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Let (Y,T) be a topological space, let $y_0 \in Y$, and let $C(Y,y_0)$ denote the set of continuous loops in Y at y_0. It has long been known that using continuous functions as relating functions on $C(Y,y_0)$ produces an equivalence relation on $C(Y,y_0)$, and that there is a natural binary operation on the resulting equivalence classes which makes the equivalence classes a group - called the fundamental group of Y at y_0, denoted by $\pi_1(Y,y_0)$. In this thesis another type of relating functions, the class of which we call an admitting homotopy relation, is defined and it is shown that these functions also produce a group, which we call the N-fundamental group of Y with respect to y_0, denoted by $N(Y,y_0)$. It is shown that this group satisfies the usual properties of the fundamental group, and that given a topological space (Y,T) and $y_0 \in Y$, there is an epimorphism from $\pi_1(Y,y_0)$ onto $N(Y,y_0)$.

NON-CONTINUOUS FUNDAMENTAL

GROUPS OF CONTINUOUS

LOOPS

by

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TABLE OF CONTENTS

INTRODUCTION

Let (Y,T) be a topological space, let $y_0 \in Y$, and let $C(Y,y_0)$ denote the set of continuous loops in Y at y_0. The idea of using continuous functions as relating functions on $C(Y,y_0)$ has long been in existence, and the resulting homotopy groups have been examined. In [1] another type of relating functions is defined and it is shown that these functions also produce a group. The purpose of this thesis is to supply the details necessary to [1].

In Chapter I, basic definitions, theorems, and notation are given.

In Chapter II, an admitting homotopy relation is defined as a class of functions which contains the class of continuous functions and certain non-continuous functions. The relation $\overset{N}{\underset{y_0}{\sim}}$ on Y is defined and shown to be an equivalence relation on $C(Y,y_0)$. This set of equivalence classes, denoted by $N(Y,y_0)$, is then shown to be a group.

In Chapter III, the fundamental group of Y modulo y_0, denoted by $\pi_1(Y,y_0)$, is defined and it is shown that there is an epimorphism from $\pi_1(Y,y_0)$ onto $N(Y,y_0)$. Also, if (X,S) and (Y,T) are topological spaces and $x_0 \in X$, $y_0 \in Y$, and H is a homeomorphism from X onto Y such that $H(x_0) = y_0$, it is shown that $N(X,x_0)$ is isomorphic to $N(Y,y_0)$. Finally, a pathwise connected topological space is defined, and it is shown that if (Y,T)

is a pathwise connected topological space and $y_0, y_1 \in Y$, then $N(Y,y_0)$ is isomorphic to $N(Y,y_1)$

The author is assuming elementary facts about set theory, functions, and the real number system. The reader is referred to [1], [2], [4], and [5] for definitions and theorems not covered in this paper.

CHAPTER I

Definition 1: A topological space is an ordered pair (X,T) such that X is a set, T is a collection of subsets of X, and each of the following is true:

(i) $\phi \in T$,

(ii) $X \in T$,

(iii) if $W \subset T$, then $\cup W = \cup \{U | U \in W\} \in T$, and

(iv) if $W \subset T$ and W is finite, then

$\cap W = \{x | \text{if } U \in W, \text{ then } x \in U\} \in T$.

The elements of T are called open subsets of X. A subset C of X is said to be closed provided that $X - C = \{x | x \in X \text{ and } x \notin C\}$ is open.

Definition 2: If $M \subset X$ and $p \in X$, then p is said to be a limit point of M provided that if U is an open subset of X containing p, then U contains a point of M different from p.

Definition 3: Let (X,T) be a topological space and let $M \subset X$. The closure of M, denoted by $\overline{M}$, is the set to which p belongs only in case $p \in M$ or p is a limit point of M. The interior of M, denoted by $M°$, is the set to which p belongs only in case there is a $U \in T$ such that $p \in U \subset M$. The boundary of M is $\overline{M} - M°$.

Definition 4: Let (X,S) and (Y,T) be topological spaces, let $f: (X,S) \to (Y,T)$ be a function, let $A \subset S$, and let $B \subset T$. Then $f(A) = \{p | p \in Y \text{ and there exists an } a \in A \text{ such that } f(A) = p\}$ and

$f^{-1}(B) = \{q \mid q \in X$ and $f(q) \in B\}$.

Definition 5: Let (X,S) and (Y,T) be topological spaces. A function $f\colon (X,S) \to (Y,T)$ is said to be continuous provided that if $U \in T$ then $f^{-1}(U) \in S$. A function $f\colon (X,S) \to (Y,T)$ is said to be a homeomorphism provided that f is one-to-one, onto, if $\sigma \in T$, then $f^{-1}(\sigma) \in S$, and if $V \in S$, then $f(V) \in T$.

Definition 6: Let R be the set of real numbers. Let λ be the collection of subsets of R to which U belongs provided that if $p \in U$, then there are elements $a,b \in R$ such that $a < p < b$ and if $x \in R$ and $a < x < b$, then $x \in U$.

Theorem 1: The space (R,λ) is a topological space.

Proof: Since there are no elements in ϕ, then $\phi \in \lambda$. Let $p \in R$. Then $p-1 < p < p+1$. Let $x \in R$ such that $p-1 < x < p+1$. Then $x \in R$. Hence, $R \in \lambda$. Let $W \subset \lambda$. Let $p \in \cup W$. Then there exists a set $U \in W$ such that $p \in U$. Since $U \in \lambda$ there exist $a,b \in R$ such that $a < p < b$ and if $x \in R$ such that $a < x < b$, then $x \in U$. Let $x \in R$ such that $a < x < b$. Then $x \in U$. So $x \in \cup W$. Hence, $\cup W \in \lambda$. Let $W \subset \lambda$ such that W is a non-empty finite set. Let $p \in \cap W$. Then for each $U \in W$ there are numbers a_U and b_U such that $a_U < p < b_U$ and if $x \in R$ and $a_U < x < b_U$ then $x \in U$. Define $A = \{a_U \mid U \in W\}$ and $B = \{b_U \mid U \in W\}$. Let a be the greatest element of A and let b be the least element of B. Hence, $a < p < b$. Let $x \in R$ such that $a < x < b$. Then for each $U \in W$, $a_U \leq a < x < b \leq b_U$ and hence, $x \in U$. Thus, $x \in \cap W$. Hence, $\cap W \in \lambda$. Hence (R,λ) is a topological space.

The topology λ is called the <u>usual topology for the reals</u>.

<u>Definition 7</u>: Let (X,T) be a topological space and let $A \subset X$. Let T_A be the collection of subsets of A to which U belongs provided that there is an element $V \in T$ such that $U = V \cap A$.

<u>Theorem 2</u>: If (X,T) is a topological space and $A \subset X$, then (A,T_A) is a topological space.

<u>Proof</u>: Since $\phi \in T$ and $A \cap \phi = \phi$, then $\phi \in T_A$. Since $A \subset X$ and $X \in T$ and $A \cap X = A$, then $A \in T_A$. Let $W \subset T_A$. If $U \in W$, then there exists $\mathcal{O}_U \in T$ such that $\mathcal{O}_U \cap A = U$. Let $\mathcal{O} = \cup \left\{\mathcal{O}_U | U \in W\right\}$. Then $\mathcal{O} \in T$ and $\mathcal{O} \cap A \in T_A$. Let $q \in \mathcal{O} \cap A$. Then $q \in A$ and $q \in \mathcal{O}$. Since $q \in \mathcal{O}$, then there exists $U \in W$ such that $q \in \mathcal{O}_U$. Now $\mathcal{O}_U \cap A = U$ and $q \in U$. So $q \in \cup W$. Let $q \in \cup W$. Then there is a $U \in W$ such that $q \in U$. Since $U = \mathcal{O}_U \cap A$, then $q \in \mathcal{O}_U$ and $q \in A$. Since $q \in \mathcal{O}_U$, $q \in \mathcal{O}$, and $q \in A$, then $q \in \mathcal{O} \cap A$. Hence, $\cup W = \mathcal{O} \cap A$ and $\cup W \in T_A$. Let $W \subset T_A$ such that W is a non-empty finite set. If $U \in W$ then there is an $\mathcal{O}_U \in T$ such that $\mathcal{O}_U \cap A = U$. Let $B = \left\{\mathcal{O}_U | U \in W\right\}$ and let $P = \cap B$. Then $P \in T$ and $P \cap A \in T_A$. Let $p \in P \cap A$. Then $p \in P$ and $p \in A$. Let $U \in W$. Then $U = \mathcal{O}_U \cap A$. Since $p \in P$, $p \in \mathcal{O}_U$, and since $p \in A$, then $p \in \mathcal{O}_U \cap A = U$. So $p \in \cap W$. Let $p \in \cap W$. Let $M \in B$. Then there is a $U \in W$ such that $\mathcal{O}_U = M$. Since $p \in \cap W$ and $U \in W$, then $p \in U$. Since $U = \mathcal{O}_U \cap A$, then $p \in \mathcal{O}_U = M$. Since for each $M \in B$, $p \in M$, then $p \in \cap B = P$. Since $W \neq \phi$, there is a $V \in W$. Since $V \in W$, then $V = \mathcal{O}_V \cap A$. Since $p \in \cap W$, $p \in V$ and hence $p \in A$. Thus $p \in P \cap A$. Thus, $\cap W = P \cap A$. Hence, (A,T_A) is a topological

space.

The topology T_A is called the <u>relative topology for</u> A <u>induced by</u> T.

<u>Definition 8</u>: Let (X,S) and (Y,T) be topological spaces. Then $X \times Y = \{(x,y) | x \in X \text{ and } y \in Y\}$. The <u>product space</u> of (X,S) and (Y,T) is a set $Z = X \times Y$ and a topology W such that $\mathcal{O} \in W$ provided that if $(p,q) \in \mathcal{O}$, then there exists $U \in S$ and $V \in T$ such that $(p,q) \in U \times V \subset \mathcal{O}$. A set M is called a <u>basic open set</u> in W provided that there exists $U \in S$ and $V \in T$ such that $M = U \times V$.

<u>Notation</u>: Let (X,S) and (Y,T) be topological spaces, let $f\colon (X,S) \to (Y,T)$ be a function and let $A \subset X$. Then $f|_A$ is a function from (A,S_A) into (Y,T) defined by if $x \in A$, then $f|_A(x) = f(x)$.

If a and b are real numbers and $a < b$, then $[a,b] = \{x | a \leq x \leq b\}$. Unless otherwise stated, we will assume that the topology we are considering on $[a,b]$ to be $\lambda_{[a,b]}$.

Also, if a, b, c, and d are real numbers such that $a < b$ and $c < d$, then $[a,b] \times [c,d]$ will be assumed to have the usual product topology.

CHAPTER II

Definition 9: Let N be the class of functions with the following four properties:

(i) N contains the class of all continuous functions,

(ii) if (X,S), (Y,T), and (Z,W) are topological spaces, and $f\colon (X,S) \to (Y,T)$ is in N, and $g\colon (Y,T) \to (Z,W)$ is a homeomorphism, then $gf\colon (X,S) \to (Z,W)$ is in N,

(iii) if (X,S), (Y,T), and (Z,W) are topological spaces and $f\colon (X,S) \to (Y,T)$ is a homeomorphism, and $g\colon (Y,T) \to (Z,W)$ is in N, then $gf\colon (X,S) \to (Z,W)$ is in N, and

(iv) if a, b, c, d, α, and β are real numbers such that $a < b < c < d$ and $\alpha < \beta$, and (Y,T) is a topological space, and $f\colon [a,d] \times [\alpha,\beta] \to (Y,T)$ is a function such that $f|_{[a,b] \times [\alpha,\beta]}$ and $f|_{[c,d] \times [\alpha,\beta]}$ are in N, and $f|_{[b,c] \times [\alpha,\beta]}$ is continuous, then f is in N.

Then N is called an admitting homotopy relation.

Theorem 3: Let (X,S), (Y,T), and (Z,W) be topological spaces. Let $f\colon (X,S) \to (Y,T)$ and $g\colon (Y,T) \to (Z,W)$ be functions. Let $U \in W$. Then $f^{-1}(g^{-1}(U)) = (gf)^{-1}(U)$.

Proof: By Definition 4,
$f^{-1}(g^{-1}(U)) = \{q \mid q \in X \text{ and } f(q) \in g^{-1}(U)\}$ and
$(gf)^{-1}(U) = \{p \mid p \in X \text{ and } gf(p) \in U\}$. Let $x \in f^{-1}(g^{-1}(U))$.

Then $x \in X$ and $f(x) \in g^{-1}(U)$. Since $f(x) \in g^{-1}(U)$, then $f(x) \in Y$ and $gf(x) \in U$. But if $x \in X$ and $gf(x) \in U$, then $x \in (gf)^{-1}(U)$. Let $x \in (gf)^{-1}(U)$. Then $x \in X$ and $gf(x) \in U$. Since $gf(x) \in U$, then $f(x) \in g^{-1}(U)$. But if $x \in X$ and $f(x) \in g^{-1}(U)$, then $x \in f^{-1}(g^{-1}(U))$. Hence, $f^{-1}(g^{-1}(U)) = (gf)^{-1}(U)$.

Theorem 4: Let (X,S), (Y,T), and (Z,W) be topological spaces. Let $f: (X,S) \to (Y,T)$ be continuous and let $g: (Y,T) \to (Z,W)$ be continuous. Then gf is continuous.

Proof: Let U be an open set in W. Then $g^{-1}(U)$ is an open set in T. Then $f^{-1}(g^{-1}(U))$ is an open set in S. So $(gf)^{-1}(U) = f^{-1}(g^{-1}(U))$ is an open set in S. Hence, gf is continuous.

Theorem 5: Let (X,S) and (Y,T) be topological spaces. Let $f: (X,S) \to (Y,T)$ be a homeomorphism. Let $A \subset X$ and let $B = f(A)$. Then $f|_A: (A,S_A) \to (B,T_B)$ is a homeomorphism.

Proof: Since $f: (X,S) \to (Y,T)$ is one-to-one and onto, and since $B = f(A)$, then $f|_A: (A,S_A) \to (B,T_B)$ is one-to-one and onto. Let $U \in T_B$. Then there exists $\mathcal{O} \in T$ such that $U = \mathcal{O} \cap B$. Let $x \in (f|_A)^{-1}(U)$. Since $x \in (f|_A)^{-1}(U)$, there exists $a \in U$ such that $f(x) = a$. Since $U \in T_B$, then $a \in B$. Since f is one-to-one, and $B = f(A)$, then $x \in A$. Also, $a \in U = \mathcal{O} \cap B \subset \mathcal{O}$. Since $a \in \mathcal{O}$, $x = f^{-1}(a) \in f^{-1}(\mathcal{O})$. Thus, $x \in f^{-1}(\mathcal{O}) \cap A$. Hence $(f|_A)^{-1}(U) \subset f^{-1}(\mathcal{O}) \cap A$. Let $x \in f^{-1}(\mathcal{O}) \cap A$. Then $x \in f^{-1}(\mathcal{O})$ and $x \in A$. Since $x \in f^{-1}(\mathcal{O})$, There exists $p \in \mathcal{O}$ such that $f(x) = p$. Since $x \in A$, then $p \in B$. Since $p \in \mathcal{O}$ and $p \in B$, then

$p \in \mathcal{O} \cap B = U$. So $x = f^{-1}(p) \in f^{-1}(\mathcal{O} \cap B)$. Thus, $x \in (f|_A)^{-1}(U)$. Hence, $f^{-1}(\mathcal{O}) \cap A \subset (f|_A)^{-1}(U)$. Therefore, $f^{-1}(\mathcal{O}) \cap A = (f|_A)^{-1}(U)$. Since f is a homeomorphism and $\mathcal{O} \in T$, then $f^{-1}(\mathcal{O}) \in S$. Thus, $(f|_A)^{-1}(U) = f^{-1}(\mathcal{O}) \cap A \in S_A$. Let $V \in S_A$. Then there exists $\mathcal{O} \in S$ such that $V = \mathcal{O} \cap A$. Let $x \in f|_A(V)$. Since $x \in f|_A(V)$, there exists $a \in V$ such that $f(a) = x$. Since $V \in S_A$, $a \in A$. Since f is one-to-one and $B = f(A)$, then $x \in B$. Also, $a \in V = \mathcal{O} \cap A \subset \mathcal{O}$. Since $a \in \mathcal{O}$, then $x = f(a) \in f(\mathcal{O})$. Thus, $x \in f(\mathcal{O}) \cap B$. Hence $f|_A(V) \subset f(\mathcal{O}) \cap B$. Let $x \in f(\mathcal{O}) \cap B$. Then $x \in f(\mathcal{O})$ and $x \in B$. Since $x \in f(\mathcal{O})$, there exists $p \in \mathcal{O}$ such that $f(p) = x$. Since $x \in B$, $p \in A$. Since $p \in \mathcal{O}$ and $p \in A$, then $p \in \mathcal{O} \cap A = V$. So $x = f(p) \in f(\mathcal{O} \cap A)$. Thus, $x \in f|_A(V)$. Hence $f(\mathcal{O}) \cap B \subset f|_A(V)$. Therefore, $f(\mathcal{O}) \cap B = f|_A(V)$. Since f is a homeomorphism and $\mathcal{O} \in S$, then $f(\mathcal{O}) \in T$. Thus, $f|_A(V) = f(\mathcal{O}) \cap B \in T_B$. Hence, $f|_A\colon (A,S_A) \to (B,T_B)$ is a homeomorphism.

<u>Theorem 6</u>: Let N be an admitting homotopy relation. Let a, b, c, d, α, and β be real numbers such that $a < b < c < d$ and $\alpha < \beta$, and let (Y,T) be a topological space. Let $f\colon [\alpha,\beta] \times [a,d] \to (Y,T)$ be a function such that $f|_{[\alpha,\beta] \times [a,b]}$ and $f|_{[\alpha,\beta] \times [c,d]}$ are in N and $f|_{[\alpha,\beta] \times [b,c]}$ is continuous. Then f is in N.

<u>Proof</u>: Define a function $g\colon [a,d] \times [\alpha,\beta] \to [\alpha,\beta] \times [a,d]$ by if $(x,y) \in [a,d] \times [\alpha,\beta]$, then $g(x,y) = (y,x)$. Let (x_1,y_1) and (x_2,y_2) be elements of $[a,d] \times [\alpha,\beta]$ such that $g(x_1,y_1) = g(x_2,y_2)$. Then $(y_1,x_1) = g(x_1,y_1) = g(x_2,y_2) = (y_2,x_2)$. Thus, $y_1 = y_2$ and

$x_1 = x_2$. Hence, $(x_1,y_1) = (x_2,y_2)$ and g is one-to-one. Let $(p,q) \in [\alpha,\beta] \times [a,d]$. Then $(q,p) \in [a,d] \times [\alpha,\beta]$, and $g(q,p) = (p,q)$ and g is onto. Let U be an open subset of $[\alpha,\beta] \times [a,d]$. Let $(u,v) \in g^{-1}(U)$ such that (u,v) is not on the boundary of $[a,d] \times [\alpha,\beta]$. Then $(v,u) = g(u,v) \in U$. There exist real numbers h, i, j, and k such that $h < v < i$, $j < u < k$, and $(h,i) \times (j,k) \subset U$. Then $(u,v) \in (j,k) \times (h,i) \subset g^{-1}(U)$. Thus (u,v) is an element of a basic open set in the product topology which is contained in $g^{-1}(U)$. If $(u,v) \in g^{-1}(U)$ and (u,v) is on the boundary of $[a,d] \times [\alpha,\beta]$, in a similar manner a basic open set containing (u,v) contained in $g^{-1}(U)$ can be found. Hence, $g^{-1}(U)$ is the union of open sets and hence open. Let V be an open subset of $[a,d] \times [\alpha,\beta]$. Let $(r,s) \in g(V)$ such that (r,s) is not on the boundary of $[\alpha,\beta] \times [a,d]$. Then $(s,r) = g^{-1}(r,s) \in V$. There exist real numbers h_1, i_1, j_1, and k_1 such that $h_1 < s < i_1$, $j_1 < r < k_1$, and $(h_1,i_1) \times (j_1,k_1) \subset V$. Then $(r,s) \in (j_1,k_1) \times (h_1,i_1) \subset g(V)$. Thus (r,s) is an element of a basic open set in the product topology which is contained in $g(V)$. If $(r,s) \in g(V)$ and (r,s) is on the boundary of $[\alpha,\beta] \times [a,d]$, in a similar manner a basic open set containing (r,s) contained in $g(V)$ can be found. Hence, $g(V)$ is the union of open sets and hence open. Hence, g is a homeomorphism. By Theorem 5, $g|_{[a,b] \times [\alpha,\beta]}: [a,b] \times [\alpha,\beta] \to [\alpha,\beta] \times [a,b]$ and $g|_{[c,d] \times [\alpha,\beta]}:[c,d] \times [\alpha,\beta] \to [\alpha,\beta] \times [c,d]$ are homeomorphisms. Since $f|_{[\alpha,\beta] \times [a,b]}$ and $f|_{[\alpha,\beta] \times [c,d]}$ are in N, then by Definition 9(iii), $f|_{[\alpha,\beta] \times [a,b]}\, g|_{[a,b] \times [\alpha,\beta]}$ and

$f\big|_{[\alpha,\beta] \times [c,d]}\, g\big|_{[c,d] \times [\alpha,\beta]}$ are in N. Since $f\big|_{[\alpha,\beta] \times [b,c]}$ is continuous, then by Theorem 4, $f\big|_{[\alpha,\beta] \times [b,c]}\, g\big|_{[b,c] \times [\alpha,\beta]}$ is continuous. Hence by Definition 9(iv), fg is in N. Since g^{-1} is a homeomorphism, then by Definition 9(iii), $f = fgg^{-1}$ is in N.

Definition 10: Let (Y,T) be a topological space and let $y_0 \in Y$. Let I be the closed interval [0,1] and let $C(Y,y_0)$ be the set of all continuous $f\colon I \to Y$ such that $f(0) = y_0 = f(1)$. Let N be an admitting homotopy relation. Let $f,g \in C(Y,y_0)$. We say that f *is* N-*homotopic* *to* g *modulo* y_0, denoted by $f \underset{y_0}{\overset{N}{\approx}} g$, provided there is a function $F\colon I \times I \to Y$ in N such that if $x \in I$ and $t \in I$, then $F(x,0) = f(x)$, $F(x,1) = g(x)$, and $F(0,t) = y_0 = F(1,t)$.

Definition 11: Let S be a set. Then a relation ~ on S is said to be an *equivalence* *relation* if each of the following is true:

(i) if $x \in S$, then $x \sim x$,

(ii) if $x,y \in S$ and $x \sim y$, then $y \sim x$, and

(iii) if $x,y,z \in S$ and $x \sim y$ and $y \sim z$, then $x \sim z$.

<u>Theorem 7</u>: Let S be a set. Let $\sim$ be an equivalence relation on S. Then there exists a collection C of disjoint non-empty subsets of S such that $\cup C = S$ and

(i) if $c \in C$ and $s_1, s_2 \in c$, then $s_1 \sim s_2$,

(ii) if $c, d \in C$ and $c \neq d$ and $s_1 \in c$ and $s_2 \in d$, then it is not the case that $s_1 \sim s_2$,

(iii) if $s_1, s_2 \in S$ and $s_1 \sim s_2$, then there is a $c \in C$ such that $s_1 \in c$ and $s_2 \in c$,

(iv) if $s_1, s_2 \in S$ and it is not the case that $s_1 \sim s_2$, then there does not exist $c \in C$ such that $s_1 \in c$ and $s_2 \in c$.

Moreover, there is only one such collection.

<u>Proof</u>: Define C by $c \in C$ if and only if c is not empty and if $s_1, s_2 \in c$, then $s_1 \sim s_2$ and if $s_1 \in c$ and $s_2 \notin c$, then it is not the case that $s_1 \sim s_2$. Let $a \in S$. Let $c = \{s | s \in S \text{ and } s \sim a\}$. Since $a \sim a$, $a \in c$ and c is non-empty. Let $s_1, s_2 \in c$. Then $s_1 \sim a$, $s_2 \sim a$, and hence $s_1 \sim s_2$. Let $s_1 \in c$ and let $s_2 \notin c$. Assume $s_1 \sim s_2$. Then $s_2 \in c$, which is bad. Thus, it is not the case that $s_1 \sim s_2$. Hence, $c \in C$. Let $s \in \cup C$. Then there exists $c \in C$ such that $s \in c$. Since $c \subset S$, $s \in S$. Hence, $\cup C \subset S$. Let $s \in S$. Since $s \sim s$, then $s \in c$. Hence, $c \in C$ and $s \in C$. Thus, $S \subset \cup C$. Therefore, $\cup C = S$. By the definition of C, it is clear that (i) is true. Let $c, d \in C$ such that $c \neq d$. Let $s_1 \in c$ and let $s_2 \in d$. Assume $s_1 \sim s_2$. Then if $x \in d$, $s_1 \sim x$ and $x \in c$. Thus $d \subset c$. Also, if $y \in c$,

$s_2 \sim y$ and $y \in d$. Thus $c \subset d$. Hence $c = d$, which is bad. Therefore, it is not the case that $s_1 \sim s_2$. Thus (ii) is true. Since $c = \{s \mid s \in S$ and $s \sim a\}$, it is clear that (iii) is true. Let $s_1, s_2 \in S$ such that it is not the case that $s_1 \sim s_2$. Assume there exists $c \in C$ such that $s_1 \in c$ and $s_2 \in c$. Then $s_1 \sim s_2$, which is bad. Hence, there does not exist $c \in C$ such that $s_1 \in c$ and $s_2 \in c$. Thus (iv) is true. Let $a, b \in C$ such that $a \neq b$. Assume $a \cap b \neq \phi$. Then there exists some $s \in S$ such that $s \in a$ and $s \in b$. Hence if $x \in a$, $s \sim x$, and if $y \in b$, $s \sim y$. Thus $x \sim y$ and $a = b$, which is bad. Therefore $a \cap b = \phi$ and the elements of C are disjoint. Assume that there are two collections of disjoint non-empty subsets of S, C_1 and C_2, such that $C_1 \neq C_2$, $\cup C_1 = S$, $\cup C_2 = S$, and (i), (ii), (iii), and (iv) hold. Let $s \in S$. Then there exists $a \in C_1$ and $b \in C_2$ such that $s \in a$ and $s \in b$. If $x \in a$, then $s \sim x$. If $y \in b$, $s \sim y$. Thus $x \sim y$ and hence, $a = b$. Therefore, $C_1 = C_2$, which is bad. Hence, there is only one such collection.

Definition 12: The set C in Theorem 7 is called the set of equivalence classes on S given by $\sim$.

Theorem 8: Let (Y,T) be a topological space and let $y_0 \in Y$. The relation $\overset{N}{\underset{y_0}{\approx}}$ is an equivalence relation on $C(Y,y_0)$.

Proof: Let $f \in C(Y,y_0)$. Define a function $F: I \times I \to Y$ by if $x \in I$ and $t \in I$, then $F(x,t) = f(x)$. Define a function $g: I \times I \to I$ by if $x \in I$ and $t \in I$, then $g(x,t) = x$. Let Θ be an open subset of I. Then $g^{-1}(\Theta) = \Theta \times I$ is open and in $I \times I$.

So g is continuous. Now $F(x,t) = f(g(x,t)) = f(x)$ and since f is continuous, then by Theorem 4, F is continuous. Hence, F is in N. Hence, by Definition 10, $f \underset{y_0}{\overset{N}{\approx}} f$.

Let $f,g \in C(Y,y_0)$ and suppose that $f \underset{y_0}{\overset{N}{\approx}} g$. Then there is an element $F: I \times I \to Y$ in N such that if $x \in I$ and $t \in I$, then $F(x,0) = f(x)$, $F(x,1) = g(x)$, and $F(0,t) = y_0 = F(1,t)$. Define $G: I \times I \to Y$ by if $(x,t) \in I \times I$, then $G(x,t) = F(x,1-t)$. If $x \in I$, then $G(x,0) = F(x,1) = g(x)$ and $G(x,1) = F(x,0) = f(x)$, and if $t \in I$, then $G(0,t) = F(0,1-t) = y_o = F(1,1-t) = G(1,t)$. Define $K: I \times I \to I \times I$ by if $(x,t) \in I \times I$, then $K(x,t) = (x,1-t)$. Let (x_1,y_1) and (x_2,y_2) be elements of $I \times I$ such that $K(x_1,y_1) = K(x_2,y_2)$. Then $(x_1,1-y_1) = K(x_1,y_1) = K(x_2,y_2) = (x_2,1-y_2)$. Thus, $x_1 = x_2$ and $1-y_1 = 1-y_2$ and $y_1 = y_2$. Hence, $(x_1,y_1) = (x_2,y_2)$ and K is one-to-one. Let $(p,q) \in I \times I$. Then $(p,1-q) \in I \times I$ and $K(p,1-q) = (p,q)$ and K is onto. Let U be an open subset of $I \times I$. Let $(u,v) \in K^{-1}(U)$ such that (u,v) is not on the boundary of $I \times I$. Then $(u,1-v) = K(u,v) \in U$. There exist real numbers h, i, j, and k such that $h < u < i$, $j < 1-v < k$, and $(h,i) \times (j,k) \subset U$. Then $(u,v) \in (h,i) \times (1-k,1-j) \subset K^{-1}(U)$. Thus (u,v) is an element of a basic open set in the product topology which is contained in $K^{-1}(U)$. If $(u,v) \in K^{-1}(U)$ and (u,v) is on the boundary of $I \times I$, in a similar manner a basic open set containing (u,v) contained in $K^{-1}(U)$ can be found. Hence, $K^{-1}(U)$ is the union of open sets and hence open. Let V be an open set of $I \times I$. Let $(r,s) \in K(V)$ such that (r,s) is not on the boundary of $I \times I$. Then $(r,1-s) = K^{-1}(r,s) \in V$. There exist real numbers h_1, i_1, j_1, and k_1 such that $h_1 < r < i_1$, $j_1 < 1-s < k_1$, and $(h_1,i_1) \times (j_1,k_1) \subset V$. Then $(r,s) \in (h_1,i_1) \times (1-k_1,1-j_1) \subset K(V)$. Thus (r,s) is an element of a basic open set in the product topology which is contained in $K(V)$. If $(r,s) \in K(V)$ and (r,s) is on the boundary of $I \times I$, in a similar manner a basic open set containing (r,s) contained in $K(V)$ can be found. Hence, $K(V)$ is the union of open sets and hence open. Hence, K is a homeomorphism. Thus, by Definition 9(iii), FK is in N. But if $(x,t) \in I \times I$, then $G(x,t) = F(x,1-t) = FK(x,t)$. Hence, by Definition 9(iii), G is in N. Hence, $g \underset{y_0}{\overset{N}{\approx}} f$.

Let $f,g,h \in C(Y,y_0)$ and suppose that $f \underset{y_0}{\overset{N}{\approx}} g$ and $g \underset{y_0}{\overset{N}{\approx}} h$. Then there are elements $F,G\colon I \times I \to Y$ in N such that if $x \in I$ and $t \in I$, then $F(x,0) = f(x)$, $F(x,1) = g(x)$, and $F(0,t) = y_0 = F(1,t)$, and $G(x,0) = g(x)$, $G(x,1) = h(x)$, and $G(0,t) = y_0 = G(1,t)$. Define $H\colon I \times I \to Y$ by if $(x,t) \in I \times I$, then

$$H(x,t) = \begin{cases} F(x,4t) & \text{if } x \in I,\ 0 \leq t \leq 1/4 \\ g(x) & \text{if } x \in I,\ 1/4 \leq t \leq 3/4. \\ G(x,4t-3) & \text{if } x \in I,\ 3/4 \leq t \leq 1 \end{cases}$$

Define $\alpha\colon I \times [0,1/4] \to I \times I$ by if $(x,t) \in I \times [0,1/4]$ then $\alpha(x,t) = (x,4t)$, and define $\beta\colon I \times [3/4,1] \to I \times I$ by if $(x,t) \in I \times [3/4,1]$ then $\beta(x,t) = (x,4t-3)$. Then $H|_{I \times [0,1/4]} = F\alpha$ and $H|_{I \times [3/4,1]} = G\beta$. Let (x_3,y_3) and (x_4,y_4) be elements of $I \times [0,1/4]$ such that $\alpha(x_3,y_3) = \alpha(x_4,y_4)$.

Then $(x_3, 4y_3) = \alpha(x_3, y_3) = \alpha(x_4, y_4) = (x_4, 4y_4)$. Thus, $x_3 = x_4$ and $y_3 = y_4$. Hence, $(x_3, y_3) = (x_4, y_4)$ and α is one-to-one. Let $(p_1, q_1) \in I \times I$. Then $(p_1, q_1/4) \in I \times [0,1/4]$, and $\alpha(p_1, q_1/4) = (p_1, q_1)$ and α is onto. Let U be an open subset of $I \times I$. Let $(u_1, v_1) \in \alpha^{-1}(U)$ such that (u_1, v_1) is not on the boundary of $I \times [0,1/4]$. Then $(u_1, 4v_1) = \alpha(u_1, v_1) \in U$. There exist real numbers h_2, i_2, j_2, and k_2 such that $h_2 < u_1 < i_2$, $j_2 < 4v_1 < k_2$, and $(h_2, i_2) \times (j_2, k_2) \subset U$. Then $(u_1, v_1) \in (h_2, i_2) \times (j_2/4, k_2/4) \subset \alpha^{-1}(U)$. Thus (u_1, v_1) is an element of a basic open set in the product topology which is contained in $\alpha^{-1}(U)$. If $(u_1, v_1) \in \alpha^{-1}(U)$ and (u_1, v_1) is on the boundary of $I \times [0,1/4]$, in a similar manner a basic open set containing (u_1, v_1) contained in $\alpha^{-1}(U)$ can be found. Hence, $\alpha^{-1}(U)$ is the union of open sets and hence open. Let V be an open subset of $I \times [0,1/4]$. Let $(r_1, s_1) \in \alpha(V)$ such that (r_1, s_1) is not on the boundary of $I \times I$. Then $(r_1, s_1/4) = \alpha^{-1}(r_1, s_1) \in V$. There exist real numbers h_3, i_3, j_3, and k_3 such that $h_3 < r_1 < i_3$, $j_3 < s_1/4 < k_3$, and $(h_3, i_3) \times (j_3, k_3) \subset V$. Then $(r_1, s_1) \in (h_3, i_3) \times (4j_3, 4k_3) \subset \alpha(V)$. Thus (r_1, s_1) is an element of a basic open set in the product topology which is contained in $\alpha(V)$. If $(r_1, s_1) \in \alpha(V)$ and (r_1, s_1) is on the boundary of $I \times I$, in a similar manner a basic open set containing (r_1, s_1) contained in $\alpha(V)$ can be found. Hence, $\alpha(V)$ is the union of open sets and hence open. Hence, α is a homeomorphism. In a similar manner it can be shown that β is a homeomorphism. Since F and G are in N

and α and β are homeomorphisms, then by Definition 9(iii), $H|_{I \times [0,1/4]} = F\alpha$ and $H|_{I \times [3/4,1]} = G\beta$ are in N. Since g is continuous, then $H|_{I \times [1/4,3/4]}$ is continuous. Hence by Theorem 6, H is in N. Now if $x \in I$ then $H(x,0) = F(x,0) = f(x)$ and $H(x,1) = G(x,1) = h(x)$. Also, if $t \in I$ then

$$H(0,t) = \begin{cases} F(0,4t) & \text{if } 0 \leq t \leq 1/4 \\ g(0) & \text{if } 1/4 \leq t \leq 3/4 = y_0, \\ G(0,4t-3) & \text{if } 3/4 \leq t \leq 1 \end{cases}$$

and

$$H(1,t) = \begin{cases} F(1,4t) & \text{if } 0 \leq t \leq 1/4 \\ g(1) & \text{if } 1/4 \leq t \leq 3/4 = y_0. \\ G(1,4t-3) & \text{if } 3/4 \leq t \leq 1 \end{cases}$$

Therefore, $f \underset{y_0}{\overset{N}{\approx}} h$. Hence, $\underset{y_0}{\overset{N}{\approx}}$ is an equivalence relation on $C(Y,y_0)$.

<u>Definition 13</u>: A <u>group</u> is an ordered pair $(G,\cdot)$ such that G is a non-empty set, $\cdot$ is a binary operation on G, and each of the following is true:

(i) if $a,b,c \in G$, then $(a\cdot b)\cdot c = a\cdot(b\cdot c)$,

(ii) if $a \in G$, then there is $e \in G$ such that $a\cdot e = a$, and

(iii) if $a \in G$, then there is $a^{-1} \in G$ such that $a\cdot a^{-1} = e$.

<u>Definition 14</u>: Let $(G,\cdot)$ and $(H,*)$ be groups. A <u>homomorphism</u> from $(G,\cdot)$ into $(H,*)$ is a function α from $(G,\cdot)$ to $(H,*)$ such that if $g_1,g_2 \in G$, then $\alpha(g_1\cdot g_2) = \alpha(g_1) * \alpha(g_2)$.

<u>Definition 15</u>: Let $(G,\cdot)$ and $(H,*)$ be groups and let α be a homomorphism from $(G,\cdot)$ into $(H,*)$.

(i) If α is onto, then α is called an <u>epimorphism</u>.

(ii) If α is one-to-one and α is an epimorphism, then α is called an isomorphism.

Definition 16: Let (Y,T) be a topological space, let $y_0 \in Y$, and let $f,g \in C(Y,y_0)$. Then $f*_N g$ is the function in $C(Y,y_0)$ defined by if $x \in I$, then

$$(f*_N g)(x) = \begin{cases} f(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4. \\ g(4x-3) & \text{if } 3/4 \leq x \leq 1 \end{cases}$$

By Theorem 7, the equivalence relation $\underset{y_0}{\overset{N}{\sim}}$ breaks $C(Y,y_0)$ into disjoint classes. Let $N(Y,y_0)$ be this set of equivalence classes on Y. Let $[f],[g] \in N(Y,y_0)$. Then we define $[f]\cdot[g]$ to be $[f*_N g]$.

Theorem 9: If (Y,T) is a topological space, $y_0 \in Y$, and $[f],[g] \in N(Y,y_0)$, then $[f]\cdot[g]$ is well-defined.

Proof: Let (Y,T) be a topological space and let $y_0 \in Y$. Let $[f],[g] \in N(Y,y_0)$. Let $f_1,f_2 \in [f]$ and let $g_1,g_2 \in [g]$. Since $f_1,f_2 \in [f]$, there a function $F\colon I \times I \to Y$ in N such that if $x \in I$ and $t \in I$, then $F(x,0) = f_1(x)$, $F(x,1) = f_2(x)$, and $F(0,t) = y_0 = F(1,t)$. Since $g_1,g_2 \in [g]$, there is a function $G\colon I \times I \to Y$ in N such that if $x \in I$ and $t \in I$, then $G(x,0) = g_1(x)$, $G(x,1) = g_2(x)$, and $G(0,t) = y_0 = G(1,t)$. Define a function $H\colon I \times I \to Y$ by if $(x,t) \in I \times I$, then

$$H(x,t) = \begin{cases} F(4x,t) & \text{if } 0 \leq x \leq 1/4,\ t \in I \\ y_0 & \text{if } 1/4 \leq x \leq 3/4,\ t \in I. \\ G(4x-3,t) & \text{if } 3/4 \leq x \leq 1,\ t \in I \end{cases}$$

Then if $x \in I$,

$$H(x,0) = \begin{cases} F(4x,0) \\ y_0 \\ G(4x-3,0) \end{cases} = \begin{cases} f_1(4x) \\ y_0 \\ g_1(4x-3) \end{cases} = (f_1 *_N g_1)(x),$$

and

$$H(x,1) = \begin{cases} F(4x,1) \\ y_0 \\ G(4x-3,1) \end{cases} = \begin{cases} f_2(4x) \\ y_0 \\ g_2(4x-3) \end{cases} = (f_2 *_N g_2)(x),$$

and if $t \in I$, then $H(0,t) = F(0,t) = y_0 = G(1,t) = H(1,t)$. Since $F(1,t) = y_0$ and $G(0,t) = y_0$, H is well-defined. Define $h: [0,1/4] \times I \to I \times I$ by if $0 \leq x \leq 1/4$ and $t \in I$, then $h(x,t) = (4x,t)$, and define $k: [3/4,1] \times I \to I \times I$ by if $3/4 \leq x \leq 1$ and $t \in I$, then $k(x,t) = (4x-3,t)$. Then $H|_{[0,1/4] \times I} = Fh$ and $H|_{[3/4,1] \times I} = Gk$. Let (x_1,y_1) and (x_2,y_2) be elements of $[0,1/4] \times I$ such that $h(x_1,y_1) = h(x_2,y_2)$. Then $(4x_1,y_1) = h(x_1,y_1) = h(x_2,y_2) = (4x_2,y_2)$. Thus, $x_1 = x_2$ and $y_1 = y_2$. Hence $(x_1,y_1) = (x_2,y_2)$ and h is one-to-one. Let $(p,q) \in I \times I$. Then $(p/4,q) \in [0,1/4] \times I$, and $h(p/4,q) = (p,q)$ and h is onto. Let U be an open subset of $I \times I$. Let $(u,v) \in h^{-1}(U)$ such that (u,v) is not on the boundary of $[0,1/4] \times I$. Then $(4u,v) = h(u,v) \in U$. There exist real numbers h_1, i_1, j_1, and k_1 such that $h_1 < 4u < i_1$, $j_1 < v < k_1$, and $(h_1,i_1) \times (j_1,k_1) \subset U$. Then $(u,v) \in (h_1/4,i_1/4) \times (j_1,k_1) \subset h^{-1}(U)$. Thus (u,v) is an element of a basic open set in the product topology which is contained in $h^{-1}(U)$. If $(u,v) \in h^{-1}(U)$ and (u,v) is on the boundary of $[0,1/4] \times I$, in a similar manner a basic open set containing (u,v) contained in $h^{-1}(U)$ can be found. Hence, $h^{-1}(U)$ is the union of open sets and hence open. Let V be an open subset of $[0,1/4] \times I$. Let $(r,s) \in h(V)$ such that (r,s) is not on the boundary of $I \times I$. Then $(r/4,s) = h^{-1}(r,s) \in V$. There exist real numbers h_2, i_2, j_2, and k_2 such that $h_2 < r/4 < i_2$, $j_2 < s < k_2$, and $(h_2,i_2) \times (j_2,k_2) \subset V$. Then $(r,s) \in (4h_2,4i_2) \times (j_2,k_2) \subset h(V)$. Thus (r,s) is an element of a basic open set in the product topology which is contained in $h(V)$. If $(r,s) \in h(V)$ and (r,s) is on the boundary of $I \times I$, in a similar manner a basic open set containing (r,s) contained in $h(V)$ can be found. Hence, $h(V)$ is the union of open sets and hence open. Hence, h is a homeomorphism. In a similar manner it can be shown that k is a homeomorphism. Since F and G are in N and h and k are homeomorphisms, then by Definition 9(iii), $H|_{[0,1/4] \times I} = Fh$ and $H|_{[3/4,1] \times I} = Gk$ are in N. Since $H|_{[1/4,3/4] \times I} = y_0$, then $H|_{[1/4,3/4] \times I}$ is continuous. Hence, by Definition 9(iv), H is in N. Thus, $f_1 *_N g_1 \underset{y_0}{\overset{N}{\simeq}} f_2 *_N g_2$. Therefore, $[f_1 *_N g_1] = [f_2 *_N g_2]$ and hence, $[f]\cdot[g]$ is well-defined.

<u>Theorem 10</u>: Let (X,S) and (Y,T) be topological spaces. Let $F: (X,S) \to (Y,T)$ be a function. Let A and B be closed subsets of X such that $A \cup B = X$. Let $F|_A: (A,S_A) \to (Y,T)$ and $F|_B: (B,S_B) \to (Y,T)$ be continuous. Then F is continuous.

<u>Proof</u>: Let $U \in T$. Since $F|_A: (A,S_A) \to (Y,T)$ and $F|_B: (B,S_B) \to (Y,T)$ are continuous, then $(F|_A)^{-1}(U) \in S_A$ and $(F|_B)^{-1}(U) \in S_B$. Since $(F|_A)^{-1}(U) \in S_A$, there exists $\mathcal{O}_1 \in S$ such

that $\mathcal{O}_1 \cap A = (F|_A)^{-1}(U)$. Since $(F|_B)^{-1}(U) \in S_B$, there exists $\mathcal{O}_2 \in S$ such that $\mathcal{O}_2 \cap B = (F|_B)^{-1}(U)$. Since A is closed, $X - A \in S$, and since B is closed, $X - B \in S$. Let $p \in (F|_A)^{-1}(U) \cup (F|_B)^{-1}(U)$. If $p \in A$ but $p \notin B$, then $p \in X - B \in S$. Hence there exists $V_1 \in S$ such that $p \in V_1 \subset X - B$. Since $(F|_A)^{-1}(U) = \mathcal{O}_1 \cap A$, then $p \in \mathcal{O}_1$. Thus $p \in V_1 \cap \mathcal{O}_1 \subset F^{-1}(U)$. If $p \in B$ but $p \notin A$, then $p \in X - A \in S$. Hence there exists $V_2 \in S$ such that $p \in V_2 \subset X - A$. Since $(F|_B)^{-1}(U) = \mathcal{O}_2 \cap B$, then $p \in \mathcal{O}_2$. Thus $p \in V_2 \cap \mathcal{O}_2 \subset F^{-1}(U)$. If $p \in A \cap B$, then $p \in \mathcal{O}_1 \cap \mathcal{O}_2 \subset F^{-1}(U)$. Thus $(F|_A)^{-1}(U) \cup (F|_B)^{-1}(U) \subset F^{-1}(U)$. Let $q \in F^{-1}(U)$. Since $X = A \cup B$, $q \in A \cup B$. If $q \in A$, then $q \in (F|_A)^{-1}(U)$. If $q \in B$, then $q \in (F|_B)^{-1}(U)$. Hence, $q \in (F|_A)^{-1}(U) \cup (F|_B)^{-1}(U)$. Thus, $F^{-1}(U) \subset (F|_A)^{-1}(U) \cup (F|_B)^{-1}(U)$. Therefore, $F^{-1}(U) = (F|_A)^{-1}(U) \cup (F|_B)^{-1}(U)$. Hence, $F^{-1}(U) \in S$ and F is continuous.

Corollary 1: Let (X,S) and (Y,T) be topological spaces, let $1 < n \leq 5$, and let $F\colon (X,S) \to (Y,T)$ be a function. Let $A_1, A_2, \ldots, A_n$ be closed subsets of X such that $A_1 \cup A_2 \cup \ldots \cup A_n = X$ and $F|_{A_1}\colon (A_1, S_{A_1}) \to (Y,T)$, $F|_{A_2}\colon (A_2, S_{A_2}) \to (Y,T)$, $F|_{A_3}\colon (A_3, S_{A_3}) \to (Y,T)$, $F|_{A_4}\colon (A_4, S_{A_4}) \to (Y,T)$, and $F|_{A_5}\colon (A_5, S_{A_5}) \to (Y,T)$ are continuous. Then F is continuous.

Proof: Let $B = A_1 \cup A_2$. By Theorem 10, $F|_B\colon (B, S_B) \to (Y,T)$ is continuous. Let $C = B \cup A_3$. By Theorem 10, $F|_C\colon (C, S_C) \to (Y,T)$ is continuous. Let $D = C \cup A_4$. By Theorem 10, $F|_D\colon (D, S_D) \to (Y,T)$ is continuous. Let $E = D \cup A_5$. By Theorem 10, $F|_E\colon (E, S_E) \to (Y,T)$

is continuous. Therefore, F is continuous.

Definition 17: Let (Y,T) be a topological space and let $y_0 \in Y$. The identity element of $N(Y,y_0)$, denoted by $[e]$, is the equivalence class which contains the function $e: I \to Y$ defined by if $x \in I$, then $e(x) = y_0$.

Theorem 11: If (Y,T) is a topological space, $y_0 \in Y$, and $[f] \in N(Y,y_0)$, then $[f]\cdot[e] = [f]$.

Proof: Let (Y,T) be a topological space, let $y_0 \in Y$, and let $[f] \in N(Y,y_0)$. Let $A = \left\{(x,t) \mid (x,t) \in I \times I \text{ and } x \leq \frac{3t+1}{4}\right\}$. Let $B = \left\{(x,t) \mid (x,t) \in I \times I \text{ and } x \geq \frac{3t+1}{4}\right\}$. Let S be the usual product topology on $I \times I$. Define a function $F: I \times I \to Y$ by if $(x,t) \in I \times I$, then

$$F(x,t) = \begin{cases} f\left(\frac{4x}{3t+1}\right) & \text{if } x \leq \frac{3t+1}{4} \\ y_0 & \text{otherwise} \end{cases}.$$

Define $h: (A,S_A) \to (I,\lambda_I)$ by if $(x,t) \in I \times I$, then $h(x,t) = \frac{4x}{3t+1}$. Let $U \in \lambda_I$. Let $(p,q) \in h^{-1}(U)$. Suppose (p,q) is not on the boundary of A. Let $z = h(p,q)$. Since (p,q) is not on the boundary of A, $z \neq 0$ and $z \neq 1$. Since $z \in U \in \lambda_I$, there is an element $\mathcal{O} \in \lambda$ such that $U = \mathcal{O} \cap I$. Since $z \in \mathcal{O} \in \lambda$, then by Definition 6, there are numbers $a,b \in I$ such that $z \in (a,b) \subset \mathcal{O}$. Let $\theta = \min\left\{z-a,\ b-z,\ \frac{z}{2},\ \frac{1-z}{2}\right\}$. Then $\theta > 0$, $z-\theta > 0$, and $z \in (z-\theta, z+\theta) \subset U$. Since $p > 0$ and $z-\theta > 0$, if $q \geq 1/3$, then $(3q-1)(z-\theta)+4p \geq 0$. Let $0 < q < 1/3$. Then $9q^2-1 < 0$. Assume $(3q-1)(z-\theta)+4p < 0$. Since $z = h(p,q) = \frac{4p}{3q+1}$, if $(3q-1)(z-\theta)+4p < 0$, then

$$(3q-1)\left(\frac{4p}{3q+1} - \theta\right) + 4p < 0, \text{ and } \frac{4p(3q-1)}{3q+1} - \theta(3q-1) + 4p < 0, \text{ and}$$

$$\frac{12pq-4p}{3q+1} - \theta(3q-1) + \frac{4p(3q+1)}{3q+1} < 0, \text{ and}$$

$$\frac{24pq}{3q+1} - \theta(3q-1) < 0, \text{ and } \frac{24pq-\theta(3q-1)(3q+1)}{3q+1} < 0, \text{ and}$$

$$24pq-\theta(3q-1)(3q+1) < 0, \text{ and } 24pq-\theta(9q^2-1) < 0, \text{ which is bad.}$$

Hence, if $0 < q < 1/3$, it is not the case that $(3q-1)(z-\theta)+4p < 0$. Hence, in either case $(3q-1)(z-\theta)+4p \geq 0$. Let

$$J_{(p,q)} = \left(\frac{(3q+1)(z-\theta)+4p}{8}, \frac{(3q+1)(z+\theta)+4p}{8}\right) \text{ and let}$$

$$K_{(p,q)} = \left(\frac{(3q-1)(z+\theta)+4p}{6(z+\theta)}, \min\left\{1, \frac{(3q-1)(z-\theta)+4p}{6(z-\theta)}\right\}\right).$$

Since $z = h(p,q) = \frac{4p}{3q+1}$, then it is the case that

$$\frac{(3q+1)(z-\theta)+4p}{8} = \frac{(3q+1)\left(\frac{4p}{3q+1} - \theta\right) + 4p}{8} = \frac{4p-\theta(3q+1)+4p}{8} =$$

$$\frac{8p-\theta(3q+1)}{8} < p < \frac{8p+\theta(3q+1)}{8} = \frac{4p+\theta(3q+1)+4p}{8} =$$

$$\frac{(3q+1)\left(\frac{4p}{3q+1} - \theta\right) + 4p}{8} = \frac{(3q+1)(z+\theta)+4p}{8}.$$

Hence, $p \in J_{(p,q)}$.

Since $\theta > 0$, $q > 0$, and $p > 0$, then $q - \frac{\theta(3q+1)(3q+1)}{24p+6\theta(3q+1)} < q$. Also, since $z = h(p,q) = \frac{4p}{3q+1}$, then it is the case that

$$\frac{(3q-1)(z+\theta)+4p}{6(z+\theta)} = \frac{(3q-1)\left(\frac{4p}{3q+1} + \theta\right) + 4p}{6\left(\frac{4p}{3q+1} + \theta\right)} =$$

$$\frac{(3q-1)\left(\frac{4p+\theta(3q+1)}{3q+1}\right) + \frac{4p(3q+1)}{3q+1}}{6\left(\frac{4p}{3q+1} + \frac{\theta(3q+1)}{3q+1}\right)} = \frac{\frac{(3q-1)(4p+\theta(3q+1))+4p(3q+1)}{3q+1}}{6\left(\frac{4p+\theta(3q+1)}{3q+1}\right)} =$$

$$\frac{(3q-1)(4p+\theta(3q+1))+4p(3q+1)}{6(4p+\theta(3q+1))} = \frac{12pq-4p+\theta(3q-1)(3q+1)+12pq+4p}{24p+6\theta(3q+1)} =$$

$$\frac{24pq+\theta(3q-1)(3q+1)}{24p+6\theta(3q+1)} = \frac{24pq}{24p+6\theta(3q+1)} + \frac{\theta(3q-1)(3q+1)}{24p+6\theta(3q+1)} =$$

$$q - \frac{6q\theta(3q+1)}{24p+6\theta(3q+1)} + \frac{\theta(3q+1)(3q-1)}{24p+6\theta(3q+1)} = q + \frac{\theta(3q+1)(3q-1-6q)}{24p+6\theta(3q+1)} =$$

$$q + \frac{\theta(3q+1)(-1-3q)}{24p+6\theta(3q+1)} = q - \frac{\theta(3q+1)(3q+1)}{24p+6\theta(3q+1)} < q.$$

Since $(p,q) \in A$, then $4p \leqslant 3q+1$, and thus $24p-6\theta(3q+1) \leqslant 6(3q+1)-6\theta(3q+1)$. Also, since $\theta < 1$ and $q > 0$, then $6(3q+1)-6\theta(3q+1) > 0$.

If $1 = \min\left\{1, \frac{(3q-1)(z-\theta)+4p}{6(z-\theta)}\right\}$, then since (p,q) is not on the boundary of A, $q < 1$ and thus $q < \min\left\{1, \frac{(3q-1)(z-\theta)+4p}{6(z-\theta)}\right\}$.

If $\frac{(3q-1)(z-\theta)+4p}{6(z-\theta)} = \min\left\{1, \frac{(3q-1)(z-\theta)+4p}{6(z-\theta)}\right\}$, then since $z = h(p,q) = \frac{4p}{3q+1}$, it is the case that

$$\frac{(3q-1)(z-\theta)+4p}{6(z-\theta)} = \frac{(3q-1)\left(\frac{4p}{3q+1} - \theta\right) + 4p}{6\left(\frac{4p}{3q+1} - \theta\right)} =$$

$$\frac{(3q-1)\left(\frac{4p-\theta(3q+1)}{3q+1}\right) + \frac{4p(3q+1)}{3q+1}}{6\left(\frac{4p}{3q+1} - \frac{\theta(3q+1)}{3q+1}\right)} = \frac{\frac{(3q-1)(4p-\theta(3q+1))+4p(3q+1)}{3q+1}}{6\left(\frac{4p-6\theta(3q+1)}{3q+1}\right)} =$$

$$\frac{(3q-1)(4p-\theta(3q+1))+4p(3q+1)}{6(4p-\theta(3q+1))} = \frac{12pq-4p-\theta(3q+1)(3q-1)+12pq+4p}{24p-6\theta(3q+1)} =$$

$$\frac{24pq-\theta(3q+1)(3q-1)}{24p-6\theta(3q+1)} = \frac{24pq}{24p-6\theta(3q+1)} - \frac{\theta(3q+1)(3q-1)}{24p-6\theta(3q+1)} =$$

$$q + \frac{6q\theta(3q+1)}{24p-6\theta(3q+1)} - \frac{\theta(3q+1)(3q-1)}{24p-6\theta(3q+1)} = q + \frac{\theta(3q+1)(1-3q+6q)}{24p-6\theta(3q+1)} =$$

$$q + \frac{\theta(3q+1)(3q+1)}{24p-6\theta(3q+1)} \geqslant q + \frac{\theta(3q+1)(3q+1)}{6(3q+1)-6\theta(3q+1)} > q, \text{ and thus,}$$

$$q < \min\left\{1, \frac{(3q-1)(z-\theta)+4p}{6(z-\theta)}\right\}.$$

Hence, $q < \min\left\{1, \frac{(3q-1)(z-\theta)+4p}{6(z-\theta)}\right\}$. Thus, $q \in K_{(p,q)}$. Therefore, $(p,q) \in J_{(p,q)} \times K_{(p,q)}$. Since $q > 0$, $z-\theta > 0$, and $p > 0$, then $(3q+1)(z-\theta)+4p > 0$.

Let $(a,b) \in J_{(p,q)} \times K_{(p,q)}$. Then $h(a,b) = \frac{4a}{3b+1}$, $4a \leq 3b+1$,

$\frac{(3q+1)(z-\theta)+4p}{8} < a < \frac{(3q+1)(z+\theta)+4p}{8}$, and

$\frac{(3q-1)(z+\theta)+4p}{6(z+\theta)} < b < \min\left\{1, \frac{(3q-1)(z-\theta)+4p}{6(z-\theta)}\right\}$.

If $\frac{(3q-1)(z-\theta)+4p}{6(z-\theta)} = \min\left\{1, \frac{(3q-1)(z-\theta)+4p}{6(z-\theta)}\right\}$, then

$\left[\frac{(3q+1)(z-\theta)+4p}{2}\right] < 4a < \left[\frac{(3q+1)(z+\theta)+4p}{2}\right]$, and

$\left[\frac{(3q-1)(z+\theta)+4p}{2(z+\theta)}\right] < 3b < \left[\frac{(3q-1)(z-\theta)+4p}{2(z-\theta)}\right]$, and

$\left[\frac{(3q-1)(z+\theta)+4p+2(z+\theta)}{2(z+\theta)}\right] < 3b+1 < \left[\frac{(3q-1)(z-\theta)+4p+2(z-\theta)}{2(z-\theta)}\right]$, and

$\left[\frac{\frac{(3q+1)(z-\theta)+4p}{2}}{\frac{(3q+1)(z-\theta)+4p}{2(z-\theta)}}\right] < \frac{4a}{3b+1} < \left[\frac{\frac{(3q+1)(z+\theta)+4p}{2}}{\frac{(3q+1)(z+\theta)+4p}{2(z+\theta)}}\right]$, and

$\left[\frac{(3q+1)(z-\theta)+4p}{2} \cdot \frac{2(z-\theta)}{(3q+1)(z-\theta)+4p}\right] < \frac{4a}{3b+1} <$

$\left[\frac{(3q+1)(z+\theta)+4p}{2} \cdot \frac{2(z+\theta)}{(3q+1)(z+\theta)+4p}\right]$,

and $(z-\theta)\left[\frac{(3q+1)(z-\theta)+4p}{(3q+1)(z-\theta)+4p}\right] < \frac{4a}{3b+1} < \left[\frac{(3q+1)(z+\theta)+4p}{(3q+1)(z+\theta)+4p}\right](z+\theta)$, and

$(z-\theta) < \frac{4a}{3b+1} < (z+\theta)$.

Since $b < 1$, then $3b+1 < 4$.

If $1 = \min\left\{1, \frac{(3q-1)(z-\theta)+4p}{6(z-\theta)}\right\}$, then $1 < \frac{(3q-1)(z-\theta)+4p}{6(z-\theta)}$, and

$$3b+1 < 4 < \left[\frac{(3q-1)(z-\theta)+4p+2(z-\theta)}{2(z-\theta)}\right] = \left[\frac{(3q+1)(z-\theta)+4p}{2(z-\theta)}\right], \text{ and}$$

$$\left[\frac{\frac{(3q+1)(z-\theta)+4p}{2}}{\frac{(3q+1)(z-\theta)+4p}{2(z-\theta)}}\right] < \left[\frac{\frac{(3q+1)(z-\theta)+4p}{2}}{4}\right] < \frac{4a}{3b+1} < \left[\frac{\frac{(3q+1)(z+\theta)+4p}{2}}{\frac{(3q+1)(z+\theta)+4p}{2(z+\theta)}}\right], \text{ and}$$

$$(z-\theta) < \left[\frac{(3q+1)(z-\theta)+4p}{8}\right] < \frac{4a}{3b+1} < (z+\theta), \text{ and thus}$$

$$(z-\theta) < \frac{4a}{3b+1} < (z+\theta).$$

Hence, in either case, $(z-\theta) < \frac{4a}{3b+1} < (z+\theta)$. Hence, $(z-\theta) < h(a,b) < (z+\theta)$ and $h(a,b) \in U$. Thus, $h(J_{(p,q)} \times K_{(p,q)}) \subset U$. Hence, $(p,q) \in J_{(p,q)} \times K_{(p,q)} \subset h^{-1}(U)$. If (p,q) is on the boundary of A, then similarly there is an element $V_{(p,q)}$ of S_A such that $(p,q) \in V_{(p,q)} \subset h^{-1}(U)$. Thus, $h^{-1}(U)$ is the union of elements of S_A. Hence, $h^{-1}(U)$ is in S_A. Hence, $h: (A,S_A) \to (I,\lambda_I)$ is continuous. Hence, by Theorem 4, $fh: (A,S_A) \to Y$ is continuous. Define $g: (B,S_B) \to Y$ by if $b \in B$, then $g(b) = y_0$. Let $U \in T$. Then either $y_0 \in U$ or $y_0 \notin U$. If $y_0 \in U$, then $g^{-1}(y_0) = B$ and $g^{-1}(U) = g^{-1}(y_0) = B \in S_B$. If $y_0 \notin U$, then $g^{-1}(U) = \phi \in S_B$. In either case, $g^{-1}(U) \in S_B$. Hence, g is continuous. Hence, $F|_A = fh$ is continuous and $F|_B = g$ is continuous. By Theorem 10, F is continuous. Hence, F is in N. Also, if $x \in I$ then

$$F(x,0) = \begin{cases} f(4x) & \text{if } 0 \le x \le 1/4 \\ y_0 & \text{otherwise} \end{cases} = (f*_N e)(x),$$

$F(x,1) = f(x)$, and if $t \in I$, $F(0,t) = f(0) = y_0 = f(1) = F(1,t)$. Thus, $f*_N e \underset{y_0}{\overset{N}{\approx}} f$. Therefore, $[f]\cdot[e] = [f]$.

Definition 18: Let (Y,T) be a topological space and let $y_0 \in Y$. If $[f] \in N(Y,y_0)$, then $[f]^{-1}$ is the element of $N(Y,y_0)$ containing the function $g: I \to Y$ defined by if $t \in I$, then $g(t) = f(1-t)$.

Theorem 12: If (Y,T) is a topological space, $y_0 \in Y$, and $[f] \in N(Y,y_0)$, then $[f]\cdot[f]^{-1} = [e]$.

Proof: Let (Y,T) be a topological space, let $y_0 \in Y$, and let $[f] \in N(Y,y_0)$. Let $A = \left\{(x,t)\,|\,(x,t) \in I \times I \text{ and } x \leq \frac{1-t}{4}\right\}$. Let $B = \left\{(x,t)\,|\,(x,t) \in I \times I \text{ and } x \geq \frac{t+3}{4}\right\}$. Let Let $C = \left\{(x,t)\,|\,(x,t) \in I \times I \text{ and } \frac{1-t}{4} \leq x \leq \frac{t+3}{4}\right\}$. Let S be the usual product topology on $I \times I$. Define a function $G: I \times I \to Y$ by if $(x,t) \in I \times I$, then

$$G(x,t) = \begin{cases} f\left(\frac{4x}{1-t}\right) & \text{if } x \leq \frac{1-t}{4},\ 0 \leq x \leq 1/4,\ 0 \leq t < 1 \\ y_0 & \text{otherwise} \\ f\left(\frac{4x-4}{t-1}\right) & \text{if } x \geq \frac{t+3}{4},\ 3/4 \leq x \leq 1,\ 0 \leq t < 1 \end{cases}.$$

Define $g: (A,S_A) \to (I,\lambda_I)$ by if $(x,t) \in A$, then $g(x,t) = \frac{4x}{1-t}$. Define $h: (B,S_B) \to (I,\lambda_I)$ by if $(x,t) \in B$, then $h(x,t) = \frac{4x-4}{t-1}$. In a manner similar to that of Theorem 11 it can be shown that g and h are continuous. Hence, $g: (A,S_A) \to (I,\lambda_I)$ is continuous and $h: (B,S_B) \to (I,\lambda_I)$ is continuous. Thus, by Theorem 4, $fg: (A,S_A) \to Y$ is continuous and $fh: (B,S_B) \to Y$ is continuous. Define $\alpha: (C,S_C) \to Y$ by if $c \in C$, then $\alpha(c) = y_0$. Let $U \in T$. Then either $y_0 \in U$ or $y_0 \notin U$. If $y_0 \in U$, then $\alpha^{-1}(y_0) = C$ and $\alpha^{-1}(U) = \alpha^{-1}(y_0) = C \in S_C$. If $y_0 \notin U$, then $\alpha^{-1}(U) = \phi \in S_C$. In either case, $\alpha^{-1}(U) \in S_C$. Hence, α is continuous. Hence,

$G|_A = fg$ and is continuous, $G|_B = fh$ and is continuous, and $G|_C = \alpha$ and is continuous. By Corollary 1, G is continuous. Hence, G is in N. Also, if $x \in I$, then

$$G(x,0) = \begin{cases} f(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4 \\ f(4-4x) & \text{if } 3/4 \leq x \leq 1 \end{cases} = (f *_N f^{-1})(x),$$

$G(x,1) = e(x)$, and if $t \in I$,

$G(0,t) = f(0) = y_0 = f(1) = f^{-1}(0) = G(1,t)$. Thus, $f *_N f^{-1} \underset{y_0}{\overset{N}{\sim}} e$.

Therefore, $[f]\cdot[f]^{-1} = e$.

<u>Theorem 13</u>: If (Y,T) is a topological space, $y_0 \in Y$, and N is an admitting homotopy relation, then $(N(Y,y_0),\cdot)$ is a group.

<u>Proof</u>: Let (Y,T) be a topological space, let $y_0 \in Y$, and let N be a admitting homotopy relation. Let $[f],[g],[h] \in N(Y,y_0)$.

Let $A = \left\{(x,t)\,|\,(x,t) \in I \times I \text{ and } x \leq \frac{3t+1}{16}\right\}$.

Let $B = \left\{(x,t)\,|\,(x,t) \in I \times I \text{ and } \frac{3t+1}{16} \leq x \leq \frac{9t+3}{16}\right\}$.

Let $C = \left\{(x,t)\,|\,(x,t) \in I \times I \text{ and } \frac{9t+3}{16} \leq x \leq \frac{9t+4}{16}\right\}$.

Let $D = \left\{(x,t)\,|\,(x,t) \in I \times I \text{ and } \frac{9t+4}{16} \leq x \leq \frac{3t+12}{16}\right\}$.

Let $E = \left\{(x,t)\,|\,(x,t) \in I \times I \text{ and } x \geq \frac{3t+12}{16}\right\}$. Let S be the usual product topology on $I \times I$. By Definition 16, if $x \in I$ then

$$[(f *_N g) *_N h](x) = \begin{cases} f(16x) & \text{if } 0 \leq x \leq 1/16 \\ y_0 & \text{if } 1/16 \leq x \leq 3/16 \\ g(16x-3) & \text{if } 3/16 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4 \\ h(4x-3) & \text{if } 3/4 \leq x \leq 1 \end{cases}, \text{ and}$$

$$[f *_N (g *_N h)](x) = \begin{cases} f(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4 \\ g(16x-12) & \text{if } 3/4 \leq x \leq 13/16 \\ y_0 & \text{if } 13/16 \leq x \leq 15/16 \\ h(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases} .$$

Define a function $H: I \times I \to Y$ by if $(x,t) \in I \times I$, then

$$H(x,t) = \begin{cases} f\left(\frac{16x}{3t+1}\right) & \text{if } x \leq \frac{3t+1}{16} \\ y_0 & \text{if } \frac{3t+1}{16} \leq x \leq \frac{9t+3}{16} \\ g(16x-9t-3) & \text{if } \frac{9t+3}{16} \leq x \leq \frac{9t+4}{16} \\ y_0 & \text{if } \frac{9t+4}{16} \leq x \leq \frac{3t+12}{16} \\ h\left(\frac{16x-3t-12}{4-3t}\right) & \text{if } x \geq \frac{3t+12}{16} \end{cases} .$$

Define $\alpha_1: (A,S_A) \to (I,\lambda_I)$ by if $(x,t) \in A$, then $\alpha_1(x,t) = \frac{16x}{3t+1}$. Define $\alpha_2: (C,S_C) \to (I,\lambda_I)$ by if $(x,t) \in C$, then $\alpha_2(x,t) = 16x-9t-3$. Define $\alpha_3: (E,S_E) \to (I,\lambda_I)$ by if $(x,t) \in E$, then $\alpha_3(x,t) = \frac{16x-3t-12}{4-3t}$. In a manner similar to that of Theorem 11, it can be shown that α_1, α_2, and α_3 are continuous. Hence, $\alpha_1: (A,S_A) \to (I,\lambda_I)$, $\alpha_2: (C,S_C) \to (I,\lambda_I)$, and $\alpha_3: (E,S_E) \to (I,\lambda_I)$ are continuous. Thus by Theorem 4, $f\alpha_1: (A,S_A) \to Y$ is continuous, $g\alpha_2: (C,S_C) \to Y$ is continuous, and $h\alpha_3: (E,S_E) \to Y$ is continuous. Define $\beta: (B,S_B) \to Y$ by if $b \in B$, then $\beta(b) = y_0$. Let $U \in T$. Then either $y_0 \in U$ or $y_0 \notin U$. If $y_0 \in U$, then $\beta^{-1}(y_0) = B$ and $\beta^{-1}(U) = \beta^{-1}(y_0) = B \in S_B$. If $y_0 \notin U$, then $\beta^{-1}(U) = \phi \in S_B$. In either case, $\beta^{-1}(U) \in S_B$. Hence, β is continuous. Define $\sigma: (D,S_D) \to Y$ by if $d \in D$, then $\sigma(d) = y_0$. In a manner similar to the above, it can be shown that σ is continuous. Hence, σ is continuous. Thus, $H|_A = f\alpha_1$ is continuous, $H|_B = \beta$ is continuous, $H|_C = g\alpha_2$ is continuous, $H|_D = \sigma$ is continuous, and $H|_E = h\alpha_3$ is continuous. By Corollary 1, H is continuous. Hence, H is in N. Also, if $x \in I$ then

$$H(x,0) = \begin{cases} f(16x) & \text{if } 0 \leq x \leq 1/16 \\ y_0 & \text{if } 1/16 \leq x \leq 3/16 \\ g(16x-3) & \text{if } 3/16 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4 \\ h(4x-3) & \text{if } 3/4 \leq x \leq 1 \end{cases} = [(f*_N g)*_N h](x),$$

$$H(x,1) = \begin{cases} f(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4 \\ g(16x-12) & \text{if } 3/4 \leq x \leq 13/16 \\ y_0 & \text{if } 13/16 \leq x \leq 15/16 \\ h(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases} = [f*_N(g*_N h)](x),$$

and if $t \in I$, $H(0,t) = f(0) = y_0 = h(1) = H(1,t)$. Thus $(f*_N g)*_N h \underset{y_0}{\overset{N}{\approx}} f*_N(g*_N h)$. Therefore, $([f]\cdot[g])\cdot[h] = [f]\cdot([g]\cdot[h])$. Therefore, $(N(Y,y_0),\cdot)$ is a group.

From now on, the symbol $N(Y,y_0)$ will mean the set $N(Y,y_0)$ together with the operation $\cdot$ and $N(Y,y_0)$ will be called the N-<u>fundamental group of</u> Y <u>with respect to</u> y_0.

CHAPTER III

<u>Definition 19</u>: Let (Y,T) be a topological space and let $y_0 \in Y$. Let M be the class of all continuous functions. Then M is an admitting homotopy relation. The <u>fundamental group of Y modulo</u> y_0, denoted by $\pi_1(Y,y_0)$, is $M(Y,y_0)$.

<u>Definition 20</u>: Let (Y,T) be a topological space and let $y_0 \in Y$. Let $f,g \in C(Y,y_0)$. Define $f*g$ by if $x \in Y$, then

$$(f*g)(x) = \begin{cases} f(2x) & \text{if } 0 \leq x \leq 1/2. \\ g(2x-1) & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

Then $[f]\odot[g]$ is defined to be $[f*g]$.

<u>Theorem 14</u>: Let (Y,T) be a topological space, let $y_0 \in Y$, and let N be an admitting homotopy relation. Let μ be the natural function from $\pi_1(Y,y_0)$ into $N(Y,y_0)$ defined by if $[f] \in \pi_1(Y,y_0)$, then $\mu([f])$ is the equivalence class in $N(Y,y_0)$ which contains f. Then $\mu: \pi_1(Y,y_0) \to N(Y,y_0)$ is an epimorphism.

<u>Proof</u>: Let $[f],[g] \in \pi_1(Y,y_0)$. Since $\mu[f]$ is the equivalence class containing f in $N(Y,y_0)$, and since $\mu[g]$ is the equivalence class containing g in $N(Y,y_0)$, then by Definition 16 $\mu[f]\cdot\mu[g]$ is the equivalence class in $N(Y,y_0)$ containing $f*_N g$. Hence, $\mu[f]\cdot\mu[g] = [f*_N g]$. Also, $\mu([f]\odot[g])$ is the equivalence class in $N(Y,y_0)$ containing $[f]\odot[g]$. Hence, by Definition 20, $\mu([f]\odot[g]) = \mu([f*g])$. Let $A = \left\{(x,t) \mid (x,t) \in I \times I \text{ and } x \leq \frac{t+1}{4}\right\}$. Let $B = \left\{(x,t) \mid (x,t) \in I \times I \text{ and } x \geq \frac{3-t}{4}\right\}$. Let

$C = \left\{(x,t) \mid (x,t) \in I \times I \text{ and } \frac{t+1}{4} \leqslant x \leqslant \frac{3-t}{4}\right\}$. Let S be the usual product topology on $I \times I$. Define a function $F: I \times I \to Y$ by if $(x,t) \in I \times I$, then

$$F(x,t) = \begin{cases} f\left(\frac{4x}{t+1}\right) & \text{if } x \leqslant \frac{t+1}{4} \\ y_0 & \text{otherwise} \\ g\left(\frac{4x+t-3}{t+1}\right) & \text{if } x \geqslant \frac{3-t}{4} \end{cases} .$$

Define $\alpha: (A,S_A) \to (I,\lambda_I)$ by if $(x,t) \in A$, then $\alpha(x,t) = \frac{4x}{t+1}$.

Define $\beta: (B,S_B) \to (I,\lambda_I)$ by if $(x,t) \in B$, then $\beta(x,t) = \frac{4x+t-3}{t+1}$.

In a manner similar to that of Theorem 11 it can be shown that α and β are continuous. Hence, $\alpha: (A,S_A) \to (I,\lambda_I)$ is continuous and $\beta: (B,S_B) \to (I,\lambda_I)$ is continuous. Thus by Theorem 4, $f\alpha: (A,S_A) \to Y$ is continuous and $g\beta: (B,S_B) \to Y$ is continuous. Define $\gamma: (C,S_C) \to Y$ by if $c \in C$, then $\gamma(c) = y_0$. Let $U \in T$. Then either $y_0 \in U$ or $y_0 \notin U$. If $y_0 \in U$, then $\gamma^{-1}(y_0) = C$ and $\gamma^{-1}(U) = \gamma^{-1}(y_0) = C \in S_C$. If $y_0 \notin U$, then $\gamma^{-1}(U) = \phi \in S_C$. In either case, $\gamma^{-1}(U) \in S_C$. Hence, γ is continuous. Hence, $F|_A = f\alpha$ is continuous, $F|_B = g\beta$ is continuous, and $F|_C = \gamma$ is continuous. By Corollary 1, F is continuous. Hence, F is in N. Also, if $x \in I$, then

$$F(x,0) = \begin{cases} f(4x) & \text{if } 0 \leqslant x \leqslant 1/4 \\ y_0 & \text{if } 1/4 \leqslant x \leqslant 3/4 \\ g(4x-3) & \text{if } 3/4 \leqslant x \leqslant 1 \end{cases} = (f *_N g)(x),$$

$$F(x,1) = \begin{cases} f(2x) & \text{if } 0 \leqslant x \leqslant 1/2 \\ g(2x-1) & \text{if } 1/2 \leqslant x \leqslant 1 \end{cases} = (f*g)(x),$$

and if $t \in I$, then $F(0,t) = f(0) = y_0 = g(1) = F(1,t)$. Thus, $f *_N g \overset{N}{\underset{y_0}{\approx}} f * g$. Therefore, $\mu([f] \odot [g]) = \mu[f] \cdot \mu[g]$. Hence, μ is a homomorphism.

Let $M \in N(Y,y_0)$. Let $f \in M$. Then $[f] \in \pi_1(Y,y_0)$ and $\mu([f]) = M$. Hence, μ is onto. Hence, μ is an epimorphism.

Theorem 15: Let (X,S) and (Y,T) be topological spaces, let $x_0 \in X$ and let $y_0 \in Y$, let H be a homeomorphism from X onto Y such that $H(x_0) = y_0$, and let N be an admitting homotopy relation. Then $N(X,x_0)$ is isomorphic to $N(Y,y_0)$.

Proof: Let $f \in C(X,x_0)$. Then $f: I \to X$ and since $H: X \to Y$, then $Hf: I \to Y$. Since Hf is continuous and $(Hf)(0) = H(f(0)) = H(x_0) = y_0 = H(f(1)) = (Hf)(1)$, then $Hf \in C(Y,y_0)$. Define $\sigma: N(X,x_0) \to N(Y,y_0)$ by if $[f] \in N(X,x_0)$, then $\sigma([f]) = [Hf]$. Let $[f] \in N(X,x_0)$ and let $f,g \in [f]$. By Definition 16, $f \overset{N}{\underset{x_0}{\approx}} g$ and there is a function $F: I \times I \to X$ in N defined by if $x \in I$ and $t \in I$, then $F(x,0) = f(x)$, $F(x,1) = g(x)$, and $F(0,t) = x_0 = F(1,t)$. Hence, $HF: I \times I \to Y$ and by Definition 9(ii), HF is in N. Also, if $x \in I$, then

$(HF)(x,0) = H(F(x,0)) = H(f(x)) = (Hf)(x)$,

$(HF)(x,1) = H(F(x,1)) = H(g(x)) = (Hg)(x)$, and if $t \in I$, then

$(HF)(0,t) = H(F(0,t)) = H(x_0) = y_0 = H(x_0) = H(F(1,t)) = (HF)(1,t)$.

Hence, $Hf \overset{N}{\underset{y_0}{\approx}} Hg$ and $[Hf] = [Hg]$. Thus, σ is well-defined.

Let $[f],[g] \in N(X,x_0)$. By Definition 16,

$\sigma([f] \cdot [g]) = \sigma([f *_N]) = [H(f *_N g)]$ and

$\sigma[f] \cdot \sigma[g] = [Hf] \cdot [Hg] = [Hf *_N Hg]$. Also, if $f *_N g \in C(X,x_0)$, then

$$H(f *_N g)(x) = \begin{cases} H(f(4x)) \\ H(x_0) \\ H(g(4x-3)) \end{cases} = \begin{cases} Hf(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{otherwise} \\ Hg(4x-3) & \text{if } 3/4 \leq x \leq 1 \end{cases}, \text{ and}$$

$$(Hf *_N Hg)(x) = \begin{cases} Hf(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{otherwise} \\ Hg(4x-3) & \text{if } 3/4 \leq x \leq 1 \end{cases}.$$

Hence, $H(f *_N g) = Hf *_N Hg$. Hence, $[H(f *_N g)] = [Hf *_N Hg]$. Thus, $\sigma([f]\cdot[g]) = \sigma[f]\cdot\sigma[g]$. Therefore, σ is a homomorphism.

Let $[f] \in N(Y,y_0)$. Then $f \in C(Y,y_0)$. Since f is continuous and

$(H^{-1}f)(0) = H^{-1}(f(0)) = H^{-1}(y_0) = x_0 = H^{-1}(y_0) = H^{-1}(f(1)) = (H^{-1}f)(1)$, then $H^{-1}f \in C(X,x_0)$. Hence, $[H^{-1}f] \in N(X,x_0)$. Hence, $\sigma([H^{-1}f]) = [HH^{-1}f] = [f]$. Therefore, σ is an epimorphism.

Let $[f],[g] \in N(X,x_0)$ such that $\sigma([f]) = \sigma([g])$. Then $[Hf] = [Hg]$ and $Hf \underset{y_0}{\overset{N}{\sim}} Hg$. By Definition 16, there is a function $G: I \times I \to Y$ in N such that if $x \in I$ and $t \in I$, then $G(x,0) = (Hf)(x)$, $G(x,1) = (Hg)(x)$, and $G(0,t) = y_0 = G(1,t)$. By Definition 9(ii), $H^{-1}G: I \times I \to X$ is in N. Also, if $x \in I$, then $(H^{-1}G)(x,0) = H^{-1}(G(x,0)) = H^{-1}(Hf(x)) = f(x)$, and $(H^{-1}G)(x,1) = H^{-1}(G(x,1)) = H^{-1}(Hg(x)) = g(x)$. If $t \in I$, then $(H^{-1}G)(0,t) = H^{-1}(G(0,t)) = H^{-1}(y_0) = x_0$, and $(H^{-1}G)(1,t) = H^{-1}(G(1,t)) = H^{-1}(y_0) = x_0$. Thus, by Definition 10, $f \underset{x_0}{\overset{N}{\sim}} g$. Hence, $[f] = [g]$. Therefore, σ is one-to-one. Hence, σ is an isomorphism and $N(X,x_0)$ is isomorphic to $N(Y,y_0)$.

<u>Definition 21</u>: A topological space (Y,T) is said to be <u>pathwise connected</u> provided that if $a,b \in Y$, then there exists a continuous function $f\colon I \to Y$ such that $f(0) = a$ and $f(1) = b$.

<u>Definition 22</u>: Let (Y,T) be a topological space, let $y_0 \in Y$, and let $f\colon I \to Y$ and $g\colon I \to Y$ be continuous functions such that $f(1) = g(0)$. Then $f *_N g$ is the continuous function defined by if $x \in I$, then

$$(f*_N g)(x) = \begin{cases} f(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4. \\ g(4x-3) & \text{if } 3/4 \leq x \leq 1 \end{cases}$$

<u>Theorem 16</u>: Let (Y,T) be a pathwise connected topological space, let $y_0, y_1 \in Y$, and let N be an admitting homotopy relation. Then $N(Y,y_0)$ is isomorphic to $N(Y,y_1)$.

<u>Proof</u>: Since (Y,T) is pathwise connected, there is a continuous function $p\colon I \to Y$ such that $p(0) = y_0$ and $p(1) = y_1$. Define $\bar{p}\colon I \to Y$ by if $x \in I$, then $\bar{p}(x) = p(1-x)$. Since p is continuous, then $\bar{p}$ is continuous. Also, $p(1) = \bar{p}(0)$ and $p(0) = \bar{p}(1)$. Define $e_0\colon I \to Y$ by if $x \in I$, then $e_0(x) = y_0$. Define $e_1\colon I \to Y$ by if $x \in I$, then $e_1(x) = y_1$. By Definition 22, $(p*_N\bar{p})(0) = p(0) = y_0 = \bar{p}(1) = (p*_N\bar{p})(1)$. Hence, by Definition 10, $p*_N\bar{p} \in C(Y,y_0)$. Also by Definition 22, $(\bar{p}*_N p)(0) = \bar{p}(0) = y_1 = p(1) = (\bar{p}*_N p)(1)$. Hence, by Definition 10, $\bar{p}*_N p \in C(Y,y_1)$. Define a function $H_1\colon I \times I \to Y$ by if $(x,t) \in I \times I$, then

$$H_1(x,t) = \begin{cases} p\left(\frac{4x}{1-t}\right) & \text{if } x \le \frac{1-t}{4},\ 0 \le x \le 1/4,\ 0 \le t < 1 \\ y_0 & \text{otherwise} \\ p\left(\frac{4x-4}{t-1}\right) & \text{if } x \ge \frac{t+3}{4},\ 3/4 \le x \le 1,\ 0 \le t < 1 \end{cases} .$$

In a manner similar to that of Theorem 11, it can be shown that H_1 is continuous. Hence, H_1 is in N. Also, if $x \in I$, then

$$H_1(x,0) = \begin{cases} p(4x) & \text{if } 0 \le x \le 1/4 \\ y_0 & \text{if } 1/4 \le x \le 3/4 \\ p(4-4x) & \text{if } 3/4 \le x \le 1 \end{cases}$$

$$= \begin{cases} p(4x) & \text{if } 0 \le x \le 1/4 \\ y_0 & \text{if } 1/4 \le x \le 3/4 \\ \bar{p}(4x-3) & \text{if } 3/4 \le x \le 1 \end{cases} = (p *_N \bar{p})(x),$$

$H_1(x,1) = y_0 = e_0(x)$, and if $t \in I$, then $H_1(0,t) = p(0) = y_0 = \bar{p}(1) = H_1(1,t)$. Thus, $p *_N \bar{p} \underset{y_0}{\overset{N}{\sim}} e_0$. Define a function $H_2\colon I \times I \to Y$ by if $(x,t) \in I \times I$, then

$$H_2(x,t) = \begin{cases} p\left(\frac{1-t-4x}{1-t}\right) & \text{if } x \le \frac{1-t}{4},\ 0 \le x \le 1/4,\ 0 \le t < 1 \\ y_1 & \text{otherwise} \\ p\left(\frac{3-4x}{t-1}\right) & \text{if } x \ge \frac{t+3}{4},\ 3/4 \le x \le 1,\ 0 \le t < 1 \end{cases} .$$

In a manner similar to that of Theorem 11, it can be shown that H_2 is continuous. Hence, H_2 is in N. Also, If $x \in I$, then

$$H_2(x,0) = \begin{cases} p(1-4x) & \text{if } 0 \le x \le 1/4 \\ y_1 & \text{if } 1/4 \le x \le 3/4 \\ p(4x-3) & \text{if } 3/4 \le x \le 1 \end{cases}$$

$$= \begin{cases} \bar{p}(4x) & \text{if } 0 \le x \le 1/4 \\ y_1 & \text{if } 1/4 \le x \le 3/4 \\ p(4x-3) & \text{if } 3/4 \le x \le 1 \end{cases} = (\bar{p} *_N p)(x),$$

$H_2(x,1) = y_1 = e_1(x)$, and if $t \in I$, then
$H_2(0,t) = \overline{p}(0) = y_1 = p(1) = H_2(1,t)$. Thus, $\overline{p}*_N p \underset{y_1}{\overset{N}{\approx}} e_1$. Hence,
$[p*_N\overline{p}] = [e_0]$ and $[\overline{p}*_N p] = [e_1]$. Define $\lambda\colon N(Y,y_0) \to N(Y,y_1)$
by if $[f] \in N(Y,y_0)$, then $\lambda([f]) = [\overline{p}*_N(f*_N p)]$. Let $f,g \in C(Y,y_0)$
such that $f \underset{y_0}{\overset{N}{\approx}} g$. By Definition 10, there is a function $F\colon I \times I \to Y$
in N such that if $x \in I$ then $F(x,0) = f(x)$, $F(x,1) = g(x)$, and
if $t \in I$ then $F(0,t) = y_0 = F(1,t)$. By Definition 22, if $x \in I$
then

$$(\overline{p}*_N(f*_N p))(x) = \begin{cases} \overline{p}(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4 \\ f(16x-12) & \text{if } 3/4 \leq x \leq 13/16 \\ y_0 & \text{if } 13/16 \leq x \leq 15/16 \\ p(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases} \text{, and}$$

$$(\overline{p}*_N(g*_N p))(x) = \begin{cases} \overline{p}(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4 \\ g(16x-12) & \text{if } 3/4 \leq x \leq 13/16 \\ y_0 & \text{if } 13/16 \leq x \leq 15/16 \\ p(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases} .$$

Define $G\colon I \times I \to Y$ by if $(x,t) \in I \times I$, then

$$G(x,t) = \begin{cases} \overline{p}(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4 \\ F(16x-12,t) & \text{if } 3/4 \leq x \leq 13/16,\ t \in I. \\ y_0 & \text{if } 13/16 \leq x \leq 15/16 \\ p(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases}$$

In a manner similar to that of Theorem 11 it can be shown that

$G|_{[0,1/4] \times I}$, $G|_{[1/4,3/4] \times I}$, $G|_{[13/16,15/16] \times I}$, and $G|_{[15/16,1] \times I}$ are continuous. Since F is in N, then $G|_{[3/4,13/16] \times I}$ is in N. Hence, by Definition 9(iv), G is in N. Also, if $x \in I$, then

$$G(x,0) = \begin{cases} \bar{p}(4x) \\ y_0 \\ F(16x-12,0) \\ y_0 \\ p(16x-15) \end{cases} = \begin{cases} \bar{p}(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4 \\ f(16x-12) & \text{if } 3/4 \leq x \leq 13/16 \\ y_0 & \text{if } 13/16 \leq x \leq 15/16 \\ p(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases}$$

$$= (\bar{p} *_N (f *_N p))(x),$$

and

$$G(x,1) = \begin{cases} \bar{p}(4x) \\ y_0 \\ F(16x-12,1) \\ y_0 \\ p(16x-15) \end{cases} = \begin{cases} \bar{p}(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_0 & \text{if } 1/4 \leq x \leq 3/4 \\ g(16x-12) & \text{if } 3/4 \leq x \leq 13/16 \\ y_0 & \text{if } 13/16 \leq x \leq 15/16 \\ p(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases}$$

$$= (\bar{p} *_N (g *_N p))(x),$$

and if $t \in I$, then $G(0,t) = \bar{p}(0) = y_1 = p(1) = G(1,t)$. Hence, $\bar{p} *_N (f *_N p) \underset{y_1}{\overset{N}{\approx}} \bar{p} *_N (g *_N p)$. Hence, if $f \underset{y_0}{\overset{N}{\approx}} g$, then $\bar{p} *_N (f *_N p) \underset{y_1}{\overset{N}{\approx}} \bar{p} *_N (g *_N p)$. Let $h,k \in C(Y,y_1)$ such that $h \underset{y_1}{\overset{N}{\approx}} k$. By Definition 10, there is a function $F_1 : I \times I \to Y$ in N such that if $x \in I$, then $F_1(x,0) = h(x)$, $F_1(x,1) = k(x)$, and if $t \in I$, then $F_1(0,t) = y_1 = F_1(1,t)$. Also,

$$(p *_N (h *_N \bar{p}))(x) = \begin{cases} p(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_1 & \text{if } 1/4 \leq x \leq 3/4 \\ h(16x-12) & \text{if } 3/4 \leq x \leq 13/16 \\ y_1 & \text{if } 13/16 \leq x \leq 15/16 \\ \bar{p}(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases} \quad , \text{ and}$$

$$(p *_N (k *_N \bar{p}))(x) = \begin{cases} p(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_1 & \text{if } 1/4 \leq x \leq 3/4 \\ k(16x-12) & \text{if } 3/4 \leq x \leq 13/16 \\ y_1 & \text{if } 13/16 \leq x \leq 15/16 \\ \bar{p}(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases} \quad .$$

Define $G_1 : I \times I \to Y$ by if $(x,t) \in I \times I$, then

$$G_1(x,t) = \begin{cases} p(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_1 & \text{if } 1/4 \leq x \leq 3/4 \\ F_1(16x-12) & \text{if } 3/4 \leq x \leq 13/16 \\ y_1 & \text{if } 13/16 \leq x \leq 15/16 \\ \bar{p}(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases} \quad .$$

In a manner similar to that of Theorem 11 it can be shown that $G_1|_{[0,1/4] \times I}$, $G_1|_{[1/4,3/4] \times I}$, $G_1|_{[13/16,15/16] \times I}$, and $G_1|_{[15/16,1] \times I}$ are continuous. Since F_1 is in N, then $G_1|_{[3/4,13/16] \times I}$ is in N. Hence, by Definition 9(iv), G_1 is in N. Also, if $x \in I$, then

$$G_1(x,0) = \begin{cases} p(4x) \\ y_1 \\ F_1(16x-12,0) \\ y_1 \\ \bar{p}(16x-15) \end{cases} = \begin{cases} p(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_1 & \text{if } 1/4 \leq x \leq 3/4 \\ h(16x-12) & \text{if } 3/4 \leq x \leq 13/16 \\ y_1 & \text{if } 13/16 \leq x \leq 15/16 \\ \bar{p}(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases}$$

$$= (p *_N (h *_N \bar{p}))(x),$$

and

$$G_1(x,1) = \begin{cases} p(4x) \\ y_1 \\ F_1(16x-12,1) \\ y_1 \\ \bar{p}(16x-15) \end{cases} = \begin{cases} p(4x) & \text{if } 0 \leq x \leq 1/4 \\ y_1 & \text{if } 1/4 \leq x \leq 3/4 \\ k(16x-12) & \text{if } 3/4 \leq x \leq 13/16 \\ y_1 & \text{if } 13/16 \leq x \leq 15/16 \\ \bar{p}(16x-15) & \text{if } 15/16 \leq x \leq 1 \end{cases}$$

$$= (p *_N (k *_N \bar{p}))(x),$$

and if $t \in I$, then $G_1(0,t) = p(0) = y_0 = \bar{p}(1) = G_1(1,t)$. Hence, $p *_N (h *_N \bar{p}) \underset{y_0}{\overset{N}{\approx}} p *_N (k *_N \bar{p})$. Hence, if $h \underset{y_1}{\overset{N}{\approx}} k$, then $p *_N (h *_N \bar{p}) \underset{y_0}{\overset{N}{\approx}} p *_N (k *_N \bar{p})$.

Let $[f] \in N(Y,y_0)$ Then by Corollary 1, $\bar{p} *_N (f *_N p)$ is continuous. Also, $(\bar{p} *_N (f *_N p))(0) = \bar{p}(0) = y_1$ and $(\bar{p} *_N (f *_N p))(1) = p(1) = y_1$. Hence $\bar{p} *_N (f *_N p) \in C(Y,y_1)$. Therefore, $\lambda([f]) \in N(Y,y_1)$ and hence λ is into. Let $[f] \in N(Y,y_0)$ and let $f,g \in [f]$. Then $f \underset{y_0}{\overset{N}{\approx}} g$. Thus $\bar{p} *_N (f *_N p) \underset{y_1}{\overset{N}{\approx}} \bar{p} *_N (g *_N p)$. So $[\bar{p} *_N (f *_N p)] = [\bar{p} *_N (g *_N p)]$ and λ is well-defined.

Let $[f],[g] \in N(Y,y_0)$. Then

$\lambda([f]\cdot[g]) = \lambda([f *_N g]) = [\bar{p} *_N f *_N g *_N p] = [\bar{p} *_N f *_N e_0 *_N g *_N p] =$

$[\bar{p} *_N f *_N p *_N \bar{p} *_N g *_N p] = [\bar{p} *_N f *_N p] \cdot [\bar{p} *_N g *_N p] = \lambda[f] \cdot \lambda[g]$.

Hence, λ is a homomorphism.

Let $[f] \in N(Y,y_1)$. Then $[p*_N(f*_N\bar{p})] \in N(Y,y_0)$ and $\lambda([p*_N(f*_N\bar{p})]) = [\bar{p}*_N p*_N(f*_N\bar{p})*_N p] = [e_1*_N f*_N e_1] = [f]$. Hence, λ is an epimorphism.

Let $[f],[g] \in N(Y,y_0)$ such that $\lambda([f]) = \lambda([g])$. Then $[\bar{p}*_N(f*_N p)] = [\bar{p}*_N(g*_N p)]$ and thus, $\bar{p}*_N(f*_N p) \underset{y_1}{\overset{N}{\sim}} \bar{p}*_N(g*_N p)$. Hence, $p*_N\bar{p}*_N(f*_N p)*_N\bar{p} \underset{y_0}{\overset{N}{\sim}} p*_N\bar{p}*_N(g*_N p)*_N\bar{p}$, and $e_0*_N f*_N e_0 \underset{y_0}{\overset{N}{\sim}} e_0*_N g*_N e_0$. Therefore, $f \underset{y_0}{\overset{N}{\sim}} g$. Hence, $[f] = [g]$ and λ is one-to-one.

Therefore, λ is an isomorphism. Thus, $N(Y,y_0)$ is isomorphic to $N(Y,y_1)$.

SUMMARY

In Definition 9, the concept of an admitting homotopy relation is defined. It is clear that if N is an admitting homotopy relation, then N is a class of functions which at least contains the continuous functions. In [2], and [4], it is shown that there are indeed classes of functions other than the class of continuous functions which are admitting homotopy relations, and in [1], it is shown that there is an admitting homotopy relation which actually produces different, non-trivial groups. The fundamental group is known in some sense to count the number of holes in a space (i.e., the number of different types of loops at a point in the space) and it is hoped that there will be admitting homotopy relations other than the class of continuous functions which would produce more information about topological spaces. This is currently an unsolved problem.

It has been shown in Theorem 13 that an admitting homotopy relation is sufficient to insure a group. It is an interesting question, and currently unsolved, that if you have a class of functions which always produces a group in the described manner leading to Theorem 13, does this class of functions have to be an admitting homotopy relation.

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