HESTER, SARA OWEN. Connected Topologies for the Unit Interval. (1971) Directed by: Dr. Hughes B. Hoyle, III. Pp. 29.

The author has presented, with proofs, the solution of Hildebrand [2] to the problem proposed by Stallings [1] on the connectedness of the unit interval. It has been shown through Hildebrand's solution that Stallings' question has a negative answer, while, by considering two added conditions, the author has shown an affirmative one.

CONNECTED TOPOLOGIES FOR THE UNIT INTERVAL

by

Sara Owen Hester

A Thesis Submitted to
the Faculty of the Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
June, 1971

Approved by

Hughes B. Hoyle, III
Thesis Adviser

APPROVAL PAGE

This thesis has been approved by the following committee of the Faculty of the Graduate School at The University of North Carolina at Greensboro.

Thesis Adviser Hughes B. Hoyle, III

Oral Examination Committee Members Karl Ray Gentry

Andrew F. Long, Jr.

Gaylord T. Hageseth

April 19, 1971
Date of Examination

ACKNOWLEDGMENT

The author wishes to express sincere appreciation to Dr. Hughes B. Hoyle, III for his assistance and encouragement in the preparation of this thesis.

TABLE OF CONTENTS

INTRODUCTION

In [1] Stallings asks the following question. Let $I = [0,1]$, let T_0 be the usual topology on I, and let T be a topology on I such that $T_0 \subset T$ and (I,T) is connected. Let T_L be the topology on I generated by T and $\{[a,b) \mid 0 \leq a < b \leq 1\}$ and let T_R be the topology generated by T and $\{(a,b] \mid 0 \leq a < b \leq 1\}$. Let L and R be subsets of I such that $I = L \cup R$, $0 \in L$, $1 \in R$, L is T_L-open, and R is T_R-open. Then is it necessarily true that $L \cap R \neq \emptyset$?

An example of a topology on I in which $L \cap R = \emptyset$ was provided by Hildebrand [2] in 1967. It is the purpose of this paper to exhibit, with proofs, the example of Hildebrand.

In Chapter I two conditions are added to the topology T which enable the author to prove that $L \cap R \neq \emptyset$.

In Chapter II several examples of connected topologies on I are exhibited.

In Chapter III Hildebrand's example is given.

The reader is expected to have a working knowledge of point set topology.

Throughout this paper I will denote the closed unit interval $[0,1]$ and T_0 the subspace topology on I inherited from the usual topology on the reals. Also, if (X,T) is a topological space and $A \subset X$, then $T\text{-Cl}(A)$ will denote the closure of A in (X,T).

CHAPTER I

A THEOREM

<u>Theorem 1</u>. [3] Let T be a connected topology for I such that T is finer than T_0. Let T_L be the topology generated by T and $\{[a,b) \mid 0 \leq a < b \leq 1\}$ and let T_R be the topology generated by T and $\{(a,b] \mid 0 \leq a < b \leq 1\}$. Let L and R be subsets of I such that $L \cup R = I$, $0 \in L$, $1 \in R$, L is T_L-open, and R is T_R-open. Suppose that in addition to the above properties T also satisfies the following two conditions:

(1) There is a T_0-dense subset A of I such that if $a \in A$ and U is a T-open set containing a, then there is a T_0-open set V such that $a \in V \subset U$, and

(2) If $p \in I$ and U is a T-open set containing p, then there is a T_0-open interval V containing p such that U is T_0-dense in V.

Then $L \cap R \neq \emptyset$.

Proof: Suppose $L \cap R = \emptyset$. $1 \in R$ and R is T_R-open. Then since T_R is generated by T and $\{(a,b] \mid 0 \leq a < b \leq 1\}$, there is a T-open set U and a real number r such that $U \cap (r,1]$ is T_R-open and $1 \in U \cap (r,1] \subset R$. By condition 2, since $1 \in U$ and U is

T-open, then there is a real number q such that U is T_0-dense in $(q,1]$. Let $s = \max\{q,r\}$. Therefore U is T_0-dense in $(s,1]$. Now $U \cap (s,1] \subset U$, so the $T_0 - Cl(U \cap (s,1]) \cap (s,1] \subset T_0 - Cl(U) \cap (s,1]$. Clearly, any point of U in $(s,1]$ is a point of the $T_0 - Cl(U \cap (s,1]) \cap (s,1]$. Let h be a T_0-limit point of U in $(s,1]$, and let W be a T_0-open set containing h. There is a T_0-open interval J such that $h \in J \subset (s,1]$. Then since $h \in J \cap W$, $J \cap W$ is T_0-open, and h is a T_0-limit point of U in $(s,1]$, there exists $x \in J \cap W$ such that $x \in U \cap (s,1]$. Hence h is a T_0-limit point of $U \cap (s,1]$. Thus the $T_0 - Cl(U) \cap (s,1] \subset T_0 - Cl(U \cap (s,1]) \cap (s,1]$ and therefore the $T_0 - Cl(U \cap (s,1]) \cap (s,1] = T_0 - Cl(U) \cap (s,1]$. But since U is T_0-dense in $(s,1]$, the $T_0\text{-}Cl(U) \supset (s,1]$. Thus the $T_0 - Cl(U \cap (s,1]) \cap (s,1] = T_0 - Cl(U) \cap (s,1] = (s,1]$. Hence the $T_0 - Cl(U \cap (s,1]) \supset (s,1]$ and therefore $U \cap (s,1]$ is dense in $(s,1]$. But since $U \cap (s,1] \subset R$, R is T_0-dense in $(s,1]$. By condition 1 there is a T_0-dense subset A of I such that if $a \in A$ and U is a T-open set containing a, then there is a T_0-open set V such that $a \in V \subset U$. Since A is T_0-dense in I and $(s,1]$ is T_0-open, there is a point $a \in A \cap (s,1]$. For suppose $a' \in (s,1]$ and $a' \notin A$. Then a' is a T_0-limit point of A. Hence the T_0-open set $(s,1]$ containing a' contains a point of A. Suppose $a \in L$. Since

L is T_L-open, there is a T-open set U' and a number t such that $a \in U' \cap [a,t) \subset L$. By condition 1 there are numbers c and s' such that $a \in (c,s') \subset U'$. Hence $[a,s') \subset U'$. If $t \leq s'$, then $[a,t) \subset [a,s') \subset U'$. Thus $[a,t) = U' \cap [a,t) \subset L$. If $t > s'$, then $[a,s') \subset [a,t)$ and since $U' \cap [a,t) \subset L$, $U' \cap [a,s') \subset L$. But $[a,s') \subset U'$, so $[a,s') = U' \cap [a,s') \subset L$. Let $t' = \min \{t,s'\}$. Then $[a,t') \subset L$. Since $a \in (s,1]$, $[a,t') \subset (s,1]$. Then because R is T_0-dense in $(s,1]$, R is T_0-dense in $[a,t')$. Thus there is a point of R in (a,t'). But $L \cap R = \emptyset$ and this is impossible. Thus $a \in R$ and therefore $A \cap (s,1] \subset R$. Suppose there exists a point $y \in L \cap (s,1]$. Since L is T_L-open, there is a T-open set U'' and a number b such that $y \in U'' \cap [y,b) \subset L$. By condition 2, since $y \in U''$ and U'' is T-open, there are numbers m and n such that $y \in (m,n)$ and U'' is T_0-dense in (m,n). Let $d = \min \{n,b\}$. Thus U'' is T_0-dense in $[y,d)$. Now A is T_0-dense in I, so there is a point $p \in A$ such that $p \in (y,d)$. Thus $p \neq y$. But since $y \in (s,1]$, $[y,d) \subset (s,1]$. Thus $p \in A \cap (s,1]$ and $p \in [y,d)$. Since $A \cap (s,1] \subset R$, $p \in R$. Because $p \in R$ and R is T_R-open, there is a T-open set V'' and there is a number e such that $p \in V'' \cap (e,p] \subset R$. By condition 1, since $p \in A$, there are numbers f and f' such that $p \in (f,f') \subset V''$. Hence $(f,p] \subset V''$. Suppose $e \geq f$. Then $(e,p] \subset (f,p] \subset V''$ and thus $(e,p] = V'' \cap (e,p] \subset R$. Suppose $f > e$. Then

$(f,p] \subset (e,p]$. But since $V'' \cap (e,p] \subset R$, $V'' \cap (f,p] \subset R$. However, $(f,p] \subset V''$, so $(f,p] = V'' \cap (f,p] \subset R$. Suppose $y \geq e$ and $y \geq f$. Then $(y,p] \subset (e,p]$ and $(y,p] \subset (f,p]$. If, in addition, $e \geq f$, from above we may conclude that $(y,p] \subset (e,p] \subset R$. But if $f > e$, from above we may conclude that $(y,p] \subset (f,p] \subset R$. Let $g = \max \{e,f,y\}$. Then $(g,p] \subset R$. Since U'' is T_0-dense in $[y,d)$ and $(g,p) \subset [y,d)$, there is a point of $U'' \cap [y,d)$ which is in (g,p) and hence in R. But $U'' \cap [y,d) \subset L$ and $L \cap R = \emptyset$, so this is impossible. Hence $(s,1] \subset R$.

Let $k = \text{g.l.b.} \{x \mid (x,1] \subset R\}$. We wish to show that $k \in R$. Suppose $k \in L$. Since L is T_L-open, $R = I - L$ is T_L-closed. Since R contains its T_L-limit points, then k is not a T_L-limit point of R. Obviously, any number greater than k is in R. Suppose k is not a T-limit point of $(k,1]$. Then there is a T-open set M containing k such that M contains no point of $(k,1]$. Since $T_0 \subset T$, $[0,k)$ and $(k,1]$ are T-open and thus $[0,k] = [0,k) \cup M$ is T-open. Hence $[0,k]$ and $(k,1]$ are disjoint T-open subsets of I whose union is I. But (I,T) is connected. Therefore k is a T-limit point of $(k,1]$. Thus k is a T_L-limit point of $(k,1]$ and hence, since $(k,1] \subset R$, k is a T_L-limit point of R. But this is impossible. Hence $k \in R$. Therefore $[k,1] \subset R$. Since $0 \in L$, $k \neq 0$. Performing exactly the same procedure as in the first part of the proof, we find that there is a number k' such that $(k',k) \subset R$. Thus $(k',1] \subset R$. But $k' < k = \text{g.l.b}\left\{x \mid (x,1] \subset R\right\}$ and this is impossible. Hence $L \cap R \neq \emptyset$.

CHAPTER II

SOME CONNECTED TOPOLOGIES ON I

Example 1. [3] The topology T_0 is a connected topology for I and T_0 also satisfies the following two conditions:

(1) There is a T_0-dense subset A of I such that if $a \in A$ and U is a T_0-open set containing a, then there is a T_0-open set V such that $a \in V \subset U$, and

(2) If $p \in I$ and U is a T_0-open set containing p, then there is a T_0-open interval V containing p such that U is T_0-dense in V.

Proof: The topology T_0 is a connected topology for I. For suppose there are non-empty T_0-open sets M and N such that $M \cap N = \emptyset$ and $I = M \cup N$ with $1 \in M$. There is a number a such that $1 \in (a,1] \subset M$. Let $m = g.l.b.\{x \mid (x,1] \subset M\}$. Suppose $(m,1] \not\subset M$. Then there is a number $y \in (m,1]$ such that $y \notin M$. But $1/2(m+y) > m$, so $(1/2(m+y),1] \subset M$ and $y \in (1/2(m+y),1]$ which is impossible. So $(m,1] \subset M$. Let $(m-\epsilon, m+\epsilon)$ be a basic T_0-open set about m. Now $m + \epsilon/2 \in (m,m+\epsilon) \subset (m,1] \subset M$ and $m + \epsilon/2 \neq m$. Thus m is a T_0-limit point of M. Since N is T_0-open, $M = I - N$ is T_0-closed. Hence $m \in M$ and thus $[m,1] \subset M$. Now $0 \neq m$ since N is not empty. Since $m \in M$ and M is T_0-open, there is a number b such that $(b,m] \subset M$.

Therefore $(b,1] \subset M$ with $b < m$ which is impossible. Hence I is not disconnected by M and N.

The set of rationals in I is a T_0-dense subset of I. It is easy to see that conditions 1 and 2 are satisfied since the collection of open intervals of I form a base for T_0.

Corollary 1. Let T_L be the topology generated by T_0 and $\{[a,b) \mid 0 \leq a < b \leq 1\}$ and let T_R be the topology generated by T_0 and $\{(a,b] \mid 0 \leq a < b \leq 1\}$. Let L and R be subsets of I such that $L \cup R = I$, $0 \in L$, $1 \in R$, L is T_L-open, and R is T_R-open. Then $L \cap R \neq \emptyset$.

Proof: Since T_0 satisfies all the conditions of T of Theorem 1, from the same theorem it can be concluded that $L \cap R \neq \emptyset$.

Example 2. [3] Let X be the set of all rational numbers in I. Let T be the topology generated by T_0 and $\{X\}$. Then T is a connected topology for I finer than T_0 and T also satisfies the following two conditions:

(1) There is a T_0-dense subset A of I such that if $a \in A$ and U is a T-open set containing a, then there is a T_0-open set V such that $a \in V \subset U$, and

(2) If $p \in I$ and U is a T-open set containing p, then there is a T_0-open interval V containing p such that U is T_0-dense in V.

Proof: Obviously $T_0 \subset T$. Suppose that $I = U \cup V$ where $0 \in U$, U and V are T-open, and $U \cap V = \emptyset$. Since

$0 \in X$ there is a number a such that $0 \in [0,a) \cap X \subset U$. Let $y \in [0,a)$. If $y \in X$, then any basic T-open set about y is of the form $(y-\varepsilon, y+\varepsilon) \cap X$. Let $a' = \min \{a, y+\varepsilon\}$. There is a rational $q \neq y$ such that $y < q < a'$. Thus $q \in (y-\varepsilon, a') \cap X \subset (y-\varepsilon, y+\varepsilon) \cap X$ and since $(y-\varepsilon, a') \cap X \subset [0,a) \cap X \subset U$, $q \in U$. If $y \notin X$, then any basic T-open set about y is of the form $(y-\varepsilon', y+\varepsilon')$. Let $a'' = \min \{a, y+\varepsilon'\}$. There is a rational $q' \neq y$ such that $y < q' < a''$. Thus $q' \in (y-\varepsilon', a'') \subset (y-\varepsilon', y+\varepsilon')$ and since $q' \in X$ and $(y-\varepsilon', a'') \cap X \subset [0,a) \cap X \subset U$, $q' \in U$. Hence, if $y \in [0,a)$, any basic T-open set about y contains a point of U different from y. Thus y is a T-limit point of U. Now since V is T-open, $U = I - V$ is T-closed. Hence $y \in U$ and therefore $[0,a) \subset U$. Let $c = \text{l.u.b.}\{z \mid [0,z) \subset U\}$. Any basic T-open set containing c is of the form $(c-\varepsilon, c+\varepsilon)$ if c is not rational and of the form $(c-\varepsilon, c+\varepsilon) \cap X$ if c is rational. In either case there is a rational $p \neq c$ such that $c - \varepsilon < p < c$. Therefore any basic T-open set containing c must intersect U in a point different from c. Thus c is a T-limit point of U. Since U is T-closed, $c \in U$. Since $V \neq \emptyset$, $c \neq 1$. Suppose c is rational. Since $c \in U$ and U is T-open, there is a number b such that $[c,b) \cap X \subset U$. Let $d \in [c,b)$. If d is rational, a basic T-open set about d is of the form $(d-\varepsilon, d+\varepsilon) \cap X$ and thus contains a rational of $[c,b)$

different from d. Then d is a T-limit point of $[c,b) \cap X$ and hence d is a T-limit point of U. If d is not rational, a basic T-open set about d is of the form $(d-\epsilon, d+\epsilon)$ and thus contains a rational of $[c,b)$ different from d. Again d is a T-limit point of U. But since U is T-closed, $d \in U$. Thus $[c,b) \subset U$. Hence $[0,b) \subset U$. But $b > c$ and this is impossible. Hence (I,T) is connected.

I - X is T_0-dense in I. Let a be an irrational and U be a T-open set containing a. Since T is generated by T_0 and $\{X\}$, there are numbers f and g such that $a \in (f,g) \subset U$. Thus T satisfies condition 1. Let $p \in I$ and U' be a T-open set containing p. If p is irrational, as seen above, there is a T_0-open interval V' such that $p \in V' \subset U'$. Hence U' is T_0-dense in V'. If p is rational, there are numbers h and j such that $p \in (h,j) \cap X \subset U'$. If U' is T_0-dense in (h,j), then condition 2 is met by T. Now $(h,j) \cap X \subset U'$ implies that the $T_0\text{-Cl}((h,j) \cap X) \subset T_0\text{-Cl}(U')$. But the $T_0\text{-Cl}((h,j) \cap X) = (h,j)$, so the $T_0\text{-Cl}(U') \supset (h,j)$ and U' is T_0-dense in (h,j).

<u>Corollary 2</u>. Let X be the set of all rational numbers in I. Let T be the topology generated by T_0 and $\{X\}$. Let T_L be the topology generated by T and $\{[a,b) \mid 0 \leq a < b \leq 1\}$ and let T_R be the topology generated by T and $\{(a,b] \mid 0 \leq a < b \leq 1\}$. Let L and R be

subsets of I such that $L \cup R = I$, $0 \in L$, $1 \in R$, L is T_L-open, and R is T_R-open. Then $L \cap R \neq \emptyset$.

Proof: Observe that T satisfies all the conditions of the topology T of Theorem 1. Thus $L \cap R \neq \emptyset$.

Example 3. [3] Let X be the set of all rational numbers in I. For each $x \in I$, let $A_x = X \cup \{x\}$. Let T be the topology generated by T_0 and $\{A_x\}$, $x \in I$. Then T is a connected topology for I finer than T_0 and T satisfies the condition that if $p \in I$ and U is a T-open set containing p, then there is a T_0-open interval V containing p such that U is T_0-dense in V. But T does not satisfy the condition that there is a T_0-dense subset A of I such that if $a \in A$ and U is a T-open set containing a, then there is a T_0-open set V such that $a \in V \subset U$.

Proof: Obviously, $T_0 \subset T$. Let $p \in I$ and U be a T-open set containing p. There are numbers f, g, and h such that $p \in (f,g) \cap A_h \subset U$. Then $(f,g) = T_0\text{-Cl}((f,g) \cap A_h) \subset T_0\text{-Cl}(U)$. Thus U is T_0-dense in (f,g).

Let $x \in I$ such that $x \neq 0$ and $x \neq 1$. Then $(X \cap (0,1)) \cup \{x\} = (X \cup \{x\}) \cap (0,1)$ is a T-open subset of I containing x. Therefore x has the property that there is a T-open subset of I containing x such that no T_0-open subset of I containing x is a subset of this T-open set, and this is true for any $x \in (0,1)$.

It remains to show that (I,T) is connected. Suppose that $I = U \cup V$ where $0 \in U$, U and V are T-open, and $U \cap V = \emptyset$. Then there are numbers a and w such that $0 \in [0,a) \cap A_w \subset U$. But since $X \subset A_w$, $[0,a) \cap X \subset U$. Let $y \in [0,a)$ and $\varepsilon > 0$. Let $a' = \min\{a, y+\varepsilon\}$. There is a rational $q \neq y$ such that $y < q < a'$. Thus $q \in (y-\varepsilon, a') \cap X \subset (y-\varepsilon, y+\varepsilon) \cap X$. But $(y-\varepsilon, a') \cap X \subset [0,a) \cap X \subset U$, so $q \in U$. Thus any T-basic open set $(y-\varepsilon, y+\varepsilon) \cap A_y$ about y must intersect U. Therefore y is a T-limit point of U. Now, since V is T-open and $U \cap V = \emptyset$, $y \notin V$. Thus $y \in U$. Hence $[0,a) \subset U$. Let $c = \text{l.u.b.}\{z \mid [0,z) \subset U\}$. Let B be a T-basic open set containing c. Then there are numbers d, e, and f such that $B = (d,e) \cap A_f$. There is a rational r such that $d < r < c$. Thus $r \in B$ and $r < c$. But since $c = \text{l.u.b.}\{z \mid [0,z) \subset U\}$, $[0,r) \subset U$ and hence $B \cap U \neq \emptyset$. Therefore any T-basic open set containing c must intersect U. Suppose $c \in V$. Since V is T-open, there is a T-basic open set B' containing c such that $B' \subset V$. But this is impossible since $U \cap V = \emptyset$. Thus $c \in U$. Since $V \neq \emptyset$, $c \neq 1$. There are numbers g, h, and i such that $c \in (g,h) \cap A_i \subset U$. But since $[c,h) \subset (g,h)$ and $X \subset A_i$, $[c,h) \cap X \subset U$. Let $y' \in [c,h)$ and $\varepsilon' > 0$. Let $h' = \min\{h, y'+\varepsilon\}$. There is a rational s such that

$y' < s < h'$. Thus $s \in (y'-\varepsilon', h') \cap X \subset (y'-\varepsilon', y'+\varepsilon') \cap X$. But $[c,h) \cap X \subset U$, so $s \in U$. Thus any T-basic open set $(y'-\varepsilon, y'+\varepsilon) \cap A_{y'}$ about y' must intersect U. Therefore y' is a T-limit point of U. Now, since V is T-open and $U \cap V = \emptyset$, $y' \notin V$. Thus $y' \in U$. Hence $[c,h) \subset U$. Thus $[0,h) \subset U$. But since $c = \text{l.u.b.}\{z \mid [0,z) \subset U\}$, $h > c$ is impossible. Hence (I,T) is connected.

Theorem 2. [3] Let X be the set of all rational numbers in I. For each $x \in I$, let $A_x = X \cup \{x\}$. Let T be the topology generated by T_0 and $\{A_x\}$, $x \in I$. Let T_L be the topology on I generated by T and $\{[a,b) \mid 0 \leq a < b \leq 1\}$ and let T_R be the topology generated by T and $\{(a,b] \mid 0 \leq a < b \leq 1\}$. Let L and R be subsets of I such that $I = L \cup R$, $0 \in L$, $1 \in R$, L is T_L-open, and R is T_R-open. Then $L \cap R \neq \emptyset$.

Proof: Suppose $L \cap R = \emptyset$. Now $1 \in R$ and R is T_R-open. So there is a number z and a T-open set U such that $(z,1] \cap U \subset R$. But since T is generated by T_0 and $\{A_x\}$, $x \in I$, there is a number z' such that $X \cap (z',1] \subset U$. Let $x = \max\{z,z'\}$. Then $(x,1] \cap X \subset R$. Suppose that $y \in (x,1]$ and that $y \notin X$. Let $\varepsilon > 0$. Then $[y,y+\varepsilon) \cap A_y$ is a basic T_L-open set containing y. There is a rational r such that $y < r < y+\varepsilon$. Hence $r \in [y,y+\varepsilon) \cap A_y$, $r \in (x,1] \cap X$, and $r \neq y$. Thus y is a T_L-limit point of $(x,1] \cap X$ and, since $(x,1] \cap X \subset R$, y is a T_L-limit point of R. Then since L is T_L-open, $R = I - L$ is T_L-closed and thus R contains its T_L-limit points. Therefore $y \in R$. Hence $(x,1] \subset R$. Let $c = \text{g.l.b.}\{m \mid (m,1] \subset R$. If $(c,1] \not\subset R$, then there is a number $m \in (c,1]$ such that $m \notin R$. But $1/2(c+m) > c$, so $(1/2(c+m),1] \subset R$ and $m \in (1/2(c+m),1]$, a contradiction. Thus $(c,1] \subset R$. Since a basic T_L-open set about c is of the form $[c,c+\varepsilon) \cap A_c$, every basic T_L-open set about c contains a point of $(c,1]$. Thus c is a T_L-limit point of $(c,1]$ and since $(c,1] \subset R$, c is a T_L-limit point of R. But since R is T_L-closed, R contains its T_L-limit points. Hence $c \in R$ and thus $[c,1] \subset R$. Since $0 \in L$, $0 < c$. Since $c \in R$ and R is T_R-open, there is a number $w < c$ such that $(w,c] \cap A_c \subset R$. Therefore $(w,c] \cap X \subset R$. Suppose $a \in (w,c]$ and $a \notin X$. Let $\varepsilon' > 0$. Then $[a,a+\varepsilon') \cap A_a$ is a basic T_L-open set about a. There is a rational r' such that $a < r' < a + \varepsilon'$. Hence $r' \in [a,a+\varepsilon') \cap A_a$, $a \in (w,c] \cap X$, and $r' \neq a$. Thus a is a T_L-limit point of $(w,c] \cap A_c$ and since $(w,c] \cap A_c \subset R$, a is a T_L-limit point of R. But since R is T_L-closed, $a \in R$. Hence $(w,c] \subset R$ and thus $(w,1] \subset R$. But $w < c = \text{g.l.b.}\{m \mid (m,1] \subset R\}$ and this is impossible. Hence $L \cap R \neq \emptyset$.

CHAPTER III

HILDEBRAND'S TANGLED TOPOLOGY

Theorem 3. [2] If $x \in I$ and T is a connected topology for I finer than T_0, then for each $\varepsilon > 0$ every T-open set containing x must have elements in $(x-\varepsilon, x)$ for $x \neq 0$ and in $(x, x+\varepsilon)$ for $x \neq 1$.

Proof: Let $x \in I$ and $\varepsilon > 0$. Let U be a T-open set containing x. Suppose $x \neq 0$. Now $(x-\varepsilon, x)$ is T-open since $T_0 \subset T$. Suppose $(x-\varepsilon, x)$ contains no point of U. Then x is not a T-limit point of $[0,x)$ and if $y \in (x,1]$, y is not a T-limit point of $[0,x)$ since $(x,1]$ is T-open. Thus $[0,x) \cap T\text{-Cl}([x,1]) = \emptyset$. Now since $[0,x)$ is T-open, if $y \in [0,x)$ then y is not a T-limit point of $[x,1]$. Thus the $T\text{-Cl}([0,x)) \cap [x,1] = \emptyset$. Hence I is the union of two T-mutually separated subsets of I, which is impossible since T is a connected topology for I. Thus if $x \neq 0$, $(x-\varepsilon, x)$ contains a point of U. Similarly, if $x \neq 1$, $(x, x+\varepsilon)$ contains a point of U.

Corollary 3. If $x \in I$ and T is a connected topology for I finer than T_0, then no T-neighborhood N of x can contain a half closed interval J where the end point of J is not 0 or 1 and is a positive distance from $N - J$.

Proof: Let $x \in I$ and N be a T-neighborhood of x. Suppose N contains a half closed interval J where the end point p of J is not 0 or 1. From Theorem 3, if $\varepsilon > 0$, then $(p-\varepsilon, p)$ contains points of N and $(p, p+\varepsilon)$ contains points of N. Thus p is not a positive distance from $N - J$.

Theorem 4. [2] Let T be a connected topology defined on I which is finer than T_0 and let x be an element of I. If the T_0-neighborhoods of x do not form a local T-base at x, then x must have a T-neighborhood, call it N, such that $(0,x] \cap C_x = \{x\}$ or $[x,1) \cap C_x = \{x\}$ where C_x is the T_0-component of N containing x.

Proof: Assume the T_0-neighborhoods of x do not form a local T-base at x. Then there is a T-neighborhood N of x such that for every $\varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \not\subset N$. Let C_x be the T_0-component of N containing x.

Suppose $x \neq 0$ and $x \neq 1$. Then C_x is of the form $[a,x]$ or $[x,b]$ for some a, b, $0 \leq a < x$, $x < b \leq 1$. If $C_x = [a,x]$, then $[x,1) \cap C_x = \{x\}$ and if $C_x = [x,b]$, then $(0,x] \cap C_x = \{x\}$.

Now suppose either $x = 0$ or $x = 1$. If $x = 0$, then $C_x = \{x\}$ and hence $(0,x] \cap C_x = \{x\}$. If $x = 1$, then $C_x = \{x\}$ and thus $[x,1) \cap C_x = \{x\}$.

Theorem 5. [2] Let T be any connected topology on I which is finer than T_0, such that for every element x of I, with the exception of a set of elements P, the T_0-neighborhoods of x form a local T-base at x. Let T_L be the topology generated by T and $\{[a,b) \mid 0 \leq a < b \leq 1\}$ and let T_R be the topology generated by T and $\{(a,b] \mid 0 \leq a < b \leq 1\}$. Let L and R be subsets of I such that $L \cup R = I$, $0 \in L$, $1 \in R$, L is T_L-open, and R is T_R-open and suppose that $L \cap R = \emptyset$. Then P contains a non-denumerable number of elements of I.

Proof: Let U be the T_0-interior of L. It follows that U is the union of at most a denumerable number of disjoint T_0-open intervals. Let U' be the set of left end points of the intervals of U. Let U'' be the set of right end points of the intervals of U and $U^* = U' \cup U''$. Let V be the T_0-interior of R. Define V^*, V', and V'' similarly for V. Observe that $U \subset L$. Assume that $a \in U''$ and $a \in R$. Now $a \in U''$ implies that there exists a $W = (b,a) \subset U$ for some $b < a$. Hence, due to the connectedness of T and Theorem 3, every T-neighborhood of a must have elements in W. Since T_R is generated by T and the left closed intervals of I, every T_R-neighborhood of a must contain elements of W, hence of $U \subset L$. This leads to the contradiction that R is not T_R-open since $L \cap R = \emptyset$. Therefore $a \in U''$ implies $a \in L$. Hence $U'' \subset L$. Similarly $V' \subset R$.

Let $\alpha \in U'$. Then $\alpha \in L$. Observe that α is a T_0-limit point of R. Let J be the interval in U with right end point α. Now L is T_L-open. Therefore there is a number r and a T-open set E such that $[\alpha,r) \cap E \subset L$. Let β be the left end point of J. Let $F = (a,\alpha) \cup ([\alpha,r) \cap E)$. Then $F \subset L$. Clearly, $F - \{\alpha\}$ is T-open. Now (a,r) is a T-open set such that $\alpha \in (a,r) \subset F$. Thus F is T-open. Clearly, F contains no T_0-neighborhood of α because α is a T_0-limit point of R and $F \subset L$.

Choose an element of I, call it y, such that $y \in I-(U \cup V \cup U^* \cup V^*)$. Assume $y \notin P$. Now either $y \in L$ or $y \in R$. It shall be assumed that $y \in L$. The proof is similar if it is assumed that $y \in R$. Since $y \in L$ and L is T_L-open, there is a T-neighborhood S of y and a number $c > y$ such that $y \in S \cap [y,c) \subset L$. Since $y \notin P$, there is an $\varepsilon > 0$ such that $(y-\varepsilon, y+\varepsilon) \subset S$. Hence $y \in (y-\varepsilon, y+\varepsilon) \cap [y,c) \subset L$. Since $y \notin P$, $y \notin U''$. Therefore either there exists a T_0-open interval contained in L containing y or y must be the left end point of an open interval contained in L. However, this implies $y \in U$ or $y \in U^*$, a contradiction. Hence $y \in I-(U \cup V \cup U^* \cup V^*)$ implies $y \in P$. Now if $U = V = \emptyset$, then the previous statement says if $y \in I$ then $y \in P$, or equivalently, $I \subset P$. But since I is non-denumerable, if $U = V = \emptyset$, then P is non-denumerable, ending the proof. Hence assume that $U \neq \emptyset$ and observe that the proof follows similarly if it is assumed that $V \neq \emptyset$.

Define $A = U \cup V$. The number of T_0-components in A is at most denumerable since A is the union of at most a denumerable number of disjoint T_0-open intervals. If A consists of a finite number of T_0-components, there exists a T_0-open interval of U which is closer to 1 than any other T_0-open interval of U. Let d be the right end point of this maximal T_0-open interval of U. Since $d \in U''$ and $U'' \subset L$, $d \in L$, and hence $d \neq 1$. Now either there is no maximal T_0-open interval of V to the right of d, or there is a first one. Suppose there is no maximal open interval of V to the right of d. Then $(d,1]$ contains no point of $A \cup U^* \cup V^*$ and thus $(d,1] \subset P$. But since $d \in U'' \subset P$, $[d,1] \subset P$. Suppose there is a first T_0-open interval of V to the right of d. Call its left end point e. Then since $e \in V'$ and $V' \subset R$, $e \in R$. Now (d,e) contains no point of $A \cup U^* \cup V^*$ and thus $(d,e) \subset P$. But since $d \in U'' \subset P$ and $e \in V' \subset P$, $[d,e] \subset P$. Thus in either case, either $[d,1] \subset P$ or $[d,e] \subset P$. Hence either P is non-denumerable and the proof is complete or A must contain a denumerable number of T_0-components.

If A consists of a denumerable number of T_0-components, form a set B as follows. Let $B \supset A$. Place $1 \in B$ if 1 is in U^* or V^*. Place 0 in B if 0 is in U^* or V^*. Let $q \in B$, for $q \in I$, if q is the end point of two distinct T_0-components of A. Notice that B is an open set under T_0. It follows that $I - B$ is a closed set under T_0.

Now assume that z is an isolated point of $I - B$ under T_0. The case where $z \in (0,1)$ shall be considered since the proof is similar if $z = 0$ or 1. Then there exists a T_0-open interval, say (k,h) where $k < z < h$, such that $(k,h) - \{z\} \subset B$. Therefore each element of $(k,h) - \{z\}$ is an element of A or an end point of two distinct T_0-components of A. Observe that since $z \notin B$, z is not in A and is not an end point of two distinct T_0-components of A. It follows that for some m, where $k < m < z$ or $z < m < h$, that either (m,z) or (z,m) must contain a denumerable number of non-overlapping T_0-components of A, say K_i, $i = 1,2,\cdot\cdot\cdot$, such that K_i is closer to z than K_j if $i > j$, m is an end point of K_1, and such that exactly one point is between K_i and K_{i+1}, $i = 1,2,\cdot\cdot\cdot$. Without loss of generality it may be assumed that $K_i \subset (m,z)$, $i = 1,2,\cdot\cdot\cdot$. Notice that since U and V are the union of disjoint T_0-open intervals, the T_0-components of A are elements of U or V and hence each is a T_0-open interval. Now for some i suppose that K_i and K_{i+1} are both elements of U. Then K_i can be written in the form (r,s) and K_{i+1} in the form (s,t). Therefore $\{s\} \subset R$ since $\{s\} \subset L$ implies that (r,t) is in the T_0-interior of L, hence in U, and therefore a T_0-component of A. Then (r,t) being a T_0-component of A implies that K_i is not a T_0-component of A, which is a contradiction. However $\{s\} \subset U'' \subset L$. Therefore K_i and

K_{i+1} are not both elements of U for any i. Similarly it can be shown that K_i and K_{i+1} are not both elements of V for any i. Hence there exists a j such that $K_j \subset U$ and $K_{j+1} \subset V$. Let c be the common end point of K_j and K_{j+1}. Since $\{c\} \subset U''$ and $U'' \subset L$, $\{c\} \subset L$ and since $\{c\} \subset V'$ and $V' \subset R$, $\{c\} \subset R$. But this contradicts the hypothesis that $L \cap R = \emptyset$. It follows that I - B contains no isolated points under T_0.

It now follows that I - B is perfect since I - B is T_0-closed and contains no isolated points under T_0. Since I - B is a compact Hausdorff space which is perfect, either I - B is the empty set or contains a non-denumerable number of elements. [4, Th. 2-80, p. 88] Since the number of T_0-components in A is at most denumerable, the number of elements of $U' \cup V''$ is at most denumerable. If $x \in I - B$, then $x \in I - (U \cup V)$. But if $x \in I - (U \cup V \cup U^* \cup V^*)$, then $x \in P$. Thus every element of I - B, with the exception of at most a denumerable number of elements of $U' \cup V''$, is in P. If I - B is not empty, this results in P being non-denumerable.

Suppose that I - B is the empty set. Since $U \neq \emptyset$, pick one maximal T_0-open interval of U. Call its right end point g. Then $g \in U'' \subset L$ and thus $g \neq 1 \in R$. Now g is not in V' for if it were, $g \in R$ since $V' \subset R$. But $g \in L$. Hence $g \notin B$ which implies $g \in I - B$ and thus I - B is not empty.

Therefore it follows that P consists of a non-denumerable number of elements and the proof is complete.

Definition 1. [2] Remove the middle 1/3 intervals of I as in the formation of the Cantor set and label them as follows:

$$(a_1,b_1)=(1/3,2/3),$$
$$(c_1,d_1)=(1/9,2/9),(c_2,d_2)=(7/9,8/9),$$
$$(a_2,b_2)=(1/27,2/27),(a_3,b_3)=(7/27,8/27),(a_4,b_4)=(19/27,20/27),$$
$$(a_5,b_5)=(25/27,26/27)$$

⋮

Continue the above process, taking out the middle 1/3 of intervals not yet labeled at the nth stage and calling them (a_i,b_i) if n is odd or (c_i,d_i) if n is even. Define M, N, and K as follows:

$$M = \bigcup_{i=1}^{\infty} I_i \quad \text{where } I_i = (a_i,b_i),$$
$$N = \bigcup_{i=1}^{\infty} J_i \quad \text{where } J_i = (c_i,d_i), \text{ and}$$
$$K = I - \bigcup_{i=1}^{\infty} T_0\text{-}Cl(I_i) - \bigcup_{i=1}^{\infty} T_0\text{-}Cl(J_i) - \{0\} - \{1\}.$$

Let T' be the topology for I which has as a subbase

$$T_0 \cup \left\{ M \cup \{p\} \,\middle|\, p \in K \cup \{0\} \cup \bigcup_{i=1}^{\infty} \{b_i\} \right\} \cup \left\{ N \cup \{p\} \,\middle|\, p \in \{1\} \cup \bigcup_{i=1}^{\infty} \{c_i\} \right\}.$$

Then T' will be called the tangled topology on I.

<u>Theorem 6</u>. [2] If T' is the tangled topology on I, then (I,T') is connected.

Proof: Assume that I is not connected under T'. Then there exist sets A and B such that $A \cup B = I$, $A \cap B = \emptyset$, A and B are both open and closed proper subsets of I. Without loss of generality, assume $0 \in B$. Therefore the set A must have a greatest lower bound, call it a.

Case I. Assume $a \in A$. It is apparent, since $0 \in B$, that $a > 0$. If every T'-subbasic open set which contains a contains points of [0,a), then every T'-open set which contains a contains points of [0,a). Let S be a T'-subbasic open set containing a.

Suppose S is a T_0-open interval. Then obviously S contains points of [0,a) and hence points of B.

Suppose $S = M \cup \{p\}$ where $p \in K \cup \{0\} \cup \bigcup_{i=1}^{\infty} \{b_i\}$. If $a \in M$, then this situation is essentially the same as the previous one. Thus assume that $S = M \cup \{a\}$ where $a \in K \cup \{0\} \cup \bigcup_{i=1}^{\infty} \{b_i\}$. Now $a \neq 0$. If $a \in K$, there is an interval $(a_i, b_i) \subset M$ with $a_i < b_i < a$. Thus S contains points of [0,a) and hence points of B.

Suppose $S = N \cup \{p\}$ where $p \in \{1\} \cup \bigcup_{i=1}^{\infty} \{c_i\}$. Again we may assume $a = p$ without loss of generality. Now $a \neq 1$, for if $a = 1$, $A = \{1\}$ which is not a T'-open set. Thus $a \in \bigcup_{i=1}^{\infty} \{c_i\}$. There is an interval $(c_j, d_j) \subset N$ with $c_j < d_j < a$. Thus S contains points of [0,a) and hence of B.

Therefore a is a T'-limit point of B such that $a \notin B$ and B is not a T'-closed subset of I, which contradicts the assumption.

Case II. Assume $a \in B$. Either

$a \in M \cup N \cup \bigcup_{i=1}^{\infty}\{a_i\} \cup \bigcup_{i=1}^{\infty}\{c_i\} \cup \bigcup_{i=1}^{\infty}\{d_i\}$,

$a \in K \cup \{0\} \cup \bigcup_{i=1}^{\infty}\{b_i\}$, or $a = 1$.

If $a \in M \cup N \cup \bigcup_{i=1}^{\infty}\{a_i\} \cup \bigcup_{i=1}^{\infty}\{c_i\} \cup \bigcup_{i=1}^{\infty}\{d_i\}$, then each T'-neighborhood of a contains an interval of the form $[a,b)$ for some $b > a$. But B is an open set of (I,T') and then must contain an interval $[a,b)$ for some $b > a$. Therefore a would not be, as defined, the greatest lower bound of A.

If $a \in K \cup \{0\} \cup \bigcup_{i=1}^{\infty}\{b_i\}$, then since $a \in B$ and B is T'-open, for some $\mathcal{E} > 0$,

$(a,a+\mathcal{E}) \cap M \subset (a-\mathcal{E},a+\mathcal{E}) \cap (M \cup \{a\}) \subset B$. Thus the end points of the intervals of M that are in $(a,a+\mathcal{E})$ are in B. Let $b \in K$ and $b \in (a,a+\mathcal{E})$. Suppose $b \in A$. Then since A is T'-open, $b \in (a,a+\mathcal{E}) \cap (M \cup \{b\}) \subset A$, a contradiction. So for any point $b \in K$, b must be in B if it is in $(a,a+\mathcal{E})$. Hence, if $x \in (a,a+\mathcal{E})$ and $x \notin$ T'-Cl(N), then $x \in B$. Therefore, since a is the greatest lower bound of A, A must contain a sequence of intervals of N approaching a from the right. Choose one of the intervals in this sequence, call it (c,d), such that $(c,d) \subset (a,a+\mathcal{E}/2)$. Every T'-neighborhood of d contains an interval of the form (e,f)

where $e < d < f$. Hence, from the construction of M and the fact that $(a,a+\varepsilon) \cap M \subset B$, the interval (d,f) contains points of B for every f. Therefore d is a T'-limit point of B and, since B is T'-closed, $d \in B$. But $(a,a+\varepsilon) \cap N \subset A$ and from the construction of N, the interval (d,f) contains points of A for every f. Thus d is also a T'-limit point of A, a contradiction.

If $a = 1$, $A = \emptyset$. However, A was restricted to be non-empty, a contradiction.

Therefore it follows that T' leaves I connected.

Theorem 7. [2] If T is a finer connected topology for I than T_0, then any connected subset of I under T_0 will be a connected subset of I under T.

Proof: Observe that from the statement of the theorem it follows that I is connected under T; hence, it is necessary only to consider proper subsets of I. Let J be a proper subset of I which is connected under T_0, that is, J is an interval contained in I.

Case I. Assume J is a closed interval. Let $J = [a,b]$ where $0 \leq a < b \leq 1$. Assume J is not a connected subset of I under T. Therefore there exist A and B such that $A \cup B = J$, $A \neq \emptyset$, $B \neq \emptyset$, $T\text{-Cl}(A) \cap B = \emptyset$, and $A \cap T\text{-Cl}(B) = \emptyset$. Now either $a \in A$ or $a \in B$ and either $b \in A$ or $b \in B$.

Assume $a \in A$ and $b \in B$. The proof is similar if it is assumed that $b \in A$ and $a \in B$. Define $[0,a) \cup A = C$ and

$(b,1] \cup B = D$. Observe that $C \cup D = I$, $C \neq \emptyset$, and $D \neq \emptyset$. Now the $T\text{-}Cl(C) \cap D = T\text{-}Cl([0,a) \cup A) \cap ((b,1] \cup B)$. Since $C \cap D = \emptyset$, the $T\text{-}Cl(C) \cap D = \emptyset$ unless D contains a T-limit point of C. But $B \cap T\text{-}Cl(A) = \emptyset$ and since $a \in A$, $B \cap T\text{-}Cl([0,a)) = \emptyset$. Since $(b,1]$ is T-open and $(b,1] \cap ([0,a) \cup A) = \emptyset$, then $(b,1] \cap T\text{-}Cl([0,a) \cup A) = \emptyset$. Thus the $T\text{-}Cl(C) \cap D = T\text{-}Cl([0,a) \cup A) \cap ((b,1] \cup B) = \emptyset$. Similarly, $C \cap T\text{-}Cl(D) = \emptyset$. So C and D are T-mutually separated sets whose union is I, a contradiction.

Now assume $a \in B$ and $b \in B$. The proof is similar if it is assumed that $a \in A$ and $b \in A$. Define $E = [0,a) \cup B \cup (b,1]$. Observe that $E \cup A = I$, $E \neq \emptyset$, and $A \neq \emptyset$. Now the $T\text{-}Cl(E) \cap A = T\text{-}Cl([0,a) \cup B \cup (b,1]) \cap A$. But the $T\text{-}Cl(B) \cap A = \emptyset$ and since $a,b \in B$, the $T\text{-}Cl([0,a)) \cap A = \emptyset$ and the $T\text{-}Cl((b,1]) \cap A = \emptyset$. Thus the $T\text{-}Cl(E) \cap A = \emptyset$. In a similar way it can be shown that $E \cap T\text{-}Cl(A) = \emptyset$. So E and A are T-mutually separated sets whose union is I, a contradiction.

Hence, if J is a closed interval of I, J is connected under T.

Case II. Assume J is a half closed interval. Assume that J does not contain its right end point. The proof is similar if it is assumed that J does not contain its left end point. Let $J = [a,b)$ where $0 \leq a < b \leq 1$. Let i be the smallest positive integer for which $b - 1/i > a$. Then J can be written as $\bigcup_{n=1}^{\infty} [a,b-1/n]$. By Case I of this theorem, for each n, $[a,b-1/n]$ is connected under T. Since a is in each of the intervals, it follows that $J = \bigcup_{n=1}^{\infty} [a,b-1/n]$ is a connected subset of I under T.

Case III. Assume J is an open interval, that is, J contains neither of its end points. Let $J = (a,b)$ where $0 \leq a < b \leq 1$. Then $J = (a,c) \cup [c,b)$ where $a < c < b$. It follows from Case II of this theorem that both $(a,c]$ and $[c,b)$ are connected subsets of I under T. Hence, since they have the point c in common, this union is a connected subset of I under T.

Corollary 4. Any connected subset of I under T_0 is a connected subset of I under the tangled topology T'.

Proof: Theorem 6 establishes that T' is a connected topology for I. Observe that in the definition of T' all open sets of I under T_0 are also open sets of I under T'. Hence T' is a finer topology for I than T_0. Therefore it follows from Theorem 7 that any connected subset of I under T_0 is a connected subset of I under T'.

Theorem 8. [2] Let T' be the tangled topology for I. Let T_L' be the topology generated by T' and $\{[a,b) \mid 0 \leq a < b \leq 1\}$ and let T_R' be the topology generated by T' and $\{(a,b] \mid 0 \leq a < b \leq 1\}$. Then $T_0 \subset T'$, (I,T') is connected, and there exist sets L' and R' such that

$L' \cup R' = I$, $0 \in L'$, $1 \in R'$, L' is T_L'-open, R' is T_R'-open, and $L' \cap R' = \emptyset$.

Proof: Observe that Theorem 6 insures that T' is a connected topology for I. From the definition of T' it is obvious that T' is finer than T_0. Now define the following sets where the symbols $M, N, K, b_i, c_i, a_i, d_i$ have the same meaning as in the definition of T'.

$$P = M \cup \{0\} \cup K \cup \bigcup_{i=1}^{\infty} \{b_i\},$$

$$Q = N \cup \{1\} \cup \bigcup_{i=1}^{\infty} \{c_i\},$$

$$L' = P \cup \bigcup_{i=1}^{\infty} \{a_i\},$$

$$R' = Q \cup \bigcup_{i=1}^{\infty} \{d_i\}.$$

Let $x \in P$. Notice that if $x \in M$, then $x \in (M \cup \{0\}) \cap (0,1) \subset P$ and that $M \cup \{0\}$ and $(0,1)$ are subbasic elements of T'. If $x = 0$, then $x \in (M \cup \{0\}) \cap [0,1) \subset P$ and $[0,1)$ is a subbasic element of T'. If $x \in K$ or $x \in \bigcup_{i=1}^{\infty} \{b_i\}$, then $x \in (M \cup \{x\}) \cap (0,1) \subset P$ and $M \cup \{x\}$ is a subbasic element of T'. Thus P is an open set in I under T' and hence in I under T_L'. Now for any i, $[a_i, b_i)$ is T_L'-open. Thus $L' = P \cup \bigcup_{i=1}^{\infty} \{a_i\} = P \cup \bigcup_{i=1}^{\infty} \{[a_i, b_i)\}$ is T_L'-open.

Let $y \in Q$. If $y \in N$, then $y \in (N \cup \{1\}) \cap (0,1) \subset Q$ and $N \cup \{1\}$ and $(0,1)$ are T'-subbasic elements. If $y = 1$, then $y \in (N \cup \{1\}) \cap (0,1] \subset Q$ and $(0,1]$ is a subbasic element of T'. If $y \in \bigcup_{i=1}^{\infty}\{c_i\}$, then $y \in (N \cup \{y\}) \cap (0,1) \subset Q$ and $N \cup \{y\}$ is a subbasic element of T'. Thus Q is a T'-open set in I and hence a T_R'-open set in I. Now for any i, $(c_i, d_i]$ is T_R'-open. Thus

$R' = Q \cup \bigcup_{i=1}^{\infty}\{d_i\} = Q \cup \bigcup_{i=1}^{\infty}\{(c_i, d_i]\}$ is T_R'-open.

Observe that $L' \cup R' = I$, $0 \in L'$, and $1 \in R'$. It is quite apparent that $L' \cap R' = \emptyset$.

SUMMARY

The author has exhibited a topology T' which provides a negative answer to Stallings [1] question. Observe that T' is as uncomplicated a topology as any which answers Stallings' problem. Theorems 3 and 4, along with Corollary 3, tend to describe the additional open sets which were added to T_0 to form T' while Theorem 5 assures that it is necessary for I under a topology satisfying Stallings' conditions to have a non-denumerable number of points at which the T_0-neighborhood system does not form a local base.

BIBLIOGRAPHY

1. J. Stallings, _Fixed point theorems for connectivity maps_, Fundamenta Mathematicae, vol. 47 (1959), pp. 249-263.

2. S. K. Hildebrand, _A connected topology for the unit interval_, Fundamenta Mathematicae, vol. 61 (1967), pp. 133-140.

3. Hughes B. Hoyle, III, _Two topologies on I_, Dissertation, University of North Carolina at Chapel Hill, Chapel Hill, N. C., 1967.

4. John G. Hocking and Gail S. Young, _Topology_, Addison Wesley, Reading, Mass., 1961.