HOMEOMORPHIC SUBSPACES IN THE PLANE

by

Mustafa Dahir

A Thesis Submitted to
the Faculty of the Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
February, 1972

Approved by

Hughes B. Hoyle, III
Thesis Advisor

DAHIR, MUSTAFA. Homeomorphic Subspaces in the Plane. (1972)
Directed by: Dr. Hughes B. Hoyle, III pp. 27

In this thesis the author discusses the topological equivalence of spaces imbedded in the plane. The material is divided into three categories: locally compactness, locally connectedness, and connected im kleinen. Examples of non-homeomorphic spaces are presented in each category.

APPROVAL SHEET

This thesis has been approved by the following committee of the Faculty of the Graduate School at The University of North Carolina at Greensboro.

Thesis Advisor Hughes B. Hoyle, III

Oral Examination Committee Members Karl Ray Gentry

E E Posey

March 22, 1972
Date of Examination

ACKNOWLEDGMENT

The author wishes to express his appreciation to Dr. Hughes B. Hoyle for his patience and guidance during the preparation of this thesis.

TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

The idea of topological equivalence, or homeomorphic, is one of the basic considerations in any study of topology. Mrs. Yandell [3], in her master's thesis, compared pairs of spaces in the plane to determine whether or not they were homeomorphic. Decisions as to whether or not a pair of spaces were homeomorphic were based on several topological properties, including compactness and connectedness. In this thesis three additional topological properties are defined and used for the purpose of increasing the number of decisions that could be made with only the topological properties discussed in [3].

In Chapter I, locally compact is defined and general theorems are proved concerning this property. In addition localy compactness is shown to be a topological property.

In Chapter II, locally connected is defined, general theorems are proved, and local connectedness is shown to be a topological property.

In Chapter III, connected im kleinen is defined, related to local connectedness, and shown to be a topological property.

In Chapter IV, examples are given to show that indeed the studies in [3] have been extended.

If (X, T) is a topological space, then (X, T) is said to be <u>regular</u> provided if C is a closed subset of X and $p \in X - C$, then there exist disjoint open sets U and V such that $p \in U$ and $C \subset V$.

If M is a subset of the real numbers and b is a real number, then b is said to be an upper bound of M provided if $m \in M$, then $m \leq b$. If h is a real number, then h is said to be the least upper bound of M provided h is an upper bound of M and if b is an upper bound of M, then $h \leq b$.

If D is a subset of the plane, then D is said to be a disc provided there is a point p in the plane and a positive number r such that D is the set of all points x in the plane such that the distance from p to x is less than r.

CHAPTER I

LOCALLY COMPACT SPACES

Definition 1: Let (X, T) be a topological space. Then $\S$ is said to be an open cover of X provided $\S$ is a collection of open subsets of X, and if $x \in X$, then there exists a $U \in \S$ such that $x \in U$.

Definition 2: If $\S$ and Σ are both open covers of X, then Σ is said to be a subcover of $\S$ provided $\Sigma \subset \S$.

Definition 3: The space (X, T) is said to be compact provided if $\S$ is an open cover of X, then there exists a finite subcover of $\S$.

Definition 4: Let (X, T) be a topological space, and $A \subset X$. Then the relative topology T_A for A is the collection to which U belongs only in case there exists $V \in T$ such that $U = V \cap A$.

Theorem 1: If (X, T) is a compact topological space and A is a closed subset of X, then (A, T_A) is compact.

Proof: Let $\S$ be an open cover of A. For each $U \in \S$, let V_U be an element of T such that $U = V_U \cap A$. Now A is closed. Then $X - A$ is open. Let $\Sigma = \{V_U \mid U \in \S\} \cup \{X - A\}$. Then Σ is an open cover of X and since X is compact, there exists a finite subcover Ψ of Σ. Let $\Theta = \{M \cap A \mid M \in \Psi\}$. Then Θ is a finite subcover of $\S$ and hence (A, T_A) is compact.

Definition 5: A topological space (X, T) is said to be a Hausdorff Space provided if p and q are points of X, then there are open sets U and V such that $p \in U$, $q \in V$ and $U \cap V = \phi$.

Theorem 2: Every compact subspace of a Hausdorff space (X, T) is closed.

Proof: Let (A, T_A) be a compact subspace of the Hausdorff space (X, T) and let $p \in (X - A)$. For each $x \in A$, there exist sets U_x and V_x in T such that $x \in U_x$, $p \in V_x$, and $U_x \cap V_x = \phi$. Let $\S = \{U_x \mid x \in A\}$. Let $\Sigma = \{A \cap U \mid U \in \S\}$. Then Σ is a cover of A by elements in T_A. Since (A, T_A) is compact, there is a finite subcover, $A \cap U_{x_1}$, $A \cap U_{x_2}$, . . ., $A \cap U_{x_n}$, of Σ. Let $V = \bigcap_{i=1}^{n} V_{x_i}$. Clearly $p \in V$, and V does not intersect A. To show $V \cap A = \phi$, suppose $x \in V \cap A$. Then $x \in A$, and $A \cap U_{x_1}$, $A \cap U_{x_2}$, . . ., $A \cap U_{x_n}$ covers A. So there is a positive integer i such that $x \in A \cap U_{x_i}$. Therefore $x \in U_{x_i}$ and $x \in V_{x_i}$, which implies that $x \in V$. But this is impossible. Hence $V \cap A = \phi$. Since no point in X - A is a limit point of A, A is closed.

Definition 6: A topological space (X, T) is said to be locally compact provided if $p \in X$, then there is an open set U such that $p \in U$ and $\overline{U}$ is compact.

Lemma 1: If (X, T) is a topological space and $A \subset X$, C is a closed subset of X, and $C \subset A$, then C is closed in (A, T_A).

Proof: Let (X, T) be a topological space, $A \subset X$ and C be a closed subset of X so that $C \subset A$. Then $X - C \in T$.

Therefore, $(X - C) \cap A \in T_A$. Thus $A - [(X - A) \cap A]$ is closed in (A, T_A). Since $C = A - [(X - C) \cap A]$, C is closed in (A, T_A).

<u>Lemma 2</u>: If (X, T) is a topological space, and A and B are subsets of X such that $A \subset B$, then $(T_B)_A = T_A$.

<u>Proof</u>: Let (X, T) be a topological space. Let A and B be subsets of X such that $A \subset B$. Let $W \in (T_B)_A$. Then there is a $V \in T_B$ such that $W = V \cap A$. There is a $U \in T$ such that $V = B \cap U$. Then $W = V \cap A = (B \cap U) \cap A = (A \cap B) \cap U = A \cap U$, and hence $W \in T_A$. Now let $W \in T_A$. Then there is a $U \in T$ such that $W = A \cap U$. But $A \cap U = (A \cap B) \cap U = A \cap (B \cap U)$, and thus $W \in (T_B)_A$. Thus $(T_B)_A = T_A$.

<u>Lemma 3</u>: Let (X, T) be a topological space. Let A and B be subsets of X such that $A \subset B$. If $A \in T_B$ and $B \in T$, then $A \in T$.

<u>Proof</u>: Since $A \in T_B$, there is a $U \in T$ such that $A = U \cap B$. Since $U, B \in T$ and T is a topology, A is in T.

<u>Definition 7</u>: If (X, T) is a topological space and A and B are subsets of X such that $A \subset B$, then to say that A is <u>closed</u> [<u>open</u>] in B means that A is a closed [open] subset of (B, T_B).

<u>Lemma 4</u>: Let (X, T) be a topological space. Let A and B be subsets of X such that A is closed [open] in B and B is closed [open] in X. Then A is closed [open] in X.

<u>Proof</u>: By Lemma 3, if A is open in B and B is open in X, then A is open in X. Since the complement of an open set is closed, the remainder of Lemma 4 follows immediately.

Definition 8: Let (X, T) be a topological space, and let $A \subset X$. Then $\overline{A}^T$ is the closure of A in the topological space (X, T).

Theorem 3: Let (X, T) be a locally compact space, and let A be a closed subset of X. Then (A, T_A) is locally compact.

Proof: Let $x \in A$. Since (X, T) is locally compact, there exists a $U \in T$ such that $x \in U$ and $(\overline{U}^T, T_{\overline{U}^T})$ is compact. Let $V = U \cap A$. Then $V \in T_A$ which contains x. Thus it remains to show that $(\overline{V}^{T_A}, T_{A_{\overline{V}^{T_A}}})$ is compact. Let $C = \overline{U}^T \cap A$. Since C is closed in X and $C \subset \overline{U}^T$, by Lemma 1, C is a closed subset of $(\overline{U}^T, T_{\overline{U}^T})$. Thus by Theorem 1, $(C, (T_{\overline{U}^T})_C)$ is compact. By Lemma 1, C is a closed subset of (A, T_A). Since $V \subset C$, $\overline{V}^{T_A} \subset C$. Thus by Theorem 1, $(\overline{V}^{T_A}, ((T_{\overline{U}^T})_C)_{\overline{V}^{T_A}})$ is compact. By Lemma 2, $((T_{\overline{U}^T})_C)_{\overline{V}^{T_A}} = (T_C)_{\overline{V}^{T_A}} = T_{\overline{V}^{T_A}}$. Hence $(\overline{V}^{T_A}, T_{\overline{V}^{T_A}})$ is compact and hence (A, T_A) is locally compact.

Remark 1: The author, having worried about in which spaces the closures are being taken and having worried about the transitivity of subspace topologies and having discovered (see Theorem 3) that these ideas do indeed work as one would expect, will no longer be concerned with such esoteric problems.

<u>Theorem 4</u>: Let (A, T_A) and (B, T_B) be locally compact subspaces of a locally compact space (X, T). Then $A \cap B$ is also locally compact.

<u>Proof</u>: Let $x \in A \cap B$. Then there are elements $U \in T_A$ and $V \in T_B$ such that $x \in U$ and $x \in V$, and $\overline{U}^{T_A}$ and $\overline{V}^{T_B}$ are compact in A and in B respectively. Now $U \cap V \in T_{A \cap B}$. And clearly since the intersection of compact sets is compact, $(U \cap V^{T_{A \cap B}}, (T_{A \cap B})_{\overline{U}}{}^{T_{A \cap B}})$ is compact. Thus $A \cap B$ is locally compact.

<u>Remark 2</u>: Since the notation for closures with respect to a particular subapace topology is very unwieldy, such notation will be omitted in the future when it is clear in what spaces closures are being taken.

<u>Theorem 5</u>: Let (X, T) be regular and let $U \in T$ and $x \in U$. Then there is an open set V containing x such that $x \in V \subset \overline{V} \subset U$.

<u>Proof</u>: Since U is open, $X - U$ is closed and does not contain x. Hence there exist open sets W and V such that $(X - U) \subset W$, $x \in V$ and $W \cap V = \phi$. Since $(X - U) \subset W$, it follows that $(X - W) \subset U$. Since $W \cap V = \phi$, $V \subset (X - W)$. Thus $x \in V \subset \overline{V} \subset (X - W) \subset U$.

<u>Example 1</u>: Let X be the plane and T the usual topology for X. Let $A = \{(x,y) \mid x > 0\}$ and $B = \{(0, 0)\}$. Then (A, T_A) and (B, T_B) are locally compact spaces but $(A \cup B, T_{A \cup B})$ is not locally compact.

Proof: Let $p \in A$ and let U be an open set containing p. Since A is an open subset of X, there exists a disc D such that $p \in D \subset U$. Since (X, T) is regular, there exists a disc E such that $p \in \bar{E} \subset D \subset U$. But E is closed and bounded and hence compact. Hence (A, T_A) is locally compact. Obviously (B, T) is locally compact. Let U be an open set containing $(0, 0)$. Then there is a disc D such that $(0, 0) \in D$ and $D \cap A \subset U$. It is clear that D cannot be compact because points on the y-axis close to $(0, 0)$ must be limit points of D which are not in D. Thus U cannot be compact.

Theorem 6: Every compact Hausdorff space is regular.

Proof: Let (X, T) be a compact Hausdorff space. Let C be a closed subset of X, and let $p \in (X - C)$. Then for each $x \in C$, there are open sets $U_x \in T$ and $V_x \in T$ such that $p \in U_x$, $x \in V_x$, and $U_x \cap V_x = \phi$. The collection $\{V_x \cap C \mid x \in C\}$ is a collection of elements of T_C covering C. It follows from Theorem 1 that (C, T_C) is compact. Hence there exists a finite subcover $C \cap V_{x_1}$, $C \cap V_{x_2}$, . . . , $C \cap V_{x_n}$ such that $\{V_{x_i} \cap C \mid 0 \leq i \leq n\}$ is a covering of C. Let $V = \bigcup_{i=1}^{n} V_{x_i}$ and $U = \bigcap_{i=1}^{n} U_{x_i}$. Then $p \in U$, $C \subset V$, $U \in T$, $V \in T$, and $U \cap V = \phi$.

Theorem 7: Let (X, T) be locally compact and regular, and let A be an open subset of X. Then (A, T_A) is locally compact.

Proof: Let $x \in A$. There is a $U \in T$ so that $x \in U$ and U is compact. Since $x \in U \cap A$, and $U \cap A \in T$, and (X, T) is regular, there is a $V \in T$ so that $x \in V \subset \overline{V} \subset U \cap A$. Since U is compact, V is compact and hence (A, T_A) is locally compact.

Definition 9: Let (X, T) be a topological space and let Θ be a collection of subsets of X. Then Θ is a base (or a neighborhood system) for T if and only if T consists of the empty set and those sets that are unions of sets in Θ.

Definition 10: Let (X, T) be a topological space. Let Θ be a collection of subsets of X, and let Ψ be the collection of all sets that are intersections of finitely many sets in Θ. Then Ψ is a subbase for T if and only if Θ is a base for T.

Definition 11: Let R denote the set of real numbers. For each $p \in R$ and $\epsilon > 0$, let $N_\epsilon(p) = \{q \mid q \in R \text{ and } |p - q| < \epsilon\}$. Let $\S = \{N_\epsilon(p) \mid p \in R, \ \epsilon > 0\}$ and let Ψ be that topology for R such that $\S$ is a base for Ψ. Then Ψ is called the Euclidean topology (or the usual topology) for R.

Definition 12: If $\{X_\alpha \mid \alpha \in \Gamma\}$ is a collection of nonempty sets, then the Cartesian product $X\{X_\alpha \mid \alpha \in \Gamma\}$ of $\{X_\alpha \mid \alpha \in \Gamma\}$ is the set of all functions $x : \Gamma \rightarrow \cup\{X_\alpha \mid \alpha \in \Gamma\}$ such that $x(\alpha) \in X_\alpha$ for each $\alpha \in \Gamma$.

Definition 13: Let $\{X_\alpha \mid \alpha \in \Gamma\}$ be a collection of nonempty sets. For each $\beta \in \Gamma$, the β^{th} projection for $X\{X_\alpha \mid \alpha \in \Gamma\}$

is the function π_β from $X\{X_\alpha \mid \alpha \in \Gamma\}$ onto X_β defined by $\pi_\beta(x) = x(\beta)$ for each $x \in X\{X_\alpha \mid \alpha \in \Gamma\}$.

<u>Definition 14</u>: Let $\{(X_\alpha, T_\alpha) \mid \alpha \in \Gamma\}$ be a collection of topological spaces. Let $X = X\{X_\alpha \mid \alpha \in \Gamma\}$. For each $\alpha \in \Gamma$ let π_α be the α^{th} projection for X. Let Θ be the collection for those sets S such that $S = \pi_\alpha^{-1}(U_\alpha)$, where $\alpha \in \Gamma$ and $U_\alpha \in X_\alpha$. Let T be the topology for X such that Θ is a subbase for T. Then (X, T) is called the <u>product space</u> of $\{(X_\alpha, T_\alpha) \mid \alpha \in \Gamma\}$ and is denoted by $X\{(X_\alpha, T_\alpha) \mid \alpha \in \Gamma\}$. Also for each $\alpha \in \Gamma$, the α^{th} <u>coordinate space</u> of (X, T) is (X_α, T_α).

<u>Theorem 8</u>: Every finite product of locally compact spaces is locally compact.

<u>Proof</u>: Let (X_1, T_1), (X_2, T_2), . . ., (X_n, T_n) be locally compact spaces. Let (X, T) denote the product space, and let $x \in X$. Then $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in X_i$ for $1 \leq i \leq n$. For each i there exists a $U_i \in T_i$ so that $x_i \in U_i$, and $\overline{U}_i$ is compact in X_i. Let $U = X\{U_i \mid 1 \leq i \leq n\}$. Then $x \in U$, and also $\overline{U} = X\{\overline{U}_i \mid 1 \leq i \leq n\}$ is compact. Therefore (X, T) is locally compact.

<u>Definition 15</u>: Let (X, S) and (Y, T) be topological spaces. The function $f: (X, S) \to (Y, T)$ is <u>continuous</u> provided if $U \in T$, then $f^{-1}(U) \in S$.

<u>Definition 16</u>: Let (X, S) and (Y, T) be topological spaces. The function $f: (X, S) \to (Y, T)$ is a <u>homeomorphism</u>

provided f is a one-to-one, onto, and continuous function, and f^{-1} is also a continuous function.

Theorem 9: Let (X, S) and (Y, T) be topological spaces such that X is compact. Then if $f: (X, S) \to (Y, T)$ is onto and continuous, (Y, T) is also compact.

Proof: Let $f: (X, S) \to (Y, T)$ be continuous. Let $\S$ be an open cover of Y. Then since $f(X) = Y \subset \cup \{U \mid U \in \S\}$, it follows that $X \subset f^{-1}(f(X)) \subset f^{-1}(\cup \{U \mid U \in \S\}) = \cup\ f^{-1}(\{U \mid U \in \S\})$. Thus $\Psi = \{f^{-1}(U) \mid U \in \S\}$ is an open cover of X and hence has a finite subcover U_1, U_x, . . . ,U_n. So
$Y = f(X) \subset f[f^{-1}(U_1) \cup f^{-1}(U_2) \cup \ldots \cup f^{-1}(U_n)] \subset U_1 \cup U_2 \cup \ldots \cup U_n$.
Thus Y is compact.

Theorem 10: Let (X, S) and (Y, T) be topological spaces. Let $f: (X, S) \to (Y, T)$ be a homeomorphism. If (X, S) is locally compact, then (Y, T) is locally compact.

Proof: Suppose (X, S) is locally compact. Let $p \in Y$. Then there is a $q \in X$ such that $f(q) = p$. Since (X, S) is locally compact, there is a $U \in T$ so that $q \in U$ and $\overline{U}$ is compact. Then by Theorem 9, $f(\overline{U})$ is compact. It remains to show that $f(\overline{U}) = \overline{f(U)}$. Let K be a closed subset of Y containing $f(U)$. Then $f^{-1}(K)$ is a closed subset of X containing U. Hence $\overline{U} \subset f^{-1}(K)$. Thus $f(\overline{U}) \subset K$. Therefore $f(\overline{U}) \subset \overline{f(U)}$. Since $\overline{U}$ is closed, $f(\overline{U})$ is closed, and clearly $f(U) \subset f(\overline{U})$. Thus $\overline{f(U)} \subset f(\overline{U})$. So $f(\overline{U}) = \overline{f(U)}$, which is compact. Thus $\overline{f(U)}$ is compact and hence (Y, T) is locally compact.

CHAPTER II

LOCALLY CONNECTED SPACES

Definition 17: Let (X, T) be a topological space. Then (X, T) is connected provided X is not the union of two nonempty disjoint open sets.

Definition 18: Let (X, T) be a topological space, and let A and B be subsets of X. Then A and B are said to be mutually separated in (X, T) if and only if $\bar{A} \cap B = \phi = A \cap \bar{B}$.

Remark 3: Clearly a topological space (X, T) is connected if and only if X is not the union of two nonempty mutually separated subsets.

Definition 19: Let X be a nonempty set. Let $T = \{A \mid A \subset X\}$. Then clearly T is a topology for X, and T is called the discrete topology for X.

Example 2: Let (X, T) be a topological space such that X consists of two or more elements. Let T be the discrete topology for X. Then (X, T) is not connected.

Proof: Let $x \in X$. Then clearly the sets $\{x\}$ and $X - \{x\}$ are nonempty disjoint open sets. It is also immediately evident that $X = \{x\} \cup (X - \{x\})$.

<u>Definition 20</u>: A subset A of a topological space (X, T) is connected provided (A, T_A) is connected.

<u>Lemma 5</u>: Let (X, T) be a topological space such that $X = U \cup V$ where U and V are mutually separated. If C is a connected subset of X, then either $C \subset U$ or $C \subset V$.

<u>Proof</u>: Suppose $C \cap U \neq \phi$ and $C \cap V \neq \phi$. Then $C = (C \cap U) \cup (C \cap V)$ where $C \cap U$ and $C \cap V$ are nonempty mutually separated subsets of (C, T_C). Thus either $C \subset U$ or $C \subset V$.

<u>Theorem 11</u>: Let (X, T) be a topological space and let A be a connected subset of X. Let Y be a subset of X such that $A \subset Y \subset \overline{A}$. Then Y is connected.

<u>Proof</u>: Suppose (Y, T_Y) is not connected. Then $Y = U \cup V$, where U and V are nonempty mutually separated subsets of Y. Since A is a connected, by Lemma 5, either $A \subset U$ or $A \subset V$, say $A \subset U$. Since $V \neq \phi$, there exists a point $p \in V$. Thus $p \notin A$. Since U and V are mutually separated, p is not a limit point of A in (Y, T_Y). Thus p is not a limit point of A in (X, T). Therefore $p \notin \overline{A}$. But this is impossible. Hence (Y, T_Y) is connected.

<u>Definition 21</u>: Let (X, T) be a topological space, and let $C \subset X$. Then C is said to be a <u>component</u> of X provided C is connected and is not properly contained in any other connected subset of X.

Theorem 12: Let (X, T) be a topological space, and let C be a component of X. Then C is closed.

Proof: By Theorem 10 since C is connected, $\overline{C}$ is connected. But $C \subset \overline{C}$, and since C is the largest connected subset of X, $C = \overline{C}$. Hence C is closed.

Theorem 13: Let (X, T) be a topological space, and let $\{C_i \mid i \in I\}$ be a collection of connected subsets of X such that $X = \cup C_i$ and $\cap C_i \neq \phi$. Then (X, T) is connected.

Proof: Suppose X is not connected. Then $X = U \cup V$, where U and V are nonempty mutually separated subsets of X. Since $U \neq \phi$ and $V \neq \phi$, there exist $i, j \in I$ so that $C_i \subset U$ and $C_j \subset V$. But since $C_i \cap C_j \neq \phi$, this is impossible. So X is connected.

Definition 22: Let R denote the set of real numbers. Let $a, b \in R$. Then the open interval from a to b, denoted by (a, b), is the set of all real numbers x such that $a < x < b$. The closed interval from a to b, denoted by $[a, b]$, is the set of all real numbers x such that $a \leq x \leq b$. The interval half-open on the left, denoted by $(a, b]$, is the set of all real numbers x such that $a < x \leq b$. The interval half-open on the right, denoted by $[a, b)$, is the set of all real numbers x such that $a \leq x < b$. The notation (a, ∞) denotes the set of all real numbers greater than a. The notation $[a, \infty)$ denotes the set of all real numbers greater than or equal to a.

The notation $(-\infty, a)$ denotes the set of all real numbers less than a. The notation $(-\infty, a]$ denotes the set of all real numbers less than or equal to a.

<u>Theorem 14</u>: Any interval is connected.

<u>Proof</u>: Let X be an interval, and suppose that X is not connected. Than $X = A \cup B$, where A and B are open in X, neither A nor B is the empty set, and $A \cap B = \phi$. Let $a \in A$, and $b \in B$. Since A and B are disjoint, $a \neq b$. So either $a < b$ or $b < a$. Assume that $a < b$. Let $V = \{x \mid x \in [a, b] \text{ and } [a, x) \subset A\}$, and let $v = \text{lub } V$. Since A is open, A has no largest element. Thus $v = a$. So $a < v \leq b$. Now $v \in \bar{A}$, and A is the complement in X of B. But B is open, so A is closed. Thus $A = \bar{A}$, and $v \in A$. Now A is also open, so there exists a real number $c > 0$ such that $(v - c, v + c) \subset A$. Then $[a, v + c) \subset A$, and hence $v + c \in V$. But then $v + c$ is an element of V which is greater than the least upper bound for V. This is impossible. So any interval is connected.

<u>Definition 23</u>: Let (X, T) be a topological space. Then (X, T) is <u>locally connected</u> at a point $x \in X$ provided, if U is an open set containing x, then there exists a connected open set V such that $x \in V \subset U$. The space (X, T) is said to be <u>locally connected</u> provided it is locally connected at each point.

<u>Theorem 15</u>: A space (X, T) is locally connected if and only if each component of each open set is open.

<u>Proof</u>: Let (X, T) be a topological space. Suppose that (X, T) is connected. Let $U \in T$. Let C be a component of U and let x be a point of C. Since U is open and X is locally connected, there is an open and connected set V such that $x \in V \subset U$. Then by Theorem 13, $C \cup V$ is connected. But $C \subset C \cup V$. Thus $C = C \cup V$, and so $x \in V \subset C$. Therefore C is open.

Suppose each component of each open set is open. Let $x \in X$ and $U \in T$ such that $x \in U$. Let C be the component of U which containins x. Then C is open and connected, and $x \in C \subset U$. Thus (X, T) is locally connected.

<u>Theorem 16</u>: Let (X, S) and (Y, T) be topological spaces, and let $f\colon (X, S) \to (Y, T)$ be continuous and onto. If (X, S) is connected, then (Y, T) is connected.

<u>Proof</u>: Let (X, S) be connected. Suppose (Y, T) is not connected. Then there exist disjoint nonempty open sets U and V so that $Y = U \cup V$. Then $X = f^{-1}(Y) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. Since f is continuous $f^{-1}(U)$, $f^{-1}(V) \in S$. Since f is onto $f^{-1}(U) \neq \phi \neq f^{-1}(V)$ and clearly $f^{-1}(U) \cap f^{-1}(V) = \phi$. But this is impossible since X is connected. Thus (Y, T) is connected.

<u>Theorem 17</u>: Let (X, S) and (Y, T) be topological spaces, and let $f\colon (X, S) \to (Y, T)$ be a homeomorphism. Then if (X, S) is locally connected, (Y, T) is locally connected.

Proof: Suppose (X, S) is locally connected. Let $y \in Y$, and let U be an open set containing y. Then $f^{-1}(y) \in X$ and $f^{-1}(U)$ is an element of S containing $f^{-1}(y)$. Since X is locally connected, there exists a connected open subset V of X such that $f^{-1}(y) \in V \subset f^{-1}(U)$. But then $y \in f(V) \subset U$. Since f is a homeomorphism, f(V) is open. By Theorem 16, f(V) is connected. Thus (Y, T) is locally connected.

Example 3: Let X denote the rational numbers. Let T be the discrete topology for X, and let T be the usual topology for X as a subspace of the real numbers. Let $f: (X, S) \to (X, T)$ be defined by $f(x) = x$ for each $x \in X$. Then (X, S) is locally connected, (X, T) is not locally connected, and f is continuous.

Proof: Let $x \in X$. Since $x \in S$, and x is connected, (X, S) is locally connected. However, since no interval of rational numbers is connected in the usual subspace topology, (X, T) is not locally connected. Clearly f is a continuous function.

Theorem 18: Every finite product of locally connected spaces is locally connected.

Proof: Let (X_1, T_1), (X_2, T_2), . . . , (X_n, T_n) be locally connected spaces. Let (X, T) denote the product space, and let $x \in X$. Then $x = (x_1, x_2, \ldots, x_n)$, where $x_i \in X_i$. Let $U \in T$ so that $x \in U$. Then there exists a basic open set V so so that $x \in V \subset U$. Now $V = X\{V_i \mid 1 \leq i \leq n\}$, where each $V_i \in T_i$. Since each T_i is locally connected, there is a

$W_i \in T_i$ so that W_i is connected and $x_i \in W_i \subset V_i$. But then $X\{W_i \mid 1 \leq i \leq n\}$ is both open and connected, and $x \in X\{W_i \mid 1 \leq i \leq n\} \subset X\{V_i \mid 1 \leq i \leq n\} \subset V \subset U$. Thus X is locally connected.

Definition 24: Let X_i, where $1 \leq i \leq n$, denote the real numbers. Let $E^n = X\{X_i \mid 1 \leq i \leq n\}$. Then E^n is called Euclidean n-space, and the topology for E^n is the usual product topology.

Theorem 19: For each natural number n, E^n is locally connected.

Proof: Since, from Theorem 14, any interval is connected, clearly the real numbers are locally connected. Thus from Theorem 18, E^n, for any natural number n, is locally connected.

Theorem 20: Let (X, T) be a locally connected space, and let $L \in T$. Then (L, T_L) is locally connected.

Proof: Let $p \in L$ and let U be an element of T_L containing p. Since $L \in T$, then $U \in T$. Thus there is a connected element V of T such that $p \in V \subset U$. Since $V \subset L$, $V \in T_L$. Since V is a connected subset of X, V is a connected subset of L. Hence (L, T_L) is locally connected.

Example 4: Let X be the subset of the plane defined by $X = \{(x, y) \mid -1 \leq y \leq 1,\ x = 0\} \cup \{(x, y) \mid y = \sin(1/x),\ 0 < x \leq 1\}$. Let T be the relative topology for X induced by the usual plane topology. Then (X, T) is connected but not locally connected.

Proof: Let $A = \{(x, y) \mid -1 \leq y \leq 1, \ x = 0\}$ and let $B = \{(x, y) \mid y = \sin(1/x), \ 0 < x \leq 1\}$. Since A is an interval, A is connected. Since the sine funtion is continuous and $(0, 1]$ is connected, B is connected. But every point of A is a limit point of B. Thus X is connected. Let $p \in A$. A basic open set in X containing p is of the form of a disc D intersected with X. But the only connected subset of X containing p and contained in D is $D \cap A$, and this set is not open in X. Hence (X, T) is not locally connected at p.

Example 5: Let $X = (0, 1) \cup (2, 3)$ and let T be the relative topology for X induced by the usual topology for the reals. Then (X, T) is locally connected but not connected.

Proof: Since (0, 1) and (2, 3) are nonempty disjoint elements of T, (X, T) is not connected. By Theorem 19, the reals are locally connected. Since X is an open subset of the reals, by Theorem 20, (X, T) is locally connected.

CHAPTER III

CONNECTED IM KLEINEN SPACES

Definition 25: A topological space (X, T) is said to be connected im kleinen at p provided if $U \in T$ and $p \in U$, there is a $V \in T$ such that $p \in V \subset U$ and if $a, b \in V$, there is a connected subset C of X such that $\{a, b\} \subset C \subset U$. If (X, T) is connected im kleinen at each point, then it is connected im kleinen.

Theorem 21: Let (X, T) be a topological space which is connected im kleinen. Then (X, T) is locally connected.

Proof: Let U be an open subset of X, and let C be a component of U. Let x be a point of C. Then there is an open set V_x containing x and lying in U, such that each point y of V_x is in a connected set C_{xy} lying in U. Then C_{xy} is a subset of C, so V_x lies in C. Thus $C = \cup\{V_x \mid x \in C\}$ is open. Then by Theorem 15, (X, T) is locally connected.

Remark 4: If (X, T) is locally connected at p, then (X, T) is connected im kleinen at p.

Definition 26: Let (X, T) be a topological space. Let C be a subset of X and let $p \in C$. Then p is said to be an interior point of C provided there exists an element $U \in T$ such that $p \in U \subset C$.

Theorem 22: Let (X, T) be a topological space, let $p \in X$, and let C be the component of X containing p. If X is connected im kleinen at p, then p is an interior point of C.

Proof: Suppose X is connected im kleinen at p. Since X is an open set containing p and X is connected im kleinen at p, there exists an open set V such that $p \in V \subset X$ and if a and b are in V, then there is a connected subset D of X containing a and b. Let $x \in V$. Then there is a connected subset D of X containing x and p. Since $p \in D \cap C$, and D and C are connected, by Theorem 13, $D \cup C$ is connected. Since C is a component of X, $D \cup C \subset C$. Thus $x \in C$ and hence $V \subset C$. Hence p is an interior point of C.

Theorem 23: Let (X, S) and (Y, T) be topological spaces. Let $f\colon (X, S) \to (Y, T)$ be a homeomorphism, and let $p \in X$. If X is connected im kleinen at p, then Y is connected im kleinen at $f(p)$.

Proof: Suppose X is connected im kleinen at p. Let U be an open set containing $f(p)$. Then since f is continuous, $f^{-1}(U)$ is an open set containing p. Since X is connected im kleinen at p, there exists an element V of S such that $p \in V \subset f^{-1}(U)$, and if a and b are in V, then there is a connected subset C of X such that $\{a, b\} \in C \subset f^{-1}(U)$. Since f^{-1} is continuous, $f(V)$ is open. Clearly $f(p) \in f(V) \subset U$. Let a and b be elements of $f(V)$. Then $f^{-1}(a)$ and $f^{-1}(b)$ are elements of V. Thus there exists a connected subset C of X such that $\{f^{-1}(a), f^{-1}(b)\} \subset C \subset f^{-1}(U)$. Clearly $\{a, b\} \subset f(C) \subset U$ and since f is continuous, by Theorem 16, $f(C)$ is connected. Hence Y is connected im kleinen at $f(p)$.

Example 6: For each ordered pair (i, j) of positive integers let $I_{(i, j)}$ be the subset of the plane which is the closed interval with end points $(1/i, 0)$ and $(1/(i+1), 1/(i+j))$. That is $I_{(i,j)} = \{(x,y) \mid 1/(i+1) \leq x \leq 1/i \text{ and } y=[-(i(i+1))/(i+j)][x-(1/i)]\}$. Let $X = \{I_{(i,j)} \mid i, j \text{ are positive integers}\} \cup [0, 1]$. [See Figure 1.] Let T be the relative topology for X induced by the usual plane topology. Then (X, T) is connected im kleinen at $(0, 0)$, but (X, T) is not locally connected at $(0, 0)$.

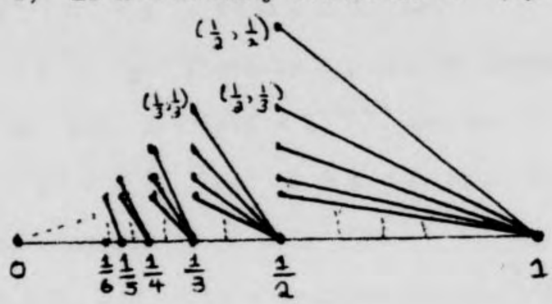

FIGURE 1

Proof: Let D be the subset of the plane defined by $D = \{(x, y) \mid x^2 + y^2 < 1/4\}$ and let $U = D \cap X$. Then $U \in T$, and it is only necessary to exhibit that there is no open connected subset of X which contains $(0, 0)$ and is contained in U. Suppose there exists a connected element $V \in T$ such that $(0, 0) \in V \subset U$. Since $(0, 0)$ is a usual plane limit point of $\{(1/n, 0) \mid n \text{ is a positive integer}\}$, there exists a positive integer n such that $(1/n, 0) \in V$. Let K be the smallest positive integer such that $(1/K, 0) \in V$. Clearly $K > 2$, and $(1/(K-1), 0) \in V$. Since $(1/K, 0)$ is a usual plane limit point of

$\{(1/K, 1/(K+j)) \mid j$ is a positive integer$\}$, there exists a positive integer L such that $(1/K, 1/(K+L)) \in V$. Then $V = (I_{(K, L)} \cap V) \cup (V - I_{(K, L)})$ and $I_{(K, L)} \cap V$ and $V - I_{(K, L)}$ are nonempty mutually separated subsets of X. Hence V is not connected. Thus (X, T) is not locally connected at $(0, 0)$.

Let $U \in T$ so that $(0, 0) \in U$. Then there exists a $r > 0$ such that $\{(x, y) \mid x^2 + y^2 < r\} \cap X \subset U$. Let $D = \{(x, y) \mid x^2 + y^2 < r\}$. There is a positive integer K such that $(1/K, 0) \in D$. Let $S = 1/(K + 1)^{1/2}$, and let $E = \{(x, y) \mid x^2 + y^2 < S^2\}$. Then $E \cap X \in T$, and $(0, 0) \in E \cap X \subset U$. Let a and b be elements of $E \cap X$. Let $C = \{I_{(i, j)} \mid i \geq K$, and j is a positive integer$\} \cup [0, 1/K]$. Then C is a connected subset of U which contains $\{a, b\}$. Hence (X, T) is connected im kleinen.

CHAPTER IV

EXAMPLES

Definition 27: Let (X, S) and (Y, T) be topological spaces. Then (X, S) is said to be homeomorphic to (Y, T) provided there exists a homeomorphism $f: (X, S) \to (Y, T)$.

Example 7: Let X be the subset of the plane defined by $X = \{(x, y) \mid x > 0\}$ and let $Y = X \cup \{(0, 0)\}$. [See Figure 2.] Let S and T be the relative topologies for X and Y, respectively, induced by the usual plane topology. Then (X, S) and (Y, T) are not homeomorphic.

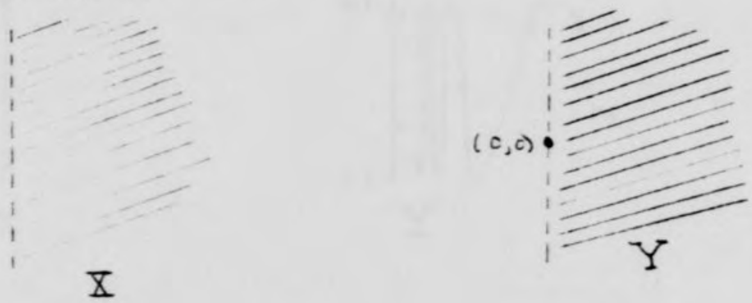

FIGURE 2

Proof: Clearly (X, S) is locally compact, since closed discs are compact. Now (Y, T) is not locally compact at (0, 0) since any open set containing (0, 0) has the property that its closure has limit points in the plane not in its closure, namely points on the y-axis which are close to 0. Since (X, S) is locally compact and (Y, T) is not locally compact, by Theorem 10, (X, S) and (Y, T) are not homeomorphic.

Remark 5: It is worth observing that in Example 7, the use of the properties of compact or connected instead of locally compact would not prove that (X, S) and (Y, T) are not homeomorphic, since both (X, S) and (Y, T) are connected, while neither is compact.

Example 8: Let X be the subset of the plane defined by $X = \{(x, y) \mid x = 0, -1 \leq y \leq 1\}$ and let $Y = X \cup \{(x, y) \mid 0 < x \leq 1, \ y = \sin(1/x)\}$. [See Figure 3.] Let S and T be the relative topologies for X and Y, respectively, induced by the usual plane topology. Then (X, S) and (Y, T) are not homeomorphic.

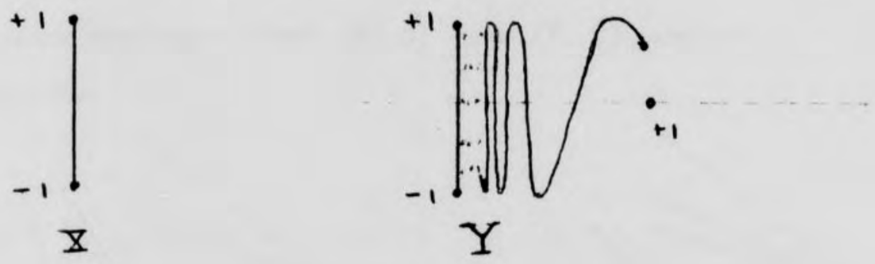

FIGURE 3

Proof: Since open intervals are connected, (X, S) is locally connected. Since basic open sets about (0, 0) in Y are discs intersected with Y, basic open sets about (0, 0) are not connected. Hence (Y, T) is not locally connected at (0, 0). Since (X, S) is locally connected and (Y, T) is not locally connected, by Theorem 17, (X, S) and (Y, T) are not homeomorphic.

Remark 6: It is worth observing in Example 8 that the use of the properties of compact, connected, or locally compact instead of locally connected would not prove that (X, S) and (Y, T) are not homeomorphic, for both (X, S) and (Y, T) are compact, connected, and locally compact.

Example 9: Let A be the space X in Example 6. Let $B = A - [0, 1]$. For each positive integer K, let

$$C_K = \left\{(x, y) \mid \left[x - \frac{2K + 1}{(K + 1)(K)}\right]^2 + y^2 = \left[\frac{1}{2(K + 1)(K)}\right]^2 ; y \leq 0\right\}.$$

Let $X = B \cup [\, \cup \{C_K \mid K \text{ is a positive integer}\}]$ and let $Y = B \cup [0, 1/2] \cup C_1$. [See Figure 4.] Let S and T be the relative topologies for X and Y, respectively, induced by the usual plane topology. Then (X, S) and (Y, T) are not homeomorphic.

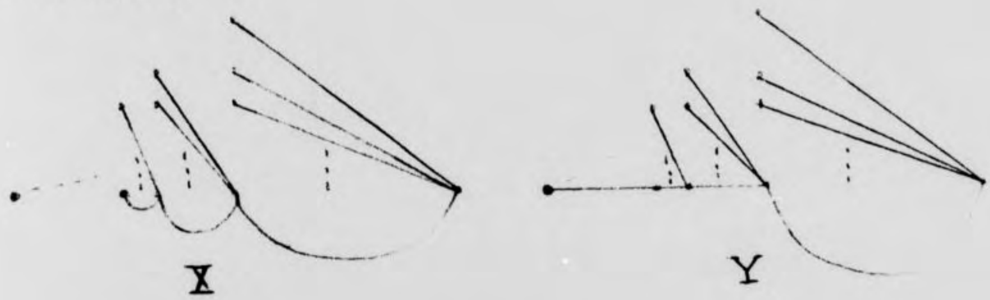

FIGURE 4

Proof: Similarly to Example 6, it is easily shown that (X, S) is connected im kleinen. However (Y, T) is not connected im kleinen at any point of (0, 1/2). Thus by Theorem 23, (X, S) and (Y, T) are not homeomorphic.

Remark 7: It is worth observing that in Example 9, that using the properties of compact, connected, locally compact, or locally compact instead of connected im kleinen would not prove that (X, S) and (Y, T) are not homeomorphic, for both (X, S) and (Y, T) are connected and locally compact, but neither is compact or locally connected.

SUMMARY

In conclusion, the author has defined locally compact, locally connected, and connected im kleinen, and using these ideas, he has distinguished between certain subspaces of the plane. In addition, the examples used to distinguish between pairs of subspaces, have been of such a nature that the properties of compact and connected could not perform the task.

BIBLIOGRAPHY

1. Hocking, John G. and Gail S. Young, Topology, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1961.

2. Willard, Stephen, General Topology, Addison-Wesley, Reading, Massachusetts, 1968.

3. Yendell, Ann Lewis, Homeomorphic Spaces in the Plane, a Master's Thesis from The University of North Carolina at Greensboro, August 1971.