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It was the purpose of this study to investigate various pure strategies for the dice game Pig. Two basic approaches were considered for formulating an optimal strategy: the maximum number of rolls per turn that a player should take and the maximum number of points per turn that a player should attempt to accumulate. Basically, an optimal strategy for Pig will be one which allows a player to accumulate a maximum number of points in a minimum number of turns in order to achieve a goal of 100 or more points. Computer simulation of the game was used to verify the results and to attempt to distinguish subtle differences among the competing strategies which could not be determined through a purely theoretical formulation of the game.

It was found that an optimal roll-per-turn strategy will be for a player to toss no less than two times per turn and no more than three times per turn. The optimal point-per-turn strategy from initial position of zero points is to attempt to accumulate at least 25 points. Through the computer simulation of the game, it was found that optimally a player should attempt to accumulate from 22 to 26 points on any turn if he is to attempt to accumulate the same number on each turn.

Although an optimal strategy from a roll-per-turn approach and for a point-per-turn approach can be stated,

these strategies will in no way guarantee that a player will win every game by assuming one of these strategies; however, a player will be guaranteed of winning more games by assuming one of these strategies than any other roll-per-turn or point-per-turn strategy.

A MATHEMATICAL APPROACH TO AN OPTIMAL STRATEGY FOR THE DICE GAME PIG

by

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TABLE OF CONTENTS

CHAPTER I
INTRODUCTION

There is very little available information concerning the dice game Pig. An extensive search through game theory texts, encyclopedias on games, and mathematical and statistical journals led to only two sources which contained any information about Pig. Richard A. Epstein's book entitled _The Theory of Gambling and Statistical Logic_ [1], and John Scarne's book entitled _Scarne on Dice_ [3] contain very brief sketches concerning the rules and actual play of the game. Communication with Martin Gardner of _Scientific American_ revealed no additional sources. No information concerning the history or origin of the game was found. Pig may appear to be a most unusual name for a dice game, however, the name proves to be very descriptive since a player tends to become very greedy as play progresses.

Essential to the makeup of any game are the rules by which the game is played. The set of rules greatly determines the actions and decisions of the participants. It is important that the rules be so complete and so precisely defined that any disagreements among the players concerning the consequence of any specific event which occurs during play are eliminated.

The game Pig is played with a pair of dice and with any number of players. Each player tries to accumulate a total of 100 or more points before his opponents. The play of the game is divided into turns. A player's turn consists of one or more rolls of a pair of dice, where his score is determined by summing the figures which appear on the dice after each roll. When a player terminates his turn, the dice are passed to the next player who begins his turn. A player's turn may be terminated in three distinct ways, each having a different effect on the player's score. The first way that a player's turn may be terminated results from the occurrence of a single one on the toss of the two dice. That is, a player's turn is terminated whenever a one appears on one of the dice but not both. The player's score is then zero for that particular turn, but the score accumulated on the previous turns is retained. Secondly, a player's turn is terminated whenever the roll of the dice results in an outcome of double ones ("snake eyes"); that is, whenever a one appears on both dice. The player's total score is then zero for the turn and, in addition, the player loses <u>all</u> of his points for the game. The final way that a player may terminate his turn is by choice. A player may stop rolling at any time he chooses prior to the appearance of a one. The player's score for that particular turn is then added to his previous score. The final turn is determined when one of the players accumulates 100 or more points and stops rolling. Each of his opponents then has a chance to roll and attempt to surpass his score. The winner will be the player who has accumulated at least 100 points and more points than any of the opposing players.

Now that the rules of the game have been established, a solution to the game will be investigated. A solution is sought by means of a pure strategy. A pure strategy is a prescribed plan of action so complete and so well-defined that it will specify what the player will do in any position with which he may be confronted during the play of the game.

In relation to forming a strategy the simplicity of the rules of this game is very misleading. Various components contribute to the complexity of this game. For instance, any number of players can play. A player's chances for winning a game are altered by the number of people competing. Generally, with m players, where the dice are being passed from player 1 to 2 to ... to m-1 to m and so on, until one player wins, the probability that the k^{th} player wins the game approaches the limit of 1/m which makes the game equitable for all players, [1,p.159]. Therefore, a player's chances of winning the game are decreased as the number of people playing the game is increased. Also, there is no restriction to the possible number of turns that can be required by any one player in this game, therefore, theoretically this game can continue indefinitely. The distinct number of positions with which a player may be confronted grows enormously large as the number of players increases. Considering only two players, it is possible that each player may be confronted with at least as many as $\binom{97}{1}\binom{109}{1}$ or 10,573 different positions. A pure strategy will require a prespecified choice of action for each and every one of these positions.

As games grow more complex, it becomes almost impossible to define a specific action for each and every event that may occur during play. This research will be limited to investigating various strategies for the accumulation of 100 or more points (within the framework of the game Pig) by a single player. For a particular strategy to be of value in the actual play of the game, it should be one that allows the player to accumulate the maximum number of points in a minimum number of turns. It is realized that an optimal single-player strategy for the accumulation of 100 or more points may not necessarily be an optimal strategy in a multiplayer game of Pig. This is due to the fact that the actual game situation would necessitate many "point-position" decisions. That is, the specific strategy used by a player under multi-player game conditions would depend somewhat on the relative scores of the various players and make it unwise for a player to always maintain a specified pre-game strategy oriented toward the optimal accumulation of 100 points. Obviously, if an opponent has 98 points and a player with 30 points is beginning his turn, it would be unwise for the player to stay with his original strategy since more than likely this will be his next to last turn. It will be wise for the rolling player to attempt to accumulate as many points as possible, obtaining a score very close to 100 points.

No attempt will be made to investigate "point-position" strategies due to the enormous number of distinct point positions with which a player may be confronted, even for a two-player game. Instead, strategies will be investigated which specify what decisions a player should make, per turn, relative to his own goal of obtaining 100 points rather than the comparative scores of other players. It is felt that such strategies will be advantageous in that they may be easily specified and yet will be applicable, to a large extent, in multiplayer games--particularly in early stages of the game and in games in which the player's score is close to the higher scores of his opponents. Two basic approaches will be used in investigating single-player strategies. The first approach will be to consider the maximum number of tosses per turn that a player should take, and the second approach is to consider the maximum number of points that a player should attempt to accumulate per turn. Before discussing a specific strategy to Pig certain mathematical preliminaries need to be established.

CHAPTER II

MATHEMATICAL PRELIMINARIES

Before considering specific strategies to the game Pig, a probabilistic formulation of the game will be presented. The number of tosses that a player can expect to take before the occurrence of an outcome which contains a one, the number of points that a player can expect to have accumulated after i tosses of the dice, and the effect of the increasing probability of an outcome which contains a one will be investigated. The number of tosses that a player can expect to take before the occurrence of an outcome which contains a one is important because a player would like a strategy which minimizes the probability of the appearance of a one. The number of points that a player can expect to have accumulated after i tosses of the dice will be an essential factor to consider when establishing a strategy. Naturally, a player is interested in accumulating as many points as possible in as few turns as possible. Thus a player will be interested in knowing the number of points he can anticipate accumulating after i consecutive tosses of the dice. This information will be helpful in establishing the number of rolls to be taken on any particular turn and in weighing the risk involved with this number of rolls versus the gain in points the player can expect to achieve.

There are 36 distinct possible outcomes for any single toss of two dice. Note that out of the 36 possible outcomes there is only one possible outcome of double ones, there are ten possible outcomes of a single one, and there are 25 possible outcomes which contain no occurrence of a one. Therefore, the probability that double ones occur is 1/36, the probability that a single one occurs is 10/36, the probability that either a single one or double ones occur is 11/36, and the probability that no one occurs is 25/36. Again, the effect of these outcomes on a player's score needs to be stressed. The occurrence of double ones sends the total score back to zero and terminates the turn. The occurrence of a single one sends the score for that single turn to zero and terminates the turn. If there is no occurrence of a one then the points for that particular toss are added to the total number of points for that particular turn, and the player may roll again if desired.

Now that the probabilities of the outcomes which contain a one have been established, the probabilities of specific outcomes which do not contain a one will be investigated. It will then be possible to determine the probability of accumulating a specific number of points after i, $i=1,2,3,\ldots$, tosses of the dice on any turn. Let

A_i denote the set of possible points which may be accumulated after i=1,2,3,... tosses of the dice. If i = 1, then $A_1 = \{0,4,5,6,7,8,9,10,11,12\}$, where the zero element of A_1 represents a score of zero resulting from the appearance of a one on the dice and the non-zero elements of A_1 represent scores which were obtained without the occurrence of a one. Likewise, if i = 2, then $A_2 = \{0,8,9,\ldots,23,24\}$, and if i = 3, then $A_3 = \{0,12,13,\ldots,35,36\}$. In general $A_n = \{0,4n,4n+1,\ldots,12n\}$, where the zero element represents the occurrence of a one on or before the n^{th} toss and the non-zero elements represent scores which were obtained without any previous occurrence of a one. Let X_i denote the random variable which is defined on A_i, and let $Pr(X_i=m)$, $m\epsilon A_i$, denote the probability that m points have been accumulated after i tosses of the dice. After the first toss, this probability will be conditional on what has occurred on the preceding tosses. However, each toss of the dice is independent of the preceding tosses, and thus, the outcome appearing on the i^{th} toss of the dice is independent of the outcomes which have appeared on the i-1 preceding tosses. Let X_1 denote a random variable which is defined on A_1, and let $Pr(X_1=m)$, $m\epsilon A_1$, denote the probability that m points have been accumulated after one toss. Due to the independence of the tosses, $Pr(X_1=m)$ will also denote the probability that m points are accumulated on any specific toss of the dice. Thus if X_i^* denotes the number of points obtained on the i^{th} toss, then $X_1^* = X_1$, and X_i^* takes values in A_1. The probabilities of having a certain score after i tosses are then defined as follows:

For m = 0:

If i = 1, then $Pr(X_1=0) = 11/36$.

If i ≥ 1, then

$$Pr(X_i=0) = Pr(X_1=0) + Pr(X_2=0|X_1\neq 0) + \ldots + Pr(X_i=0|\ X_1\neq 0, X_2\neq 0, \ldots, X_{i-1}\neq 0)$$

$$= Pr(X_1=0) + Pr(X_2=0)Pr(X_1\neq 0) + Pr(X_3=0)Pr(X_2\neq 0)Pr(X_1\neq 0) + \ldots + Pr(X_i=0)\cdot Pr(X_{i-1}\neq 0)\ \ldots\ Pr(X_1\neq 0)$$

$$= 11/36 + (11/36)(25/36) + (11/36)(25/36)(25/36) + \ldots + (11/36)(25/36)^{i-1}$$

$$= \sum_{j=0}^{i-1}(11/36)(25/36)^j \qquad (2.1)$$

For m > 0:

If i = 1, then $Pr(X_1=m)$ is:

$Pr(X_1=4) = 1/36$	$Pr(X_1=9) = 4/36$
$Pr(X_1=5) = 2/36$	$Pr(X_1=10) = 3/36$
$Pr(X_1=6) = 3/36$	$Pr(X_1=11) = 2/36$
$Pr(X_1=7) = 4/36$	$Pr(X_1=12) = 1/36$
$Pr(X_1=8) = 5/36$	

If i > 1, then

$$Pr(X_i=m) = \sum Pr(X_i^*=k)Pr(X_{i-1}=\ell) \qquad (2.2)$$

where the sum is extended over all $k \epsilon A_1$, $\ell \epsilon A_{i-1}$, $k + \ell = m$, and $k, \ell \neq 0$.

For example: let $i = 2$, and $m = 10$.

$$Pr(X_2=10) = Pr(X_1=4)Pr(X_1^*=6) + Pr(X_1=6)Pr(X_1^*=4) + Pr(X_1=5)Pr(X_1^*=5) = (1/36)(3/36) + (3/36)(1/36) + (2/36)(2/36) = 10/36^2.$$

The determination of the probabilities, $Pr(X_i=m), m \epsilon A_i$, for a large number of tosses is a laborious arithmetical task due to the recursive nature of equation (2.2), and the fact that the summation in this equation is extended over all possible pairs $k \epsilon A_1$, and $\ell \epsilon A_{i-1}$, with $k \neq 0$, $\ell \neq 0$, and $k + \ell = m$. The probabilities, $Pr(X_i=m)$, for $m \epsilon A_i$, $i=1,2,3,4$, were computed using the IBM 370/125 computer and are given in Appendix A. A listing and brief discription of the Fortran source program used in these computations appear in Appendix F.

Although the recursive property of equation (2.2) enables computation of $Pr(X_i=m)$ for all $m \epsilon A_i$ and for any desired number of tosses, i, it also prevents the equation from being an efficient means of doing such computation. Since $Pr(X_i=m)$ is defined in terms of X_i^* and X_{i-1}, the use of equation (2.2) necessitates the computation of the probabilities associated with all X_k, $k=1,2,\ldots,i-1$, in order to determine $Pr(X_i=m)$. This also makes equation (2.2) unattractive for additional theoretical work, such as calculation of expectations, and indicates the desirability of an alternate expression for the probability density function of the random variable X_i.

Let $f(x_i) = Pr(X_i=x_i)$, $x_i \varepsilon A_i$, denote the probability density function of X_i. Epstein, [1,p.155], gives a general formula for finding the probability, P(s), of obtaining a given sum, s, of the outcomes of n throws of one die. A modification of this result to the game of Pig will be used to determine $f(x_i)$, since X_i represents the score (sum of certain "non-one" outcomes) after i throws of a pair of dice.

In the game of Pig the number of points X_i after i throws of a pair of dice will be zero if a one has occurred on any die thrown, and X_i will be equal to the sum of the outcomes of the i throws of a pair of dice provided no one has occurred on any die. Since i throws of two dice is equivalent to 2i throws of a single die, $X_i = 0$ if a one has occurred on any of the 2i throws, and $X_i = s$ where $s = b_1 + b_2 + \ldots + b_{2i}$ with b_k representing the outcome on the k^{th} throw of the die, $b_k \varepsilon (2,3,4,5,6)$, if no one has occurred on any of the 2i throws of the single die.

The result of equation (2.1) is still applicable to specify $f(0) = Pr(X_i = 0)$. Hence,

$$f(0) = \sum_{j=0}^{i-1} (11/36)(25/36)^j. \qquad (2.3)$$

Attention will now be focused on determining $f(x_i), x_i \neq 0$.

It is possible to obtain 6^{21} different configurations on the 21 tosses of a single die. The number of configurations summing to x_i and containing no one divided by 6^{21} will be $f(x_i) = Pr(X_i = x_i)$. The problem of finding $f(x_i)$ is now reduced to finding the number of configurations of the 21 tosses which sum to x_i. This is equivalent to determining the number of solutions of the equation

$$b_1 + b_2 + \ldots + b_{21} = x_i \qquad (2.4)$$

with $b_k \varepsilon (2,3,4,5,6)$. Equation (2.4) may be rewritten as

$$(b_1 - 1) + (b_2 - 1) + \ldots + (b_{21} - 1) = x_i - 21$$

where $b_k \varepsilon (2,3,4,5,6)$ or

$$a_1 + a_2 + \ldots + a_{21} = s \qquad (2.5)$$

with $s = x_i - 21$, $a_k = b_k - 1$ and $a_k \varepsilon (1,2,3,4,5)$.

A theorem from number theory, [2,p.124-5], states that the number of solutions to equation (2.5) corresponds to the coefficient of y^s in the expansion of the polynomial $(y+y^2+y^3+y^4+y^5)^{21}$. Factoring the polynomial yields

$$(y+y^2+y^3+y^4+y^5)^{21} = \left[\frac{y-y^6}{1-y}\right]^{21}.$$

Letting $a = y$ and $b = -y^6$ in the binomial expansion of

$$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} b^n a^{k-n}$$

yields

$$(y-y^6)^{2i} = \sum_{n=0}^{2i} \binom{2i}{n} (-y^6)^n y^{2i-n} = \sum_{n=0}^{2i} \binom{2i}{n} (-1)^n y^{5n+2i}$$

Also, $(1-y)^{-2i} = \sum_{k=0}^{\infty} \binom{2i+k-1}{2i-1} y^k$.

Then $\left[\frac{y-y^6}{1-y}\right]^{2i} = (y-y^6)^{2i}(1-y)^{-2i}$

$$= \sum_{n=0}^{2i} \binom{2i}{n} (-1)^n y^{5n+2i} \sum_{k=0}^{\infty} \binom{2i+k-1}{2i-1} y^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{2i}{n} (-1)^n y^{5n+2i} \binom{2i+k-1}{2i-1} y^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{2i}{n} (-1)^n \binom{2i+k-1}{2i-1} y^{k+5n+2i}$$

$$= \sum_{s=0}^{\infty} \left[\sum_{n=0}^{\left[\frac{s-2i}{5}\right]} \binom{2i}{n} (-1)^n \binom{s-5n-1}{2i-1} \right] y^s,$$

where $\left[\frac{s-2i}{5}\right]$ represents the integral part of $\frac{s-2i}{5}$.

Thus the coefficient of y^s in the expansion of

$(y+y^2+y^3+y^4+y^5)^{2i}$ is

$$\sum_{n=0}^{\left[\frac{s-2i}{5}\right]} \binom{2i}{n}(-1)^n \binom{s-5n-1}{2i-1}, \qquad (2.6)$$

and this represents the number of solutions of equation (2.5). However, in equation (2.5) $s = x_i - 2i$. Hence it may be concluded that the number of solutions of equation (2.4) is given by (2.6) with $s = x_i - 2i$. Thus the number of configurations which sum to x_i is

$$\sum_{n=0}^{\left[\frac{x_i-4i}{5}\right]} \binom{2i}{n}(-1)^n \binom{x_i-2i-5n-1}{2i-1}. \qquad (2.7)$$

The number of "non-zero" configurations, $N(x_i)$, which sum to x_i (specified by equation (2.7)), for $i = 1,2,3,4$ tosses is given in Appendix B.

It is now possible to write, for $x_i \neq 0$

$$f(x_i) = Pr(X_i=x_i) = \frac{\sum_{n=0}^{\left[\frac{x_i-4i}{5}\right]} \binom{2i}{n}(-1)^n \binom{x_i-2i-5n-1}{2i-1}}{6^{2i}}. \qquad (2.8)$$

For example, consider the probability of having a sum of 11 after 2 tosses.

$$f(11) = Pr(X_2=11) = \frac{\sum_{n=0}^{\left[\frac{11-4(2)}{5}\right]} (-1)^n \binom{4}{n} \binom{11-4-5n-1}{4-1}}{36^2}$$

$$= \frac{1 \cdot 1 \cdot \binom{6}{3}}{36^2} = \frac{20}{36^2}.$$

Thus, the probability density function $f(x_i)$ of the random variable X_i is given by (2.3) when $x_i = 0$ and (2.8) when $x_i \neq 0$.

One of the reasons for determining this probability density function was to provide a more attractive method for representing expectations. As was stated previously, a player will be interested in knowing the number of points that he may anticipate accumulating after i tosses of the dice. Define $E[X_i]$ to be the expected number of points accumulated at the end of the i^{th} toss where, as before, X_i denotes the random variable defined on A_i. Then $E[X_i] = \sum x_i f(x_i)$ where the sum is over all $x_i \epsilon A_i$ and $f(x_i)$ represents the probability that a player has a total of x_i points after i tosses. Hence

$$E[X_i] = 0 \cdot \sum_{j=0}^{i-1} (11/36)(25/36)^j$$

$$+ \sum_{x_i=4i}^{12i} \left[x_i \left(\left(\sum_{n=0}^{\left[\frac{x_i-4i}{5}\right]} (-1)^n \binom{2i}{n} \binom{x_i-2i-5n-1}{2i-1} \right) \Big/ 6^{2i} \right) \right]. \quad (2.9)$$

Needless to say, calculating these expectations is a tedious arithmetical task. The values of $E[X_i]$, for $i=1,2,\ldots,8$, were calculated on the computer by using the program given in Appendix F. A table of these computations may be found in Appendix C.

The next factor to be investigated is the expected number of tosses of the dice that it will take to achieve a particular event which has constant probability, p. From this result, the expected number of tosses that it will take for a player to receive an outcome which contains double ones, a single one, and double ones or a single one can be determined. A generalized theorem concerning the expected number of trials that it will take to achieve a specific event is stated by Epstein, [1,p.158]. A generalized theorem which conforms to the needs of this investigation is stated in the following way.

<u>Theorem 2.1</u>. For any series of tosses of the dice for which the probability of the occurrence of a particular

event, A, is constantly p, the expected number of tosses, E[Y], to achieve the event A is the reciprocal of the probability of its occurrence.

Proof: It will be shown that $E[Y] = \sum_{y=1}^{\infty} py(1-p)^{y-1} = \frac{1}{p}$. Let p be the constant probability of the occurrence of event A. Let 1-p=q. Consider the number of tosses, Y, required for the first occurrence of A. Note that Y is a discrete random variable, taking values 1,2,3,..., and has a geometric distribution. Hence, if the first occurrence of Y appears on the y^{th} toss of the dice, then the preceding y-1 tosses demonstrate nonoccurrences of event Y. Therefore the probability that A occurs for the first time on the y^{th} toss is $p(y) = pq^{y-1}$, y=1,2,3,... . It follows that

$$E[Y] = \sum_{y=1}^{\infty} y[pq^{y-1}]$$

$$= \sum_{y=1}^{\infty} yp(1-p)^{y-1}$$

$$= p + \sum_{y=2}^{\infty} yp(1-p)^{y-1}$$

$$= p + \sum_{y=1}^{\infty} (y+1)(1-p)^{y}$$

$$= p + (1-p) \sum_{y=1}^{\infty} (y+1)(1-p)^{y-1}$$

$$= p + (1-p)[E[Y]+1]$$

Then,

$$E[Y] = p + E[Y] + 1 - pE[Y] - p.$$

Solving this equation for E[Y] yields:

$$E[Y] = 1/p.$$

Corollary 2.1. The expected number of tosses required to achieve an occurrence of double ones, an occurrence of a single one, and an occurrence of double ones or a single one is 36, 3.6, 3.2727..., respectively.

Proof: The probability that double ones occur on any toss is 1/36, the probability that a single one occurs on any toss is 10/36, and the probability that double ones or a single one occurs on any toss is 11/36. Therefore, from Theorem 2.1, $E[Y] = \frac{1}{1/36} = 36$, $E[Y] = \frac{1}{10/36} = 36/10 = 3.6$, and $E[Y] = \frac{1}{11/36} = 36/11 = 3.2727\ldots$.

It may now be concluded that, on the average, a double one will occur once in 36 tosses, a single one will occur once in 3.6 tosses, and a double one or a single one will occur once in 3.2727... tosses.

Another important question to consider is the increasing probabilities of the occurrences of double ones, of a single one, and of double ones or a single one.

Theorem 2.2. If a specific outcome has constant probability p of occurrence, then the probability that this outcome appears on or before the n^{th} toss of the dice is $1 - q^n$ where $q = 1 - p$.

<u>Proof</u>: The probability that an outcome which has constant probability, p, appears on or before the n^{th} toss is $\sum_{i=0}^{n-1} p(1-p)^i = \sum_{i=0}^{n-1} pq^i$. However, $\sum_{i=0}^{n-1} pq^i$ represents the $n-1^{th}$ partial sum of a geometric series. Thus

$$\sum_{i=0}^{n-1} pq^i = p\left[\frac{1-q^n}{1-q}\right] = p\left[\frac{1-q^n}{p}\right] = 1 - q^n.$$

<u>Corollary 2.2</u>: If a specific outcome has constant probability p of occurrence, then the probability that this outcome has not occurred by the n^{th} toss of the dice is q^n where $q = 1 - p$.

<u>Proof</u>: In Theorem 2.2 it was found that the probability that the event occurred by the n^{th} toss is $1 - q^n$. Therefore, the probability that the event has not occurred is $1 - (1-q^n) = q^n$.

<u>Corollary 2.3</u>: The probability that double ones occur on or before the n^{th} toss is $1 - (35/36)^n$. The probability that a single one occurs on or before the n^{th} toss is $1 - (26/36)^n$. The probability that either double ones or a single one occurs on or before the n^{th} toss is $1 - (25/36)^n$.

<u>Proof</u>: The probability that double ones do not occur on any toss is 35/36, the probability that a single one does

not occur on any toss is 26/36, and the probability that double ones or a single one does not occur on any toss is 25/36. Hence, by Theorem 2.2, the probability that double ones occur on or before the n^{th} toss is $1 - (35/36)^n$, the probability that a single one occurs on or before the n^{th} toss is $1 - (26/36)^n$, and the probability that double ones or a single one occurs on or before the n^{th} toss is $1 - (25/36)^n$.

A table of these probabilities for $n = 1,2,\ldots,36$ can be found in Appendix D. Note that the probability of the occurrence of double ones on or before n tosses increases very slowly, while the probability of the occurrence of a single one on or before n tosses increases very rapidly.

These basic facts will now be used to determine a specific approach to the game Pig. In Chapter III, the approach concerning the maximum number of tosses per turn that a player should take will be investigated, and in Chapter IV, the approach concerning the maximum number of points per turn that a player should accumulate will be investigated.

CHAPTER III

ROLL PER TURN STRATEGIES

The first approach considered in finding an optimal strategy to the game Pig is that of determining the maximum number of tosses per turn that a player should take with the understanding that he will maintain this strategy throughout the game. The basic factors concerning the increasing probability of the occurrence of an outcome which contains a one and the number of points that a player can expect to accumulate in a fixed number of rolls are essential in establishing a roll per turn strategy.

The increasing probability of the occurrence of an outcome which contains a one will affect a player's decision concerning the number of times that he should roll per turn. The more a player decides to toss on any turn, the more he increases his chances of losing points which have already been accumulated. Obviously, the number of rolls per turn which minimizes this point loss will be an important fact to consider in the determination of a strategy.

<u>Theorem 3.1</u>. After the first toss of any single turn, the probability of the occurrence of an outcome which contains either double ones or a single one is greater

than the probability of the nonoccurrence of such an outcome.

Proof: The probability that either double ones or a single one has occurred on or before the i^{th} toss of the dice is $\sum_{j=0}^{i-1}(11/36)(25/36)^j$, and the probability that double ones or a single one has not occurred on or before the i^{th} toss is $1 - \sum_{j=0}^{i-1}(11/36)(25/36)^j$. It will be shown that $1 - \sum_{j=0}^{i-1}(11/36)(25/36)^j > \sum_{j=0}^{i-1}(11/36)(25/36)^j$ for $i = 1$. In Corollary 2.3, it was shown that $\sum_{j=0}^{i-1}(11/36)(25/36)^j = 1 - (25/36)^i$. Hence, it will be shown that $1 - (1-(25/36)^i) > 1 - (25/36)^i$ for $i = 1$. This problem can be reduced to finding the values of i which satisfy the inequality $(25/36)^i > 1 - (25/36)^i$.

$$(25/36)^i > 1 - (25/36)^i$$

$$2(25/36)^i > 1$$

$$(25/36)^i > 1/2$$

$$i \cdot \log (25/36) > \log (1/2)$$

$$i < \log (1/2)/\log(25/36)$$

$$i < 1.900892008462$$

Obviously, the only value of i which satisfies this inequality is 1. Hence, it can be concluded that after the first toss of any single turn the probability of the occurrence of an outcome which contains either double ones or a single one is greater than the probability of the nonoccurrence of such an outcome.

Theorem 3.2. After the second toss of any single turn, the probability of the occurrence of an outcome which contains a single one is greater than the probability of the nonoccurrence of such an outcome.

Proof: The probability that a single one has occurred on or before the i^{th} toss of the dice is $\sum_{j=0}^{i-1}(10/36)(26/36)^j$, and the probability that a single one has not occurred on or before the i^{th} toss of the dice is $1 - \sum_{j=0}^{i-1}(10/36)(26/36)^j$. In Corollary 2.3, it was shown that

$\sum_{j=0}^{i-1}(10/36)(26/36)^j = 1 - (26/36)^i$. Hence, it will be shown that $1 - \sum_{j=0}^{i-1}(10/36)(26/36)^j > \sum_{j=0}^{i-1}(10/36)(26/36)^j$ for $i = 1$ or 2. This problem can be reduced to finding the values of i which satisfy the inequality $1 - (1-(26/36)^i) > 1 - (26/36)^i$.

$$1 - (1-(26/36)^i) > 1 - (26/36)^i$$
$$(26/36)^i > 1 - (26/36)^i$$
$$2(26/36)^i > 1$$
$$(26/36)^i > 1/2$$
$$i \cdot \log(26/36) > \log(1/2)$$

$$i < \log(1/2)/\log(26/36)$$

$$i < 2.129992218219$$

Clearly, the only values of i which satisfy this inequality are 1 and 2. Hence, it can be concluded that after the second toss of any single turn, the probability of the occurrence of an outcome which contains a single one is greater than the probability of the nonoccurrence of such an outcome.

Obviously, the probability that a single one has occurred on or before the i^{th} toss increases much more rapidly than the probability that double ones have occurred on or before the i^{th} toss. For small values of i, the probability that double ones will occur is very small. Hence, if i is small, the chance that the total score will be zero after i tosses on any turn is small. However, comparatively, the probability that the score is zero for any particular turn after i tosses of the dice increases quite rapidly as the values of i increase. That is, the probability that an event which contains a single one occurs is increasing rapidly even for small values of i. It has been shown that after two tosses, the probability that a single one has occurred is greater than the probability that a single one has not occurred

The results of Theorem 3.1 and Theorem 3.2 demonstrate that a player's chances of losing points which have been accumulated are greater than his chances of retaining and increasing his score for the game after the first toss, and a player's chances of losing points which have been accumulated on any single turn are greater than his chances of retaining and increasing his score for any single turn after the second roll. However, it should not be concluded from the above discussion that a player should never roll more than once or twice per turn. Other factors will enter into this decision, one of which is the fact that the number of points which can be accumulated on additional rolls is large enough to partially offset the risk involved in continuing to toss.

Consider now the number of points which a player can expect to have accumulated after i tosses of the dice. Recall that X_i represents the random variable defined on A_i, and $E[X_i]$ denotes the number of points which a player can expect to have accumulated after the i^{th} toss of the dice. The values of $E[X_i]$ for i = 1,2,3,4,5 (to three decimal places) were found to be the following: $E[X_1]$ = 5.555, $E[X_2]$ = 7.716, $E[X_3]$ = 8.037, $E[X_4]$ = 7.442, and $E[X_5]$ = 6.460.

The greatest values for $E[X_i]$ occur when i = 2 or 3. Therefore a player can expect to have accumulated the greatest number of points after the second or third toss. It is interesting to note that $E[X_i]$ increases for i = 1, 2, and 3 and decreases for i > 3. So, the more a player rolls after the third roll the fewer points he can expect to accumulate. This fact is due to the increasing probability of the occurrence of an outcome which contains a one. The values of $E[X_i]$ are extremely close when i = 2, 3, or 4. Consider now the minimum number of points that a player can accumulate after 2, 3 or 4 tosses. These values are 8, 12, and 16 respectively. A comparison of $E[X_i]$ with the minimum number of points that can be accumulated after i tosses leads to the fact that $E[X_i]$ is considerably less than this respective minimum value except when $i = 2$. When $i = 2$, $E[X_i]$ is close to the minimum number of points that can be accumulated after two tosses, but is still less than this minimum value. Actually, $E[X_i]$ is greater than the minimum number of points after i tosses only when $i = 1$. This fact is also a consequence of the increasing probability of an occurrence of an outcome which contains a one.

It has been brought out that a player can expect to accumulate the greatest number of points on any turn after the third toss. A decision with which a player may be confronted is whether or not he should roll a second time if after the first toss he has accumulated more than the number of points which he can expect to accumulate at the completion of the third toss. That is, should a player continue rolling if he receives more than 8 points on the first toss. It has been stated that $E[X_i]$ increases for

$i = 1, 2,$ and 3, and decreases for $i > 3$. Also, it has been brought out that each toss of the dice is independent of every other toss of the dice. Hence, on the second roll, the probability that a one will appear is 11/36, and the probability that a one will not appear is 25/36. Clearly, it will be to the player's advantage to roll again. However, this will not be the case if a player has already rolled three or more times on a particular turn and is trying to decide whether or not he should roll again. Obviously, if a player has successfully rolled three times then the number of points which have been accumulated at the completion of the third roll is greater than the number of points which he can expect to accumulate on any succeeding roll. This is due to the fact that after the third roll of any single turn, the number of points that a player can expect to accumulate begins to decrease. Thus, if a player decides to toss again in this situation, it is likely that his score will decrease rather than increase.

As was stated previously, the objective in a strategy for the game Pig is to accumulate a maximum number of points in a minimum number of turns. It was also pointed out that the more a player chooses to roll per turn, the more he chances losing points which have thus far been accumulated. A comparison of the probabilities of an event which contains a one on or before the i^{th} toss with the expectations of the number of points that can be accumulated on the i^{th}

toss, for $i > 3$, demonstrates that as i increases, the probabilities are increasing whereas the expectations are decreasing. Obviously, rolling only one time per turn will be the safest approach. However, on the average it will take 18 turns to achieve a score of 100 points provided double ones do not occur, whereas if a player rolls either two or three times per turn it will take, on the average, 13 turns to achieve a score of 100 points provided double ones do not occur. Also a player can expect to accumulate a greater number of points if he chooses to roll two or three times per turn than if he chooses to roll only once per turn. Hence it can be concluded that a player should roll more than one time per turn. Rolling more than three times per turn will not be a good approach due to the large probability of the occurrence of an outcome which contains a one. Also, $E[X_i]$ is decreasing for $i > 3$. Hence a player can expect to accumulate more points rolling two or three times per turn than rolling more than three times per turn. Thus, a player should roll at least two times per turn, but not more than three times per turn.

It was not determined in this investigation whether or not a three-roll strategy is better than a two-roll strategy. It is true that $E[X_3] > E[X_2]$, however, the probability that an event which contains a one has occurred after two tosses is less than the probability that an event which contains a one has occurred after three tosses. These facts cause the two-roll-per-turn and the three-roll-per-turn strategies to be practically indistinguishable.

The following is a list of outcomes of games concerning per turn strategies which were simulated on the computer by using the program which appears in Appendix G. Due to the theoretical closeness of several of the roll-per-turn strategies, computer simulation was implemented to attempt to distinguish subtle differences in the strategies. This program simulated the play of the game Pig by two persons, each assuming some specified roll-per-turn strategy. In order to restrict this game from continuing indefinitely, play was stopped if a player had completed more than 100 turns. It is interesting to note that some outcome for the game was always determined before a player completed 100 turns. The configurations which would occur on the toss of two dice were randomly generated. The computer played blocks of 1,000 games at a time. The results obtained from the computer simulation correspond to the theoretical results which have been established. It was determined that a two-roll-per-turn strategy and a three-roll-per-turn strategy are virtually indistinguishable. The computer results show that a two-roll-per-turn strategy won 82 (or 1.64%) games more than a three-roll-per-turn strategy after the play of 5,000 games. It is possible that this 82-game difference will deteriorate or reverse itself if more games are simulated. For several of the

1,000 block games, a three-roll-per-turn strategy won more games than a two-roll-per-turn strategy. In the table below, Player 1 Strategy and Player 2 Strategy represent the respective number of times per turn that each player chose to roll. Player 1 Wins and Player 2 Wins represent the respective number of games won by each player.

Player 1 Strategy	Player 2 Strategy	Player 1 Wins	Player 2 Wins
1	2	378	622
1	3	331	699
1	4	386	614
1	5	403	597
2	3	2541	2459
2	4	2688	2312
2	5	563	437
3	4	2692	2308
3	5	573	427
4	5	584	446

In the following table of results, Player 1 had the added option to stop rolling if his score exceeded 8 points. That is, Player 1 would roll either two or three times per turn or until he had accumulated 8 points. As was discussed previously and as the table demonstrated, a player should definitely attempt to accumulate more than 8 points per turn.

Player 1 Strategy	Player 2 Strategy	Player 1 Wins	Player 2 Wins
Rolls 2 times or until score≥8	2	455	545
Rolls 3 times or until score≥8	2	422	578
Rolls 2 times or until score≥8	3	446	554
Rolls 3 times or until score≥8	3	444	556

In conclusion, the best roll-per-turn strategy will be to roll no less than two times per turn and no more than three times per turn. These two strategies are virtually indistinguishable. An approach to the game by using one of these strategies will in no way guarantee that a player will win every time, but will guarantee that a player will win more games by using one of these approaches than any other roll-per-turn strategy.

CHAPTER IV

POINT PER TURN STRATEGIES

The final approach to an optimal strategy for Pig is the consideration of a point-per-turn strategy. In this consideration the number of points that a player should attempt to accumulate on any single turn will be specified. In order to mathematically formulate the problem of finding a point-per-turn strategy, it will be advantageous to introduce some basic terminology of game theory.

During the course of any game, a player is required to make various decisions by selecting a strategy from a group of alternative strategies. This selection should be made on the basis of the comparative effectiveness of the strategies in achieving the player's goal. A utility function will define a measure of efficiency for a strategy, thereby allowing for an objective appraisal of its effectiveness. A function of utility will be defined for the game Pig by requiring a player to pay one unit at the termination of each turn before the achievement of his stated pre-game goal. The expected cost of achieving the pre-game goal by playing according to a given strategy will specify the utility for that strategy. Comparatively, strategies with lower expected cost will be assigned higher utilities since these strategies will require, on the average, fewer turns to achieve the stated pre-game goal and therefore be most efficient.

It will now be beneficial to introduce a means by which a compact description of the player's position in the game may be represented. A plateau will designate the total score that a player has accumulated at the termination of any turn. A player may voluntarily declare a plateau by choosing to terminate his turn before the occurrence of a one. In this instance, the player's plateau will be equal to the sum of the previous plateau and the number of points accumulated during this turn. If a single one occurs before a player declares a plateau, his turn is automatically terminated and his plateau will be the same as the plateau for the previous turn. If a double one occurs before a player declares a plateau, then his plateau will be zero.

A player's position at the end of any toss during the game may now be represented by means of an ordered pair (a,b) where the first coordinate, a, represents the total cumulative score at the end of that toss, and the second coordinate, b, represents the player's plateau. For example, the position (52,40) indicates that the player had 40 points at the completion of his last turn and has thus far successfully accumulated 12 points during the current turn.

Before investigating a point-per-turn strategy for the game Pig as defined in this investigation, a less complicated but similar problem examined by Epstein, [1,p.164-6], will be considered. The version of the game considered by Epstein is such that the appearance of a double one does not automatically revert the score to zero, but rather has the same consequence as the appearance of a single one. Hence, a player's turn may be terminated in one of two ways. First, a player will be required to terminate his turn upon the occurrence of any outcome which contains a one and the player's score is then equal to his previous plateau. Secondly, a player may voluntarily terminate his turn and increase his previous plateau by the total number of points accumulated during that turn. The specific problem which will now be investigated is that of accumulating a total of 24 or more points.

According to the utility function which has been defined, a player will be required to pay one unit at the completion of each turn before achieving 24 points. It is now possible to compare two competing strategies for the accumulation of 24 points by means of a comparison of the respective expected costs to achieve the goal of 24 points. Specifically, the two competing strategies will be that of attempting to accumulate 24 points without voluntarily declaring a plateau and that of declaring a plateau at some position (a,0) where $a < 24$. In order to show that the strategy of never declaring a plateau is the optimal strategy in this game, it will suffice to show that a plateau should not be declared at (23,0).

If a player has position (23,0) and decides to declare a plateau, then his position will become (23,23), and he must pay one unit for declaring the plateau. On his next turn, two events are possible. First, the player could, with probability 11/36, obtain an outcome which contains a one and remain at (23,23) with a cost of one additional unit, or secondly, the player could, with probability 25/36, obtain an outcome which does not contain a one and achieve his goal with no additional cost. In the former case, the player must continue tossing until he obtains a nonoccurrence of a one, paying one unit for each turn required before the achievement of 24 points. Since the probability of a nonoccurrence of a one is 25/36, Theorem 2.1 implies that the expected number of tosses before the nonoccurrence of a one is $\frac{1}{(25/36)} = 36/25$. Thus, the expected cost, C, under the strategy of declaring a plateau at (23,0) is

$$C = 1 + (11/36)(36/25), \qquad (4.1)$$

where "1" represents the cost of declaring the plateau at (23,0) and reaching the position (23,23), and (11/36)(36/25) represents the expected additional cost of getting from (23,23) to the goal of 24 points. Now, the expected cost if a plateau is not declared at (23,0) will be considered. Let W represent the expected cost of reaching the goal of 24 points from the (0,0) position under a strategy of never declaring a plateau. Hence, W represents the expected number of times that a one will appear before achieving a score of 24 points or more by continuously rolling the dice. The probability of accumulating 24 points or more assumes some constant value, p. Thus, by Theorem 2.1, $W = 1/p$. Let C^* represent the expected cost at position (23,0) if the player decides not to declare a plateau at this point. The player then risks getting a one on the next toss, costing him one unit and reverting back to position (0,0). Hence

$$C^* = (11/36)(1+W) \qquad (4.2)$$

It will be shown that a player should not declare a plateau at (23,0) by showing the expected cost C^*, under the strategy of not declaring a plateau at (23,0) is less than the expected cost, C, under the stragegy of declaring a plateau at this point. The inequality $C^* < C$ is equivalent to $(11/36)(1+W) < 36/25$. Simplifying this inequality yields

$$11/36 + (11/36)W < 36/25$$

$$(11/36)W < 36/25 - 11/36$$

$$W < (36^2/11 \cdot 25) - 1$$

$$W < 3.71.$$

Thus, $C^* < C$ if and only if $W < 3.71$ where W is the reciprocal of the probability of accumulating 24 points or more. A numerical value for W can be found by calculating the probability of achieving 24 or more points, and taking its reciprocal. The computer was implemented to find the probability for accumulating x points or more on any single turn by using the source program which appears in Appendix F. The value of these probabilities for $x = 24, 25, \ldots, 30$ can be found in Appendix E. Thus, it was found that the value of W is 3.465. Since $3.465 < 3.71$ it can be concluded that a player does not want to declare a plateau at (23,0) if his initial goal is 24 points. In the single-player game under the modification of double ones having the same consequence as single ones where the goal is 100 points, it may now be concluded that a plateau should not be declared at any position before the accumulation of 24 points. This agrees with the result in Epstein [1,p.165]; however, Epstein does not indicate the exact position at which the plateau should be declared if the goal is 100 points. The problem just considered, with initial goal of 24 points and current position (23,0), may be generalized to the problem with initial goal of $d+1$ points and current position (d,0). For this more general problem the functions C and C^* remain identical to those previously derived in equations (4.1) and (4.2). However, W will now represent the expected cost of accumulating a total of d+1 or more points under a strategy of never declaring a plateau. The problem of deciding the position for declaring the first plateau is now that of finding the smallest number of points, d, for which C* exceeds C since this will indicate that the expected cost of not declaring a plateau at (d,0) exceeds the expected cost of declaring a plateau at (d,0) when the goal is d+1 points. This is equivalent to finding d such that W exceeds 3.71 where W is the reciprocal of the probability, p, of the accumulation of d+1 or more points under a strategy of never declaring a plateau. Reference to Appendix E indicates that if d = 25, then the probability, p, of accumulating d + 1 = 26 or more points is .26324. This gives a value of W = 3.799 > 3.71. It may then be concluded that if a player's goal is 26 points, a plateau should be declared at 25 points.

Extending this to a game in which the player's goal is 100 points, it may be stated that the optimal strategy is to declare a plateau at (25,0) which indicates that a player should try to accumulate at least 25 points on his first turn before declaring a plateau. Thus the optimal point-per-turn strategy for the first toss is to declare a plateau only after 25 points have been accumulated.

Attention will now be focused on investigating an optimal point-per-turn strategy for the game of Pig with the rules as originally stated in this research. The previously discussed problem was for the particular game in which the appearance of double ones on a toss had the same consequence as the appearance of single ones. However, in the originally stated rules for Pig the appearance of double ones on any toss reverted the total score to zero and gave the player a position of (0,0). With this consequence for double ones, the problem of finding the optimal points-per-turn strategy for the position (0,0) will now be considered. Obviously, this change in the consequence of double ones will affect the declaration of a plateau, since the occurrence of double ones will now revert a player's plateau to zero regardless of his plateau on the previous turn. That is, a player is no longer assured that the declaration of a plateau at d points will permanently assure him of having a score of at least d points.

It will be assumed that the player has a goal of d+1 points and has reached the position of (d,0) and is now faced with the decision of either declaring a plateau at (d,0) or continuing to toss in an attempt to reach his goal. The respective expected costs of the two alternate strategies will now be considered. If the player declares a plateau at (d,0), pays one unit, and accepts the position of (d,d), then on his next turn there are three distinct possibilities:

(1) a double one can occur costing the player one unit and reverting his position to (0,0);

(2) a single one can occur costing the player one unit and retaining his position (d,d);

(3) a "non-one" outcome occurs and the player adds the points received to his score of d with no cost to the player.

Let W(d+1) represent the expected cost of getting from (0,0) to the goal of d+1 points under the strategy of declaring a plateau at d points. Then the expected cost, C(d+1), of getting from d points to d+1 points under the strategy of declaring a plateau at (d,0) is

$$C(d+1) = 1 + (10/36)[1+C(d+1)] + (1/36)[1+W(d+1)]. \quad (4.3)$$

In equation (4.3) the "1" represents the cost of declaring a plateau at (d,0) and accepting the position (d,d). On his next turn, the probability of a single one occurring is 10/36 and the expected cost resulting from this is 1 + C(d+1); the probability of double ones occurring is 1/36, costing one unit and reverting the player's position to (0,0) where the expected cost of reaching d+1 is W(d+1). In the case of a "non-one" outcome, with probability 25/36 of occurring, the player automatically achieves his goal of d+1 points and has no further additional cost.

Under the strategy of not declaring a plateau at (d,0), with a goal of d+1 points, a player either achieves his goal on the next toss with the occurrence of a "non-one" outcome, or the player's score is reverted to the (0,0) position upon the occurrence of either a single or a double one with a cost of one unit. Let W*(d+1) denote the expected cost of getting from (0,0) to the goal of d+1 under the strategy of never declaring a plateau. Then the expected cost, C*(d+1) of getting from (d,0) to the goal of d+1 under the strategy of not declaring a plateau

$$C^*(d+1) = (11/36)[1+W^*(d+1)]. \qquad (4.4)$$

If $C^*(d+1) < C(d+1)$ then the optimal strategy will be that of not declaring a plateau at (d,0); otherwise, the optimal strategy is that of declaring a plateau at (d,0). The determination of the numerical values of C(d+1) and C*(d+1), while theoretically possible, requires extensive computations including the enumeration of all possible branchings from (0,0) to d+1 points. However, further conclusions may be drawn from a comparison of equations (4.3) and (4.4) with equations (4.1) and (4.2) without an actual numerical evaluation of C(d+1) and C*(d+1).

If a player's goal is 24 points, then d = 23 and equation (4.4) is equivalent to equation (4.2) since W*(24) in each of the equations represents the expected cost of getting from (0,0) to the goal of 24 without the declaration of a plateau. It should be noted that under such a strategy with a player's initial position of (0,0) the occurrence of double ones and a single one have the same consequence regardless of the rules of the game. Also, it is obvious that when $d = 23$, $C(d+1)$ in equation (4.4) will be less than C in equation (4.1) since the expected cost of getting from (23,23) to the goal of 24 points under a strategy of declaring a plateau at 23 points will be greater in the game in which double ones revert the score to zero rather than maintain the previous plateau of (23,23). In the previous problem it was found that if the goal is 24 points then $C^* < C$. This establishes the following inequality

$$C^*(24) = C^* < C < C(24).$$

From this it may be concluded that the optimal strategy in the game with goal of 24 points and with double ones reverting the total score to zero is to never declare a plateau. Arguing in a similar manner for the case when $d = 25$ leads to the conclusion that the optimal point-per-turn strategy for a player whose initial position is (0,0) is to attempt to accumulate at least 25 points before declaring a plateau. The exact number of points that should be accumulated before declaring a plateau may exceed 25 since it is at 25 that C^* exceeds C. However, this does not imply that $C^*(25)$ exceeds $C(25)$. The determination of the exact value of d for which $C^*(d+1) > C(d+1)$ would require computation of $C^*(d+1)$ and $C(d+1)$ for various values of d.

Due to the complexity involved in calculating the above-mentioned cost functions, an optimal point-per-turn strategy was considered only for a player whose initial position was (0,0). A determination of an optimal point-per-turn strategy for other positions in the game would be exceedingly difficult due to the fact that such a strategy would now depend on previous plateaus, the scores of other players, and the number of players competing. However, a generalized point-per-turn strategy was investigated by a simulation of the game for two players on the computer by using the program which appears in Appendix G. Each player assumed a pure strategy of accumulating x points or more on each turn with the understanding that x points would be the goal for every turn.

In the following table of outcomes of the game Pig played by two people, Player 1 Strategy and Player 2 Strategy represent the respective per game specified number of points that each would attempt to accumulate. Player 1 Wins and Player 2 Wins designate the number of games won by using these specific strategies.

Player 1 Strategy	Player 2 Strategy	Player 1 Wins	Player 2 Wins
24	18	519	481
24	20	536	464
24	22	496	504
24	26	500	500
24	28	519	481
24	30	560	440
25	18	515	485
25	20	505	495
25	22	509	491
25	24	499	501

Player 1 Strategy	Player 2 Strategy	Player 1 Wins	Player 2 Wins
25	26	1045	855
25	28	530	470
25	30	539	461
26	18	541	459
26	20	514	486
26	22	522	478
26	28	519	481
26	30	528	472

The results of this simulation show that the pure strategy of accumulating at least 22, 23, 24, 25 or 26 points is relatively indistinguishable. Thus, if a player is required to designate a number of points that he will attempt to accumulate per turn, then this number should not be less than 22 nor greater than 26.

In summary, it can be concluded that a player should attempt to accumulate no less than 25 points from the initial position of (0,0). Also, if a player is required to designate a number of points which he will attempt to accumulate on any turn regardless of his initial position, then it appears that the player should set a goal at no less than 22 points per turn and no more than 26 points per turn.

CHAPTER V

CONCLUSION

In this investigation an optimal strategy for the game Pig was sought through the means of two basic approaches. The first approach considered the maximum number of rolls per turn that a player should take and the second approach considered the maximum number of points per turn that a player should take.

It was found that an optimal roll-per-turn strategy is for a player to toss no less than two times per turn and no more than three times per turn. It was not determined in this investigation whether or not a two-roll-per-turn strategy is better than a three-roll-per-turn strategy. These two strategies have very similar characteristics causing the results obtained by using either one of these strategies to be essentially the same.

The computations involved in investigating an optimal point-per-turn strategy are tedious arithmetical tasks. It is for this reason that the optimal number of points to be accumulated only when a player's initial position is (0,0) was considered. It was found that a player should attempt to reach a plateau of at least 25 points from this position. It was implied previously that a player will want to set as few plateaus as possible in this game since he is not guaranteed of being able to retain the plateaus declared throughout the game. Several pure point-per-turn strategies were investigated by a simulation of the game on the computer. It was found that a player setting a plateau at from 22 points to 26 points won more games than a player who chose to set a plateau at some number less than 22 or greater than 26. Thus, it appears an optimal point-per-turn strategy will be for a player to attempt to accumulate around 22-26 points per turn.

In the previous chapters, each simulation of the game involved testing strategies which were of the same type. That is, roll-per-turn strategies were played against roll-per-turn strategies and point-per-turn strategies were played against point-per-turn strategies. The source program found in Appendix G was used to test several roll-per-turn strategies against several point-per-turn strategies. Player 1 Strategy and Player 2 Strategy represent the strategies chosen by the respective players, and Player 1 Wins and Player 2 Wins relate the number of games won by each player under the respective strategy chosen.

Player 1 Strategy	Player 2 Strategy	Player 1 Wins	Player 2 Wins
12 points per turn	2 rolls per turn	531	469
12 " " "	3 " " "	474	526
18 " " "	2 " " "	539	461
18 " " "	3 " " "	508	492
24 " " "	2 " " "	518	482
24 " " "	3 " " "	524	476
30 " " "	2 " " "	467	533
30 " " "	3 " " "	482	518

It appears that the "best" strategy is a point-per-turn strategy which sets as the goal 24 points per turn. It was found in Chapter IV that 22 through 26 points achieved similar results and it might be concluded that any of these will perform as well as 24-points-per-turn strategy versus a roll-per-turn strategy.

As was stated previously the simplicity of the rules of this game is very misleading in relation to the formulation of a strategy for this game. This research was limited to investigating strategies for a single-player game of Pig in which the player's goal is the optimal accumulation of 100 points. Only strategies which consisted of the same action on each and every turn of the game were considered. Of the competing strategies which were investigated, it may be concluded that the best strategy is to attempt to accumulate at least 25 points on the first turn and to attempt to accumulate from 22 to 26 points on each additional turn. In multiplayer games of Pig, this may not be the optimal strategy since such an optimal strategy will depend on the number of players in the game, the relation between the scores of the players, and the relation of the player to his goal. An optimal strategy would then be one which included all these factors but such a strategy would be difficult to state and much more difficult to mathematically justify. It is felt that the "best" strategy derived in this investigation is easily stated yet, even in multiplayer games, will be competitive with an optimal strategy especially in games where the player's score is close to the highest scores of the opponents.

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APPENDIXES

APPENDIX A

The following is a list of the probabilities Pr(X=m), of accumulating m points without the appearance of a one for the first time on the i^{th} toss of the dice on any single turn.

i=1

m	Pr(X=m)
4	0.0277
5	0.0555
6	0.0833
7	0.1111
8	0.1361
9	0.1111
10	0.0833
11	0.0555
12	0.0277

i=2

m	Pr(X=m)
8	0.0008
9	0.0031
10	0.0077
11	0.0154
12	0.0270
13	0.0401
14	0.0525
15	0.0617
16	0.0656
17	0.0617
18	0.0525
19	0.0401
20	0.0270
21	0.0154
22	0.0077
23	0.0031
24	0.0008

i=3

m	Pr(X=m)
12	0.0000
13	0.0001
14	0.0005
15	0.0012
16	0.0027
17	0.0053
18	0.0091
19	0.0143
20	0.0204
21	0.0267
22	0.0323
23	0.0361
24	0.0375
25	0.0361
26	0.0323
27	0.0267
28	0.0204
29	0.0143
30	0.0091
31	0.0053
32	0.0027
33	0.0012
34	0.0005
35	0.0001
36	0.0000

i=4

m	Pr(X=m)
16	0.0000
17	0.0000
18	0.0000
19	0.0001
20	0.0002
21	0.0005
22	0.0010
23	0.0019
24	0.0033
25	0.0052
26	0.0078
27	0.0109
28	0.0143
29	0.0175
30	0.0202
31	0.0221
32	0.0227
33	0.0221
34	0.0202
35	0.0175
36	0.0143
37	0.0109
38	0.0078
39	0.0052
40	0.0033
41	0.0019
42	0.0010
43	0.0005
44	0.0002
45	0.0001
46	0.0000
47	0.0000
48	0.0000

APPENDIX B

The following is a list of the number of outcomes, $N(x_i)$, with a total of x_i points without the appearance of a one after i consecutive tosses of the dice on any single turn. The probability of having x_i points after i tosses will be $N(x_i)/36^i$.

i=1

x_1	$N(x_1)$
4	1
5	2
6	3
7	4
8	5
9	4
10	3
11	2
12	1

i=2

x_2	$N(x_2)$
8	1
9	4
10	10
11	20
12	35
13	52
14	68
15	80
16	85
17	80
18	68
19	52
20	35
21	20
22	10
23	4
24	1

i=3

x_3	$N(x_3)$
12	1
13	6
14	21
15	56
16	126
17	246
18	426
19	666
20	951
21	1246
22	1506
23	1686
24	1757
25	1686
26	1506
27	1246
28	951
29	666
30	426
31	246
32	126
33	56
34	21
35	6
36	1

i=4

x_4	$N(x_4)$
16	1
17	8
18	36
19	120
20	330
21	784
22	1652
23	3144
24	5475
25	8800
26	13140
27	18320
28	23940
29	29400
30	34000
31	37080
32	38165
33	37080
34	34000
35	29400
36	23940
37	18320
39	13140
39	8800
40	5475
41	3144
42	1652
43	784
44	330
45	120
46	36
47	8
48	1

APPENDIX C

The following is a list of the expected number of points, $E[X_i]$, after i consecutive tosses of the dice.

i	$E[X_i]$
1	5.555555
2	7.716049
3	8.037551
4	7.442177
5	6.460223
6	5.383502
7	4.361626
8	3.461604

APPENDIX D

The following is a list of the probabilities that an outcome which contains a double one (A), a single one (B), and a double one or a single one (C), has occurred after n tosses of the dice.

n	Pr(A)	Pr(B)	Pr(C)
1	0.02778	0.27778	0.30556
2	0.05478	0.47840	0.51775
3	0.08104	0.62329	0.66510
4	0.10657	0.72793	0.76743
5	0.13138	0.80350	0.83849
6	0.15551	0.85809	0.88784
7	0.17897	0.89751	0.92211
8	0.20178	0.92598	0.94591
9	0.22395	0.94654	0.96244
10	0.24551	0.96139	0.97392
11	0.26646	0.97211	0.98189
12	0.28684	0.97986	0.98742
13	0.30665	0.98545	0.99126
14	0.32591	0.98950	0.99393
15	0.34464	0.99241	0.99579
16	0.36284	0.99452	0.99707
17	0.38054	0.99604	0.99797
18	0.39775	0.99714	0.99859
19	0.41448	0.99794	0.99902
20	0.43074	0.99851	0.99932
21	0.44655	0.99892	0.99953
22	0.46193	0.99922	0.99967
23	0.47687	0.99944	0.99977
24	0.49140	0.99959	0.99984
25	0.50553	0.99971	0.99989
26	0.51927	0.99979	0.99992
27	0.53262	0.99985	0.99995
28	0.54560	0.99989	0.99996
29	0.55823	0.99992	0.99997
30	0.57050	0.99994	0.99998
31	0.58243	0.99996	0.99999
32	0.59403	0.99997	0.99999
33	0.60530	0.99998	0.99999
34	0.61627	0.99998	1.00000
35	0.62693	0.99999	1.00000
36	0.63729	0.99999	1.00000

APPENDIX E

The following is a list of the probabilities, Pr(X≥x), of accumulating x points or more for the first time without the appearance of a one on the n^{th} toss of the dice.

n	Pr(X≥24)	Pr(X≥25)	Pr(X≥26)	Pr(X≥27)	Pr(X≥28)	Pr(X≥29)	Pr(X≥30)
1	0	0	0	0	0	0	0
2	0.00077	0	0	0	0	0	0
3	0.18567	0.14868	0.11254	0.08026	0.05356	0.03317	0.01890
4	0.09963	0.12243	0.14229	0.15688	0.16452	0.16442	0.15683
5	0.00250	0.00475	0.00836	0.01371	0.02111	0.03064	0.04211
6	0.00000	0.00001	0.00003	0.00009	0.00021	0.00046	0.00094
7	0	0	0	0	0.00000	0.00000	0.00000
8	0	0	0	0	0	0	0

It should be noted that 0 indicates an impossible event while 0.00000 indicates a possible event with actual probability less than 10^{-5}.

APPENDIX F

The following is the source program used to compute the number of points which a player can expect to accumulate after i tosses, the probability of accumulating x_i points for the first time on the i^{th} toss, and the probability of accumulating x_i points or more for the first time on the i^{th} toss. The input, LMAX, represents the largest number of tosses for which these values were to be calculated.

```
      DIMENSION P2(30),PROB2(30),P(10,200),PROB(10,200)
      DIMENSION EXPX(10),P1(10,200),CUMPRO(10,200)
      PROB2A = (11.0/36.0)+((25.0/36.0)*(11.0/36.0))
      DO 1 I=4,24
    1 P2(I)=0.0
      DO 2 I=4,12
      DO 2 J=4,12
      K=I+J
    2 P2(K)=P2(K)+(5-IABS(8-I))*(5-IABS(8-J))
      WRITE(3,3)
    3 FORMAT(1H1,'NO. TOSSES     SCORE    PROBABILITY')
      L=2
      LX=0
      WRITE(3,4) L,LX,PROB2A
    4 FORMAT(1H0,4X,I2,8X,I2,6X,E12.5)
      DO 5 I=8,24
    5 PROB2(I)=P2(I)/(36**2)
      DO 7 I=8,24
    7 WRITE(3,4) L,I,PROB2(I)
      DO 8 I=8,24
    8 P(2,I)=P2(I)
    9 READ(1,10) LMAX
   10 FORMAT(I2)
      DO 100 L=3,LMAX
      LL=L-1
      PROB0=11.0/36.0
      DO 11 I=2,L
   11 PROB0=PROB0 +((11./36.)*((25./36.)**(I-1)))
      JJ=(L-1)*4
      JJJ=(L-1)*12
      II=L*4
      III=L*12
      DO 12 I=II,III
      P(L,I)=0.0
   12 PROB(L,I)=0.0
      DO 13 I=4,12
      DO 13 J=JJ,JJJ
      K=I+J
   13 P(L,K)=P(L,K)+P(LL,J)*(5-IABS(8-I))
      DO 14 I=II,III
   14 PROB(L,I)=P(L,I)/(36.**L)
      EXPX(L)=0.0
      WRITE(3,3)
      WRITE(3,4) L,LX,PROB0
      DO 16 I=II,III
      EXPX(L)=EXPX(L)+I*PROB(L,I)
   16 WRITE(3,4) L,I,PROB(L,I)
      DO 90 KK = 24,30
      IF(JJ-KK) 59,90,90
   59 M = MAX0(KK-12,JJ)
      MM = MIN0(KK-1,JJJ)
      MMM = MIN0(KK+11,JJJ+12)
      DO 60 I = KK,MMM
      P1(L,I)=0.0
   60 CUMPRO(L,I) = 0.0
      DO 70 I = 4,12
      DO 70 J = M,MM
      K = I+J
      IF (K-KK) 70,61,61
   61 P1(L,KK)=P1(L,KK)+(P(LL,J)*(5-IABS(8-I)))/(36.**L)
```

```
70 CONTINUE
   KK1 = MAXO(KK,II)
   DO 80 I = KK1,MMM
80 CUMPRO(L,KK) = CUMPRO(L,KK) + P1(L,I)
   WRITE(3,81) L,KK,CUMPRO(L,KK)
81 FORMAT(1H0,'NUMBER OF TOSSES = ',I2,'PROB. OF SCORE OF ',I2,' OR
  *REATER = ',E16.8)
90 CONTINUE
100 WRITE(3,17) EXPX(L)
17 FORMAT(1H0,'EXPECTED SCORE = ',E16.8)
   STOP
   END
$DATA
```

APPENDIX G

The following is the source program used for the simulation of the game Pig by two players. In order to restrict this game from continuing indefinitely, each player was restricted to accumulating no more than 100 turns. The subroutine entitled "TOSS" determined the outcomes which would appear on the toss of two dice. The random number generator entitled "RANDU" was used to generate a real number in the interval (0,1]. This interval was divided into subintervals. The lengths of these subintervals were calculated to correspond to the magnitude of the probabilities of specific point outcomes occurring in the game Pig. The input consisted of the four variables NGAME, NPLAY1, NPLAY2, and NSTART. NGAME represents the number of games that were played, NPLAY1 and NPLAY2 represent the specified strategies of player 1 and player 2 respectively, and NSTART represents the random three-digit odd number used to initiate the generating of the random numbers. In the following program, roll-per-turn strategies were used by each of the players. A slight modification in this program will enable a player to choose a point-per-turn strategy.

```
      READ(1,1) NGAME, NPLAY1, NPLAY2, NSTART
    1 FORMAT(I4,3I3)
      WRITE(3,850) NPLAY1,NPLAY2,NSTART
  850 FORMAT(1H0,I5,I5,I5)
      N1WIN = 0
      N2WIN = 0
      NTIES = 0
      NOWIN = 0
      DO 20 N=1, NGAME
      ISCOR1=0
      ISCOR2=0
   45 DO 11 K=1,100
   25 NTEMP1 = 0
      DO 110 I = 1, NPLAY1
      CALL TOSS (NSTART,NOUT,NPOINT)
      NSTART = NOUT
      IF (NPOINT) 2,3,4
    2 ISCOR1=0
      GO TO 5
    3 ISCOR1=ISCOR1
      GO TO 5
    4 NTEMP1=NTEMP1+NPOINT
      LL = ISCOR1 + NTEMP1
      IF (LL.GE.100) GO TO 500
  110 CONTINUE
  500 ISCOR1 = ISCOR1 + NTEMP1
      IF (LL.GE.100) GO TO 14
    5 NTEMP=0
      DO 400 I = 1,NPLAY2
      CALL TOSS (NSTART, NOUT, NPOINT)
      NSTART = NOUT
      IF (NPOINT) 6,7,8
    6 ISCOR2=0
      GO TO 9
    7 ISCOR2=ISCOR2
      GO TO 9
    8 NTEMP=NTEMP+NPOINT
      LL = ISCOR2 + NTEMP
      IF (LL.GE.100) GO TO 420
  400 CONTINUE
  420 ISCOR2 = ISCOR2 + NTEMP
    9 IF(ISCOR1-100) 10,12,12
   10 IF (ISCOR2 - 100) 11,17,17
   11 CONTINUE
      GO TO 90
   12 IF (ISCOR1-ISCOR2) 80,13,13
   13 IF (NPOINT) 70,70,14
   14 LL = ISCOR2
      DO 16 I = 1,28
      IF (ISCOR1 - LL) 80,15,15
   15 CALL TOSS (NSTART,NOUT,NPOINT)
      NSTART = NOUT
      IF (NPOINT) 70,100,16
   16 LL = LL + NPOINT
   17 LL = ISCOR1
      DO 19 J = 1,28
      IF (LL-ISCOR2) 18,18,70
   18 CALL TOSS (NSTART,NOUT,NPOINT)
      NSTART = NOUT
      IF (NPOINT) 80,100,19
```

```
   19 LL = LL + NPOINT
   70 N1WIN = N1WIN + 1
      GO TO 20
   80 N2WIN = N2WIN + 1
      GO TO 20
   90 NOWIN = NOWIN + 1
      GO TO 20
  100 IF (ISCOR1-ISCOR2) 80,120,70
  120 NTIES = NTIES + 1
   20 CONTINUE
      WRITE(3,30) NGAME, N1WIN, N2WIN, NTIES, NOWIN
   30 FORMAT(1H0,5(I5,5X))
      STOP
      END
      SUBROUTINE TOSS (NSTART,NOUT,NPOINT)
      A1=1./36.
      A2=11./36.
      A3=12./36.
      A4=14./36.
      A5=17./36.
      A6=21./36.
      A7=26./36.
      A8=30./36.
      A9=33./36.
      A10=35./36.
    1 CALL RANDU (NSTART,NOUT,D)
      IF (D-A1) 2,29,3
    2 NPOINT = -2
      GO TO 30
    3 IF (D-A2) 4,29,5
    4 NPOINT = 0
      GO TO 30
    5 IF (D-A3) 6,29,7
    6 NPOINT = 4
      GO TO 30
    7 IF (D-A4) 8,29,9
    8 NPOINT = 5
      GO TO 30
    9 IF (D-A5) 10,29,11
   10 NPOINT = 6
      GO TO 30
   11 IF (D-A6) 12,29,13
   12 NPOINT = 7
      GO TO 30
   13 IF (D-A7) 14,29,15
   14 NPOINT = 8
      GO TO 30
   15 IF (D-A8) 16,29,17
   16 NPOINT = 9
      GO TO 30
   17 IF (D-A9) 18,29,19
   18 NPOINT = 10
      GO TO 30
   19 IF (D-A10) 27,29,28
   27 NPOINT = 11
      GO TO 30
   28 NPOINT = 12
      GO TO 30
   29 NSTART = NOUT
      GO TO 1
```

```
   30 RETURN
      END
$DATA
```