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A topological space (X,T') is said to be an expansion of the topological space (X,T) if $T \subset T'$. A connected topological space is said to be maximally connected if every proper expansion of the space is disconnected. During recent years there has been much discussion in the literature concerning maximally connected topological spaces, especially maximally connected Hausdorff spaces. This thesis is a survey of some of the recently published results and some of the folklore results on this topic.

MAXIMALLY CONNECTED

TOPOLOGICAL

SPACES

by

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TABLE OF CONTENTS

INTRODUCTION

A connected topological space is called maximally connected if every proper expansion of this space is disconnected. The purpose of this thesis is to investigate the existance of and the properties of maximally connected topological spaces.

For many years it was unknown whether or not there existed maximally connected Hausdorff spaces. Particular interest concerning maximally connected Hausdorff spaces settled around the existance of a maximally connected expansion of the usual topology on the set of real numbers. Recently, in fact during the writing of this thesis, this problem was solved independently by J. A. Guthrie, H. E. Stone, and Wage. All three men have constructed examples of maximally connected Hausdorff spaces. Regretably, none of these have been published at the time of this writing.

In chapter I, certain definitions and results regarding the relationship between ultrafilters and maximally connected topologies are considered. It is true that an ultrafilter on a set always defines a maximally connected topology on the set. However, the converse of this statement is not true, for otherwise there cannot exist maximally connected Hausdorff spaces.

In chapter II, properties not possessed by maximally connected topological spaces are considered. It is proved that no metric space is maximally connected. Special emphesis is given to Hausdorff spaces and it is shown that maximally connected Hausdorff spaces cannot possess basic properties such as being first countable, compact, or locally compact.

Finally in chapter III, properties possessed by maximally connected topological spaces in general are considered. It is shown that all maximally connected topological spaces are T_0 and submaximal, and that product and quotient spaces of maximally connected topological spaces are not necessarily maximally connected. It is also shown that maximal connectedness is hereditary to connected subspaces, and that maximal connectedness is preserved by open, connected, monotone functions.

A working knowledge of set theory and elementary topology is assumed. The reader is referred to [1], [2], [3], and [6] for definitions and theorems not covered in this thesis. The closure of a set A with respect to a topology T will be denoted by $cl(A,T)$. Also, Z^+ will be used to denote the set of positive integers.

CHAPTER I

ULTRAFILTERS AND MAXIMALLY CONNECTED TOPOLOGICAL SPACES

Definition 1.1: An ordered pair (X,R) is said to be partially ordered provided X is a set and R is a subset of $X \times X$ such that:

(1) if $x \in X$, then $(x,x) \in R$;

(2) if $x,y \in X$, then if $(x,y) \in R$ and $(y,x) \in R$, then $x = y$; and

(3) if $x,y,z \in X$, then if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$.

The relation R is called a partial ordering on X and X is said to be a partially ordered set. If (X,R) is partially ordered, then $(a,b) \in R$ will be denoted by aRb.

Lemma 1.1: If C is a collection of sets, then the inclusion relation, $\subset$, is a partial ordering on C.

Definition 1.2: An ordered pair (X,R) is said to be linearly ordered provided X is a set and R is a subset of $X \times X$ such that:

(1) if $x \in X$, then $(x,x) \notin R$;

(2) if $x,y \in X$ and $x \neq y$, then either $(x,y) \in R$ or $(y,x) \in R$; and

(3) if x,y $z \in X$, then if $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$.

The relation R is called a linear ordering on X and X is said to be a linearly ordered set.

Definition 1.3: Let X be a partially ordered set and let R be a partial ordering on X. An element m in X is said to be maximal if n in X, $n \neq m$, implies mRn is false.

Zorn's Lemma 1.2: If in a nonempty partially ordered set P, each linearly ordered subset of P has an upper bound, then P has a maximal element.

Definition 1.4: Let X be a nonempty set. A filter, F, on X is a nonempty set of nonempty subsets of X, which satisfies the following properties:

(1) any intersection of a finite number of sets in F is also in F; and

(2) if $A \in F$ and B is a subset of X which contains A, then $B \in F$.

Definition 1.5: A topology (filter) T′ on a set X is said to be finer than a topology (filter) T on X if $T \subset T'$. If in addition we have $T \neq T'$ we say that T′ is strictly finer than T.

Definition 1.6: Let X be a nonempty set. An ultrafilter, U, on X is a filter on X which has the property that no filter on X is strictly finer than U.

Theorem 1.1: Every filter on a nonempty set X is contained in an ultrafilter on X.

Proof: Let X be a nonempty set and let F be a filter on X. Let C be the set of all filters on X which contain F. Since $F \in C$, then C is nonempty. By lemma 1.1, C is partially ordered by the set inclusion relation. Let L be a linearly ordered subset of C. Let $K = \cup\{W | W \in L\}$. Since any W in L is a filter on X, then

$\phi \notin W$; hence $\phi \notin K$. If A and B are distinct elements of K, then there exists filters F_A and F_B in L such that $A \in F_A$ and $B \in F_B$. Since L is a linearly ordered set, then either $F_A \subseteq F_B$ or $F_B \subseteq F_B$. Let $F_A \subseteq F_B$. Hence, $A \in F_B$ and $B \in F_B$ and since F_B is a filter, then $A \cap B \in F_B$. Also, since $F_B \in L$, then $A \cap B \in K$.

Let $P \in K$ and let $G \subset X$ such that $P \subset G$. Since $P \in K$, then there exists a filter $F_P \in L$ such that $P \in F_P$. Since F_P is a filter, then $G \in F_P$. Also, since $F_P \in L$, then $G \in K$. Hence, K is a filter on X.

Since $K = \cup\{W | W \in L\}$, then K contains every filter in L; hence K is an upper bound for L. Therefore, by Zorn's lemma, C has a maximal element, M. Since $M \in C$, then M is a filter containing F. Suppose H is a filter on X such that $M \subseteq H$. Since $F \subset M$, then $F \subset H$, which implies that $H \in C$. But this contradicts the fact that M is the maximal element of C. Therefore, H must be M. Since there is no filter on X which is strictly finer than M, then M must be an ultrafilter. Hence, F is contained in an ultrafilter.

<u>Remark 1.1</u>: Notice that theorem 1.1 guarantees the existance of an ultrafilter.

<u>Definition 1.7</u>: Let X be a set and let F be a filter on X. A class $B \subset F$ is called a <u>filter base</u> for F if and only if every set in F contains a set in B.

<u>Theorem 1.2</u>: Let B be a nonempty class of subsets of a nonempty set X. Then B is a base for some filter on X if and only if it possesses the following two properties:

(1) the empty set is not in B; and

(2) for any $A, C \in B$, there is a $D \in B$ such that $D \subset A \cap C$.

Proof: Let X be a nonempty set. Let F be a filter on X. Let B be a nonempty class of subsets of X such that B is a filter base for F. Then $B \subset F$ and every set in F contains a set in B. Since F is a filter, then $\phi \notin F$; hence $\phi \notin B$. Let $G_1, G_2 \in B$. Since $B \subset F$, then G_1 and G_2 are elements of F. Since F is a filter, then $G_1 \cap G_2 \in F$. Hence, there exists a set $Z \in B$ such that $Z \subset G_1 \cap G_2$. Therefore, B satisfies properties (1) and (2) above.

Let X be a nonempty set and let B be a nonempty class of subsets of X such that $\phi \notin B$ and for any $A, C \in B$, there is a $D \in B$ such that $D \subset A \cap C$. Let $F = \{K \subset X \mid \text{there is a } G \in B \text{ such that } G \subset K\}$. Since B is nonempty, then there exists a $H \in B$ and since $H \subset X$ and $H \subset H$, then $H \in F$; hence F is a nonempty set. Let $M, N \in F$. Then there exists $U, V \in B$ such that $U \subset M$ and $V \subset N$. Since $U, V \in B$, then there is a $P \in B$, such that $P \subset (U \cap V) \subset (M \cap N)$. Since $P \subset M \cap N$ and $P \in B$, then $M \cap N \in F$. Let $A \in F$ and let $S \subset X$ such that $A \subset S$. Since $A \in F$, then there is a $W \in B$ such that $W \subset A$ and since $A \subset S$, then $W \subset S$; hence $S \in F$. Thus, F is a filter on X. By the definition of F it is guaranteed that every set in F contains a set in B. Hence, B is a filter base for the filter F.

Theorem 1.3: Let X be a nonempty set. A filter F on X is an ultrafilter if and only if for each $A \subset X$, either $A \in F$ or $X-A \in F$.

Proof: Let X be a nonempty set. Let F be a filter on X such that if $A \subset X$, then either $A \in F$ or $X-A \in F$. Suppose F' is a filter on X such that F' is strictly finer than F. Then there is an $A' \in F'$ such that $A' \notin F$. Since $A' \notin F$, then $X-A' \in F \subseteq F'$. But then A' and $X-A'$ are both members of F'. Since F' is a filter then $A' \cap (X-A') \in F'$. But $A' \cap (X-A') = \phi$, which contradicts the supposition that F' is a filter. Hence, there is no filter on X which is strictly finer than F. Thus, F is an ultrafilter on X.

Let X be a nonempty set and let F be an ultrafilter on X. Assume that A is a nonempty subset of X such that neither A nor $X-A$ is a member of F. Let $B = \{A \cap C | C \in F\}$. Since F is a filter, then F is nonempty; so let $U \in F$. Since F is a filter, then $U \neq \phi$. Suppose $U \cap A = \phi$. Then $U \subset X-A$. Since F is a filter and $X-A$ is a superset of U, then $X-A \in F$. But this contradicts the fact that neither A nor $X-A$ is a member of F. Hence, $U \cap A \neq \phi$. But $U \cap A \in B$; hence B is nonempty. Since $\phi \notin F$ and $A \neq \phi$ and $V \cap A \neq \phi$ for any $V \in F$, then $\phi \notin B$. Let $J, K \in B$. Then there are $C_1, C_2 \in F$ such that $J = C_1 \cap A$ and $K = C_2 \cap A$. Hence, $J \cap K = (C_1 \cap A) \cap (C_2 \cap A) = (C_1 \cap C_2) \cap A$. Since F is a filter, then $C_1 \cap C_2 \in F$. Hence, $J \cap K = (C_1 \cap C_2) \cap A \in B$. Therefore, B is a filter base for some filter on X. In fact, by theorem 1.2, B is a filter base for the filter

$F' = \{Y \subset X \mid \text{there is a } D \in B \text{ such that } D \subset Y\}$. Let $P \in F$. Then $P \cap A \in B \subset F'$. Since F' is a filter and $P \cap A \subset P$, then $P \in F'$. Hence, $F \subset F'$. But $P \cap A$ is also contained in A which implies that $A \subset F'$. Now $A \not\in F$, which implies that $F \neq F'$. Hence, F' is strictly finer than F, which is a contradiction to the fact that F is an ultrafilter. Thus, for every $A \subset X$, either $A \in F$ or $X-A \in F$.

<u>Theorem 1.4</u>: Let X be a nonempty set. If F is a filter on X, then $S = F \cup \{\phi\}$ is a topology on X.

Proof: Let X be a nonempty set and let F be a filter on X. Let $S = F \cup \{\phi\}$. By definition, $\phi \in S$. Since F is a filter, then F is nonempty and $\phi \not\in F$. Hence, there is a nonempty set $G \subset X$ such that $G \in F$. Since F is a filter, then $X \in F$, which implies that $X \in S$. Let C be any nonempty set of elements of S. Let $K = \cup\{U \mid U \in C\}$. If ϕ is the only element of C, then $K = \phi \in S$. If ϕ is not the only element of C or if $\phi \not\in C$, then there exists a $V \in C$ such that $V \neq \phi$ and $V \in F$. Since $V \in C$, then $V \subset K$. Since F is a filter and $V \in F$, then $K \in F$; hence $K \in S$. Let $A,B \in S$. If both A and B are in F, then $A \cap B \in F$, which implies that $A \cap B \in S$. If either A or B is empty, then $A \cap B = \phi \in S$. Hence, $A \cap B \in S$. Thus, $S = F \cup \{\phi\}$ is a topology on X and S is said to be defined by the filter F.

<u>Definition 1.8</u>: A topological space (X,T) is said to be <u>connected</u> if X is not the union of two nonempty, disjoint, open sets. If X is a set with topology T we will say that T is connected if (X,T) is a connected topological space.

Theorem 1.5: Let X be a nonempty set. If F is a filter on X and $S = F \cup \{\phi\}$, then (X,S) is a connected topological space.

Proof: Let X be a nonempty set and let F be a filter on X. Let $S = F \cup \{\phi\}$. By theorem 1.4, S is a topology on X. Assume (X,S) is disconnected. Then X can be expressed as the union of two nonempty, disjoint, open sets, A and B, from S. Since A and B are nonempty then A and B are elements of F. Since F is a filter, then $A \cap B \in F$. But $A \cap B = \phi$, which contradicts the fact that F is a filter. Hence, (X,S) is a connected topological space.

Definition 1.9: A topological space (X,T) is said to be maximally connected if (X,T) is connected and if T' is any topology on X which is strictly finer than T, then (X,T') is not connected.

Theorem 1.6: Let X be a nonempty set. If U is an ultrafilter on X and if $S = U \cup \{\phi\}$, then (X,S) is a maximally connected topological space.

Proof: Let X be a nonempty set. Let U be an ultrafilter on X. Let $S = U \cup \{\phi\}$. By theorems 1.4 and 1.5, S is a connected topology on X. Let T be a topology on X such that T is strictly finer than S. Then there is a $V \in T$ such that $V \notin S$. Since $V \notin S$, then $V \notin U$. Hence, by theorem 1.3, $X-V \in U$. Since $U \subset S$ and $S \subset T$, then $X-V \in T$. Now $V \cap (X-V) = \phi$ and $V \cup (X-V) = X$. Since S and T are topologies, then both S and T contain ϕ and X. Now, $V \notin S$ implies that $V \neq \phi$ and $V \neq X$. Since $V \neq X$, then $X-V \neq \phi$. Hence, V and $X-V$ are nonempty, disjoint, open sets in T. Hence, T is disconnected. Thus, any topology on X which is strictly finer than S is disconnected. Therefore, (X,S) is a maximally connected topological space.

Remark 1.2: A topological space (X,T) having the property that if $A \subset X$, then either $A \in T$ or $X-A \in T$ is called a door space. This is appropriate since every subset of X has the property that it must be either open or closed. Notice that in theorem 1.3 it is determined that if X is a set and U is an ultrafilter on X, then for each $A \subset X$, either $A \in U$ or $X-A \in U$. Hence, a topology defined by an ultrafilter is a door space and, in fact, these are equivalent notions. Therefore, since any topology defined by an ultrafilter is maximally connected, as determined in theorem 1.6, then any door space is maximally connected. However, it is not true that every maximally connected topological space is a door space, as is exhibited in example 1.3.

Example 1.1: MAXIMALLY CONNECTED T_0-SPACE.

Let X be a nonempty set and let $p \in X$. Let $U = \{V \subset X | p \in V\}$. Since $p \notin \phi$, then $\phi \notin U$. Since $p \in X$ and $p \subset \{p\} \subset X$, then $\{p\} \subset U$. Hence, U is nonempty. Let $A,B \in U$. Then $p \in A$ and $p \in B$; hence $p \in A \cap B \subset X$, which implies that $A \cap B \in U$. Let $D \in U$ and let E be a subset of X which contains D. Since $D \in U$, then $p \in D$ and since $D \subset E$, then $p \in E$. Hence, $E \in U$. Thus, U is a filter on X. Let $F \in X$. Then either $p \in F$ or $p \notin F$. If $p \in F$, then $F \in U$. If $p \notin F$, then $p \in X-F$; hence $X-F \in U$. Therefore, either $F \in U$ or $X-F \in U$. Thus, by theorem 1.3, U is an ultrafilter on X. Therefore, by theorems 1.4, 1.5, and 1.6, $T = U \cup \{\phi\}$ is a maximally connected topology on X.

Let x and y be distinct points in X. Then $\{x,p\}$ and $\{y,p\}$ are open sets in T. So, $\{x,p\}$ is an open set which contains x but not y. Hence, for every pair of distinct points in X, there is an open set containing one of the points which does not contain the other. Thus, (X,T) is a T_0-space. Therefore, (X,T) is a maximally connected T_0-space.

Example 1.2: MAXIMALLY CONNECTED T_1-SPACE.

Let X be an infinite set. Let $F = \{V \subset X | X-V \text{ is finite}\}$. Let $x \in X$. Then $\{x\} \subset X$ and $\{x\} = X-(X-\{x\})$ is finite. Hence, $X-\{x\} \in F$, which implies that F is nonempty. Since $X-\phi = X$ and X is infinite, then $\phi \notin F$. Let $A,B \in F$. Then, by DeMorgan's laws, $X-(A \cap B) = (X-A) \cup (X-B)$. Since A and B are in F, then $X-A$ and $X-B$ are finite subsets of X. Hence, $(X-A) \cup (X-B)$ is finite. Therefore, $A \cap B \in F$. Let $D \in F$ and let E be any subset of X such that $D \subset E$. Let $y \in X-E$. Then $y \notin E$. Since $D \subset E$, then $y \notin D$, which implies that $y \in X-D$. Hence, $X-E \subset X-D$. Since $D \in F$, then $X-D$ is finite. Therefore, $X-E$ is finite, which implies that $E \in F$. Thus, F is a filter on X. Now, theorem 1.1 guanantees that there exists an ultrafilter, U, on X which contains F. By theorems 1.4, 1.5, and 1.6, $S = U \cup \{\phi\}$ is a maximally connected topology on X.

Let a and b be distinct points in X. Then, $\{a\}$ and $\{b\}$ are finite subsets of X. Since $\{a\} = X-(X-\{a\})$ and $\{b\} = X-(X-\{b\})$, then $X-\{a\}$ and $X-\{b\}$ are elements of F. Since

$F \subset U \subset S$, then $X-\{a\}$ and $X-\{b\}$ are elements of S. Now, $a \in X-\{b\}$ and $b \in X-\{a\}$, so for any pair of distinct points in X, each point is contained in an open set in S which does not contain the other. Hence, (X,S) is a T_1-space. Therefore, (X,S) is a maximally connected T_1-space.

Theorem 1.7: Let X be a nondegenerate set. If S is a topology on X which is defined by a filter, then (X,S) cannot be Hausdorff.

Proof: Let X be a nondegenerate set. Let F be a filter on X and let $S = F \cup \{\phi\}$. Then, by theorem 1.4, S is the topology on X which is defined by F. Let x and y be distinct elements of X. Assume that (X,S) is Hausdorff. Then there exist disjoint open sets, W and V, in S such that $x \in W$ and $y \in V$. Hence, W and V are nonempty. Therefore, W and V are elements of F. Since F is a filter, then $W \cap V$ must be in F. But, $W \cap V = \phi$, which contradicts the fact that F is a filter. Hence, distinct points do not have disjoint open sets in (X,S). Thus, (X,S) is not a Hausdorff space.

Corollary 1.7.1: Let X be a nondegenerate set. If (X,S) is a connected Hausdorff space, then S cannot be defined by a filter.

Proof: This is a direct consequence of theorem 1.7.

Corollary 1.7.2: Let X be a nondegenerate set. If (X,S) is a maximally connected Hausdorff space, then S cannot be defined by an ultrafilter.

Proof: This is a direct consequence of theorem 1.7.

Remark 1.3: In theorem 1.6 it is shown that if X is a nonempty set and U is an ultrafilter on X, then U defines a maximally connected topology on X. If the converse of this statement were true, then, by corollary 1.7.2, there could not exist a maximally connected Hausdorff space. The following example demonstrates that not all maximally connected topologies are defined by ultrafilters.

Example 1.3: Let $X = \{a,b,c\}$. Let $T = \{X,\phi,\{a\},\{b\},\{a,b\}\}$. Since X and ϕ are in T and all unions and intersections of elements of T are in T, then T is a topology on X. Also, there are not two nonempty, disjoint, open sets in T whose union is X. Hence, (X,T) is connected. Only three of the 29 topologies on X are strictly finer than T, namely,

(1) $T_i = \{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\}\}$,

(2) $T_j = \{X,\phi,\{a\},\{b\},\{a,b\},\{b,c\}\}$, and

(3) the discrete topology on X.

Since $\{a\} \cup \{b,c\}$ and $\{b\} \cup \{a,c\}$ are both equal to X, then neither T_i nor T_j nor the discrete topology on X is connected. Hence, (X,T) is a maximally connected topological space. Let $F = T-\{\phi\} = \{X,\{a\},\{b\},\{a,b\}\}$. Since $\{a\} \cap \{b\} = \phi$, then $\{a\} \cap \{b\} \notin F$. Hence, F is not a filter, so F is not an ultrafilter. Thus, $T = F \cup \{\phi\}$ is a maximally connected topology on X which is not defined by an ultrafilter.

Theorem 1.8: If all maximally connected T_1-spaces are defined by ultrafilters, then there does not exist a maximally connected Hausdorff space.

Proof: Let X be a nondegenerate set. Assume that all T_1-spaces are defined by ultrafilters. Let T be a topology on X such that (X,T) is a T_2-space. Since (X,T) is T_2, then (X,T) is T_1. Assume (X,T) is a maximally connected topological space. Since (X,T) is T_1, then by assumption (X,T) must be defined by an ultrafilter. But, (X,T) is also T_2 and by corollary 1.7.2, cannot be defined by an ultrafilter. Therefore, this contradicts the assumption that (X,T) is maximally connected. Hence, (X,T) is not maximally connected.

Remark 1.4: Since maximally connected Hausdorff spaces have recently been discovered by Guthrie, Wage, and Stone, then it must be true that not all maximally connected T_1-spaces are defined by ultrafilters.

CHAPTER II

PROPERTIES NOT POSSESSED BY MAXIMALLY CONNECTED TOPOLOGICAL SPACES

Theorem 2.1: Let (X,S) be a topological space. If $A \subset X$, and $C = S \cup \{V \cap A | V \in S\}$, and T is the set of all possible unions of elements of C, then T is a topology on X which contains S, and C is a basis for T.

Definition 2.1: Let (X,S) be a topological space and let A be a nonempty subset of X. The topology T, as constructed in theorem 2.1, is finer than S and is said to be the topology generated by $S \cup \{A\}$. The topology T is called a simple expansion of S and is denoted by $T = \langle S \cup \{A\} \rangle$.

Theorem 2.2: Let (X,T) be a topological space and let D be a subset of X. If $G \in T' = \langle T \cup \{D\} \rangle$, then there exist $U_1, U_2 \in T$ such that $G = U_1 \cup (U_2 \cap D)$.

Proof: Let (X,T) be a topological space and let $D \subset X$. Let Let $T' = \langle T \cup \{D\} \rangle$ be the topology generated by $T \cup \{D\}$. Let $G \in T'$. By theorem 2.1, $C = T \cup \{V \cap D | V \in T\}$ is a basis for T', and T' is the set of all possible unions of elements of C. Hence, there exists a subcollection C_G of elements of C such that $\cup\{W|W \in C_G\} = G$. For all $W \in C_G$, either $W \in T$ or there is a $Y \in T$ such that $W = Y \cap D$. Let $C_1 = \{K \in C_G | K \in T\}$ and let $C_2 = \{U \cap D | U \in T$ and $U \cap D \in C_G\}$. Then $C_G = C_1 \cup C_2$. Hence, $G = \cup\{W|W \in C_G\} = (\cup C_1) \cup (\cup C_2)$. Since T is a topology on X, then

$U_1 = \cup C_1$ is an element of T. Now,

$\cup C_2 = \cup\{U \cap D | U \in T$ and $U \cap D \in C_G\} =$

$(\cup\{U | U \in T$ and $U \cap D \in C_G\}) \cap D$.

Hence, $U_2 = \cup\{U | U \in T$ and $U \cap D \in C_G\}$ is an element of T. Thus, $G = (\cup C_1) \cup (\cup C_2) = U_1 \cup (U_2 \cap D)$ and U_1 and U_2 are elements of T.

Definition 2.2: Let (X,T) be a topological space. A subset D of X is said to be dense in (X,T) if and only if $cl(D,T) = X$.

Lemma 2.1: Let (X,T) be a connected topological space. Let D be a subset of X. If D is dense in (X,T), then $T' = \langle T \cup \{D\}\rangle$ is connected.

Proof: Let (X,T) be a connected topological space. Let D be a subset of X such that D is dense in (X,T). Let $T' = \langle T \cup \{D\}\rangle$.

Assume that (X,T') is not connected. Then there exist $A,B \in T'$ such that $A,B \neq \phi$, and $A \cap B = \phi$, and $A \cup B = X$. By theorem 2.2, there exist $U_1,U_2,V_1,V_2 \in T$ such that

$A = U_1 \cup (U_2 \cap D) = [(U_1 \cup U_2) \cap (U_1 \cup D)]$ and

$B = V_1 \cup (V_2 \cap D) = [(V_1 \cup V_2) \cap (V_1 \cup D)]$. Since $U_1,U_2,V_1,V_2 \in T$, then $U = (U_1 \cup U_2) \in T$ and $V = (V_1 \cup V_2) \in T$.

Assume that $U_1 \cap V_1 = \phi$, and $U_2 \cap V_1 = \phi$, and $U_1 \cap V_2 = \phi$, and $U_2 \cap V_2 = \phi$. Let $x \in U_1$. Since $U_1 \cap V_1$ and $U_1 \cap V_2$ are empty, then $x \notin V_1$ and $x \notin V_2$; hence $x \notin V_1 \cup V_2$. In a similar manner it can be shown that if $y \in U_2$, then $y \notin V_1 \cup V_2$. Therefore, if $z \in U = (U_1 \cup U_2)$, then $z \notin V = (V_1 \cup V_2)$. Hence, $U \cap V = \phi$. Since $X = A \cup B$ and $A \cap B = \phi$, then any element in X is either an element of A or an element of B, but not both. Let $p \in A$. Since $A = [(U_1 \cup U_2) \cap (U_1 \cup D)]$, then $p \in (U_1 \cup U_2) = U$; hence $A \subset U$. Let $q \in B$. Since $B = [(V_1 \cup V_2) \cap (V_1 \cup D)]$, then $q \in (V_1 \cup V_2) = V$; hence $B \subset V$. Let $u \in U$. Since $U \cap V = \phi$, then $u \notin V$. Since $B \subset V$, then $u \notin B$; hence $u \in X-B = A$. Therefore, $U \subset A$. Since $A \subset U$ and $U \subset A$, then $A = U \in T$. In a similar manner it can be shown that $B = V \in T$. Hence, A and B are elements of T, which contradicts the fact that (X,T) is connected. Hence, either $U_1 \cap V_1 \neq \phi$ or $U_2 \cap V_1 \neq \phi$ or $U_1 \cap V_2 \neq \phi$ or $U_2 \cap V_2 \neq \phi$.

Now A and B are disjoint sets and

$$\phi = A \cap B = [U_1 \cup (U_2 \cap D)] \cap [V_1 \cup (V_2 \cap D)] =$$
$$[(U_1 \cup (U_2 \cap D)) \cap V_1] \cup [(U_1 \cup (U_2 \cap D)) \cap (V_2 \cap D)] =$$
$$[(U_1 \cap V_1) \cup (U_2 \cap D \cap V_1)] \cup [(U_1 \cap V_2 \cap D) \cup (U_2 \cap D \cap V_2 \cap D)] =$$
$$(U_1 \cap V_1) \cup [(U_2 \cap V_1) \cap D] \cup [(U_1 \cap V_2) \cap D] \cup [(U_2 \cap V_2) \cap D].$$

If $U_1 \cap V_1 \neq \phi$, then $A \cap B \neq \phi$. Since D is dense in (X,T), then $cl(D,T) = X$. Let $W \in T$ and let $w \in W$. If $w \in D$, then $W \cap D \neq \phi$. If $w \notin D$, then w is a limit point of D and since W is an open set containing w, then W must contain a point $t \in D$ such that $t \neq w$. Hence, $W \cap D \neq \phi$. Therefore, the intersection of any nonempty set in T and D is nonempty. Since $U_1, U_2, V_1, V_2 \in T$, then $U_2 \cap V_1$, and $U_1 \cap V_2$, and $U_2 \cap V_2$ are elements of T. If $U_2 \cap V_1 \neq \phi$, then $(U_2 \cap V_1) \cap D \neq \phi$. If $U_1 \cap V_2 \neq \phi$, then $(U_1 \cap V_2) \cap D \neq \phi$. If $U_2 \cap V_2 \neq \phi$, then $(U_2 \cap V_2) \cap D \neq \phi$. Since it is true that at least one of the sets $U_1 \cap V_1$, $U_2 \cap V_1$, $U_1 \cap V_2$, $U_2 \cap V_2$ is nonempty, then $A \cap B \neq \phi$. But this contradicts A and B being disjoint. Hence, (X,T) is connected.

<u>Difinition 2.3</u>: Let (X,S) be a topological space and let M be a subset of X. Then M is said to be <u>nowhere dense</u> in (X,S) if and only if any open set G of S contains a nonempty open set U of S such that $U \cap M = \phi$.

<u>Lemma 2.2</u>: Let (X,T) be a topological space. If A is a subset of X such that A is nowhere dense in (X,T), then X-A is dense in (X,T).

Proof: Let (X,T) be a topological space. Let A be a subset of X such that A is nowhere dense in (X,T). Let $x \in X$ such that $x \notin X-A$. Let $G \in T$ such that $x \in G$. Since A is nowhere dense, then there exists a $U \in T$ such that $U \neq \phi$, $U \subset G$, and $U \cap A = \phi$. Hence, $U \subset X-A$, which implies that $x \notin U$. Since $U \neq \phi$ and $U \subset G$, then there exists a $p \in U \subset G$ such that $p \neq x$ and $p \in X-A$. Hence, x is a limit point of X-A. Since any point of X which is not in X-A is a limit point of X-A, then cl(X-A,T) = X. Hence, X-A is dense in (X,T).

<u>Lemma 2.3</u>: Let X be a nondegenerate set. If (X,T) is a connected, Hausdorff, topological space, then if V is a nonempty open set in T, then V contains an infinite number of points.

<u>Definition 2.4</u>: Let (X,T) be a topological space and let $p \in X$. A <u>local base</u>, B_p, at p is any set of open sets such that p is an element of every set of B_p and every open set which contains p contains a set of B_p.

<u>Definition 2.5</u>: Let (X,T) be a topological space and let $p \in X$. Then (X,T) is said to satisfy the <u>first axiom of countability at p</u> if and only if p has a countable local base.

<u>Lemma 2.4</u>: Let (X,T) be a topological space and let $p \in X$. If (X,T) satisfies the first axiom of countability at p, then there exists a countable local base, $D = \{D_i \mid i \in Z^+\}$, at p such that if $j,k \in Z^+$ and $j > k$, then $D_j \subset D_k$.

<u>Lemma 2.5</u>: Let X be a nondegenerate set and let (X,T) be a Hausdorff topological space. If p and q are distinct elements of X and $U \in T$ such that $p,q \in U$, then there exist disjoint open sets W_p and W_q such that $p \in W_p \subseteq U$ and $q \in W_q \subseteq U$.

<u>Theorem 2.3</u>: Let X be a nondegenerate set and let (X,T) be a connected, Hausdorff, topological space. If $p \in X$ such that (X,T) satisfies the first axiom of countability at p, then (X,T) is not maximally connected.

Proof: Let X be a nondegenerate set and let (X,T) be a connected Hausdorff space. Let $p \in X$ such that (X,T) satisfies the first axiom of countability at p. Then, by lemma 2.4, there exists a "nested" countable local base, $D = \{D_i \mid i \in Z^+\}$, at p such that if $j,k \in Z^+$ and $j > k$, then $D_j \subset D_k$. Also, if G is any open set which contains p, then $p \in D_i \subset G$, for some integer i.

Since X is nondegenerate, then there exists $b \in X$ such that $b \neq p$. Let $x_1 = b$. Since (X,T) is Hausdorff, then there exist disjoint open sets U_1 and V_1 such that $p \in U_1$ and $x_1 \in V_1$. Since U_1 is open, then there is an integer i such that

$p \in D_i \subset U_1$. Since the positive integers are well-ordered, then let $i(1)$ be the first positive integer for which $p \in D_{i(1)} \subset U_1$.

Let N be any positive integer. For each positive integer n such that $n \leq N$ we will construct a point $x_n^N \in X$, and open sets U_n^N and V_n^N, and a local base element $D_{j(n)}^N \in D$ with the following properties:

(I) If $n \in Z^+$, then if $N = n = 1$, then let $x_1^1 = x_1$, $U_1^1 = U_1$, $V_1^1 = V_1$, and $D_{j(1)}^1 = D_{i(1)}$; and

(II) If $n,m \in Z^+$, then if $N > 1$, then

(1) if $n \leq m < N$, then $x_n^n = x_n^m = x_n^N$, and $U_n^n = U_n^m = U_n^N$, and $D_{j(n)}^n = D_{j(n)}^m = D_{j(n)}^N$,

(2) if $n,m \leq N$ and $n \neq m$, then $V_n^N \cap V_m^N = \phi$ and $x_n^N \neq x_m^N$,

(3) if $n \leq N$, then $x_n^N \neq p$, and $x_n^N \in V_n^N$, and $U_n^N \cap V_n^N = \phi$, and

(4) if $n < m \leq N$, then $p \in D_{j(m)}^N \subset U_m^N \subseteq D_{j(n)}^N \subset U_n^N$ and $x_m^N \in V_m^N \subseteq D_{j(n)}^N \subset U_n^N$.

Mathematical induction is used to prove this construction is valid.

Let R be the set of all positive integral values of N for which the above construction is true. Since $x_1^1 = x_1$, $U_1^1 = U_1$, $V_1^1 = V_1$, and $D_{j(1)}^1 = D_{i(1)}$, then the construction is true for $N = 1$. Let $N = 2$. Let $x_1^2 = x_1^1$, $U_1^2 = U_1^1$, $V_1^2 = V_1^1$, and $D_{j(1)}^2 = D_{j(1)}^1$. Therefore, property 1 is satisfied. Since $D_{j(1)}^2$ is open and since (X,T) is connected and Hausdorff and since X is nondegenerate, then, by lemma 2.3, there is a point $x_2^2 \in D_{j(1)}^2$ such that $x_2^2 \neq p$.

Since $D^2_{j(1)} \subset U^2_1$ and $x^2_1 \in V^2_1$ and $U^2_1 \cap V^2_1 = \phi$, then $x^2_2 \neq x^2_1$. Since X is nondegenerate and (X,T) is Hausdorff, then, by lemma 2.5, there exist disjoint open sets U^2_2 and V^2_2 such that $p \in U^2_2 \subseteq D^2_{j(1)}$ and $x^2_2 \in V^2_2 \subseteq D^2_{j(1)}$. Since $p \in U^2_2$ and U^2_2 is open, then there is a base element $D^2_{j(2)} \subset U^2_2$ such that $p \in D^2_{j(2)}$. Since $p \in D^2_2 \subset U^2_2 \subseteq D^2_{j(1)} \subset U^2_1$ and $x^2_2 \in V^2_2 \subseteq D^2_{j(1)} \subset U^2_1$, then property 4 is satisfied. Since $V^2_2 \subset U^2_1$ and $U^2_1 \cap V^2_1 = \phi$, then $V^2_2 \cap V^2_1 = \phi$. Also, $x^2_1 \neq x^2_2$. Hence, property 2 is satisfied. Since $x^2_2 \neq p$, and $x^1_1 \neq p$, and $U^2_2 \cap V^2_2 = \phi$, and $U^1_1 \cap V^1_1 = \phi$, then property 3 is satisfied. Hence the construction is valid for $N = 2$.

Assume that the construction is true for $N = K$. Thus, for each positive integer $n \leq K$, there exist a point x^K_n, open sets U^K_n and V^K_n, and a base element $D^K_{j(n)}$ such that properties 1,2,3, and 4 are satisfied. Let n and m be positive integers such that $n \leq m < K+1$ and let $U^{K+1}_n = U^n_n = U^m_n$, and $V^{K+1}_n = V^n_n = V^m_n$, and $x^{K+1}_n = x^n_n = x^m_n$, and $D^{K+1}_{j(n)} = D^n_{j(n)} = D^m_{j(n)}$; hence property 1 is satisfied. Now $p \in D^K_{j(K)} \subset U^K_K$. Since $D^K_{j(K)}$ is open and since (X,T) is T_2 and connected, then, by lemma 2.3, there exists a point $x^{K+1}_{K+1} \in D^K_{j(K)} \subset U^K_K$ such that $p \neq x^{K+1}_{K+1}$. Since X is nondegenerate and (X,T) is Hausdorff, then, by lemma 2.5, there exist disjoint open sets U^{K+1}_{K+1} and V^{K+1}_{K+1} such that $p \in U^{K+1}_{K+1} \subseteq D^K_{j(K)} = D^{K+1}_{j(K)}$ and $x^{K+1}_{K+1} \in V^{K+1}_{K+1} \subseteq D^K_{j(K)}$. Since $p \in U^{K+1}_{K+1}$ and U^{K+1}_{K+1} is open, then there is a base element $D^{K+1}_{j(K+1)} \subset U^{K+1}_{K+1}$ such that $p \in D^{K+1}_{j(K+1)}$. Since $x^{K+1}_{K+1} \neq p$, and $x^{K+1}_{K+1} \in V^{K+1}_{K+1}$, and $U^{K+1}_{K+1} \cap V^{K+1}_{K+1} = \phi$, and property 3 holds for $N = K$, then property 3 holds for $N = K+1$. Since

$V^{K+1}_{K+1} \subset D^K_{j(K)} \subset U^K_K$ and since if $n \in Z^+$ such that $n \leq K$, then $U^K_K \cap V^K_n = \phi$, then $V^{K+1}_{K+1} \cap V^K_n = \phi$. Since $x^{K+1}_{K+1} \in V^{K+1}_{K+1}$, then if $n \leq K$, then $x^{K+1}_{K+1} \neq x^K_n$. Hence, property 2 is satisfied for $N = K+1$. Since $p \in D^{K+1}_{j(K+1)} \subset U^{K+1}_{K+1} \subseteq D^K_{j(K)} \subset U^K_K$, and $x^{K+1}_{K+1} \in V^{K+1}_{K+1} \subseteq D^K_{j(K)} \subset U^K_K$, and property 4 holds for $N = K$, then property 4 holds for $N = K+1$. Thus, if the construction is valid for $N = K$, then it is valid for $N = K+1$. Hence, R must contain all the positive integers. Therefore, the construction is valid for all positive integers N, by mathematical induction.

For any positive integer i, let $x_i = x^i_i$, and $U_i = U^i_i$, and $V_i = V^i_i$, and $D_i = D^i_{j(1)}$. Let $Y = \{x_i | i \in Z^+\}$. Since for each i, $x_i \neq p$, then $p \notin Y$. Let $W \in T$ such that $p \in W$. Then there exists a positive integer t such that $p \in D_t \subset W$. If z and r are positive integers such that $z < r$, then $D^z_{j(z)} \supset D^r_{j(r)}$ and $D^z_{j(z)} \neq D^r_{j(r)}$ and $j(z) \neq j(r)$. Since $D^N_{j(n)}$ exists for any positive integer $n \leq N$, then $D^N_{j(N)}$ exists for any positive integer N. Hence, $\{j(r) | r \in Z^+\}$ is unbounded. Therefore, there exists a $j(r)$ such that $j(r) > t$, which implies that $D^r_{j(r)} \subset D_t$. Hence, $p \in D^r_{j(r)} \subset D_t \subset W$. Thus, $x_r = x^r_r \subset D^r_{j(r)} \subset D_t \subset W$. Therefore, every open set containing p contains an element of Y different from p. Hence, p is a limit point of Y which is not in Y. Therefore, Y is not closed, which implies that X-Y is not open.

Let F be an open set such that $F \cap Y \neq \phi$. Then there is a point $x_q \in Y$ such that $x_q \in F$. By property 3, $x_q \in V_q$; hence $x_q \in V_q \cap F \subset F$. By property 2, x_q is the only element of Y that is in V_q; hence $V_q \cap Y = \{x_q\}$, which implies that $(V_q \cap F) \cap Y = \{x_q\}$. Since V_q and F are open, then $V_q \cap F$ is open. Since $x_q \in V_q \cap F$ and since (X,T) is Hausdorff and connected, then, by lemma 2.3, there exists a point $y \neq x_q$ such that $y \in V_q \cap F$. Since (X,T) is Hausdorff, then, by lemma 2.5, there exist disjoint open sets, W_{x_q} and W_y, contained in $V_q \cap F$ such that $x_q \in W_{x_q}$ and $y \in W_y$. Since $(V_q \cap F) \cap Y = x_q$ and $x_q \notin W_y$, then $W_y \cap Y = \phi$. Hence, every open set of T contains an open set H such that $H \cap Y = \phi$. Hence, Y is nowhere dense. Thus, by lemma 2.2, $X-Y$ is dense in (X,T).

Let $T' = \langle T \cup \{X-Y\} \rangle$. Since (X,T) is connected and since $X-Y$ is dense, then, by lemma 2.1, (X,T') is connected. Since T' is a simple expansion of T, then T' is finer than T. Now $X-Y \in T'$, but since $X-Y$ is not open in (X,T), then $X-Y \notin T$. Hence, T' is strictly finer than T, and since (X,T') is connected, then (X,T) is not maximally connected.

<u>Definition 2.6</u>: A topological space (X,T) is said to satisfy the <u>first axiom of countability</u> or to be <u>first countable</u> if and only if each point x of X has a countable local base.

<u>Corollary 2.3.1</u>: Let X be a nondegenerate set. If (X,T) is a first countable, Hausdorff, topological space, then (X,T) is not maximally connected.

Proof: This is a direct conclusion of theorem 2.3.

<u>Theorem 2.4</u>: Let X be a nondegenerate set and let (X,T) be a maximally connected topological space. If D is a subset of X such that D is dense in (X,T), then D is open.

Proof: Let X be a nondegenerate set. Let (X,T) be a maximally connected topological space. Let D be a dense subset of X. Assume that D is not open. Let $T' = \langle T \cup \{D\} \rangle$. Since (X,T) is connected and since D is dense in (X,T), then, by lemma 2.1, (X,T′) is connected. Since T′ is a simple expansion of T, then T′ is finer than T. Also, since $D \in T'$ and $D \notin T$, then T′ is strictly finer than T. But this contradicts the fact that (X,T) is maximally connected. Hence, D must be open.

<u>Definition 2.7</u>: Let X be a set. A function d from $X \times X$ into the set of real numbers is said to be a <u>metric for</u> X provided:

(1) if $x \in X$, then $d(x,x) = 0$ and if $x,y \in X$ and $d(x,x) = 0$, then $x = y$;

(2) if $x,y \in X$, then $d(x,y) = d(y,x)$; and

(3) if $x,y,z \in X$, then $d(x,y) + d(y,z) \geq d(x,z)$.

A set X with metric d is called a <u>metric space</u>, denoted (X,d).

<u>Definition 2.8</u>: Let (X,d) be a metric space. Let $x \in X$ and let p be a positive real number. Then the <u>d-p-neighborhood</u> of X is defined to be the set of all points y of X such that $d(x,y) < p$, denoted $N(x,p)$.

<u>Definition 2.9</u>: Let (X,d) be a metric space. Let U be a subset of X. Then U is said to be <u>d-open</u> if given any point x of U, there is a positive real number p, such that the d-p-neighborhood

of X is a subset of U.

Lemma 2.6: Let (X,d) be a metric space. Let T be the set of all d-open subsets of X. Let

B = $\{N(x,p) \mid x \in X$ and p is a positive real number$\}$. Then T is a topology on X and B is a basis for T.

Remark 2.1: If (X,d) is a metric space and T is a topology on X which is constructed as in lemma 2.6, then T is called the topology on X defined by the metric d.

Lemma 2.7: Let (X,d) be a metric space. If T is the topology on X defined by d, then (X,T) is Hausdorff.

Lemma 2.8: Let (X,d) be a metric space. If T is the topology on X defined by d, then (X,T) is first countable.

Theorem 2.5: Let X be a nondegenerate set. Let (X,d) be a metric space. If T is the topology on X defined by d, then (X,T) is not maximally connected.

Proof: Let X be a nondegenerate set. Let (X,d) be a metric space and let T be the topology on X defined by d. Then, by lemma 2.7, (X,T) is Hausdorff and, by lemma 2.8, (X,T) is first countable. Hence, by corollary 2.3.1, (X,T) is not maximally connected.

Definition 2.10: Let (X,T) be a topological space. Let B be a subset of X and let $p \in X$. Then p is said to be an accumulation point of B, if every open set containing p contains infinitely many points of B.

Definition 2.11: Let (X,T) be a topological space and let $E \subset X$. Let C be a collection of open sets in T. If $E \subset \cup\{W | W \in C\}$, then C is an open cover of E.

Definition 2.12: Let (X,T) be a topological space and let $A \subset X$. Then A is compact if and only if each open cover of A has a finite subcovering.

Lemma 2.9: Let (X,T) be a compact topological space. If B is an infinite subset of X, then there is an accumulation point of B in X.

Proof: Let (X,T) be a compact topological space. Let B be an infinite subset of X. Assume that B does not have an accumulation point in X. Let $C = \{x_i | i \in Z^+, x_i \in B\}$ be a countably infinite sequence of distinct points from B. Let n be a positive integer and let $A_n = \{x_i \in C | i \geq n\}$ and let $D_n = X - A_n$. Then if j and k are integers and $j < k$, then $D_j \subset D_k$. Let $V_n = \cup\{W \in T | W \subset D_n\}$. Therefore $V_n \subset D_n$. Since $D_j \subset D_k$ for $j < k$, then $V_j \subset V_k$ for $j < k$. Let $M = \{V_i | i \in Z^+\}$. Let y be an element of X. Then y is not an accumulation point of B. Hence, there exists an open set U_y containing y such that $U_y \cap B$ contains at most finitely many elements. Since $C \subset B$, then U_y contains at most finitely many elements of C. Since the positive integers are well-ordered, then there is a positive integer t such that $x_t \notin U_y$ and if f is a positive integer such that $f \geq t$, then $x_f \notin U_y$. Let n always be larger than t. Then $U_y \cap A_n = \phi$; hence $y \in U_y \subset D_n$. Since U_y is open, then $y \in U_y \subset V_n$. Therefore, we can always choose n to be sufficiently large enough to allow U_y to be contained in V_n for any $y \in X$. Since V_n is the union of open sets, then V_n is open. Hence, for any $y \in X$, there exists an open set $V_i \in M$ such that $y \in V_i$, which implies that $X = \bigcup_{i=1}^{\infty} V_i$. Thus, M is an open cover of X. But since (X,T) is compact, then there must be a finite subcover of X in M, say $K = \{V_{i(1)}, V_{i(2)}, \ldots, V_{i(m)}\}$, where if r and s are positive integers and $r < s$, then $V_{i(r)} \subset V_{i(s)}$. Since $x_{i(m+1)} \notin D_{i(m)}$, then $x_{i(m+1)} \notin \bigcup_{k=1}^{m} D_{i(k)}$. Since $V_p \subset D_p$, for $p \in Z^+$, then $x_{i(m+1)} \notin \bigcup_{k=1}^{m} V_{i(k)}$, which is a contradiction since K is a subcover of X. Consequently, B must have an accumulation point.

<u>Theorem 2.6</u>: Let X be a nondegenerate set. If (X,T) is a connected, compact, Hausdorff, topological space, then (X,T) cannot be maximally connected.

Proof: Let X be a nondegenerate set. Let (X,T) be a compact, connected, Hausdorff space. Since X is nondegenerate, there exist distinct points p and x_1 in X. Since (X,T) is Hausdorff, then there exist disjoint open sets U_1 and V_1 such that $p \in U_1$ and $x_1 \in V_1$. Since (X,T) is connected and Hausdorff, then, by lemma 2.3, there is a point $x_2 \in U_1$ such that $x_2 \neq p$ and since $U_1 \cap V_1 = \phi$, then $x_2 \neq x_1$. Since (X,T) is Hausdorff, then, by lemma 2.5, there exist disjoint open sets U_2 and V_2 such that $p \in U_2 \subseteq U_1$ and $x_2 \in V_2 \subseteq U_1$. Again, by lemma 2.3, there is an $x_3 \in U_2$ such that $x_3 \neq p$ and since $U_2 \cap V_2 = \phi$, then $x_3 \neq x_2$ and since $U_2 \subset U_1$ and $U_1 \cap V_1 = \phi$, then $x_3 \neq x_1$. The above process can be continued a countably infinite number of times in a manner similar to that used in theorem 2.3. Hence, for any positive integer n, there exist a point x_n and open sets U_n and V_n with the following properties:

(1) if $i \in Z^+$, then $p \in U_i$, and $x_i \in V_i$, and $x_i \neq p$;

(2) if $i,j \in Z^+$ and $i \neq j$, then $x_i \neq x_j$, and $V_i \cap V_j = \phi$, and $x_i \notin V_j$; and

(3) if $i,j \in Z^+$ and $i < j$, then $U_j \cap V_i = \phi$ and $V_j \subseteq U_i$.

Let $Y = \{x_i | i \in Z^+\}$. Since for each i, $x_i \neq p$, then $p \notin Y$. Since if $i \neq j$, then $x_i \neq x_j$, then the elements of Y are distinct; hence Y contains a countably infinite number of elements. Since (X,T) is compact, then, by lemma 2.8, there exists an accumulation point of Y in X. Since for all $i \in Z^+$, V_i is open and $x_i \in V_i$ and $V_i \cap V_j = \phi$ and if $j \in Z^+$ such that $j \neq i$, then $x_i \in V_j$, then any element of Y is contained in an open set which contains no other elements of Y. Hence, for $i \in Z^+$, no $x_i \in Y$ is an accumulation point of Y. Hence, Y does not contain its accumulation point, which by definition is a limit point of Y. Therefore, Y is not closed, which implies that $X-Y$ is not open.

Let F be an open set such that $F \cap Y \neq \phi$. Then there is an $x_q \in Y$ such that $x_q \in F$. Now, $x_q \in V_q$, so $x_q \in V_q \cap F \subset F$. By property 2, x_q is the only element of Y that is in V_q; hence $V_q \cap Y = \{x_q\}$, which implies that $(V_p \cap F) \cap Y = \{x_q\}$. Since V_q and F are open, then $V_q \cap F$ is open. Since $x_q \in V_q \cap F$ and since (X,T) is Hausdorff and connected, then by lemma 2.3, there exists $y \neq x_q$ such that $y \in V_q \cap F$. Since (X,T) is Hausdorff, then by lemma 2.5, there exist disjoint open sets W_{x_q} and W_y

contained in $V_q \cap F \subset F$ such that $x_q \in W_{x_q}$ and $y \in W_y$. Since $(V_q \cap F) \cap Y = \{x_q\}$ and $x_q \notin W_y$, then $W_y \cap Y = \phi$. Hence, every open set of T contains an open set H such that $H \cap Y = \phi$. Hence, Y is nowhere dense. Thus, by lemma 2.2, X-Y is dense in (X,T).

Let $T' = \langle T \cup \{X-Y\} \rangle$. Since (X,T) is connected and since X-Y is dense, then by lemma 2.1, (X,T′) is connected. Since T′ is a simple expansion of T, then T′ is finer than T. Now $X-Y \in T'$, but since X-Y is not open in (X,T), then $X-Y \notin T$. Hence, T′ is strictly finer than T, and since (X,T′) is connected, then (X,T) is not maximally connected.

<u>Definition 2.13</u>: Let (X,T) be a topological space and let $A \subset X$. The <u>interior</u> of A, denoted intA, is the union of all open sets which are contained in A.

<u>Definition 2.14</u>: A topological space (X,T) is said to be <u>locally compact</u>, if given any $x \in X$ and any open set U containing x, there is a compact set A such that $x \in \text{int}A \subset A \subset U$.

<u>Lemma 2.10</u>: Let (X,T) be a topological space and let $A \subset X$. Let T_A be the topology induced on A by T. Let $S \subset A$ and let $p \in A$. If p is a limit point of S in the subspace (A,T_A), then p is a limit point of S in (X,T).

<u>Theorem 2.7</u>: Let X be a non-degenerate set. If (X,T) is a connected, locally compact, Hausdorff topological space, then (X,T) cannot be maximally connected.

Proof: Let X be a non-degenerate set. Let (X,T) be a connected, locally compact, Hausdorff topological space. Let $p \in X$.

Now X is an open set containing p and since (X,T) is locally compact, then there is a compact set A contained in X such that $p \in \text{int}A \subset A$. Since $\text{int}A$ is the union of open sets in T, then $\text{int}A$ is open in T. Since (X,T) is connected and Hausdorff, then, by lemma 2.3, there is a point $x_1 \in \text{int}A$ such that $x_1 \neq p$. Since (X,T) is Hausdorff, then, by lemma 2.5, there exist disjoint open sets U_1 and V_1 such that $p \in U_1 \subseteq A$ and $x_1 \in V_1 \subseteq \text{int}A$. Again, by lemma 2.3, there is an $x_2 \in U_1$ such that $x_2 \neq p$ and since $U_1 \cap V_1 = \phi$, then $x_2 \neq x_1$. Again, by lemma 2.5, there exist disjoint open sets U_2 and V_2 such that $p \in U_2 \subseteq U_1$ and $x_2 \in V_2 \subseteq U_1$. Since U_2 is open, then again, by lemma 2.3, there is an $x_3 \in U_2$ such that $x_3 \neq p$. Since $U_2 \cap V_2 = \phi$, then $x_3 \neq x_2$ and since $U_2 \subset U_1$ and $U_1 \cap V_1 = \phi$, then $x_3 \neq x_1$. The above process is continued in a manner similar to that of theorem 2.3. Hence, for any positive integer n, there exist a point x_n and open sets U_n and V_n with the following properties:

(1) if $i \in Z^+$, then $p \in U_i$, and $x_i \in V_i$, and $x_i \neq p$, and $x_i \in A$, and $U_i \subset A$, and $V_i \subset A$;

(2) if $i,j \in Z^+$ and $i \neq j$, then $x_i \neq x_j$, and $V_i \cap V_j = \phi$, and $x_i \notin V_j$; and

(3) if $i,j \in Z^+$ and $i < j$, then $U_j \cap V_i = \phi$ and $V_j \subseteq U_i$.

Let $Y = \{x_i \mid i \in Z^+\}$. Since for each i, $x_i \neq p$, then $p \notin Y$. Since $x_i \neq x_j$ for $i \neq j$, then the elements of Y are distinct. Hence, Y contains a countably infinite number of elements. Let T_A be the topology induced on A by T. Since A is compact in (X,T), then the subspace (A,T_A) is a compact topological space. Since for all $i \in Z^+$, $x_i \in A$, then $Y \subset A$. Since (A,T_A) is compact and Y is an infinite set, then by lemma 2.8, there exists an accumulation point of Y in (A,T_A). Since for any $x_i \in Y$ there is an open set V_i in T, such that $x_i \in V_i$, which contains no other points of Y, then $V_i \cap A$ is an open set in T_A which contains x_i and no other points of Y. Therefore no element of Y is an accumulation point of Y in (A,T_A). Hence, Y does not contain its accumulation point in (A,T_A). It follows by lemma 2.10, Y does not contain all its accumulation points in (X,T). Thus Y is not closed in (X,T). Hence, $X-Y$ is not open in (X,T).

Let F be an open set such that $F \cap Y \neq \phi$. Then there is an $x_q \in Y$ such that $x_q \in F$. Now, $x_q \in V_q$, so $x_q \in V_q \cap F \subset F$. By property 2, x_q is the only element of Y that is in V_q; hence $V_q \cap Y = \{x_q\}$, which implies that $(V_q \cap F) \cap Y = \{x_q\}$. Since V_q and F are open, then $V_q \cap F$ is open. Since $x_q \in V_q \cap F$ and since (X,T) is Hausdorff and connected, then by lemma 2.3, there exists $y \neq x_q$ such that $y \in V_q \cap F$. Since (X,T) is Hausdorff, then by lemma 2.5, there exist disjoint open sets W_{x_q} and W_y contained in $V_q \cap F \subset F$ such that $x_q \in W_{x_q}$ and $y \in W_y$. Since $(V_q \cap F) \subset Y = \{x_q\}$ and $x_q \notin W_y$, then $W_y \cap Y = \phi$. Hence, every open set of T contains an open set H such that $H \cap Y = \phi$. Hence, Y is nowhere dense. Thus by lemma 2.2, $X-Y$ is dense in (X,T).

Let $T' = \langle T \cup \{X-Y\} \rangle$. Since (X,T) is connected and since $X-Y$ is dense, then by lemma 2.1, (X,T') is connected. Since T' is a simple expansion of T, then T' is finer than T. Now $X-Y \in T'$, but since X-Y is not open in (X,T), then $X\text{-}Y \notin T$. Hence, T′ is strictly finer than T, and since (X,T′) is connected, then (X,T) is not maximally connected.

CHAPTER III

PROPERTIES OF MAXIMALLY CONNECTED TOPOLOGICAL SPACES

Theorem 3.1: If (X,T) is a maximally connected topological space, then (X,T) is T_0.

Proof: Let (X,T) be a maximally connected topological space. Assume that (X,T) is not T_0. Then there exist distinct points $x,y \in X$ such that $x \in cl(\{y\}, T)$ and $y \in cl(\{x\},T)$. Since $x \in cl(\{y\},T)$, then $\{y\}$ is not closed; hence $X-\{y\}$ is not open in (X,T). Since every open set containing y also contains x and since $x \in X-\{y\}$, then $\{y\}$ is a limit point of $X-\{y\}$. Therefore, $cl(X-\{y\},T) = X$; hence $X-\{y\}$ is dense in (X,T). Let $T' = \langle T \cup \{X-\{y\}\} \rangle$. Since $X-\{y\}$ is nonopen and dense in (X,T), then by lemma 2.1, (X,T') is connected. Since $X-\{y\} \notin T$ and $X-\{y\} \in T'$, then the simple expansion T' is strictly finer than T. But this contradicts the fact that (X,T) is maximally connected.

Definition 3.1: Let (X,T) be a topoligical space. Then (X,T) is said to be submaximal if every dense subset of X is open.

Theorem 3.2: If (X,T) is a maximally connected topological space, then (X,T) is submaximal.

Proof: Let (X,T) be a maximally connected topological space. Let D be a dense subset of X. Then, by theorem 2.5, D must be open. Hence, every dense subset of X is open in (X,T), which implies that (X,T) is submaximal.

Definition 3.2: Let (X,T) be a topological space and let $A \subset X$. Let T_A be the topology induced on A by T. Then A is a connected subset of (X,T) if and only if the subspace (A,T_A) is connected.

Definition 3.3: Let (X,T) be a topological space and let $M \subset X$. The boundary of M is said to be $cl(M,T)$-$intM$ and is denoted bdM.

Definition 3.4: Let (X,T) be a connected topological space. A point $p \in X$ is said to be a cut point of X if the subspace $(X-\{p\},T_{X-\{p\}})$ is disconnected.

Theorem 3.3: If (X,T) is a maximally connected topological space, then every nonempty, nondense, open set in T has a cut point of X in its boundary.

Proof: Let (X,T) be a maximally connected topological space. Let V be a nonempty, nondense, open set in (X,T). Assume, for any $q \in bdV$, that $V \cup \{q\}$ is open. Since $intV \subset V$, then $V \cup bdV = cl(V,T)$; hence $cl(V,T) = \cup\{V \cup \{q\} | q \in bdV\}$. Therefore, since $cl(V,T)$ is the union of open sets in (X,T), then $cl(V,T) \in T$. But $cl(V,T)$ is a nonempty proper subset of X which is both open and closed, which contradicts the fact that (X,T) is connected. Hence, there exists some $p \in bdV$ such that $V \cup \{p\}$ is not open.

Let $T' = \langle T \cup \{V \cup \{p\}\}\rangle$. Since $V \cup \{p\} \notin T$ and $V \cup \{p\} \in T'$ and T' is a simple expansion of T, then T' is strictly finer than T. Since (X,T) is maximally connected, then (X,T') is disconnected. Hence, there exist nonempty, disjoint, open sets A and B in T'

such that $A \cup B = X$. By theorem 2.2, there exist $U_1, U_2, V_1, V_2 \in T$ such that $A = U_1 \cup (U_2 \cap (V \cup \{p\}))$ and $B = V_1 \cup (V_2 \cap (V \cup \{p\}))$.

Since A and B disconnect (X,T'), then either $p \in A$ or $p \in B$, but $p \notin A \cap B$. Let $p \in A$. Then $p \notin B$. Since $p \notin B = V_1 \cup (V_2 \cap (V \cup \{p\}))$, then $p \notin V_1$ and $p \notin V_2$; hence $p \notin V_2 \cap (V \cup \{p\})$. Thus, $V_2 \cap (V \cup \{p\}) = V_2 \cap V$, which implies that $B = V_1 \cap (V_2 \cup V)$. Since V_1, V_2, and V are elements of T, then $V_1 \cap (V_2 \cup V) \in T$; hence $B \in T$. Therefore, $B \cap X-\{p\}$ is open in the subspace $(X-\{p\}, T_{X-\{p\}})$. Since $p \notin B$, then $B \subset X-\{p\}$; hence $B \cap (X-\{p\}) = B$. Thus, $B \in T_{X-\{p\}}$. Since $A \cup B = X$, and $A \cap B = \phi$, and $A,B \neq \phi$, and $B \in T$, and (X,T) is connected, then $A \notin T$. Now,

$$A \cap X-\{p\} = [U_1 \cup (U_2 \cap (V \cup \{p\}))] \cap (X-\{p\}) =$$
$$[U_1 \cap (X-\{p\})] \cup [(U_2 \cap (V \cup \{p\})) \cap (X-\{p\})] =$$
$$[U_1 \cap (X-\{p\})] \cup [((U_2 \cap V) \cup (U_2 \cap \{p\})) \cap (X-\{p\})] =$$
$$[U_1 \cap (X-\{p\})] \cup [(U_2 \cap V) \cap (X-\{p\})] \cup [(U_2 \cap \{p\}) \cap (X-\{p\})] =$$
$$[U_1 \cap (X-\{p\})] \cup [(U_2 \cap V) \cap (X-\{p\})] \cup \phi.$$

Since $U_1 \in T$, then $U_1 \cap (X-\{p\}) \in T_{X-\{p\}}$. Since $U_2, V \in T$, then $U_2 \cap V \in T$; hence $(U_2 \cap V) \cap (X-\{p\}) \in T_{X-\{p\}}$. Therefore, $[U_1 \cap (X-\{p\}] \cup [(U_2 \cap V) \cap (X-\{p\})] \in T_{X-\{p\}}$; hence $A \cap (X-\{p\}) \in T_{X-\{p\}}$. Since $p \in A$, then $A \cap (X-\{p\}) = A-\{p\}$; so $A-\{p\} \in T_{X-\{p\}}$. Since $A \cap B = \phi$, then $(A-\{p\} \cap B = \phi$. Since $A \cup B = X$ and $p \notin B$, then $(A-\{p\}) \cup B = X-\{p\}$.

Assume that $\{p\} \in T'$. Then there exist $W_1, W_2 \in T$ such that $p = W_1 \cup (W_2 \cap (V \cup \{p\}))$. Since $V \in T$, then $V \subset \text{int}V$; hence $V = \text{int}V$. Since $p \in \text{bd}V = \text{cl}(V,T)-\text{int}V$, then $p \notin V$; hence p is a limit point of V. Since p is a limit point of V in (X,T), then any open set in T which contains V must contain a point of V different from p. Hence, $\{p\} \notin T$. Therefore, $W_1 \neq \{p\}$ and $W_2 \neq \{p\}$. Since $\{p\} = W_1 \cup (W_2 \cap (V \cup \{p\}))$, then either $p \in W_1$ or $p \in W_2$.

If $p \in W_1$, then there is an $r \in W_1$ such that $r \neq p$, which implies that $\{p\} \neq W_1 \cup (W_2 \cap (V \cup \{p\}))$. If $p \in W_2$, then since $W_2 \in T$ and p is a limit point of V, then there is an $s \in W_1$ such that $s \in V$ and $s \neq p$; hence $s \in W_2 \cap (V \cup \{p\})$, which implies that $\{p\} \neq W_1 \cup (W_2 \cap (V \cup \{p\}))$. In either case this is a contradiction; hence $\{p\} \notin T'$.

Since $A \in T'$ and $\{p\} \notin T'$, then $A \neq \{p\}$. Since A is nonempty, then $A-\{p\}$ is nonempty. Also $B \neq \phi$. Hence, $A-\{p\}$ and B disconnect $(X-\{p\}, T_{X-\{p\}})$. Since $X-\{p\}$ is disconnected and (X,T) is connected, then p is a cut point of X which is in the boundary of V.

<u>Theorem 3.4</u>: If (X,T) is a maximally connected topological space, then whenever A and X-A are both connected subsets of (X,T) it follows that A is either open or closed.

Proof: Let (X,T) be a maximally connected topological space. Let $A \subset X$ such that A and X-A are both connected. Assume that A is neither open or closed in (X,T). Let $T' = \langle T \cup \{A\} \rangle$. Since $A \in T'$ and $A \notin T$ and T' is a simple expansion of T, then T' is strictly finer than T. Since T is maximally connected, then T' is disconnected. Hence, there exist nonempty disjoint open sets C and

D in T' such that $C \cup D = X$. Now, by theorem 2.2, there exist $U_1, U_2, V_1, V_2 \in T$ such that $C = U_1 \cup (U_2 \cap A)$ and $D = V_1 \cup (V_2 \cap A)$.

Let T_A be the topology induced on A by T. Assume that there are points x and y in A such that $x \in C$ and $y \in D$. Since $C \cap D = \phi$, then $x \neq y$. Now
$C \cap A = (U_1 \cup (U_2 \cap A)) \cap A = (U_1 \cap A) \cup (U_2 \cap A)$ and
$D \cap A = (V_1 \cup (V_2 \cap A)) \cap A = (V_1 \cap A) \cup (V_2 \cap A)$. Since $U_1, U_2 \in T$, then $U_1 \cap A$ and $U_2 \cap A$ are elements of T_A; hence $C \cap A \in T_A$. Similarly $D \cap A \in T_A$. Since $C \cup D = X$ and $C \cap D = \phi$, then $(C \cap A) \cup (D \cap A) = A$ and $(C \cap A) \cap (D \cap A) = \phi$. Since $x, y \in A$ and $x \in C$ and $y \in D$, then $x \in C \cap A$ and $y \in D \cap A$; hence $C \cap A$ and $D \cap A$ are nonempty. But this implies that $C \cap A$ and $D \cap A$ disconnect (A, T_A) which contradicts (A, T_A) being connected. Hence, either $A \subset C$ or $A \subset D$, say $A \subset C$. Then $A \cap D = \phi$. Since $A \cap D = \phi$, then $V_2 \cap A = \phi$; hence $D = V_1$.

Let T_{X-A} be the topology induced on $X-A$ by T. Now
$C \cap X\text{-}A = (U_1 \cup (U_2 \cap A)) \cap X\text{-}A = U_1 \cap X\text{-}A$ and $D \cap X\text{-}A = V_1 \cap X\text{-}A$. Since U_1 and V_1 are elements of T, then $E = U_1 \cap X\text{-}A$ and $F = V_1 \cap X\text{-}A$ are both in T_{X-A}. Assume that $p, q \in X\text{-}A$ such that $p \in E$ and $q \in F$. Since $C \cup D = X$ and $C \cap D = \phi$, then $E \cup F = X\text{-}A$ and $E \cap F = \phi$. Since $p \in E$ and $q \in F$, then E and F are nonempty. But this implies that E and F disconnect $(X\text{-}A, T_{X-A})$ which controdicts $X\text{-}A$ being connected. Hence either $X\text{-}A \subset E$ or $X\text{-}A \subset F$.

If $X\text{-}A \in E$, then $X\text{-}A \subset U_1$ which implies that $X\text{-}A \subset C$. Since

$A \subset C$, then $X-A \subset C$ implies that $C = X$. But $C = X$ implies that $D = \phi$ which is a contradiction.

If $X-A \subset F = V_1 \cap X-A$, then $X-A \subset V_1$. Since A is not closed in (X,T), then there exists a limit point z of A such that $z \notin A$; hence $z \in X-A$. Since $X-A \subset V_1$, then $z \in V_1$. But since $V_1 \subset D$ and $A \subset C$ and $D \cap C = \phi$, then $V_1 \cap A = \phi$. Since $V_1 \in T$ and $z \in V_1$ and $V_1 \cap A = \phi$, then this contradicts z being a limit point of A which contradicts A being closed. In either case we have a contradiction; hence A must be either open or closed in (X,T).

<u>Definition 3.5</u>: Let (X,T) be a maximally connected topological space. Let (X,S) be a disconnected topological space such that S is strictly finer than T. Let E and F be nonempty disjoint open sets in S such that $E \cup F = X$. Then x is called a <u>T-interior point</u> of E if there exists a $V \in T$ such that $x \in V \subset E$.

<u>Lemma 3.1</u>: Every open connected subspace of a maximally connected topological space is maximally connected.

Proof: Let (X,T) be a maximally connected topological space. Let A be an open subset of X such that the subspace (A,T_A) is connected, where T_A is the topology induced on A by T. Assume that (A,T_A) is not maximally connected. Hence, A has a subset D not in T_A such that the simple expansion $S = \langle T_A \cup \{D\} \rangle$ is strictly finer than T_A and (A,S) is also connected. Since $D \subset A$, then $D \cap A = D$. Since $T_A = \{U \cap A \mid U \in T\}$ and $D = D \cap A \notin T_A$, then $D \notin T$. Let $S' = \langle T \cup \{D\} \rangle$. Since S' is a simple expansion of T

and $D \not\in T$ and $D \in S'$, then S' is strictly finer than T. Since T is maximally connected, then S' is disconnected. Hence, there exist nonempty disjoint open sets E and F in S' such that $E \cup F = X$. Since $E, F \in S'$, then, by lemma 2.2, there exist $U_1, U_2, V_1, V_2 \in T$ such that $E = U_1 \cup (U_2 \cap D)$ and $F = V_1 \cup (V_2 \cap D)$. Hence, $E \cap A = (U_1 \cup (U_2 \cap D)) \cap A = (U_1 \cap A) \cup ((U_2 \cap D) \cap A)$ and $F \cap A = (V_1 \cup (V_2 \cap D)) \cap A = (V_1 \cap A) \cup ((V_2 \cap D) \cap A)$. Since $U_1 \in T$, then $U_1 \cap A \in T_A$; hence $U_1 \cap A \in S$. Now $(U_2 \cap D) \cap A = (U_2 \cap A) \cap D$. Since $U_2 \in T$, then $U_2 \cap A \in T_A$; hence $U_2 \cap A \in S$ and since $D \in S$, then $(U_2 \cap A) \cap D \in S$. Since S is a topology, then $(U_1 \cap A) \cup ((U_2 \cap D) \cap A) = E \cap A$ is an element of S. Similarly, $F \cap A \in S$. Since $E \cup F = X$ and $A \subset X$, then $(E \cap A) \cup (F \cap A) = A$. Also, since $E \cap F = \phi$, then $(E \cap A) \cap (F \cap A) = \phi$. Since $F \cap A$ and $E \cap A$ are in S and (A, S) is connected, then either $E \cap A = \phi$ or $F \cap A = \phi$; otherwise $E \cap A$ and $F \cap A$ would disconnect (A, S). If $E \cap A = \phi$, then $F \cap A = A$; hence $A \subset F$. If $F \cap A = \phi$, then $E \cap A = A$; hence $A \subset E$. So either $A \subset E$ or $A \subset F$.

Let $A \subset E$. Hence, $A \cap F = \phi$. Now $F = V_1 \cup (V_2 \cap D)$ and since $D \subset A$ and $A \cap F = \phi$, then $V_2 \cap D = \phi$; hence $F = V_1$. Since $V_1 \in T$, then $F \in T$. Let $x \in E$. Since $E = U_1 \cup (U_2 \cap D)$, then either $x \in U_1$ or $x \in U_2 \cap D$. If $x \in U_1 \subset E$, then x is a T-interior point of E. Since $D \subset A$, then $U_2 \cap D \subset U_2 \cap A$. Since $U_2 \in T$ and $A \in T$, then $U_2 \cap A \in T$. Also, since $A \subset E$, then $U_2 \cap A \subset E$. Therefore, if $x \in U_2 \cap D$, then $x \in U_2 \cap A \subset A$; hence x is again a T-interior point of E. Since every element of E is a T-interior point of E, then E is the union of open sets of T, which implies that $E \in T$. Hence, E and F are in T. But since E and F disconnect X, this contradicts the fact that (X,T) is connected. Thus, (A,T_A) must be maximally connected.

Lemma 3.2: Every closed connected subspace of a maximally connected topological space is maximally connected.

Proof: Let (X,T) be a maximally connected topological space. Let A be a closed subset of X such that the subspace (A,T_A) is connected, where T_A is the topology induced on A by T. Assume that (A,T_A) is not maximally connected. Hence, A has a subset D not in T_A such that the simple expansion $S = \langle T_A \cup \{D\} \rangle$ is strictly finer than T_A and (A,S) is connected. Let $M = X-(A-D)$. Since $D \subset A$, then $M \cap A = D$. Since $T_A = \{U \cup A | U \in T\}$ and $M \cap A = D$ and $D \notin T_A$, then $M \notin T$. Let $S' = \langle T \cup \{M\} \rangle$. Since S' is a simple expansion of T and $M \notin T$ and $M \in S'$, then S' is strictly finer than T. Since T is maximally connected, then S' is disconnected. Hence, there exist nonempty disjoint open sets F and G in S' such that $F \cup G = X$. Since $E,F \in S'$, then, by theorem 2.2, there exist $U_1,U_2,V_1,V_2 \in T$ such that $F = U_1 \cup (U_2 \cap M)$ and $G = V_1 \cup (V_2 \cap M)$. Now $F \cap A = (U_1 \cup (U_2 \cap M)) \cap A = (U_1 \cap A) \cup (U_2 \cap M \cap A)$ and since $U_1 \in T$, then $U_1 \cap A \in T_A$; hence $U_1 \cap A \in S$. Now $U_2 \cap M \cap A = (U_2 \cap A) \cap M$ and since $U_2 \in T$, then $U_2 \cap A \in T_A$; hence $U_2 \cap A \in S$. Let $a \in A$. Since $D \subset A$, then either $a \in D$ or $a \in A-D$. If $a \in D$, then $a \notin M-D$. If $a \in A-D$, then $a \notin M$. Hence, $A \cap M-D = \phi$. Therefore, $A \cap M = A \cap D$. Since $U_2 \cap A \subset A$, then

$(U_2 \cap A) \cap M = (U_2 \cap A) \cap D$ and since $U_2 \cap A \in S$ and $D \in S$, then $(U_2 \cap A) \cap D \in S$. Hence, $(U_2 \cap A) \cap M \in S$. Since S is a topology on A, then $(U_1 \cap A) \cup (U_2 \cap M \cap A) = F \cap A \in S$. Similarly $G \cap A \in S$. Since $F \cup G = X$ and $F \cap G = \phi$ and $A \subset X$, then $(F \cap A) \cup (G \cap A) = A$ and $(F \cap A) \cap (G \cap A) = \phi$. Since $F \cap A$ and $G \cap A$ are in S and (A,S) is connected, then either $F \cap A = \phi$ or $E \cap A = \phi$; otherwise $F \cap A$ and $G \cap A$ would disconnect (A,S). If $F \cap A = \phi$, then $G \cap A = A$ and $A \subset G$. If $G \cap A = \phi$, then $F \cap A = A$ and $A \subset F$. So either $A \subset F$ or $A \subset G$.

Let $A \subset F$. Then $G \cap A = \phi$. Let $x \in G = V_1 \cup (V_2 \cap M)$. Then either $x \in V_1$ or $x \in V_2 \cap M$. If $x \in V_1 \subset G$, then $V_1 \in T$ implies that x is a T-interior point of G. Since $A \subset F$, then $G = X\text{-}F \subset X\text{-}A$. Therefore, if $x \in V_2 \cap M \subset G$, then since $x \in G \subset X\text{-}A \subset M$, then $x \in [V_2 \cap (X\text{-}A)] \subset (V_2 \cap M) \subset G$. But since A is closed in (X,T), then $X\text{-}A \in T$. Hence, $x \in (V_2 \cap X\text{-}A) \in T$; which again implies that x is a T-interior point of G. Since every element of G is a T-interior point of G, then G is the union of open sets of T, which implies that $G \in T$. Let $y \in F = U_1 \cup (U_2 \cap M)$. Then either $y \in U_1$ or $y \in U_2 \cap M$. If $y \in U_1 \subset F$, then $U_1 \in T$ implies that y is a T-interior point of F. Now $U_2 \cap (X\text{-}M) \subset X\text{-}M = A\text{-}D \subset A \subset F$. Therefore, if $y \in U_2 \cap M$, then since $U_2 \cap X\text{-}M \subset F$, then $U_2 = [U_2 \cap ((X\text{-}M) \cup M)] \subset F$. But since $y \in U_2 \in T$, then y is again a T-interior point of F. Hence, F is the union of elements of T, which implies that $F \in T$. Hence, F and G are elements of T. But since F and G disconnect X, then this contradicts the fact that (X,T) is connected. Thus, (A,T_A) must be maximally connected.

Lemma 3.3: Let (X,T) be a topological space and let $A \subset X$. If A is a connected subset of X in (X,T), then $cl(A,T)$ is a connected subset of X in (X,T).

Proof: Let (X,T) be a topological space. Let A be a subset of X such that (A,T_A) is connected. Assume that $K = (cl(A,T),T_{cl(A,T)})$ is disconnected. Then there exist nonempty disjoint sets F and G in $T_{cl(A,T)}$ such that $cl(A,T) = F \cup G$. Since $A \subset cl(A,T)$, then $A \cap F$ and $A \cap G$ are both open in (A,T_A). Since $F \cup G = cl(A,T)$ and $F \cap G = \phi$, then $(A \cap F) \cup (A \cap G) = A$ and $(A \cap F) \cap (A \cap G) = \phi$. Since (A,T_A) is connected, then either $A \subset F$ or $A \subset G$. Let $A \subset F$. Then $A \cap G = \phi$. Since $A \subset F$, then $cl(A,T) \subset cl(F,T)$. But since $cl(A,T) = F \cup G$, then $G \subset cl(F,T)$. Let $p \in F$ and $q \in G$. Since $F \cap G = \phi$, then $p \notin G$ and $q \notin F$. Since $F,G \in T_{cl(A,T)}$, then p is not a limit point of G and q is not a limit point of F in K. Hence, $F \cap cl(G,T_{cl(A,T)}) = \phi$ and $G \cap cl(F,T_{cl(A,T)}) = \phi$. But $cl(F,T_{cl(A,T)}) = [cl(A,T) \cap cl(F,T)]$. Therefore, since $G \subset cl(A,T)$, then
$\phi = [G \cap cl(F,T_{cl(A,T)})] = [G \cap cl(A,T) \cap cl(F,T)] = [G \cap cl(F,T)]$.
Since $G \cap cl(F,T) = \phi$ and $G \subset cl(F,T)$, then $G = \phi$. But this contradicts the fact that G is nonempty. Thus, K is connected.

Theorem 3.5: A subspace of a maximally connected topological space is maximally connected if and only if the subspace is connected.

Proof: Let (X,T) be a maximally connected topological space.

Let A be a subset of X such that the subspace (A,T_A) is connected. Then by lemma 3.3, the subspace $K = (cl(A,T),T_{cl(A,T)})$ is connected. Since $cl(A,T)$ is closed in (X,T) and (X,T) is maximally connected, then, by lemma 3.2, K is maximally connected. Since K is maximally connected, then, by theorem 3.2, K is submaximal. Since A is dense in K and K is submaximal, then A is open in K. Since K is maximally connected, then, by lemma 3.1, $(A,T_{cl(A,T)(A)})$ is maximally connected, where $T_C = T_{cl(A,T)(A)}$ is the topology induced on A by $T_{cl(A,T)}$. Let $U \in T_A$. Then there is a $U_1 \in T$ such that $U = U_1 \cap A$. Since $A \subset cl(A,T)$, then $U_1 \cap A = (U_1 \cap cl(A,T)) \cap A$. Hence, $U \in T_C$. Let $V \in T_C$. Then there is a $V_1 \in T_{cl(A,T)}$ such that $V = (V_1 \cap A)$. Since $V_1 \in T_{cl(A,T)}$, then there is a $V_2 \in T$ such that $V_1 = V_2 \cap cl(A,T)$. Hence, $V = (V_2 \cap cl(A,T)) \cap A = V_2 \cap A$. Hence, $V \in T_A$. Therefore, $T_C = T_A$ and since (A,T_C) is maximally connected, then (A,T_A) is maximally connected.

Remark 3.1: The product space of a collection of maximally connected topological spaces need not be maximally connected as the next example exhibits.

Example 3.1: Let $X = \{a,b\}$. Then $T = \{\phi,X,\{b\}\}$ is a topology on X. Now T is connected and since $\{a\}$ is the only subset of X which is not in T and $\{a\} \cup \{b\} = X$, then (X,T) is maximally connected. Let X' be the product set of X with itself. Hence, $X' = X \times X = \{(a,a),(a,b),(b,a),(b,b)\}$. Now $B = \{\phi,X,\{(a,b),(b,b)\},\{(b,a),(b,b)\},\{(b,b)\}\}$ is a subbasis for the product topology

$S = \{X, \phi, \{(a,b),(b,b)\}, \{(b,a),(b,b)\}, \{(b,b)\}, \{(a,b),(b,a),(b,b)\}\}$.

By inspection it can be seen that (X',S) is connected. Let

$S' = \langle S \cup \{\{(b,a)\}\} \rangle$. Hence,

$S' = \{X, \phi, \{(a,b),(b,b)\}, \{(b,a),(b,b)\}, \{(b,b)\}, \{(a,b),(b,a),(b,b)\}, \{(b,a)\}\}$.

Now (X',S') is also connected and since S' is strictly finer than S, then the product space (X',S) is not maximally connected.

Definition 3.8: Let (X,T) be a topological space and let R be an equivalence relation on X. Let $X|R$ denote the set of R-equivalence classes. Define the function f from X into $X|R$ by $f(x) = \bar{x}$, where x is any element of X and $\bar{x}$ is the R-equivalence class of x. Then f is called the quotient mapping from X into $X|R$. Define a subset U of $X|R$ to be open if $f^{-1}(U)$ is open in X. The topology T' thus obtained on $X|R$ is called the quotient topology on $X|R$ and $(X|R,T')$ is the quotient space.

Remark 3.2: The quotient space defined by an equivalence relation on a maximally connected topological space is not necessarily maximally connected as the next example exhibits.

Example 3.2: Let $X = \{a,b,c,d\}$. Then

$T = \{X, \phi, \{b\}, \{c\}, \{b,c\}, \{a,b,c\}, \{b,d\}, \{b,c,d\}\}$ is a topology on X.

In fact (X,T) is connected and any simple expansion of T which is strictly finer than T is disconnected; hence (X,T) is maximally connected. Let $R = \{(a,b),(b,a),(c,d),(d,c),(a,a),(b,b),(c,c),(d,d)\}$.

Then $R \subset X \times X$ is a relation on X. Now R satisfies the reflexive, symmetric, and transitive properties; hence R is an equivalence relation on X. Let $\bar{b} = \{y \in X | yRb\} = \{a,b\}$ and let $\bar{d} = \{y \in X | yRd\} = \{c,d\}$. Hence, the set of equivalence classes on X is $X|R = \{\bar{b},\bar{d}\}$ and $X|R$ partitions X. Define $f : X \to X|R$ by $f(x) = \bar{x}$, where x is any element of X and $\bar{x}$ is the equivalence class of x. Hence, $f(a) = \bar{b}$, $f(b) = \bar{b}$, $f(c) = \bar{d}$, and $f(d) = \bar{d}$. The quotient topology on $X|R$ is $T' = \{U \subset X|R \,|\, f^{-1}(U) \in T\}$. Now $f^{-1}(\{\bar{b}\}) = \{a,b\} \notin T$, and $f^{-1}(\{\bar{d}\}) = \{c,d\} \notin T$, and $f^{-1}(\{\bar{b},\bar{d}\}) = f^{-1}(X|R) = X \in T$, and $f^{-1}(\phi) = \phi \in T$. Hence, $T' = \{X,\phi\}$, the trivial topology on $X|R$. But $(X|R,T')$ is not T_0; hence, by the contrapositive of theorem 3.1, the quotient space $(X|R,T')$ is not maximally connected.

Definition 3.9: Let (X,S) and (Y,T) be topological spaces and let f be a function from X into Y. Then f is said to be open if and only if the image of every element of S is an element of T.

Definition 3.10: Let (X,S) and (Y,T) be topological spaces and let f be a function from X into Y. Then f is said to be a connected function if and only if the image of every connected subset of X in (X,S) is connected in (Y,T).

Definition 3.11: Let (X,S) and (Y,T) be topological spaces and let f be a function from X onto Y. Then f is said to be a monotone function if and only if $y \in Y$ implies that $f^{-1}(\{y\})$ is connected in (X,S).

Theorem 3.8: If (X,S) is a maximally connected topological space and (Y,T) is a topological space and f is an open, connected, monotone function from (X,S) onto (Y,T), then (Y,T) is maximally connected.

Proof: Let (X,S) be a maximally connected topological space. Let (Y,T) be a topological space. Let f be an open, connected, monotone function from (X,S) onto (Y,T). Since (X,S) is connected and f is an onto connected function, then (Y,T) is connected. Assume that (Y,T) is not maximally connected. Then there exists a $D \subset Y$ such that $D \in T$ and the simple expansion $T' = \langle T \cup \{D\} \rangle$ is is connected and T' is strictly finer than T. Let $A = f^{-1}(D) = \{x \in X | f(x) \in D\}$. Let $S' = \langle S \cup \{A\} \rangle$. Since $A = f^{-1}(D)$, then $f(A) = f(f^{-1}(D)) = D$. Since f is open and $f(A) = D \notin T$, then $A \notin S$. Since $A \notin S$ and $A \in S'$ and S' is a simple expansion of S, then S' is strictly finer than S. Since (X,S) is maximally connected, then (X,S') is disconnected. Hence, there exist nonempty disjoint sets C and E in S' such that $C \cup E = X$. By theorem 2.2, there exist $U_1, U_2, V_1, V_2 \in S$ such that $C = U_1 \cup (U_2 \cap A)$ and $E = V_1 \cup (V_2 \cap A)$. Now $f(C) = f(U_1 \cup (U_2 \cap A)) = f(U_1) \cup f(U_2 \cap A)$ and $f(E) = f(V_1 \cup (V_2 \cap A)) = f(V_1) \cup f(V_2 \cap A)$. Since $A = f^{-1}(D)$, then $f(U_2 \cap A) = f(U_2 \cap f^{-1}(D))$ and $f(V_2 \cap A) = f(V_2 \cap f^{-1}(D))$.

Let $p \in f(U_2 \cap A)$. Then there exists an $x_1 \in U_2 \cap A$ such that $f(x_1) = p$. Since $x_1 \in U_2 \cap A$, then $x_1 \in U_2$ and $x_1 \in A$. Therefore, $f(x_1) = p \in f(U_2)$ and $f(x_1) = p \in f(A)$. So $p \in f(U_2) \cap f(A)$. Hence, $f(U_2 \cap A) \subset f(U_2) \cap f(A)$. Let $d \in f(U_2) \cap f(A) = f(U_2) \cap f(f^{-1}(D)) = f(U_2) \cap D$. Then $d \in f(U_2)$ and $d \in D$. Since $d \in f(U_2)$, then there exists an $x_2 \in U_2$ such that $f(x_2) = d$. Since $f^{-1}(D) = \{x \in X | f(x) \in D\}$ and $f(x_2) = d \in D$, then

$x_2 \in f^{-1}(D) = A$. Since $x_2 \in U_2$ and $x_2 \in A$, then $x_2 \in U_2 \cap A$; hence $d = f(x_2) \in f(U_2 \cap A)$. Hence, $f(U_2) \cap f(A) \subset f(U_2 \cap A)$. Thus, $f(U_2 \cap A) = f(U_2) \cap f(A) = f(U_2) \cap D$. Therefore, $f(C) = f(U_1) \cup (f(U_2) \cap D)$. Similarly $f(E) = f(V_1) \cup (f(V_2) \cap D)$.

Since $U_1, U_2 \in S$ and f is open, then $f(U_1)$ and $f(U_2)$ are elements of T; hence $f(U_1), f(U_2) \in T'$. Since $D \in T'$, then $f(C) = f(U_1) \cup (f(U_2) \cap D) \in T'$. Similarly $f(E) \in T'$. Since $E \cup C = X$ and $f(X) = Y$, then $f(E \cup C) = Y$; hence $f(E) \cup f(C) = Y$.

Assume that $y \in f(C) \cap f(E)$. Since $y \in f(C)$ and $y \in f(E)$, then there exist $r \in C$ and $t \in E$ such that $f(s) = y$ and $f(t) = y$. Hence, $\{r,t\} \subset f^{-1}(\{y\})$. Let $K = f^{-1}(\{y\})$. Since f is monotone, then K is connected in S. Thus, the subspace (K, S_K) is connected. Now $t \in K \cap E \neq \phi$ and $r \in K \cap C \neq \phi$. Since $E \cap C = \phi$, then $(K \cap E) \cap (K \cap C) = \phi$. Since $E \cup C = X$, then $(K \cap E) \cup (K \cap C) = K$. Now $K \cap C = [K \cap (U_1 \cup (U_2 \cap A)] = [(K \cap U_1) \cup (K \cap U_2 \cap A)]$ and $K \cap E = [(K \cap V_1) \cup (K \cap V_2 \cap A)]$. Since $f(C) = [f(U_1) \cup (f(U_2) \cap D)]$ and $f(E) = [f(V_1) \cup (f(V_2) \cap D)]$, then either $y \in D$ or $y \notin D$.

If $y \in D$, then $K = f^{-1}(\{y\}) \subset f^{-1}(D) = A$. Since $K \subset A$, then $K \cap A = K$, so $(K \cap A \cap V_2) = K \cap V_2$ and $(K \cap A \cap U_2) = K \cap U_2$. Therefore, $K \cap E = [(K \cap V_1) \cup (K \cap V_2)]$ and $K \cap C = [(K \cap U_1) \cup (K \cap U_2)]$. But since $U_1, U_2 \in S$, then $K \cap U_1$ and $K \cap U_2$ are elements of the relative topology S_K. Hence, $K \cap C = [(K \cap U_1) \cup (K \cap U_2)] \in S_K$. Similarly $K \cap E \in S_K$. Therefore, $K \cap C$ and $K \cap E$ disconnect (K, S_K), which contradicts the fact that

(K,S_K) is connected.

If $y \notin D$, then $y \notin f(U_2) \cap D$ and $y \notin f(V_2) \cap D$. Now $K \cap C = [(K \cap U_1) \cup (K \cap (U_2 \cap A))]$. Assume that $w \in (K \cap (U_2 \cap A))$. Then $w \in K$ and $w \in (U_2 \cap A)$. Since $w \in K$, then $f(w) = y$. Since $w \in U_2 \cap A$, then $y = f(w) \in f(U_2 \cap A) = [f(U_2) \cap f(A)] = [f(U_2) \cap D]$. But this contradicts the fact that $y \notin D$. Hence, $K \cap (U_2 \cap A) = \phi$. Similarly $K \cap (V_2 \cap A) = \phi$. Hence, $K \cap C = K \cap U_1$ and $K \cap E = K \cap V_1$. So $K \cap E \in S_K$ and $K \cap C \in S_K$. Since $K \cap C$ and $K \cap E$ disconnect (K,S_K), then this again contradicts the fact that (K,S_K) is connected. Therefore, in both cases we have a contradiction; hence $f(C) \cap f(E) = \phi$.

Since $E \neq \phi$ and $C \neq \phi$, then $f(E) \neq \phi$ and $f(C) \neq \phi$. Therefore, $f(E)$ and $f(C)$ disconnect (Y,T'), which contradicts the fact that (Y,T') is connected. Therefore, the assumption that (Y,T) is not maximally connected must be false.

SUMMARY

In chapter 1, the concept of a maximally connected topological space is defined. It has been shown that if X is a nonempty set and U is an ultrafilter on X, then $U \cup \{\phi\}$ is a maximally connected topology on X, but a maximally connected Hausdorff space cannot be generated by an ultrafilter. It was also shown that maximally connected Hausdorff spaces cannot be compact, locally compact, or first countable. It was decided that connected subspaces of maximally connected topological spaces inherit maximal connectedness but that this is not necessarily true for product and quotient spaces. If (X,T) is a maximally connected topological space, then every nondense open set has a cut point in its boundary. It is also true that the image of a maximally connected topological space under an open, connected, monotone function is maximally connected. It is known that there exist maximally connected Hausdorff spaces, but to the author's knowledge it is unknown whether there is a countable set X with topology T such that (X,T) is a maximally connected Hausdorff space.

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