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Hex is a tree game with several interesting properties. It can be shown that Hex can never end in a draw, and that the player who moves first can win if he plays correctly. Also the player who moves first can lose if he plays his first move incorrectly. The problem is thus to determine the correct moves for this player to win.

Small games of Hex are analyzed, and the possibilities for the first move investigated. A winning strategy for 7 x 7 Hex is given.

AN INTRODUCTION TO THE GAME OF HEX

by

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Introduction

Hex is a tree game (defined in chapter II) with several interesting properties. Hex never ends in a draw. Either White, the player who moves first, or Black, the other player, can force a win. If Black can force a win, then White can force a win. This leads to the conclusion that White can force a win. All of this is established in chapter II.

Although it is known that White can force a win if he plays correctly, a winning strategy for the general n x n Hex game, where n is any positive integer, has yet to be discovered. Chapter III investigates some basic principles of strategy and gives paired strategies for Hex games of sizes 2 x 2, 3 x 3, 4 x 4, and 5 x 5.

In chapter IV some winning and losing opening moves for White are explored. And finally a winning strategy for White in a 7 x 7 Hex game is developed.

CHAPTER II

The Outcome of Hex and Beck's Hex

Definition: A hexagon for the purposes of this paper is a regular hexagon as defined in the usual Euclidean geometry, together with the area enclosed by it.

Definition: A Hex board is a set of n^2 hexagons arranged in the form of a rhombus, where n is a positive integer (see figure 1 below). For the sake of this discussion the Hex board will be positioned with the top, bottom, left and right edges as shown so that an acute corner is located in the lower left.

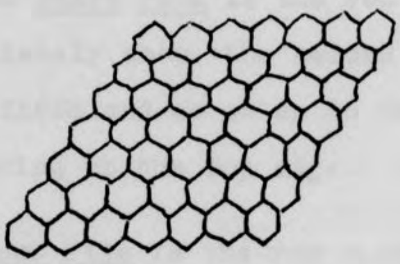

Figure 1. A 7 x 7 Hex Board.

Definition: A move consists of a player's placement of a marker on one of the hexagons of the Hex board, which then remains there for the remainder of the game.

<u>Definition</u>: A set of two or more hexagons is said to be <u>connected</u> (or equivalently <u>joined</u> or <u>linked</u>) provided it is connected as a point set with the usual plane topology. That is a set of two or more hexagons is said to be <u>connected</u> if it is not the union of two mutually separated sets.

<u>Definition</u>: A <u>white chain</u> is a set of connected hexagons, each containing a white marker.

<u>Definition</u>: A <u>black chain</u> is a set of connected hexagons, each containing a black marker.

<u>Definition</u>: The <u>first rank</u> is the row of hexagons which makes up the bottom edge of the Hex board. The <u>second rank</u> is the row of hexagons adjacent to and immediately above the first rank. The <u>third rank</u> is the row of hexagons adjacent to and immediately above the second rank. Similarly define the fourth, fifth and so on up to the n^{th} rank, which is the row making up the top edge.

<u>Definition</u>: The <u>first file</u> is the row which makes up the left edge of the Hex board. The <u>second file</u> is the row of hexagons adjacent to and immediately to the right of the first file. The <u>third file</u> is the row of hexagons adjacent to and immediately to the right of the second file. Similarly define the fourth, fifth on up to the n^{th} file, which is the right edge of the Hex board.

To specify a particular hexagon, it is only necessary to give the numbers of its rank and file respectively. Ordered pair notation will be used to give the rank and file numbers of any hexagons under consideration. For example (3, 5) refers to the hexagon located in the third rank and fifth file of some Hex board.

Rules of Hex: There are two players and they alternate moves throughout the game. The player who moves first is called White, and the other player is called Black. White starts the game by placing a white marker on any hexagon of the Hex board. Black then moves a black marker to any of the remaining hexagons. Then White moves; then Black, and so forth for rest of the game. No hexagon can have more than one marker played on it.

White's purpose is to construct a white chain which includes one hexagon from the first rank and one hexagon from the n^{th} rank. Black's purpose is to form a black chain which includes one hexagon from the first file and one hexagon from the n^{th} file.

The game is over as soon as either player accomplishes his purpose or when there are no more vacant hexagons on which to play.

Lemma 1: White and Black cannot both succeed in forming their respective chains. That is White and Black cannot both win.

Proof: The rules stipulate the game is concluded as soon as either player has succeeded in forming his chain to join his two edges. Since the players alternate moves, it is clear that one player must win at least one move before the other.

Lemma 2: If every hexagon on the Hex board is covered with a white or black marker there is either a white chain joining a hexagon in the first rank to a hexagon in the n^{th} rank, or there is a black chain joining a hexagon in the first file with a hexagon in the n^{th} file.

Proof: Suppose the Hex board is completely covered with white and black markers and there is neither a white chain joining one of the hexagons of the top rank to one of the hexagons of the bottom rank, nor a black chain joining one of the hexagons of the first file to one of the hexagons of the n^{th} file.

For the sake of argument, add another row of hexagons to the left side of the Hex board, to form another file, and place a black marker in each of these additional hexagons. Denote the Hex board together with this additional file of black markers as the augmented Hex board.

Define a pair as a set of two connected hexagons, one of which is covered by a white marker, called the left hexagon, the other of which is covered by a black marker, called the right hexagon.

There are at most two markers adjacent to both of the members of the pair. Using the orientation determined by the pair, the marker in front of the pair is called the preceding marker, if it exists. Denote the other marker which is adjacent to both of the markers of the pair as the following marker, if it exists.

Note that every pair has associated with it either a preceding marker or a following marker, or both. And every pair need not have associated with it both a preceding and a following marker.

Suppose a given pair has associated with it a preceding marker. A new pair can now be formed by using the preceding marker and the marker of the opposite color in the given pair. This new pair is called the preceding pair relative to the given pair. Similarly, if the given pair has associated with it a following marker, a following pair is determined relative to the given pair.

Now consider the black markers in the top rank of the Hex board. There is at least one black marker which is joined to the left side of the augmented Hex board by a black chain. Choose the right most such black marker in the top rank. This marker is not on the right side of the board since by assumption there does not exist a black chain joining the left side to the right side of the Hex board. Therefore the marker immediately to the right of the black marker under consideration is a white marker. These two markers form a pair with a preceding marker, but with no following marker. Denote this pair as the <u>initial pair</u>.

Since the initial pair has a marker preceding it, there is a pair preceding the initial pair; denote this pair as the <u>second pair</u>. Note that the black marker(s) in the initial and second pairs are connected as defined above, and thus form a black chain. The white markers of the initial and second pairs are also connected and thus form a white chain. To continue this process, if the i^{th} pair is defined, and has a marker preceding it, the $(i + 1)^{st}$ pair is defined as the pair preceding the i^{th} pair.

Suppose that k pairs have been defined as indicated above, where k is some positive integer, with the property that the set of white markers of the k pairs form a white chain, and the set of black markers form a black chain. If there is a marker preceding the k^{th} pair, then the $(k + 1)^{st}$ pair is the pair preceding the k^{th} pair, as above, and has the property that the white markers of the $(k + 1)$ pairs form a white chain, and the black markers of the $(k + 1)$ pairs form a black chain.

Suppose the k^{th} pair does not have a preceding marker. Then the k^{th} pair must be on an edge. The k^{th} pair cannot be on the top edge to the left of the initial pair, since by assumption the black marker of the initial pair is joined to the left edge by a black chain. Also the k^{th} pair cannot be on the top edge to the right of the initial pair, since the black marker of the k^{th} pair is connected to the black marker of the initial pair with a black chain and the black marker of the initial pair is in turn connected to the left edge by a black chain, thus violating the assumption that the black marker of the initial pair is the right most black marker connected to the left edge of the augmented Hex board by a black chain. The k^{th} pair cannot be on the left side of the augmented Hex board since both markers of the k^{th} pair would then have to be black contrary to the definition of a pair. The k^{th} pair cannot be on the bottom edge, since the white marker of the initial pair would then be connected to the white marker of the k^{th} pair and would violate the assumption that there does not exist a white chain joining top and bottom edges of the Hex board. Finally, the k^{th} pair cannot be on the right edge since the black marker of the k^{th} pair would then be joined to the left edge of the augmented Hex board through the black marker of the initial pair. This would violate the original assumption that there does not exist a black chain connecting left and right edges of the Hex board. Therefore, the k^{th} pair cannot be on any edge, and thus must have a preceding marker. Therefore, for any positive integer, k, there is always a $(k + 1)^{st}$ pair.

Suppose the markers of the initial pair can be identified by two different integers. That is, suppose there is a positive integer, r, greater than one so that the markers of the initial pair are identically the markers of the r^{th} pair. Since the r^{th} pair is defined in terms of the $(r - 1)^{st}$ pair, the r^{th} pair must have a following marker. But as noted above the initial pair does not have a following marker; so the markers of the initial and r^{th} pair cannot be identically the same. Therefore there is no such number r different from one which identifies the initial pair.

Suppose there is a pair of markers which is identified by the above system with two different integers, say h and j so that $h \neq j$. From the definition of pair, the h^{th} and j^{th} pairs have the same preceding and following markers. Therefore the $(h - 1)^{st}$ and the $(j - 1)^{st}$ pairs must be identically the same markers. Let q be the greatest positive integer so that $(h - q)$ and $(j - q)$ represent the same pair. The $(h - q)^{th}$ and the $(j - q)^{th}$ pair does not have a following marker, since if it did it would determine a following pair identified by $(h - q - 1)$ and also by $(j - q - 1)$ contradicting the definition of q. Therefore the $(h - q)^{th}$ and the $(j - q)^{th}$ pair must be the initial pair since it is the only pair with no following marker. But this is impossible since the initial pair cannot be represented by two different integers. Thus each pair can be included exactly once in the sequence of pairs. Since there can be only a finite number of pairs on an n x n Hex board, there is a positive integer, m, so that the m^{th} pair is defined and exists but with no pair preceding it. This contradicts the above argument that every pair has a pair preceding it.

Therefore there is either a white chain joining the top to the bottom of the Hex board, or there is a black chain joining the left side to the right side of the Hex board.

Theorem 1: A game of Hex never ends in a draw.

Proof: Since both players cannot complete their respective chains, and the game must continue until either a player has won or the board has no more vacant hexagons, by lemma 2 a draw is impossible.

Definition: A game is a tree game provided:

1) At each turn, the number of possible moves a player can make is finite.

2) There are exactly two players, and they alternate turns.

3) There is a certain number of turns a game can last, called the order of a game, and when that number is reached the outcome of the game is decided.

4) Each player knows the moves of the other player.

Definition: By the outcome of a game for a given player is meant upon the completion of the game, the assignment of one of the values: win, lose or draw to the player.

Definition: Denote the player who moves first in a tree game as White; denote the other player as Black.

Definition: The natural outcome of a game is:

1) "White to win" if White can force a win, despite any possible moves Black can make.

2) "Black to win" if Black can force a win, despite any possible moves White can make.

3) "draw" if neither White nor Black can force a win.

Lemma 3: Every tree game has a natural outcome.

Proof: Every tree game of order one has a natural outcome, since White alone moves and wins if possible, and draws if he can't win, and allows Black to win if he can neither win nor draw.

Assume all tree games of orders 1, 2, 3, ..., n - 1, have a natural outcome. Consider a tree game of order n, and suppose White has k possible first moves. After White has chosen a first move, say alternative j, the remaining portion of the game is a tree game of order n - 1, with Black assuming White's role in moving first. Therefore every tree game of order n has a natural outcome, and by induction every tree game must have a natural outcome.

Lemma 4: In Hex, either White or Black can force a win.

<u>Proof</u>: Hex being a tree game has a natural outcome. By theorem 1 above, a game of Hex cannot end in a draw; therefore either "White to win" or "Black to win" is the natural outcome of Hex.

<u>Lemma 5</u>: If Black can force a win in Hex, then White can force a win.

<u>Proof</u>: If Black has a winning strategy, White can use this strategy by simply moving anywhere and pretending he hasn't moved. So now Black assumes White's role by moving first, by ignoring the move White has made. If at some point White needs to move to the hexagon occupied by his first marker, he merely moves anywhere else and pretends he has moved to the hexagon occupied by his first marker.

Thus if not moving first is an advantage, White can essentially throw away a move and put himself in Black's preferred position.

<u>Theorem 2</u>: The natural outcome for Hex is "White to win".

<u>Proof:</u> Since either White can force a win or Black can force a win, and if Black can force a win then White can force a win, it must be the case that White can always force a win.

<u>Definition;</u> <u>Beck's</u> <u>Hex</u> is the game of Hex with the restrictions

that the n x n board have the integer n at least 2, and that Black is given the choice of where White must play his first marker.

Lemma 6: If White can force a win in Beck's Hex, then Black can force a win in Beck's Hex.

Proof: Let Black move a white marker to an acute corner of the Hex board. And then let Black play a black marker next to the white marker and on a black edge (i.e. either the right or left edge, depending on which acute corner Black has chosen to place White's marker). Now it is necessary only to show that Black's position is at least as strong as White's position before the first move.

Black now plays as if both of the markers so far played are black, and thus follows White's supposed winning strategy until he thinks he has won, still believing the marker in the acute corner is black.

If Black has joined the left side of the board to the right side with a black chain, then Black has won. If, however the White marker in question is part of Black's winning chain, it must be the case that Black has moved a black marker into the only other hexagon adjacent to the white marker in the acute corner, or the black chain contains the black marker Black played next to the white marker he played for White's first move. In the first case the black marker adjacent to the first white marker is also adjacent to the black marker on the Black edge, so the white marker in question is not necessary to the black chain. In the second case the black chain already includes the black marker which Black played next to the white marker in the acute corner, and thus the white marker is still unnecessary to the black chain, since Black's first marker is already part of Black's edge.

So if White does have a winning strategy for Beck's Hex, then Black can use it to force a win for himself.

Theorem 3: White can lose a game of n x n Hex, where n is 2 or larger, if he plays incorrectly on his first move.

Proof: This follows directly from the previous lemma since White's move to an acute corner is clearly incorrect.

CHAPTER III

Some Principles of Strategy and Paired Strategies

Definition: A paired strategy consists of a hexagon called the initial hexagon together with a collection of pairs of hexagons with the property that White wins if he plays as follows:

1) White's first move is to the initial hexagon.

2) If Black plays to one half of a pair of hexagons, then White plays in the other hexagon.

3) If Black does not play in a hexagon which is paired with another hexagon, then White can play anywhere.

Principle 1: A marker in a hexagon adjacent to two vacant hexagons, which are on an edge of the Hex board, can be linked to the edge with a chain of the same color.

Proof: By assumption the marker is adjacent to two hexagons which are on the edge. The opponent can cover at most one of them on a given move, leaving the other alternative to the other player.

Principle 2: Two markers of the same color bordering on the same two vacant hexagons can always be connected.

Proof: As in principle 1 there are two possible ways a paired strategy for this situation. The hexagon marked "w" is the initial hexagon with a white marker in it. Each hexagon has a number in it which corresponds to another hexagon with the same number.

To show that White can force a connection between w and the edge shown, it is necessary to demonstrate that Black cannot connect the left and right edges, which is the only way Black could prevent the connection.

If Black can form a black chain from left to right, he must play a black marker to either d or g. Suppose Black has played a marker to hexagon d. White responds by playing a white marker to e. This forces Black to play to h. Now it is clear Black cannot join the left and right sides, since White now moves to f which is connected to w through e and to the edge by principle 1. Suppose Black has moved to g and forces White to move to c. If Black moves to d the above argument holds. If Black moves to h, White moves to f which is connected to w through c and to the edge using principle 1.

<u>Principle 4</u>: Given the situation illustrated in figure 4, with a white (or equivalently a black) marker in the hexagon marked "w", White (Black) can join the marker w to the edge indicated, provided White (Black) has a white (black) marker in one of hexagons c, d or e, and Black (White) has not moved in any of the ten hexagons and it is Black's (White's) first move.

a paired strategy for this situation. The hexagon marked "w" is the initial hexagon with a white marker in it. Each hexagon has a number in it which corresponds to another hexagon with the same number.

To show that White can force a connection between w and the edge shown, it is necessary to demonstrate that Black cannot connect the left and right edges, which is the only way Black could prevent the connection.

If Black can form a black chain from left to right, he must play a black marker to either d or g. Suppose Black has played a marker to hexagon d. White responds by playing a white marker to e. This forces Black to play to h. Now it is clear Black cannot join the left and right sides, since White now moves to f which is connected to w through e and to the edge by principle 1. Suppose Black has moved to g and forces White to move to c. If Black moves to d the above argument holds. If Black moves to h, White moves to f which is connected to w through c and to the edge using principle 1.

Principle 4: Given the situation illustrated in figure 4, with a white (or equivalently a black) marker in the hexagon marked "w", White (Black) can join the marker w to the edge indicated, provided White (Black) has a white (black) marker in one of hexagons c, d or e, and Black (White) has not moved in any of the ten hexagons and it is Black's (White's) first move.

Figure 4. Principle 4.

Figure 5. A paired strategy for Principle 4.

Proof: If White has a marker in hexagon w and one in hexagon d, then he can connect w to the edge with a white chain by using principle 2 to join w to d and principle 1 to link d to the edge.

Suppose White has markers at w and e (the case for markers at w and c follows symmetrically). Figure 5 shows a paired strategy for this case. If Black can prevent White's connection, he must form a black chain which joins one of the hexagons on the left side of the diagram to either of hexagons b or i. Suppose Black has moved to b and White moves to a according to the above strategy. Black must move to d to which White answers with c. But now it is clear White has won, since c is connected to w through a, and to the edge by principle 1. Black cannot make a connection from the left side to hexagon i, because White responds to i with h and the white markers at h and e isolate the black marker at i. So White can always connect w to the edge provided he has a marker in any of hexagons c, d or e.

Black can use White's strategy if he has a black marker at w and another black marker in any of hexagons c, d or e. The proof of this is identical to the White version except for the substitution of "black" for "white" and vice versa everywhere in the above argument.

<u>Strategy 1</u>: Figure 6 gives a paired strategy for 2 x 2 Hex.

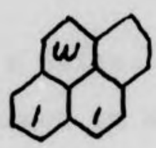

Figure 6. A Paired Strategy for 2 x 2 Hex.

<u>Proof</u>: The hexagon marked w can be connected to the first rank by principle 1.

From the diagram it is clear that White can win by moving to hexagon (1, 2) for his first move, since he has only to rotate the Hex board 180° and use the above strategy.

If White makes his first move to an acute corner, Black can win as in Beck's Hex.

Strategy 2: Figure 7 gives a paired strategy for 3 x 3 Hex with White's initial hexagon in the center of the shorter diagonal of the Hex board, (2, 2).

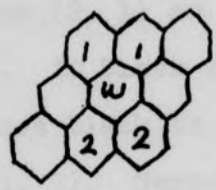

Figure 7. A Paired Strategy for 3 x 3 Hex.

Proof: White's initial hexagon, w, is connected to top and bottom edges by principle 1.

Strategy 3: Figure 8 gives a paired strategy for 3 x 3 Hex with White's initial hexagon in an obtuse corner of the Hex board; i.e. either of hexagons (3, 1) or (1, 3) are possible initial hexagons.

Figure 8. A Paired Strategy for 3 x 3 Hex.

Proof: Figure 8 shows the strategy for White's initial hexagon at (3, 1). The strategy for White's initial hexagon at (1, 3) is the same as for (3, 1) if the Hex board is rotated through 180°.

If it is possible for Black to defeat this strategy, he must connect either of hexagons (2, 1) or (1, 1) to the right edge with a black chain. If Black moves to (2, 1) White moves to (2, 2) and forces Black to move to (1, 2), and White wins by moving to (1, 3). Suppose Black starts by moving to (1, 1). White responds with (2, 3). If Black moves to (2, 1) the above argument holds. So Black must move to (1, 2), and White counters with (1, 3). Now it is clear that White has won since (2, 3) is connected to the bottom edge of the board through (1, 3) and to the top of the board by principle 1. Black cannot connect the right edge to either (1, 1) or (2, 1) of the left edge, and so White must win.

Strategy 4: Figure 9 gives a paired strategy for 3 x 3 Hex where White's initial hexagon is either (2, 1) or (2, 3) which are not located on the shorter diagonal of the Hex board.

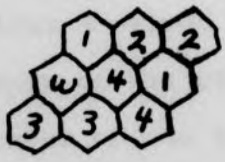

Figure 9. A Paired Strategy for 3 x 3 Hex.

Proof: Figure 9 gives the strategy with White's initial hexagon at (2, 1). By rotating the Hex board 180°, figure 9 gives the strategy for (2, 3).

Black cannot connect (1, 1) to the right edge of the Hex board since White's initial hexagon at (2, 1) is connected to the bottom by principle 1. Thus if Black is to prevent a white chain from top to bottom of the Hex board, he must link one of the three right hexagons to (3, 1). Suppose Black moves to (2, 3) and White answers with (3, 1). White has won since (2, 1) is connected to the top through (3, 1) and to the bottom by principle 1. If Black started by moving to (3, 3), White's response at (3, 2) forces Black to (2, 3) and the above argument holds. If Black moves to (1, 3), again White's response to (2, 2) forces Black to (2, 3) which has been shown to be a losing position for Black. Therefore Black cannot prevent White from winning.

Strategy 4 demonstrates that White's initial move does not have to be along either diagonal of the Hex board for White to win, for at least one size Hex board.

Strategy 5: Figure 10 gives a paired strategy for 4 x 4 Hex with White's initial hexagon on the short diagonal, in a hexagon nearest to the center of the board. That is, White's initial hexagon may be either (3, 2) or (2, 3).

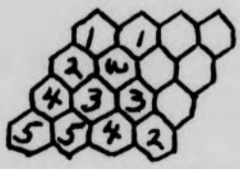

Figure 10. A Paired Strategy for 4 x 4 Hex.

Proof: Figure 10 gives a paired strategy for White's initial hexagon at (3, 2); the case for White's initial hexagon at (2, 3) is the same as figure 10 with the Hex board rotated through 180°.

White's initial marker at (3, 2) is connected to the top of the Hex board by principle 1, and to the bottom by principle 3.

Strategy 6: Figure 11 gives a paired strategy for 5 x 5 Hex with White's initial hexagon in the center of the Hex board, (3, 3).

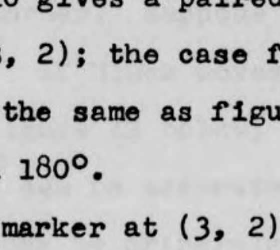

Figure 11. A Paired Strategy for 5 x 5 Hex.

Proof: White's initial marker at (3, 3) is connected to the top of the Hex board by principle 3 and likewise to the bottom.

CHAPTER IV

Some Winning and Losing Opening Moves for White

Strategy 7: In 4 x 4 Hex, White can win by moving his first marker to an obtuse corner. Suppose White moves to (4, 1) for his initial move. If Black moves to any of the hexagons with an x in it, in figure 12 below, White can move to (2, 2) and win, since (2, 2) can be connected to (4, 1) by principle 2 and to the bottom edge by principle 1.

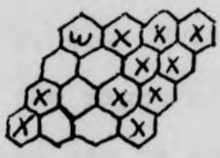

Figure 12. Some Possibilities for Black.

If Black answers White's initial move to (4, 1) with any of (3, 1), (2, 2) or (1, 2), White can move to (2, 3) which can be connected to the bottom edge by principle 1, and to the top edge either by moving to (3, 2) and linking with (4, 1) or by moving to (3, 3) and connecting to the top by principle 1.

That leaves two other possibilities for Black to counter White's initial move to (4, 1), namely (1, 3) and (3, 2). Suppose Black chooses (1, 3). White can answer by moving to (3, 2). If Black does not move to any of the five remaining hexagons to the left of the short diagonal, White can move to (2, 1) and win, since (2, 1) can be connected to the bottom by principle 1 and to (3, 2) by principle 2 and thus to the top. If Black moves to one of the five vacant hexagons to the left of the short diagonal, White can move to (1, 4). Black must now move to (2, 3) to prevent White from moving there and completing his chain to the top. So White now moves to (2, 4) and wins, since Black cannot block both (3, 3) which completes White's chain, and (3, 4) which can be connected to the top by principle 1.

Suppose Black decides to move to (3, 2) to counter White's initial move to (4, 1). White now moves to (2, 3). If Black makes a move to the right of the short diagonal, White moves to (3, 1) which can be connected to the bottom edge by either moving to (2, 1) and using principle 1 to link to the bottom, or moving to (2, 2) and linking to (2, 3) which can be connected to the bottom by principle 1. If Black moves to the left of the short diagonal with the exception of (1, 3) which is part of the principle 1 connection from (2, 3) to the bottom edge, White moves to (3, 3) which can be connected to the top by principle 1.

Therefore White can win by moving to an obtuse corner of the Hex board in a 4 x 4 game.

Strategies 5 and 7 show that White can win a 4 x 4 Hex game by moving anywhere on the short diagonal for his first move. Unlike the 3 x 3 Hex game, a 4 x 4 game can not be won by White starting from a hexagon not on the short diagonal. Suppose White makes (3, 1) his first move in a 4 x 4 Hex game. Black can win by moving to (2, 3) which is linked to the right edge by principle 1. Black has two ways of winning from this position: either by moving to (1, 2) and linking to the right edge by principle 1 and to (2, 3) by principle 2, or by moving to (4, 1) and connecting to the right edge by principle 4. White cannot stop both of these and so loses.

If White's initial move is to any of the remaining five hexagons to the left of the shorter diagonal, Black can move to (3, 2) which can be connected to the left edge by principle 1 and to the right edge by principle 3. So White cannot win if his first move is to the left of the short diagonal, and by rotating the Hex board through 180° it can be seen that White cannot move anywhere to the right of the short diagonal for his first move and expect to win. Thus the only first moves for White to win are along the short diagonal in a 4 x 4 Hex game.

This differs from the 3 x 3 game since strategy 4 demonstrates that White can win if he plays his first marker to (2, 1) which is not on the short diagonal. However (2, 1) and its counterpart (2, 3) are the only places which are not on the short diagonal from which White can win by playing his first move there. The acute corners (1, 1) and (3, 3) are losing first moves for White as noted in the discussion of Beck's Hex. The only other possibilities for White's first move are (1, 2) and symmetrically (3, 2). If White moves to either of these hexagons for his first move, Black can move to (2, 2) which can be connected to the left and right edges of the board by principle 1, and White loses. Figure 13 below shows the hexagons for White's first move which guarantee White can win if he plays correctly from then on. These are marked with a "w". If White plays first in any of the hexagons marked "l", White loses if Black plays correctly. A possible generalization of figure 13

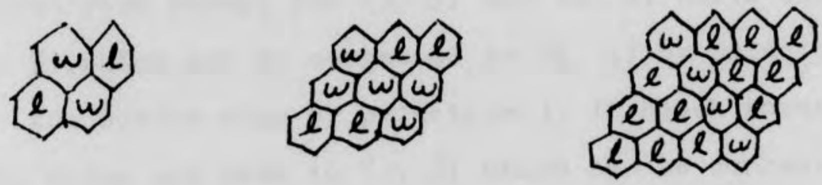

Figure 13. Winning and Losing First Moves for White.

to n x n Hex is that White can win if his first move is to any hexagon along the short diagonal, but this has not been proven.

Strategy 8: A strategy for White to win a 7 x 7 Hex game.

White's first move is to (4, 4). White would like to connect (4, 4) to the top and bottom edges, and certainly Black would like to prevent this. If Black moves to any hexagon in the first three ranks, or in the fourth rank to the left of (4, 4), White will consider Black's move as an attack against White's link from (4, 4) to the bottom edge; if Black moves to any other hexagon, White will consider the move as an attack against White's link from (4, 4) to the top edge.

Suppose Black decides to try and block White's connection from (4, 4) to the bottom edge. If Black moves anywhere in the first rank except for (1, 5) and (1, 6) White can move to (2, 5) which can be connected to (4, 4) by principle 2, and to the bottom edge by principle 1. If Black moves to (1, 6), White can move to (3, 3) which can be connected to (4, 4) by principle 2, and to the bottom edge by principle 3. If Black moves to (1, 5), White can move to (3, 4) and connect to the bottom edge either by moving to (2, 3) or to (2, 6) and using principles 1 and 2. Black cannot block both of these moves, and so White can complete his link to the bottom edge. Thus Black cannot block White's connection from (4, 4) to the bottom by moving anywhere in the first rank.

If Black moves anywhere in the second rank except for (2, 5), White can move to (2, 5) and connect to the bottom, using principles 1 and 2. If Black moves to (2, 5), White moves to (3, 3) and connects to the bottom edge using principles 2 and 3. Thus Black cannot block White's connection to the bottom by moving anywhere in the second rank.

If Black moves anywhere in the third rank other than (3, 4) or (3, 5), or in the fourth rank to the left of (4, 4), White can connect (4, 4) to the bottom edge by moving to (2, 5) and using principles 1 and 2. If Black moves to (3, 5), White can move to (3, 3) and connect to the bottom edge using principles 2 and 3. Therefore if Black can block White's connection to the bottom edge, Black must play to (3, 4). Similarly, at this point in the game the only move Black can make to prevent White from connecting (4, 4) to the top edge is to (5, 4).

Suppose Black moves to (3, 4). White now moves to (2, 4). Black can now connect the black marker at (3, 4) to either the left or right edges. White's strategy is to permit one such connection and deny the other. If Black moves to any hexagon in the intersection of the first three files and the first four ranks, White can connect (4, 4) to the bottom edge by moving to (3, 5) and then either to (2, 5) and linking to (2, 4) or to (2, 6) and using principle 1; Black cannot block both of these possibilities.

So suppose Black moves to one of the hexagons in the intersection of the first three ranks and the last three files, but not in (1, 5). White must protect his principle 1 connection from (2, 4) to the bottom if Black moves to either (1, 4) or (1, 5), by moving to the other. If Black moves to either (2, 5) or (3, 5), White moves to (4, 2); if Black moves to any other hexagon in this area, White moves to (3, 5) forcing Black to (2, 5) and then moves to (4, 2).

Suppose Black has moved to (3, 5) and White has moved to (4, 2). The white marker at (4, 2) is connected to the bottom edge by principle 4. So Black must prevent White from connecting (4, 2) to the top edge.

If Black moves to any hexagon in the first file other than (7, 1), or to any hexagon in the intersection of the last three files and the last four ranks, White can move to (5, 3) and win by using principles 2 and 3.

If Black moves to either (7, 1) or (6, 2), White moves to (5, 3) and connects to the top by either (6, 3) and principle 1, or by moving to (5, 5) and using principle 2 twice and principle 3. Black cannot prevent both possibilities for White's connection to the top.

If Black moves to (4, 3), White moves to (5, 2) which gives Black two possibilities to prevent White from connecting (5, 2) to the top by either (7, 1) and principle 2 or (6, 2) and principle 1, and these are for Black to move to (7, 1) or (6, 2). White now moves to (5, 3) which forces Black to (6, 3). Now White wins by moving to (5, 5) and connecting to (4, 4) with principle 2 and to the top by principle 3. Therefore Black's original counter to White's (4, 2), (4, 3) is a poor choice for Black.

If Black moves to (7, 2), White moves to (7, 1). Black must move to either (6, 1) or (5, 2) to prevent White from making a principle 2 connection to (7, 1). If Black moves to (6, 1), White moves to (5, 3) and wins by either moving to (6, 2) and connecting to (7, 1) or by moving to (5, 5) and connecting to the top by principle 3, and to (4, 4) by principle 2. If on the other hand, Black moves to (5, 2), White moves to (4, 3). Now Black must prevent White from connecting (4, 4) to the top. By an argument similar to the discussion of Black's first move, the only possibility for Black to block White's connection from (4, 4) to the top of the Hex board is for Black to move to (5, 4). White's response is to (6, 4). Now if Black moves to any hexagon in the intersection of files 5, 6 and 7 with ranks 5, 6 and 7, White wins by moving to (5, 3) and connecting to either (7, 1) or to (6, 4). So Black must move to one of hexagons (6, 2), (6, 3) or (5, 3). White now moves to (4, 6) which is connected to the top of the board by principle 4. This forces Black to (4, 5). White now moves to (2, 7) which forces Black to (1, 7). Then White moves to (2, 6) forcing Black to (1, 6), and finally White moves to (2, 5) and connects to (2, 4). If Black had moved a marker to (2, 6) earlier, and White responded by moving to (3, 5) before moving to (4, 2), White's marker at (4, 6) would be connected to (4, 4) and thus to the bottom, and by principle 4 to the top.

If Black moves to (7, 3) to counter White's move to (4, 2), White moves to (6, 2) which forces Black to (5, 2). Now White moves to (4, 3) forcing Black to (5, 3). White can now win by moving to (5, 4) and either connecting to (6, 2) or moving to (6, 5) and using principles 1 and 2, in order to connect to the top.

If Black moves to counter White's (4, 2) to (7, 4), White moves to (6, 3) which is connected to the top edge by principle 1, and to (4, 4) by principle 2. If Black does not move to any of hexagons (6, 2), (5, 2), (5, 3), (4, 3) or (5, 4), White can connect (4, 2) to (6, 3) by moving to (5, 2) or (5, 3) and using principle 2. If Black moves to (6, 2) or (5, 2) White moves to (4, 3) and connects to (4, 4) which in turn is already connected to the top through (6, 3) and principles 1 and 2. If Black moves to (4, 3), (5, 3) or (5, 4), White moves to (6, 1) and connects to the top edge through (6, 3) by moving to (6, 2) or moves directly to (7, 1). So White would win.

If Black moves to (6, 3) to counter White's move to (4, 2), White moves to (5, 3) which forces Black to move to (6, 2). Now White can win by moving to (5, 5) and using principle 2 twice and principle 3. And again White would win.

If Black moves to (6, 4) to counter White's (4, 2), White moves to (6, 3) and wins as before if Black had moved to (7, 4).

If Black counters White's move to (4, 2) by moving to (5, 2), White moves to (4, 3). The only way Black can prevent White from directly connecting (4, 4) to the top of the board is for Black to move to (5, 4). White answers with (6, 4). Now White can win by one of two ways. If White moves to (5, 3), then this marker can be connected to (6, 4) and to the top by principle 1, or to (6, 2) and use principle 1. If White moves to (4, 6), he can win by connecting this marker to the top by principle 4 and to the bottom by either connecting to (4, 4) or by moving to (2, 7) and connecting to (2, 4) as before.

If Black moves to counter (4, 2) by playing to (5, 3), White moves to (5, 4), which is connected to the top of the board by principle 3. This forces Black to (4, 3). White now moves to (6, 1) which is connected to (4, 2) by principle 2 and forces Black to (7, 1). White's next move is to (6, 2) which forces Black to (7, 2). White's next move is to (6, 3) which forces Black to (7, 3). White now wins by moving to (6, 5) and connecting to the top by principles 1 and 2.

Black's only other serious counter to White's (4, 2) is (5, 4). White responds to this by moving to (6, 4). If Black moves to the intersection of the last four ranks and the last three files, White can go to (5, 3) which can be connected to the top by moving to (6, 2) and using principle 1 or moving to (6, 3) and connecting to (6, 4) which is itself connected to the top by principle 1. If Black moves to either (7, 3) or (7, 4), White moves to the other to insure his principle 1 connection from (6, 4) to the top.

If Black moves to any of (7, 1), (7, 2), (6, 1) or (6, 2), White moves to (5, 3) and forces Black to move to (6, 3). Then White moves to (4, 6) and wins by connecting it to the top by principle 4, and to the bottom by moving to (4, 5) and connecting to (4, 4) or by moving to (2, 7) and connecting to (2, 4) as before.

If Black moves to either of (4, 3) or (5, 3), White moves to (6, 1) which forces Black to (7, 1). White then moves to (6, 2) which forces Black to (7, 2). Then White moves to (6, 3) and connects to (6, 4) which is in turn connected to the top by principle 1 and wins.

If Black moves to (6, 3), White moves to (5, 3) which forces Black to one of (7, 1), (7, 2) or (6, 2). (4, 2) is now linked to (4, 4) and so White can move to (4, 6) and win as before by linking (4, 6) to the top by principle 4 and to the bottom by either connecting to (4, 4) or by moving to (2, 7) and eventually connecting to (2, 4).

If Black moves to (5, 2), White moves to (4, 3) and connects to (4, 4). White can win by moving to (5, 3) and connecting to (6, 4) or to (2, 6) and to the top by principle 1, or White can win by moving to (4, 6) as before, and forming a connection to the top by principle 4, and to the bottom by either connecting to (4, 4) or moving to (2, 7) and eventually connecting to (2, 4).

Therefore once White has moved to (4, 2), Black can not stop White from winning. So suppose on his second move, Black moves to one of the hexagons in the intersection of the first three files and the first four ranks. This permits White to move to (3, 5) and connect to the bottom edge by either moving to (2, 6) and connecting to (2, 4) which is connected to the bottom edge by principle 1, or by moving to (2, 6) and connecting to the bottom edge by principle 1.

Black must try and prevent White from connecting (4, 4) to the top, and this can only be possible if Black moves to (5, 4). White responds with (6, 4). If Black tries to connect (5, 4) to the right edge, then White moves to (5, 3) and wins by either moving to (6, 3) and connecting to (6, 4) which can be connected to the top by principle 1, or by moving to (6, 2) and connecting to the top by principle 1. If Black tries to link (5, 4) to the left side of the board, White moves to (4, 6) which can be connected to the top by principle 4 and to (3, 5) by principle 2.

Therefore White wins the 7 x 7 Hex game if he plays a marker to (4, 4) for his opening move, and follows this strategy.

CHAPTER V

Summary

It has been shown that the game of Hex never ends in a draw, and that the fact that Hex is a tree game of order n^2 implies either White or Black can force a win. And if Black can force a win, then White can force a win. Thus the natural outcome of Hex is White to win.

So White can win if he plays correctly. The discussion of Beck's Hex shows not only that White can lose, but that he can lose if he makes a mistake on his first move. The strategies for 3 x 3, 4 x 4, 5 x 5, and 7 x 7 Hex games show that White has an advantage if he moves in or near the middle of the Hex board. The short diagonal of the Hex board seems to be important for White's first move as can be seen from the discussion of 4 x 4 Hex, since White can only be sure of winning if he moves to a hexagon on the short diagonal of the 4 x 4 Hex board.

Finally, the strategy for the 7 x 7 Hex game shows how White can win a non-trivial Hex game.

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