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The basic properties of a pure subgroup of an abelian group are examined, and are then used to prove the Fundamental Theorem of Finitely Generated Abelian Groups. A number of examples of pure subgroups are included. Further, the concept of purity in abelian groups is generalized to a concept of R-purity in modules over a commutative ring R. Several properties of an R-pure submodule of a module are then proved.

ON PURITY IN ABELIAN GROUPS AND IN MODULES

by

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INTRODUCTION

The purpose of this thesis is to study the elementary properties of pure subgroups of an abelian group, to use these properties to prove the Fundamental Theorem of Finitely Generated Abelian Groups, and to generalize the abelian group concept of purity to modules over a commutative ring. Many of the results of this paper are found in Kaplansky [4].

Chapter I defines a pure subgroup of an abelian group, and considers the behavior of purity with respect to direct summands, to divisible groups, to the operations of union and intersection, to homomorphisms, and to torsion and torsionfree groups. An easy method of obtaining pure subgroups of primary groups is discussed. Also, a characterization of an abelian group in which every subgroup is a pure subgroup is developed.

In Chapter II, groups of bounded order are introduced. Pure subgroups which are direct sums of cyclic groups are then considered, and are used to prove that any group of bounded order is a direct sum of cyclic groups. The last major result of this chapter is using the concept of purity to prove the Fundamental Theorem of Finitely Generated Abelian Groups.

Chapter III defines R-purity for modules over a commutative ring, defines injective modules, and defines the injective envelope of a module. The concept of a module being absolutely R-pure is introduced and is characterized. Several of the properties of pure subgroups of an abelian group are generalized to the module concept of R-purity.

CHAPTER I

ELEMENTARY PROPERTIES OF PURE SUBGROUPS

Throughout this paper, it is understood that the term "group" will mean "abelian group," Z^+ will represent the set of positive integers, and Z^* will represent the set of nonzero integers.

We present three definitions of a pure subgroup of a group that are easily seen to be equivalent. Thus, we will feel free to use the most convenient definition in our later proofs.

Definition 1.1. A subgroup H of a group G is pure in G provided if $n \in Z^*$, $y \in G$ such that $ny \in H$, then $ny = nh$ for some $h \in H$.

Definition 1.2. A subgroup H of a group G is pure in G if $nH = nG \cap H$ for every $n \in Z^*$.

Definition 1.3. A subgroup H of a group G is pure in G if for $h \in H$, $h = ny$ $(n \in Z^*, y \in G)$, then this implies $h = nh_1$ with $h_1 \in H$. In other words, if $h \in H$ is divisible by n in G, h is already divisible by n in H. Note that for $h \in H$ to be divisible by n in G, there must exist $y \in G$ such that $h = ny$.

Example 1.4. For an example of a nonpure subgroup, let G be the additive group of integers mod 4, Z_4. Then $G = \{0, 1, 2, 3\}$. Consider the subgroup $H = \{0, 2\}$. Choose $2 \in Z^*$, $1 \in G$ such that $2 \cdot 1 = 2 \in H$. Then there does not exist $h \in H$ such that $2 \cdot h = 2 \cdot 1$, since h must be either 0 or 2 and in either case $2 \cdot h = 0$. Therefore H is not a pure subgroup of G.

Example 1.5. Consider the four nontrivial subgroups of Z_{12}. Let $H_1 = \{0, 6\}$, let $H_2 = \{0, 4, 8\}$, let $H_3 = \{0, 3, 6, 9\}$, and let $H_4 = \{0, 2, 4, 6, 8, 10\}$. Then one can tediously check that H_2 and H_3 are pure subgroups of Z_{12}. However, H_1 and H_4 are not pure subgroups of Z_{12}, since for $2 \in Z^*$, $\{0\} = 2H_1 \neq 2Z_{12} \cap H_1 = \{0, 2, 4, 6, 8, 10\} \cap \{0, 6\} = \{0, 6\}$ and, for $6 \in Z^*$, $\{0\} = 6H_4 \neq 6Z_{12} \cap H_4 = \{0, 6\} \cap \{0, 2, 4, 6, 8, 10\} = \{0, 6\}$.

The following propositions contain some important properties of pure subgroups.

Proposition 1.6. Any direct summand of a group G is a pure subgroup of G.

Proof: Let G be a group such that $G = S \oplus T$. We wish to show that S is a pure subgroup of G. Let $n \in Z^*$, $y \in G$ such that $ny \in S$. Since $G = S \oplus T$, $y \in G$ implies $y = y_1 + y_2$ for $y_1 \in S$, $y_2 \in T$. Thus $ny = n(y_1 + y_2) = ny_1 + ny_2 \in S$. But $ny_1 \in S$ and $ny_2 \in T$, so $ny_2 = ny - ny_1 \in S \cap T = 0$. Therefore $ny_2 = 0$, so $ny = ny_1$ for $y_1 \in S$. Hence S is a pure subgroup of G.

Remark 1.7. The converse of Proposition 1.6 is not necessarily true. As we will see in Example 1.33, the purity of a subgroup T of a group G does not imply T is a direct summand of G. However, Theorem 1.18 and Theorem 2.12 give two special cases where a pure subgroup of a group G is a direct summand of G.

The next proposition shows an easy way of obtaining pure subgroups by using Proposition 1.6. Recall that for integers m and n, $(m, n) = 1$ means that m and n are relatively prime; i.e., the greatest common divisor of m and n is 1.

Proposition 1.8. If $(m, n) = 1$ for $m, n \in Z^+$, then $Z_{mn} \cong Z_m \oplus Z_n$. Furthermore, Z_m and Z_n are pure subgroups of Z_{mn}.

Proof: Let $(m, n) = 1$ for $m, n \in Z^+$. Choose $(1_m, 1_n) \in Z_m \oplus Z_n$, where 1_m generates Z_m and 1_n generates Z_n. Consider the cyclic subgroup generated by $(1_m, 1_n)$, denoted $<(1_m, 1_n)>$. Now 1_m has order m and 1_n has order n, so the order of $(1_m, 1_n)$ is the least common multiple of m and n. Since $(m, n) = 1$, the order of $(1_m, 1_n)$, and hence the order of $<(1_m, 1_n)>$, is mn. Thus $Z_m \oplus Z_n$, with order mn, has a cyclic subgroup $<(1_m, 1_n)>$ of order mn, so $Z_m \oplus Z_n \cong <(1_m, 1_n)>$. But Z_{mn} is a cyclic group of order mn, so $<(1_m, 1_n)> \cong Z_{mn}$. Therefore $Z_m \oplus Z_n \cong Z_{mn}$.

Finally, by Proposition 1.6, Z_m and Z_n are pure subgroups of Z_{mn} with orders n and m respectively, since we may consider Z_m and Z_n as direct summands of Z_{mn}.

We remark that Proposition 1.8 gives us a quick method of seeing that H_2 and H_3 are pure subgroups of Z_{12} in Example 1.5, because $(3, 4) = 1$ and hence $Z_{12} \cong Z_3 \oplus Z_4 \cong H_2 \oplus H_3$.

Proposition 1.9. If H is a pure subgroup of a group G, and if K is a pure subgroup of H, then K is a pure subgroup of G.

Proof: Let $n \in Z^*$. Then $nK = K \cap nH = K \cap (H \cap nG) = (K \cap H) \cap nG = K \cap nG$. Thus K is a pure subgroup of G.

An alternate method of proof follows: Let $n \in Z^*$, $y \in G$ such that $ny \in K$. Since H is a pure subgroup of G and $K \subseteq H$, $ny \in H$ and $ny = nh$ for some $h \in H$. Now K is a pure subgroup of H, so $nh = nk$ for some $k \in K$. Therefore $ny = nk$ for some $k \in K$. Thus K is a pure subgroup of G.

<u>Remark 1.10</u>. Recall the following definitions for a set A. A relation $\leq$ on a set A is called a <u>partial order</u> provided for a, b, c $\in$ A, the relation $\leq$ is (i) reflexive: $a \leq a$; (ii) anti-symmetric: If $a \leq b$ and if $b \leq a$, then $a = b$; and (iii) transitive: If $a \leq b$ and if $b \leq c$, then $a \leq c$. A <u>partially ordered set</u> $(A, \leq)$ is a set A together with a partial order $\leq$ on A. If $(A, \leq)$ is a partially ordered set, a nonempty subset C of A is called a <u>chain</u> provided if c, d $\in$ C, then either $c \leq d$ or $d \leq c$.

In our use of the above definitions, we will be concerned with subsets of a set, and our partial order will be set inclusion.

<u>Proposition 1.11</u>. The union of an ascending chain of pure subgroups of a group G is a pure subgroup of G.

<u>Proof</u>: Let $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots \subseteq H_i \subseteq \cdots$ be an ascending chain of pure subgroups of G. We must show $\bigcup_I H_i$ is a pure subgroup of G. Let $n \in Z^*$, $y \in G$ such that $ny \in \bigcup_I H_i$. Then $ny \in H_i$ for some $i \in I$. Since each H_i is a pure subgroup of G, there exists $h \in H_i$ such that $ny = nh$. Therefore there exists $h \in \bigcup_I H_i$ such that $ny = nh$. Thus $\bigcup_I H_i$ is a pure subgroup of G.

An alternate method of proof follows: Let $n \in Z^*$, and let $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots \subseteq H_i \subseteq \cdots$ be an ascending chain of pure subgroups of G. Then $nG \cap (\bigcup_I H_i) = \bigcup_I (nG \cap H_i) = \bigcup_I (nH_i) = n(\bigcup_I H_i)$. Therefore $\bigcup_I H_i$ is a pure subgroup of G.

<u>Remark 1.12</u>. Due to this property (Proposition 1.11) of pure subgroups, one often finds it more convenient to work with pure subgroups instead of with direct summands, since the union of an ascending chain of direct summands is not necessarily a direct summand.

Example 1.13. In general, the union of pure subgroups of a group G is not a pure subgroup of G. Consider $G = Z_6 = \{0, 1, 2, 3, 4, 5\}$. Then $S = \{0, 3\}$ and $T = \{0, 2, 4\}$ are pure subgroups of G by Propositions 1.6 and 1.8. However, $S \cup T = \{0, 2, 3, 4\}$ is not even a subgroup of G.

Definition 1.14. A group G is divisible if $G = nG$ for all $n \in Z^*$; i.e., for all $x \in G$ and $n \in Z^*$, there exists $y \in G$ such that $x = ny$.

Proposition 1.15. A divisible subgroup H of a group G is a pure subgroup of G.

Proof: Let H be a divisible subgroup of a group G, and let $n \in Z^*$. Since $nH \subseteq nG$, $nH = nG \cap nH = nG \cap H$. Therefore H is a pure subgroup of G.

Proposition 1.16. A subgroup H of a divisible group G is a pure subgroup of G if and only if H is divisible.

Proof: $(\rightarrow)$ Assume H is a pure subgroup of a divisible group G, and let $n \in Z^*$. Then $nH = nG \cap H = G \cap H = H$. Therefore $nH = H$.

$(\leftarrow)$ Assume H is a divisible subgroup of a divisible group G. By Proposition 1.15, H is a pure subgroup of G.

The following is a fundamental lemma on pure subgroups.

Lemma 1.17. Let G be a group, let H be a pure subgroup of G, and let $y \in G/H$. Then there exists an element z in G, with $z + H = y$, such that z has the same order as y.

Proof: Let H be a pure subgroup of a group G, and let $y \in G/H$. Suppose y has infinite order. Then any choice of an element z in G, with $z + H = y$, will have the desired property, because the order of $z + H$ is less than or equal to the order of z.

Suppose y has finite order n. Then $y \in G/H$ implies $y = y_1 + H$ for $y_1 \in G$, and $ny = n(y_1 + H) = ny_1 + H = H$, so $ny_1 \in H$. By the purity of H, there exists $h \in H$ such that $ny_1 = nh$. So $ny_1 - nh = n(y_1 - h) = 0$. Let $z = y_1 - h$. Then $z + H = (y_1 - h) + H = y_1 + H = y$. Now $n(z) = n(y_1 - h) = 0$, so the order of z is less than or equal to n. But clearly z cannot have order less than n, because then $y = z + H$ would have order less than n. Therefore z has the same order as y.

The next theorem, stating that a pure subgroup is sometimes a direct summand, follows from Lemma 1.17.

Theorem 1.18. Let G be a group, and let H be a pure subgroup of G such that G/H is a direct sum of cyclic groups. Then H is a direct summand of G.

Proof: Let H be a pure subgroup of a group G such that G/H is a direct sum of cyclic groups. For each cyclic summand of G/H, pick a generator y_i. By Lemma 1.17, we can select elements x_i in G, with $x_i + H = y_i$, such that x_i has the same order as y_i. Let K be the subgroup of G generated by the elements x_i.

Claim: $G = H \oplus K$

Subproof: We must show (i) $G = H + K$, and (ii) $H \cap K = 0$.

(i) Clearly $H + K \subseteq G$. Let $t \in G$, and let $t + H = t^* \in G/H$. Since G/H is a direct sum of cyclic groups, we may write t^* as a finite sum $\Sigma\, a_i y_i$, where $a_i \in Z$. Then $(t - \Sigma\, a_i x_i) + H = t^* - \Sigma\, a_i y_i$, which is zero in G/H, so $t - \Sigma a_i x_i \in H$. Since $\Sigma a_i x_i \in K$, we have $t = (t - \Sigma a_i x_i) + \Sigma a_i x_i \in H + K$. Thus $G \subseteq H + K$. Therefore $G = H + K$.

(ii) Let $w \in H \cap K$, say $w = \Sigma a_i x_i$. Then if $w + H = \Sigma a_i y_i$, we have $\Sigma a_i y_i = 0$ since $w \in H$. If each y_i has infinite order, then $\Sigma a_i y_i = 0$ implies $a_i = 0$. If some y_i has finite order n_i, then a_i must be a multiple of n_i. In either case, $w = \Sigma a_i x_i = 0$. Thus $H \cap K = 0$.

Therefore $G = H \oplus K$, from which we conclude that H is a direct summand of G.

The next two lemmas are concerned with the behavior of purity with respect to a homomorphism.

Lemma 1.19. Let G be a group, let S be a pure subgroup of G, and let T be a subgroup of G containing S such that T/S is a pure subgroup of G/S. Then T is a pure subgroup of G.

Proof: Let $n \in Z^*$, $x \in G$ such that $nx \in T$. Then $x + S$ is the homomorphic image of x in G/S, so $n(x + S) = nx + S \in T/S$. By the purity of T/S in G/S, there exists $t + S \in T/S$ such that $n(x + S) = n(t + S)$. Thus $nx + S = nt + S$, so $nx - nt = n(x - t) \in S$. Now S is a pure subgroup of G, so $n(x - t) = ns$ for some $s \in S$. Thus $nx = nt + ns = n(t + s)$ for $t + s \in T$. Therefore T is a pure subgroup of G.

The converse of Lemma 1.19 is also true. If S and T are pure subgroups of G, with $S \subseteq T$, then T/S is a pure subgroup of G/S. However, the purity of S is not essential. Therefore we state it as follows.

Proposition 1.20. Let S be a subgroup of G, and let T be a pure subgroup of G, with $S \subseteq T$. Then T/S is a pure subgroup of G/S.

Proof: Let $n \in Z^*$ and $g + S \in G/S$ for $g \in G$, such that $n(g + S) \in T/S$. We wish to show that there exists $c + S \in T/S$ such that $n(g + S) = n(c + S)$. Let $n(g + S) = t + S \in T/S$. Then $ng + S = t + S$, or $ng = t + s \in T$ for some $s \in S$. By the purity of T in G, there exists $c \in T$ such that $ng = nc \in T$. So $ng = nc = t + s$, or $nc + S = t + S = n(g + S)$. Thus $n(c + S) = n(g + S)$. Therefore T/S is a pure subgroup of G/S.

Lemma 1.21. Let $n \in Z^*$, and let S be a pure subgroup of G with $nS = 0$. Then $(S + nG)/nG$ is a pure subgroup of G/nG.

Proof: Let $m \in Z^*$, $y \in G/nG$ such that $my = x \in (S + nG)/nG$. Now $y = t + nG$ for some $t \in G$, and $x = s + nG$ for some $s \in S$. So $x = my$ implies $s + nG = m(t + nG) = mt + nG$, or $s = mt + nz$ for some $z \in G$. Let $(m, n) = r$, then $m = rm_1$ and $n = rn_1$. Then $s = rm_1t + rn_1z = r(m_1t + n_1z)$. Since $(m_1, n_1) = 1$, there exist $a, b \in Z$ such that $am_1 + bn_1 = 1$. Now $s = r(m_1t + n_1z)$ where $(m_1t + n_1z) \in G$, and S is a pure subgroup of G, so $s = rs_1$ for some $s_1 \in S$. Therefore $s = rs_1 = r(am_1 + bn_1)s_1 = ram_1s_1 + rbn_1s_1 = ams_1 + bns_1 = ams_1$, since $ns_1 \in nS = 0$. Thus $my = x = s + nG = ams_1 + nG = m(as_1 + nG)$, where $(as_1 + nG) \in (S + nG)/nG$. Hence $(S + nG)/nG$ is a pure subgroup of G/nG.

Definition 1.22. The torsion subgroup T of a group G is the set of all elements in G having finite order; i.e., if $a \in G$, then $a \in T$ provided there exists a positive integer n such that $na = 0$. It is easy to prove that T is, in fact, a subgroup of the abelian group G.

Proposition 1.23. The torsion subgroup T of a group G is a pure subgroup of G.

Proof: Let $n \in Z^*$. We must show $nT = nG \cap T$. Clearly $nT \subseteq nG \cap T$. Choose $x \in nG \cap T$. Then $x \in nG$, so $x = ny$ for some $y \in G$. Since $x = ny \in T$, ny has finite order, say m. Thus $0 = m(ny) = (mn)y$, which implies $y \in T$, or $x = ny \in nT$. Therefore $nG \cap T \subseteq nT$. Hence $nT = nG \cap T$.

Definition 1.24. The group G is said to be torsionfree provided all the elements (except zero) of G have infinite order. In other words, the torsion subgroup of G is $\{0\}$.

Remark 1.25. If T is the torsion subgroup of a group G, then G/T is torsionfree.

Proof: Let T be the torsion subgroup of a group G. Consider $g + T \in G/T$ for $g \in G$. Assume n is a positive integer such that $n(g + T) = T$, where T is the identity of G/T. Then $T = n(g + T) = ng + T$ implies $ng \in T$. Thus there exists a positive integer m such that $m(ng) = 0$, so $(mn)g = 0$, which implies $g \in T$. Hence $g + T = T$, and the torsion part of G/T is zero. Therefore G/T is torsionfree.

Proposition 1.26. If S is a subgroup of a group G such that G/S is torsionfree, then S is a pure subgroup of G.

Proof: Let $n \in Z^*$, $y \in G$ such that $ny \in S$. Consider $y + S \in G/S$. Then $n(y + S) = ny + S = S$. Since $n \neq 0$ and G/S is torsionfree, $y + S = S$, which implies $y \in S$. Thus, if $x \in nG \cap S$, then $x = ny \in S$ $(n \in Z^*, y \in G)$ and $y \in S$. So $x \in nS$ and $nG \cap S \subseteq nS$. Clearly $nS \subseteq nG \cap S$. Therefore $nS = nG \cap S$.

Proposition 1.27. Let G be a torsionfree group, and let S be a pure subgroup of G. Then

(i) S is closed (within G) under division by integers; i.e., for all $s \in S$ and $n \in Z^*$, $ny = s \in S$ for some $y \in G$ implies that $y \in S$; and

(ii) this division by integers is unique; i.e., y is unique.

Proof: Let S be a pure subgroup of a torsionfree group G.

(i) Let $n \in Z^*$, $y \in G$ such that $ny \in S$. Since S is a pure subgroup of G, $ny = ns$ for some $s \in S$. Thus $0 = ny - ns = n(y - s)$. But $n \neq 0$, so $y - s = 0$, or $y = s$. Therefore $y \in S$.

(ii) Assume for all $s \in S$ and $n \in Z^*$ that n divides s implies there exist $y, w \in G$ such that $ny = s$ and $nw = s$. Then $nw = ny$ implies $nw - ny = n(w - y) = 0$. Now $n \neq 0$ and G is torsionfree, so $w - y = 0$. Thus $w = y$. Therefore y is unique.

From the results of Proposition 1.27, we obtain two corollaries, valid only for a torsionfree group.

Corollary 1.28. If G is a torsionfree group, then the intersection of pure subgroups of G is a pure subgroup of G.

Proof: Let G be a torsionfree group, and let $\{H_i\}_I$ be a collection of pure subgroups of G. Let $n \in Z^*$, $y \in G$ such that $ny \in \bigcap_I H_i$. Then $ny \in H_i$ for all $i \in I$. Since G is torsionfree and each H_i is a pure subgroup of G, $ny \in H_i$ for all i implies $y \in H_i$ for all i. Thus $y \in \bigcap_I H_i$, so $ny \in n(\bigcap_I H_i)$. Therefore $nG \cap (\bigcap_I H_i) \subseteq n(\bigcap_I H_i)$. Clearly $n(\bigcap_I H_i) \subseteq nG \cap (\bigcap_I H_i)$. Hence $n(\bigcap_I H_i) = nG \cap (\bigcap_I H_i)$, so $\bigcap_I H_i$ is a pure subgroup of G.

Corollary 1.29. Any subset K of a torsionfree group G is contained in a unique smallest pure subgroup.

Proof: Let $\{H_i\}_I$ be the collection of pure subgroups of a torsion-free group G which contain K. By Corollary 1.28, $\bigcap_I H_i$ is also a pure subgroup of G. Since $K \subseteq H_i$ for all i, then $K \subseteq \bigcap_I H_i$. Furthermore, if P is a pure subgroup of G containing K, then $\bigcap_I H_i \subseteq P$. Thus $\bigcap_I H_i$ is the unique smallest pure subgroup of G which contains K.

Theorem 1.30. Let G be a torsionfree group, let $x \in G$, and let $S = \{(a/b)x \mid a, b \in Z, b \neq 0, \text{ and } (a/b)x = (1/b)(ax) \text{ is defined in } G\}$. Then S is the smallest pure subgroup of G containing x.

Proof: Let G be a torsionfree group, let $x \in G$, and let $S = \{(a/b)x \mid a, b \in Z, b \neq 0, \text{ and } (a/b)x = (1/b)(ax) \text{ is defined in } G\}$.

Claim 1: S is a subgroup of G.

Subproof: Clearly $0 \in S$ and S contains inverses. Thus we only need to consider closure. Choose $u, v \in S$; so $u = (a/b)x$ and $v = (c/d)x$ for $a, b, c, d \in Z$, $b \neq 0$ and $d \neq 0$. Then $u = (1/b)(ax)$, so $bu = ax$; and $v = (1/d)(cx)$, so $dv = cx$. Thus $bdu = adx$ and $bdv = bcx$, so $bdu + bdv = adx + bcx$, or $bd(u + v) = (ad + bc)x$. Therefore $u + v = [(ad + bc)/bd]x \in S$. We have shown that if $(a/b)x$ and $(c/d)x$ are defined in G, then $u + v = (a/b)x + (c/d)x = [(ad + bc)/bd]x \in S$. Thus S is closed, and hence is a subgroup of G.

Claim 2: S is a pure subgroup of G.

Subproof: Let $n \in Z^*$. Clearly $nS \subseteq nG \cap S$. Choose $y \in nG \cap S$. Then $y = nz$ for some $z \in G$ and $y = (a/b)x$ for $a, b \in Z$, $b \neq 0$. Thus $nz = (a/b)x$, so $bnz = ax$ or $z = (1/bn)(ax) = (a/bn)x \in S$. Therefore $y = nz \in nS$. Hence $nG \cap S \subseteq nS$, so $nS = nG \cap S$.

Finally, let P be the smallest pure subgroup of G containing x; i.e., P is the intersection of all the pure subgroups of G which contain x. Since S is one of the subgroups over which this intersection takes place, then $P \subseteq S$. To prove that $S \subseteq P$, choose $(a/b)x \in S$. Then there exists $y \in S$ with $by = ax$. But $x \in P$ implies $ax \in P$, so $by \in P \cap bG = bP$ since P is a pure subgroup of G. Hence $by = bp$ for some $p \in P$, or $by - bp = b(y - p) = 0$. Since $b \neq 0$ and G is torsionfree, we have $y - p = 0$, so $y = p \in P$. Thus $y = (a/b)x \in P$. Therefore $S \subseteq P$, so $S = P$. Hence S is the smallest pure subgroup of G containing x.

Definition 1.31. A torsion group G is said to be primary (or p-primary) if, for a prime number p, every element of G has order a power of p.

Proposition 1.32. Let $m \in Z^*$, let p be a prime number, and let H be a subgroup of a p-primary group G. If $(m, p) = 1$, then $mH = H$ and also $mH = mG \cap H$. Finally, $mG = G$.

Proof: Let H be a subgroup of a p-primary group G, and let $m \in Z^*$ such that $(m, p) = 1$. Choose $x \in H$; then x has order p^k. Since $(m, p^k) = 1$, there exist $a, b \in Z$ such that $am + bp^k = 1$. Therefore $x = 1 \cdot x = (am + bp^k)x = amx + bp^k x = amx = m(ax)$. Hence there exists $y \in H$ such that $my = x$, namely $y = ax$. Thus H is divisible by m, so $mH = H$. Therefore $mG \cap H = mG \cap mH = mH$. If $H = G$, then we have shown that $mG = G$.

Example 1.33. If G is not torsionfree, then Corollary 1.28 is not necessarily true; i.e., the intersection of pure subgroups of G may not be a pure subgroup of G.

Consider $G = Z_{p^2} \oplus Z_p$ for a prime number p, where Z_{p^2} is generated by a, so $Z_{p^2} = \langle a \rangle$, and Z_p is generated by b, so $Z_p = \langle b \rangle$. Let $S = \langle (a, 0) \rangle$ and $T = \langle (a, b) \rangle$. Then, by Proposition 1.6, S is a pure subgroup of G since S is a direct summand of G.

Claim: T is a pure subgroup of G.

Subproof: Since G is a p-primary group, by Proposition 1.32, if $(m, p) = 1$ for $m \in Z^*$, then $mT = mG \cap T$. Assume $(m, p) \neq 1$. Then m is a multiple of p, so $m = np$ for $n \in Z^*$. Choose $x \in mG \cap T$. For some $(ka, \ell b) \in G$, $x = m(ka, \ell b) = (mka, m\ell b) = (npka, np\ell b) = (npka, 0) = np(ka, 0) = np(ka, kb) = m(ka, kb) \in mT$. Thus $mG \cap T \subseteq mT$. Clearly $mT \subseteq mG \cap T$, so we have $mT = mG \cap T$ for any $m \in Z^*$.

Now we know that S and T are pure subgroups of G. Consider the subgroup $S \cap T = \langle (pa, 0) \rangle$, and let $m = p$. Then $\{0\} = p(S \cap T) \neq pG \cap (S \cap T) = (S \cap T) \cap (S \cap T) = S \cap T$. Therefore $S \cap T$ is not a pure subgroup of G.

For example, choose $p = 3$. Then $G = Z_9 \oplus Z_3 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2), (5, 0), (5, 1), (5, 2), (6, 0), (6, 1), (6, 2), (7, 0), (7, 1), (7, 2), (8, 0), (8, 1), (8, 2)\}$. So $S = \langle (1, 0) \rangle = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6, 0), (7, 0), (8, 0)\}$ and $T = \langle (1, 1) \rangle = \{(0, 0), (1, 1), (2, 2), (3, 0), (4, 1), (5, 2), (6, 0), (7, 1), (8, 2)\}$. Thus $S \cap T = \langle (3, 0) \rangle = \{(0, 0), (3, 0), (6, 0)\}$. Now $S \cap T$ is a subgroup of G. However, for $3 \in Z^*$, $\{0\} = 3(S \cap T) \neq 3G \cap (S \cap T) = \{(0, 0), (3, 0), (6, 0)\} \cap (S \cap T) = S \cap T$. Therefore $S \cap T$ is not a pure subgroup of G.

Proposition 1.34. If G is a torsionfree group, and if S is a pure subgroup of G, then G/S is torsionfree.

Proof: Let S be a pure subgroup of a torsionfree group G. Let $y + S \in G/S$ for $y \in G$, and assume $n \in Z^*$ such that $n(y + S) = S$. Then $S = ny + S$, so $ny \in S$. Since G is torsionfree and S is a pure subgroup of G, $ny \in S$ implies $y \in S$ by Proposition 1.27. Therefore $y + S = S$, and the torsion part of G/S is zero. Thus G/S is torsionfree.

From Propositions 1.26 and 1.34 we obtain the following corollary.

Corollary 1.35. Let G be a torsionfree group. Then S is a pure subgroup of G if and only if G/S is a torsionfree group.

Finally, we wish to show that G is a torsion group in which every element has square-free order if and only if every subgroup of G is a pure subgroup of G. To do so, we need some preliminary information.

Theorem 1.36. Any torsion group is a direct sum of primary groups.

Proof: Let G be a torsion group. For every prime p, let G_p be the subset of G consisting of elements with order a power of p. Then G_p is a subgroup of G, and G_p is primary. In order to show $G \cong \oplus \Sigma_I G_{p_i}$, we need (i) $G = \Sigma_I G_{p_i}$, and (ii) for any $x \in G$, the expression for x is unique.

(i) Choose $x \in G$ with order n. By factoring n into prime powers, we have $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, and we can write $n_i = n/p_i^{r_i}$ for $i = 1, 2, \cdots, k$. Then the greatest common divisor of $n_1, n_2, \cdots, n_k$ is 1, so there exist integers $a_1, a_2, \cdots, a_k$ such that $a_1n_1 + a_2n_2 + \cdots + a_kn_k = 1$. Thus

$x = 1 \cdot x = (a_1n_1 + a_2n_2 + \cdots + a_kn_k)x = a_1n_1x + a_2n_2x + \cdots + a_kn_kx$, where $n_i x$ has order $p_i^{r_i}$ since $0 = nx = (p_i^{r_i} n_i)x = p_i^{r_i}(n_i x)$. Therefore $n_i x \in G_{p_i}$, and for $x \in G$, x is the sum of elements in the subgroups G_{p_i}.

(ii) Let $x = y_1 + y_2 + \cdots + y_k$ and $x = z_1 + z_2 + \cdots + z_k$, where $y_i, z_i \in G_{p_i}$. Then $y_1 + y_2 + \cdots + y_k = z_1 + z_2 + \cdots + z_k$. Consider $y_1 - z_1 = (z_2 + z_3 + \cdots + z_k) - (y_2 + y_3 + \cdots + y_k)$. Now $y_1 - z_1$ has order a power of p_1, but the order of $(z_2 + z_3 + \cdots + z_k) - (y_2 + y_3 + \cdots + y_k)$ is a product of powers of $p_2, p_3, \cdots, p_k$. Thus $y_1 - z_1$ must equal 0, which implies $y_1 = z_1$. In a similar way, $y_i = z_i$ for all i. Therefore the expression for x is unique.

Hence $G \cong \oplus \Sigma_I G_{p_i}$.

Remark 1.37. Recall from Remark 1.10 the definitions of a partially ordered set and of a chain. In addition, note the following definitions. If C is a chain of a partially ordered set $(A, \leq)$, an element $a \in A$ is called an upper bound of C provided $c \leq a$ for all $c \in C$. We call an element $x \in A$ maximal (with respect to $\leq$) provided if $a \geq x$ for some $a \in A$, then $a = x$.

Zorn's Lemma 1.38. Let $(A, \leq)$ be a nonempty partially ordered set in which every chain in A has an upper bound in A. Then A has a maximal element.

Proposition 1.39. Let H and K be subgroups of a group G, let $H \subseteq K$, and let H be a pure subgroup of G. Then H is a pure subgroup of K.

Proof: If $n \in Z^*$, then $nK \cap H \subseteq nG \cap H = nH$. Clearly $nH \subseteq nK \cap H$, so $nH = nK \cap H$.

Fact 1.40. No nontrivial subgroup of Z is a pure subgroup of Z. For let H be a nontrivial subgroup of Z; say $H = nZ$ for $n \in Z^+$. Then $nH = n^2Z$, and $nZ \cap H = H$. So $nZ \cap H = H = nZ \not\subseteq n^2Z = nH$. Hence $nH \subsetneq nZ \cap H$.

Theorem 1.41. Let G be a group in which every subgroup of G is a pure subgroup of G. Then G is a torsion group in which every element has square-free order.

Proof: Let G be a group in which every subgroup of G is a pure subgroup of G.

Claim 1: G is a torsion group.

Subproof: Let $x \in G$, and assume x is torsionfree. Then $Zx \cong Z$. If H is a nontrivial subgroup of Zx, then H is a pure subgroup of G. Thus by Proposition 1.39, since $H \subseteq Zx \subseteq G$, H is a pure subgroup of Zx, which contradicts Fact 1.40. Therefore every element of G is a torsion element, so G is a torsion group.

Claim 2: If $x \in G$, then x has square-free order.

Subproof: Let n be the order of $x \in G$, and assume $n = p^2k$ for a prime p and $k \in Z^+$. Then $nx = (p^2k)x = p^2(kx) = 0$, so kx is an element of order p^2. Thus $K = \langle kx \rangle \cong Z_{p^2}$. Let $H = \langle pkx \rangle \subseteq K$. Since H is a pure subgroup of G, H is a pure subgroup of K by Proposition 1.39. But $pK \cap H = H \not\subseteq pH = 0$, so $pH \subsetneq pK \cap H$, which contradicts H being a pure subgroup of K. Thus $x \in G$ has square-free order; i.e., no element of G has order kp^2 for some $p \in Z^+$, $p \geq 2$.

Theorem 1.42. Let G be a torsion group in which every element has square-free order. Then every subgroup of G is a pure subgroup of G.

Proof: Let G be a torsion group in which every element has square-free order. Then $G = \oplus \Sigma_I P_i$ where P_i are p_i-primary groups, by Theorem 1.36. Fix $i \in I$, and consider the p_i-primary group P_i, where p_i is a prime number. If $x \in P_i$, then the order of x is a power of p_i. But since x must have square-free order, this means that $p_i x = 0$, so x has order p_i. Thus $p_i P_i = 0$.

It follows from this fact that P_i can be considered a vector space over the field Z_{p_i}. For if $\alpha \in Z_{p_i}$ and $x \in P_i$, we can define αx to be $x + x + \cdots + x$ (α - times). This is well-defined, since if $\alpha \equiv \beta \pmod{p_i}$, then $\alpha = \beta + kp_i$, so $\alpha x = (\beta + kp_i)x = \beta x + kp_i x = \beta x$ since $p_i x = 0$. Now every vector space has a basis. If $y \in P_i$ is a "basis vector," then $Z_{p_i} y \cong Z_{p_i}$, where the isomorphism is both as vector spaces and as groups. Hence P_i is isomorphic to a direct sum of groups Z_{p_i}. These are simple groups since they contain no non-trivial subgroups, so P_i is a direct sum of simple groups. Hence G is a direct sum of simple groups.

We have $G = \oplus \Sigma_I S_i$ is a direct sum of simple groups for some index set I. Let H be a subgroup of G.

Claim: H is a direct summand of G.

Subproof: If $H = G$, then we are finished. Assume $H \neq G$; then there exists some simple group S_i with $S_i \not\subseteq H$. Hence $S_i \cap H = 0$, since $S_i \cap H$ is a subgroup of S_i and S_i is a simple group. Let $\mathcal{S} = \{S \subseteq G \mid S$ is a subgroup of G and $S \cap H = 0\}$. Then $\mathcal{S} \neq \emptyset$, so by Zorn's Lemma 1.38, $\mathcal{S}$ has a maximal element L. Thus L is a subgroup of G maximal with respect to $L \cap H = 0$. Hence $H \oplus L \subseteq G$. We wish to show that $H \oplus L = G$. Suppose $G \not\subseteq H \oplus L$; then there exists some simple group S_j with $S_j \nsubseteq H \oplus L$. Thus $(H \oplus L) \cap S_j = 0$ as before, so $H \oplus L \oplus S_j \subseteq G$. But $H \cap (L \oplus S_j) = 0$, which contradicts the maximality of L. Therefore $H \oplus L = G$, so H is a direct summand of G.

Hence every subgroup of G is a pure subgroup of G, since it is a direct summand of G.

In conclusion, from Theorem 1.41 and Theorem 1.42 we obtain the following corollary.

Corollary 1.43. Every subgroup of a group G is a pure subgroup of G if and only if G is a torsion group in which every element has square-free order; or, alternately, if and only if every subgroup of G is a direct summand of G.

CHAPTER II

PURITY IN FINITELY GENERATED GROUPS

In this chapter, we will use purity to show that any group of bounded order is a direct sum of cyclic groups. We will then work towards proving the Fundamental Theorem of Finitely Generated Abelian Groups, which states that every finitely generated abelian group is the direct sum of cyclic groups.

Definition 2.1. A torsion group G is said to be of bounded order if there is a fixed upper bound to the orders of the elements; i.e., if there exists a positive integer n such that $nx = 0$ for all $x \in G$, or equivalently, such that $nG = 0$.

Remark 2.2. Any finite group is of bounded order. However, an infinite group can also be of bounded order. For example, consider the direct sum of an infinite number of finite cyclic groups, having an upper bound on the orders of the summands; such as $G = Z_3 \oplus Z_3 \oplus Z_3 \oplus \cdots$.

Remark 2.3. Although it is difficult to construct a cyclic direct summand in a given group of bounded order, the next lemma helps us to obtain a pure cyclic subgroup. So, once again, we see that a pure subgroup can be a good temporary substitute for a direct summand.

Lemma 2.4. Let G be a primary group satisfying $p^rG = 0$. Let $x \in G$ have order p^r. Then the cyclic subgroup K generated by x is a pure subgroup of G.

Proof: Let G be a primary group satisfying $p^rG = 0$, let $x \in G$ have order p^r, and let K be the cyclic subgroup generated by x. We know that every element in a primary group is divisible by any integer prime to p by Proposition 1.32. Thus, if $n = p^i t$ with $(p, t) = 1$, then $tK = K$ and $tG = G$. Hence $nK = p^i tK = p^i K$ and $nG = p^i G$. We wish to show that $p^i K = p^i G \cap K$ for $1 \leq i < r$. Clearly $p^i K \subseteq p^i G \cap K$. Let $y \in G$ and $p^i y \in p^i G \cap K$, so $p^i y = kx$ for $0 \leq k < p^r$. Let $k = p^j t'$ for $0 \leq j < r$ and for $(p, t') = 1$. Then $kx = p^j(t'x)$ and $t'x$ also generates K since $(t', p^r) = 1$. Let $t'x = z$, so z has order p^r. Thus $p^i y = p^j z$. If $i \leq j$, then $p^i y = p^i(p^{j-i}z)$, and $p^i y \in p^i K$ since $p^{j-i}z \in K$. So $p^i G \cap K \subseteq p^i K$. If $i > j$, then $0 = p^r y = p^{r-i} p^i y = p^{r-i} p^j z = p^{r+j-i} z$, and $0 \leq r-i+j < r$. This contradicts the hypothesis that z has order p^r. Therefore, for a group G of bounded order, the cyclic subgroup K generated by $x \in G$ is a pure subgroup of G.

We wish to construct pure subgroups which are direct sums of cyclic groups. The following lemma will insure the independence of these pure subgroups.

Lemma 2.5. Let S be a subgroup of a group G, and let x be an element of G with $x + S = y \in G/S$. Suppose x and y have the same order. Let K be the cyclic subgroup generated by x. Then the sum $S + K$ is a direct sum.

Proof: We must show that $S \cap K = 0$. Assume $rx \in S \cap K$. Then $r(x + S) = rx + S = S$ since $rx \in S$. Also $r(x + S) = ry$, which implies $ry = S$, or $ry = 0$. Thus r is a multiple of the order of y. By the hypothesis, x and y have the same order, so $rx = 0$. Therefore $S \cap K = 0$.

To make the proof of the next theorem more concise, we make the following definitions.

Definition 2.6. A subset X of a group G is said to be pure provided it generates a pure subgroup of G.

Definition 2.7. Let $\{S_i\}$ be any set of subgroups of G. If the sum ΣS_i of these subgroups is a direct sum, we call the subgroups independent.

Definition 2.8. The elements $\{x_i\}_I \subseteq G$ are said to be independent provided the cyclic subgroups they generate are independent. Equivalently, the elements $\{x_i\}_I \subseteq G$ are independent provided in any finite sum $\Sigma n_i x_i = 0$ for $n_i \in Z$, this implies each $n_i x_i = 0$.

Remark 2.9. Thus, a subset S of a group G is a pure, independent set provided S is independent and the subgroup generated by S is a pure subgroup of G.

Theorem 2.10. A group G of bounded order is a direct sum of cyclic groups.

Proof: Let G be a group of bounded order. Then G is a torsion group, so by Theorem 1.36, we may assume G is primary. We wish to construct a direct sum of cyclic groups, preserving purity as we go. Now we know that the subsets of G form a partially ordered set under the partial order of set inclusion. Let $x \in G$ be of maximal order; we know that x exists because G is bounded. Assume x has order p^r, so $p^r x = 0$. Then every other element of G has order a power of p less than or equal to r, so $p^r G = 0$. Hence the cyclic subgroup generated by x is pure, by Lemma 2.4. Thus $\{x\}$ is a pure, independent subset of G.

Let $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots \subseteq H_i \subseteq \cdots$ be a chain of pure, independent subsets of G. If $\langle H_i \rangle$ denotes the subgroup generated by H_i, then $\langle H_i \rangle$ is pure in G. Also $\langle \cup_I H_i \rangle = \cup_I \langle H_i \rangle$, so $\langle \cup_I H_i \rangle$ is a pure subgroup of G by Proposition 1.11. Let x be an element in the cyclic subgroups generated by $\cup_I H_i$. Then x is a finite sum $x = x_1 + x_2 + \cdots + x_n$ of elements in the cyclic subgroups generated by $\cup_I H_i$, and so each of these elements x_j belongs to the cyclic subgroups generated by some H_k. Since H_k is independent, the expression $x = x_1 + x_2 + \cdots + x_n$ is unique. Thus $\cup_I H_i$ is independent. We have shown that $\cup_I H_i$ is a pure, independent subset of G. Therefore $\cup_I H_i$ is an upper bound of the chain in G. Hence, by Zorn's Lemma 1.38, we may select a maximal pure, independent subset $\{x_i\}_I$ of G. Let S be the subgroup generated by the elements x_i. Then S is a pure, independent subgroup of G, and $S = \oplus \Sigma_I \langle x_i \rangle$. If S is all of G, our proof is complete.

Assume S is not all of G, and consider G/S. Now G/S is a primary group of bounded order. We may select an element y of G/S of maximal order, say $p^r y = 0$. Then $p^r(G/S) = 0$ since G/S is a primary group and if $z \in G/S$, then $p^t z = 0$ where $0 \leq p^t \leq p^r$. So by Lemma 2.4, the cyclic subgroup generated by y is a pure subgroup of G/S. By Lemma 1.17, there exists $x \in G$, with $x + S = y \in G/S$, such that x has the same order as y.

<u>Claim</u>: $\{x, \{x_i\}_I\}$ is a pure, independent set.

<u>Subproof</u>: Let T be the subgroup generated by $\{x, \{x_i\}_I\}$. Then $S \subseteq T$ and $T/S = \langle y \rangle$, which is a pure subgroup of G/S. By Lemma 1.19, T is a pure subgroup of G, so $\{x, \{x_i\}_I\}$ is pure. Now let K be the

subgroup generated by $x \in G$. By Lemma 2.5, the sum $S + K$ is a direct sum. Thus S and K are independent subgroups, which implies $\{x, \{x_i\}_I\}$ is independent.

Therefore $\{x, \{x_i\}_I\}$ is a pure, independent set containing $\{x_i\}_I$, which contradicts the maximality of $\{x_i\}_I$. Hence S is indeed all of G. Therefore G is the direct sum of cyclic groups.

<u>Lemma 2.11</u>. Let S and T be subgroups of a group G with $S \cap T = 0$. Suppose $(S + T)/T$ is a direct summand of G/T. Then S is a direct summand of G.

<u>Proof</u>: Let S and T be subgroups of a group G with $S \cap T = 0$, and let $(S + T)/T$ be a direct summand of G/T. Suppose R/T is a complementary summand to $(S + T)/T$ in G/T. Then $G/T = (S + T)/T \oplus R/T$, so $R + (S + T) = G$ and $R \cap (S + T) = T$.

<u>Claim</u>: $G = S \oplus R$

<u>Subproof</u>: Since $T \subseteq R$, we have $S + R = S + T + R = G$. Also $(R \cap S) \subseteq R \cap (S + T) = T$, so $(R \cap S) \subseteq (T \cap S) = 0$. Therefore $R \cap S = 0$, and $G = S \oplus R$. Thus S is a direct summand of G.

<u>Theorem 2.12</u>. Let G be a group, and let S be a pure subgroup of bounded order. Then S is a direct summand of G.

<u>Proof</u>: Let S be a pure subgroup of a group G. Assume $nS = 0$ for some positive integer n. Then, by Lemma 1.21, $(S + nG)/nG$ is a pure subgroup of G/nG. Also, G/nG and all its homomorphic images are groups of bounded order; thus, by Theorem 2.10, they are direct sums of cyclic groups. Thus $(G/nG)/(S + nG/nG)$ is a direct sum of cyclic groups. By Theorem 1.18, $(S + nG)/nG$ is a direct summand of G/nG. Now

since S is a pure subgroup of G and $nS = 0$, we have $S \cap nG = nS = 0$. Therefore, by Lemma 2.11, S is a direct summand of G.

Theorem 2.13. A divisible subgroup H of a group G is a direct summand of G.

Proof: See Kaplansky [4, p. 8].

Remark 2.14. We have shown, in Proposition 1.23, that the torsion subgroup T of a group G is a pure subgroup of G. Now as a special case of Theorem 2.12, if the torsion subgroup T of a group G is of bounded order, we can show that T is a direct summand of G. In fact, with the aid of Theorem 2.13, we can carry this statement even further.

Theorem 2.15. Let T be the torsion subgroup of a group G. Suppose T is the direct sum of a divisible group D and a group B of bounded order. Then T is a direct summand of G.

Proof: Let T be the torsion subgroup of a group G. Suppose $T = B \oplus D$ for a group B of bounded order and a divisible group D. Since B is a direct summand of T, B is a pure subgroup of T. Now T is a pure subgroup of G, so by the transitivity of pure subgroups (Proposition 1.9), B is a pure subgroup of G. By Theorem 2.12, since B is a pure subgroup of bounded order, B is a direct summand of G, say $G = B \oplus G'$. Now $D \cong T/B \subseteq G/B \cong G'$, so G' has a subgroup D' isomorphic to D. Thus D' is a divisible subgroup of G'. By Theorem 2.13, D' is a direct summand of G', so $G' = D' \oplus G''$, or $G' \cong D \oplus G''$. Therefore $G = B \oplus G' \cong B \oplus D' \oplus G'' \cong B \oplus D \oplus G'' \cong T \oplus G''$. Hence T is a direct summand of G.

Division Algorithm 2.16. Let a and b be integers with $b > 0$, then there exist unique integers q and r such that $a = qb + r$ for $0 \leq r < b$.

Lemma 2.17. Let G be a torsionfree group, let $x \in G$, and assume that the minimal pure subgroup of G which contains x, $S = \{(a/b)x \mid a, b \in Z, b \neq 0, \text{ and } (a/b)x = (1/b)(ax) \text{ is defined in } G\}$, is a finitely generated subgroup of G. Then S is a cyclic subgroup of G.

Proof: Let G be a torsionfree group, let $x \in G$, and let $S = \{(a/b)x \mid a, b \in Z, b \neq 0, \text{ and } (a/b)x = (1/b)(ax) \text{ is defined in } G\}$ be a finitely generated subgroup of G. Let us first assume that S has two generators, y_1 and y_2. Then $S = Zy_1 + Zy_2$. Now $y_1, y_2 \in S$, so $y_1 = (a/b)x$ and $y_2 = (c/d)x$ for $a, b, c, d \in Z$, $b \neq 0$, and $d \neq 0$. Choose $s \in S$; then $s = m_1y_1 + m_2y_2$ where $m_1, m_2 \in Z$. Hence $s = m_1y_1 + m_2y_2 = (m_1a/b)x + (m_2c/d)x = [(m_1ad + m_2bc)/bd]x$ since S is a subgroup of G.

Let k be the smallest positive integer such that $(k/bd)x$ is defined in G, so $k \geq 1$. Clearly k exists, since $(a/b)x + (c/d)x = [(ad + bc)/bd]x$ and $-(a/b)x - (c/d)x = [-(ad + bc)/bd]x$, and either $ad + bc$ or $-(ad + bc)$ is positive. Now there exists $y \in G$ such that $bdy = kx$, so $bdy \in bdG \cap S = bdS$ since S is pure in G. Hence $bdy = bdt$ for some $t \in S$, so $bdy - bdt = bd(y - t) = 0$, or $y = t$ since G is torsionfree and $bd \neq 0$. But $y = (k/bd)x$, so $(k/bd)x \in S$.

Claim: $(k/bd)x$ generates S.

Subproof: Since $(k/bd)x \in S$, then $Z(k/bd)x = \langle (k/bd)x \rangle \subseteq S$. By the Division Algorithm 2.16, there exist integers q and r such that $m_1ad + m_2bc = qk + r$ where $0 \leq r < k$. Hence for $s \in S$, $s = [(m_1ad + m_2bc)/bd]x = [(qk + r)/bd]x$, so $bds = (qk + r)x = qkx + rx$, or $bds - qkx = rx$. Now $(1/bd)(bds)$ is defined, $(1/bd)(qkx)$ is defined, so $(1/bd)(bds - qkx) = (1/bd)(rx) = (r/bd)x$ is defined. But $0 \leq r < k$, so by the choice of k, we must have $r = 0$. Thus $m_1ad + m_2bc = qk$, so $s = (qk/bd)x = q[(k/bd)x] \in Z(k/bd)x$. Hence $S \subseteq Z(k/bd)x$, so $S = Z(k/bd)x$.

Now we can complete our proof by generalizing the preceding argument. Assume S is generated by $y_1, y_2, \cdots, y_n$. Then if $y_1 = (a_1/b_1)x$, $y_2 = (a_2/b_2)x, \cdots, y_n = (a_n/b_n)x$, we let k be the smallest positive integer such that $(k/b_1b_2\cdots b_n)x$ is defined in G. Arguing exactly as before, we can show that $(k/b_1b_2\cdots b_n)x$ generates S. Therefore S is a cyclic subgroup of G.

Theorem 2.18. Any finitely generated abelian group is a direct sum of cyclic groups.

Proof: Assume first that G is a finitely generated, torsionfree abelian group. We wish to prove by induction that G is a direct sum of cyclic groups. Clearly, if G is generated by one element, then G is a cyclic group. Assume that any group generated by $k - 1$, or fewer elements, is a direct sum of cyclic groups. Let G be generated by $\{x_1, x_2, \cdots, x_k\}$. By Theorem 1.30,
$S = \{(a/b)x_1 \mid a, b \in Z, b \neq 0, \text{ and } (a/b)x_1 = (1/b)(ax_1) \text{ is defined in } G\}$
is the smallest pure subgroup of G containing x_1. Then G/S is generated by $\{x_2 + S, x_3 + S, \cdots, x_k + S\}$, so G/S is generated by $k - 1$ elements. Thus, by the induction hypothesis, G/S is a direct sum of cyclic groups. By Theorem 1.18, S is a direct summand of G. Therefore $G = S \oplus C_1 \oplus C_2 \oplus \cdots \oplus C_n$ where C_i are cyclic groups. Now S is finitely generated since the homomorphic image of a finitely generated group is finitely generated. Thus, by Lemma 2.17, S is a cyclic subgroup of G. Hence G is the direct sum of cyclic groups.

We now turn to the general case for which G is not necessarily torsionfree. Let T be the torsion subgroup of G. Since G/T is torsionfree, we have just proven that G/T is a direct sum of cyclic groups. By Theorem 1.18, since T is pure in G, then T is a direct summand of G, so T is finitely generated. Thus T has bounded order since it is a torsion group, so by Theorem 2.10, T is a direct sum of cyclic groups. Therefore $G \cong T \oplus G/T$ is a direct sum of cyclic groups.

Fundamental Theorem of Finitely Generated Abelian Groups 2.19. Let G be a finitely generated abelian group. There exist unique finite index sets A and B such that $G \cong \oplus \Sigma_A Z_{n_i} \oplus \oplus \Sigma_B Z$, where each n_i is a nontrivial power of a prime member. Furthermore, this expression for G is unique, except for order.

Proof: Let G be a finitely generated abelian group. Then, by Theorem 2.18 and its proof, G is a direct sum of cyclic groups, and $G \cong T \oplus G/T$ where T is the torsion subgroup of G. By Theorem 1.36, T is a direct sum of a finite number of primary groups P_i, and each P_i is a homomorphic image of G, hence is finitely generated and torsion and so is of bounded order. Thus, by Theorem 2.10, every P_i is a direct sum of finitely many cyclic groups. Now each cyclic group must be of prime power order, so we have $T \cong \oplus \Sigma_A Z_{n_i}$ for a finite index set A and each n_i is a nontrivial power of a prime number. Since G/T is a direct sum of cyclic groups, is torsionfree, and is finitely generated, we have $G/T \cong \oplus \Sigma_B Z$ for a finite index set B. Thus $G \cong T \oplus G/T \cong \oplus \Sigma_A Z_{n_i} \oplus \oplus \Sigma_B Z$.

Furthermore, this expression for G is unique, except for order. (We do not present a proof of uniqueness since it does not involve the use of pure subgroups. If the reader wishes to see a proof of uniqueness, he may consult either Kaplansky [4] for a proof using the Ulm invariant, or Bernhardt [1].)

CHAPTER III

PURITY IN MODULES

In this chapter, we will generalize the concept of purity in abelian groups to the concept of R-purity in modules. It will be understood that the term "ring" represents a commutative ring with unity, unless otherwise explicitly stated.

Definition 3.1. If R is a (not necessarily commutative) ring with unity, a unital left R-module, denoted ${}_RM$, is an abelian group $(M, +)$ together with a function $\mu: R \times M \to M$, where we denote $\mu(r, x)$ by $r\mu x$, such that:

(i) $(r + s)\mu x = r\mu x + s\mu x$;

(ii) $r\mu(x + y) = r\mu x + r\mu y$;

(iii) $(rs)\mu x = r\mu(s\mu x)$;

(iv) $1\mu x = x$;

for all $r, s \in R$ and $x, y \in M$. We will denote $r\mu x$ by rx.

The reader can easily see how to define a unital right R-module, M_R, where $\mu: M \times R \to M$. However, since we are working with a commutative ring, every right R-module is also a left R-module, and conversely. Thus we will simply use the term "module."

Definition 3.2. A submodule ${}_RN$ of a module ${}_RM$ is a subgroup N of M with the property that for each $n \in N$ and $r \in R$, $rn \in N$.

Remark 3.3. Recall the definitions of a subring of a ring R and an ideal of a ring R. A nonempty subset S of a ring R is called a subring of R provided S together with $+$ and $\cdot$ restricted to S

forms a ring. In other words, S is a subring of R provided $a - b \in S$ and $ab \in S$ for all $a, b \in S$. If I is a subring of R, we call I a <u>left ideal</u> of R if $ra \in I$ for all $a \in I$ and $r \in R$. Again the reader can see how to define a right ideal of R. A <u>two-sided ideal</u> of R, or just an <u>ideal</u> of R, is both a left and a right ideal of R. However, in a commutative ring, a left ideal is also an ideal of R. Thus we do not need the adjectives "left" or "right."

<u>Definition 3.4</u>. If I is an ideal of a ring R, and if ${}_RM$ is a module, then IM denotes all finite sums of the form ax, where $a \in I$ and $x \in M$.

<u>Proposition 3.5</u>. If I is an ideal of a ring R, and if ${}_RM$ is a module, then IM is a submodule of M.

<u>Proof</u>: Let I be an ideal of R, and let ${}_RM$ be a module.

<u>Claim 1</u>: IM is a subgroup of M.

<u>Subproof</u>: Choose $\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{m} b_i y_i \in IM$ for $a_i, b_i \in I$ and $x_i, y_i \in M$. Then $\sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{m} b_i y_i =$
$(a_1x_1 + a_2x_2 + \cdots + a_nx_n) - (b_1y_1 + b_2y_2 + \cdots + b_my_m) =$
$a_1x_1 + a_2x_2 + \cdots + a_nx_n - b_1y_1 - b_2y_2 - \cdots - b_my_m =$
$a_1x_1 + a_2x_2 + \cdots + a_nx_n + b_1(-y_1) + b_2(-y_2) + \cdots + b_m(-y_m) \in IM$ since $-y_i \in M$ for all i. Therefore IM is a subgroup of M.

<u>Claim 2</u>: If $r \in R$, $\sum_{i=1}^{n} a_i x_i \in IM$, then $r(\sum_{i=1}^{n} a_i x_i) \in IM$.

<u>Subproof</u>: Let $r \in R$ and $\sum_{i=1}^{n} a_i x_i \in IM$ for $a_i \in I$, $x_i \in M$. Then
$r(\sum_{i=1}^{n} a_i x_i) = r(a_1x_1 + a_2x_2 + \cdots + a_nx_n) =$
$(ra_1)x_1 + (ra_2)x_2 + \cdots + (ra_n)x_n \in IM$ since $ra_i = a_ir \in I$ for all i.

Therefore IM is a submodule of M.

Definition 3.6. Let $_RN$ be a submodule of a module $_RM$. Then N is an R-pure submodule of M provided $IN = IM \cap N$ for all ideals I of the ring R.

Theorem 3.7. If $Z = R$, and if $_RL$ is a submodule of a module $_RM$, then L is an R-pure submodule of M if and only if L is a pure subgroup of M.

Proof: Let $_RL$ be a submodule of a module $_RM$. Let $Z = R$; then the nonzero ideals of Z are of the form Zn for $n \in Z^*$. Also $ZL = RL = L$, and $ZM = RM = M$.

($\rightarrow$) Assume L is an R-pure submodule of a module M. Then $ZnL = ZnM \cap L$ for all ideals Zn of Z. Let $n \in Z^*$. Then $nL = nZL = ZnL = ZnM \cap L = nZM \cap L = nM \cap L$. Therefore $nL = nM \cap L$, and L is a pure subgroup of M.

($\leftarrow$) Assume L is a pure subgroup of M. Then $nL = nM \cap L$ for all $n \in Z^*$. Choose a nonzero ideal $I = Zn$ of Z. Then $IL = ZnL = nZL = nL = nM \cap L = nZM \cap L = ZnM \cap L = IM \cap L$. Hence L is an R-pure submodule of M.

Remark 3.8. By this theorem, we can see that this concept of R-purity generalizes the abelian group concept of purity.

Proposition 3.9. If $_RN$ is an R-pure submodule of a module $_RM$, and if $_RL$ is an R-pure submodule of N, then L is an R-pure submodule of M.

Proof: Let I be an ideal of R, let $_RN$ be an R-pure submodule of a module $_RM$, and let $_RL$ be an R-pure submodule of N. Then $IL = IN \cap L = (IM \cap N) \cap L = IM \cap (N \cap L) = IM \cap L$. Thus L is an R-pure submodule of M.

Proposition 3.10. If ${}_RL$ is an R-pure submodule of a module ${}_RM$, and if ${}_RN$ is a submodule of M containing ${}_RL$, then L is an R-pure submodule of N.

Proof: Let ${}_RL$ be an R-pure submodule of a module ${}_RM$, let ${}_RN$ be a submodule of M containing L, and let I be an ideal of R. Then $IN \cap L \subseteq IM \cap L = IL$. Clearly $IL \subseteq IN \cap L$. Therefore $IL = IN \cap L$.

Proposition 3.11. Any direct summand of a module ${}_RM$ is an R-pure submodule of M.

Proof: Let ${}_RM$ be a module such that $M = L \oplus N$ for submodules ${}_RL$ and ${}_RN$ of M. We wish to show that L is an R-pure submodule of M. Let I be an ideal of R. Choose $y \in IM \cap L$; so $y = \sum_{i=1}^{m} a_i x_i$ for $a_i \in I$, $x_i \in M$. Then $y = \sum_{i=1}^{m} a_i x_i \in L$. Since $M = L \oplus N$, $x_i \in M$ implies $x_i = \ell_i + n_i$ for $\ell_i \in L$ and $n_i \in N$. Thus
$\sum_{i=1}^{m} a_i x_i = \sum_{i=1}^{m} a_i(\ell_i + n_i) = \sum_{i=1}^{m}(a_i\ell_i + a_i n_i) = \sum_{i=1}^{m} a_i\ell_i + \sum_{i=1}^{m} a_i n_i \in L$. But $\sum_{i=1}^{m} a_i\ell_i \in L$ and $\sum_{i=1}^{m} a_i n_i \in N$, so $\sum_{i=1}^{m} a_i n_i = \sum_{i=1}^{m} a_i x_i - \sum_{i=1}^{m} a_i\ell_i \in L \cap N = 0$. Therefore $\sum_{i=1}^{m} a_i n_i = 0$, or $y = \sum_{i=1}^{m} a_i x_i = \sum_{i=1}^{m} a_i\ell_i \in IL$. Hence $IM \cap L \subseteq IL$. Since $IL \subseteq IM \cap L$, we have $IL = IM \cap L$. Therefore L is an R-pure submodule of M.

Proposition 3.12. The union of an ascending chain of R-pure submodules of a module ${}_RM$ is an R-pure submodule of M.

Proof: Let $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots \subseteq N_k \subseteq \cdots$ be an ascending chain of R-pure submodules of a module ${}_RM$. We must show that $\bigcup_A N_k$ for $k \in A$ is an R-pure submodule of M. Let I be an ideal of R. Then

$IM \cap (\bigcup_A N_k) = \bigcup_A (IM \cap N_k) = \bigcup_A (IN_k)$.

Claim: $\bigcup_A (IN_k) = I(\bigcup_A N_k)$

<u>Subproof</u>: Choose $y \in \bigcup_A (IN_k)$. Then $y \in IN_k$ for some $k \in A$. So $y = \sum_{i=1}^m a_i x_i$ for $a_i \in I$, $x_i \in N_k$. Since $x_i \in N_k$ for some $k \in A$, $x_i \in \bigcup_A N_k$. Therefore $y = \sum_{i=1}^m a_i x_i \in I(\bigcup_A N_k)$, so $\bigcup_A (IN_k) \subseteq I(\bigcup_A N_k)$. Choose $z \in I(\bigcup_A N_k)$. Then $z = \sum_{i=1}^m a_i x_i$ for $a_i \in I$, $x_i \in \bigcup_A N_k$. Pick $k \in A$ such that $i \leq k$ for $x_i \in \bigcup_A N_k$. Then $x_i \in N_k$, so $\sum_{i=1}^m a_i x_i \in IN_k$. Hence $z = \sum_{i=1}^m a_i x_i \in \bigcup_A (IN_k)$, and $I(\bigcup_A N_k) \subseteq \bigcup_A (IN_k)$. Thus $\bigcup_A (IN_k) = I(\bigcup_A N_k)$.

Therefore $IM \cap (\bigcup_A N_k) = I(\bigcup_A N_k)$, so $\bigcup_A N_k$ is an R-pure submodule of M.

<u>Proposition 3.13</u>. If ${}_RL$ is an R-pure submodule of a module ${}_RM$, and if ${}_RN$ is a submodule of M containing L such that N/L is an R-pure submodule of M/L, then N is an R-pure submodule of M.

<u>Proof</u>: Let I be an ideal of R, let ${}_RL$ be an R-pure submodule of a module ${}_RM$, and let ${}_RN$ be a submodule of M containing L such that N/L is an R-pure submodule of M/L. Choose $y \in IM \cap N$. Then $y + L \in N/L \cap (IM + L)/L = N/L \cap I(M/L) = I(N/L)$, so $y + L \in I(N/L) = (IN + L)/L$. Hence $y + L = z + L$ where $z \in IN$. Thus $y - z \in L$. Also $y \in IM$ and $z \in IN \subseteq IM$, so $y - z \in IM \cap L = IL$. Thus $y - z \in IL \subseteq IN$. Let $y - z = w \in IN$. Then $y = z + w \in IN$. Hence $IM \cap N \subseteq IN$; clearly $IN \subseteq IM \cap N$. Therefore $IM \cap N = IN$.

<u>Proposition 3.14</u>. If ${}_RL$ is a submodule of a module ${}_RM$, and if ${}_RN$ is an R-pure submodule of M containing L, then N/L is an R-pure submodule of M/L.

<u>Proof</u>: Let I be an ideal of R, let ${}_RL$ be a submodule of a module ${}_RM$, and let ${}_RN$ be an R-pure submodule of M containing L. Clearly $I(N/L) \subseteq I(M/L) \cap (N/L)$. Choose $y \in I(M/L) \cap (N/L)$. Then

$y = \sum_{i=1}^{m} a_i(x_i + L) \in I(M/L)$ for $a_i \in I$, $x_i + L \in M/L$. We must show there exists $b_j \in I$, $c_j + L \in N/L$ such that $\sum_{i=1}^{m} a_i(x_i + L) = \sum_{j=1}^{k} b_j(c_j + L) \in I(N/L)$. Since $\sum_{i=1}^{m} a_i(x_i + L) \in N/L$, there exists $n \in N$ such that $\sum_{i=1}^{m} a_i(x_i + L) = n + L$, or $\sum_{i=1}^{m} a_i x_i = n + \ell \in N$ for some $\ell \in L$. By the R-purity of N in M, there exists $b_j \in I$, $c_j \in N$ such that $\sum_{i=1}^{m} a_i x_i = \sum_{j=1}^{k} b_j c_j = n + \ell$, or $\sum_{i=1}^{m} a_i x_i + L = n + L = \sum_{j=1}^{k} b_j c_j + L$. Thus $y = \sum_{i=1}^{m} a_i(x_i + L) = \sum_{j=1}^{k} b_j(c_j + L) \in I(N/L)$ for $c_j + L \in N/L$, so $I(M/L) \cap (N/L) \subseteq I(N/L)$. Therefore $I(N/L) = I(M/L) \cap (N/L)$, and N/L is an R-pure submodule of M/L.

Remark 3.15. If ${}_RL$ and ${}_RQ$ are modules, the function h from L into Q is called an R-homomorphism provided $h(x + y) = h(x) + h(y)$ and $h(rx) = rh(x)$ for all $r \in R$ and $x, y \in L$. Recall that a sequence of modules $0 \to {}_RL \xrightarrow{f} {}_RM$ is called exact provided the R-homomorphism f from L into M is one-to-one.

Definition 3.16. Let ${}_RQ$ be a module. We call Q injective provided if $0 \to {}_RL \xrightarrow{f} {}_RM$ is an exact sequence of modules and if g is an R-homomorphism from L into Q, then there exists an R-homomorphism h from M into Q such that the diagram

$$\begin{array}{ccccc} 0 & \to & {}_RL & \xrightarrow{f} & {}_RM \\ & & g\downarrow & \swarrow h & \\ & & {}_RQ & & \end{array}$$

commutes; i.e., such that $h \circ f = g$.

Definition 3.17. Let ${}_RM$ be a module. If M is an R-pure submodule of every module which contains it as a submodule, then we say M is absolutely R-pure.

Example 3.18. Every injective module is absolutely R-pure.

Proof: Let ${}_RQ$ be an injective module, and let Q be a submodule of a module ${}_RM$. One can prove that an injective module is a direct summand of any module which contains it as a submodule; see [1, p. 148]. Hence Q is a direct summand of M, so Q is an R-pure submodule of M. Therefore Q is absolutely R-pure.

Definition 3.19. Let ${}_R N$ be a submodule of a module ${}_R M$. We say that N is essential in M provided if ${}_R L \neq 0$ is a submodule of M, then $L \cap N \neq 0$. If N is an essential submodule of M, we call M an essential extension of N.

Definition 3.20. Let ${}_R M$ be a module. The injective envelope of M, denoted $E_R(M)$, is an injective module which is an essential extension of M.

One can prove that every module has an injective envelope, which is unique up to isomorphism. See [1, p. 153].

Theorem 3.21. Let ${}_R M$ be a module. Then M is absolutely R-pure if and only if M is an R-pure submodule of its injective envelope $E_R(M)$.

Proof: Let ${}_R M$ be a module.

($\rightarrow$) Assume M is absolutely R-pure. Since $E_R(M)$ is the injective envelope of M, we have $M \subseteq E_R(M)$. But M is absolutely R-pure, so M is an R-pure submodule of $E_R(M)$.

($\leftarrow$) Assume M is an R-pure submodule of its injective envelope $E_R(M)$. Let M be a submodule of the module ${}_R N$; then $E_R(M) \subseteq E_R(N)$. But $E_R(M)$ is injective, and hence is an R-pure submodule of $E_R(N)$. Since M is an R-pure submodule of $E_R(M)$, then M is an R-pure submodule of $E_R(N)$ by Proposition 3.9. Thus M is an R-pure submodule of N by Proposition 3.10. Hence M is absolutely R-pure.

Theorem 3.22. Let $R = Z$, so that the R-modules are just abelian groups. Then the abelian group M is absolutely pure if and only if M is divisible.

Proof: (→) Assume that M is absolutely pure, so that M is a pure subgroup of its injective envelope E(M) by Theorem 3.21. But an injective abelian group is just a divisible abelian group; see [1, p. 144]. Hence M is a pure subgroup of a divisible group E(M). Thus M is divisible, since $nM = nE(M) \cap M = E(M) \cap M = M$ for every $n \in Z^*$.

(←) Assume that M is a divisible abelian group. Then one can prove that M is an injective R-module; see [1, p. 144]. Hence M is absolutely pure by Example 3.18.

SUMMARY

In this thesis, we examined the basic properties of a pure subgroup of an abelian group, and, subsequently, we proved the Fundamental Theorem of Finitely Generated Abelian Groups by using the concept of purity. Several examples of pure subgroups were exhibited. Finally, we generalized the concept of purity in abelian groups to a concept of R-purity in modules over a commutative ring R.

There are several ways of generalizing purity in abelian groups to modules over a ring, and we would have liked to study more of these, given sufficient time. Another interesting question is the following: for what rings, other than the integers, is it true that the absolutely pure modules are the injective modules?

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