected. We say that a continuum $\mathscr{C} \subset \Sigma$ bifurcates from infinity at $\lambda_\infty \in [0, +\infty)$ if there exists a sequence $(\lambda_n, u_n) \in \mathscr{C}$ such that $\lambda_n \to \lambda_\infty$ and $\|u_n\|_\infty \to +\infty$ as $n \to +\infty$. Then, we prove the following.

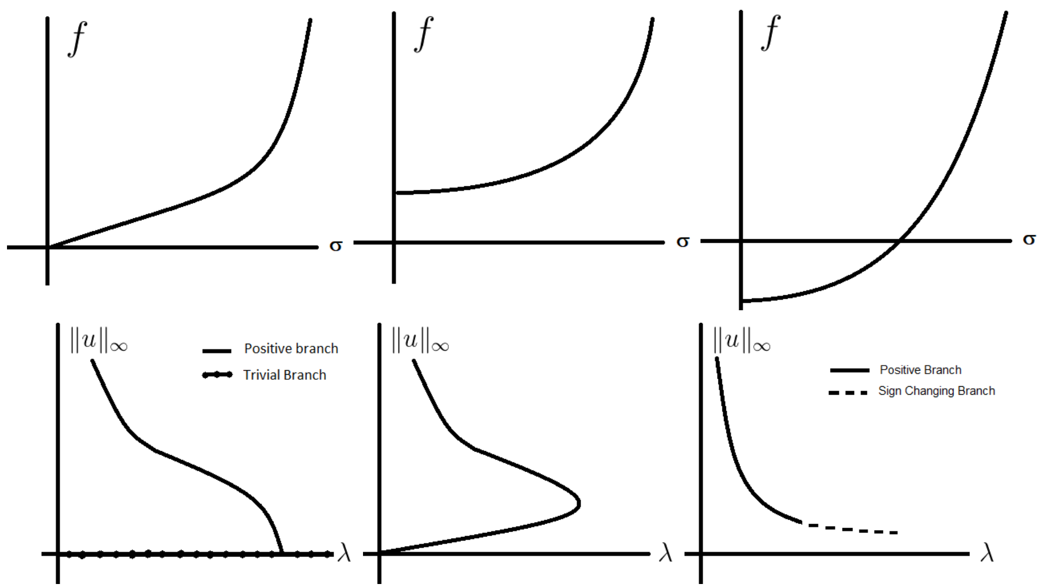


Figure 4.1. Nonlinearities and Bifurcation Diagrams for Theorem 4.1

Theorem 4.1. *Let f satisfy $(G1)$. Then there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$, (4.1) has a positive weak solution u such that $\|u\|_\infty \to \infty$ as $\lambda \to 0^+$. Moreover, there exists a continuum $\mathscr{C} \subset \Sigma$, bifurcating from infinity at $\lambda = 0$, such that λ takes all values in $(0, \lambda_0]$ along $\mathscr{C}$ and $u > 0$ whenever $(\lambda, u) \in \mathscr{C}$ and $\lambda \in (0, \lambda_0]$.*

The shapes of the nonlinearity f and the expected bifurcation diagrams corresponding to Theorem 4.1 are given in Figure 4.1. Theorem 4.1 establishes the existence of the continuum bifurcating from infinity at $\lambda = 0$. The complete bifurcation diagrams, as shown in Figure 4.1, require additional hypotheses to $(G1)$, and such results are not yet known. However, if there exists $a_2 > 0$ such that $f(\sigma) \geqslant a_2\,\sigma$ for $\sigma \geqslant 0$, then Theorem 5.7 (ii) implies that (4.1) has no nonnegative nontrivial solution for $\lambda > \frac{\lambda_1}{a_2}$. A similar nonexistence result was established in [ROS14b, Prop 1.2] for C^1 non-decreasing f satisfying $f(0) > 0$ and f superlinear at infinity.

Examples of nonlinearities satisfying the hypotheses of Theorem 4.1 are $f(\sigma) = \sigma^p$, $f(\sigma) = 3(1+\sigma)^{\frac{1}{3}} + \sigma^p$, $f(\sigma) = \sigma + \sigma^p$, and $f(\sigma) = \sigma + \sigma^p - 1$ for $\sigma \geqslant 0$ with $1 < p < 2_s^* - 1$.

In the Laplacian case ($s = 1$), similar existence results for superlinear problems were discussed in [Lio82, Section 1.1 & 2.1] for the case $f(0) \geqslant 0$, and in [ANZ92,AAB94, Uns88] for the case $f(0) < 0$. In the Laplacian case, $C^{1,\eta}$ regularity of solutions were crucial in establishing the positivity of solutions obtained using variational methods or degree theory, especially when $f(\sigma) < 0$ for some $\sigma \geqslant 0$. However, for the fractional Laplacian case one cannot expect better than $C^{0,s}(\overline{\Omega})$ regularity for any solution of (4.1), see [RO16, Sec. 7.1]. Therefore, we carefully analyze the behavior of solutions near the boundary to achieve the positivity of weak solutions for λ small.

The main tool in the proof of Theorem 4.1 is degree theory. The following L^∞ a priori bound result is crucial in applying degree theory.

Proposition 4.1. *([BDPGMQ18, Thm. 3.1]) Let $\Omega \subset \mathbb{R}^N$ be a C^2 bounded domain. Let $g(x,\sigma) := \sigma^p + h(x,\sigma)$ with $1 < p < 2_s^* - 1$, where h satisfies $|h(x,\sigma)| \leqslant c(1+|\sigma|^r)$ for a.e. $x \in \Omega$, for all $\sigma \in \mathbb{R}$, for some $0 < r < p$, and for some $c > 0$. Then, there exists a constant $M > 0$ such that every positive viscosity solution u of* (4.2) *satisfies*

$$\|u\|_\infty \leqslant M\,.$$

Remark 4.1. Let g be as in Proposition 4.1. Then, Lemma 4.1 implies that every weak solution u of (4.2) belongs to $L^\infty(\Omega)$. This in turn implies that u is a viscosity solution of (4.2) (see [SV14, Thm. 1]). Hence, Proposition 4.1 holds for every positive weak solution of (4.2).

The proofs of Theorem 4.1 and Lemma 4.1 rely on the following regularity result.

Proposition 4.2. *[BWZ17, Lem 2.5] Let $\ell \in L^q(\Omega)$ for some $q \geqslant \frac{2N}{N+2s}$. Then the linear problem* (2.6) *has a unique weak solution v. In addition, the following assertions hold:*

(a) *If $q > \frac{N}{2s}$, then $v \in L^\infty(\Omega)$ and there exists a positive constant $C_1 = C_1(N, s, q)$ such that $\|v\|_\infty \leqslant C_1 \|\ell\|_{L^q(\Omega)}$.*

(b) *If $\frac{2N}{N+2s} \leqslant q \leqslant \frac{N}{2s}$, then $v \in L^{\tilde{q}}(\Omega)$ for every $\tilde{q}$ satisfying $q \leqslant \tilde{q} < \frac{Nq}{N-2sq}$ and there exists a positive constant C_2 such that $\|v\|_{L^{\tilde{q}}(\Omega)} \leqslant C_2 \|\ell\|_{L^q(\Omega)}$.*

In Section 4.2, we prove Theorem 4.1 in the spirit of [ANZ92, AAB94]. More precisely, we use degree theory to establish the existence of a weak solution for a corresponding re-scaled problem. Positivity of the solution is then achieved by carefully analyzing solutions near the boundary. Then, using the Leray-Schauder continuation theorem, we conclude that there exists a continuum of positive weak solutions bifurcating from infinity at $\lambda = 0$. In Section 4.3, we prove Lemma 4.1 using a bootstrap argument.

4.2. Proof of Theorem 4.1

We first extend f to all of $\mathbb{R}$ by setting $f(\sigma) = f(|\sigma|)$ for any $\sigma \in \mathbb{R}$ and consider the modified problem

$$\begin{cases} (-\Delta)^s u &= \lambda f(|u|) \quad \text{in} \quad \Omega\,; \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Omega\,. \end{cases} \tag{4.3}$$

Then, for $\lambda > 0$, u satisfies (4.3) if and only if the re-scaled function $w := \lambda^{\frac{1}{p-1}} u$ satisfies

$$(-\Delta)^s w = \lambda^{\frac{1}{p-1}} (-\Delta)^s u = \lambda^{\frac{p}{p-1}} f(\lambda^{\frac{1}{1-p}} |w|).$$

Since $p > 1$, using the continuity of f for $\sigma_0 = 0$ and $(G1)$ for $\sigma_0 \neq 0$, for any $\sigma_0 \in \mathbb{R}$, there holds

$$\lim_{\substack{\lambda \to 0^+ \\ \sigma \to \sigma_0}} \lambda^{\frac{p}{p-1}} f(\lambda^{\frac{1}{1-p}} |\sigma|) = b|\sigma_0|^p\,.$$

Therefore, the argument above shows that the modified nonlinearity $F : [0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$F(\lambda, \sigma) := \lambda^{\frac{p}{p-1}} f(\lambda^{\frac{1}{1-p}} |\sigma|) = \lambda^{\frac{p}{p-1}} (f(\lambda^{\frac{1}{1-p}} |\sigma|) - b\lambda^{\frac{p}{1-p}} |\sigma|^p) + b|\sigma|^p$$

is continuous by setting $F(0, \sigma) := b|\sigma|^p$ for $\sigma \in \mathbb{R}$. For $\lambda \geqslant 0$, we study the following problem

$$\begin{cases} (-\Delta)^s w &= F(\lambda, w) \quad \text{in} \quad \Omega\,; \\ w &= 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Omega\,. \end{cases} \tag{4.4}$$

We note that, for $\lambda = 0$, equation (4.4) reduces to the following limiting problem

$$\begin{cases} (-\Delta)^s w &= b|w|^p \quad \text{in} \quad \Omega\,; \\ w &= 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Omega\,. \end{cases} \tag{4.5}$$

To define an operator equation corresponding to (4.4), we note that the solution operator $(-\Delta)^{-s}\colon L^\infty(\Omega) \to L^\infty(\Omega)$ given by $\ell \mapsto v$ is well-defined, continuous, and compact as discussed in Section 2.2.

Then, for any $\gamma > \frac{pN}{2s}$ fixed, $(-\Delta)^{-s} : L^{\gamma/p}(\Omega) \to L^\infty(\Omega)$ is continuous by Proposition 4.2(a).

Since $L^\infty(\Omega) \hookrightarrow L^\gamma(\Omega)$, the Nemytskii operator $\tilde{F}(\lambda, w)(x) := F(\lambda, w(x))$ is continuous as a mapping from $[0, +\infty) \times L^\infty(\Omega)(\hookrightarrow [0, +\infty) \times L^\gamma(\Omega)) \to L^{\gamma/p}(\Omega)$, see [Vai64, Sec. 19]. Therefore, it follows that the map $S : [0, +\infty) \times L^\infty(\Omega) \longrightarrow L^\infty(\Omega)$, defined by

$$S(\lambda, w) := \left((-\Delta)^{-s} \circ \tilde{F}\right)(\lambda, w)$$

is continuous. We note that for any $K > 0$, the value

$$\max\left\{|F(\lambda, \sigma)| \;\middle|\; \lambda \in [0, K], \sigma \in [-K, K]\right\}$$

is achieved since the function F is continuous. Hence, the Nemytskii operator $\tilde{F}$ takes bounded sets in $[0, +\infty) \times L^\infty(\Omega)$ into bounded sets in $L^\infty(\Omega)$. Then, the compactness of $(-\Delta)^{-s}$ implies that S is compact. Clearly w is a solution of the operator equation $w = S(\lambda, w)$ if and only if w is a weak solution of (4.4) for $\lambda \geqslant 0$.

The following lemma, concerning the limiting problem (4.5), was established in [BDPGMQ18, proof of Thm. 1.2] for a more general right hand side than that of

(4.5) for viscosity solutions within a different functional framework using the cone of nonnegative functions. Since we do not assume any sign condition on f, we cannot work with the cone of nonnegative solutions. However, thanks to Remark 4.1, the result still holds within our functional framework for weak solutions using Proposition 4.1. In fact, we give a more simple proof since the existence of the principal eigenvalue of $(-\Delta)^s$ is now known, see [SV13, Prop. 9].

Lemma 4.2. *There exist* $0 < r < R$ *and* $\psi \in L^\infty(\Omega)$ *with* $\psi \geqslant 0$ *in* Ω *such that*

(*a*) $w \neq \theta S(0, w)$ *for all* $\theta \in [0, 1]$ *and all* $w \in L^\infty(\Omega)$ *with* $\|w\|_\infty = r$, *and* $\deg(I - S(0, \cdot), B_r, 0) = 1$.

(*b*) $w \neq S(0, w) + t\psi$ *for all* $t \geqslant 0$ *and all* $w \in L^\infty(\Omega)$ *with* $\|w\|_\infty = R$, *and* $\deg(I - S(0, \cdot), B_R, 0) = 0$.

Proof of Lemma 4.2. Suppose by contradiction that for any $r > 0$, there exist $\theta \in [0, 1]$ and $w \in L^\infty(\Omega)$ such that $\|w\|_\infty = r$ and $w = \theta S(0, w)$. That is, w satisfies (4.5) with $\theta b \geqslant 0$ in place of $b > 0$. If $\theta = 0$, then $w :\equiv 0$ contradicts $\|w\|_\infty = r > 0$. If $\theta \in (0, 1]$ and $\|w\|_\infty = r$, then we have $\theta b|w(x)|^p \leqslant b|w(x)|^p \leqslant b\|w\|_\infty^p = br^p$ a.e. in Ω and thus $\|\theta b|w|^p\|_\infty \leqslant br^p$. By (2.7) applied to (2.6) with $\ell = \theta b|w|^p$, we get

$$\|w\|_\infty \leqslant \|w\|_{C^{0,s}(\overline{\Omega})} \leqslant Cbr^p = Cbr^{p-1}\|w\|_\infty . \tag{4.6}$$

We get a contradiction to (4.6) by choosing $r > 0$ sufficiently small such that $Cbr^{p-1} < 1$ since $p > 1$. Therefore, there exists $r > 0$ such that $w \neq \theta S(0, w)$ for all $\theta \in [0, 1]$ satisfying $\|w\|_\infty = r$. Using the homotopy invariance of degree with θ as the homotopy parameter and the fact that $\deg(I, B_r, 0) = 1$, we conclude $\deg(I - S(0, \cdot), B_r, 0) = \deg(I - \theta S(0, \cdot), B_r, 0) = \deg(I, B_r, 0) = 1$. Hence (*a*) holds.

Now we prove part (b). Let $e \in H_0^s(\Omega)$ be the positive weak solution of (2.9). It follows from (2.7) that $e \in L^\infty(\Omega)$. Moreover, for $t \geqslant 0$, solutions of $w = S(0,w) + te$ satisfy

$$\begin{cases} (-\Delta)^s w &= b|w|^p + t \quad \text{in} \quad \Omega\,; \\ w &= 0 \quad\quad\quad\quad \text{in} \quad \mathbb{R}^N \backslash \Omega\,. \end{cases} \tag{4.7}$$

Note, since $b|\sigma|^p + t \geqslant 0$ for $t \geqslant 0$, any weak solution of (4.7) is nonnegative for $t \geqslant 0$. In fact, any nontrivial weak solution of (4.7) is positive by Proposition 2.5. First, we show that (4.7) has no weak solution for t large. For this, let $\lambda_1 > 0$ be the principal eigenvalue and $0 < \varphi_1$ be the corresponding eigenfunction of the fractional eigenvalue problem (2.15).

Let $\mu > \lambda_1$ be fixed. Then, since $p > 1$ and $t \geqslant 0$, there exists a constant $\tilde{C} > 0$ such that $b\sigma^p + t \geqslant \mu\sigma + t - \tilde{C}$. Suppose w is a nonnegative nontrivial weak solution of (4.7) with $t > \tilde{C}$. Then, using $0 < \varphi_1$ as a test function in the weak formulation of (4.7), we arrive at a contradiction to $\mu > \lambda_1$:

$$\begin{aligned} &\lambda_1 \int_\Omega w\,\varphi_1 \mathrm{d}x = \mathcal{E}(w,\varphi_1) \\ &= \int_\Omega [bw^p + t]\varphi_1 \mathrm{d}x \geqslant \int_\Omega [\mu\, w + t - \tilde{C}]\varphi_1 \mathrm{d}x \geqslant \mu \int_\Omega w\varphi_1 \mathrm{d}x\,. \end{aligned} \tag{4.8}$$

Hence, (4.7) has no weak solution for $t > \tilde{C}$, and in particular, for $t = \tilde{C} + 1$. Therefore,

$$\deg(I - S(0,\cdot) + (\tilde{C}+1)e, B_\varrho, 0) = 0 \text{ for any } \varrho > 0\,.$$

Then, by Proposition 4.1 combined with Remark 4.1 with $h(x,\sigma) = t$ for $0 \leqslant t \leqslant \tilde{C}+1$, there exists $M > 0$ (depending only on Ω, N, s and $\tilde{C}$) such that $\|w\|_\infty \leqslant M$ for any positive weak solution w of (4.7). Setting $R > \max\{M, r\}$, we conclude that (4.7) has no solution for $0 \leqslant t \leqslant \tilde{C} + 1$ satisfying $\|w\|_\infty = R$. Using $0 \leqslant t \leqslant \tilde{C} + 1$ as the homotopy parameter, we conclude that

$$\begin{aligned} \deg(I - S(0, \cdot), B_R, 0) &= \deg(I - S(0, \cdot) + te, B_R, 0) \\ &= \deg(I - S(0, \cdot) + (\tilde{C} + 1)e, B_R, 0) = 0\,. \end{aligned}$$

This establishes part (b), completing the proof of Lemma 4.2. ■

It is clear from the construction of $r > 0$ and $R > 0$ above that $0 < r < R$ holds. Then, using the excision property of Leray-Schauder degree, it follows from Lemma 4.2 that

$$\deg(I - S(0, \cdot), B_R \backslash \overline{B}_r, 0) = -1 \neq 0\,.$$

Therefore, there exists a solution w_0 of (4.5) (not necessarily unique). Moreover, since $(-\Delta)^s w_0 \geqslant 0$ in Ω and $\|w_0\|_\infty > r > 0$, it follows from Proposition 2.5 that $w_0 > 0$ in Ω. Now using $\lambda \geqslant 0$ as a homotopy parameter and Lemma 4.2, we prove the existence of a positive weak solution of the re-scaled problem (4.4) for λ small.

Lemma 4.3. *There exists $\lambda_0 > 0$ such that*

(i) $\deg(I - S(\lambda, \cdot), B_R \backslash \overline{B}_r, 0) = -1$ *for all* $0 \leqslant \lambda \leqslant \lambda_0$.

(ii) If $w_\lambda = S(\lambda, w_\lambda)$ with $0 \leqslant \lambda \leqslant \lambda_0$ and $r < \|w_\lambda\|_\infty < R$, then it follows that $w_\lambda > 0$ in Ω.

Proof of Lemma 4.3. To prove (i) using the homotopy invariance of Leray-Schauder degree, it suffices to show that there exists $\lambda_0 > 0$ such that $S(\lambda, w_\lambda) \neq w_\lambda$ for all $\|w_\lambda\|_\infty \in \{r, R\}$ and all $0 \leqslant \lambda \leqslant \lambda_0$. If not, there exists a sequence $(\lambda_n, w_{\lambda_n})$ in

$[0,+\infty) \times L^\infty(\Omega)$ with $\lambda_n \to 0^+$, $\|w_{\lambda_n}\|_\infty \in \{r, R\}$, and $S(\lambda_n, w_{\lambda_n}) = w_{\lambda_n}$. Since S is compact and continuous, $(\lambda_n, w_{\lambda_n}) \to (0, w_0)$ (up to a subsequence) for some $w_0 \in L^\infty(\Omega)$, $\|w_0\|_\infty \in \{r, R\}$, and $S(0, w_0) = w_0$, a contradiction with Lemma 4.2. Hence, (i) holds.

Now we prove (ii). If $f(\sigma) \geqslant 0$ for all $\sigma \geqslant 0$, then $F(\lambda, \sigma) \geqslant 0$ for all $\sigma \in \mathbb{R}$. Then, using $\|w_\lambda\|_\infty > r > 0$, it follows from Proposition 2.5 that $w_\lambda > 0$ in Ω. On the other hand, if $f(\sigma) < 0$ for some $\sigma \geqslant 0$, then Proposition 2.5 does not apply. In this case, we proceed by contradiction. Suppose there exist sequences $(\lambda_n, w_{\lambda_n}) \in [0,+\infty) \times L^\infty(\Omega)$ and $x_n \in \Omega$ satisfying $\lambda_n \to 0^+$, $w_{\lambda_n}(x_n) \leqslant 0$ such that $r < \|w_{\lambda_n}\|_\infty < R$ and $S(\lambda_n, w_{\lambda_n}) = w_{\lambda_n}$. First we show that w_{λ_n} is bounded in $C^{0,s}(\overline{\Omega})$. Since $\lambda_n \to 0^+$, we may assume that $0 \leqslant \lambda_n \leqslant R$. Letting $\ell_n := F(\lambda_n, w_{\lambda_n})$ and using the facts that F is continuous, and w_{λ_n} is measurable, we conclude that ℓ_n is measurable for each n. Moreover,

$$
\begin{aligned}
|\ell_n(x)| = |F(\lambda_n, w_{\lambda_n}(x))| &\leqslant \max\{|F(\lambda, \sigma)| \;\big|\; 0 \leqslant \lambda \leqslant R,\, 0 \leqslant \sigma \leqslant R\} \\
&:= \text{const.} < +\infty \text{ a.e. in } \Omega\,.
\end{aligned}
$$

This gives $\|\ell_n\|_\infty \leqslant$ const., independent of n. It then follows from (2.7) that $\|w_{\lambda_n}\|_{C^{0,s}(\overline{\Omega})} \leqslant C\|\ell_n\|_\infty \leqslant$ const., again independent of n, as desired. Then, using the compactness of the embedding $C^{0,s}(\overline{\Omega}) \hookrightarrow C^{0,s'}(\overline{\Omega})$ with $0 < s' < s$, we conclude that $w_{\lambda_n} \to w_0$ in $C^{0,s'}(\overline{\Omega})$ (up to a subsequence) for some $w_0 \in C^{0,s'}(\Omega)$, and $S(0, w_0) = w_0$ by the continuity argument as above. Now since $w_0 > 0$ in Ω with $(-\Delta)^s w_0(x) \geqslant 0$ in Ω, by Proposition 2.5 there exists $c > 0$, such that

$$
w_0(x) \geqslant c\,\delta^s(x) \text{ for all } x \in \Omega\,.
$$

Since $x_n \in \Omega$ and Ω is bounded, $x_n \to x_0 \in \overline{\Omega}$ (up to a subsequence). Using the facts that $w_{\lambda_n} \to w_0$ in $C^{0,s'}(\overline{\Omega})$, $w_{\lambda_n}(x_n) \leqslant 0$, and $w_0 > 0$ in Ω, it follows that $x_0 \in \partial\Omega$. Let $x_n^{\mathrm{bdd}} \in \partial\Omega$ be the point on $\partial\Omega$ nearest to the point $x_n \in \Omega$. Since $\partial\Omega \in C^2$, this point is unique for each n sufficiently large (see [GT15, App. 14.6]) and clearly $x_n \neq x_n^{\mathrm{bdd}}$. Therefore, for $0 < \epsilon < c$ fixed, there exists n sufficiently large such that

$$\begin{aligned}
\epsilon > [w_0 - w_{\lambda_n}]_{C^{0,s'}(\overline{\Omega})} &\geqslant \frac{|w_0(x_n) - w_{\lambda_n}(x_n) - (w_0(x_n^{\mathrm{bdd}}) - w_{\lambda_n}(x_n^{\mathrm{bdd}}))|}{|x_n - x_n^{\mathrm{bdd}}|^{s'}} \\
&\geqslant \frac{w_0(x_n)}{|x_n - x_n^{\mathrm{bdd}}|^{s'}} \\
&> \frac{w_0(x_n)}{|x_n - x_n^{\mathrm{bdd}}|^{s}}
\end{aligned}$$

since $w_0(x_n) > 0$ and $w_{\lambda_n}(x_n) \leqslant 0$ in Ω, $w_{\lambda_n}(x_n^{\mathrm{bdd}}) = 0 = w_0(x_n^{\mathrm{bdd}})$, and $0 < s' < s$. This yields

$$\epsilon > \frac{w_0(x_n)}{|x_n - x_n^{\mathrm{bdd}}|^{s}} = \frac{w_0(x_n)}{[\mathrm{dist}(x_n, \partial\Omega)]^{s}} \geqslant c\,,$$

a contradiction to $\epsilon < c$. Therefore, $w_\lambda > 0$ for all $0 \leqslant \lambda \leqslant \lambda_0$ with $r < \|w_\lambda\|_\infty < R$ (with λ_0 possibly smaller). This concludes the proof of Lemma 4.3. ∎

Now we complete the proof of Theorem 4.1. By Lemma 4.3, for each $0 \leqslant \lambda \leqslant \lambda_0$, there exists a positive weak solution w_λ of (4.4) with $r < \|w_\lambda\|_\infty < R$. This in turn implies that (4.1) has a positive weak solution $u = \lambda^{\frac{1}{1-p}} w_\lambda$ for $0 < \lambda \leqslant \lambda_0$. Owing to the facts that $p > 1$ and $\|w_\lambda\|_\infty > r > 0$, we can infer that $\|u\|_\infty \to +\infty$ as $\lambda \to 0^+$.

Finally, we use the following Leray-Schauder continuation theorem to prove the existence of a connected branch of positive solutions for λ small.

Proposition 4.3. *([Maw99, Thm. 2.2]) Let X be a Banach space and $Y \subset X$ a bounded open set and $[a, b] \subset \mathbb{R}$. Suppose $T : [a, b] \times \overline{Y} \to X$ is continuous and compact. Define $\mathscr{S} := \{(\mu, z) \in [a, b] \times \overline{Y} : z = T(\mu, z)\}$ and assume the following conditions hold:*

(a) $\mathscr{S} \cap ([a, b] \times \partial Y) = \varnothing$, and

(b) $deg(I - T(a, \cdot), Y, 0) \neq 0$.

Then $\mathscr{S}$ contains a continuum $\mathcal{C}$ along which μ takes all values in $[a, b]$.

Indeed, S satisfies the hypotheses of Proposition 4.3 via Lemma 4.3 with $[a, b] = [0, \lambda_0]$, $X = L^\infty(\Omega)$, and $Y = B_R \backslash \overline{B}_r$. Therefore, there exists a continuum $\mathscr{D}$ of positive weak solutions w_λ of (4.4) along which λ takes all values in $[0, \lambda_0]$. This in turn implies, using the relation, $u = \lambda^{\frac{1}{1-p}} w_\lambda$ for $0 < \lambda \leqslant \lambda_0$, there exists a continuum $\mathscr{C}$ of positive weak solutions of (4.1) bifurcating from infinity at $\lambda_\infty = 0$. Moreover, λ takes all values in $(0, \lambda_0]$ along $\mathscr{C}$ and $u > 0$ in Ω whenever $u \in \mathscr{C}$ and $\lambda \in (0, \lambda_0]$. This completes the proof of Theorem 4.1. ■

4.3. Regularity of Weak Solutions

Here we give the proof of Lemma 4.1 by using Proposition 4.2 and a bootstrap argument.

Proof of Lemma 4.1. Let u be a weak solution of (4.2). Then $u \in H_0^s(\Omega) \hookrightarrow L^{2_s^*}(\Omega)$, see [DNPV12, Thm. 6.7].

The Nemytskii operator, defined as

$$\tilde{g}(u)(x) := g(x, u(x)),$$

is continuous as a mapping from $L^{2_s^*}(\Omega)$ to $L^{\frac{2_s^*}{p}}(\Omega)$, see [Vai64, Sec. 19]. Keeping in mind that $N \geqslant 2$, we distinguish three cases:

(i) $N < 6s$ and $1 < p < \frac{4s}{N-2s}$.

(ii) $N < 6s$ and $\frac{4s}{N-2s} \leqslant p < 2_s^* - 1$.

(iii) $6s \leqslant N$ and $1 < p < 2_s^* - 1$.

In case (i), $\frac{2_s^*}{p} = \frac{2N}{N-2s} \cdot \frac{1}{p} > \frac{N}{2s}$ holds since $N < 6s$ yields $\frac{4s}{N-2s} > 1$. Then we are done by Proposition 4.2(a).

For cases (ii) and (iii), we use a bootstrap argument by employing Proposition 4.2(b). We observe that $\frac{2N}{N+2s} \leqslant \frac{2N}{N-2s} \cdot \frac{1}{p} \leqslant \frac{N}{2s}$ for both (ii) and (iii). Indeed, the left inequality $\frac{2N}{N-2s} \cdot \frac{1}{p} > \frac{2N}{N+2s}$ holds since $p < 2_s^* - 1$. The right inequality holds for (ii) and for (iii) since $p \geqslant \frac{4s}{N-2s}$ and $N \geqslant 6s$, respectively.

Define $q_0 := \frac{2N}{N-2s} \cdot \frac{1}{p}$ and $q_1 := \frac{1}{p}\left(\alpha q_0 + \beta \frac{N q_0}{N - 2q_0 s}\right)$, where $\alpha := \frac{Np - Np^2 + 2ps + 2p^2 s}{8s} \in (0,1)$ and $\beta := 1 - \alpha$. Then,

$$q_1 - q_0 = \frac{N(N - Np + 2(1+p)s)}{p(N-2s)(Np - 2(2+p)s)} > 0\,.$$

If $q_1 > \frac{N}{2s}$, we are done. Otherwise, we continue with the bootstrap argument as follows.

For each $k \in \mathbb{N}$, let $\alpha, \beta \in (0,1)$ be as above and $\frac{2N}{p(N-2s)} \leqslant q_k \leqslant \frac{N}{2s}$. Define $q_{k+1} := \frac{1}{p}\left(\alpha q_k + \beta \frac{Nq_k}{N-2q_k s}\right)$. Then,

$$
\begin{aligned}
q_{k+1} - q_k &= \frac{q_k \left(N(p-1)\left(pq_k - 4\right) - 2(p-3)psq_k\right)}{4p\left(N - 2sq_k\right)} \\
&\geqslant \frac{N(N - Np + 2(p+1)s)}{p(N-2s)(Np - 2(p+2)s)} > 0
\end{aligned}
$$

is independent of $k \in \mathbb{N}$. Hence, $q_{k+1} > \frac{N}{2s}$ can be achieved in finitely many steps. Then, by Proposition 4.2(a), $\|u\|_\infty \leqslant C_1 \|\tilde{g}(u)\|_{L^{\frac{2_s^*}{p}}(\Omega)} < +\infty$, and we are done. ■

CHAPTER V

SUBLINEAR, ASYMPTOTICALLY LINEAR, AND LOGISTIC PROBLEMS

5.1. Introduction and Statement of Results

We consider a nonlinear problem of the form

$$\begin{cases} (-\Delta)^s u &= \lambda f(u) \quad \text{in} \quad \Omega; \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Omega\,, \end{cases} \tag{5.1}$$

where $\lambda > 0$ is a bifurcation parameter and $\Omega \subset \mathbb{R}^N$ is a bounded $C^{1,1}$ domain if $N \geqslant 2$ and a bounded open interval if $N = 1$. Throughout this chapter, $f : [0, +\infty) \to \mathbb{R}$ is a Hölder continuous function, unless stated otherwise.

It is well known that the the existence as well as nonexistence of positive solutions of problems like (5.1), with local operators such as the classical Laplacian instead of $(-\Delta)^s$, with respect to the parameter λ, depends heavily on the behavior of the nonlinearity f near the origin as well as at infinity. For the Laplacian case ($s = 1$), see [Lio82] for an excellent review for the case $f(0) \geqslant 0$, and see [CMS00] for the case $f(0) < 0$ (*semipositone*).

Here we discuss several existence results and a simple nonexistence result of (5.1) depending both on the behavior of the nonlinearity near the origin and at infinity. Existence results in this chapter are established using the sub- and supersolution theorem, Theorem 3.1, established in Chapter III.

First, we consider classes of nonlinearity f that satisfy the sublinear at infinity condition

$$(F1) \qquad \lim_{\sigma \to +\infty} \frac{f(\sigma)}{\sigma} = 0\,.$$

For the local case, based on the Laplacian or p-Laplacian, existence results for sublinear problems are well studied. The paper [LSY09] provides a nice review of the development from the point of view of the sub- and supersolution method.

Our first result deals with the *positone* case ($f > 0$).

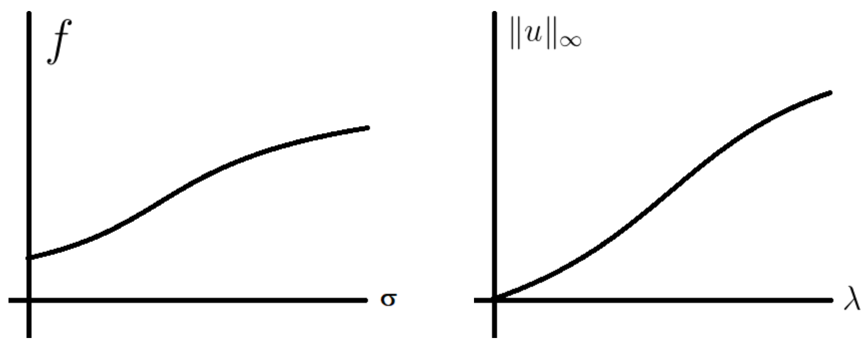


Figure 5.1. Nonlinearity and Bifurcation Diagram for Theorem 5.1

Theorem 5.1. *Suppose $f(\sigma) > 0$ for $\sigma \geqslant 0$ and satisfies* $(F1)$*. Then,* (5.1) *has a positive weak solution for each $\lambda > 0$.*

Figure 5.1 gives a typical example of the nonlinearity f and the expected bifurcation diagram ($\|u\|_\infty$ vs. λ diagram). An example satisfying the hypotheses of Theorem 5.1 is the reaction term $f(\sigma) = e^{\frac{\kappa\sigma}{\kappa+\sigma}}$ for $\sigma \geqslant 0$ with $\kappa > 0$, referred to in the literature as the *perturbed Gelfand problem* when considered with the Laplacian operator, see [BE89, Chap. 2]. For a nonlinearity like $f(\sigma) = e^{\frac{5\sigma}{5+\sigma}}/\sigma^q,\ q \in (0,1)$, it was shown in [GMS19] that there is a range of λ for which there exists three positive solutions and a unique positive solution for λ large. Their result suggests the existence of an S-shaped bifurcation diagram with an additional assumption on the shape of the nonlinearity. In fact, our numerical experiments in Chapter VII show that for $f(\sigma) = e^{\frac{5\sigma}{5+\sigma}}$, the numerical bifurcation diagram is S-shaped. For an existence result and bifurcation diagram for the Laplacian case ($s = 1$), see [Lio82, Sec. 2.2].

Our next result deals with the case $f(0) = 0$. Let λ_1 be the principle eigenvalue of the fractional eigenvalue problem (2.11).

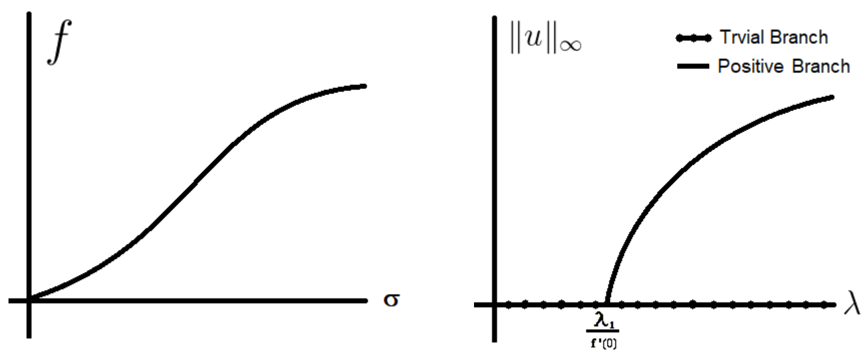


Figure 5.2. Nonlinearity and Bifurcation Diagram for Theorem 5.2

Theorem 5.2. *Suppose* $f : [0,\infty) \to [0,\infty)$ *is a* C^1 *function such that* $f(0) = 0$, $f'(0) > 0$ *with* $f(\sigma) > 0$ *for all* $\sigma > 0$, *and* $(F1)$ *is satisfied. Then,* (5.1) *has a positive weak solution for any* $\lambda > \frac{\lambda_1}{f'(0)}$.

Figure 5.2 gives the typical example of the nonlinearity f and the expected bifurcation diagram corresponding to Theorem 5.2. An example satisfying the hypotheses of Theorem 5.2 is the reaction term $f(\sigma) = 3(1+\sigma)^{1/3} - 3$ for $\sigma \geqslant 0$.

To the best of our knowledge, this simple existence result is not known for the fractional Laplacian case. The Laplacian case was discussed in [Lio82, Sec. 2.2].

Next, we consider a nonlinearity f which includes semipositone behavior near the origin ($f(0) < 0$) and establish the following existence result.

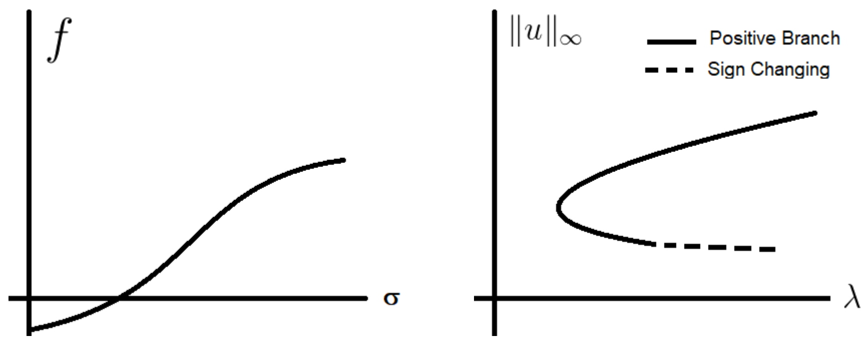


Figure 5.3. Nonlinearity and Bifurcation Diagram for Theorem 5.3

Theorem 5.3. *Let* $f : [0, \infty) \to \mathbb{R}$ *satisfy* $(F1)$. *If*

$$(F2) \qquad \lim_{\sigma \to \infty} f(\sigma) = \infty\,,$$

then (5.1) *has a positive weak solution for* λ *large.*

Figure 5.3 gives the typical shape of the nonlinearity f and the expected bifurcation diagram corresponding to Theorem 5.3. An example satisfying the hypotheses of Theorem 5.3 is the reaction term $f(\sigma) = \ln(1 + \sigma) - 0.5$ for $\sigma \geqslant 0$. A multi-parameter, sublinear semipositone problem was considered with pure powers in [DT19] to establish the existence of a positive solution. Theorem 5.3 extends their result to general semipositone nonlinearities satisfying $(F1)$. For the Laplacian case, see [LSY09]. The proof of Theorem 5.3 combines the ideas from [LSY09] and [DT19] to construct a positive weak subsolution.

Next, we consider classes of nonlinearities f that are asymptotically linear at infinity

$$(F3) \qquad \lim_{\sigma\to+\infty} \frac{f(\sigma)}{\sigma} = m_\infty > 0 \quad \text{for some} \quad 0 < m_\infty < \infty \,,$$

and establish the following existence result.

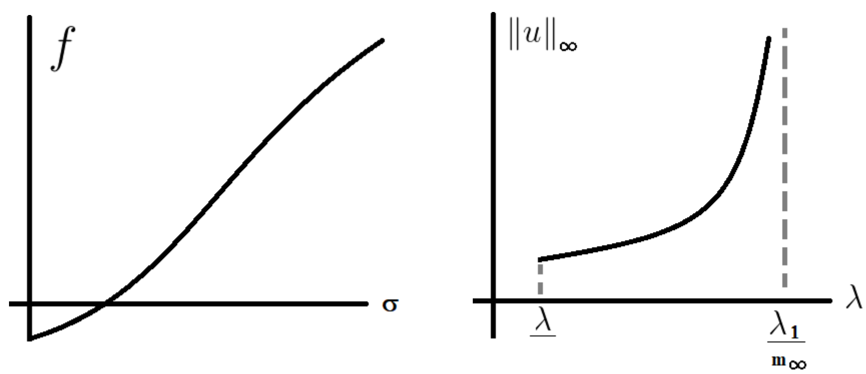


Figure 5.4. Nonlinearity and Bifurcation Diagram for Theorem 5.4

Theorem 5.4. *Let $f : [0, \infty) \to \mathbb{R}$ be a function satisfying $(F3)$. Then, there exists $\underline{\lambda} > 0$ with $\underline{\lambda} < \frac{\lambda_1}{m_\infty}$ such that* (5.1) *has a positive weak solution for $\lambda \in [\underline{\lambda}, \frac{\lambda_1}{m_\infty})$.*

Figure 5.4 gives an example of the shape of the nonlinearity f and the expected bifurcation diagram corresponding to Theorem 5.4. A simple example satisfying the hypothesis of Theorem 5.4 is the reaction term $f(\sigma) = \frac{1}{2}\sigma + 3(1+\sigma)^{\frac{1}{3}} - 4$ with $m_1 = 1$ and $m_\infty = \frac{1}{2}$. It is clear from the hypothesis (5.13) that the result above is independent of the sign of f near the origin. The exact shape of the bifurcation diagram will further depend on the precise information of the nonlinearity f near the origin.

Using bifurcation theory, the authors in [CG20] discussed an existence result for the fractional Laplacian in the left neighborhood of $\frac{\lambda_1}{m_\infty}$. Our result, Theorem 5.4, extends the range of λ for existence of a positive solution farther to the left of $\frac{\lambda_1}{m_\infty}$. Existence results for such problems for the local case were discussed in [AAB94] using bifurcation theory and in [Hai12, HSS12, KS20] using sub- and supersolution methods.

Next, we consider several classes of logistic problems. For a derivation of the time dependent fractional logistic model $u_t + (-\Delta)^s u = \lambda u(1-u)$ with $u = u(x,t)$ and $(x,t) \in \mathbb{R}^2$ for a simple two particle reaction scheme, see [BH04]. The authors in [CDV17] study logistic problems involving the fractional Laplacian and it serves as an excellent resource in this topic. They argue that under certain conditions, a nonlocal diffusion strategy ($s \in (0,1)$ with $s \approx 0$) may be advantageous for species in a confined environment with a hostile surrounding area which corresponds to the homogeneous Dirichlet external condition assumed here. In [MV17], the authors show that the the nonlocal strategy is advantageous for a diffusing population, and in [CDV17] the authors show that nonlocal populations may better adapt to sparse resources and small environments with hostile surrounding area.

First, we consider a weighted logistic problem and establish the following existence result.

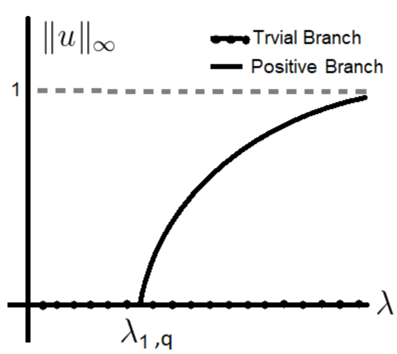


Figure 5.5. Bifurcation Diagram for Theorem 5.5

Theorem 5.5. *Let $q \in L^{\infty}(\Omega)$ be such that $0 \leqslant q \leqslant 1$ a.e. in Ω and $q(x) > \frac{1}{2}$ on a set of positive measure, and $\lambda_{1,q}$ is the principle eigenvalue of the weighted eigenvalue problem* (2.13). *The fractional logistic problem*

$$\begin{cases} (-\Delta)^s u = \lambda u(q(x) - u) & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{5.2}$$

has a positive weak solution for any $\lambda > \lambda_{1,q}$.

Figure 5.5 gives the expected bifurcation diagram corresponding to Theorem 5.5. In [CDV17], authors prove existence results that generalizes Theorem 5.5 using an energy minimization existence result. Therefore, our contribution is in the different approach of establishing this result using sub- and supersolution methods. See [SS03] for a precise bifurcation diagram for the logistic equation in the Laplacian case.

Next, we prove the following existence results for logistic problems with constant yield harvesting:

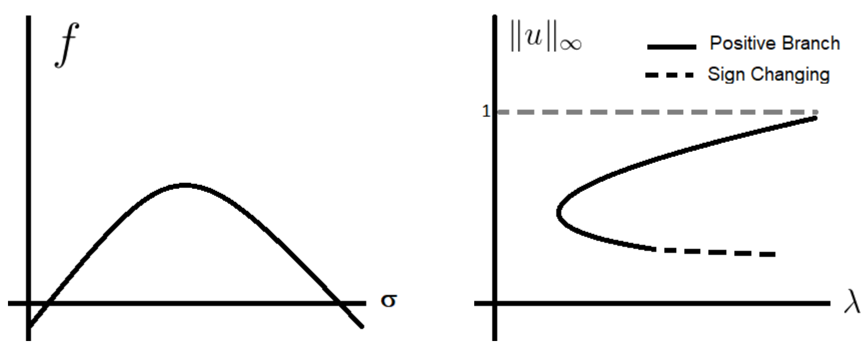


Figure 5.6. Nonlinearity and Bifurcation Diagram for Theorem 5.6

Theorem 5.6. *For any $\lambda > \lambda_1$, there exists $a^* = a^*(\lambda) > 0$ such that the fractional logistic problem with constant yield harvesting*

$$\begin{cases} (-\Delta)^s u &= \lambda\,[\,u(1-u) - a\,] \quad \text{in } \Omega; \\ u &= 0 \quad \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{5.3}$$

has a positive weak solution for $a \in (0, a^*)$.

Figure 5.6 gives the shape of the nonlinearity $f(\sigma) = \sigma(1-\sigma) - a$ and the expected bifurcation diagram corresponding to Theorem 5.6.

We conclude this section with the following nonexistence result that applies to several classes of the nonlinearity f considered in this chapter and Chapter IV.

Theorem 5.7.

(i) If there exist $a_1 > 0$ and $b_1 \geqslant 0$ such that $f(\sigma) \leqslant a_1\sigma - b_1$ for all $\sigma \geqslant 0$, then there is no nonnegative nontrivial weak solution of (5.1) for $\lambda < \frac{\lambda_1}{a_1}$.

(ii) If there exist $a_2 > 0$ and $b_2 \geqslant 0$ such that $f(\sigma) \geqslant a_2\sigma + b_2$ for all $\sigma \geqslant 0$, then there is no nonnegative nontrivial weak solution of (5.1) for $\lambda > \frac{\lambda_1}{a_2}$.

It follows from Theorem 5.7 (i) that if the nonlinear reaction term f satisfies the hypotheses of Theorem 5.2, Theorem 5.3, Theorem 5.5, or Theorem 5.6, then, in each case, the considered fractional Laplacian problem has no nonnegative nontrivial solution for λ small. Similarly, if in addition to $(G1)$, f satisfies the hypothesis of Theorem 5.7 (ii), then (4.1) has no nonnegative nontrivial solution for λ large.

In the rest of this chapter, we prove Theorem 5.3, Theorem 5.4, Theorem 5.5 and Theorem 5.6 by constructing a suitable ordered pair of weak sub- and supersolutions of (5.1) and employing Theorem 3.1. Constructions are motivated by what is known for the local cases (both Laplacian and p-Laplacian) and adapted to the fractional Laplacian case. To prove Theorem 5.7, we utilize the principal eigenvalue, corresponding positive eigenfunction, and weak formulation of problem (2.11).

The proofs of Theorem 5.1 and Theorem 5.2 are rather simple and turn out to be similar to the proofs of local cases. Therefore, we give these proofs in Appendix B for completeness.

5.2. Proof of Theorem 5.3

As in the local case ([LSY09]), we will construct a positive weak subsolution as a multiple of φ_1^2, where $0 < \varphi_1 \in H_0^s(\Omega)$ is the eigenfunction corresponding to the principle eigenvalue λ_1 of the eigenvalue problem (2.11). However, unlike local cases, estimates of the gradient near the boundary are not available. Instead, the following estimate, established in [DT19], is crucial in the construction of a positive weak subsolution. We provide the proof in Appendix 8.2 for completeness.

Proposition 5.1 ([DT19]). *Let $\varphi_1 > 0$ be the eigenfunction corresponding to the principle eigenvalue λ_1 of the eigenvalue problem* (2.11). *There exists $\gamma > 0$ such that $\gamma < h(x) < +\infty$ for all $x \in \Omega$ where*

$$h(x) := \int_{\mathbb{R}^N} \frac{|\varphi_1(x) - \varphi_1(y)|^2}{|x-y|^{N+2s}} \mathrm{d}y\,. \tag{5.4}$$

Now we recall the computation of $(-\Delta)^s\varphi_1^2$ by using definition (1.1) (see [DT19]) as follows:

$$\begin{aligned}
(-\Delta)^s\varphi_1^2(x) &= \text{P.V.}\int\limits_{\mathbb{R}^N} \frac{\varphi_1^2(x)-\varphi_1^2(y)}{|x-y|^{N+2s}}\mathrm{d}y \\
&= \text{P.V.}\int\limits_{\mathbb{R}^N} \frac{[\varphi_1(x)+\varphi_1(y)][\varphi_1(x)-\varphi_1(y)]}{|x-y|^{N+2s}}\mathrm{d}y \\
&= 2\varphi_1(x)\text{P.V.}\int\limits_{\mathbb{R}^N} \frac{\varphi_1(x)-\varphi_1(y)}{|x-y|^{N+2s}}\mathrm{d}y - \text{P.V.}\int\limits_{\mathbb{R}^N} \frac{[\varphi_1(x)-\varphi_1(y)]^2}{|x-y|^{N+2s}}\mathrm{d}y \\
&= 2\varphi_1(x)(-\Delta)^s\varphi_1(x) - \text{P.V.}h(x)\,.
\end{aligned}$$

By Proposition 5.1, P.V. $h(x) = h(x)$ in Ω. Hence

$$(-\Delta)^s\varphi_1^2(x) = 2\lambda_1\varphi_1^2(x) - h(x) \text{ in } \Omega\,. \tag{5.5}$$

Without loss of generality, assume $\|\varphi_1\|_\infty = 1$. Then it follows from Proposition 5.1, using $\varphi_1 = 0$ in $\mathbb{R}^N\backslash\Omega$, that there exist $\eta, m, \mu > 0$ such that

$$m < h(x) - 2\lambda_1\varphi_1^2(x) \quad \text{in} \quad \Omega_\eta\,, \tag{5.6}$$

and

$$\mu \leqslant \varphi_1 \leqslant 1 \quad \text{in} \quad \Omega\backslash\overline{\Omega}_\eta\,, \tag{5.7}$$

where $\Omega_\eta := \{x \in \Omega : \delta(x) < \eta\}$. Since f is continuous on $[0, +\infty)$ and satisfies $(F2)$, there exists $b_0 > 0$ such that

$$f(\sigma) \geqslant -b_0 \text{ for all } \sigma \geqslant 0. \tag{5.8}$$

Let $\underline{u} := \frac{b_0\lambda}{m}\varphi_1^2 \in H_0^s(\Omega)$. Therefore, by Proposition 2.1, for every $\phi \in H_0^s(\Omega)$, there holds

$$\begin{aligned}\mathcal{E}(\overline{u}, \phi) &= \frac{b_0\lambda}{m}\mathcal{E}(\varphi_1^2, \phi) \\ &= \frac{b_0\lambda}{m}\int\limits_{\Omega}\left\{2\lambda_1\varphi_1^2(x) - h(x)\right\}\phi(x)\mathrm{d}x\,.\end{aligned}$$

Thus, $\underline{u}$ is a weak subsolution of (5.1) if

$$\frac{b_0\lambda}{m}\int\limits_{\Omega}\left\{2\lambda_1\varphi_1^2(x) - h(x)\right\}\phi(x)\mathrm{d}x \leqslant \lambda\int\limits_{\Omega} f\left(\frac{b_0\lambda}{m}\varphi_1^2(x)\right)\phi(x)\mathrm{d}x \tag{5.9}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \leqslant \phi$ in Ω. We split the analysis into two cases: $x \in \Omega_\eta$ and $x \in \Omega\backslash\overline{\Omega}_\eta$. If $x \in \Omega_\eta$, by (5.8) and (5.6), there holds

$$\begin{aligned}&\frac{b_0\lambda}{m}\int\limits_{\Omega_\eta}\left\{2\lambda_1\varphi_1^2(x) - h(x)\right\}\phi(x)\mathrm{d}x \\ &= -\frac{b_0\lambda}{m}\int\limits_{\Omega_\eta}\left\{h(x) - 2\lambda_1\varphi_1^2(x)\right\}\phi(x)\mathrm{d}x \\ &< \lambda\int\limits_{\Omega_\eta} -b_0\phi(x)\mathrm{d}x \\ &\leqslant \lambda\int\limits_{\Omega_\eta} f\left(\frac{b_0\lambda}{m}\varphi_1^2(x)\right)\phi(x)\mathrm{d}x \\ &= \lambda\int\limits_{\Omega_\eta} f(\underline{u}(x))\phi(x)\mathrm{d}x \end{aligned} \tag{5.10}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \leqslant \phi$ in Ω. Now let $x \in \Omega \backslash \overline{\Omega}_\eta$. Then, using (5.7), it follows from the hypothesis $(F2)$ that for $\lambda >> 1$, we have

$$\frac{2b_0\lambda_1}{m} \leqslant f\left(\frac{b_0\lambda}{m}\mu^2\right).$$

Therefore, using the fact that $h(x) > \gamma > 0$, it follows that

$$\begin{aligned}
&\frac{b_0\lambda}{m}\int\limits_{\Omega\backslash\overline{\Omega}_\eta}\left\{2\lambda_1\varphi_1^2(x) - h(x)\right\}\phi(x)\mathrm{d}x \\
&\leqslant \frac{2b_0\lambda\lambda_1}{m}\int\limits_{\Omega\backslash\overline{\Omega}_\eta}\varphi_1^2(x)\phi(x)\mathrm{d}x \\
&\leqslant \lambda\int\limits_{\Omega\backslash\overline{\Omega}_\eta} f\left(\frac{b_0\lambda}{m}\varphi_1^2(x)\right)\phi(x)\mathrm{d}x \\
&= \lambda\int\limits_{\Omega\backslash\overline{\Omega}_\eta} f(\underline{u}(x))\phi(x)\mathrm{d}x
\end{aligned} \tag{5.11}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \leqslant \phi$ in Ω. Combining (5.10) and (5.11), it follows that (5.9) holds. Therefore, $\underline{u} = \frac{b_0\lambda}{m}\varphi_1^2$ is a positive weak subsolution of (5.1).

We show that there exists $M_\lambda > 0$ such that $\overline{u} := Me$ is a weak supersolution of (5.1) for all $M \geqslant M_\lambda$, where $0 < e \in H_0^s(\Omega)$ is the weak solution of (2.9). We observe that while f is not assumed to be nondecreasing, $\overline{f}(t) := \max\limits_{\sigma\in[0,t]} f(\sigma)$ is nondecreasing. Moreover, $f(t) \leqslant \overline{f}(t)$ for all $t \geqslant 0$, and $\overline{f}$ satisfies the sublinear condition at infinity

$$\lim_{t\to+\infty}\frac{\overline{f}(t)}{t} = 0\,. \tag{5.12}$$

Therefore, since $\overline{f}$ satisfies (5.12), there exists $M_\lambda > 0$ sufficiently large such that for all $M \geqslant M_\lambda$

$$\frac{\overline{f}(M\|e\|_\infty)}{M\|e\|_\infty} \leqslant \frac{1}{\lambda\|e\|_\infty} \text{ or equivalently } \lambda\overline{f}(M\|e\|_\infty) \leqslant M\,.$$

Therefore, with $M \geqslant M_\lambda$, we get $\overline{u} = Me \in H_0^s(\Omega)$ satisfies

$$\begin{aligned}
\mathcal{E}(\overline{u}, \phi) &= M\mathcal{E}(e, \phi) \\
&= M\int_\Omega \phi(x)\mathrm{d}x \\
&\geqslant \lambda\int_\Omega \overline{f}(M\|e\|_\infty)\phi(x)\mathrm{d}x \\
&\geqslant \lambda\int_\Omega \overline{f}(Me(x))\phi(x)\mathrm{d}x \\
&\geqslant \lambda\int_\Omega f(Me(x))\phi(x)\mathrm{d}x \\
&= \lambda\int_\Omega f(\overline{u})\phi(x)\mathrm{d}x\,,
\end{aligned}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \leqslant \phi$ in Ω. Hence, $\overline{u} := Me$ is a weak supersolution of (5.1) for $M \geqslant M_\lambda$.

Finally, using the right estimate in (2.12), the left estimate in (2.10), and taking M larger, if necessary, we get

$$\underline{u} = \frac{b_0\lambda}{m}\varphi_1^2 \leqslant Me = \overline{u} \text{ a.e. in } \Omega\,.$$

Hence, by Theorem 3.1, (5.1) has a positive weak solution u such that $\underline{u} \leqslant u \leqslant \overline{u}$ a.e. in Ω for λ sufficiently large. This completes the proof of Theorem 5.3. ■

5.3. Proof of Theorem 5.4

Let $\lambda_1 > 0$ be the principal eigenvalue of (2.11) and $0 < \varphi_1 \in H_0^s(\Omega)$ its corresponding eigenfunction. As in the proof of Theorem 5.3, a suitable positive constant multiple of φ_1^2 serves as a positive weak subsolution of (5.1). Recall

$$(-\Delta)^s \varphi_1^2(x) = 2\lambda_1 \varphi_1^2(x) - \, h(x)\,,$$

where

$$h(x) := \int\limits_{\mathbb{R}^N} \frac{[\varphi_1(x) - \varphi_1(y)]^2}{|x-y|^{N+2s}} \mathrm{d}y$$

satisfies $0 < \gamma < h(x) < +\infty$ in Ω (by Proposition 5.1). Since f is continuous on $[0, \infty)$ and satisfies $(F3)$, there exist $\sigma_0, b_1 > 0$ and $m_1 > 2m_\infty$ such that

$$f(\sigma) \geqslant m_1\sigma - b_1 \quad \text{for all} \quad 0 \leqslant \sigma \leqslant \sigma_0. \tag{5.13}$$

Let $\underline{u} = k_0\varphi_1^2$, where k_0 satisfies

$$k_0 > \frac{\lambda_1 b_1}{m_\infty \gamma}\,. \tag{5.14}$$

Then, $\underline{u}$ satisfies

$$(-\Delta)^s k_0\varphi_1^2(x) = 2\lambda_1 k_0 \varphi_1^2(x) - k_0\, h(x)\,. \tag{5.15}$$

Therefore, by using (5.15) and Proposition 2.1, it follows that for all $\phi \in H_0^s(\Omega)$ $\underline{u}$ satisfies

$$\begin{aligned}\mathcal{E}(\underline{u}, \phi) &= k_0 \mathcal{E}(\varphi_1^2, \phi) \\ &= \int_\Omega \left\{2\lambda_1 \, k_0 \, \varphi_1^2(x) - k_0 \, h(x)\right\} \phi(x) \mathrm{d}x\end{aligned}$$

Setting $\sigma_0 := k_0 \|\varphi_1^2\|_\infty$, it follows from (5.13) that $\underline{u} = k_0 \varphi_1^2$ is a weak subsolution if

$$\int_\Omega \{2\lambda_1 \, k_0 \, \varphi_1^2(x) - k_0 \, h(x)\} \phi(x) \mathrm{d}x \leqslant \lambda \int_\Omega \left\{m_1 k_0 \varphi_1^2(x) - b_1\right\} \phi(x) \mathrm{d}x \tag{5.16}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \leqslant \phi$ in Ω. If $\lambda \geqslant \frac{2\lambda_1}{m_1} := \underline{\lambda}$, then

$$2\lambda_1 k_0 \varphi_1^2(x) \leqslant \lambda m_1 k_0 \varphi_1^2(x) \quad \text{ for a.e. } \quad x \in \Omega\,. \tag{5.17}$$

On the other hand, for $\lambda < \frac{\lambda_1}{m_\infty}$, it follows from the choice of k_0 in (5.14) that

$$\lambda b_1 < \frac{\lambda_1 b_1}{m_\infty} < k_0 \gamma < k_0 h(x) \quad \text{ for a.e. } \quad x \in \Omega\,. \tag{5.18}$$

Clearly $\frac{2\lambda_1}{m_1} < \frac{\lambda_1}{m_\infty}$ since $m_1 > 2m_\infty$. Then, it follows from (5.17) and (5.18) that inequality (5.16) holds for $\lambda \in [\frac{2\lambda_1}{m_1}, \frac{\lambda_1}{m_\infty})$. Hence $\underline{u} = k_0 \varphi_1^2$ is a positive weak subsolution for $\lambda \in [\frac{2\lambda_1}{m_1}, \frac{\lambda_1}{m_\infty})$.

We construct a supersolution for $\lambda < \frac{\lambda_1}{m_\infty}$. Let $\epsilon > 0$ be such that $\lambda_1 > \lambda(m_\infty + \epsilon)$. Since f is continuous on $[0, +\infty)$ and satisfies (5.13), there exists $L > 0$ such that

$$f(\sigma) \leqslant (m_\infty + \epsilon)\sigma + L \text{ for all } \sigma \geqslant 0. \tag{5.19}$$

Since e and φ_1 satisfy the estimates (2.10) and (2.12), respectively, there exists $c > 0$ such that $e \leqslant c\varphi_1$ in Ω. Let $\overline{u} := M\varphi_1 + \lambda Le$, where $M \geqslant M_\lambda := \frac{\lambda^2 cL(m_\infty+\epsilon)}{\lambda_1-\lambda(m_\infty+\epsilon)}$. Then,

$$\begin{aligned}\mathcal{E}(\overline{u}, \phi) &= M\mathcal{E}(\varphi_1, \phi) + \lambda L\mathcal{E}(e, \phi) \\ &= \int_\Omega [M\lambda_1\varphi_1(x) + \lambda L]\,\phi(x)\mathrm{d}x \end{aligned} \tag{5.20}$$

for all $\phi \in H_0^s(\Omega)$. Then by (5.19) and the choices of M and c,

$$\begin{aligned}\lambda \int_\Omega f(\overline{u}(x))\phi(x)\mathrm{d}x &\leqslant \lambda \int_\Omega [L + (m_\infty + \epsilon)\overline{u}(x)]\phi(x)\mathrm{d}x \\ &= \lambda \int_\Omega [L + (m_\infty + \epsilon)(M\varphi_1(x) + \lambda Le(x))]\,\phi(x)\mathrm{d}x \\ &\leqslant \lambda \int_\Omega [L + (m_\infty + \epsilon)(M\varphi_1(x) + \lambda Lc\varphi_1(x))]\,\phi(x)\mathrm{d}x \\ &= \int_\Omega \left[\lambda L + M\lambda(m_\infty + \epsilon) + \lambda^2 Lc(m_\infty + \epsilon)\right]\varphi_1(x)\phi(x)\mathrm{d}x \\ &\leqslant \int_\Omega [\lambda L + M\lambda_1\varphi_1(x)]\phi(x)\mathrm{d}x \end{aligned} \tag{5.21}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \leqslant \phi$ in Ω. Hence, $\overline{u}$ is a weak supersolution for $\lambda < \frac{\lambda_1}{m_\infty}$. Finally, using the right estimate in (2.12), the left estimate in (2.10), and taking M larger, if necessary, we get

$$\underline{u} = k_0\varphi_1^2 \leqslant M\varphi_1 + Le = \overline{u} \quad \text{a.e. in} \quad \Omega\,.$$

Hence, by Theorem 3.1, (5.1) has a positive weak solution u such that $\underline{u} \leqslant u \leqslant \overline{u}$ a.e. in Ω for $\lambda \in [\frac{2\lambda_1}{m_1}, \frac{\lambda_1}{m_\infty})$. This completes the proof of Theorem 5.4. ■

5.4. Proof of Theorem 5.5

First, we construct a positive weak subsolution of (5.2). Let $\lambda > \lambda_{1,q}$ be fixed. Then, by Proposition 2.3, there exists $l \in \mathbb{N}$ such that $\lambda_{1,q} \leqslant \lambda_{1,\gamma_l} < \lambda$, where λ_{1,γ_l} is the principal eigenvalue of (2.18) with $\gamma_l(x)$ defined by (2.19). Let $\varphi_{1,\gamma_l} \in H_0^s(\Omega)$ be the positive eigenfunction corresponding to λ_{1,γ_l}, and let $e \in H_0^s(\Omega)$ be the positive weak solution of (2.9).

We show there exist $m_\lambda > 0$ and $\varepsilon > 0$ such that for all $m \in (0, m_\lambda)$, $\underline{u} := m\,(\varphi_{1,\gamma_l} - \varepsilon\, e) \in H_0^s(\Omega)$ is a positive weak subsolution of (5.2). Set $\alpha := \sqrt{\frac{\lambda_{1,\gamma_l}}{\lambda}} \in (0,1)$. Then, with $q = \gamma_l$, we see that φ_{1,γ_l} satisfies (2.16), and e satisfies (2.10). Hence, there exists $\varepsilon > 0$ such that

$$\varphi_{1,\gamma_l} - \varepsilon\, e > \alpha\, \varphi_{1,\gamma_l} > 0 \quad \text{a.e. in } \Omega\,. \tag{5.22}$$

Next, define $m_\lambda := \min\left\{ \dfrac{\varepsilon}{\lambda\alpha\|\varphi_{1,\gamma_l}(\varphi_{1,\gamma_l} - \varepsilon\, e)\|_\infty}, \quad \dfrac{1-\alpha}{l\|\varphi_{1,\gamma_l} - \varepsilon e\|_\infty} \right\}$, and let $m \in (0, m_\lambda)$.

Using the weak formulation of e and φ_{1,γ_l}, we see that $\underline{u} = m\,(\varphi_{1,\gamma_l} - \varepsilon\, e) \in H_0^s(\Omega)$ satisfies

$$\mathcal{E}(\underline{u}, \phi) = m \int_\Omega (\lambda_{1,\gamma_l}\, \gamma_l(x)\, \varphi_{1,\gamma_l}(x) - \varepsilon)\, \phi(x)\mathrm{d}x$$

for all $\phi \in H_0^s(\Omega)$. Therefore, $\underline{u}$ is a weak subsolution of (5.2) if

$$\begin{aligned} m &\int_\Omega (\lambda_{1,\gamma_l}\, \gamma_l(x)\, \varphi_{1,\gamma_l}(x) - \varepsilon)\, \phi(x)\mathrm{d}x \\ &\leqslant \lambda\, m \int_\Omega (\varphi_{1,\gamma_l}(x) - \varepsilon\, e(x))\big[q(x) - m(\varphi_{1,\gamma_l}(x) - \varepsilon\, e(x))\big]\phi(x)\mathrm{d}x \end{aligned} \tag{5.23}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \leqslant \phi$ in Ω. The definition of γ_l and (5.22) yields

$$\begin{aligned}&\lambda\, m\, \alpha\, \varphi_{1,\gamma_l}(x)\big[\gamma_l(x) - m\,(\varphi_{1,\gamma_l}(x) - \varepsilon\, e(x))\big]\\ &\leqslant \lambda\, m\,(\varphi_{1,\gamma_l}(x) - \varepsilon\, e(x))\big[q(x) - m\,(\varphi_{1,\gamma_l}(x) - \varepsilon e(x))\big] \text{ a.e. in } \Omega\,.\end{aligned}$$

Therefore, (5.23) holds if

$$\lambda_{1,\gamma_l}\, \gamma_l(x)\varphi_{1,\gamma_l}(x) - \varepsilon \leqslant \lambda\, \alpha\, \varphi_{1,\gamma_l}\big[\gamma_l(x) - m\,(\varphi_{1,\gamma_l}(x) - \varepsilon\, e(x))\big] \text{ a.e. in } \Omega\,. \tag{5.24}$$

Define $\Omega_l := \{x \in \Omega : q(x) < 1/l\}$. If $x \in \Omega_l$, then $\gamma_l = 0$. In this case, (5.24) holds since

$$m < m_\lambda \leqslant \frac{\varepsilon}{\lambda\alpha\|\varphi_{1,\gamma_l}(\varphi_{1,\gamma_l} - \varepsilon\, e)\|_\infty}\,.$$

If $x \in \Omega\backslash\Omega_l$, then $\gamma_l(x) = q(x) \geqslant 1/l$. In this case, (5.24) is satisfied since the inequality

$$\lambda_{1,\gamma_l}\, \gamma_l(x)\varphi_{1,\gamma_l} \leqslant \lambda\, \alpha\, \varphi_{1,\gamma_l}\big[\gamma_l(x) - m\,(\varphi_{1,\gamma_l} - \varepsilon\, e)\big]$$

holds by choosing

$$m < m_\lambda \leqslant \frac{1-\alpha}{l\|\varphi_{1,\gamma_l} - \varepsilon e\|_\infty}\,.$$

Hence, $\underline{u} = m\,(\varphi_{1,\gamma_l}(x) - \varepsilon\, e(x))$ is a positive weak subsolution of (5.2) for any $m \in (0, m_\lambda)$.

Next, we construct a positive weak supersolution. We show a constant multiple of e is a supersolution for (5.2), where $0 < e \in H_0^s(\Omega)$ is the solution of (2.9). Since $q \in L^\infty(\Omega)$ with $0 \leqslant q(x) \leqslant 1$ a.e. in Ω, $Me(x)(1 - Me(x)) \geqslant Me(x)\big(q(x) - Me(x)\big)$ holds a.e. in Ω.

Then, noting that $\max_{\sigma \geqslant 0} \sigma(1-\sigma) = \frac{1}{4}$, define $\overline{u} := Me$ for $M \geqslant M_\lambda := \frac{\lambda}{4}$. Thus, $\overline{u}$ satisfies

$$\begin{aligned}
\mathcal{E}(\overline{u}, \phi) &= M\mathcal{E}(e, \phi) \\
&= \int_\Omega M\phi(x)\mathrm{d}x \\
&\geqslant \frac{\lambda}{4} \int_\Omega \phi(x)\mathrm{d}x \\
&\geqslant \lambda \int_\Omega \overline{u}(x)(q(x) - \overline{u}(x))\phi(x)\mathrm{d}x
\end{aligned}$$

for all $\phi \in H_0^s(\Omega)$ with $0 \leqslant \phi$ in Ω. Therefore, $\overline{u} = Me$ is a positive weak supersolution for all $M \geqslant M_\lambda$.

Finally, using the left estimate of (2.10), and the right estimate of (2.16), we can choose either $M \geqslant M_\lambda$ sufficiently large or $0 < m < m_\lambda$ sufficiently small, so that $\underline{u} = m\,(\varphi_{1,\gamma_l} - \varepsilon\, e) \leqslant m\,\varphi_{1,\gamma_l} \leqslant Me = \overline{u}$ a.e. in Ω. Hence, by Theorem 3.1, (5.2) has a positive weak solution u satisfying $\underline{u} \leqslant u \leqslant \overline{u}$ a.e. in Ω for any $\lambda > \lambda_{1,q}$. This completes the proof of Theorem 5.5. ■

5.5. Proof of Theorem 5.6

First, we construct a positive weak subsolution for (5.3). Let $\lambda > \lambda_1$ be fixed, and define $\beta := \sqrt{\frac{\lambda_1}{\lambda}} \in (0,1)$. Then, it follows from (2.10) and (2.12) that there exists $\varepsilon > 0$ such that $\varphi_1 - \varepsilon e > \beta\varphi_1 > 0$. Define $\underline{u} := m^*(\varphi_1 - \varepsilon e) \in H_0^s(\Omega)$ with fixed $m^* := \frac{1-\beta}{2\|\varphi_1 - \varepsilon e\|_\infty} > 0$.

We show that $\underline{u}$ is a positive weak subsolution of (5.3) for any $0 < a < a^* := \frac{\epsilon m^*}{\lambda}$. Since $\underline{u}$ satisfies

$$\mathcal{E}(\underline{u}, \phi) = m^* \int_\Omega (\lambda_1 \varphi_1(x) - \varepsilon)\, \phi(x) \mathrm{d}x$$

for all $\phi \in H_0^s(\Omega)$, it is a weak subsolution of (5.3) if

$$m^* (\lambda_1 \varphi_1(x) - \varepsilon) \leqslant \lambda\, m^* (\varphi_1(x) - \varepsilon\, e(x)) \big[1 - m^*(\varphi_1(x) - \varepsilon\, e(x))\big] - \lambda a\,. \tag{5.25}$$

Since $\varphi_1 - \varepsilon e > \beta\varphi_1 > 0$, (5.25) is satisfied if

$$\lambda_1 \varphi_1(x) \leqslant \lambda\ \beta\varphi_1(x))\big[1 - m^*(\varphi_1(x) - \varepsilon\, e(x))\big] + \varepsilon m^* - \lambda a\,. \tag{5.26}$$

Note, $\varepsilon m^* - \lambda a \geqslant 0$ since $\varepsilon m^* - \lambda a^* = 0$ by the choice of a^*, and $a < a^*$. Then, using $\varphi_1 > \varphi_1 - \varepsilon e > 0$ a.e. in Ω and $m^* = \frac{1-\beta}{2\|\varphi_1 - \varepsilon\, e\|_\infty}$, (5.26) follows from the inequality

$$\lambda_1 \varphi_1(x)\ \leqslant \lambda\, \beta\varphi_1(x) \big[1 - m^* \|\varphi_1 - \varepsilon\, e\|_\infty\big]\,.$$

Hence $\underline{u} = m^*(\varphi_1 - \varepsilon e)$ is a subsolution of (5.3) for $a < a^*$.

As in the proof of Theorem 5.5, $\overline{u} = M\, e \in H_0^s(\Omega)$ is a supersolution of (5.3) for any $M \geqslant M_\lambda = \frac{\lambda}{4}$ since $\frac{1}{4} \geqslant M\, e(x)\big(1 - M\, e(x)\big) - a$ for a.e. $x \in \Omega\,.$

Using the estimates (2.10) and (2.16), we can further refine the choice of $M \geqslant M_\lambda$ to be sufficiently large such that $\underline{u} = m^*(\varphi_1 - \varepsilon e) \leqslant M\, e = \overline{u}$ a.e. in Ω. Therefore, by Theorem 3.1, for any $\lambda > \lambda_1$, (5.3) has a positive weak solution u satisfying $m^* (\varphi_1 - \varepsilon\, e) \leqslant u \leqslant Me$ a.e. in Ω for $0 < a < a^*$. This completes the proof of Theorem 5.6. ■

5.6. Proof of Theorem 5.7

Let $u \in H_0^s(\Omega)$ be a nonnegative weak solution of (5.1) and $\varphi_1 \in H_0^s(\Omega)$ be the principal eigenfunction of (2.11). Taking $0 < \varphi_1$ as a test function in (2.5), we get

$$\lambda_1 \int_\Omega u\varphi_1 \mathrm{d}x = \mathcal{E}(u, \varphi_1) = \lambda \int_\Omega f(u)\varphi_1 \mathrm{d}x\,. \tag{5.27}$$

In case (i), since $f(\sigma) \leqslant a_1\sigma - b_1$ for all $\sigma \geqslant 0$, (5.27) yields

$$\lambda_1 \int_\Omega u\varphi_1 \mathrm{d}x \leqslant \lambda \int_\Omega (a_1 u - b_1)\varphi_1 \mathrm{d}x \leqslant \lambda a_1 \int_\Omega u\varphi_1 \mathrm{d}x\,,$$

a contradiction if $\lambda < \frac{\lambda_1}{a_1}$.

In case (ii), since $f(\sigma) \geqslant a_2\sigma + b_2$ for all $\sigma \geqslant 0$, (5.27) yields

$$\lambda_1 \int_\Omega u\varphi_1 \mathrm{d}x \geqslant \lambda \int_\Omega (a_2 u + b_2)\varphi_1 \mathrm{d}x \geqslant \lambda a_2 \int_\Omega u\varphi_1 \mathrm{d}x\,,$$

a contradiction if $\lambda > \frac{\lambda_1}{a_2}$. This completes the proof. ■

Remark 5.1. The simple approach in the proof of Theorem 5.7 above does not apply to the case $f(\sigma) \geqslant a\sigma - b$ for all $\sigma \geqslant 0$ with a and b positive. This case appears to be more challenging as it was in the Laplacian case (see [ANZ92]).

CHAPTER VI

RADIAL SYMMETRY

6.1. Introduction and Statement of Result

In this chapter, we study nonnegative classical solutions of a fractional Laplacian problem of the form

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } B_1\,; \\ u = 0 & \text{in } \mathbb{R}^N \backslash B_1\,, \end{cases} \tag{6.1}$$

where $B_1 := \{x \in \mathbb{R}^N : |x| < 1\}$ with $N \geqslant 2$ and the nonlinearity $f : [0, \infty) \to \mathbb{R}$ satisfies

$(G1)$ $\quad f$ is locally Lipschitz,

$(G2)$ $\quad f(0) < 0.$

The following symmetry result was established for positive solutions of (6.1).

Proposition 6.1. *([FW14, Thm. 1.1]) If f satisfies $(G1)$ and $u \in C(\mathbb{R}^N)$ is a positive classical solution to (6.1), then u must be radially symmetric and radially decreasing in $r = |x| \in (0, 1)$.*

The goal of this chapter is to extend the above symmetry result to nonnegative nontrivial classical solutions of (6.1). If $f \geqslant 0$, then by Proposition 2.6 any nonnegative nontrivial solution u of (6.1) is positive in B_1, hence Proposition 6.1 applies. However, if f satisfies $(G2)$, then Proposition 2.6 does not apply.

Our goal is to show that if f satisfies $(G2)$, then any nonnegative nontrivial solution of (6.1) is positive and hence Proposition 6.1 applies.

For the Laplacian case, a result analogous to Proposition 6.1 is contained in the celebrated paper of Gidas-Ni-Nirenberg [GNN79, Theorem 1]. Using an example in one dimension, it was conjectured in [GNN79] that if $f(0) < 0$, then $u > 0$ in B_1 cannot be replaced by $u \geqslant 0$ with $u \not\equiv 0$ in [GNN79, Theorem 1]. In [CS89], the authors made a key observation that the boundary of a ball B_1 is not connected in one dimension but connected when $N \geqslant 2$. Using this information, they proved that the conjecture is false by showing that if the nonlinearity f is smooth and satisfies $f(0) < 0$, then every nonnegative, nontrivial solution is positive, hence radially symmetric and radially decreasing (by [GNN79, Theorem 1]).

By combining the ideas from [FW14] and [CS89], we prove:

Theorem 6.1. *If f satisfies $(G1) - (G2)$ and $u \in C(\mathbb{R}^N)$ is a nonnegative nontrivial classical solution to (6.1), then u is positive in B_1 and hence radially symmetric and radially decreasing in B_1.*

To prove Theorem 6.1, we use the moving plane method developed for the fractional Laplacian in [FW14]. The main tools are the maximum principle in a ball Proposition 2.6, and the maximum principle for thin domains (see Proposition 2.7). In the Laplacian case, the smoothness of solutions played a crucial role in the analysis which is not available in the fractional Laplacian case.

6.2. Proof of Theorem 6.1

We begin with notations and terminologies that will be used throughout this chapter. Let $x = (x_1, x') \in \mathbb{R}^N$ and $\lambda \in (0,1)$. Then, define

$$\Sigma_\lambda := \{x = (x_1, x') \in B_1 \ : \ x_1 > \lambda\},$$

$$A_\lambda := \{x = (x_1, x') \in \mathbb{R}^N \ : \ x_1 > \lambda\},$$

$$T_\lambda := \{x = (x_1, x') \in B_1 \ : \ x_1 = \lambda\}.$$

For $x \in \mathbb{R}^N$, let $x_\lambda = (2\lambda - x_1, x')$ denote the reflection of x across the hyperplane T_λ. Define $u_\lambda(x) := u(x_\lambda)$ and $w_\lambda(x) := u_\lambda(x) - u(x)$. Then $u_\lambda(x_\lambda) = u(x)$, and w_λ is an antisymmetric function; that is,

$$w_\lambda(x_\lambda) = u_\lambda(x_\lambda) - u(x_\lambda) = -(u_\lambda(x) - u(x)) = -w_\lambda(x) \quad \text{for all} \quad x \in \mathbb{R}^N. \tag{6.2}$$

We establish several lemmas to prove our result.

The following lemma was proved in [FW14] for positive solutions of (6.1). We observe that the result holds for nonnegative nontrivial solutions. We give the proof below for completeness.

Lemma 6.1. *([FW14, Step 1]) If $u \in C(\mathbb{R}^N)$ is a nonnegative nontrivial classical solution of* (6.1) *and $\lambda \in (0,1)$ with $\lambda \approx 1$, then $w_\lambda \geqslant 0$ in Σ_λ.*

Proof of Lemma 6.1. Let $\Sigma_\lambda^- := \{x \in \Sigma_\lambda : w_\lambda(x) < 0\}$, and suppose to the contrary that $\Sigma_\lambda^- \neq \varnothing$ for $\lambda \in (0,1)$ with $\lambda \approx 1$. Define

$$w_\lambda^+(x) := \begin{cases} w_\lambda(x); & x \in \Sigma_\lambda^-, \\ 0; & \mathbb{R}^N \backslash \Sigma_\lambda^-, \end{cases} \qquad w_\lambda^-(x) := \begin{cases} 0; & x \in \Sigma_\lambda^-, \\ w_\lambda(x); & \mathbb{R}^N \backslash \Sigma_\lambda^-. \end{cases}$$

Then, we have that $w_\lambda^+(x) = w_\lambda(x) - w_\lambda^-(x)$ for all $x \in \mathbb{R}^N$. The plan is to first show that

$$(-\Delta)^s w_\lambda^- \leqslant 0 \text{ for all } x \in \Sigma_\lambda^- \tag{6.3}$$

for all $\lambda \in (0,1)$, and then apply Proposition 2.7 in Σ_λ^- for $\lambda \approx 1$. Let $x \in \Sigma_\lambda^-$. Using $w_\lambda^-(x) = 0$, we get

$$(-\Delta)^s w_\lambda^-(x) = \int\limits_{\mathbb{R}^N} \frac{w_\lambda^-(x) - w_\lambda^-(z)}{|x-z|^{N+2s}}\,\mathrm{d}z = -\int\limits_{\mathbb{R}^N \setminus \Sigma_\lambda^-} \frac{w_\lambda^-(z)}{|x-z|^{N+2s}}\,\mathrm{d}z\,.$$

Since $u = 0$ in $\mathbb{R}^N \setminus B_1$, $w_\lambda(x) = 0$ in $\mathbb{R}^N \setminus (B_1 \cup (B_1)_\lambda)$ which gives

$$(-\Delta)^s w_\lambda^-(x) = -\int\limits_{(B_1 \cup (B_1)_\lambda)\setminus \Sigma_\lambda^-} \frac{w_\lambda^-(z)}{|x-z|^{N+2s}}\,\mathrm{d}z\,.$$

To analyze the integral on the right hand side above, we partition $(B_1 \cup (B_1)_\lambda)\setminus\Sigma_\lambda^-$ into the disjoint union $S_1 \cup S_2 \cup S_3$ (see Figure 6.1), where

$$S_1 := (B_1 \setminus (B_1)_\lambda) \cup ((B_1)_\lambda \setminus B_1), \quad S_2 := (\Sigma_\lambda \setminus \Sigma_\lambda^-) \cup (\Sigma_\lambda \setminus \Sigma_\lambda^-)_\lambda, \quad S_3 := (\Sigma_\lambda^-)_\lambda.$$

Then,

$$\begin{aligned}(-\Delta)^s w_\lambda^-(x) &= -\int\limits_{S_1} \frac{w_\lambda(z)}{|x-z|^{N+2s}}\,\mathrm{d}z - \int\limits_{S_2} \frac{w_\lambda(z)}{|x-z|^{N+2s}}\,\mathrm{d}z - \int\limits_{S_3} \frac{w_\lambda(z)}{|x-z|^{N+2s}}\,\mathrm{d}z \\ &= -I_1 - I_2 - I_3. \end{aligned} \tag{6.4}$$

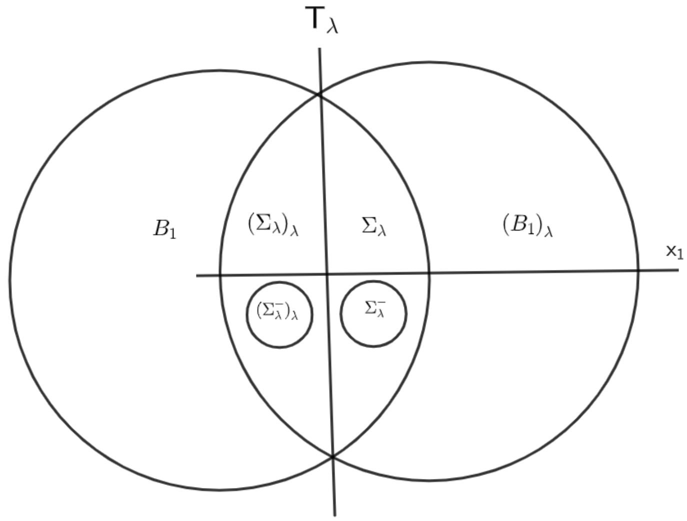


Figure 6.1. Disjoint Sets S_1, S_2, S_3

To establish (6.3), it suffices to show that $I_1, I_2, I_3 \geqslant 0$. First, we consider the integral I_1. Observe that $u = 0$ in $(B_1)_\lambda \backslash B_1$ since $u = 0$ in $\mathbb{R}^N \backslash B_1$. Further, $z \in B_1 \backslash (B_1)_\lambda$ implies $z_\lambda \in \mathbb{R}^N \backslash B_1$, and hence $u_\lambda(z) = u(z_\lambda) = 0$ in $B_1 \backslash (B_1)_\lambda$. Therefore,

$$
\begin{aligned}
I_1 &= \int\limits_{S_1} \frac{w_\lambda(z)}{|x-z|^{N+2s}} \,\mathrm{d}z \\
&= \int\limits_{(B_1)_\lambda \backslash B_1} \frac{u_\lambda(z) - u(z)}{|x-z|^{N+2s}} \,\mathrm{d}z + \int\limits_{B_1 \backslash (B_1)_\lambda} \frac{u_\lambda(z) - u(z)}{|x-z|^{N+2s}} \,\mathrm{d}z \\
&= \int\limits_{(B_1)_\lambda \backslash B_1} \frac{u_\lambda(z)}{|x-z|^{N+2s}} \,\mathrm{d}z - \int\limits_{B_1 \backslash (B_1)_\lambda} \frac{u(z)}{|x-z|^{N+2s}} \,\mathrm{d}z \\
&= \int\limits_{(B_1)_\lambda \backslash B_1} \frac{u_\lambda(z)}{|x-z|^{N+2s}} \,\mathrm{d}z - \int\limits_{(B_1)_\lambda \backslash B_1} \frac{u(z_\lambda)}{|x-z_\lambda|^{N+2s}} \,\mathrm{d}z \qquad (6.5) \\
&= \int\limits_{(B_1)_\lambda \backslash B_1} u_\lambda(z) \left[\frac{1}{|x-z|^{N+2s}} - \frac{1}{|x-z_\lambda|^{N+2s}} \right] \mathrm{d}z \\
&\geqslant 0\,. \qquad (6.6)
\end{aligned}
$$

The equality (6.5) holds by relabelling since, by symmetry, integrating $u(z)$ over $(B_1)\backslash(B_1)_\lambda$ with respect to z is the same as integrating $u(z_\lambda)$ over $(B_1)_\lambda\backslash(B_1)$ with respect to z. Inequality (6.6) follows since $|x-z_\lambda| > |x-z|$ as x and z fall on the same side of T_λ for $z \in (B_1)_\lambda\backslash B_1$, and $u_\lambda \geqslant 0$ with $u_\lambda \not\equiv 0$.

Now we consider I_2. In this case, we have

$$
\begin{aligned}
I_2 &= \int\limits_{S_2} \frac{w_\lambda(z)}{|x-z|^{N+2s}}\,\mathrm{d}z \\
&= \int\limits_{\Sigma_\lambda\backslash\Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z|^{N+2s}}\,\mathrm{d}z + \int\limits_{(\Sigma_\lambda\backslash\Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2s}}\,\mathrm{d}z \\
&= \int\limits_{\Sigma_\lambda\backslash\Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z|^{N+2s}}\,\mathrm{d}z + \int\limits_{\Sigma_\lambda\backslash\Sigma_\lambda^-} \frac{w_\lambda(z_\lambda)}{|x-z_\lambda|^{N+2s}}\,\mathrm{d}z \qquad (6.7)\\
&= \int\limits_{\Sigma_\lambda\backslash\Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z|^{N+2s}}\,\mathrm{d}z - \int\limits_{\Sigma_\lambda\backslash\Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z_\lambda|^{N+2s}}\,\mathrm{d}z \qquad (6.8)\\
&= \int\limits_{\Sigma_\lambda\backslash\Sigma_\lambda^-} w_\lambda(z)\left[\frac{1}{|x-z|^{N+2s}} - \frac{1}{|x-z_\lambda|^{N+2s}}\right]\mathrm{d}z \\
&\geqslant 0\,. \qquad (6.9)
\end{aligned}
$$

As in the computation of I_1, (6.7) holds by symmetry. The antisymmetric property (6.2) of w_λ yields (6.8). Finally, since $x \in \Sigma_\lambda^-$ and $z \in \Sigma_\lambda\backslash\Sigma_\lambda^-$, x and z fall on the same side of T_λ and thus $|x-z_\lambda| > |x-z|$. Then, using the fact that $w_\lambda \geqslant 0$ in $\Sigma_\lambda\backslash\Sigma_\lambda^-$, inequality (6.9) follows.

Finally, we consider I_3. In this case, using the symmetry of the integral and the antisymmetric property of $w_\lambda(z)$ (as used in I_2) and $w_\lambda < 0$ in Σ_λ^-, we get

$$I_3 = \int\limits_{S_3} \frac{w_\lambda(z)}{|x-z|^{N+2s}}\,\mathrm{d}z = \int\limits_{(\Sigma_\lambda^-)_\lambda} \frac{w_\lambda(z)}{|x-z|^{N+2s}}\,\mathrm{d}z = -\int\limits_{\Sigma_\lambda^-} \frac{w_\lambda(z)}{|x-z_\lambda|^{N+2s}}\,\mathrm{d}z > 0\,. \tag{6.10}$$

Then, using (6.6), (6.9), and (6.10) in (6.4), it follows that $(-\Delta)^s w_\lambda^-(x) \leqslant 0$ for all $x \in \Sigma_\lambda^-$, as claimed. Therefore, using (6.3), we get

$$\begin{aligned}
(-\Delta)^s w_\lambda^+(x) &= (-\Delta)^s w_\lambda(x) - (-\Delta)^s w_\lambda^-(x) \\
&\geqslant (-\Delta)^s w_\lambda(x) \\
&= (-\Delta)^s u_\lambda(x) - (-\Delta)^s u(x) \\
&= f(u_\lambda(x)) - f(u(x)) \\
&= \frac{f(u_\lambda(x)) - f(u(x))}{u_\lambda(x) - u(x)} w_\lambda^+(x) \quad \text{for} \quad x \in \Sigma_\lambda^-\,.
\end{aligned}$$

The last expression is well-defined since $x \in \Sigma_\lambda^-$ yields $w_\lambda^+(x) = w_\lambda(x) = u_\lambda(x) - u(x) < 0$. Then, for $x \in \Sigma_\lambda^-$, w_λ^+ satisfies

$$\begin{cases} (-\Delta)^s w_\lambda^+(x) \geqslant \varphi(x) w_\lambda^+(x) & \text{in } \Sigma_\lambda^-\,; \\ \qquad w_\lambda^+(x) = 0 & \text{in } \mathbb{R}^N \backslash \Sigma_\lambda^-\,, \end{cases}$$

where $\varphi(x) := [f(u_\lambda(x)) - f(u(x))]/[u_\lambda(x) - u(x)] \in L^\infty(\Sigma_\lambda^-)$ since f is Lipschitz continuous by $(G1)$. Now, by taking $\lambda \in (0,1)$, $\lambda \approx 1$, it is guaranteed that there exists $\xi > 0$ such that $|\Sigma_\lambda^-| < \xi$. Then, it follows from Proposition 2.7 that $w_\lambda(x) = w_\lambda^+(x) \geqslant 0$ in Σ_λ^-, a contradiction to the definition of Σ_λ^-. Therefore, $\Sigma_\lambda^- = \varnothing$ and, hence, $w_\lambda \geqslant 0$ in Σ_λ for $\lambda \in (0,1)$ with $\lambda \approx 1$. ■

For positive solutions, it suffices to show $w_\lambda \geqslant 0$ and $w_\lambda \not\equiv 0$ in Σ_λ (see [FW14]) in the lemma below. The proof is contained in their paper, but we give for completeness.

Lemma 6.2. *(cf. [FW14]) Let $u \in C(\mathbb{R}^N)$ be a nonnegative nontrivial classical solution to* (6.1)*. For any $\lambda \in (0,1)$, if $w_\lambda \geqslant 0$ and $w_\lambda \not\equiv 0$ in A_λ, then $w_\lambda > 0$ in Σ_λ.*

Proof of Lemma 6.2. We prove by contradiction. Suppose there exists $x_0 \in \Sigma_\lambda$ such that $w_\lambda(x_0) = 0$, that is, $u_\lambda(x_0) = u(x_0)$. Then,

$$(-\Delta)^s w_\lambda(x_0) = (-\Delta)^s u_\lambda(x_0) - (-\Delta)^s u(x_0) = f(u_\lambda(x_0)) - f(u(x_0)) = 0\,. \tag{6.11}$$

On the other hand, using (6.2) and $w_\lambda(x_0) = 0$, by calculating $(-\Delta)^s w_\lambda(x_0)$ according to definition (1.1) we get

$$\begin{aligned}
(-\Delta)^s w_\lambda(x_0) &= \int\limits_{\mathbb{R}^N} \frac{w_\lambda(x_0) - w_\lambda(z)}{|x_0 - z|^{N+2s}}\,\mathrm{d}z \\
&= -\int\limits_{A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}}\,\mathrm{d}z - \int\limits_{\mathbb{R}^N\setminus A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}}\,\mathrm{d}z \\
&= -\int\limits_{A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}}\,\mathrm{d}z - \int\limits_{A_\lambda} \frac{w_\lambda(z_\lambda)}{|x_0 - z_\lambda|^{N+2s}}\,\mathrm{d}z \\
&= -\int\limits_{A_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}}\,\mathrm{d}z + \int\limits_{A_\lambda} \frac{w_\lambda(z)}{|x_0 - z_\lambda|^{N+2s}}\,\mathrm{d}z \\
&= -\int\limits_{A_\lambda} w_\lambda(z)\left[\frac{1}{|x_0 - z|^{N+2s}} - \frac{1}{|x_0 - z_\lambda|^{N+2s}}\right]\mathrm{d}z\,.
\end{aligned}$$

For $z \in A_\lambda$ and $x_0 \in \Sigma_\lambda \subset A_\lambda$, it follows that x_0 and z fall on the same side of the hyperplane T_λ and hence $|x_0 - z_\lambda| > |x_0 - z|$.

Then, using the fact that $w_\lambda(z) \geqslant 0$, $w_\lambda(z) \not\equiv 0$ in A_λ, and w_λ is continuous, we get $(-\Delta)^s w_\lambda(x_0) < 0$, a contradiction to (6.11). Therefore $w_\lambda > 0$ in Σ_λ. ■

Lemma 6.3. *If* $u \in C(\mathbb{R}^N)$ *is a nonnegative nontrivial classical solution to* (6.1) *and* $\lambda \in (0, 1)$ *with* $\lambda \approx 1$*, then* $w_\lambda > 0$ *in* Σ_λ*.*

Proof of Lemma 6.3. By Lemma 6.1, $w_\lambda \geqslant 0$ for $\lambda \approx 1$. Therefore, to apply Lemma 6.2, it suffices to show $w_\lambda \not\equiv 0$ in A_λ. Since $f(0) < 0$ and $u \geqslant 0$ is a solution to (6.1), it follows that $u \not\equiv 0$. Hence, there exists $x_0 \in B_1$ such that $u(x_0) > 0$. By the continuity of u, there exists $\delta_0 > 0$ such that $u > 0$ in $B_{\delta_0}(x_0) \subset B_1$. Then, either $\partial B_{\delta_0}(x_0) \cap B_1 = \varnothing$ or $\partial B_{\delta_0}(x_0) \cap B_1 \neq \varnothing$ (see Figure 6.2).

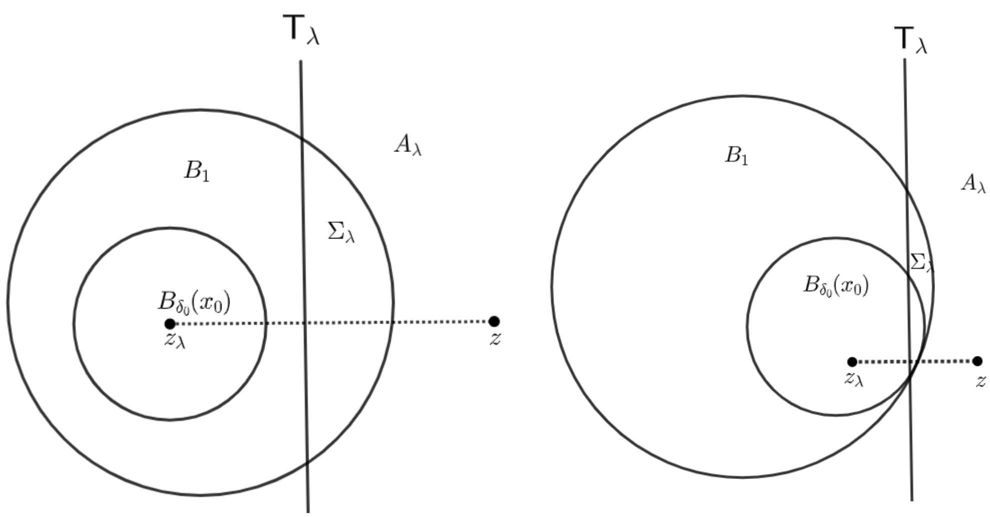


Figure 6.2. (a) $\partial B_{\delta_0}(x_0) \cap B_1 = \varnothing$ (b) $\partial B_{\delta_0}(x_0) \cap B_1 \neq \varnothing$

In either case, by taking $\lambda \approx 1$, there exists $z \in A_\lambda \backslash B_1$ such that $z_\lambda \in B_{\delta_0}(x_0)$. Then $u(z) = 0$ and $u_\lambda(z) = u(z_\lambda) > 0$, that is, $w_\lambda(z) = u(z_\lambda) - u(z) > 0$ yielding $w_\lambda \not\equiv 0$ in A_λ. Then, by Lemma 6.2, $w_\lambda > 0$ in Σ_λ for $\lambda \approx 1$. ■

Lemma 6.4. *Let $u \in C(\mathbb{R}^N)$ be a nonnegative nontrivial classical solution of* (6.1)*. If there exists $\tilde{\lambda} \in (0,1)$ such that $w_\lambda > 0$ in Σ_λ for all $\lambda \in (\tilde{\lambda}, 1)$, then $u > 0$ in $\Sigma_\lambda \cup (\Sigma_\lambda)_\lambda$ for all $\lambda \in (\tilde{\lambda}, 1)$.*

Proof of Lemma 6.4. We proceed by contradiction. Suppose $w_\lambda > 0$ in Σ_λ for all $\lambda \in (\tilde{\lambda}, 1)$ but there exists $\lambda' \in (\tilde{\lambda}, 1)$ and $x \in \Sigma_{\lambda'} \cup (\Sigma_{\lambda'})_{\lambda'}$ such that $u(x) = 0$.

If $x \in (\Sigma_{\lambda'})_{\lambda'}$, then there exists $z \in \Sigma_{\lambda'}$ such that $x = z_{\lambda'}$, the reflection of $z \in \Sigma_{\lambda'}$ about the hyperplane $T_{\lambda'}$. Then, using $u(z) \geqslant 0$, we get the contradiction

$$0 < w_{\lambda'}(z) = u(z_{\lambda'}) - u(z) = u(x) - u(z) = -u(z) \leqslant 0\,.$$

On the other hand, if $x \in \Sigma_{\lambda'}$, then there exists $\lambda'' \in (\lambda', 1)$ such that $x \in (\Sigma_{\lambda''})_{\lambda''}$ and $w_{\lambda''} > 0$ in $\Sigma_{\lambda''}$. This leads to a similar contradiction for λ'' as in the previous case for λ'. Hence, $u > 0$ in $\Sigma_\lambda \cup (\Sigma_\lambda)_\lambda$ for all $\lambda \in (\tilde{\lambda}, 1)$. ■

For the Laplacian case, the following lemma was established in [CS89].

Lemma 6.5. *There exists $\eta > 0$ such that $u > 0$ in $K_\eta := \{x \in B_1 \mid \operatorname{dist}(x, \partial B_1) < \eta\}$.*

Proof of Lemma 6.5. Let $z \in \partial B_1$. Then z defines a radial direction from the origin. Since $(-\Delta)^s$ is rotationally invariant (see [Šil20]), without loss of generality, we assume that $z = (1, 0 \ldots, 0)$. By Lemma 6.3, there exists $\lambda_z \in (0,1)$ with $\lambda_z \approx 1$ such that $w_\lambda > 0$ in Σ_λ for all $\lambda \in (\lambda_z, 1)$.

By Lemma 6.4, $u > 0$ in $B_1 \cap B_{r_z}(z) \subset \Sigma_{\lambda_z} \cup (\Sigma_{\lambda_z})_{\lambda_z}$ (see Figure 6.3), where $0 < r_z < 1 - \lambda_z$.

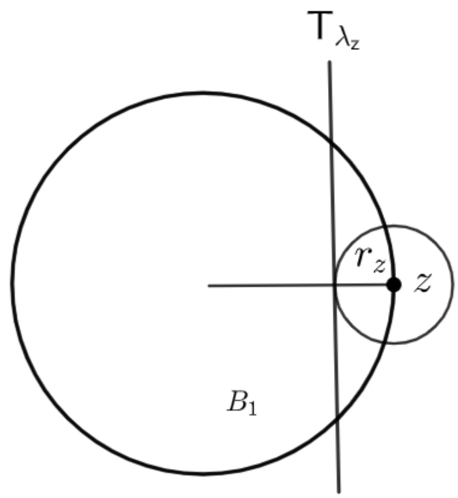


Figure 6.3. Construction of r_z

Since $\bigcup_{z \in \partial B_1} B_{r_z}(z)$ is an open cover of the compact set ∂B_1, there exist $z_1, \ldots, z_m \in \partial B_1$ and corresponding radii, $r_{z_i} > 0$, such that $W := B_{r_{z_1}}(z_1) \cup \ldots \cup B_{r_{z_m}}(z_m) \supset \partial B_1$. Since W is open and ∂B_1 is closed, there exists $\eta > 0$ such that $\{x \in \mathbb{R}^N \mid \operatorname{dist}(x, \partial B_1) < \eta\} \subset W$. Therefore, if $x \in K_\eta := \{x \in B_1 \mid \operatorname{dist}(x, \partial B_1) < \eta\}$, then $x \in B_1 \cap B_{r_{z_i}}(z_i)$ for some $i = 1, \ldots, m$, and hence $u > 0$ in K_η. ∎

The following lemma was proved in [FW14]. We give the proof below for completeness.

Lemma 6.6. *([FW14]) If $\lambda \in (0,1)$ and $w_\lambda > 0$ in Σ_λ, then there exists $\epsilon \in (0, \lambda)$ such that $w_{\lambda_\epsilon} \geqslant 0$ in $\Sigma_{\lambda_\epsilon}$, where $\lambda_\epsilon := \lambda - \epsilon$.*

Proof of Lemma 6.6. Let $D_\mu := \{x \in \Sigma_\lambda \mid \operatorname{dist}(x, \partial\Sigma_\lambda) \geqslant \mu\}$ for $\mu > 0$. Then, $D_\mu \subset B_1 \subset \mathbb{R}^n$ is closed and bounded and hence compact. Since $w_\lambda > 0$ is continuous on the compact set D_μ, there exists $\mu_0 > 0$ such that $w_\lambda \geqslant \mu_0 > 0$ in D_μ.

We claim that there exists $\epsilon > 0$ sufficiently small so that $w_{\lambda_\epsilon} \geqslant 0$ in D_μ, where $\lambda_\epsilon := \lambda - \epsilon$. Indeed, suppose to the contrary that there exists a sequence $\epsilon_n \to 0^+$ and corresponding sequence $x_n \in D_\mu$ such that $w_{\lambda_{\epsilon_n}}(x_n) < 0$. By the compactness of D_μ, $x_n \to x_0 \in D_\mu$ (up to a subsequence). Then, by the continuity of w_λ, we arrive at the contradiction

$$0 > \lim_{n\to\infty} w_{\lambda_{\epsilon_n}}(x_n) = w_\lambda(x) \geqslant \mu_0 > 0\,.$$

Hence $w_{\lambda_\epsilon} \geqslant 0$ in D_μ for $\epsilon \approx 0$. Therefore, $\Sigma^-_{\lambda_\epsilon} \subset \Sigma_{\lambda_\epsilon} \backslash D_\mu$, where $\Sigma^-_{\lambda_\epsilon} := \{x \in \Sigma_{\lambda_\epsilon} \mid w_{\lambda_\epsilon}(x) < 0\}$. Using the notation used in the proof of Lemma 6.1, it follows from (6.3) that for all $x \in \Sigma^-_{\lambda_\epsilon}$, we get

$$\begin{aligned}
(-\Delta)^s w^+_{\lambda_\epsilon}(x) &= (-\Delta)^s w_{\lambda_\epsilon}(x) - (-\Delta)^s w^-_{\lambda_\epsilon}(x) \\
&\geqslant (-\Delta)^s w_{\lambda_\epsilon}(x) \\
&= (-\Delta)^s u_{\lambda_\epsilon}(x) - (-\Delta)^s u(x) \\
&= \varphi(x) w^+_{\lambda_\epsilon}(x)\,,
\end{aligned}$$

where the Lipschitz continuity of f gives $\varphi(x) := [f(u_{\lambda_\epsilon}(x)) - f(u(x))]/[u_{\lambda_\epsilon}(x) - u(x)] \in L^\infty(\Sigma^-_{\lambda_\epsilon})$. By taking $\epsilon > 0$ and $\mu > 0$ small, we have that $|\Sigma^-_{\lambda_\epsilon}|$ is small. Moreover, $w^+_{\lambda_\epsilon} = 0$ in $\mathbb{R}^N \backslash \Sigma^-_{\lambda_\epsilon}$, and hence it follows from Proposition 2.7 that $w_{\lambda_\epsilon} \geqslant 0$ in $\Sigma_{\lambda_\epsilon}$, as desired. ■

Proof of Theorem 6.1. Let $u \in C(\mathbb{R}^N)$ be a nonnegative nontrivial classical solution to (6.1). Define

$$\lambda_0 := \inf\{\lambda \in (0,1) \mid w_t > 0 \text{ in } \Sigma_t \text{ for all } t \in (\lambda, 1)\}.$$

Clearly $w_{\lambda_0} \geqslant 0$ in Σ_{λ_0}. By Lemma 6.3, $w_\lambda > 0$ in Σ_λ for $\lambda \approx 1$. Therefore, $0 \leqslant \lambda_0 < 1$.

We show $\lambda_0 = 0$. Assume to the contrary that $\lambda_0 > 0$. To show $w_{\lambda_0} > 0$ in Σ_{λ_0} using Lemma 6.2, we must show that $w_{\lambda_0} \not\equiv 0$ in A_{λ_0}. By Lemma 6.5, there exists $\eta > 0$ such that $u > 0$ in K_η.

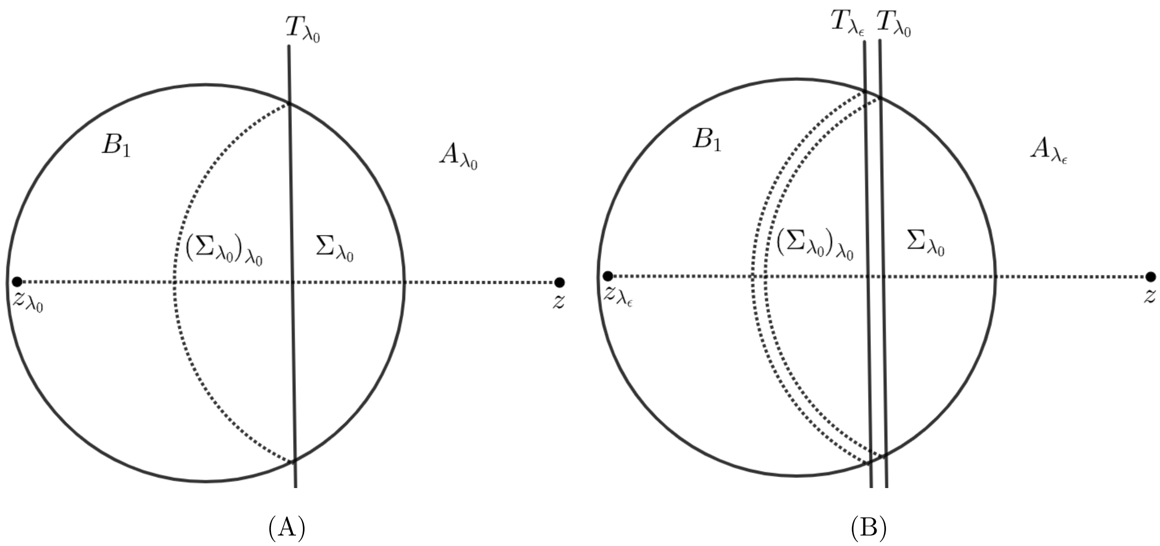


Figure 6.4. (A) Existence of z_{λ_0} and (B) Existence of z_{λ_ϵ}

Then, there exists $z_{\lambda_0} \in K_\eta$ with $u(z_{\lambda_0}) > 0$ and a corresponding $z \in A_{\lambda_0} \backslash B_1 \subset \mathbb{R}^N \backslash B_1$ (see Figure 6.4 (A)) such that $w_{\lambda_0}(z) = u(z_{\lambda_0}) - u(z) = u(z_{\lambda_0}) > 0$. Therefore, since $\lambda_0 \in (0,1)$, $w_{\lambda_0} \geqslant 0$ in Σ_{λ_0}, and $w_{\lambda_0} \not\equiv 0$ in A_{λ_0}, Lemma 6.2 yields $w_{\lambda_0} > 0$ in Σ_{λ_0}. Then, by Lemma 6.6, there exists $\epsilon > 0$ such that $0 < \lambda_\epsilon := \lambda_0 - \epsilon < \lambda_0 < 1$ and $w_{\lambda_\epsilon} \geqslant 0$ in $\Sigma_{\lambda_\epsilon}$. Repeating the previous argument for λ_ϵ, there exists $z \in A_{\lambda_\epsilon} \backslash B_1$ with corresponding $z_{\lambda_\epsilon} \in K_\eta$ such that $u_{\lambda_\epsilon}(z_{\lambda_\epsilon}) > 0$ (see Figure 6.4 (B)). Hence $w_{\lambda_\epsilon} \not\equiv 0$ in A_{λ_ϵ}.

Then, by Lemma 6.2, $w_{\lambda_\epsilon} > 0$ in $\Sigma_{\lambda_\epsilon}$. This contradicts the definition of λ_0. Hence, $\lambda_0 = 0$ and $w_\lambda > 0$ in Σ_λ for all $\lambda \in (0, 1)$. Then, by Lemma 6.4 $u > 0$ in $\Sigma_\lambda \cup (\Sigma_\lambda)_\lambda$ for every $\lambda \in (0, 1)$. If $x \in B_1$, then there exists $\lambda \in (0, 1)$ such that $x \in \Sigma_\lambda \cup (\Sigma_\lambda)_\lambda$. Therefore, $u > 0$ in B_1. Then, by Proposition 6.1, u is radially symmetric and radially decreasing in B_1. This completes the proof of Theorem 6.1. ■

CHAPTER VII

NUMERICAL EXPERIMENTS

The finite element approximation of the linear fractional Laplacian problem

$$\begin{cases} (-\Delta)^s u &= \ell(x) \quad \text{in} \quad \Omega; \\ u &= 0 \quad \text{in} \quad \mathbb{R}^N \backslash \Omega\,, \end{cases} \tag{7.1}$$

for $s \in (0,1)$ with $\Omega \subset \mathbb{R}^N (N = 1,2)$ has been investigated, including convergence results with ℓ in appropriate function spaces (see [BDP16] for $N = 1$ and [ABB17] for $N = 2$). For a complete N-dimensional finite element analysis for the fractional Laplacian, including regularity of solutions of (7.1) and the convergence for piece-wise linear elements, see [AB17]. This motivates the investigation of numerical positive weak solutions for nonlinear fractional Laplacian problems

$$\begin{cases} (-\Delta)^s u &= \lambda f(x,u) \quad \text{in} \quad \Omega; \\ u &= 0 \quad \text{in} \quad \mathbb{R} \backslash \Omega\,, \end{cases} \tag{7.2}$$

where $\lambda > 0$ and $f : \Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function. For numerical experiments, we further assume that $f_t(x,t) := \frac{\partial f}{\partial t}(x,t)$ is continuous a.e. in Ω, and f satisfies certain Hölder type conditions with respect to $x \in \Omega$, as specified below. We consider examples of nonlinearities f satisfying respective hypotheses of Theorem 4.1 and Theorems 5.1 - 5.6.

We use the finite element method (FEM) developed for linear fractional Laplacian problems of the form (7.1) in [BDP16] and [ABB17] to construct numerical solutions u (often positive) with $\lambda > 0$ of the nonlinear problem (7.2) in dimensions $N = 1, 2$. Moreover, using the branch following technique of [NSS06], we construct bifurcation diagrams $\|u\|_\infty$ vs. λ.

As in [BDP16] and [ABB17], we use the weak formulation of (7.2) to seek solution $u \in H_0^s(\Omega)$ such that

$$\frac{C_{N,s}}{2}\mathcal{E}(u, \phi) = \lambda \int_\Omega f(x, u(x))\phi(x)dx \text{ for all } \phi \in H_0^s(\Omega)\,.$$

Remark 7.1. In addition to the fact that $(-\Delta)^s \to -\Delta$ as $s \to 1^-$, it was shown in [BHS18] that the weak solution of the Poisson's equation for $(-\Delta)^s$ with homogeneous Dirichlet condition on $\mathbb{R}^N \backslash \Omega$ approaches the weak solution of Poisson's equation for $-\Delta$ with homogeneous Dirichlet condition on $\partial\Omega$ as $s \to 1^-$. We utilize this limiting behavior as a hint for the correctness of our numerical scheme. In particular, throughout this chapter, we use a finite difference or quadrature method to generate the bifurcation diagram for the Laplacian case ($s = 1$) and then compare to the fractional Laplacian case ($s = 0.99$) using the finite element method before proceeding with any $s \in (0, 1)$.

Remark 7.2. Let $\mathcal{S}$ be a closed, connected set of $(\lambda, u) \in \mathbb{R} \times L^\infty(\Omega)$ such that u is a positive weak solution of (7.2) corresponding to $\lambda > 0$. In each example below, we discuss the shape of $\mathcal{S}$ via the bifurcation diagram obtained numerically in the $\|u\|_\infty$ vs. λ plane. These bifurcation diagrams verify the results obtained in previous sections and motivate future directions. Observe that for each choice of nonlinear reaction term the bifurcation diagrams are qualitatively similar for all values of $s \in (0, 1]$.

For each of the bifurcation diagrams, we also give numerical positive solution(s) for a specific value of λ for which existence is guaranteed by the results of previous chapters. We emphasize the fact that the influence of $s \in (0,1]$ on the behavior of positive solutions near the boundary $\partial\Omega$ becomes more pronounced as $s \in (0,1)$ gets small.

We first describe the approximation method when $\Omega := (0,1) \subset \mathbb{R}$. Fix a uniform partition $0 = x_0 < x_1 < x_2 \ldots < x_{n+1} = 1$ of $[0,1]$ with step size $h = x_i - x_{i-1}$ for $i = 1, \ldots, n+1$. Let V_h be an n - dimensional subset of $H_0^s(0,1)$ spanned by $\{\phi_1, \ldots, \phi_n\}$, where

$$\phi_i(x) := \begin{cases} 1 - |x - x_i|/h & \text{if } x \in [x_{i-1}, x_{i+1}]\,, \\ 0 & \text{if } x \in \mathbb{R}\backslash[x_{i-1}, x_{i+1}] \end{cases} \tag{7.3}$$

for $i = 1, \ldots, n$. The finite element approximation $u_h \in V_h$ for a weak solution $u \in H_0^s(0,1)$ of (7.2) is expressed as

$$u_h(x) := \sum_{i=1}^{n} u_i \phi_i(x)\,,$$

where $u_i \in \mathbb{R}$ are unknowns and u_h satisfies system of n equations

$$\frac{C_{1,s}}{2}\mathcal{E}(u_h, \phi) = \lambda \int_0^1 f\left(x, u_h(x)\right) \phi_j(x) \mathrm{d}x \tag{7.4}$$

for all $j = 1, \cdots, n$. In order to implement the finite element scheme, we express (7.4) in matrix notation.

For a column vector $\mathbf{u} := [u_1, \cdots, u_n]^T$, the left hand side of (7.4) can be expressed as $\mathcal{A}\mathbf{u}$, where $\mathcal{A}$ is the $n \times n$ stiffness matrix corresponding to the left hand side of (7.4) derived in [BDP16].

To numerically compute the integral on the right hand side of (7.4), we assume that there exists $L > 0$ such that for any $y_1, y_2 \in (0,1)$ and any $t_1, t_2 \geqslant 0$,

$$|f(y_2, t_2) - f(y_1, t_1)| \leqslant L\left(|y_2 - y_1|^s + |t_2 - t_1|\right) . \tag{7.5}$$

Then, the expectation that $\|u_h\|_{C^{0,s}([0,1])} \leqslant K'$ holds (independent of h), yields

$$|f(x, u_h(x)) - f(x_j, u_h(x_j))| \leqslant L\left(|x - x_j|^s + |u_h(x) - u_h(x_j)|\right) \leqslant L(1 + K')h^s .$$

Therefore, for all $j = 1, \ldots, n$, using the definition (7.3) of ϕ_j one has

$$\begin{aligned}
&\int_0^1 f\left(x, u_h(x)\right)\phi_j(x)\mathrm{d}x = \int_{x_{j-1}}^{x_{j+1}} f\left(x, u_h(x)\right)\phi_j(x)\mathrm{d}x \\
&= \int_{x_{j-1}}^{x_{j+1}} \left[f\left(x_j, u_h(x_j)\right)\phi_j(x) + f\left(x, u_h(x)\right)\phi_j(x) - f\left(x_j, u_h(x_j)\right)\phi_j(x)\right]\mathrm{d}x \\
&\quad = f(x_j, u_j)\int_{x_{j-1}}^{x_{j+1}} \phi_j(x)\mathrm{d}x + \int_{x_{j-1}}^{x_{j+1}} \left[f\left(x, u_h(x)\right)\phi_j(x) - f\left(x_j, u_h(x_j)\right)\phi_j(x)\right]\mathrm{d}x \\
&\quad = hf(x_j, u_j) + O(h^{1+s}) ,
\end{aligned}$$

where (more precisely) $0 \leqslant |O(h^{1+s})| \leqslant L(1+K')h^{1+s}$. Then, defining the column vector $\mathbf{F}$ by

$$\mathbf{F}(\mathbf{u}) := h[f(x_1, u_1)\,, f(x_2, u_2)\,, \cdots, f(x_n, u_n)]^T\,,$$

we rewrite (7.4) as a matrix equation

$$\mathcal{A}\mathbf{u} = \lambda \mathbf{F}(\mathbf{u})\,. \tag{7.6}$$

We solve the system (7.6) for a given nonlinearity f and $\lambda > 0$ with Newton's method, provided a suitable initial guess for the iteration. A multiple of the solution of the linear problem $(-\Delta)^s e = 1$ in $(0,1)$ with $u = 0$ in $\mathbb{R}\backslash(0,1)$ is a good candidate for an initial guess in many cases.

7.1. Finite Element Algortithm

Now we describe the pseudo-code for constructing numerical solutions and numerical bifurcation diagrams.

Input:	$s \in (0, 1)$	(real parameter in $(-\Delta)^s$)
	$0 \leqslant \lambda_{min} < \lambda_{max}$	(range of values of λ in the bifurcation diagram)
	$m \in \mathbb{N}$	(number of partition of interval $[\lambda_{min}, \lambda_{max}]$)
	$n \in \mathbb{N}$	(number of interior nodes in partition of interval [0,1])
	$6 < r < 15$	(10^{-r} is the tolerance in the Newton iteration)
Output:	$\mathcal{S}$	(list of points of the form $(\lambda, \|\mathbf{u}\|_\infty)$)

Begin % Initialization

01 `Create interior nodes of the uniform partition` $\mathcal{P}$ `of` $[0, 1]$

`by setting` $x_j \leftarrow j/(n+1), \quad j = 1, \ldots, n$

02 $C_{1,s} \leftarrow \frac{2^{2s}\, s\,\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1-s)}$

03 `Assemble` $n \times n$ `stiffness matrix` $\mathcal{A}$ `for the partition` $\mathcal{P}$ and parameter s

`using the algorithm described in [BDP16, page 12]`

04 `Create a partition` Λ `of` $[\lambda_{\min}, \lambda_{\max}]$

`by setting` $\mu_i \leftarrow \lambda_{\min} + \frac{\lambda_{\max} - \lambda_{\min}}{m} i, \quad i = 0, \ldots, m$

05 $\mathbf{u}_{\text{init}} \leftarrow$ `Solution of` $\mathcal{A}\mathbf{e} = \mathbf{1}$ % Here $\mathbf{1}$ stands for $n \times 1$ column vector of 1s

06 $\mathcal{S} \leftarrow$ `Empty list` % End of Initialization

For $i := 0 : m$ do   % Main Loop

% Apply Newton iterations to: $\mathcal{A}\mathbf{u} = \mu_i \mathbf{F}(\mathbf{u})$

$\mathbf{u} \leftarrow \mathbf{u}_{\text{init}}$

% Compute $\mathbf{F}(\mathbf{u})$ componentwise (represented by column vector $\mathbf{b}$)

$[\mathbf{b}]_j \leftarrow hf(x_j, u_j)$ for $j = 1, \ldots, n$

$\mathbf{res} \leftarrow \mathcal{A}\mathbf{u} - \mu_i \mathbf{b}$

While $|\mathbf{res}|_\infty > 10^{-r}$ do   % Newton loop

% Compute $\mathbf{J_F}$, the Jacobian matrix of $\mathbf{F}(\mathbf{u})$ componentwise

$[\mathbf{J_F}]_{j,j} \leftarrow hf_t(x_j, u_j)$, and $[\mathbf{J_F}]_{i,j} \leftarrow 0$ for $i \neq j$, $i, j = 1, \ldots n$

% Compute $\mathbf{J}$, the Jacobian matrix of the system (7.6)

$\mathbf{J} \leftarrow \mathcal{A} - \mu_i \mathbf{J_F}$

$\mathbf{u} \leftarrow \mathbf{u} - \mathbf{J}^{-1}\mathbf{res}$ % Newton's update of $\mathbf{u}$

% Update of $\mathbf{F}(\mathbf{u})$ componentwise

$[\mathbf{b}]_j \leftarrow hf(x_j, u_j)$ for $j = 1, \ldots, n$

$\mathbf{res} \leftarrow \mathcal{A}\mathbf{u} - \mu_i \mathbf{b}$ % Update of $\mathbf{res}$

EndWhile   % End of Newton Loop

$\mathcal{S} \leftarrow \text{Append}(\mathcal{S}, (\mu_i, |\vec{u}|_\infty))$

EndFor   % End of Main Loop

Return $\mathcal{S}$

End

7.2. Numerical Experiments Corresponding to Theorem 4.1: N=1

Here we focus on examples of f satisfying the hypothesis of Theorem 4.1 independent of the sign of f at 0. The shape of the bifurcation diagram and nodal properties of solutions beyond $\lambda > 0$ small depends on the behavior of f away from infinity. Specifically, we explore the cases, $f(0) = 0$ with $f'(0) = 0$ (Example 1), $f(0) = 0$ with $f'(0) > 0$ (Example 2), $f(\sigma) > 0$ (Example 3), and $f(0) < 0$ (Example 4), and demonstrate that positive weak solutions indeed bifurcate from infinity at $\lambda = 0$. Observe, for each of the examples, there is a bifurcation diagram with the property $\|u\|_\infty \to +\infty$, as $\lambda \to 0^+$ confirming the result of Theorem 4.1.

Example 1: Consider $f(\sigma) = \sigma^2$ for $\sigma \geqslant 0$.

In Figure 7.1, bifurcation diagrams for the cases (A) $s = 1$, (B) $s = 0.99$, (C) $s = 0.9$, (D) $s = 0.7$, (E) $s = 0.5$, and (F) $s = 0.3$ are given. In each case, the corresponding inset gives the numerical positive solution for $\lambda = 1$. In this case, since $f(0) = 0$, $u \equiv 0$ is a solution for all $\lambda > 0$. In addition, we find a numerical positive solution for each $\lambda > 0$ satisfying $\|u\|_\infty \to +\infty$ as $\lambda \to 0^+$ and $\|u\|_\infty \to 0$ as $\lambda \to +\infty$.

Example 2: Consider $f(\sigma) = \sigma + \sigma^2$ for $\sigma \geqslant 0$.

In Figure 7.2, bifurcation diagrams for the cases (A) $s = 1$, (B) $s = 0.99$, (C) $s = 0.9$, (D) $s = 0.7$, (E) $s = 0.5$, and (F) $s = 0.3$ are given. In each case, the corresponding inset gives the numerical positive solution for $\lambda = 1$. As proved, positive solutions bifurcate from infinity at $\lambda = 0$. In addition, the solution set also bifurcates from the trivial branch at λ_1 defined by (2.11). Since $f(\sigma) \geqslant 0$, all nontrivial solutions on the bifurcation diagram are positive.

Example 3: Let $f(\sigma) = 3(1+\sigma)^{\frac{1}{3}} + \sigma^2 > 0$ for $\sigma \geqslant 0$.

In Figure 7.3, bifurcation diagrams for the cases (A) $s = 1$, (B) $s = 0.99$, (C) $s = 0.9$, (D) $s = 0.7$, (E) $s = 0.5$, and (F) $s = 0.3$ are given and two numerical positive solutions for $\lambda = 0.25$ are shown in (G) - (K). In the bifurcation diagram, we see that positive solutions bifurcate from infinity at $\lambda = 0$, turns, and connects to the origin. The existence of the minimal solution, the lower branch on the bifurcation diagram, and the *extremal* solution corresponding to the turning point was established in [ROS14b]. Theorem 4.1 guarantees the existence of the upper branch bifurcating from infinity at $\lambda = 0$.

Example 4: Let $f(\sigma) = \sigma + \sigma^2 - 1$ for $\sigma \geqslant 0$.

In Figure 7.4, bifurcation diagrams for the cases (A) $s = 1$, (B) $s = 0.99$, (C) $s = 0.9$, (D) $s = 0.7$, (E) $s = 0.5$, and (F) $s = 0.3$ are given. In the bifurcation diagrams, the solid part corresponds to positive solutions and the dashed part corresponds to the sign changing solutions. The markers, $\triangle$ and $\ast$, mark two locations of λ on the bifurcation diagrams for which a positive numerical solution exist. The solution corresponding to λ to the right of $\bigcirc$ on the bifurcation diagram are sign changing.

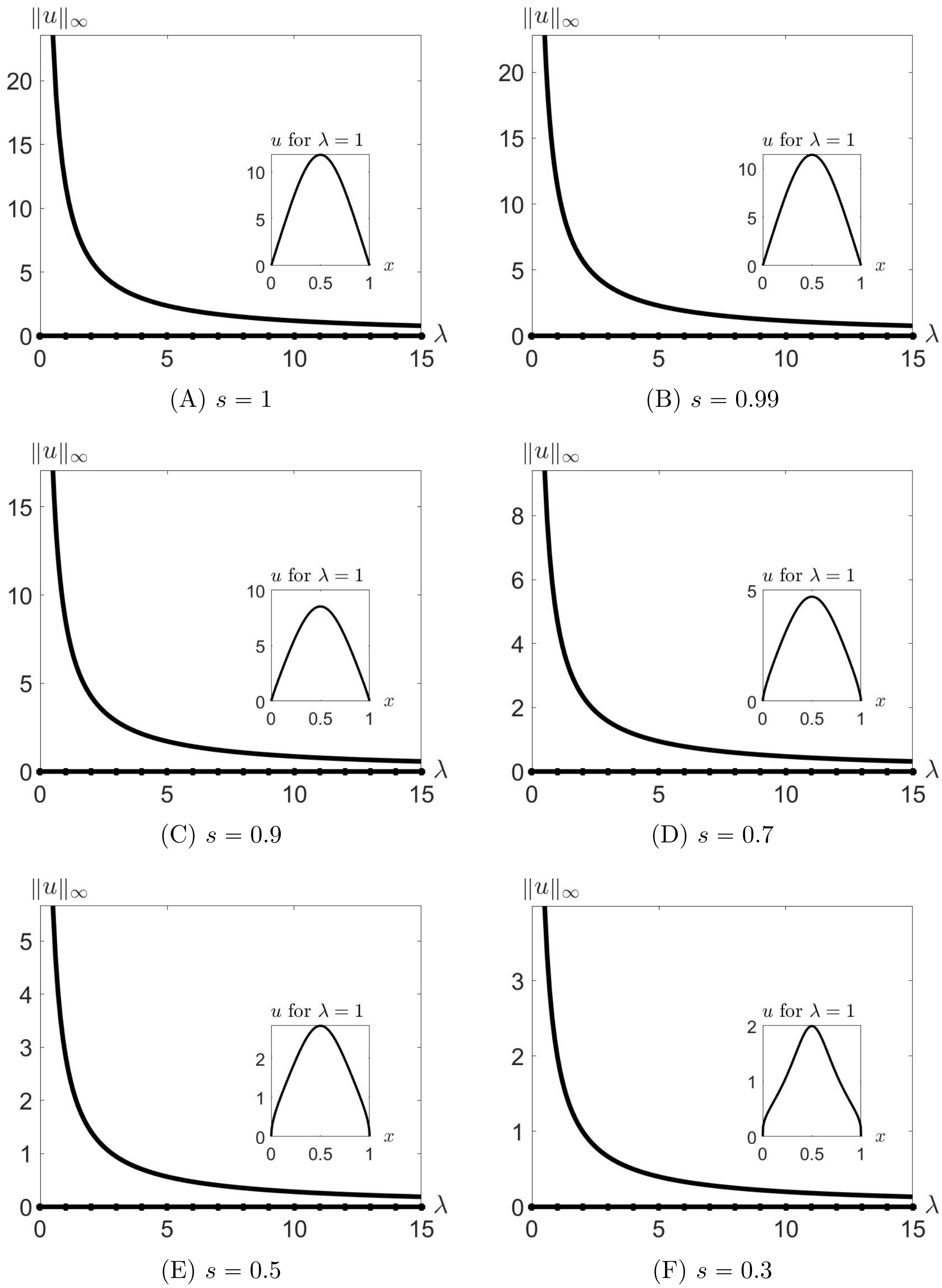


Figure 7.1. Bifurcation Diagrams for $f(\sigma) = \sigma^2$ and Positive Solutions for $\lambda = 1$

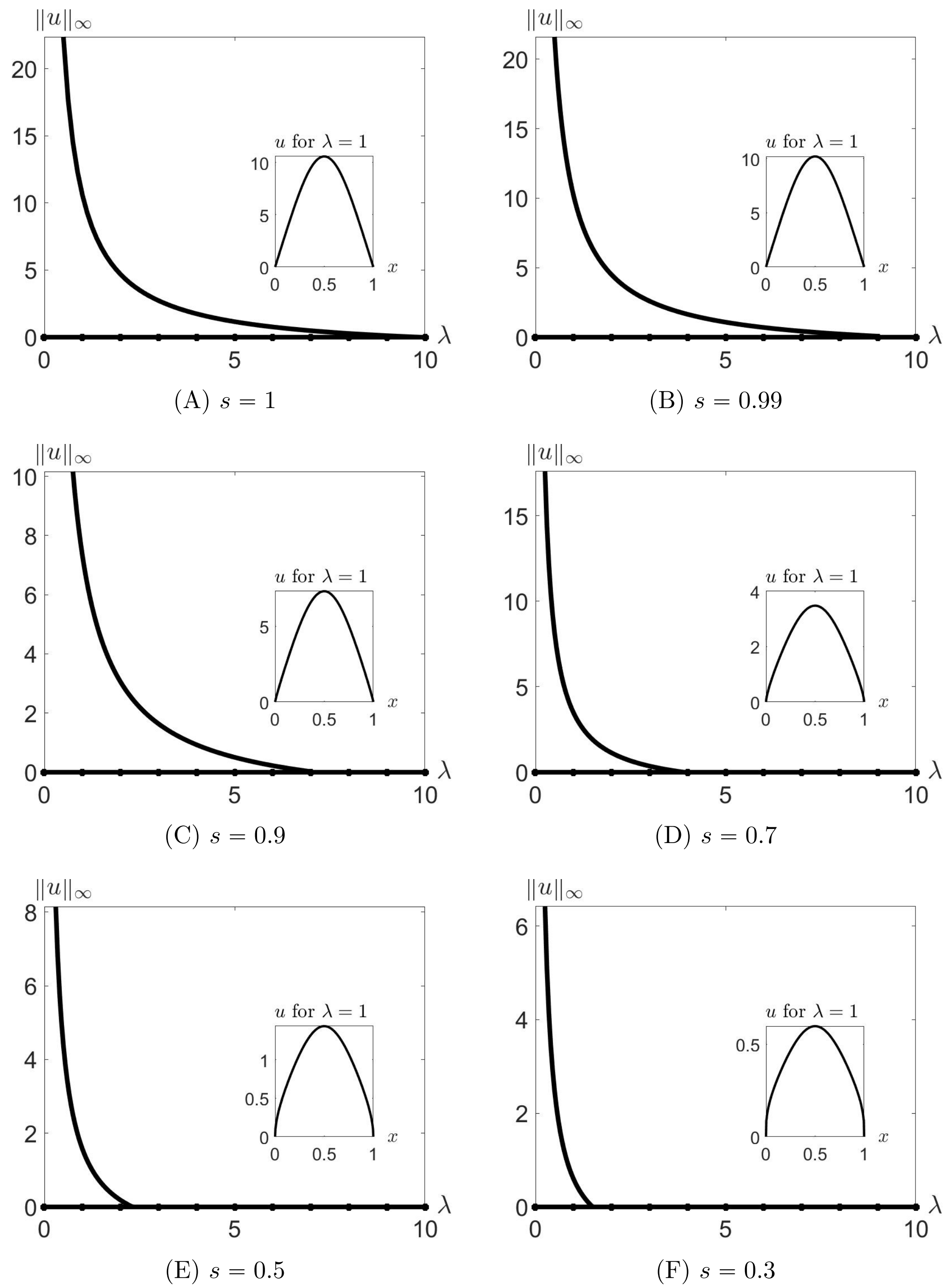
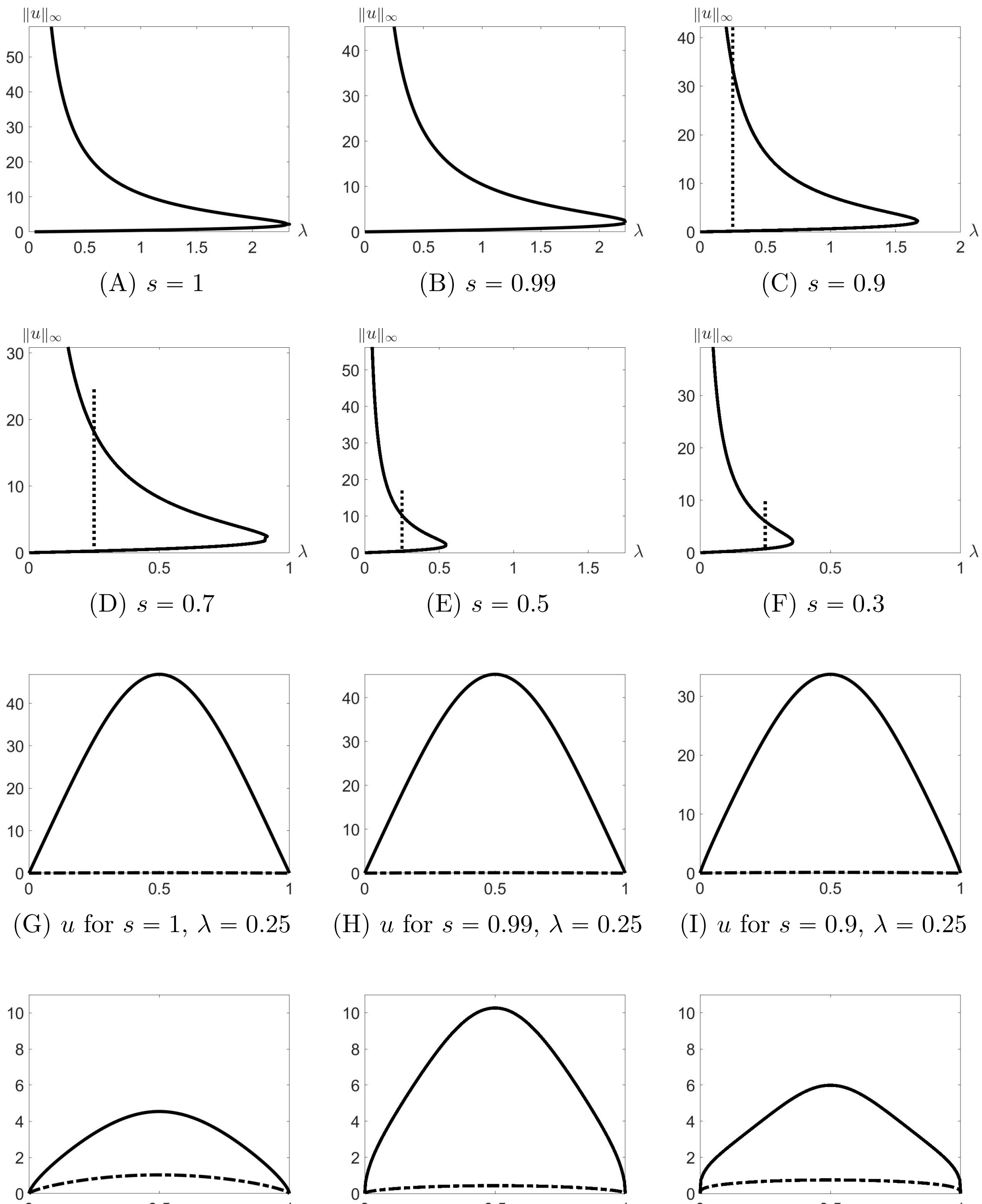


Figure 7.2. Bifurcation Diagrams for $f(\sigma) = \sigma + \sigma^2$ and Positive Solutions for $\lambda = 1$



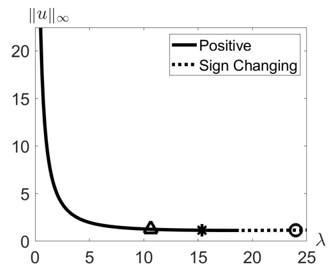
(A) $s = 1$ (B) $s = 0.99$ (C) $s = 0.9$

(D) $s = 0.7$ (E) $s = 0.5$ (F) $s = 0.3$

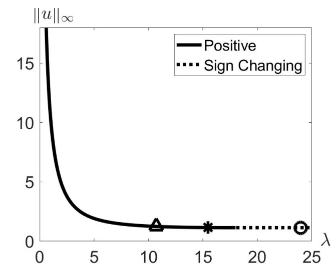
(G) u for $s = 1$, $\lambda = 0.25$ (H) u for $s = 0.99$, $\lambda = 0.25$ (I) u for $s = 0.9$, $\lambda = 0.25$

(J) u for $s = 0.7$, $\lambda = 0.25$ (K) u for $s = 0.5$, $\lambda = 0.25$ (L) u for $s = 0.3$, $\lambda = 0.25$

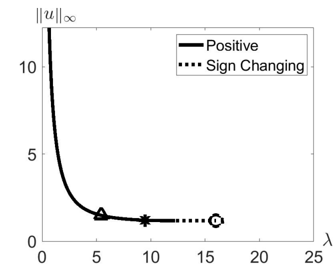
Figure 7.3. Bifurcation Diagrams for $f(\sigma) = 3(1+\sigma)^{\frac{1}{3}} + \sigma^2$ and Two Positive Solutions for $\lambda = 0.25$



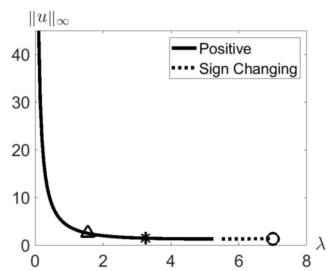
(A) $s = 1$



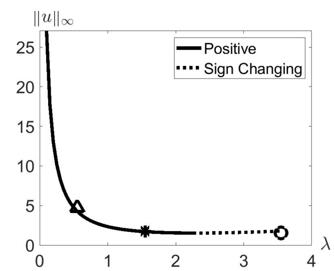
(B) $s = 0.99$



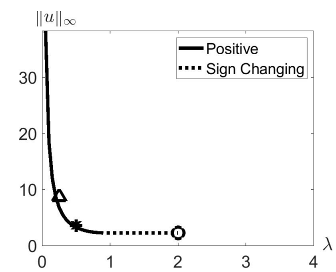
(C) $s = 0.9$



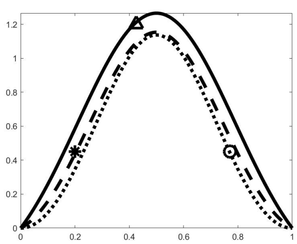
(D) $s = 0.7$

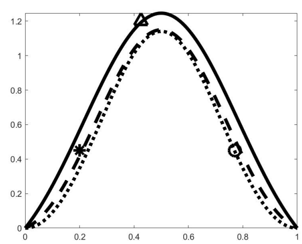


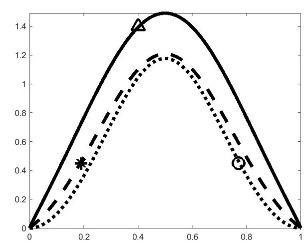
(E) $s = 0.5$

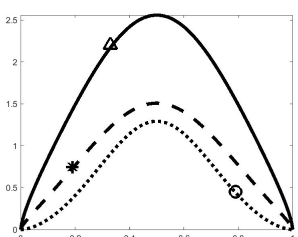


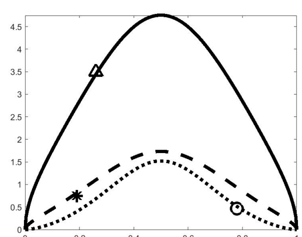
(F) $s = 0.3$


(G) $s = 1$


(H) $s = 0.99$


(I) $s = 0.9$


(J) $s = 0.7$


(K) $s = 0.5$

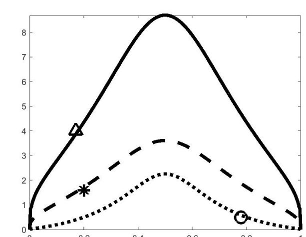
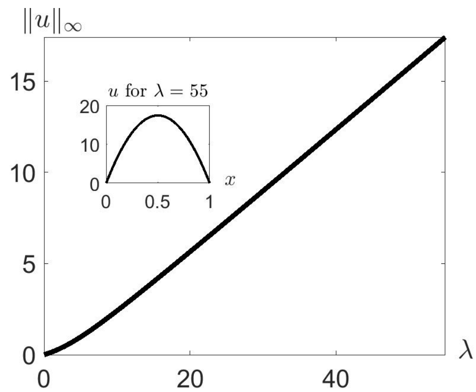

(L) $s = 0.3$

Figure 7.4. Bifurcation Diagrams for $f(\sigma) = \sigma + \sigma^2 - 1$, Two Positive Solutions for λ Corresponding to $\triangle$, $*$, and a Sign Changing Solution Corresponding to $\bigcirc$

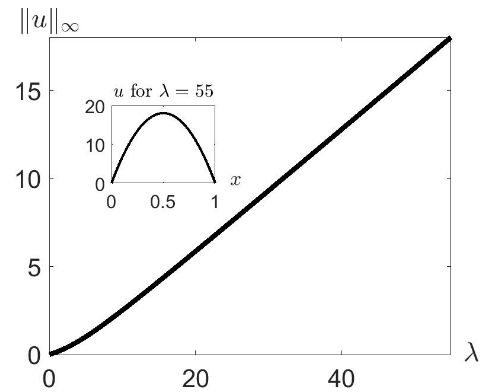
7.3. Numerical Experiments Corresponding to Theorem 5.1: N=1

Here we consider two examples satisfying the hypotheses of Theorem 5.1. First, $f(\sigma) = e^{\frac{\sigma}{1+\sigma}}$ for $\sigma \geqslant 0$ and then $f(\sigma) = e^{\frac{5\sigma}{5+\sigma}}$ for $\sigma \geqslant 0$. By Theorem 5.1, (7.2) has a positive weak solution for each $\lambda > 0$, and $u \equiv 0$ is a solution of (7.2) for $\lambda = 0$. Figure 7.5 shows the bifurcation diagrams for (A) $s = 1$ (B) $s = 0.99$ (C) $s = 0.9$ (D) $s = 0.7$ (E) $s = 0.5$ and (F) $s = 0.3$. The inset in each bifurcation diagram shows the profile of a numerical positive solution corresponding to $\lambda = 55$ and the influence of s on the behavior of the positive solution near the boundary points $x = 0$ and $x = 1$.

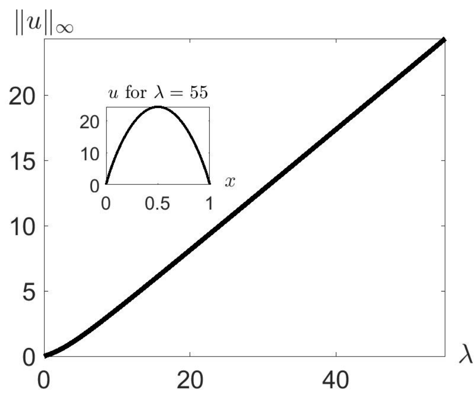
Notice in the bifurcation diagrams in Figure 7.5 that the solution set $\mathcal{S}$ emanates from the origin and increases with respect to λ (hence a unique positive solution exists for each $\lambda > 0$). Moreover, $\|u\|_\infty \to 0$ as $\lambda \to 0^+$ and $\|u\|_\infty \to +\infty$ as $\lambda \to +\infty$. In [LPPS15], the authors prove uniqueness of a positive solution if $\frac{f(\sigma)}{\sigma}$ is decreasing in σ. Note, this condition is satisfied by $f(\sigma) = e^{\frac{\sigma}{1+\sigma}}$ and the bifurcation diagram confirms the uniqueness result in [LPPS15, Thm. 20].



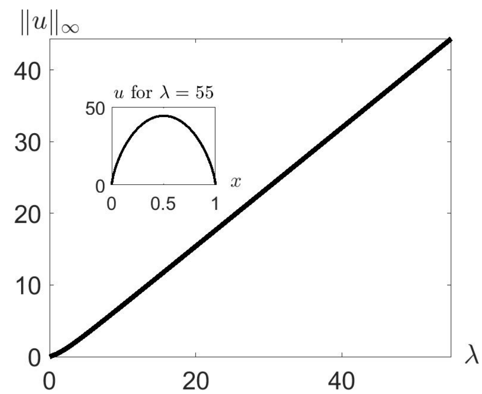
(A) $s = 1$



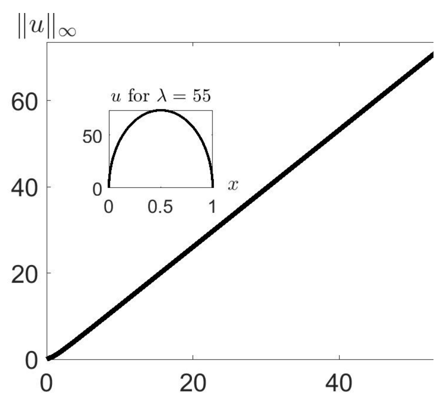
(B) $s = 0.99$



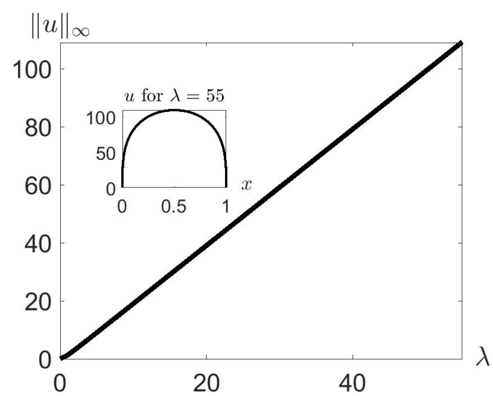
(C) $s = 0.9$



(D) $s = 0.7$



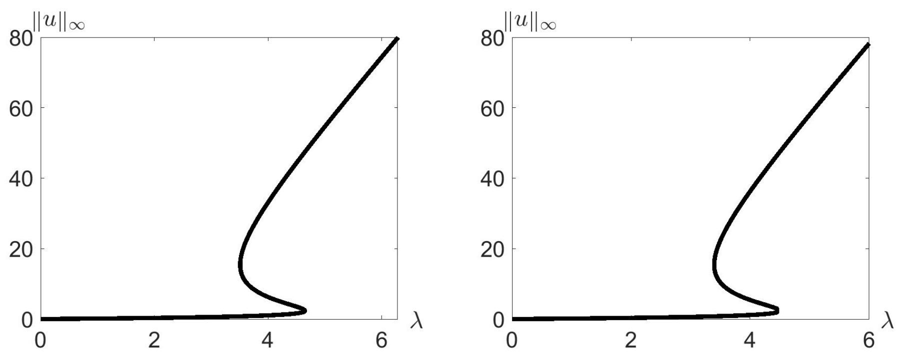
(E) $s = 0.5$



(F) $s = 0.3$

Figure 7.5. Bifurcations Diagrams for $f(\sigma) = e^{\frac{\sigma}{1+\sigma}}$ and Positive Solutions for $\lambda = 55$

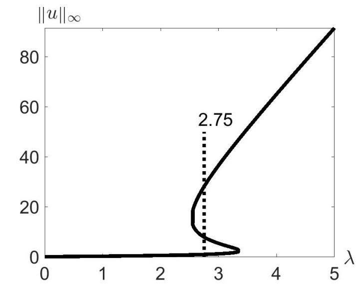
Now consider $f(\sigma) = e^{\frac{5\sigma}{5+\sigma}}$ for $\sigma \geqslant 0$. Investigation of the bifurcation diagram for the perturbed Gelfand problem $f(\sigma) = e^{\frac{\kappa\sigma}{\kappa+\sigma}}$ for $\kappa > 0$ and $\sigma \geqslant 0$ has been of interest since the paper of Keller and Cohen [KC67]. It was shown in [BIS81] that the sufficient condition for the bifurcation diagram to be S-shaped is satisfied if $\kappa \geqslant 4.07$ for the Laplacian case ($s = 1$). Indeed, we see in Figure 7.6 that the numerical bifurcation diagram is S-shaped for both $s = 1$ (obtained using quadrature method) and $s = 0.99$.



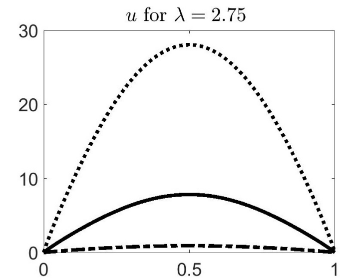
(A) $s = 1$ (B) $s = 0.99$

Figure 7.6. Comparison of Bifurcation Diagrams for $f(\sigma) = e^{\frac{5\sigma}{5+\sigma}}$ with (A) $s = 1$ and (B) $s = 0.99$

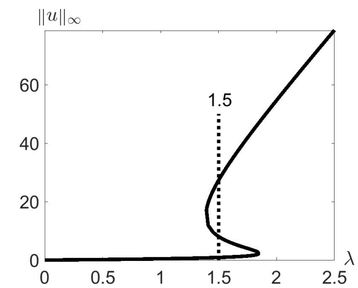
As in the case of $f(\sigma) = e^{\frac{\sigma}{1+\sigma}}$, $\|u\|_\infty \to 0$ as $\lambda \to 0^+$ and $\|u\|_\infty \to +\infty$ as $\lambda \to +\infty$. However, the solution set $\mathcal{S}$ is not monotone with respect to λ. Additionally, there is a range of λ for which we see three numerical positive solutions.



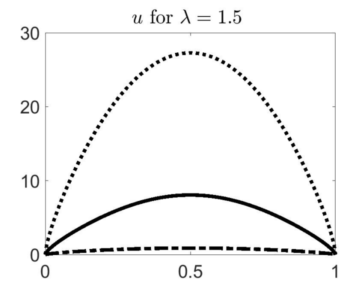
(A) $s = 0.9$



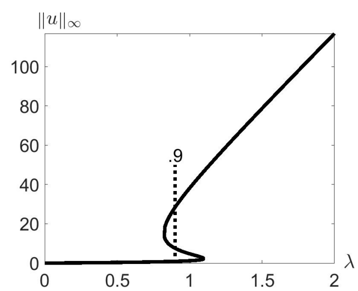
(B) 3 solns for $s = 0.9$



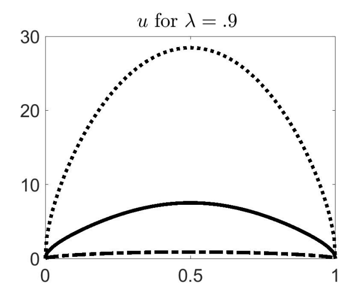
(C) $s = 0.7$



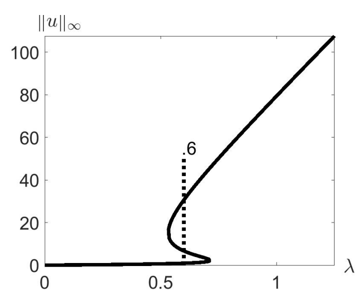
(D) 3 solns for $s = 0.7$



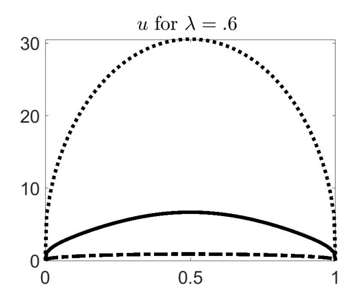
(E) $s = 0.5$



(F) 3 solns for $s = 0.5$



(G) $s = 0.3$



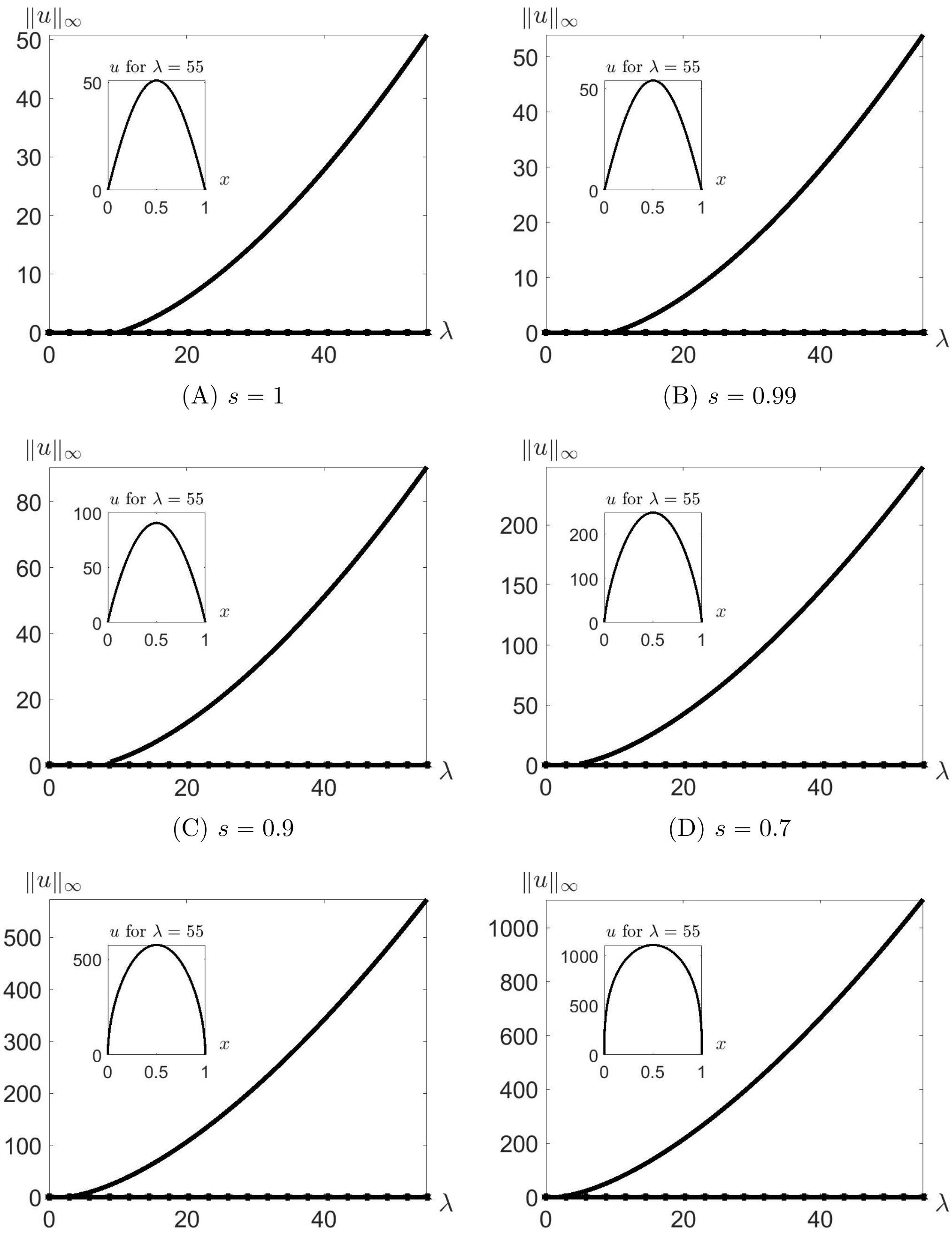
(H) 3 solns for $s = 0.3$

Figure 7.7. Bifurcation Diagrams for $f(\sigma) = e^{\frac{5\sigma}{5+\sigma}}$ and Three Positive Solutions for the λ Specified

7.4. Numerical Experiments Corresponding to Theorem 5.2: N=1

Consider $f(\sigma) = 3(1+\sigma)^{\frac{1}{3}} - 3$ for $\sigma \geqslant 0$ satisfying the hypotheses of Theorem 5.2. The bifurcation diagram for the Laplacian case ($s = 1$) was discussed in detail in [Lio82, Sec. 1.2]. In particular, if $f(0) = 0$ and $f'(0) > 0$, then the positive solution bifurcates from the line of trivial solutions at $\lambda = \frac{\lambda_1}{f'(0)}$. Here $f'(0) = 1$, so bifurcation occurs at $\lambda = \lambda_1 = \pi^2$ for $s = 1$, see Figure 7.8 (A). The inset of Figure 7.8 (A) shows a numerical positive solutions for $\lambda = 55$.

Figure 7.8, (B) - (F) shows bifurcation diagrams and the insets give the numerical positive solutions corresponding to $\lambda = 55$ for (B) $s = 0.99$, (C) $s = 0.9$, (D) $s = 0.7$, (E) $s = 0.5$, and (F) $s = 0.3$. Observe that bifurcation diagrams for any $s \in (0, 1)$ are qualitatively similar to those for $s = 1$. For $s = 0.99$, the bifurcation of positive solutions from the line of trivial solutions occurs near $\pi^2 \approx 9.8696$, see Figure 7.8 (B). The influence of $s \in (0, 1)$ is noticeable in the location of the point of bifurcation from the line of trivial solutions. This can be justified by the estimate of the principal eigenvalue of $(-\Delta)^s$ on $(0, 1)$, see [Kwa12]. Also, the profile of the numerical positive solutions corresponding to $\lambda = 55$ for values of $s \in (0, 1]$ exhibit the boundary behavior similar to δ^s.



(A) $s = 1$ (B) $s = 0.99$

(C) $s = 0.9$ (D) $s = 0.7$

(E) $s = 0.5$ (F) $s = 0.3$

Figure 7.8. Bifurcation Diagrams for $f(\sigma) = 3(1+\sigma)^{\frac{1}{3}} - 3$ and Positive Solutions for $\lambda = 55$

7.5. Numerical Experiments Corresponding to Theorem 5.3: N=1

Consider $f(\sigma) = \ln(1+\sigma) - 0.5$ for $\sigma \geqslant 0$ satisfying the hypotheses of Theorem 5.3. The bifurcation diagram for the case $s = 1$ was obtained in [CS88, Thm. 1.1(B)] using the quadrature method. For a comparison of the bifurcation diagrams for the cases $s = 1$ and $s = 0.99$, see Figure 7.9 (A) and (B). Theorem 5.3 guarantees a positive solution for λ sufficiently large. However, the solution set $\mathcal{S}$ is not monotone with respect to λ. Additionally, in Figure 7.9, there is a range of λ, depending on s, for which two positive solutions exist. The inset of Figure 7.9 (A) and (B) show the profile of two numerical positive solutions for $\lambda = 35$ for both of the cases $s = 1$ and $s = 0.99$. Figure 7.9 $(C) - (F)$ shows the bifurcation diagram for the cases (C) $s = 0.9$, (D) $s = 0.7$, (E) $s = 0.5$, (F) $s = 0.3$, and the inset shows the profiles of two numerical positive solutions for (C) $\lambda = 25$, (D) $\lambda = 15$, (E) $\lambda = 9$, and (F) $\lambda = 5.5$, respectively.

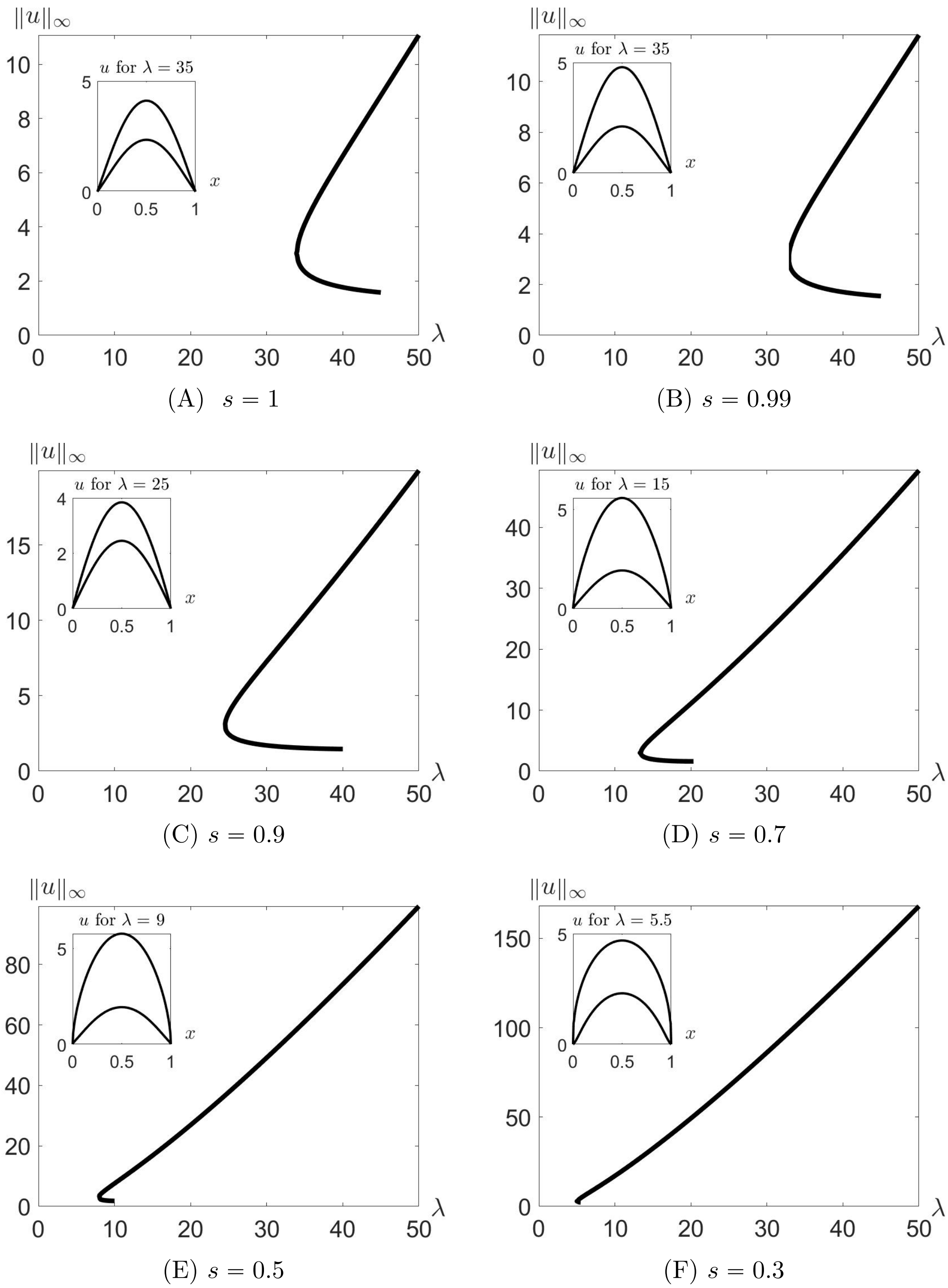


Figure 7.9. Bifurcation Diagram for $f(\sigma) = \ln(1+\sigma) - 0.5$ and Two Positive Solutions for the λ Specified

7.6. Numerical Experiments Corresponding to Theorem 5.4: N=1

Consider $f(\sigma) = 0.5\sigma + 3(1+\sigma)^{\frac{1}{3}} - 4$ for $\sigma \geqslant 0$. Here $m_\infty = 0.5$ and $m_1 = 1.5 > 1 = 2m_\infty$. Theorem 5.4 guarantees a positive weak solution for $\lambda \in [\frac{2\lambda_1}{m_1}, \frac{\lambda_1}{m_\infty})$. See Figure 7.10 (A) and (B) for the comparison of bifurcation diagrams for $s = 1$ to $s = 0.99$. In each case, the inset is a plot of the positive numerical solution for the specified $\lambda > 0$.

We see that there exists a positive numerical solution on an interval bounded away from zero and to the left of $\frac{\lambda_1}{m_\infty} \approx 2\pi^2$. However, for this example with $s = 1$ and $s = 0.99$, we find numerical positive solutions further to the left of $\frac{2\lambda_1}{m_1} = \frac{4\pi^2}{3} \approx 13.1595$. This suggests that the choice of m_1 is not optimal in Theorem 5.4. Figure 7.10 shows bifurcation diagrams for the cases (A) $s = 0.9$, (B) $s = 0.7$, (C) $s = 0.5$, and (D) $s = 0.3$. In each case, the inset is a plot of the positive numerical solution for the specified $\lambda > 0$.

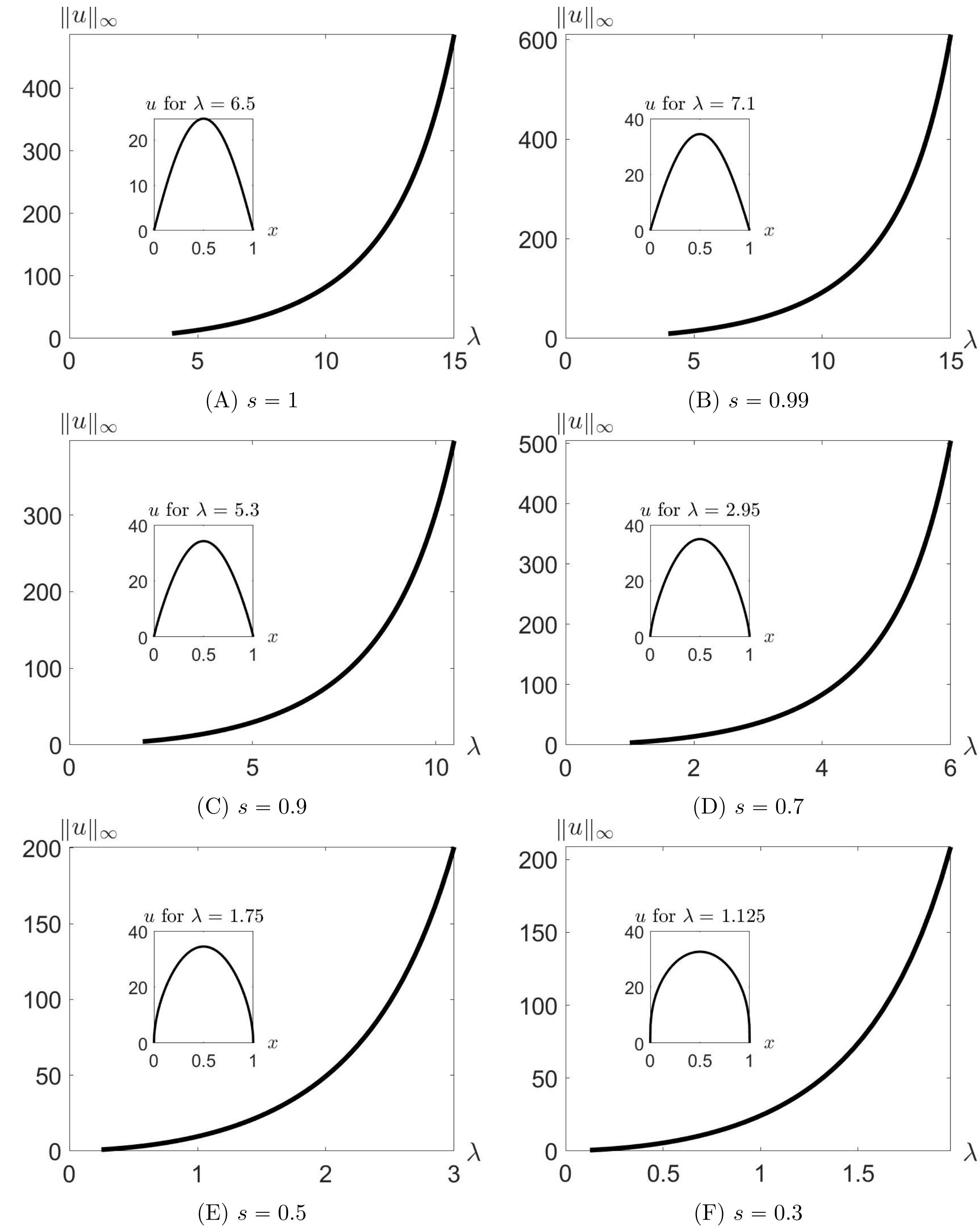
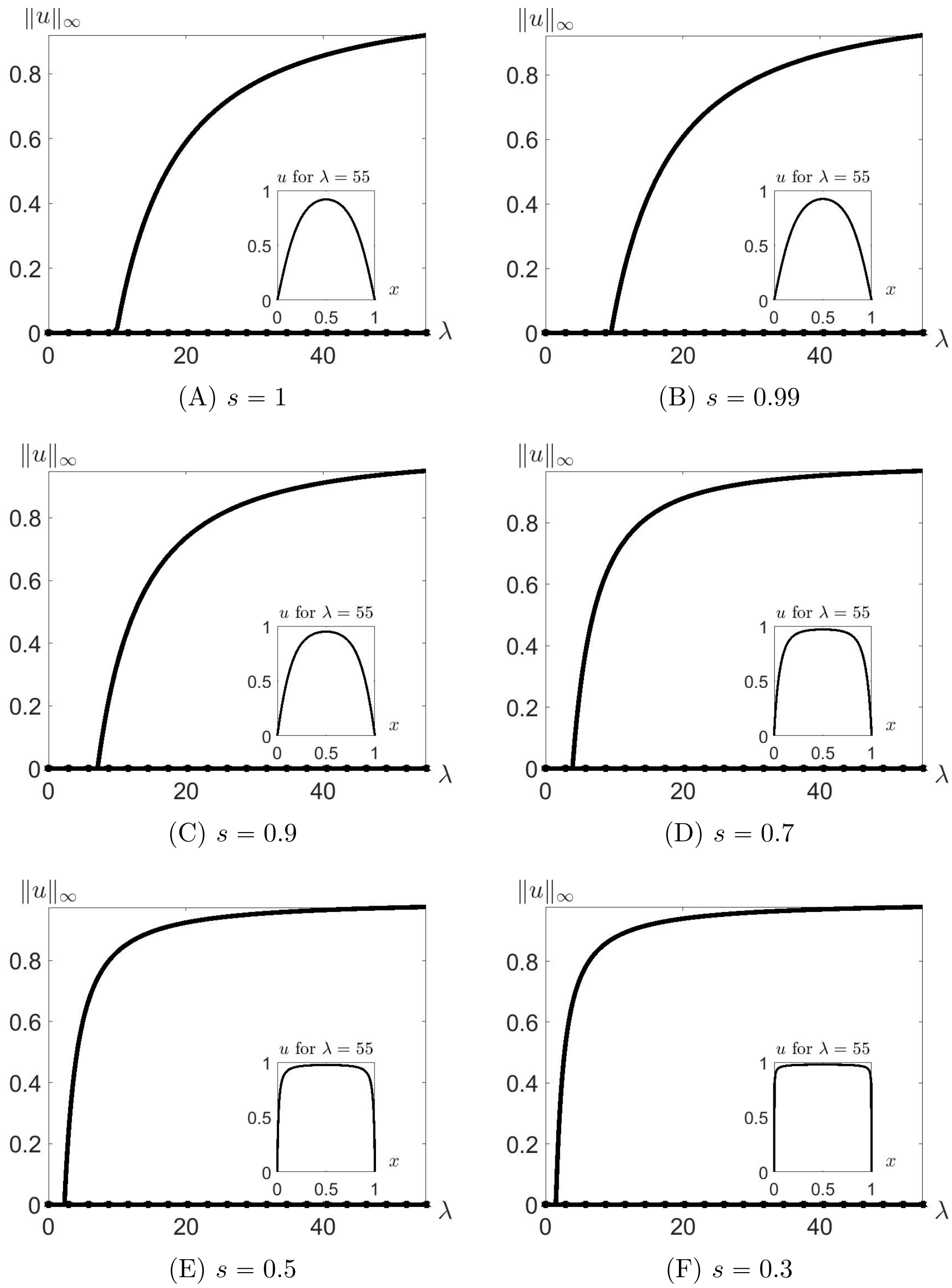


Figure 7.10. Bifurcation Diagrams for $f(\sigma) = 0.5\sigma + 3(1+\sigma)^{\frac{1}{3}} - 4$ and Positive Solutions for the λ specified

7.7. Numerical Experiments Corresponding to Theorem 5.5: N=1

First we consider the logistic reaction term $f(\sigma) = \sigma(1-\sigma)$ for $\sigma \geqslant 0$ (corresponding to $q \equiv 1$) considered here essentially behaves like a sublinear nonlinearity at infinity with $f(0) = 0$ and $f'(0) = 1$. Hence, the bifurcation diagrams in Figure 7.11 resemble those obtained in Figure 7.8 above. Here the L^∞ norm of solutions $\|u\|_\infty$ are bounded above by 1 for any $s \in (0,1]$. Therefore, to understand the influence of $s \in (0,1)$ on positive solutions, we compute L^1 norm $\|u\|_{L^1(0,1)} = \|u\|_1$ of numerical positive solutions u for $\lambda = 55$. We observe that $\|u\|_1$ increases as s decreases. It appears, numerically, that $\|u\|_1 \nearrow 1$ as $s \to 0^+$.



s	1	0.99	0.9	0.7	0.5	0.3
$\|u\|_1$	0.652287	0.660978	0.725085	0.842018	0.921431	0.960814

Figure 7.11. Bifurcation Diagrams for $f(\sigma) = \sigma(1-\sigma)$, Positive Solution, and L^1 Norm of the Solutions with $\lambda = 55$

Next we consider the weighted logistic problem $f(x, \sigma) = \sigma(q(x) - \sigma)$ for $\sigma \geqslant 0$ and $x \in (0, 1)$, where $0 \leqslant q \leqslant 1$ is as in Theorem 5.5. For the numerical experiment we consider five specific examples of q. Namely, let $q_i : [0, 1] \to \{0, 1\}$ for $i = 1, 2, 3$ be given by the following:

1. $q_1(x) = 1$ for $x \in [3/9, 8/9)$ and $q_1(x) = 0$ otherwise,

2. $q_2(x) = 1$ for $x \in [0, 1/5) \cup [2/5, 4/5)$ and $q_2(x) = 0$ otherwise,

3. $q_3(x) = 1$ for $x \in [1/5, 2/5) \cup [3/5, 4/5)$ and $q_3(x) = 0$ otherwise,

In Figures 7.12 - 7.14, (A) gives the graph of $q_i(i = 1,\ 2,\ 3)$, and (B)-(F) show numerical bifurcation diagrams for $s = 0.99$, $s = 0.9$, $s = 0.7$, $s = 0.5$, and $s = 0.3$, respectively. Each inset in (B)-(F) gives a numerical positive solution for a fixed specified λ.

Our numerical experiments appear to agree with the findings of [CDV17] that the nonlocal diffusion strategy may be advantageous to adapt to sparse resources. In particular, we see that the L^1 norm, $\|u\|_{L^1}$, increases as $s \to 0^+$ for each of the q_i considered. The table at the end of each figure provides a comparison of the $\|u\|_{L^1}$ norm for positive solutions when $s = 0.99$, $s = 0.9$, $s = 0.7$, $s = 0.5$, and $s = 0.3$, respectively.

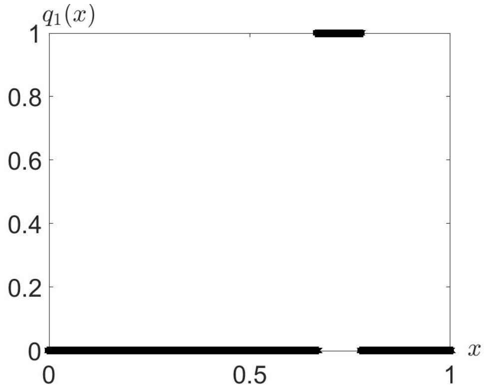
Here the nonlinearity $f(x, \sigma)$ has discontinuity at finitely many points. In this case, we partition the interval $[0, 1]$ in such a way that the points of discontinuity occur at x_j with $j \in \{1, \cdots, n\}$.

In order to compute the integral on the right hand side of (7.4), we modify the Hölder type assumption (7.5) for any $\sigma_1, \sigma_2 \geqslant 0$ to a local type assumption as follows: for all $j = 1, \ldots, n$.

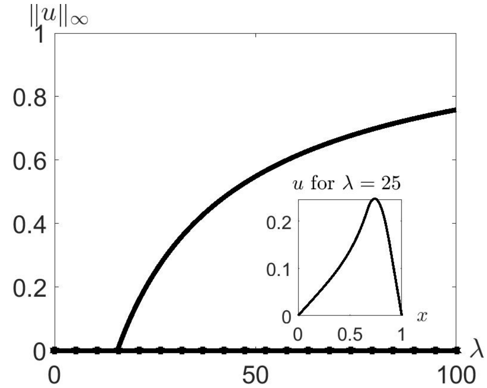
$$|f(y_2, \sigma_2) - f(y_1, \sigma_1)| \leqslant L\left(|y_2 - y_1|^s + |\sigma_2 - \sigma_1|\right) \text{ for any } y_1, y_2 \in (x_{j-1}, x_j) \tag{7.7}$$

Then, assuming f satisfies (7.7), we compute the integral on the right hand side of (7.4) as

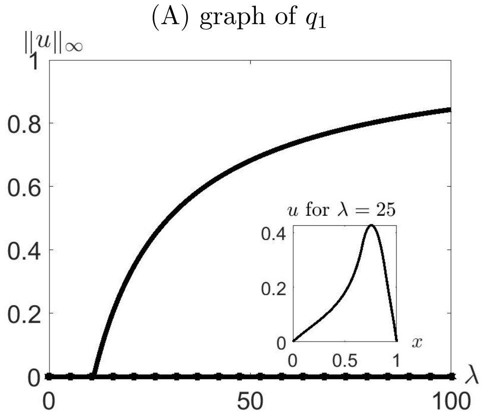
$$\begin{aligned}
\int_0^1 f\left(x, u_h(x)\right)\phi_j(x)\mathrm{d}x &= \left(\int_{x_{j-1}}^{x_j} + \int_{x_j}^{x_{j+1}}\right) f\left(x, u_h(x)\right)\phi_j(x)\mathrm{d}x \\
&= \frac{h}{2}\left(f\left(\frac{x_{j-1}+x_j}{2}, u_j\right) + f\left(\frac{x_j+x_{j+1}}{2}, u_j\right)\right) \\
&\qquad + \int_{x_{j-1}}^{x_j}\left[f\left(x, u_h(x)\right) - f\left(\frac{x_{j-1}+x_j}{2}, u_h(x_j)\right)\right]\phi_j(x)\mathrm{d}x \\
&\qquad + \int_{x_j}^{x_{j+1}}\left[f\left(x, u_h(x)\right) - f\left(\frac{x_j+x_{j+1}}{2}, u_h(x_j)\right)\right]\phi_j(x)\mathrm{d}x \\
&= \frac{h}{2}\left(f\left(\frac{x_{j-1}+x_j}{2}, u_j\right) + f\left(\frac{x_j+x_{j+1}}{2}, u_j\right)\right) + O(h^{1+s}).
\end{aligned}$$



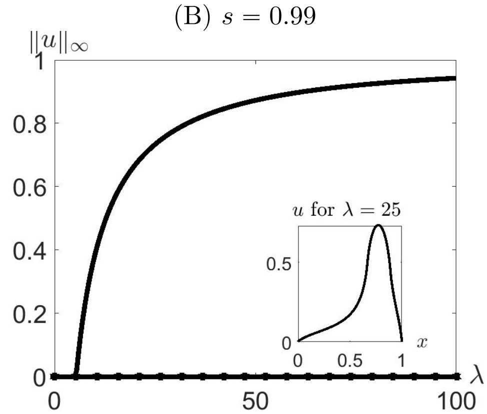
(A) graph of q_1



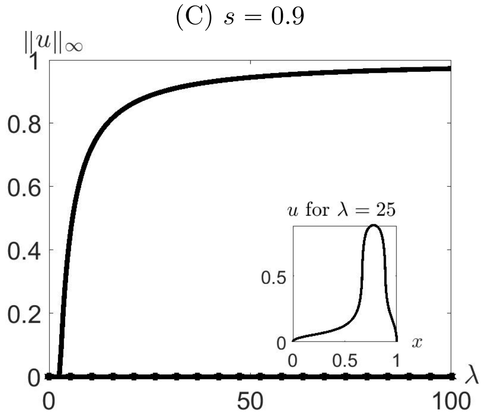
(B) $s = 0.99$



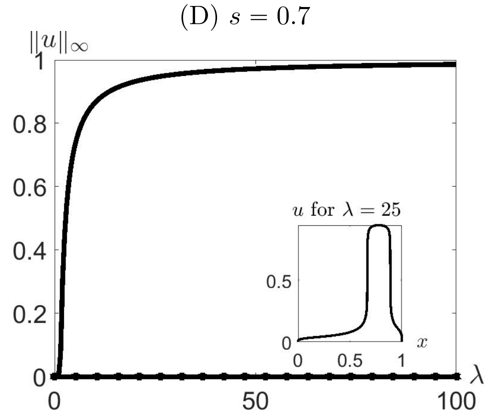
(C) $s = 0.9$



(D) $s = 0.7$



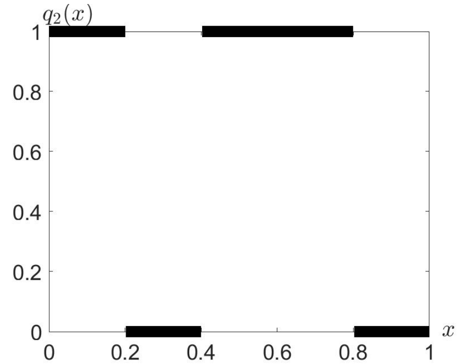
(E) $s = 0.5$



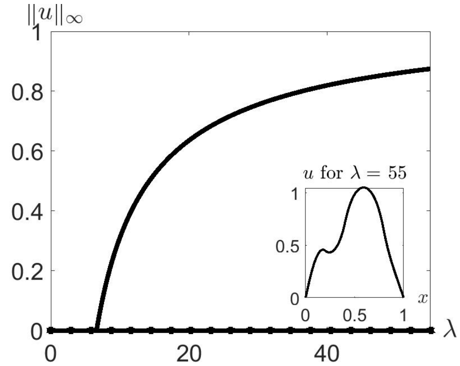
(F) $s = 0.3$

s	0.99	0.9	0.7	0.5	0.3
$\|u\|_{L^1}$	0.118602	0.183006	0.257141	0.274889	0.275915

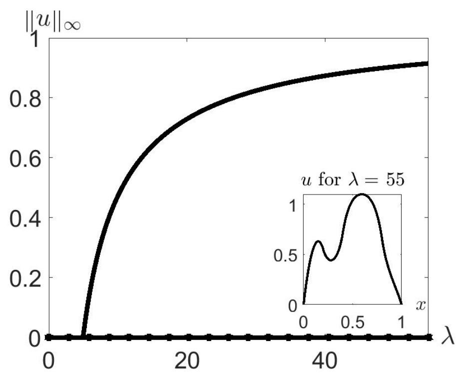
Figure 7.12. Bifurcation Diagrams for $f(x, \sigma) = \sigma(q_1(x) - \sigma)$, Positive Solution, and L^1 Norm of the Solutions with $\lambda = 25$



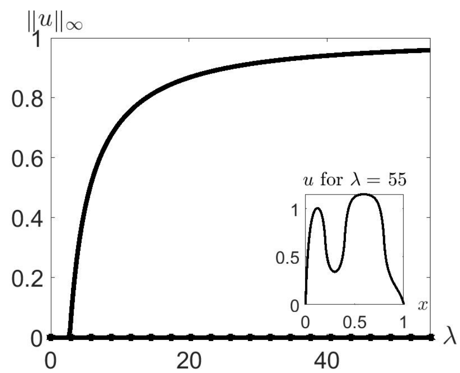
(A) graph of q_2



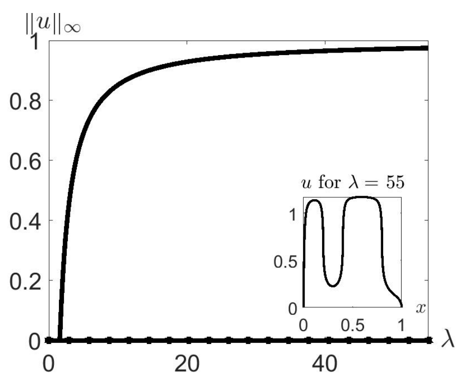
(B) $s = 0.99$



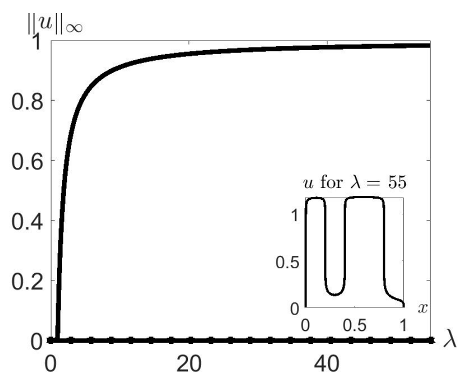
(C) $s = 0.9$



(D) $s = 0.7$



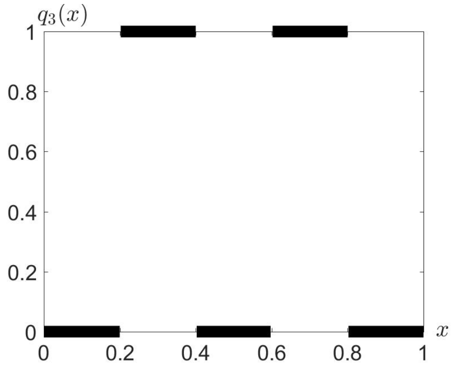
(E) $s = 0.5$



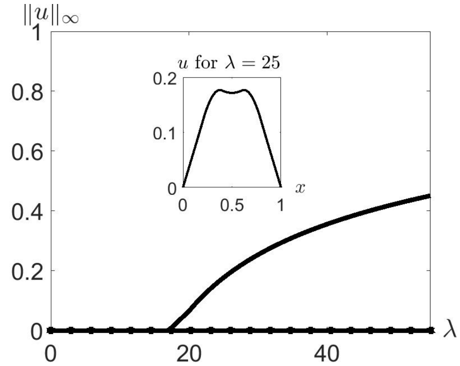
(F) $s = 0.3$

s	0.99	0.9	0.7	0.5	0.3
$\|u\|_{L^1}$	0.416029	0.463028	0.562381	0.617061	0.627710

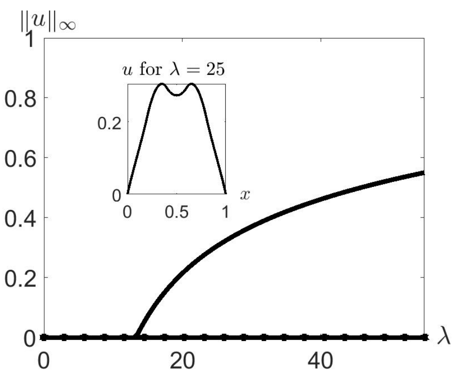
Figure 7.13. Bifurcation Diagrams for $f(x, \sigma) = \sigma(q_2(x) - \sigma)$, Positive Solutions, and L^1 Norm of the Solutions When $\lambda = 55$



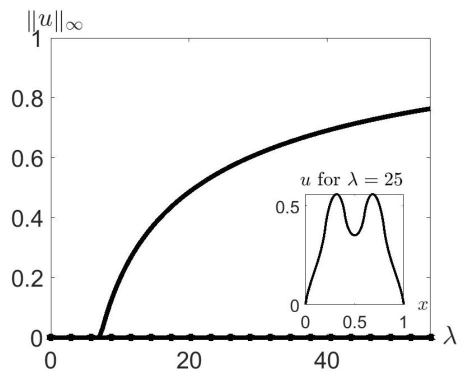
(A) graph of q_3



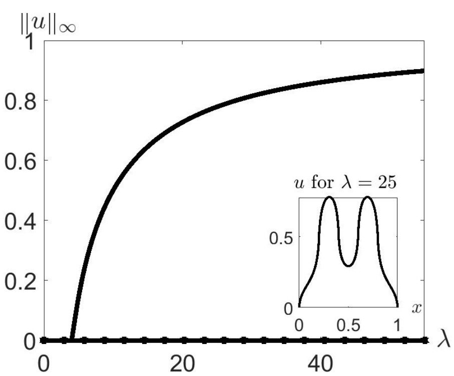
(B) $s = 0.99$



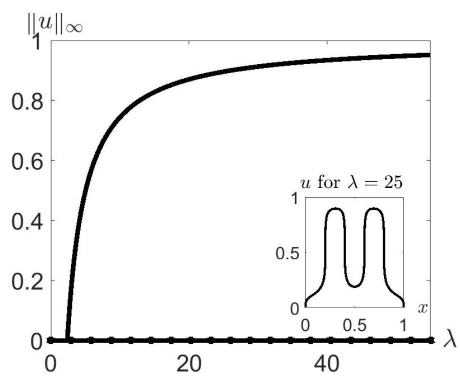
(C) $s = 0.9$



(D) $s = 0.7$



(E) $s = 0.5$



(F) $s = 0.3$

s	0.99	0.9	0.7	0.5	0.3
$\|u\|_{L^1}$	0.122209	0.211938	0.357183	0.433505	0.449811

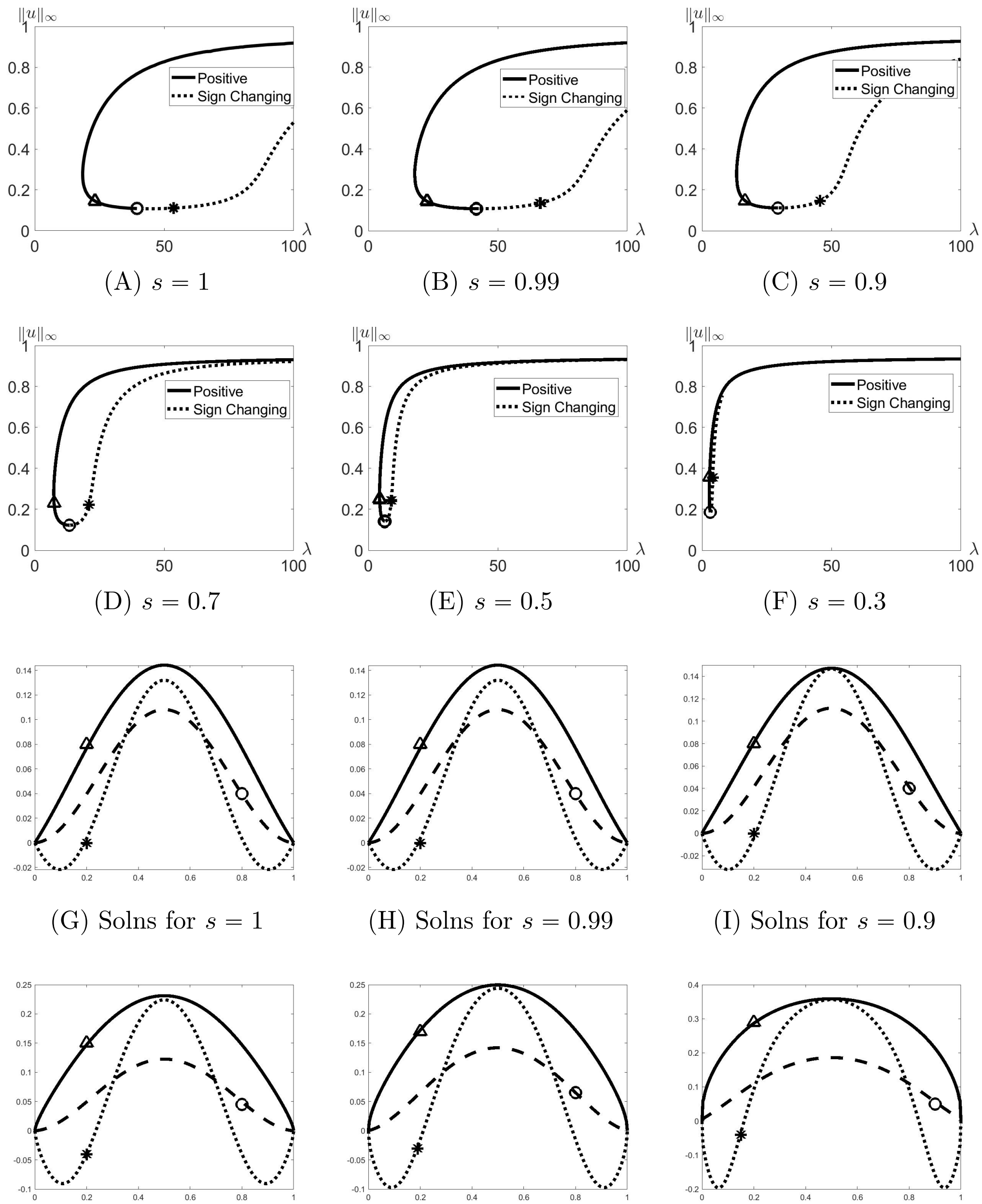
Figure 7.14. Bifurcation Diagrams for $f(x, \sigma) = \sigma(q_3(x) - \sigma)$, Positive Solutions, and L^1 Norm of the Solutions With $\lambda = 25$

7.8. Numerical Experiments Corresponding to Theorem 5.6: N=1

Consider the logistic reaction with constant effort harvesting $f(\sigma) = \sigma(1 - \sigma) - 0.05$ for $\sigma \geqslant 0$ (satisfying hypotheses of Theorem 5.6).

The bifurcation diagrams are given in Figure 7.15 for (A) $s = 1$, (B) $s = 0.99$, (C) $s = 0.9$, (D) $s = 0.7$, (E) $s = 0.5$, and (F) $s = 0.3$ which all retain the qualitative behavior observed for $s = 1$. The solid part of the solution set $\mathcal{S}$ contains positive solutions, and dashed part contains sign changing solutions. On the solution set $\mathcal{S}$, the markers $\triangle$, $\bigcirc$, and $*$ indicate the locations of a positive solution, the last positive solution in the positive λ direction on the lower branch of $\mathcal{S}$, and a sign changing solution in $(0, 1)$, respectively. The locations of $\triangle$ and $*$ are chosen so that $\|\cdot\|_\infty$ of solutions corresponding to these locations are approximately the same but greater than the one for the solution corresponding to $\bigcirc$.

Figure 7.15 (G) - (L) shows the three numerical solutions corresponding to the location of $\triangle$, $\bigcirc$ and $*$ on $\mathcal{S}$ for a given $s \in (0, 1)$. The numerical solution corresponding to $\triangle$ is plotted with a solid line, the solution corresponding to $\bigcirc$ is plotted with a long dashed line, and the solution corresponding to $*$ is plotted with a short dashed line.



(A) $s = 1$ (B) $s = 0.99$ (C) $s = 0.9$

(D) $s = 0.7$ (E) $s = 0.5$ (F) $s = 0.3$

(G) Solns for $s = 1$ (H) Solns for $s = 0.99$ (I) Solns for $s = 0.9$

(J) Solns for $s = 0.7$ (K) Solns for $s = 0.5$ (L) Solns for $s = 0.3$

Figure 7.15. Bifurcation Diagrams for $f(\sigma) = \sigma(1-\sigma) - 0.05$ and Three Solutions for λ Corresponding to $\triangle$, $\bigcirc$, and $*$

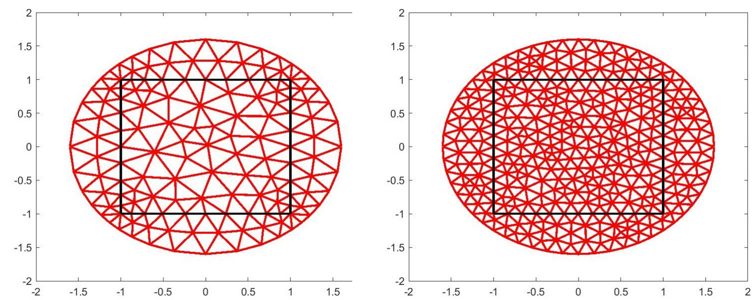
Here we let $\Omega := (-1,1) \times (-1,1) \subset \mathbb{R}^2$ and use the two dimensional finite element method (FEM) developed for linear fractional Laplacian problems of the form (7.1) in [ABB17] for $N = 2$ to construct numerical solutions u (often positive) with $\lambda > 0$ of the nonlinear problem (7.2). As with the case $N = 1$, we use the branch following technique of [NSS06] to construct bifurcation diagrams $\|u\|_\infty$ vs. λ, where $\|u\|_\infty = \|u\|_{L^\infty(\Omega)}$.

For the sake of brevity and the importance of logistic problems in modelling population dynamics, we will only consider the logistic problem and weighted logistic problems in this section. In particular, we show that the size of the population grows monotonically as $s \in (0,1)$ decreases for the cases considered by computing the L^1 norm of numerical solutions.

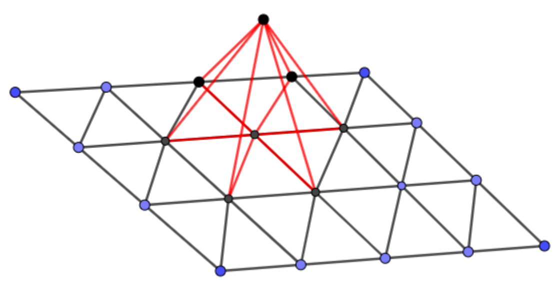
To accommodate the exterior condition on $\mathbb{R}^N \backslash \Omega$, as in [ABB17], it is useful to consider a ball $B \supset \Omega$. Consider a triangulation $\mathcal{T}$ of $B \supset \Omega$ consisting of $N_{\mathcal{T}}$ elements with interior nodes $\{x_1, \ldots, x_n\}$. Given a triangle (*element*) $T \in \mathcal{T}$, denote by h_T and ρ_T it's longest edge length and radius of the largest inscribed circle in T, respectively. Let $h := \max_{T \in \mathcal{T}} h_T$. Further, we assume $\mathcal{T}$ to be shape-regular, that is, there exists $\beta > 0$ such that $h_T \leqslant \beta \rho_T$ for all $T \in \mathcal{T}$. See Figure 7.16 for examples of the triangulation for $h = 0.4$ and $h = 0.2$.

Let V_h be an n - dimensional subset of $H_0^s(\Omega)$ spanned by the piecewise linear basis functions $\{\phi_1, \ldots, \phi_n\}$, where (see Figure 7.16 (C))

$$\phi_i(x_j) := \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{if } i \neq j\,. \end{cases}$$



(A) $h = 0.4$ (B) $h = 0.2$



(C) Basis function ϕ_j

Figure 7.16. Triangulation of $B \supset \Omega$ and Finite Element Basis Functions

The finite element approximation $u_h \in V_h$ of a weak solution $u \in H_0^s(\Omega)$ of (7.2) is expressed as

$$u_h(x) := \sum_{i=1}^{n} u_i \phi_i(x)\,,$$

where $u_i \in \mathbb{R}$ are unknowns and u_h satisfies the system of n equations

$$\frac{C_{2,s}}{2}\mathcal{E}(u_h,\phi) = \lambda \int_{\Omega} f\left(x, u_h(x)\right)\phi_j(x)\mathrm{d}x \tag{7.8}$$

for all $j = 1, \cdots, n$. In order to implement the finite element scheme, we express (7.8) in matrix notation. For a column vector, $\mathbf{u} := [u_1, \cdots, u_n]^T$, the left hand side of (7.8) can be expressed as $\mathcal{A}\mathbf{u}$, where $\mathcal{A}$ is the $n \times n$ stiffness matrix corresponding to the left hand side of (7.8) derived in [ABB17]. To compute the right hand side of (7.8), we use the linear interpolation of f defined by $\Pi f(x,\mathbf{u}) := \sum_{j=1}^{N_{\mathcal{T}}} f(x_j, u_j)\phi_i(x)$, where Π is the projection. Then define the column vector $\mathbf{F}$ with components

$$\begin{aligned}
\mathbf{F}_i := \int_{\Omega} f\left(x, u_h(x)\right)\phi_j(x)\mathrm{d}x &\approx \int_{\Omega} \Pi f\left(x, u_h(x_j)\right)\phi_i(x)\mathrm{d}x \\
&= \int_{\Omega} \sum_{i=1}^{n} f(x_i, u_h(x_j))\phi_i(x)\phi_j(x)\mathrm{d}x \\
&= \sum_{i=1}^{n} f(u_h(x_j)) \int_{\Omega} \phi_i(x)\phi_j(x)\mathrm{d}x\,.
\end{aligned}$$

Therefore $\mathbf{F} = \mathbf{M}\vec{f}$, where $\vec{f}$ is the vector with components defined by $[\vec{f}]_j := f(x_j, u_h(x_j))$, and $\mathbf{M}$ is the standard finite element mass matrix with components defined by

$$[\mathbf{M}]_{i,j} := \int\limits_{\Omega} \phi_i(x)\phi_j(x)\mathrm{d}x\,.$$

The corresponding Jacobian matrix $\mathbf{J}$ of $\mathbf{F}$ is defined componentwise by $[\mathbf{J}]_{i,j} = \mathbf{M}\mathbf{F_t}$, where $\mathbf{F_t}$ is the diagonal matrix defined by $[\mathbf{F_t}]_{i,j} = f_t(x_j, u_j)$ for $i = j$ and $[\mathbf{F_t}]_{i,j} = 0$ otherwise. We rewrite (7.8) as a matrix equation

$$\mathcal{A}\mathbf{u} = \lambda\mathbf{F}(\mathbf{u})\,. \tag{7.9}$$

Then, with $\mathcal{A}$, $\mathbf{F}$, and $\mathbf{J_F}$ defined as above, we solve (7.9) using Algorithm 7.1, adjusted for $N = 2$.

7.9. Numerical Experiments Corresponding to Theorem 5.5: N=2

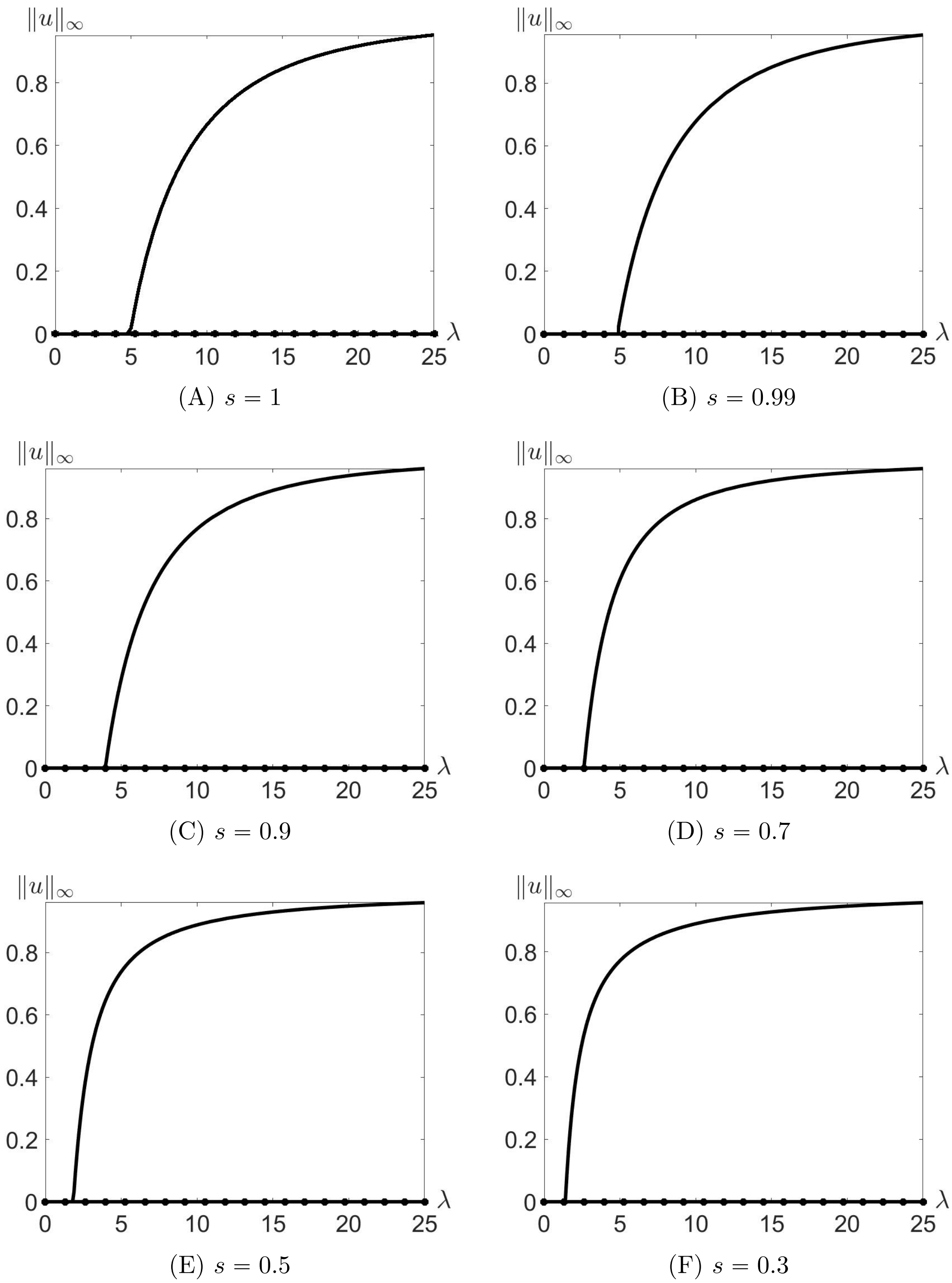
Here we consider the logistic reaction term $f(x,\sigma) = \sigma(q(x) - \sigma)$ with $x \in \Omega$ and $\sigma \geqslant 0$ for several examples of $q : \Omega \to \{0,1\}$. We consider the case $q \equiv 1$ in Ω as well as cases where $q \equiv 1$ on a subset of Ω with positive measure, and $q = 0$ otherwise. We note that for each fixed q, $\lambda_1 := \lambda_1(s,q)$ decreases as $s \in (0,1]$ decreases, where $\lambda_{1,q}$ is the principal eigenvalue of the weighted eigenvalue problem (2.13). Recall that when $s = 1$ and $q \equiv 1$, $\lambda_1 = \frac{\pi^2}{2}$.

The bifurcation diagrams in this section are qualitatively similar to the one dimensional case. In particular, positive solutions bifurcate from the trivial branch of solutions at $\lambda_1 = \lambda_1(s,q,\Omega)$. In Figure 7.17 (for $q \equiv 1$), we see bifurcation diagrams for the cases (A) $s = 1$, (B) $s = 0.99$, (C) $s = 0.9$, (D) $s = 0.7$, (E) $s = 0.5$, and (F) $s = 0.3$.

In each of Figure 7.19 (for $q = q_1$), Figure 7.21 (for $q = q_2$) and Figure 7.23 (for $q = q_3$), (A) shows graph of $q_i (i = 1, 2, 3)$ and (B) $s = 0.99$, (C) $s = 0.9$, (D) $s = 0.7$, (E) $s = 0.5$, and (F) $s = 0.3$ give bifurcation diagrams. We observe that $\|u\|_\infty \to 0^+$ as $\lambda \to \lambda_1^+$ and $\|u\|_\infty \to 1^-$ as $\lambda \to +\infty$.

In each of the Figures 7.18, 7.20, 7.22, 7.24 we plot a numerical positive solution for $\lambda = 11$, $\lambda = 20$, $\lambda = 12$, and $\lambda = 8$ corresponding to each of the bifurcation diagrams in Figures 7.17, 7.19, 7.21, and 7.23, respectively.

To further investigate the impact of $s \in (0, 1)$, we compute the L^1 norm of the numerical positive solutions. We observe that the L^1 norm increases as $s \in (0, 1)$ decreases in all cases considered.

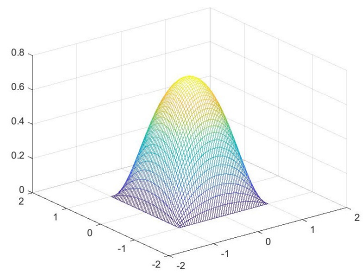


(A) $s = 1$ (B) $s = 0.99$

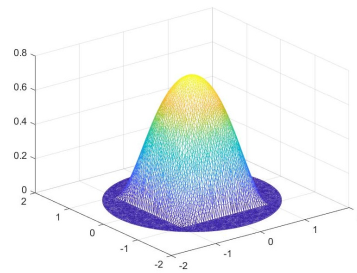
(C) $s = 0.9$ (D) $s = 0.7$

(E) $s = 0.5$ (F) $s = 0.3$

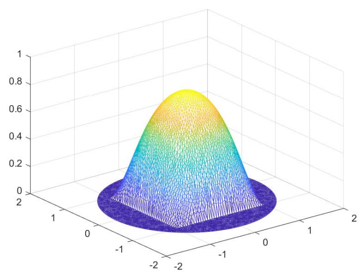
Figure 7.17. Bifurcations Diagrams for $f(\sigma) = \sigma(1 - \sigma)$



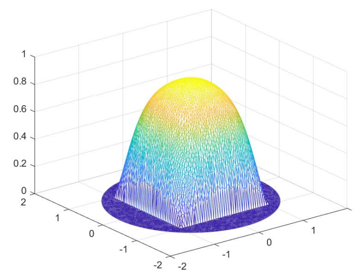
(A) $s = 1$



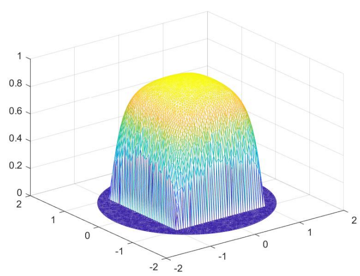
(B) $s = 0.99$



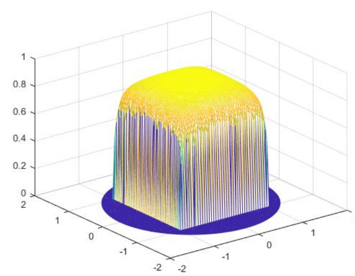
(C) $s = 0.9$



(D) $s = 0.7$



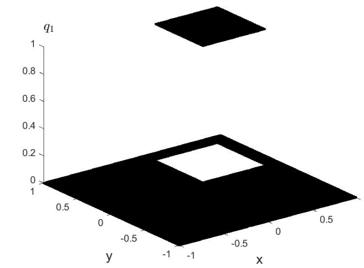
(E) $s = 0.5$



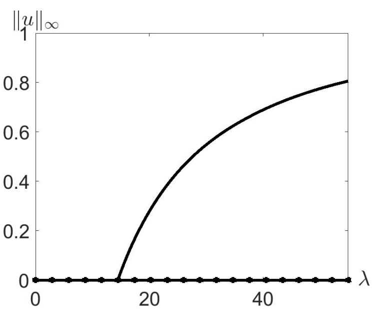
(F) $s = 0.3$

s	1	0.99	0.9	0.7	0.5	0.3
$\|u\|_{L^1}$	1.339337	1.344952	1.635926	2.219306	2.685930	2.969384

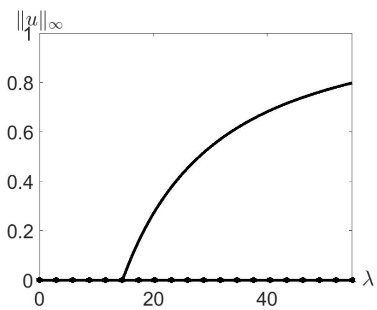
Figure 7.18. Positive Solutions for $f(\sigma) = \sigma(1-\sigma)$ with $\lambda = 11$ and Their L^1 norms



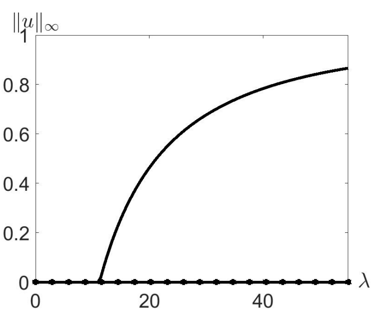
(A) Graph of $q_1(x)$



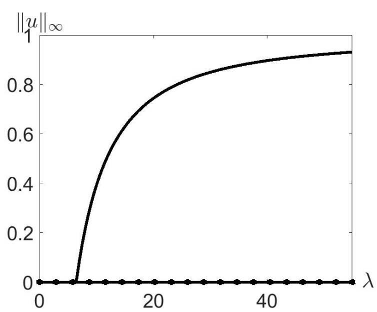
(B) $s = 1$



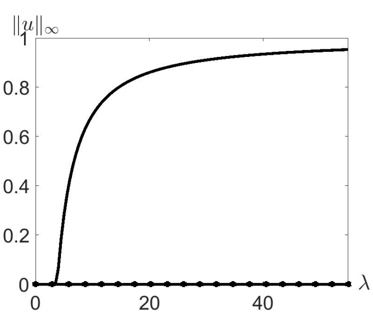
(C) $s = 0.99$



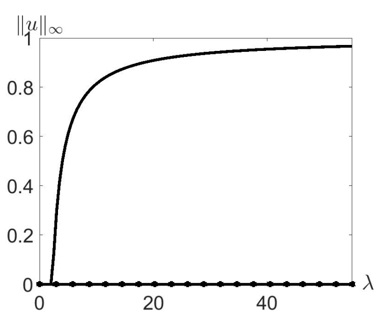
(D) $s = 0.9$



(E) $s = 0.7$

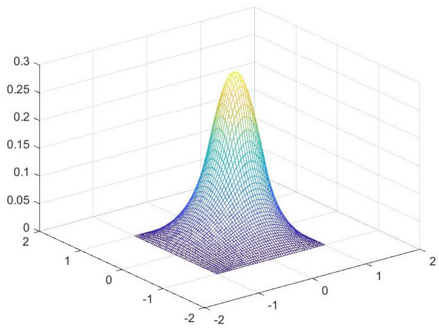


(F) $s = 0.5$

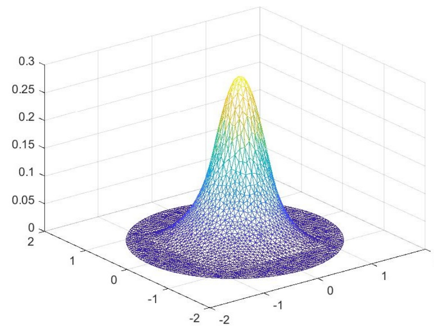


(G) $s = 0.3$

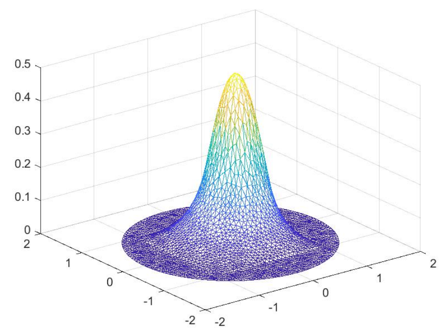
Figure 7.19. $q_1(x)$ and Bifurcation Diagrams for $f(x, \sigma) = \sigma(q_1(x) - \sigma)$



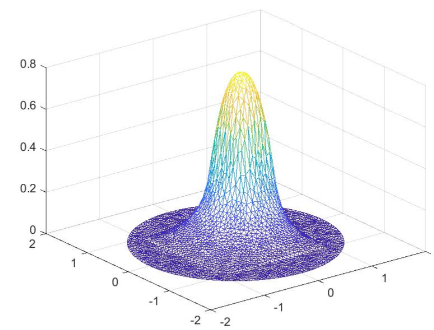
(A) $s = 1$



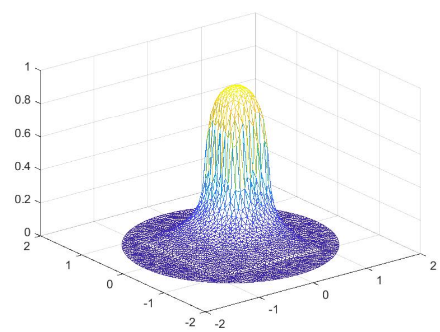
(B) $s = 0.99$



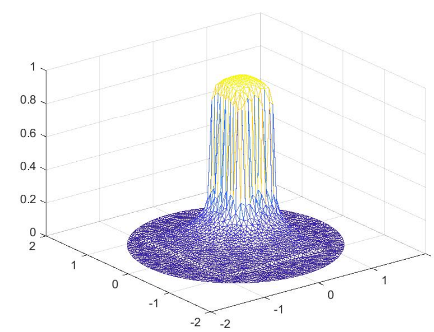
(C) $s = 0.9$



(D) $s = 0.7$



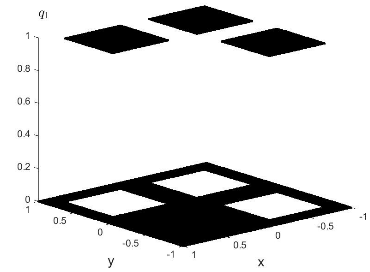
(E) $s = 0.5$



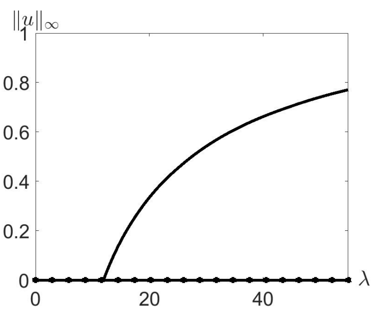
(F) $s = 0.3$

s	1	0.99	0.9	0.7	0.5	0.3
$\|u\|_{L^1}$	0.225890	0.228942	0.354075	0.510228	0.569967	0.574716

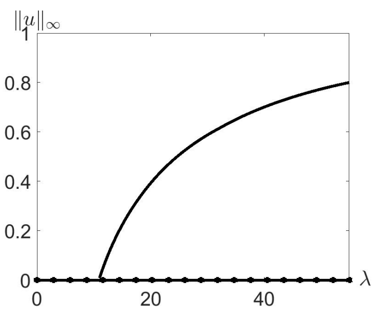
Figure 7.20. Positive Solutions for $f(x, \sigma) = \sigma(q_1(x) - \sigma)$ With $\lambda = 20$ and Their L^1 Norms



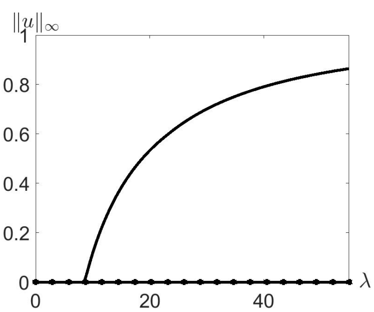
(A) Graph of $q_2(x)$



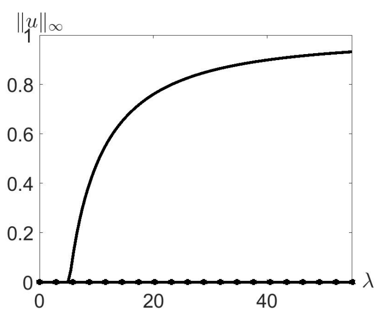
(B) $s = 1$



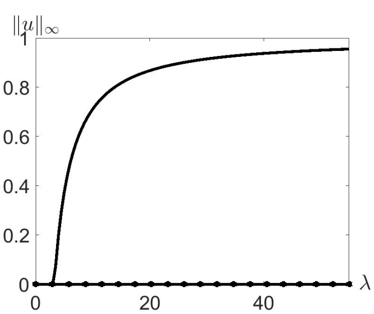
(C) $s = 0.99$



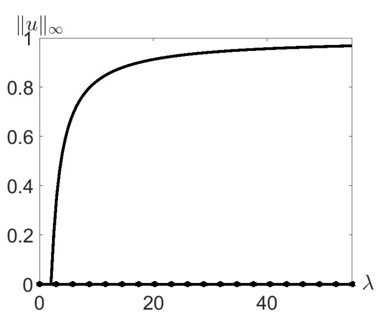
(D) $s = 0.9$



(E) $s = 0.7$

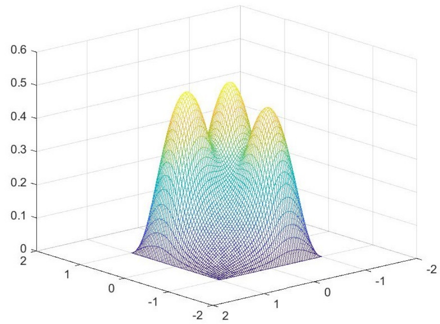


(F) $s = 0.5$

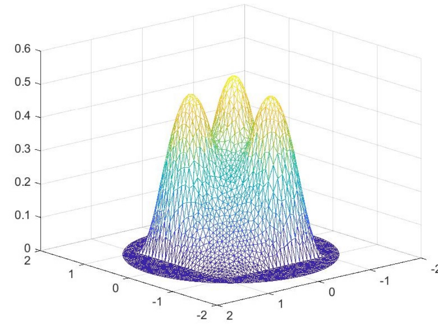


(G) $s = 0.3$

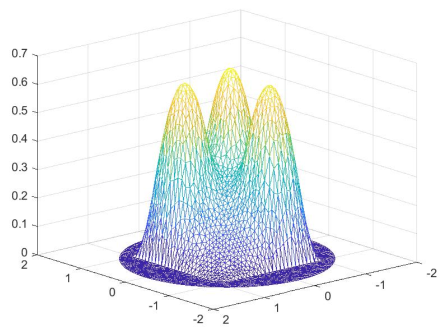
Figure 7.21. $q_2(x)$ and Bifurcation Diagrams for $f(x, \sigma) = \sigma(q_2(x) - \sigma)$



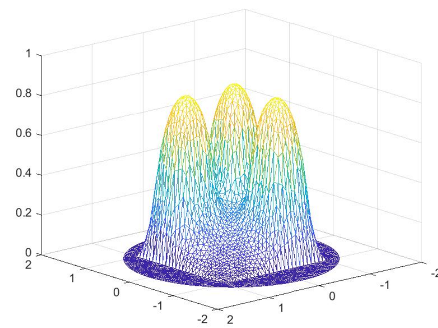
(A) $s = 1$



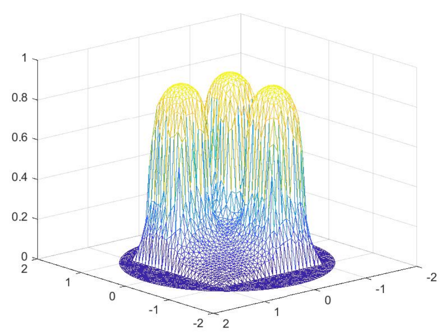
(B) $s = 0.99$



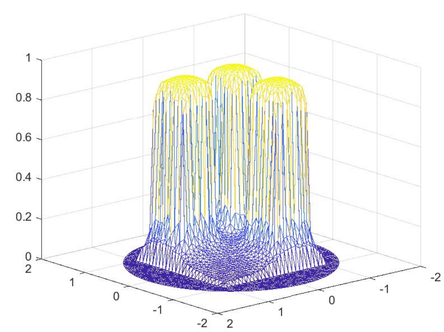
(C) $s = 0.9$



(D) $s = 0.7$



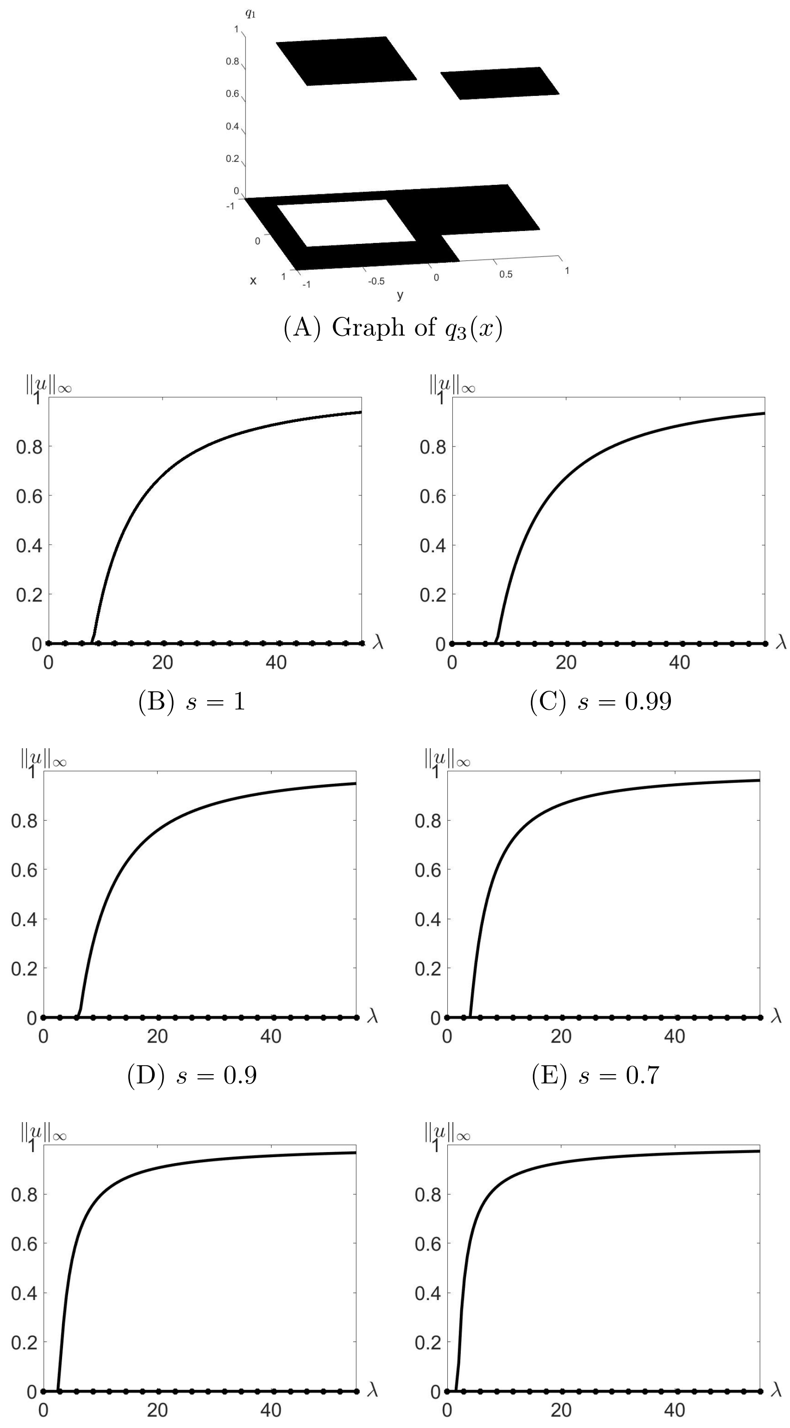
(E) $s = 0.5$



(F) $s = 0.3$

s	1	0.99	0.9	0.7	0.5	0.3
$\|u\|_{L^1}$	0.820375	0.835800	1.034661	1.328953	1.464519	1.470581

Figure 7.22. Positive Solutions for $f(x,\sigma) = \sigma(q_2(x) - \sigma)$ With $\lambda = 25$ and Their L^1 Norms



(A) Graph of $q_3(x)$

(B) $s = 1$

(C) $s = 0.99$

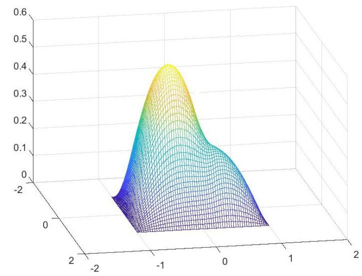
(D) $s = 0.9$

(E) $s = 0.7$

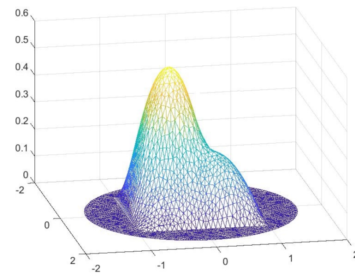
(F) $s = 0.5$

(G) $s = 0.3$

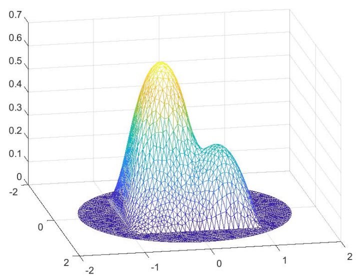
Figure 7.23. $q_3(x)$ and Bifurcation Diagrams for $f(x, \sigma) = \sigma(q_3(x) - \sigma)$



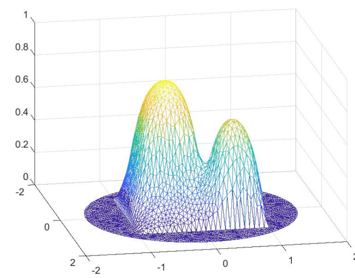
(A) $s = 1$



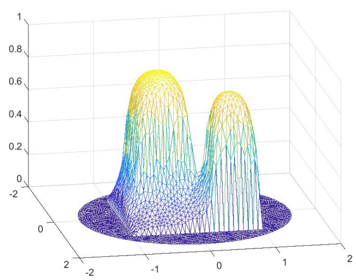
(B) $s = 0.99$



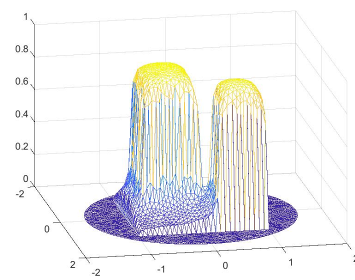
(C) $s = 0.9$



(D) $s = 0.7$



(E) $s = 0.5$



(F) $s = 0.3$

s	1	0.99	0.9	0.7	0.5	0.3
$\|u\|_{L^1}$	0.714398	1.040305	1.317497	1.849060	2.193005	2.328553

Figure 7.24. Positive Solutions for $f(x, \sigma) = \sigma(q_3(x) - \sigma)$ with $\lambda = 15$ and Their L^1 Norms

CHAPTER VIII

CONCLUSIONS AND FUTURE DIRECTIONS

8.1. Conclusions

In this dissertation, we studied nonnegative solutions of reaction-diffusion equations involving the fractional Laplacian as the diffusion operator, with respect to a bifurcation parameter. In particular, we focused on the combined impact of the nonlocal diffusion operator and the behavior of the reaction terms near the origin as well as at infinity, with regards to existence and nonexistence results. To prove our existence results, we used the sub- and supersolution theorem (established in Chapter III) as well as degree theory. We verified all theoretical results obtained in Chapter IV and Chapter V with numerical experiments in Chapter VII that used the finite element method. We gave numerical bifurcation diagrams and showed the profile of numerical positive solutions. The effect of the nonlocal nature of the fractional Laplacian operator $(-\Delta)^s$, in particular the effect of $s \in (0,1)$, is apparent in the numerical bifurcation diagrams and profiles of numerical solutions as $s \to 0^+$. In Chapter VI, we employed the moving plane method to show that every nonnegative solution in a ball is positive, and hence is radially symmetric and radially decreasing

8.2. Future Directions

Based on the theoretical results and numerical experiments discussed in this dissertation and available literature, the following problems are of interest for future work.

1. Extend Theorem 5.3 to bounded nonlinearities.

2. Study the nonexistence of positive solutions for superlinear semipositone problems, as observed in numerical experiments.

3. Investigate the uniqueness of positive solutions for sublinear problems for λ large, as observed in numerical experiments.

4. Investigate the exact global bifurcation diagrams for sublinear, logistic, and superlinear problems as observed in numerical experiments.

5. Study the stability and instability of positive solutions.

6. Investigate the monotonicity of the fractional Laplacian with respect to $s \in (0, 1]$.

7. Extend the results established in Chapter III, Chapter IV and Chapter V to the fractional p-Laplacian, and to the coupled systems of equations.

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APPENDIX A

PROOFS OF LEMMA 2.1 AND PROPOSITIONS 2.2, 5.1

Proof of Lemma 2.1: Let $s \in (0,1)$ and $v \in H_0^s(0,1)$. Then

$$\begin{aligned}\mathcal{E}(v,v) &= \|v\|^2_{H_0^s(0,1)} \\ &\leqslant \|v\|^2_{H^s(\mathbb{R})} \\ &= \int_0^1 |v(x)|^2 \mathrm{d}x + \mathcal{E}(v,v)\,. \end{aligned} \tag{A.1}$$

To establish the reverse inequality, we compute the integral below using $v = 0$ in $\mathbb{R}\backslash(0,1)$

$$\begin{aligned}\mathcal{E}(v,v) &= \int_{\mathbb{R}} \int_0^1 \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} \int_{\mathbb{R}\backslash(0,1)} \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}} \mathrm{d}x\mathrm{d}y \\ &= \left[\int_0^1 \int_0^1 + \int_{\mathbb{R}\backslash(0,1)} \int_0^1 + \int_{\mathbb{R}\backslash(0,1)} \int_{\mathbb{R}\backslash(0,1)} + \int_0^1 \int_{\mathbb{R}\backslash(0,1)} \right] \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}} \mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \int_0^1 \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}} \mathrm{d}x\mathrm{d}y + 2\int_0^1 |v(y)|^2 \, \frac{y^{-2s}+(1-y)^{-2s}}{2s} \mathrm{d}y \\ &\geqslant \int_0^1 |v(y)|^2 \omega(y) \mathrm{d}y\,, \end{aligned}$$

where $\omega(y) := \frac{y^{-2s}+(1-y)^{-2s}}{s}$. Letting $B := \left(\min_{y\in(0,1)} \omega(y)\right)^{-1} > 0$, we obtain

$$\int_0^1 |v(y)|^2 \mathrm{d}y \leqslant B\,\mathcal{E}(v,v)\,. \tag{A.2}$$

Then, combining (A.1) and (A.2), we get $\|v\|_{H^s(\mathbb{R})} \leqslant (1+B)^{1/2}\|v\|_{H_0^s(0,1)}$ as desired. Hence, the two norms are equivalent in $H_0^s(0,1)$. ■

Proof of Proposition 2.2: For $N \geqslant 2$, parts (a)-(c) can be obtained by repeating the argument of [MBRS16, Prop 3.1] with the $L^2(\Omega)$ norm replaced with the weighted L^2 norm $\int_\Omega q(x)|\phi(x)|^2 \mathrm{d}x$ in constructing the eigenvalue $\lambda_{1,q}$ as the Rayleigh quotient given by (2.14). For $N = 1$, these follow from the fact that our definition of $H_0^s(\Omega)$, via $H^s(\mathbb{R}^N)$, allows us to prove the compact embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ without considering an *extension domain* (cf. proof of [DNPV12, Thm. 7.1]). Then $q \in L^\infty(\Omega)$ gives continuous embedding $L^2(\Omega) \hookrightarrow L^2((\Omega); q)$, and hence the principal eigenvalue can be constructed as the Rayleigh quotient given in (2.14).

For part (d), the $C^{1,1}$ assumption on Ω is used to establish the inequalities of (2.16). In particular, the arguments used in establishing the left inequality in [RO16, Lem 7.3] and the right inequality in [ROS14a, Thm. 1.2] apply in this case as well, which are independent of the dimension N.

For part (e), clearly, $\leqslant$ holds in (2.17). Using the definition of the infimum and using the fact that $\varphi_{1,q} \geqslant \delta^s$ a.e. in Ω (after a suitable scaling of $\varphi_{1,q}$) due to (2.16), we find

$$\begin{aligned} \inf_{\substack{\phi \in H_0^s(\Omega) \\ \phi \geqslant \delta^s \text{ a.e. in } \Omega}} & \frac{\mathcal{E}(\phi,\phi)}{\displaystyle\int_\Omega q(x)|\phi(x)|^2 \mathrm{d}x} \\ & \leqslant \frac{\mathcal{E}(\varphi_{1,q},\varphi_{1,q})}{\displaystyle\int_\Omega q(x)|\varphi_{1,q}(x)|^2 \mathrm{d}x} = \lambda_{1,q}\,, \end{aligned}$$

which establishes $\geqslant$ in (2.17), completing part (e). ■

Proof of Proposition 5.1 First we show $0 < h(x) < \infty$. Note that $\varphi_1 > 0$ in Ω and $\varphi_1 \not\equiv \text{const}$. Then,

$$
\begin{aligned}
h(x) &= \int\limits_{\mathbb{R}^N} \frac{(\varphi_1(x) - \varphi_1(y))^2}{|x-y|^{N+2s}} \mathrm{d}y \\
&= \int\limits_{\Omega} \frac{(\varphi_1(x) - \varphi_1(y))^2}{|x-y|^{N+2s}} \mathrm{d}y + \int\limits_{\mathbb{R}^N \setminus \Omega} \frac{(\varphi_1(x) - \varphi_1(y))^2}{|x-y|^{N+2s}} \mathrm{d}y \,.
\end{aligned}
$$

If $x \in \Omega$, then $\varphi_1(x) > 0$, and hence

$$
h(x) \geqslant \int\limits_{\mathbb{R}^N \setminus \Omega} \frac{(\varphi_1(x) - \varphi_1(y))^2}{|x-y|^{N+2s}} \mathrm{d}y = \int\limits_{\mathbb{R}^N \setminus \Omega} \frac{\varphi_1^2(x)}{|x-y|^{N+2s}} \mathrm{d}y > 0 \,.
$$

On the other hand, if $x \in \partial\Omega$, then $\varphi_1(x) = 0$, and hence

$$
h(x) \geqslant \int\limits_{\Omega} \frac{(\varphi_1(x) - \varphi_1(y))^2}{|x-y|^{N+2s}} \mathrm{d}y = \int\limits_{\Omega} \frac{\varphi_1^2(y)}{|x-y|^{N+2s}} \mathrm{d}y > 0 \,.
$$

Therefore $h > 0$ in $\overline{\Omega}$.

Next, we show $h < \infty$. If $0 < s < \frac{1}{2}$, then pick $\alpha > 0$ such that $2s + \alpha < 1$. Then, $\varphi_1 \in C^{2s+\alpha}(B_\rho(x))$ (see [ROS14a, Prop. 2.2],[RO16, Sec. 8]) for $\rho > 0$ such that $B_\rho(x) \subset\subset \Omega$. Let $x \in B_\rho(x)$.

Then, there exists a $C > 0$ such that $|\varphi_1(x) - \varphi_1(y)| \leqslant C|x-y|^{2s+\alpha}$ for all $y \in B_\rho(x)$. Therefore,

$$\begin{aligned}
h(x) &= \int\limits_{\mathbb{R}^N} \frac{[\varphi_1(x) - \varphi_1(y)]^2}{|x-y|^{N+2s}} \mathrm{d}y \\
&= \int\limits_{B_\rho(x)} \frac{[\varphi_1(x) - \varphi_1(y)][\varphi_1(x) - \varphi_1(y)]}{|x-y|^{N+2s}} \mathrm{d}y + \int\limits_{\mathbb{R}^N \setminus B_\rho(x)} \frac{[\varphi_1(x) - \varphi_1(y)]^2}{|x-y|^{N+2s}} \mathrm{d}y \\
&\leqslant \int\limits_{B_\rho(x)} \frac{C|x-y|^{2s+\alpha}[\|\varphi_1\|_\infty + \|\varphi_1\|_\infty]}{|x-y|^{N+2s}} \mathrm{d}y + \int\limits_{\mathbb{R}^N \setminus B_\rho(x)} \frac{[\|\varphi_1\|_\infty + \|\varphi_1\|_\infty]^2}{|x-y|^{N+2s}} \mathrm{d}y \\
&= 2C\|\varphi_1\|_\infty \int\limits_{B_\rho(x)} \frac{1}{|x-y|^{N-\alpha}} \mathrm{d}y + 4\|\varphi\|_\infty^2 \int\limits_{\mathbb{R}^N \setminus B_\rho(x)} \frac{1}{|x-y|^{N+2s}} \mathrm{d}y \\
&= 2C\|\varphi_1\|_\infty \int\limits_{B_\rho(0)} \frac{1}{|z|^{N-\alpha}} \mathrm{d}z + 4\|\varphi_1\|_\infty^2 \int\limits_{\mathbb{R}^N \setminus B_\rho(0)} \frac{1}{|z|^{N+2s}} \mathrm{d}z \,,
\end{aligned}$$

where the last equality holds by the change of variable $z = y - x$. Converting to polar coordinates, the first integral is finite since $1 - \alpha < 1$ implies

$$\int\limits_0^\rho \frac{1}{r^{N-\alpha}} r^{N-1} \mathrm{d}r = \int\limits_0^\rho \frac{1}{r^{1-\alpha}} \mathrm{d}r < \infty \,.$$

Similarly, the second integral is finite since $1 + 2s > 1$ implies

$$\int\limits_\rho^\infty \frac{1}{r^{N+2s}} r^{N-1} \mathrm{d}r = \int\limits_\rho^\infty \frac{1}{r^{1+2s}} \mathrm{d}r < \infty \,.$$

Therefore, $h(x) < \infty$ for all $x \in \Omega$ and $0 < s < \frac{1}{2}$. Now let $\frac{1}{2} \leqslant s < 1$. We first show that φ_1 is Lipschitz continuous in $B_\rho \subset\subset \Omega$.

Without loss of generality, we assume that $0 < \rho \leqslant 1$. Then, for $x, y \in B_\rho$ with $x \neq y$,

$$\begin{aligned}
|\varphi_1(x) - \varphi_1(y)| &= \frac{|\varphi_1(x) - \varphi_1(y)|}{|x-y|^s}|x-y|^s \\
&\leqslant \max_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \left\{ \frac{|\varphi_1(x) - \varphi_1(y)|}{|x-y|^s} \right\} |x-y|^s \\
&\leqslant L|x-y|\,,
\end{aligned}$$

where $L := \max_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \left\{ \frac{|\varphi_1(x)-\varphi_1(y)|}{|x-y|^s} \right\} < \infty$ since $\varphi_1 \in C^s(\overline{\Omega})$. Thus, φ_1 is Lipshitz continuous in B_ρ. Now we show $h(x) < \infty$ for $\frac{1}{2} \leqslant s < 1$. Using the discussion above, for $x \in \Omega$, we have

$$\begin{aligned}
h(x) &= \int\limits_{B_\rho(x)} \frac{[\varphi_1(x) - \varphi_1(y)]^2}{|x-y|^{N+2s}} \mathrm{d}y + \int\limits_{\mathbb{R}^N \setminus B_\rho(x)} \frac{[\varphi_1(x) - \varphi_1(y)]^2}{|x-y|^{N+2s}} \mathrm{d}y \\
&\leqslant L \int\limits_{B_\rho(x)} \frac{|x-y|^2}{|x-y|^{N+2s}} \mathrm{d}y + \int\limits_{\mathbb{R}^N \setminus B_\rho(x)} \frac{[\|\varphi_1\|_\infty + \|\varphi_1\|_\infty]^2}{|x-y|^{N+2s}} \mathrm{d}y \\
&\leqslant L \int\limits_{B_\rho(x)} \frac{1}{|x-y|^{N+2s-2}} \mathrm{d}y + 4\|\varphi_1\|_\infty^2 \int\limits_{\mathbb{R}^N \setminus B_\rho(x)} \frac{1}{|x-y|^{N+2s}} \mathrm{d}y\,. \qquad \text{(A.3)}
\end{aligned}$$

Letting $z = y - x$ and using polar coordinates, the first integral in (A.3) is finite since $\frac{1}{2} \leqslant s < 1$ implies $1 - 2s \geqslant 0$, and hence,

$$\int\limits_0^\rho \frac{1}{r^{N+2s-2}} r^{N-1} \mathrm{d}r = \int\limits_0^\rho r^{1-2s} \mathrm{d}r < \infty\,.$$

Similarly, the second integral in the right hand side of (A.3) is finite since $1 + 2s > 1$ implies

$$\int_{\rho}^{\infty} \frac{1}{r^{N+2s}} r^{N-1} \mathrm{d}r = \int_{\rho}^{\infty} \frac{1}{r^{1+2s}} \mathrm{d}r < \infty \,.$$

Therefore, $h(x) < \infty$ for all $x \in \Omega$ and $s \in (0,1)$. Finally, we show that there exists $\gamma > 0$ such that $h(x) > \gamma$ in Ω. Suppose not. Then there exists $x_n \subset \Omega$ such that $x_n \to x_0 \in \overline{\Omega}$ with $h(x_n) \to 0$. Letting

$$z_n := \frac{[\varphi_1(x_n) - \varphi_1(y)]^2}{|x_n - y|^{N+2s}}$$

be a sequence of nonnegative measurable functions, we get $h(x_n) = \int_{\mathbb{R}^N} z_n \, \mathrm{d}y$. By Fatou's lemma, $\liminf z_n$ is measurable and $h(x_0) \leqslant \liminf \int_{\mathbb{R}^N} z_n \, \mathrm{d}y = \liminf h(x_n) = 0$, a contradiction to the fact that $h(x) > 0$ on $\overline{\Omega}$. Therefore, there exists $\gamma > 0$ such that $\gamma < h(x) < +\infty$ in Ω. This completes the proof of Proposition 5.1. ■

APPENDIX B

PROOFS OF THEOREMS 5.1 - 5.2

Proof of Theorem 5.1 Since $f(0) > 0$, it follows that $\underline{u} \equiv 0 \in H_0^s(\Omega)$ is a weak subsolution of (5.1). Now let $\lambda > 0$ be fixed and $e \in H_0^s(\Omega)$ be the positive weak solution of (2.9). Then as in the proof of Theorem 5.3, there exists $M_\lambda > 0$ such that $\overline{u} := Me$ is a weak supersolution of (5.1) for all $M \geqslant M_\lambda$.

Clearly $\overline{u} = Me \geqslant 0 = \underline{u}$ a.e. in Ω. Hence, by Theorem 3.1, there exists a positive weak solution u of (5.1) such that $0 \leqslant u \leqslant \overline{u}$ a.e. in Ω for all $\lambda > 0$. Moreover, $u \not\equiv 0$ since $f > 0$. Thus $0 < u$ in Ω by Proposition 2.5. This completes the proof of Theorem 5.1. ∎

Proof of Theorem 5.2 Let $\lambda > \frac{\lambda_1}{f'(0)}$ be fixed, where $\lambda_1 > 0$ is the principal eigenvalue of (2.11) and $0 < \varphi_1 \in H_0^s(\Omega)$ is the corresponding eigenfunction. Since $f(0) = 0$, $u \equiv 0$ is a solution and hence a subsolution of (5.1). So, to complete the proof, we must construct a positive weak subsolution. We show that an appropriate constant multiple of φ_1 is a weak subsolution of (5.1). We find this constant by analyzing the function $\Theta(\sigma) := \lambda_1\sigma - \lambda f(\sigma)$ for $\sigma \geqslant 0$. Clearly $\Theta(0) = 0$ and $\Theta'(\sigma) = \lambda_1 - \lambda f'(\sigma)$. Therefore, $\Theta'(0) < 0$, since $\lambda > \frac{\lambda_1}{f'(0)}$, and hence, there exists $\theta(\lambda) > 0$ such that $\Theta(\sigma) < 0$ for any $\sigma \in (0, \theta(\lambda))$.

We show that $\underline{u} := m\varphi_1 \in H_0^s(\Omega)$ is a positive weak subsolution of (5.1) for any $m \in (0, m_\lambda)$, where $m_\lambda := \frac{\theta(\lambda)}{\|\varphi_1\|_\infty}$.

Indeed, by using the weak formulation of the eigenvalue problem (2.11) and the discussion above, $\underline{u}$ satisfies

$$\begin{aligned}\mathcal{E}(\underline{u}, \phi) &= m\mathcal{E}(\varphi_1, \phi) \\ &= \lambda_1 \int_\Omega m\varphi_1(x)\phi(x)\mathrm{d}x \\ &\leqslant \lambda \int_\Omega f(m\varphi_1)\phi(x)\mathrm{d}x \\ &= \lambda \int_\Omega f(\underline{u})\phi(x)\mathrm{d}x\end{aligned}$$

for all $\phi \in H_0^s(\Omega)$ with $\phi \geqslant 0$ in Ω. Hence for any $\lambda > \frac{\lambda_1}{f'(0)}$ and any $m \in (0, m_\lambda)$, $\underline{u} = m\varphi_1$ is a positive weak subsolution of (5.1).

As in the proof of Theorem 5.3, for any $\lambda > \frac{\lambda_1}{f'(0)}$, there exists $M_\lambda > 0$ such that for $M \geqslant M_\lambda$, the function $\overline{u} = Me \in H_0^s(\Omega)$ is a weak supersolution of (5.1). Using the left estimate of e in (2.10) and the right estimate of φ_1 in (2.12), and by choosing either M sufficiently large or m sufficiently small, we get $\underline{u} = m\varphi_1 \leqslant Me = \overline{u}$ a.e. in Ω. Hence, by Theorem 3.1, (5.1) has a positive weak solution u for any $\lambda > \frac{\lambda_1}{f'(0)}$ such that $\underline{u} \leqslant u \leqslant \overline{u}$ a.e. in Ω. This completes the proof of Theorem 5.2. ∎