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The author has defined the concept of a bi-open set in a bi-topological space. With this concept many properties of bi-topological spaces which closely parallel the usual properties of topological spaces are defined, and some theorems which resemble theorems of topological spaces are proved.

BI-TOPOLOGICAL SPACES

by

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The concepts underlying the more familiar T_0 and T_1 topologies are very well known to everyone such as physicists, chemists, biologists, and engineers, as is partially demonstrated in the following examples and theorems. It is therefore not surprising that the study of topological spaces has led to the investigation of other topologies and their properties in bi-topological spaces. The investigation was that of bi-topological spaces. A similar line has recently been followed in [1]. It is the purpose of this paper to introduce the concept of bi-topological spaces and to demonstrate the properties of topological spaces and their bi-topological spaces.

In Chapter I, bi-topology is defined, and the bi-topological spaces are studied.

In Chapter II, bi-limit points and bi-closed sets are studied.

INTRODUCTION

Kelly initiated the study of bi-topological spaces in [3]. In this paper the concepts of pairwise- T_1 , pairwise regular, pairwise Hausdorff, quasi-pseudo-metrizable, and quasi-metrizable are introduced, and the following theorem is proved. Theorem. If (X, P, Q) is a pairwise regular bi-topological space satisfying the second axiom of countability, then (X, P, Q) is quasi-pseudo-metrizable. If in addition (X, P, Q) is pairwise Hausdorff, it is quasi-metrizable.

The concepts underlying the above definitions do not easily carry over to concepts such as pairwise compact, pairwise connected, and pairwise continuous, as is partially demonstrated in [2]. This author was therefore led to investigate other possible ways for defining properties in bi-topological spaces. One idea investigated was that of bi-open sets. A similar idea has recently been studied in [1]. It is the purpose of this paper to introduce the concept of bi-open sets in bi-topological spaces and to demonstrate how properties of topological spaces can easily be expressed for bi-topological spaces.

In Chapter I, bi-open is defined, and the bi-separation axioms are studied.

In Chapter II, bi-closure is defined, and basic theorems of bi-closed sets are studied.

In Chapter III, bi-interior is defined, and basic theorems for bi-open sets are proved.

In Chapter IV and Chapter V, bi-continuity and bi-convergence are investigated.

CHAPTER I

BASIC PROPERTIES OF BI-TOPOLOGICAL SPACES

Definition 1: Let X be a set. Let P and Q be topologies for X . Then the ordered triple (X, P, Q) is said to be a bi-topological space.

Definition 2: Let (X, P, Q) be a bi-topological space. A subset U of X is said to be bi-open provided U is the empty set, or there is an open set V in P and an open set W in Q such that V is not the empty set, W is not the empty set, $V \subset U$, and $W \subset U$.

Definition 3: Let (X, P, Q) be a bi-topological space. A subset C of X is said to be bi-closed provided its complement $X - C$ is bi-open.

Theorem 1: The union of any collection of bi-open sets is a bi-open set.

Proof: Let (X, P, Q) be a bi-topological space. Let G be a collection of bi-open subsets of X . Then $G = \{G_a \mid a \in A \text{ for some index set } A, \text{ and } G_a \text{ is bi-open}\}$. Let $O = \cup \{G_a \mid a \in A\}$. Suppose that for all $a \in A$, G_a is the empty set. Then G is the empty set, and thus G is bi-open. So suppose there is a b in A such that G_b is not the empty set. G_b is bi-open. Thus there is a set V in P and a set W in Q such that V is not the

empty set, W is not the empty set, $V \subset G_b$, and $W \subset G_b$. Thus $V \subset G_b \subset O$, and $W \subset G_b \subset O$. Therefore, by Definition 1, O is bi-open. Hence the union of any collection of bi-open sets is bi-open.

Example 1: Let $X = \{1, 2, 3\}$, $P = \{\phi, X, \{1\}, \{1, 2\}, \{2, 3\}\}$, and $Q = \{\phi, X, \{2\}, \{3\}, \{2, 3\}\}$. Let $A = \{1, 2\}$ and $B = \{1, 3\}$. Then A and B are bi-open sets whose intersection is not bi-open.

Proof: Let R denote the set of bi-open subsets of X . Then $R = \{\phi, X, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$. Thus $A \cap B = \{1, 2\} \cap \{1, 3\} = \{1\}$, which is not bi-open.

Definition 4: A bi-topological space (X, P, Q) is said to be bi-discrete provided, if $A \subset X$, then A is bi-open.

Theorem 2: A bi-topological space (X, P, Q) is bi-discrete if and only if each of (X, P) and (X, Q) is discrete.

Proof: Let (X, P, Q) be a bi-topological space. Suppose that (X, P, Q) is bi-discrete. Let $x \in X$. Since (X, P, Q) is bi-discrete, $\{x\}$ is bi-open. There is thus a set V in P and a set W in Q such that V is not the empty set, W is not the empty set, $V \subset \{x\}$, and $W \subset \{x\}$. Therefore $V = \{x\}$, and $W = \{x\}$. Then $\{x\} \in P$ and $\{x\} \in Q$. So by definition of discrete, (X, P) and (X, Q) are discrete.

Suppose that (X, P) and (X, Q) are discrete. Let A be a subset of X . Since (X, P) and (X, Q) are discrete, $A \in P$ and

$A \in \mathcal{Q}$. A is a subset of Λ . So A is bi-open. Therefore (X, P, Q) is a bi-discrete bi-topological space by Definition 4.

Definition 5: A bi- T_0 -space is a bi-topological space (X, P, Q) such that, if $p \in X$ and $q \in X$, then there is a bi-open set U which contains one point and does not contain the other.

Definition 6: A bi- T_1 -space is a bi-topological space (X, P, Q) such that, if $p \in X$, $q \in X$, and $p \neq q$, then there exists a bi-open set U with $p \in U$ and $q \notin U$.

Definition 7: A bi- T_2 -space is a bi-topological space (X, P, Q) such that, if $p \in X$, $q \in X$, and $p \neq q$, then there exist disjoint bi-open sets U and V with $p \in U$ and $q \in V$.

Definition 8: A bi- T_3 -space is a bi-topological space (X, P, Q) such that (X, P, Q) is a bi- T_1 -space and such that, if $p \in X$, and C is a bi-closed subset of X with p not in C , then there exist disjoint bi-open subsets, U and V , of X with $p \in U$ and $C \subset V$.

Definition 9: A bi- T_4 -space is a bi-topological space (X, P, Q) such that (X, P, Q) is a bi- T_1 -space and such that, if C and D are disjoint bi-closed subsets of X , then there exist disjoint bi-open subsets, U and V , of X such that $C \subset U$ and $D \subset V$.

Theorem 3: If (X, P, Q) is a bi- T_1 -space and $x \in X$, then $\{x\}$ is bi-closed.

Proof: Let (X, P, Q) be a $\text{bi-}T_1$ -space. Let $x \in X$. Let $p \in X - \{x\}$. Then p is not x . Since (X, P, Q) is a $\text{bi-}T_1$ -space, there is a bi-open set G_p such that $p \in G_p$ and x is not in G_p . Thus $p \in G_p \subset X - \{x\}$. Thus for each element p of X , there exists a bi-open set G_p which contains p and is contained in the complement of $\{x\}$.

It remains to show that $X - \{x\} = \cup \{G_p \mid p \in X - \{x\}\}$. Let $y \in X - \{x\}$. Then there is a bi-open set G_y such that $y \in G_y$, which is contained in $X - \{x\}$. Thus $y \in \cup \{G_p \mid p \in X - \{x\}\}$. Therefore $X - \{x\} \subset \cup \{G_p \mid p \in X - \{x\}\}$. Let $y \in \cup \{G_p \mid p \in X - \{x\}\}$. Thus there is a G_p such that $y \in G_p \subset X - \{x\}$. So $y \in X - \{x\}$. Therefore $\cup \{G_p \mid p \in X - \{x\}\} \subset X - \{x\}$. Hence $X - \{x\} = \cup \{G_p \mid p \in X - \{x\}\}$.

Since each G_p is bi-open, $\cup \{G_p \mid p \in X - \{x\}\}$ is bi-open. But this implies that $X - \{x\}$ is bi-open. Then by Definition 3 $\{x\} = X - (X - \{x\})$ is bi-closed.

Theorem 4: If (X, P, Q) is a $\text{bi-}T_4$ -space, then (X, P, Q) is a $\text{bi-}T_3$ -space.

Proof: Let (X, P, Q) be a $\text{bi-}T_4$ bi-topological space. Let $p \in X$. Let Q be a bi-closed subset of X which does not contain p . Since (X, P, Q) is a $\text{bi-}T_4$ -space, (X, P, Q) is a $\text{bi-}T_1$ -space, and $\{p\}$ is bi-closed by Theorem 3. Also there are disjoint bi-open sets, U and V , such that $\{p\} \subset U$ and $Q \subset V$. But $\{p\} \subset U$ implies that $p \in U$. Thus $p \in U$, $Q \subset V$, and $U \cap V = \phi$. Therefore, by Definition 8, (X, P, Q) is a $\text{bi-}T_3$ -space.

Theorem 5: If (X, P, Q) is a bi- T_3 -space, then (X, P, Q) is a bi- T_2 -space.

Proof: Let (X, P, Q) be a bi- T_3 bi-topological space. Let $p \in X$ and $q \in X$ such that p is not q . Since (X, P, Q) is a bi- T_3 -space, (X, P, Q) is a bi- T_1 -space. Then, by Theorem 3, $\{p\}$ is bi-closed. Thus there are disjoint bi-open sets U and V such that $q \in U$ and $\{p\} \subset V$. Since $\{p\} \subset V$, $p \in V$. So $q \in U$, $p \in V$, and $U \cap V = \phi$. Therefore (X, P, Q) is a bi- T_2 -space.

Theorem 6: If (X, P, Q) is a bi- T_2 -space, (X, P, Q) is a bi- T_1 -space.

Proof: Let (X, P, Q) be a bi- T_2 -space. Let $p \in X$ and $q \in X$ such that p is not q . Since (X, P, Q) is a bi- T_2 -space, there are disjoint bi-open sets U and V such that $p \in U$ and $q \in V$. But $U \cap V = \phi$ implies that q is not in U . Thus there is a bi-open set U such that $p \in U$ and $q \notin U$. Thus (X, P, Q) is a bi- T_1 -space.

Theorem 7: If (X, P, Q) is a bi- T_1 -space, then (X, P, Q) is a bi- T_0 -space.

Proof: Let (X, P, Q) be a bi- T_1 -space. Let $p \in X$ and $q \in X$ such that $p \neq q$. Since (X, P, Q) is a bi- T_1 -space, there is a bi-open set U such that $p \in U$ and $q \notin U$. Thus (X, P, Q) is a bi- T_0 -space.

Example 2: Let $X = \{1, 2, 3\}$, $P = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}$, and $Q = \{\phi, X, \{2\}, \{2, 3\}\}$. Then (X, P, Q) is not bi-discrete.

Proof: Let R denote the set of bi-open subsets of X . Then $R = \{\phi, X, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and thus $\{1\}$ is not bi-open. Hence (X, P, Q) is not a bi-discrete space.

Example 3: Let $X = \{1, 2, 3\}$, $P = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$, and $Q = \{\phi, X, \{1\}\}$. Then (X, P, Q) is a bi- T_0 -space and is not a bi- T_1 -space.

Proof: Let R denote the set of bi-open subsets of X . Then $R = \{\phi, X, \{1\}, \{1, 2\}, \{1, 3\}\}$. Thus (X, P, Q) is a bi- T_0 -space. However there is no bi-open set containing 2 and not containing 1. Hence (X, P, Q) is not a bi- T_1 -space.

Example 4: Let $X = \{1, 2, 3\}$, $P = \{\phi, X, \{1\}, \{1, 2\}, \{2, 3\}\}$, and $Q = \{\phi, X, \{2\}, \{3\}, \{2, 3\}\}$. Then (X, P, Q) is a bi- T_1 -space and is not a bi- T_2 -space.

Proof: Let R denote the set of bi-open sets. Then $R = \{\phi, X, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$. Although (X, P, Q) is a bi- T_1 -space, (X, P, Q) is not a bi- T_2 -space since there are no two disjoint bi-open sets U and V such that $1 \in U$ and $3 \in V$.

Example 5: Let $X = \{1, 2, 3, 4\}$, $P = \{\phi, X, \{1\}, \{1, 2\}\}$, and $Q = \{\phi, X, \{1\}\}$. Then (X, P, Q) is a bi- T_0 -space, and neither topological space is a T_0 -space.

Proof: The set of bi-open subsets of X is $\{\phi, X, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$. Since there is no open set containing 3 and not containing 4 or containing 4 and not containing 3, (X, P) is not a T_0 -space. Similarly, (X, Q) is not a T_0 -space since there does not exist an open set which

contains 2 and does not contain 3 or which contains 3 and does not contain 2. However (X, P, Q) is a bi- T_0 -space.

Example 6: Let $X = \{1, 2, 3, 4\}$, $P = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$, and $Q = \{\phi, X, \{2\}, \{3\}, \{2, 3\}, \{2, 3, 4\}\}$. Then (X, P, Q) is a bi- T_1 -space, and neither topological space is a T_1 -space.

Proof: Let R denote the set of bi-open subsets of X . Then $R = \{\phi, X, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}\}$. Then (X, P, Q) is a bi- T_1 -space. But (X, P) is not a T_1 -space since there is no open set containing 3 and not containing 4. Also (X, Q) is not a T_1 -space since there is no open set containing 4 and not containing 3.

Example 7: Let $X = \{1, 2, 3, 4\}$, $P = \{\phi, X, \{1\}, \{1, 2\}, \{3, 4\}, \{2\}, \{2, 3, 4\}\}$, and $Q = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3, 4\}\}$. Then (X, P, Q) is a bi- T_2 -space, and neither topological space is a T_2 -space.

Proof: Let R denote the set of bi-open subsets of X . Then $R = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}\}$. Then (X, P) is not a T_2 -space since there are not disjoint open sets U and V such that $3 \in U$ and $4 \in V$. Also (X, Q) is not a T_2 -space since there are not disjoint open sets S and T such that $3 \in S$ and $4 \in T$. However (X, P, Q) is a bi- T_2 -space.

Example 8: In Example 7, (X, P, Q) is a bi- T_4 -space, and neither topological space is a T_4 -space.

Proof: Since neither topological space is a T_3 -space, neither topological space is a T_4 -space. However (X, P, Q) is a bi- T_4 , bi-topological space.

Example 9: Let $X = \{1, 2, 3\}$, $P = \{\phi, X, \{2\}, \{1, 2\}\}$, and $Q = \{\phi, X, \{3\}, \{2, 3\}\}$. Then (X, P, Q) is not a bi- T_0 -space, but each topological space is a T_0 -space.

Proof: The set of bi-open subsets of X is $\{\phi, X, \{3\}, \{2, 3\}\}$. Although (X, P) and (X, Q) are T_0 -spaces, (X, P, Q) is not a bi- T_0 -space since there is no bi-open set containing 2 and not 3 or containing 3 and not 2.

Theorem 8: If (X, P) and (X, Q) are T_1 -spaces, then (X, P, Q) is a bi- T_1 -space.

Proof: Let (X, P) and (X, Q) be T_1 -spaces. Let $p \in X$ and $q \in X$ such that $p \neq q$. Since (X, P) is a T_1 -space, there is an open set V in Q such that $p \in V$ and $q \notin V$. Also, there is an open set U in P such that $p \in U$ and $q \notin U$. Since $U \subset U \cup V$, and $V \subset U \cup V$, $U \cup V$ is bi-open. Clearly $p \in U \cup V$, and $q \notin U \cup V$. Thus (X, P, Q) is a bi- T_1 -space.

Theorem 9: If X is a finite set, and if each of (X, P) and (X, Q) is a T_4 -space, then (X, P, Q) is a bi- T_4 -space.

Proof: Let X be a finite set. Let P and Q be topologies for X such that (X, P) and (X, Q) are T_1 -spaces. Since (X, P) and (X, Q) are finite T_1 -spaces, (X, P) and (X, Q) are discrete spaces. Then by Theorem 2, (X, P, Q) is a bi-discrete, bi-topological space. Let C and D be disjoint bi-closed

subsets of X . Since (X, P, Q) is a bi-discrete bitopological space, C and D are also bi-open subsets of X . But $C \subset C$ and $D \subset D$ and $C \cap D = \phi$. Hence (X, P, Q) is a $bi-T_4$ -space.

Corollary 1: If X is a finite set, and if each of (X, P) and (X, Q) is a T_i topological space, then (X, P, Q) is a $bi-T_i$, bi-topological space for $i = 1, 2, 3, 4$.

CHAPTER II

BI-LIMIT POINTS AND BI-CLOSED SETS

Definition 10: Let (X, P, Q) be a bi-topological space. A point p in X is a bi-limit point of a subset A of X provided every bi-open set which contains p contains a point q of A such that q is not p .

Example 10: Let $X = \{1, 2, 3\}$, $P = \{\phi, X, \{1\}, \{1, 2\}, \{2, 3\}, \{2\}\}$, and $Q = \{\phi, X, \{3\}, \{2, 3\}\}$. Then the set of bi-open subsets of X is $\{\phi, X, \{1, 3\}, \{2, 3\}\}$. If $B = \{1, 2\}$, then $\{3\}$ is the set of bi-limit points of B .

Proof: Since $\{1, 3\}$ is a bi-open set which contains 1 and contains no point of B different from 1, then 1 is not a bi-limit point of B . Since $\{2, 3\}$ is a bi-open set which contains 2 and no point of B different from 2, 2 is not a bi-limit point of B . Every bi-open set which contains 3 contains a point of B different from 3. Thus $\{3\}$ is the set of bi-limit points of B .

Theorem 10: If (X, P, Q) is a bi-topological space, then (i) X and ϕ are bi-closed; (ii) the intersection of any collection of bi-closed subsets of X is bi-closed.

Proof: Let (X, P, Q) be a bi-topological space.

(i) Clearly $(X - X) = \phi$, which is bi-open. Thus, by Definition 3, X is bi-closed. Since $(X - \phi) = X$, which is bi-open, ϕ is bi-closed.

(ii) Let $A = \{A_a \mid a \in I \text{ for some index set } I, \text{ and each } A_a \text{ is bi-closed}\}$ be a collection of bi-closed subsets of X . Then by definition of bi-closed, if $a \in I$, then $A_a = X - U_a$, where U_a is a bi-open subset of X . Thus $\cap \{A_a \mid a \in I\} = \cap \{X - U_a \mid a \in I\} = X - \cup \{U_a \mid a \in I\}$. But $\{U_a \mid a \in I\}$ is a collection of bi-open subsets of X . Thus by Theorem 1, $\cup \{U_a \mid a \in I\}$ is bi-open. By Definition 3, $X - \{U_a \mid a \in I\}$ is bi-closed. Thus $\cap \{A_a \mid a \in I\}$ is bi-closed.

Theorem 11: A subset B of a bi-topological space (X, P, Q) is bi-closed if and only if B contains all of its bi-limit points.

Proof: Let (X, P, Q) be a bi-topological space. Let $B \subset X$.

Suppose that B is bi-closed. Let M denote the set of bi-limit points of B . Suppose that $M \not\subset B$. Then there is an $m \in M$ such that $m \notin B$. Since $m \notin B$, $m \in X - B$. But B is bi-closed implies that $X - B$ is bi-open. Furthermore, $X - B$ contains no point of B . But this contradicts the definition of bi-limit point. Thus $M \subset B$, and B contains each of its bi-limit points.

Suppose that B contains each of its bi-limit points. So $M \subset B$. Let $x \in X - B$. Then $x \notin B$, and since $M \subset B$, $x \notin M$. Because x is not a bi-limit point of B , there is a bi-open subset of X which contains x and no point of B . Such a set exists for each element of X . Denote the set containing x by U_x .

It remains to show that $X - B = \cup \{U_x \mid x \in X - B\}$. Let $y \in X - B$. Then $y \in U_y \subset \cup \{U_x \mid x \in X - B\}$. Thus $X - B \subset \cup \{U_x \mid x \in X - B\}$. Let $y \in \cup \{U_x \mid x \in X - B\}$. Then there is a w

in $X - B$ such that $y \in U_w \subset X - B$. Therefore $y \in X - B$, and $\cup \{U_x \mid x \in X - B\} \subset X - B$. Thus $X - B = \cup \{U_x \mid x \in X - B\}$.

Since each U_x is bi-open, $\cup \{U_x \mid x \in X - B\}$ is bi-open by Theorem 1. Hence $(X - B)$ is bi-open. But by Definition 3, $B = X - (X - B)$ is bi-open.

Definition 11: Let (X, P, Q) be a bi-topological space. Let B be a subset of X . The bi-closure of B , denoted by \tilde{B} , is the intersection of all bi-closed subsets of X which contain B .

Example 11: Let $X = \{1, 2, 3, 4\}$, $P = \{\phi, X, \{2, 3\}, \{2\}\}$, and $Q = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. If $B = \{2\}$, then $\tilde{B} = \{1, 2\}$.

Proof: Let C denote the set of bi-closed subsets of X . Let R denote the set of bi-open subsets of X . Then $R = \{\phi, X, \{2\}, \{2, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2\}, \{2, 4\}, \{1, 2, 4\}\}$. So $C = \{\phi, X, \{1, 3, 4\}, \{1, 4\}, \{4\}, \{1\}, \{3, 4\}, \{1, 2\}, \{1, 3\}, \{3\}\}$. So $\tilde{B} = X \cap \{1, 2\} = \{1, 2\}$.

Theorem 12: If (X, P, Q) is a bi-topological space, and A and B are subsets of X such that $A \subset B$, then the set of bi-limit points of A is a subset of the set of bi-limit points of B .

Proof: Let (X, P, Q) be a bi-topological space. Let $A \subset X$ and $B \subset X$, with $A \subset B$. Let M_A be the set of bi-limit points of A , and M_B be the set of bi-limit points of B . Let $p \in M_A$. Let U be a bi-open set containing p . Since $p \in M_A$, U contains a point q of A such that $q \neq p$. But $A \subset B$, and so

$q \in B$. Hence every bi-open set containing p contains a point q of B such that $q \neq p$. Therefore $p \in M_A$. Thus $M_A \subset M_B$.

Theorem 13: If (X, P, Q) is a bi-topological space, and $B \subset X$, then (i) \tilde{B} is bi-closed; (ii) if F is a bi-closed subset of X containing B , then $B \subset \tilde{B} \subset F$; and (iii) B is bi-closed if and only if $B = \tilde{B}$.

Proof: Let (X, P, Q) be a bi-topological space. Let $B \subset X$.

(i) By definition of bi-closure, $\tilde{B} = \cap \{F \mid F \text{ is a bi-closed subset of } X, \text{ and } B \subset F\}$. Thus \tilde{B} is the intersection of bi-closed sets, and by Theorem 10, \tilde{B} is bi-closed.

(ii) Let F be a bi-closed subset of X containing B . Now $B \subset \tilde{B} = \cap \{C \mid C \text{ is a bi-closed subset of } X, \text{ and } B \subset C\}$. Since $B \subset F$, clearly $F \in \{C \mid C \text{ is a bi-closed subset of } X \text{ and } B \subset C\}$. Thus $B \subset \tilde{B} \subset F$.

(iii) Suppose that B is bi-closed. Since $B \subset B$, B is a bi-closed set containing B . Thus by (ii), $\tilde{B} \subset B$. Therefore $B = \tilde{B}$. Suppose $B = \tilde{B}$. Since \tilde{B} is bi-closed, B is bi-closed.

Theorem 14: If (X, P, Q) is a bi-topological space and $B \subset X$, then $\tilde{B} = B \cup M$, where M is the set of bi-limit points of B .

Proof: Let (X, P, Q) be a bi-topological space. Let B be a subset of X . Let M be the set of bi-limit points of B .

Let $x \in X - (B \cup M) = (X - B) \cap (X - M)$. Then $x \notin B$ and $x \notin M$. Since $x \notin M$, there is a bi-open set U which contains

no point of B different from x . Thus, $B \cap (U - \{x\}) = \phi$. But $x \notin B$. So $B \cap U = \phi$. There exists such a set U for all elements of $X - (B \cup M)$. Denote the set related to x by U_x . So $U_x \cap B = \phi$. Suppose $U_x \cap M \neq \phi$. Let $p \in U_x \cap M$. Therefore $p \in U_x$ and $p \in M$. Now, p is a bi-limit point of B implies that $B \cap (U_x - p) = \phi$. Since $p \in B$, $B \cap U_x = \phi$. Therefore $U_x \cap M = \phi$. Let $y \in X - (B \cup M)$. Then $y \in U_y \subset U\{U_x \mid x \in X - (B \cup M)\}$. Thus $y \in U_y \subset U\{U_x \mid x \in (B \cup M)\}$. So $X - (B \cup M) \subset U\{U_x \mid x \in X - (B \cup M)\}$. Let $y \in U\{U_x \mid x \in X - (B \cup M)\}$. Then there exists an x in X such that $y \in U_x$. $U_x \cap B = \phi$ and $U_x \cap M = \phi$. Thus $y \notin B$ and $y \notin M$. Therefore $y \in X - (B \cup M)$. Thus $U\{U_x \mid x \in X - (B \cup M)\} \subset X - (B \cup M)$. Hence $X - (B \cup M) = U\{U_x \mid x \in X - (B \cup M)\}$. Each U_x is bi-open, and by Theorem 1, $X - (B \cup M) = \{U_x \mid x \in X - (B \cup M)\}$ is bi-open. So by Definition 3, $B \cup M = X - (X - (B \cup M))$ is bi-closed.

Clearly $B \subset \widetilde{B}$, which is bi-closed. Let N denote the set of bi-limit points of \widetilde{B} . By Theorem 12, $M \subset N \subset \widetilde{B}$. Thus $M \subset \widetilde{B}$. But $B \cup M$ is bi-closed. Thus by Theorem 13, $\widetilde{B} \subset B \cup M$. Therefore $\widetilde{B} = B \cup M$.

Theorem 15: If (X, P, Q) is a bi-topological space, and $A \subset X$, and $B \subset X$, then $\widetilde{A \cup B} = \widetilde{A} \cup \widetilde{B}$.

Proof: Let (X, P, Q) be a bi-topological space. Let $A \subset X$ and $B \subset X$. Then by Theorem 13, $A \subset \widetilde{A}$ and $B \subset \widetilde{B}$. So $A \cup B \subset \widetilde{A} \cup \widetilde{B}$. Thus $\widetilde{A} \cup \widetilde{B}$ is a bi-closed subset of X containing $A \cup B$. So by Theorem 13, $\widetilde{A \cup B} \subset \widetilde{A} \cup \widetilde{B}$. Since $A \subset A \cup B$ and $B \subset A \cup B$,

it follows from Theorem 12 that $\widetilde{A} \subset \widetilde{A \cup B}$ and $\widetilde{B} \subset \widetilde{A \cup B}$. Therefore $\widetilde{A \cup B} \subset \widetilde{A \cup B}$. Thus $\widetilde{A \cup B} = \widetilde{A \cup B}$.

Theorem 16: If (X, P, Q) is a bi-topological space and $B \subset X$, then $\widetilde{\widetilde{B}} = \widetilde{B}$.

Proof: Let (X, P, Q) be a bi-topological space. Let $B \subset X$. Then \widetilde{B} is a bi-closed subset of X , and by Theorem 13, $\widetilde{\widetilde{B}} = \widetilde{B}$.

Theorem 17: If (X, P, Q) is a bi-topological space, and $A \subset X$, and $B \subset X$, then $\widetilde{A \cap B} \subset \widetilde{A} \cap \widetilde{B}$.

Proof: Let (X, P, Q) be a bi-topological space. Let $A \subset X$ and $B \subset X$. Then $A \cap B \subset A$. By Theorem 12, $\widetilde{A \cap B} \subset \widetilde{A}$. Likewise, $\widetilde{A \cap B} \subset \widetilde{B}$. Thus $\widetilde{A \cap B} \subset \widetilde{A} \cap \widetilde{B}$.

CHAPTER III

BI-OPEN SETS AND BI-INTERIOR

Definition 12: Let (X, P, Q) be a bi-topological space.

Let B be a subset of X . A point $x \in B$ is said to be a bi-interior point of B provided there is a bi-open set U such that $x \in U$ and $U \subset B$. The set of all bi-interior points of B is the bi-interior of B and is denoted by B^I .

Example 12: In Example 11, if $B = \{2, 3, 4\}$, then $B^I = \{2, 3, 4\}$.

Proof: Since $2 \in \{2, 3\} \subset B$, $2 \in B^I$. Also, $3 \in \{2, 3\} \subset B$, and $3 \in B^I$. Finally, $4 \in \{2, 4\} \subset B$, and $4 \in B^I$. Since there is no bi-open set which contains 1, and which is contained in B , 1 is not a bi-interior point of B .

Theorem 17: If (X, P, Q) is a bi-topological space, and $B \subset X$, then the bi-interior of B is bi-open.

Proof: Let (X, P, Q) be a bi-topological space. Let $B \subset X$. Let $x \in B^I$. From Definition 12, there is a bi-open set U_x such that $x \in U_x \subset B$. Such a set can be found for all x in B .

It remains to show that $B^I = \cup \{U_x \mid x \in B^I, \text{ and } U_x \subset B\}$.

Let $y \in B^I$. Then there exists a bi-open set U_y such that $y \in U_y \subset B$. Thus $y \in \cup \{U_x \mid x \in B^I\}$, and $B^I \subset \cup \{U_x \mid x \in B^I \text{ and } U_x \subset B\}$. Let $y \in \cup \{U_x \mid x \in B^I, \text{ and } U_x \subset B\}$. Then there is an $x \in B^I$ such that $y \in U_x \subset B$. But $y \in B^I$. Therefore

$\cup \{U_x \mid x \in B^I, \text{ and } U_x \subset B\} \subset B^I$. Thus $B^I = \cup \{U_x \mid x \in B^I, \text{ and } U_x \subset B\}$. Since each U_x is bi-open, B^I is bi-open.

Theorem 18: If (X, P, Q) is a bi-topological space, $B \subset X$, and U is a bi-open subset of X such that $U \subset B$, then $U \subset B^I$.

Proof: Let (X, P, Q) be a bi-topological space. Let $B \subset X$. Let U be a bi-open subset of X such that $U \subset B$. Let $x \in U$. Then by Definition 12, $x \in B^I$. Thus $U \subset B^I$.

Theorem 19: If (X, P, Q) is a bi-topological space, and $B \subset X$, then B is bi-open if and only if $B = B^I$.

Proof: Let (X, P, Q) be a bi-topological space. Let $B \subset X$.

Suppose B is bi-open. From the definition of interior, it follows that $B^I \subset B$. But B is a bi-open subset of X . So, by Theorem 18, $B \subset B^I$. Therefore $B = B^I$.

Suppose that $B = B^I$. From Theorem 17, B^I is bi-open. Thus B is bi-open.

Theorem 20: If (X, P, Q) is a bi-topological space, $A \subset X$ and $B \subset X$, and $A \subset B$, then $A^I \subset B^I$.

Proof: Let (X, P, Q) be a bi-topological space. Let $A \subset X$ and $B \subset X$ such that $A \subset B$. Let $x \in A^I$. Then there is a bi-open set U such that $x \in U \subset A \subset B$. Thus $x \in U \subset B$, and $x \in B^I$. Therefore $A^I \subset B^I$.

Theorem 21: If (X, P, Q) is a bi-topological space, and $B \subset X$, then $(B^I)^I = B^I$.

Proof: Let (X, P, Q) be a bi-topological space. Let $B \subset X$. Then B^I is bi-open, and by Theorem 19, $(B^I)^I = B^I$.

Example 13: In Example 5, if $A = \{1, 3\}$ and $B = \{1, 2\}$, then $A^I \cap B^I \neq (A \cap B)^I$.

Proof: Since A and B are bi-open, $A^I = A$, and $B^I = B$. But $A \cap B = \{1, 3\} \cap \{1, 2\} = \{1\}$, which is not bi-open. Thus $(A \cap B)^I = \{1\}^I = \emptyset$. So $A^I \cap B^I = \{1\}$, and $\{1\} \neq \emptyset$. Thus $A^I \cap B^I \neq (A \cap B)^I$.

CHAPTER IV
SEQUENCES AND BI-CONVERGENCE

Definition 13: Let (X, P, Q) be a bi-topological space.

Let S be a sequence in X . Then S is said to bi-converge to $x \in X$ provided, if U is a bi-open subset of X such that $x \in U$, then there exists a positive integer N such that, if $n \geq N$, then $S_n \in U$.

Example 14: Let $X = \{1, 2, 3, 4\}$, $P = \{\phi, X, \{1, 2, 3\}, \{2, 3, 4\}, \{2, 3\}\}$, and $Q = \{\phi, X, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{2\}\}$. Define S by $S_n = 2$ if n is odd and $S_n = 3$ if n is even. Then S bi-converges to 4 .

Proof: The set of bi-open sets is $\{\phi, X, \{1, 2, 3\}, \{2, 3, 4\}, \{2, 3\}\}$. There are only two bi-open sets which contain 4 , and each of these also contains 2 and 3 . Let $N = 0$. For $n \geq 0$, $S_n \in \{2, 3, 4\}$. Thus S bi-converges to 4 .

Example 15: In Example 10, define a sequence T by $T_n = 1$ if n is even and $T_n = 4$ if n is odd. Then T does not bi-converge.

Proof: The sequence T does not bi-converge to 1 because $\{1, 2, 3\}$ is a bi-open set which contains 1 and does not contain 4 . Similarly, T does not bi-converge to either 2 or 3 because $\{2, 3\}$ is a bi-open set which contains both 2 and 3 but neither 1 nor 4 . Finally, T does not bi-converge to 4 because $\{2, 3, 4\}$

is a bi-open set which contains l_4 and which does not contain l_1 . Thus T is a sequence which does not bi-converge.

Theorem 22: In a bi- T_2 -space no sequence can bi-converge to more than one point.

Proof: Let (X, P, Q) be a bi- T_2 -space. Let S be a sequence in X . Suppose that S bi-converges to x and to y and $x \neq y$. Since (X, P, Q) is a bi- T_2 -space, there are disjoint bi-open sets, U and V , such that $x \in U$ and $y \in V$. Because S bi-converges to x , there is a positive integer N such that for all positive integers n with $n \geq N$, $S_n \in U$. Since S bi-converges to y , there is a positive integer M such that for all integers m with $m \geq M$, $S_m \in V$. Let D be an integer such that $D \geq N + M$. Let d be an integer such that $d \geq D \geq N$. Since $d \geq N$, $S_d \in U$. Likewise $d \geq D \geq M$. Thus $S_d \in V$. So $S_d \in U \cap V = \phi$. But $S_d \notin \phi$. Therefore $x = y$, and S bi-converges to only one point.

Theorem 23: In a bi-discrete, bi-topological space, if S is a sequence with the property that there does not exist a positive integer N such that, if $n \geq N$, then $S_n = S_N$, then S does not bi-converge.

Proof: Let (X, P, Q) be a bi-discrete, bi-topological space. Let S be a sequence in X with the property that there does not exist a positive integer N such that, if $n \geq N$, then $S_n = S_N$. Suppose S bi-converges to $x \in X$. Since (X, P, Q) is bi-discrete, $\{x\}$ is bi-open. There is a positive integer M such that for all

integers m with $m \geq M$, $S_m \in \{x\}$. Since $S_m \in \{x\}$, $S_m = x$. For some $m \geq M$, $S_m \neq S_M$. Since $S_m \in \{x\}$, and $S_M \in \{x\}$, this is impossible. Thus S does not bi-converge.

Corollary 2: In a bi-discrete, bi-topological space (X, P, Q) a sequence S bi-converges to $x \in X$ if and only if there exists a positive integer N such that, if $n \geq N$, then $S_n = S_N$.

Proof: One-half of the proof follows from the previous theorem, and the other half is obvious.

Theorem 24: If (X, P, Q) is a bi-topological space, $B \subset X$, $x \in X$, and if there is a sequence S in B such that S bi-converges to x , then $x \in \tilde{B}$.

Proof: Let (X, P, Q) be a bi-topological space. Let $B \subset X$, and let $x \in X$. Let S be a sequence in B such that S bi-converges to x . Let U be a bi-open set such that $x \in U$. Either $x \in B$ or $x \notin B$. Suppose $x \in B$. Then $x \in B \subset \tilde{B}$. Therefore $x \in \tilde{B}$. Suppose $x \notin B$. Since S bi-converges to x , there is a positive integer N such that if n is an integer, and $n \geq N$, then $S_n \in U$. Let n be an integer such that $n \geq N$. Since $S_n \in B$ and $x \in B$, $S_n \neq x$. Thus x is a bi-limit point of B , and by Theorem 14, $x \in \tilde{B}$.

Theorem 25: If (X, P, Q) is a bi-topological space, $B \subset X$, and $x \in X$, and if there exists a sequence S of distinct points in B that bi-converges to x , then x is a bi-limit point of B .

Proof: Let (X, P, Q) be a bi-topological space, with $B \subset X$ and $x \in B$. Suppose S is a sequence of distinct points in B

such that S bi-converges to x . Let U be a bi-open set containing x . Since S bi-converges to x , there is a positive integer N such that, if n is an integer and $n \geq N$, then $S_n \in U$. Either $S_n = x$ or $S_n \neq x$. Suppose $S_n \neq x$. Then U contains a point of B which is not x , and so x is a bi-limit point of B . Suppose $S_n = x$. Since S is a sequence of distinct points, $S_n \neq S_{n+1}$. But $n+1 > n \geq N$. Thus $S_{n+1} \in U$, and $S_{n+1} \neq x$. Therefore U contains a point of B which is not x . Thus x is a bi-limit point of B .

CHAPTER V

BI-CONTINUOUS FUNCTIONS

Definition 14: Let (X, P, Q) and (Y, C, D) be bi-topological spaces. Let f be a function whose domain is X and whose range is a subset of Y . Then f is said to be bi-continuous provided, if U is a bi-open subset of Y , then $f^{-1}(U)$ is a bi-open subset of X .

Example 16: Let $X = \{1, 2, 3\}$, $P = \{\phi, X, \{1\}, \{1, 2\}, \{2, 3\}, \{2\}\}$, and $Q = \{\phi, X, \{1\}, \{3\}, \{1, 3\}\}$. Let $Y = \{4, 5, 6\}$, $C = \{\phi, Y, \{4\}, \{4, 5\}\}$, and $D = \{\phi, Y, \{4\}, \{6\}, \{4, 6\}\}$. Define f , mapping X into Y , by $f(1) = 4$; $f(2) = 5$; $f(3) = 6$. Then f is bi-continuous.

Proof: The set of bi-open subsets of X is $\{\phi, X, \{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. The set of bi-open subsets of Y is $\{\phi, Y, \{4\}, \{4, 5\}, \{4, 6\}\}$. Then $f^{-1}(\phi) = \phi$; $f^{-1}(Y) = X$; $f^{-1}(\{4\}) = \{1\}$; $f^{-1}(\{4, 5\}) = \{1, 2\}$; $f^{-1}(\{4, 6\}) = \{1, 3\}$. Thus if U is a bi-open subset of Y , then $f^{-1}(U)$ is a bi-open subset of X . Therefore f is bi-continuous.

Example 17: Let (X, P, Q) and (Y, C, D) be defined as in Example 15. Define g , mapping Y into X , by $g(4) = 2$, $g(5) = 1$, $g(6) = 3$. Then g is not bi-continuous.

Proof: By definition of inverse, $g^{-1}(\{1\}) = \{5\}$; $g^{-1}(\{1, 2\}) = \{4, 5\}$; $g^{-1}(\{1, 3\}) = \{5, 6\}$; $g^{-1}(\{2, 3\}) = \{4, 6\}$. Although

$\{1\}$ is a bi-open subset of X , $g^{-1}(\{1\}) = \{5\}$ is not a bi-open subset of Y . Thus g is not bi-continuous.

Definition 15: Let (X, P, Q) and (Y, C, D) be bi-topological spaces. Let f map X into Y . Let $x \in X$. Then f is said to be bi-continuous at x provided, if given any set V which is a bi-open subset of Y and such that $f(x) \in V$, then there exists a set U which is a bi-open subset of X , and such that $x \in U$ and $f(U) \subset V$.

Example 18: Let (X, P, Q) and (Y, C, D) be defined as in Example 15. If g is defined as in Example 17, then g is bi-continuous at a point and is not bi-continuous.

Proof: Since $\{4\}$ is bi-open, and each bi-open subset of Y which contains $g(4) = 2$ also contains $g(\{4\}) = \{2\}$, g is bi-continuous at $4 \in Y$. It was shown, however, that g is not bi-continuous.

Theorem 26: If (X, P, Q) and (Y, C, D) are bi-topological spaces, then f , mapping X into Y , is bi-continuous if and only if f is bi-continuous at each point of X .

Proof: Suppose (X, P, Q) and (Y, C, D) are bi-topological spaces. Let f map X into Y .

Suppose that f is bi-continuous. Let $x \in X$. Let V be a bi-open set such that $f(x) \in V$. Since $f(x) \in V$, $x \in f^{-1}(V)$. But f is bi-continuous. Therefore $f^{-1}(V)$ is bi-open. Thus $x \in f^{-1}(V)$ and $f(f^{-1}(V)) \subset V$. Therefore f is bi-continuous at x .

Suppose f is bi-continuous at every point $x \in X$. Let U be a bi-open subset of Y . Let $x \in f^{-1}(U)$. Then $f(x) \in U$. By Definition 15, there is a bi-open set $G_x \subset X$ such that $x \in G_x$, and $G_x \subset f^{-1}(U)$. Such a bi-open set can be found for each x in X .

It remains to show that $f^{-1}(U) = \cup \{G_x \mid x \in f^{-1}(U)\}$. Let $y \in f^{-1}(U)$. Then $f(y) \in U$. So $y \in G_y \subset \cup \{G_x \mid x \in f^{-1}(U)\}$. Thus $f^{-1}(U) \subset \cup \{G_x \mid x \in f^{-1}(U)\}$. Let $y \in \cup \{G_x \mid x \in f^{-1}(U)\}$. Then there is an x in X such that $y \in G_x \subset f^{-1}(U)$. Therefore $y \in f^{-1}(U)$. Hence $\cup \{G_x \mid x \in f^{-1}(U)\} \subset f^{-1}(U)$. So $f^{-1}(U) = \cup \{G_x \mid x \in f^{-1}(U)\}$.

Since each G_x is bi-open, $f^{-1}(U)$, which is a union of bi-open sets, is bi-open. Thus f is bi-continuous.

Theorem 27: If (X, P, Q) and (Y, P, Q) are bi-topological spaces and f , mapping X into Y , is bi-continuous, and F is a bi-closed subset of Y , then $f^{-1}(F)$ is a bi-closed subset of X .

Proof: Let (X, P, Q) and (Y, C, D) be bi-topological spaces. Let f , mapping X into Y , be bi-continuous. Let F be a bi-closed subset of Y . Since F is bi-closed, $X - F$ is bi-open. So $f^{-1}(X - F)$ is bi-open, and $f^{-1}(X - F) = f^{-1}(X) - f^{-1}(F) = Y - f^{-1}(F)$. Since $Y - f^{-1}(F) = f^{-1}(X - F)$, $Y - f^{-1}(F)$ is bi-open. But then $f^{-1}(F) = Y - (Y - f^{-1}(F))$ is a bi-closed subset of X .

Theorem 28: If (X, P, Q) and (Y, C, D) are bi-topological spaces, and f , mapping X into Y , is such that, if F is a

bi-closed subset of Y , then $f^{-1}(F)$ is a bi-closed subset of X , then f is a bi-continuous function.

Proof: Let (X, P, Q) and (Y, C, D) be bi-topological spaces. Let f , mapping X into Y , be such that, if F is a bi-closed subset of Y , then $f^{-1}(F)$ is a bi-closed subset of X . Let U be a bi-open subset of Y . Then $U = Y - (Y - U)$, and $(Y - U)$ is bi-closed. So $f^{-1}(U) = f^{-1}(Y - (Y - U)) = f^{-1}(Y) - f^{-1}(Y - U) = X - f^{-1}(Y - U)$. Since $(Y - U)$ is bi-closed, $f^{-1}(Y - U)$ is a bi-closed subset of X . Then $f^{-1}(U) = X - f^{-1}(Y - U)$, which is a bi-open subset of X . Therefore f is bi-continuous.

Definition 16: Let (X, P, Q) and (Y, C, D) be bi-topological spaces. Let f map X into Y . Then f is said to be a bi-open function provided, if U is a bi-open subset of X , then $f(U)$ is a bi-open subset of Y .

Example 19: Let (X, P, Q) and (Y, C, D) be defined as in Example 15. Define h , mapping Y into X , by $h(4) = 1$; $h(5) = 2$; $h(6) = 3$. Then h is bi-open and not bi-continuous.

Proof: Clearly $h(\phi) = \phi$; $h(Y) = X$; $h(\{4\}) = \{1\}$; $h(\{4, 5\}) = \{1, 2\}$; $h(\{4, 6\}) = \{1, 3\}$. Thus if U is a bi-open subset of Y , then $h(U)$ is a bi-open subset of X , and h is a bi-open function. However $h^{-1}(\{2, 3\}) = \{5, 6\}$, which is not a bi-open subset of Y . Therefore h is not bi-continuous.

Example 20: Let (X, P, Q) , (Y, C, D) , and f be defined as in Example 16. Then f is bi-continuous and not bi-open.

Proof: It was shown that f is bi-continuous. However $\{2, 3\}$ is a bi-open subset of X , but $f(\{2, 3\}) = \{5, 6\}$ is not a bi-open subset of Y . Thus f is not bi-open.

Definition 17: Let (X, P, Q) and (Y, C, D) be bi-topological spaces. Let f map X into Y . Then f is said to be a bi-closed function provided, if F is a bi-closed subset of X , then $f(F)$ is a bi-closed subset of Y .

Example 21: Let (X, P, Q) and (Y, C, D) be defined as in Example 16. Let h be defined as in Example 19. Then h is a bi-closed function which is not bi-continuous.

Proof: The set of bi-closed subsets of X is $\{\phi, X, \{2, 3\}, \{3\}, \{2\}, \{1\}\}$, and the set of bi-closed subsets of Y is $\{\phi, Y, \{5, 6\}, \{6\}, \{5\}\}$. It was shown that h is not bi-continuous. However $h(\phi) = \phi$; $h(Y) = X$; $h(\{5, 6\}) = \{2, 3\}$; $h(\{5\}) = \{2\}$; $h(\{6\}) = \{3\}$. Thus if F is a bi-closed subset of Y , then $h(F)$ is a bi-closed subset of X . So h is bi-closed.

Example 22: Let (X, P, Q) and (Y, C, D) and f be defined as in Example 16. Then f is bi-continuous and not bi-closed.

Proof: It was shown that f is bi-continuous. However $\{1\}$ is a bi-closed subset of X , and $f(\{1\}) = \{4\}$ is not a bi-closed subset of Y . Thus f is not a bi-closed function.

Example 23: Let (X, P, Q) and (Y, C, D) be defined as in Example 16. Define f , mapping X into Y , by if $x \in X$, then $f(x) = 6$. Then f is bi-closed and is not bi-open.

Proof: Since $\{6\}$ is bi-closed, if F is a bi-closed subset of X , then $f(F) = \{6\}$ is a bi-closed subset of Y . However $\{6\}$ is not bi-open. So if O is a bi-open subset of X , then $f(O) = \{6\}$ is not a bi-open subset of Y . Thus f is a bi-closed function which is not bi-open.

Example 24: Let (X, P, Q) and (Y, C, D) be defined as in Example 16. Define g , mapping X into Y , by if $x \in X$, then $g(x) = 4$. Then g is a function which is bi-open and not bi-closed.

Proof: Since $\{4\}$ is a bi-open subset of Y , if O is a bi-open subset of X , then $g(O) = \{4\}$, is a bi-open subset of Y . Thus g is a bi-open function. However if F is a bi-closed subset of X , then $g(F) = \{4\}$, which is not a bi-closed subset of Y . Thus g is not a bi-closed function. Hence g is a function which is bi-open and not bi-closed.

Theorem 29: If (X, P, Q) and (Y, C, D) are bi-topological spaces, S is a sequence in X which bi-converges to $x \in X$, and f , mapping X into Y , is bi-continuous, then $f(S)$ bi-converges to $f(x) \in Y$.

Proof: Let (X, P, Q) and (Y, C, D) be bi-topological spaces. Suppose f , mapping X into Y , is bi-continuous. Suppose S is a sequence in X which bi-converges to $x \in X$. Let O be a

bi-open subset of Y such that $f(x) \in O$. Then $x \in f^{-1}(O)$.

Since S bi-converges to x , there is a positive integer N such that for all integers n , with $n \geq N$, $S_n \in f^{-1}(O)$. But $S_n \in f^{-1}(O)$ implies that $f(S_n) \in O$. Therefore $f(S)$ bi-converges to $f(x)$.

SUMMARY

The author has introduced a definition of a bi-open set in a bi-topological space. It has been shown that with this definition it is possible to develop properties of bi-topological spaces which closely resemble properties of topological spaces. Some of these properties are bi-closure, bi-interior, bi-convergence, and bi-continuity. A remaining problem is how well the definition of bi-open adapts to concepts such as bi-compactness and bi-connectedness.

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