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In this thesis, sequences are shown to be inadequate to define certain concepts in general topological spaces; the idea of a net is introduced as a generalization of a sequence, and this inadequacy is overcome. Cauchy nets of real numbers are defined and a Cauchy criterion for them is proved. The usual theorems for convergence of sequences are generalized to nets and many basic topological concepts are defined in terms of convergence of nets. Finally, alternative methods for discussing convergence in topological spaces are defined and it is shown that convergence in terms of them is equivalent to convergence in terms of nets.

CONVERGENCE IN TOPOLOGICAL SPACES

by

Dargan Frierson, Jr.

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Approved by

E. E. Posey
Thesis Adviser

APPROVAL SHEET

This thesis has been approved by the following committee of the Faculty of the Graduate School at The University of North Carolina at Greensboro

Thesis Adviser E. E. Posey

Oral Examination Committee Members Gaylord T. Hageseth

Robert L. Burnhardt

Hughes B. Hoyle, III

C. A. Church, Jr.

October 19, 1970
Date of Examination

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INTRODUCTION

This paper will show the inadequacy of sequences to define certain concepts in topological spaces as fundamental as the real numbers. It introduces a generalization of a sequence, called a net, and shows that with nets it is possible to overcome this inadequacy. The idea of a Cauchy net in the real numbers R is defined, and a Cauchy criterion for nets in R is proved. Then it is shown that subnets exist (corresponding to subsequences) and generalizations of the usual theorems on sequences are given. Basic topological concepts such as Hausdorff and compact spaces, continuous functions, and the closure operator are then shown to be definable in terms of convergence of nets. Finally, alternative methods of discussing convergence in topological spaces are given and it is shown that convergence in terms of them is equivalent to convergence in terms of nets.

CHAPTER I

NETS

It is well known that sequences are adequate to define the closed sets in all first countable topological spaces. Since every metric space is first countable, sequences are of great importance in analysis; however, there are certain topics in metric spaces, even one as fundamental as the real numbers, for which sequences are inadequate. Perhaps the most significant of these topics is the Riemann integral as a "limit" of Riemann sums.

If $f : [a,b] \to \mathcal{R}$ is a function from the closed bounded interval $[a,b]$ to the reals $\mathcal{R}$, then to find the Riemann integral of f, we proceed as follows:

A partition $P = \{x_i\}_{i=0}^{n}$ of $[a,b]$ is a finite set of points such that $a = x_0 < x_1 < x_2 < \cdot\cdot\cdot < x_{n-1} < x_n = b$. If P and Q are partitions of $[a,b]$, P is said to be <u>finer</u> than Q if $Q \subset P$. Now, let $P = \{x_i\}_{i=0}^{n}$ be a partition of $[a,b]$. We say that P' is a <u>marking</u> of P if $P' = P \cup \{\xi_i \mid i = 1,2,\cdot\cdot\cdot,n \text{ and } \xi_i \in [x_{i-1},x_i]\}$. We will say that P' is a <u>marked partition</u>, where it will be understood that we mean P' is a marking of a partition P. If P' is a marked partition of $[a,b]$ and f is any function on $[a,b]$ then define the Riemann sums by $S(f,P') = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$.

In order to find the Riemann integral of f over [a,b], it is necessary to consider a "limit" of a "sequence" of an uncountable number of partitions of [a,b] and an uncountable number of markings of any given partition. Because a sequence is a function defined on the countable set of positive integers, it obviously is inadequate to handle this limiting situation. What is needed is a generalized concept of a sequence and its limit.

A sequence has as its domain the positive integers, Z^+. We want an ordered set which has those properties of Z^+ actually necessary to define the convergence of a sequence, along with additional properties necessary to define the convergence of a generalized sequence. Listed below are some properties of Z^+ which may or may not be important in our theory of convergence.

(1) Z^+ is <u>preordered</u> by $\le$; i.e.,

(i) If $n \in Z^+$, then $n \le n$.

(ii) If $n_0, n_1, n_2 \in Z^+$ such that $n_0 \le n_1$ and $n_1 \le n_2$ then $n_0 \le n_2$.

(2) Z^+ is <u>partially ordered</u> by $\le$; i.e.,

(1) holds and if $n_0, n_1 \in Z^+$ such that $n_0 \le n_1$ and $n_1 \le n_0$, then $n_0 = n_1$.

(3) Z^+ is <u>well ordered</u> by $\le$; i.e., each non-empty subset of Z^+ has a first element.

(4) Z^+ is <u>totally ordered</u> by $\le$; i.e., if $n_0, n_1 \in Z^+$, then either $n_0 \le n_1$ or $n_1 \le n_0$.

(5) If $n \in Z^+$, then n has an <u>immediate successor</u>, $n + 1$.

(6) Z^+ is <u>countable</u>.

(7) $\leq$ is <u>compositive</u> on Z^+; i.e., if $n_0, n_1 \in Z^+$, then there is an $n_2 \in Z^+$ such that $n_0 \leq n_2$ and $n_1 \leq n_2$.

Of the above seven properties of Z^+, only (1) and (7) are used in defining the limit of a sequence. The very important fact that Z^+ is well ordered is of no consequence in the theory of sequences, as is the fundamental porperty (5) concerning immediate successors; and of course, the countability of Z^+ is a definite inadequacy, as we have seen in the example above. So, in our search for a generalization, we make the following definitions.

<u>DEFINITION 1</u>: A non-empty set D is said to be <u>directed</u> by a binary relation $\prec$ iff $(D, \prec)$ is a preordered set and $\forall a, b \in D$, $\exists c \in D$ such that $a \prec c$ and $b \prec c$.

<u>REMARK 1</u>: Here and throughout the remainder of this paper, the symbols $\forall$ and $\exists$ will be used merely as abbreviations for the words "for each" and "for some" or "there exists" respectively, and not as logical operators.

<u>DEFINITION 2</u>: If $(D, \prec)$ is a directed set and X is any topological space, then a <u>net in X</u> is a map $\varphi : D \to X$.

We must now define the concept of convergence for nets. Recall that a sequence $S : Z^+ \to X$ <u>converges</u> to a point $x_0 \in X$ or has limit x_0 $(S \to x_0)$ iff $\forall$ neighborhood (nbd) U of x_0, $\exists\, n_0 \in Z^+$ such that $\forall\, n \geq n_0$, $S(n) \in U$. Also S <u>accumulates</u>

at a point $x_0 \in X$ ($S \succ x_0$) iff $\forall$ nbd U of x_0 and $\forall n \in Z^+$, $\exists$ $m \in Z^+$, $m \geq n$, and $S(m) \in U$. When $S \to x_0$ it is said that S is <u>eventually</u> in each nbd of x_0; when $S \succ x_0$, it is said that S is <u>frequently</u> in each nbd of x_0. As with sequences, we make the following definitions.

<u>DEFINITION 3</u>: A net $\varphi : D \to X$ <u>converges</u> to a point $x_0 \in X$ ($\varphi \to x_0$) iff $\forall$ nbd U of x_0, $\exists d_0 \in D$ such that $\forall d \in D$, $d_0 \prec d$, $\varphi(d) \in U$. If $\varphi \to x_0$, we say that φ is <u>eventually</u> in each nbd of x_0.

<u>DEFINITION 4</u>: A net $\varphi : D \to X$ <u>accumulates</u> at $x_0 \in X$ ($\varphi \succ x_0$) iff $\forall$ nbd U of x_0 and $\forall d \in D, \exists d' \in D$, $d \prec d'$, $\varphi(d') \in U$. If $\varphi \succ x_0$, we say that φ is <u>frequently</u> in each nbd of x_0.

We are now in a better position to discuss the Riemann integral. Let $D = \{P' \mid P'$ is a marked partition of $[a,b]\}$. It is easy to see that D is directed by $\prec$, where if $P', Q' \in D$, $P' \prec Q'$ means $P' \subset Q'$. Now for $f : [a,b] \to R$, consider $S = \{S(f,P') \mid P' \in D\}$ of Riemann sums. S is a net defined on the directed set D of marked partitions of $[a,b]$, and if $S \to r$, $r \in R$, we call r the <u>Riemann integral of f over [a,b]</u> and usually denote r by $\int_a^b f(x)dx$. Note that the above enables us to discuss the integrability of f if we know the limit of S exists; we might ask if we can decide about the convergence of a net without knowing explicitly the limit. The answer is to be found in a "Cauchy criterion" for nets.

CHAPTER II

CAUCHY NETS

Recall that in discussing the net $\varphi : D \to X$, we said $\varphi \to x_0$ iff $\forall$ nbd U of x_0, $\exists\, d_0 \in D$ such that $\forall\, d \in D$, $d_0 \prec d$, $\varphi(d) \in U$; we see that the convergence of φ is determined by the values it assumes on points of D following some point $d_0 \in D$. With this in mind we make the following definitions:

DEFINITION 5: If $(D,\prec)$ is a directed set, then $T_a = \{d \in D \mid a \prec d\}$ is called the terminal set determined by $a \in D$.

Now if T_a is a given terminal set in a directed set D, we let $T = \{T_x \mid a \prec x\} = \{T_x \mid T_x$ is terminal in $T_a\}$. It is easy to see that T is directed by $\supset$: The reflexive and transitive properties are trivially satisfied by $\supset$, and if $T_x, T_y \in T$, then $\exists\, z \in T_a$ such that $x \prec z$ and $y \prec z$. Hence $T_x \supset T_z$ and $T_y \supset T_z$. So, $T = \{T_x \mid T_x$ is terminal in $T_a\}$ is directed by $\supset$.

We can now redefine convergence of a net as follows:

DEFINITION 6: If $\varphi : D \to X$ is a net then $\varphi \to x_0 \in X$ iff $\forall$ nbd U of x_0, $\exists\, T_a$ such that $\varphi(T_a) \subset U$, where $\varphi(T_a) = \{\ \varphi(d) \mid d \in T_a\}$.

Now in a manner similar to defining a Cauchy sequence, we make the following definition:

DEFINITION 7: Let $(D,\prec)$ be a directed set and let $\varphi: D \to R$ be a net in R. φ is said to be a Cauchy net in R iff $\forall\ \epsilon > 0,\ \exists\ T_a$ such that if $d,d' \in T_a$, then $|\varphi(d) - \varphi(d')| < \epsilon$.

THEOREM 1: (i) Every convergent net in R is a Cauchy net in R.

(ii) Every Cauchy net in R converges.

PROOF: (i) Let $\varphi : D \to R$ be a net in R which converges to $r_0 \in R$. If $\epsilon > 0$ is given, then $\exists\ T_a$ such that $\forall\ d,d' \in T_a$, $|\varphi(d) - r_0| < \epsilon/2$ and $|\varphi(d') - r_0| < \epsilon/2$.

So,
$$\begin{aligned} |\varphi(d) - \varphi(d')| &= |\varphi(d) - r_0 - \varphi(d') + r_0| \\ &= |\varphi(d) - r_0 - (\varphi(d') - r_0)| \\ &\le |\varphi(d) - r_0| + |\varphi(d') - r_0| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Therefore, $|\varphi(d) - \varphi(d')| < \epsilon$, so that φ is a Cauchy net in R.

Now in order to prove part (ii) of the theorem we proceed as follows: Let $\varphi : D \to R$ be a Cauchy net in R. For fixed but arbitrary $\epsilon > 0$, we know $\exists\ T_a$ such that $\forall\ d,d' \in T_a$, $|\varphi(d) - \varphi(d')| < \epsilon$; so, we let $\Phi(a,\epsilon) = \{\varphi(d) \mid d \in T_a\}$ be the image of the terminal set T_a under φ. We now prove the following lemmas about the $\Phi(a,\epsilon)$.

LEMMA 1: For each $\epsilon > 0$, $\Phi(a,\epsilon)$ is bounded.

PROOF: Let $\epsilon > 0$ be given. Then since φ is a Cauchy net in R, $\exists\ T_a$ such that $\forall\ d,d' \in T_a$, $|\varphi(d) - \varphi(d')| < \epsilon$.
Let $\varphi(d) \in \Phi(a,\epsilon)$. Then,

$$|\varphi(d)| = |\varphi(d) - \varphi(a) + \varphi(a)|$$
$$\leq |\varphi(d) - \varphi(a)| + |\varphi(a)|$$
$$< \epsilon + |\varphi(a)|.$$

Therefore, $\Phi(a,\epsilon)$ is bounded. Next we have Lemma 2, which is simply stated without proof.

<u>LEMMA 2</u>: For each $\epsilon > 0$, the closure of $\Phi(a,\epsilon)$, $\overline{\Phi(a,\epsilon)}$, is bounded, closed and hence compact.

<u>LEMMA 3</u>: If $a,b \in D$ such that $a \prec b$, then $T_a \supset T_b$. Hence, $\overline{\Phi(a,\epsilon_1)} \supset \overline{\Phi(b,\epsilon_2)}$.

<u>PROOF</u>: By definition, if $a,b \in D$ such that $a \prec b$ then $T_a \supset T_b$. By defintion of a function and the closure operator, $\overline{\Phi(a,\epsilon_1)} \supset \overline{\Phi(b,\epsilon_2)}$.

<u>LEMMA 4</u>: $I = \{\overline{\Phi(a,\epsilon)} \mid a \in D\}$ is directed by $\supset$.

<u>PROOF</u>: The reflexive and transitive properties of $\supset$ are trivially verified. For the compositive property, let $\overline{\Phi(a,\epsilon_1)}$, $\overline{\Phi(b,\epsilon_2)} \in I$ be given. For $a,b \in D$, we know $\exists\, c \in D$ such that $a \prec c$ and $b \prec c$, since $(D,\prec)$ is directed. So, $T_a \supset T_c$ and $T_b \supset T_c$. By Lemma 3, $\overline{\Phi(a,\epsilon_1)} \supset \overline{\Phi(c,\epsilon_3)}$ and $\overline{\Phi(b,\epsilon_2)} \supset \overline{\Phi(c,\epsilon_3)}$. Let us now make the following definition:

<u>DEFINITION 8</u>: The <u>diameter</u> of a set A of real numbers, denoted $\delta(A)$, is the lub of $\{|a - b| \mid a,b \in A\}$.

<u>LEMMA 5</u>: $\delta : I \to R$ is a net in R which converges to 0.

<u>PROOF</u>: To show $\delta[\overline{\Phi(d,\epsilon)}] \to 0$, we recall that φ is a Cauchy net in R, so that given $\epsilon > 0$, $\exists\, T_a$ such that $\forall d,d' \in T_a$, $|\varphi(d) - \varphi(d')| < \epsilon/2$. Now consider

$\overline{\Phi(a,\epsilon)} \in I$; $\forall\ \overline{\Phi(d,\epsilon')} \in I$ such that $\overline{\Phi(a,\epsilon)} \supset \overline{\Phi(d,\epsilon')}$ we have

$\left|\delta[\overline{\Phi(d,\epsilon')}] - 0\right| = \left|\delta[\overline{\Phi(d,\epsilon')}]\right| = \left|\text{lub}\{|\varphi(d) - \varphi(d')| \,\middle|\, d,d' \in T_a\}\right| < \epsilon.$

Thus, $\delta[\overline{\Phi(d,\epsilon')}] \to 0$.

<u>LEMMA 6</u>: $\bigcap_{\substack{a<d \\ d\in D}} \overline{\Phi(d,\epsilon)} = \{x\}$.

<u>PROOF</u>: We first show this intersection is non-empty. Let $\epsilon > 0$ be given and consider $\overline{\Phi(a,\epsilon)}$. Then $\bigcap_{\substack{a<d \\ d\in D}} \overline{\Phi(d,\epsilon')}$ is the intersection of a family of closed subsets of a compact set, $\overline{\Phi(a,\epsilon)}$. By the finite intersection property, if $\bigcap_{\substack{a<d \\ d\in D}} \overline{\Phi(d,\epsilon')} = \emptyset$, then there must exist a finite number of subsets of $\overline{\Phi(a,\epsilon)}$, say $\overline{\Phi(d_1\epsilon_1)}$, $\overline{\Phi(d_2,\epsilon_2)}, \cdot\cdot\cdot, \overline{\Phi(d_n,\epsilon_n)}$ such that $\bigcap_{i=1}^{n} \overline{\Phi(d_i,\epsilon_i)} = \emptyset$. We will show this is a contradiction by the following construction:

$\{d_1,d_2,\cdot\cdot\cdot,d_n\} \subset T_a$, a terminal set of the directed set $(D,\prec)$.

(i) For $\{d_1,d_2\}$, $\exists\ d_2^* \in D$ such that $d_1 \prec d_2^*$ and $d_2 \prec d_2^*$.

(ii) For $\{d_1,d_2,d_3\}$, consider $d_3,d_2^* \in D$. We know $\exists\ d_3^* \in D$ such that $d_2^* \prec d_3^*$ and $d_3 \prec d_3^*$. Hence, from (i) above and the transitive property of $\prec$, we have $d_i \prec d_3^*$, $i \in \{1,2,3\}$.

(iii) For $\{d_1,d_2,d_3,d_4\}$, consider $d_4,d_3^* \in D$. We know $\exists\ d_4^* \in D$ such that $d_3^* \prec d_4^*$ and $d_4 \prec d_4^*$. Hence, from (ii) above and the transitive property of $\prec$, we have $d_i \prec d_4^*$, $i \in \{1,2,3,4\}$.

(iv) Clearly then, $\exists\ d_n^* \in D$ such that $d_i \prec d_n^*$, $i \in \{1,2,3,\cdot\cdot\cdot,n\}$. Thus

$T_{d_i} \supset T_{d_{n*}}$, $i \in \{1,2,\cdot\cdot\cdot,n\}$, so that $\bigcap_{i=1}^{n} T_{d_i} \supset T_{d*_n}$. Hence

$\varphi(T_{d_{n*}}) \subset \varphi(\bigcap_{i=1}^{n} T_{d_i}) \subset \bigcap_{i=1}^{n} \varphi(T_{d_i})$ implies $\Phi(d_{n*},\epsilon^*) \subset \bigcap_{i=1}^{n} \overline{\Phi(d_i,\epsilon_i)}$,

so that $\bigcap_{i=1}^{n} \overline{\Phi(d_i,\epsilon_i)} \neq \emptyset$.

Now, to see that $\bigcap_{\substack{a<d \\ d\in D}} \overline{\Phi(d,\epsilon)}$ is a single point, we use Lemma 5 to observe that the diameters of the $\overline{\Phi(d,\epsilon)}$ are approaching 0.

THEOREM 1: (ii) Every Cauchy net in R converges.

PROOF: Let φ be the Cauchy net under consideration and $\epsilon > 0$ be given. By Lemma 5, $\exists\, \overline{\Phi(a,\epsilon)}$ such that $\forall\, \overline{\Phi(d,\epsilon')} \in I$, $\overline{\Phi(a,\epsilon)} \supset \overline{\Phi(d,\epsilon')}$, $|\delta[\overline{\Phi(d,\epsilon')}]| < \epsilon$. By Lemma 6, we know $\exists\, x \in R$ such that $\bigcap_{\substack{a<d \\ d\in D}} \overline{\Phi(d,\epsilon')} = \{x\}$. So, $x \in \overline{\Phi(d,\epsilon')}$; this implies $\forall\, \varphi(d') \in \overline{\Phi(d,\epsilon')}$, $|\varphi(d') - x| < \epsilon$. Therefore, $\varphi \to x$ and the proof is complete.

Now that we have a Cauchy criterion for convergence of a net, we can discuss the integrability of a function f over [a,b] without specifically knowing what value $\int_a^b f(x)dx$ has, by making the following definition.

DEFINITION 9: A function $f : [a,b] \to R$ is said to be Riemann integrable over [a,b] if and only if the net $S = \{S(f,P') \mid P'$ is a marked partition of $[a,b]\}$ of Riemann sums is a Cauchy net in R.

CHAPTER III

NETS IN GENERAL TOPOLOGICAL SPACES

After the example given above, we see that a net is a useful generalization of a sequence. In searching for generalizations of some theorems about sequences and subsequences, we first need the concept of a subnet. Our initial attempt at defining a type of subnet might be as follows.

DEFINITION 10: Let $(D,\prec)$ be a directed set and $\varphi: D \to X$ a net in a topological space X. Let $D^* \subset D$ be a subset of D which is directed by $\prec$ and denote this subdirected set by $(D^*,\prec)$. Then the map $\varphi^* = \varphi|D^* : D^* \to X$ is a net called a restricted subnet of φ. We will see that restricted subnets are of little use in our scheme of having nets as generalizations of sequences. Not even the basic relationships between sequences and subsequences are true for nets and restricted subnets, as shown in the example which follows.

Let $(\omega+\omega,<)$ be a directed set of ordinals and define $\varphi: \omega + \omega \to R$ by

$$\varphi(x) = \begin{cases} 1 & \text{if } 1 \le x \le \omega \\ 1/n & \text{if } x = \omega + n,\ 1 \le n < \omega \end{cases}.$$

Obviously, $\varphi \to 0$; but consider a subdirected set $(\omega,<)$ of $(\omega+\omega,<)$ and the restricted subnet $\varphi^* : \omega \to R$. $\varphi^* \to 1$, since φ^* is constantly defined to be 1 on ω. So, we have a net converging to one point of the space and a restricted subnet of φ

converging to a different point of the space. This is an undesirable property for subnets to have. J. L. Kelley, in order to allow for a generalization of the theorem which says "If a sequence S converges to a point in a space, then every subsequence of S converges to the same point", initially defined subnets as follows.

DEFINITION 11: Let $(D,\prec)$ and $(E,<)$ be directed sets. Then if $\varphi : D \to X$ is a net we say $S : E \to X$ is a subnet of φ iff $\exists$ a function $f : E \to D$ such that (i) $S = \varphi \circ f$ and (ii) For each $m \in D$, $\exists\, n \in E$ such that if $n < p$, then $m \prec f(p)$.

Still a third way of defining a subnet (in fact, this is the definition we shall use in proving the theorems which follow) is to let the function $f : E \to D$ be a monotone increasing function as follows.

DEFINITION 12: Let $(D,\prec)$ be a directed set and $\varphi : D \to X$ a net in a topological space X. If $(E,<)$ is a directed set and $f : E \to D$ is a function such that (i) If $e_1 < e_2$, then $f(e_1) \prec f(e_2)$ and (ii) If $d, d' \in D$, then $\exists\, e \in E$ such that $d \prec f(e)$ and $d' \prec f(e)$, then the composite $\varphi \circ f : E \to X$ is said to be a subnet of the net φ.

Since the net $\varphi : D \to X$ may be denoted $\{\varphi(d) | d \in D\} = \{\varphi_d | d \in D\}$, the subnet $\varphi \circ f : E \to X$ may be denoted $\{\varphi(f(e)) | e \in E\} = \{\varphi_f(e)\ e \in E\}$.

THEOREM 2: If $\varphi : D \to X$ is a net in a topological space X such that $\varphi \to x$, then each subnet of φ, $\varphi_f : E \to X$, converges to x.

PROOF: Suppose $\varphi \to x$. Then if U is a given nbd of x, we know $\exists\, T_a \subset D$ such that if $d \in T_a$, $\varphi(d) \in U$. Now suppose $\varphi_f : E \to X$ is a subnet of φ; by definition of a subnet, we have for $a \in T_a$, $\exists\, b \in E$ such that $a \prec f(b)$. Consider the terminal set determined by the above $b \in E$, T_b, and let $e \in T_b$. Then $b < e$, so that $f(b) \prec f(e)$. Thus, $a \prec f(e)$, so $f(e) \in T_a$. Therefore, $\varphi(f(e)) \in U$ and $\varphi_f \to x$.

THEOREM 3: If X is a Hausdorff space, then each net which converges in X, converges to exactly one point.

PROOF: Assume X is Hausdorff and let $\varphi: D \to X$ be a net in X such that $\varphi \to x$ and $\varphi \to y$, $x \neq y$. Since X is T_2, we know $\exists$ nbds U of x and V of y such that $U \cap V = \emptyset$. Since $\varphi \to x$ we know $\exists\, T_a \subset D$ such that $\forall\, d \in T_a$, $\varphi(d) \in U$; and since $\varphi \to y$, we know $\exists\, T_b \subset D$ such that $\forall d' \in T_b$, $\varphi(d') \in V$. So, given $d, d' \in D$, we know $\exists\, d^* \in D$ such that $d \prec d^*$ and $d' \prec d^*$, which implies $\varphi(d^*) \in U$ and $\varphi(d^*) \in V$, a contradiction. Therefore, φ converges to a unique point in X, and the theorem is proved.

In order to establish the converse of the above theorem, we need to recall the concept of the neighborhood system of $x \in X$; that is, the family of all neighborhoods of the point $x \in X$, denoted N_x. Note that N_x is directed by $\supset$: the reflexive and transitive properties are trivial to verify and if $A, B \in N_x$, then certainly $A \cap B \in N_x$ such that $A \supset A \cap B$ and $B \supset A \cap B$. We also need to notice that given two directed sets $(D, \prec)$ and $(E, <)$, we may direct the cross product $D \times E$ by defining a relation $\leq$ on $D \times E$, where for (d_1,e_1), $(d_2,e_2) \in D \times E$, we want $(d_1,e_1) \leq (d_2,e_2)$ to mean $d_1 \prec d_2$ and $e_1 < e_2$. We now are able to prove the following theorem.

THEOREM 4: If a convergent net $\varphi: D \to X$ converges to exactly one point of X, then X is T_2.

PROOF: Suppose X is not Hausdorff; then $\exists$ $x,y \in X$, $x \neq y$, such that if U is a nbd of x and V is a nbd of y, then $U \cap V \neq \emptyset$. Let N_x and N_y be the nbd systems of x and y respectively. We have seen that N_x and N_y are directed by $\supset$ and that $(N_x \times N_y, \leq)$ is a directed set when $\leq$ is defined as above. Now define the net $\varphi: N_x \times N_y \to X$ by $\varphi(U,V) \in U \cap V$, and notice that $\varphi \to x$ and $\varphi \to y$, since each nbd of x meets each nbd of y. Under the assumption that X is not a Hausdorff space, we have exhibited a net in X which converges to two distinct points of X, so the theorem is proved.

THEOREM 5: If $\varphi : D \to X$ is a net in X and $\varphi_f : E \to X$ is a subnet of φ such that $\varphi_f \succ\!\!\!\to x$, then $\varphi \succ\!\!\!\to x$.

PROOF: Let U be a nbd of x and $e \in E$; since $\varphi_f \succ\!\!\!\to x$, we know $\exists$ $a \in E$, $e \prec a$, such that $\varphi_f(a) \in U$. Now $e \prec a$ implies $f(e) \prec f(a)$, so that given nbd U of x and $f(e) \in D$, we have $f(a) \in D$, $f(e) \prec f(a)$, with $\varphi(f(a)) \in U$. This says that $\varphi \succ\!\!\!\to x$, and the theorem is proved.

THEOREM 6: A net $\varphi : D \to X$ accumulates at a point $x \in X$ iff there is a subnet of φ, $\varphi_f : E \to X$, which converges to x.

<u>PROOF</u>: (i) Suppose $\exists$ a subnet $\varphi_f : E \to X$ of a net $\varphi : D \to X$ such that $\varphi_f \to x$. Then φ_f certainly accumulates at x, so by the above theorem, $\varphi \succ\!\!\circ\, x$.

(ii) Let $\varphi : D \to X$ be a net which accumulates at a point $x \in X$. Then we know $\forall$ nbd U of x and $\forall\, d \in D$, $\exists\, a \in D$, $d \prec a$, such that $\varphi(a) \in U$. Let $D^* = \{a \in D \mid \varphi(a) \in U,\ U$ a nbd of $x\}$ and $N_x = \{U \subset X \mid U$ is a nbd of $x\}$. Now D^* is directed by $\prec$, since D is directed by $\prec$ and since $\varphi \succ\!\!\circ\, x$; we have seen previously that $(N_x, \supset)$ is a directed set. Also we know that the cross product $D^* \times N_x$ with the relation $\leq$ on $D^* \times N_x$, where $(a,U) \leq (b,V)$ means $a \prec b$ and $U \supset V$, is a directed set. Let $f : D^* \times N_x \to D$ be defined by $f(a,U) = a$.

The function f satisfies properties (i) and (ii) in the definition of a subnet because D is directed and because $\varphi \succ\!\!\circ\, x$. We will now show that $\varphi \circ f \to x$. Let U be a nbd of x; since $\varphi \succ\!\!\circ\, x$, we know if $d \in D$, $\exists\, a \in D$ such that $d \prec a$ and $\varphi(a) \in U$. So, consider the point $(a,U) \in D^* \times N_x$; if $(b,V) \in D^* \times N_x$ is such that $(a,U) \leq (b,V)$, then we have $a \prec b$ and $U \supset V$. Therefore, $\varphi \circ f[(b,V)] = \varphi(b) \in V \subset U$; that is $\varphi \circ f[(b,V)] \in U$, so that $\varphi \circ f \to x$.

This theorem is not true for sequences in a general topological space; R. Arens gives the following example in which a sequence accumulates to a point and yet has no subsequence which converges to that point.

Let $X = Z^+ \times Z^+ \cup \{(0,0)\}$, where Z^+ is the set of positive integers; let $C_n = n \times Z^+$. Define a topology on X as follows:

(i) If $(0,0) \notin A$, $A \subset X$, then A is open.

(ii) If $(0,0) \in U$, then U is open iff $\exists\, N \in Z^+$ such that $\forall\, n \geq N$, U contains all but a finite number of points of the n^{th} column C_n.

Note that X is Hausdorff and that $Z^+ \times Z^+ \subset X$ is discrete. Now consider the diagonal sequence $f : Z^+ \to Z^+ \times Z^+$ defined by $f(n) = (n,n)$. It is easy to see that f accumulates at $(0,0)$. Let U be any nbd of $(0,0)$ and let $k \in Z^+$ be given. There is an $N \in Z^+$ such that $\forall\, n \geq N$, U contains all but a finite number of points of C_n. Thus there is an $n \geq k$ such that U contains all but a finite number of points of C_n and $f(n) = (n,n) \in U$. We make the following claim: No sequence in $Z^+ \times Z^+$ converges to $(0,0)$.

Let $f : Z^+ \to Z^+ \times Z^+$ be a sequence. There are two possibilities for the range of f, $f(Z^+)$:

(i) $f(Z^+)$ has a finite number of points from the columns C_n; i.e., $f(Z^+) \cap C_n = A_n$ is finite, for each $n \in Z^+$.

(ii) There is at least one $n \in Z^+$ for which $C_n \cap f(Z^+)$ is infinite.

If (i) is the case, then the complement in X of $\bigcup_{i=1}^{\infty} A_i$ is a nbd of (0,0), call it U, and $U \cap f(Z^+) = \emptyset$. If (ii) is the case, then the complement in X of C_n is a nbd of (0,0) and obviously f is not eventually in that nbd. So, in either of the cases above, f does not converge to (0,0), since we have exhibited nbds of (0,0) with f not eventually in those nbds. Therefore, no sequence converges to (0,0); in particular, no subsequence of the diagonal sequence can converge to (0,0).

Another example of the same type might be of interest. Let $X = \{f \mid f : \mathcal{R} \to \mathcal{R}$ is a function$\}$. Define a topology on X as follows. For $f \in X$, let F be any finite subset of $\mathcal{R}$ and p any positive real number. Then define $U(f,F,p) =$ $\{g \in X \mid |g(x) - f(x)| < p, \forall x \in F\}$, and let $\mathcal{N}_f = \{U(f,F,p) \mid \forall F \subset \mathcal{R},$ and $\forall p > 0\}$. Then, for all $f \in X$, the family of $\mathcal{N}_f$ defines a topology for X.

Now let $A = \{f \in X \mid f(x) = 0$ or $f(x) = 1, \forall x \in \mathcal{R}$, and $f(x) = 0$ for at most countably many $x \in \mathcal{R}\}$. If $g : \mathcal{R} \to \mathcal{R}$ is defined by $g(x) = 0, \forall x \in \mathcal{R}$, it is easy to see that $g \in \bar{A}$; however no sequence of functions in A converges to g. Thus we have a topological space X and a subset A of X, such that there is a point $g \in \bar{A}$ having no sequence of points of A converging to it.

The space X above does not have the property that each of its points has a <u>countable</u> <u>neighborhood</u> <u>basis</u>. A space with this property is said to be <u>first</u> <u>countable</u>, and only if a space is first countable will be able to define the closed sets by using sequences. We will prove in the next theorem that nets are adequate to define the closed sets in any topological space.

THEOREM 7: Let X be a topological space and $A \subset X$. Then $x \in \bar{A}$ iff there is a net in A which converges to x.

PROOF: (i) Suppose $\varphi: D \to A$ is a net in A such that $\varphi \to x$. Then $\forall$ nbd U of x, $\exists\, d \in D$ such that $\forall d' \in D$, $d \prec d'$, $\varphi(d') \in U$. This implies $\forall$ nbd U of $x, U \cap A \neq \emptyset$, so that $x \in \bar{A}$.

(ii) Suppose now that $x \in \bar{A}$ and let N_x be the nbd system of x directed by $\supset$. Then since $x \in \bar{A}$, $\forall U \in N_x$, $U \cap A \neq \emptyset$, so define a net $\varphi: N_x \to A$ by $\varphi(U) \in U \cap A$. Then $\varphi \to x$, since if $U \in N_x$, then $\forall V \in N_x$ such that $U \supset V$, $\varphi(V) \in V \cap A \subset U \cap A$, so the theorem is proved.

Note that since X is any topological space, there may not exist a countable nbd base or countable nbd system of the points of X, so that a net, rather than a sequence, is needed to prove the above theorem.

It is also possible to define the continuity of a function $f : X \to Y$ in terms of convergence of nets. First note that if $\varphi: D \to X$ is a net in X and $f : X \to Y$ is any map, then $f \circ \varphi : D \to Y$ is a net in Y. We have the following theorem.

THEOREM 8: A function $f : X \to Y$ is continuous at $x_0 \in X$ iff $f \circ \varphi \to f(x_0)$ where $\varphi : D \to X$ is any net which converges to x_0.

PROOF: (i) Suppose $f : X \to Y$ is a map and $\varphi : D \to X$ is a net in X such that $\varphi \to x_0$ and $f \circ \varphi \to f(x_0)$. If f is not continuous at x_0, then $\exists$ nbd W of $f(x_0)$ such that $\forall$ nbd U of x_0, $f(U) \not\subset W$. But this is a contradiction, since by assumption φ is eventually in each nbd of x_0 and $f \circ \varphi$ is eventually in each nbd of $f(x_0)$.

(ii) Suppose now that $f : X \to Y$ is continuous at x_0 and that $\varphi : D \to X$ is a net such that $\varphi \to x_0$ but $f \circ \varphi \not\to f(x_0)$. Let W be a nbd of $f(x_0)$; then f continuous at x_0 implies $\exists$ nbd U of x_0 such that $f(U) \subset W$. But by assumption φ is eventually in each nbd of x_0 and $f \circ \varphi$ is not eventually in each nbd of $f(x_0)$. This is a contradiction to the fact that f is continuous at x_0.

We now have the following corollary:

COROLLARY 9: A map $f : X \to Y$ is continuous on X iff $f \circ \varphi : D \to Y$ converges to $f(x)$, for each $x \in X$ and for each net $\varphi : D \to X$ which converges to x.

In order to prove the next theorem, we need to recall the following definition concerning product spaces.

DEFINITION 13: Let $\{X_\alpha \mid \alpha \in A\}$ be a family of sets. The cartesian product $\prod_{\alpha \in A} X_\alpha$ is the set of all maps $c : A \to \bigcup_{\alpha \in A} X_\alpha$ having the property that $\forall \alpha \in A$, $c(\alpha) \in X_\alpha$.

Note that c is a choice function, so that $\prod_{\alpha \in A} X_\alpha$ is the set of all choice functions defined on $\{X_\alpha \mid \alpha \in A\}$. An element $c \in \prod_{\alpha \in A} X_\alpha$ is written $\{x_\alpha\}$, meaning that $c(\alpha) = x_\alpha$, $\forall \alpha \in A$, and x_α is called the α^{th} coordinate of $\{x_\alpha\}$. For each $\beta \in A$, the function $p_\beta : \prod_{\alpha \in A} X_\alpha \to X_\beta$ defined by $p_\beta(\{x_\alpha\}) = x_\beta$ is the projection map of $\prod_{\alpha \in A} X_\alpha$ onto the β^{th} factor.

If the sets X_α, $\alpha \in A$, happen to be topological spaces with topologies T_α, $\alpha \in A$, we have the following definition for a topology on $\prod_{\alpha \in A} X_\alpha$.

DEFINITION 14: Let $\{(X_\alpha, T_\alpha) \mid \alpha \in A\}$ be a family of topological spaces. The cartesian product topology in $\prod_{\alpha \in A} X_\alpha$ is that having as basic open sets those of the form

$\mathcal{U} = U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\beta \notin \{\alpha_1, \ldots, \alpha_n\}} X_\beta$, where n is finite and each U_{α_i} is an open set in X_{α_i}. Denote the open set $\mathcal{U}$ by $< U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} >$.

THEOREM 10: A net $\varphi : D \to \prod_{\alpha \in A} X_\alpha$ converges to a point $\{x_\alpha\} \in \prod_{\alpha \in A} X_\alpha$ iff $p_\beta \circ \varphi \to x_\beta$, for each fixed $\beta \in A$.

PROOF: (i) If $U_\beta \subset X_\beta$ is open, then $p_\beta^{-1}(U_\beta) = U_\beta \times \prod_{\alpha\varepsilon A-\{\beta\}} X_\alpha$ is open in $\prod_{\alpha\varepsilon A} X_\alpha$, so the projection map p_β is continuous for each $\beta \varepsilon A$. Therefore if $\varphi: D \to \prod_{\alpha\varepsilon A} X_\alpha$ is a net such that $\varphi \to \{x_\alpha\}$, then $p_\beta \circ \varphi \to x_\beta$, for each $\beta \varepsilon A$, by the preceeding corollary.

(ii) Now suppose $p_\beta \circ \varphi \to x_\beta$, $\beta \varepsilon A$. If $\varphi \not\to \{x_\alpha\}$, then $\exists$ nbd $< U_{\alpha_1}, \cdot \cdot \cdot, U_{\alpha_n} >$ of $\{x_\alpha\}$ such that $\forall\, d \varepsilon D$, $\exists\, d^*$, $d \prec d^*$ and $\varphi(d^*) \not\varepsilon <U_{\alpha_1}, \cdot \cdot \cdot, U_{\alpha_n}>$. This means that $\varphi(d^*) \not\varepsilon <U_{\alpha_j}>$, for some $j \varepsilon \{1,2,\cdot \cdot \cdot,n\}$. But this would imply that $p_{\alpha_j} \circ \varphi(d^*) \not\varepsilon U_{\alpha_j}$, which is a contradiction to the assumption that $p_\beta \circ \varphi \to x_\beta, \forall \beta \varepsilon A$. Therefore, $\varphi \to \{x_\alpha\}$, and the theorem is proved.

In Theorem 6 above, we saw that if a net accumulates at a point, then there is a subnet which converges to that point. We might ask whether there is a type of net in a space such that if it accumulates at a point, then it also converges to that point. The answer lies in the concept of a <u>maximal net</u> or <u>ultranet</u>.

<u>DEFINITION 15</u>: Let $\varphi: D \to X$ be a net in a topological space X and let $A \subset X$. Then we say that φ is an <u>ultranet in X</u> iff φ is eventually in A or is eventually in $X - A$.

One example of an ultranet is the constant net $\varphi: D \to X$ defined by $\varphi(d) = a$, $\forall\, d \varepsilon D$. Another large family of examples can be constructed by letting D be a directed set with a last element, such as the set of ordinals $\{1,2,\cdots,\alpha,\alpha+1,\cdots,\omega\}$. Then any map $\varphi: D \to X$ is eventually in any set A or eventually in $X - A$, since if $A \subset X$, $\varphi(\omega)$ is either in A or in $X - A$. We have the following theorem.

THEOREM 11: Let $\varphi: D \to X$ be an ultranet in X. If $\varphi \succ\!\!\!\circ\, x$, then $\varphi \to x$.

PROOF: Suppose $\varphi: D \to X$ is an ultranet in X and let U be a nbd of $x \in X$. Since $\varphi \succ\!\!\!\circ\, x$, we know $\forall\, d \in D$, $\exists\, d' \in D$, $d \prec d'$ and $\varphi(d') \in U$, which implies that φ could not eventually be in $X - U$. Since φ is an ultranet, φ must eventually be in U. Thus $\varphi \to x$, and the theorem is proved.

Our examples above have shown the existence of ultranets; even more interesting is the fact that we can find an "ultra-subnet" of any given net. We use the following fundamental lemma on subnets due to Kelley.

LEMMA 7: Given a net $\varphi: D \to X$ and a family Q of subsets of X such that:

(i) φ is frequently in each element of Q, and

(ii) The intersection of two members of Q is a member of Q.

Then there is a subnet of φ which is eventually in each member of Q.

PROOF: Note that $(Q,\supset)$ is a directed set, so that we may consider the cross product of $(D,\prec)$ with $(Q,\supset)$ and have a directed product set, $(D\times Q,\leq)$, where $(d,A) \leq (d',A')$ means $d \prec d'$ and

$A \supset A'$. If $(d,A) \in D \times Q$, then let $f : D \times Q \to D$ be a map such that $f(d,A) = d'$ where $d \prec d'$ and $\varphi \circ f(d,A) \in A$. By definition of the directed cross product of two directed sets, it is easy to see that f satisfies properties (i) and (ii) in the definition of subnet, so that $\varphi \circ f$ is a subnet of φ. Now if $A \in Q$ and $(d,A) \in D \times Q$, then $\forall (p,B) \in D \times Q$, where $(d,A) \leq (p,B)$, we have $\varphi \circ f(p,B) \in B \subset A$. So, $\varphi \circ f(p,B) \in A$ and we have found a subnet of φ which is eventually in each member of Q.

THEOREM 12: Each net $\varphi : D \to X$ has a subnet which is an ultranet.

PROOF: Let $\varphi : D \to X$ be a net in a topological space X. In view of Lemma 7, we will have proved the theorem if we can show there is a family Q of subsets of X with the following three properties:

(i) If $A \subset X$, then either $A \in Q$ or $X - A \in Q$,

(ii) The intersection of a finite number of elements of Q is an element of Q, and

(iii) φ is frequently in each element of Q.

Suppose $\mathcal{Q}$ is the collection of all families of subsets of X which satisfy properties (ii) and (iii) above. It is easy to show the following about $\mathcal{Q}$:

(a) $\mathcal{Q} \neq \emptyset$, since $\emptyset \in \mathcal{Q}$.

(b) $\mathcal{Q}$ is partially ordered by $\subset$, where $Q_\alpha \subset Q_\beta$ means if $A \in Q_\alpha$, then $A \in Q_\beta$.

(c) If $\{Q_\alpha \mid \alpha \in \mathcal{A}\}$ is any totally ordered subset of $\mathcal{Q}$, then $Q' = \bigcup_{\alpha\in\mathcal{A}} Q_\alpha$ is an upper bound for $\{Q_\alpha \mid \alpha \in \mathcal{A}\}$ and Q' satisfies (ii) and (iii) above.

Thus we may apply Zorn's Lemma and have the existence of a maximal family of $\mathcal{Q}$, call it Q^*. We claim that Q^* also satisfies (i) above. Suppose $Q^* = \{B_\alpha \mid B_\alpha \subset X, \alpha \in \mathcal{A}\}$ and let A be any subset of X. If $A \cap B_\alpha$ satisfies (iii) above $\forall\, \alpha \in \mathcal{A}$, then $A \in Q^*$, since Q^* is maximal. Hence if $A \notin Q^*$, $\exists\, \beta \in \mathcal{A}$ such that $A \cap B_\beta$ does not satisfy (iii) above; i.e., φ is eventually in $X - (A \cap B_\beta)$. Thus, φ is frequently in $X - (A \cap B_\beta)$, so that $X - (A \cap B_\beta) \in Q^*$. By (i) above, $B_\beta \cap [X - (A \cap B_\beta)] \in Q^*$. But $B_\beta \cap [X - (A \cap B_\beta)] = B_\beta - (A \cap B_\beta)$ and $X - A \supset B_\beta - (A \cap B_\beta)$. Therefore, $X - A \in Q^*$ since φ frequently in $B_\beta - (A \cap B_\beta)$ implies φ frequently in $X - A$, and since Q^* is maximal. We have shown that if $A \notin Q^*$, then $X - A \in Q^*$, which is property (i) above. So, by the above lemma, there is a subnet $\varphi \circ f$ of φ which is eventually in each member of Q^*. But for each subset A of X, either $A \in Q^*$ or $X - A \in Q^*$, so that $\varphi \circ f$ is an ultranet, and the theorem is proved.

The very important concept of compactness can also be discussed easily in terms of nets. Recall the following definition.

DEFINITION 16: A topological space X is said to be compact iff each open covering has a finite subcovering.

We have the following characterizations of compactness.

<u>THEOREM 13</u>: A topological space X is compact iff for each family $\{F_\alpha \mid \alpha \in A\}$ of closed sets in X for which $\bigcap_{\alpha \in A} F_\alpha = \emptyset$, there exists a finite subfamily $\{F_{\alpha_1}, F_{\alpha_2}, \cdots, F_{\alpha_n}\}$ such that $\bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset$.

<u>PROOF</u>: The proof is standard and is omitted.

<u>THEOREM 14</u>: A topological space X is compact iff each net φ in X has an accumulation point.

<u>PROOF</u>: (i) Suppose X is compact and let $\varphi: D \to X$ be any net in X. For each $d \in D$, let $\varphi(T_d) = \{\varphi(d') \mid d \prec d'\}$ be the image of the terminal set T_d under φ. Then $\Phi = \{\overline{\varphi(T_d)} \mid d \in D\}$ is a family of closed sets such that the intersection of any finite number of elements of Φ is non-empty. Since X is compact, $\bigcap_{d \in D} \overline{\varphi(T_d)} \neq \emptyset$. Let $x_0 \in \bigcap_{d \in D} \overline{\varphi(T_d)}$. Then $\varphi \succ\!\!\!\!\to x_0$, since if U is any nbd of x_0, then $\forall d \in D$, $U \cap \varphi(T_d) \neq \emptyset$.

(ii) Suppose X is a topological space and let $F = \{F_\alpha \mid \alpha \in A\}$ be any family of closed subsets of X such that each finite subfamily of F has a non-empty intersection. Suppose further that each net in X has an accumulation point, Let $G = \{\bigcap_{i=1}^{n} F_{\alpha_i} \mid F_{\alpha_i} \in F \text{ and } n \text{ finite}\}$. Note that since $\mathcal{F} \subset \mathcal{G}$ and since each finite subfamily of $\mathcal{G}$ has a non-empty intersection, we will have proved that X is compact if we can show that $\bigcap_{\beta \in \mathcal{B}} G_\beta \neq \emptyset$, where $\mathcal{G} = \{G_\beta \mid \beta \in \mathcal{B}$ and G_β is a finite intersection of elements of $\mathcal{F}\}$. It is easy to see that $(\mathcal{G},\supset)$ is a directed set, since the intersection of any two elements of $\mathcal{G}$ is an element of $\mathcal{G}$. Let $\varphi: \mathcal{G} \to X$ be a choice function; that is, $\varphi(G_\beta) \in G_\beta$, $\forall \beta \in \mathcal{B}$. By hypothesis φ has an accumulation point, call it x_0. If G_β, $G_\alpha \in \mathcal{G}$ such that $G_\beta \supset G_\alpha$, then $\varphi(G_\alpha) \in G_\alpha \subset G_\beta$, so that φ is eventually in each closed set $G_\beta \in \mathcal{G}$. Thus $x_0 \in G_\beta, \forall \beta \in \mathcal{B}$, so that $\bigcap_{\beta \in \mathcal{B}} G_\beta \neq \emptyset$ and the theorem is proved.

<u>THEOREM 15</u>: A topological space X is compact iff each ultranet in X converges.

<u>PROOF</u>: (i) Suppose X is a compact topological space and let $\varphi: D \to X$ be an ultranet. Then by Theorem 14, φ has an accumulation point $x_0 \in X$ and since φ is an ultranet, $\varphi \to x_0$.

(ii) Suppose each ultranet in a topological space X converges. Let $\varphi: D \to X$ be any net in X. By Theorem 12, there is a subnet $\varphi \circ f$ of φ which is an ultranet. By hypothesis, $\varphi \circ f \to x_0$, for some $x_0 \in X$. Thus x_0 is an accumulation point of φ, so by Theorem 14, X is compact.

CHAPTER IV

FILTERS AND FILTERBASES

A net is one of three equivalent methods of discussing convergence in general topological spaces. The others, filters and filterbases, are the subject of what follows. We begin with the basic definitions.

DEFINITION 17: A filter $\mathcal{F}$ on a set X is a family of non-empty subsets of X such that:

(i) $\mathcal{F} \neq \emptyset$

(ii) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and

(iii) If $A \in \mathcal{F}$ and $A \subset B \subset X$, then $B \in \mathcal{F}$.

DEFINITION 18: If X is a topological space and $\mathcal{F}$ is a filter on X, then $\mathcal{F}$ converges to $x_0 \in X$, $(\mathcal{F} \to x_0)$ iff $\forall$ nbd U of x_0, $U \in \mathcal{F}$. The filter $\mathcal{F}$ accumulates at $x_0 (\mathcal{F} \succ\!\!\!\to x_0)$ iff $\forall$ nbd U of x_0 and $\forall A \in \mathcal{F}$, $U \cap A \neq \emptyset$.

Given a net, it is easy to construct a filter from it, which is said to be generated by the net; conversely, given a filter, we may construct a net from it which is said to be based on the filter. We will also see that these two ideas lead to an equivalent notion of convergence.

Let $\varphi: D \to X$ be a net in a topological space X and $\forall d \in D$, let T_d be a terminal set in D. Consider the family of all images of the T_d under φ, call it $\mathcal{B} = \{\varphi(T_d) \mid d \in D\}$.

Then $F = \{A \subset X \mid \varphi(T_d) \subset A$, for some $d \in D\}$ is a filter on X, called the filter generated by the net φ. The family $\mathcal{B}$ is called the basis of the filter F. Because D is a directed set, it follows that if $\varphi(T_d), \varphi(T_{d'}) \in \mathcal{B}$, then there is a $\varphi(T_{d''}) \in \mathcal{B}$ such that $\varphi(T_{d''}) \subset \varphi(T_d) \cap \varphi(T_{d'})$. Any family $\mathcal{B} = \{B_\alpha \mid \alpha \in A\}$ with the above property is a basis for the filter $F = \{A \subset X \mid B_\alpha \subset A$, for some $\alpha \in A\}$.

Conversely, suppose F is a filter on a topological space X. It is trivial to verify that $(F, \supset)$ is a directed set. Since each element of F is non-empty, there is a choice function $\varphi: F \to X$ such that $\varphi(A) \in A$, $\forall A \in F$. Then φ is a net in X, called the net based on the filter F.

THEOREM 16: Let $\varphi: D \to X$ be a net in a space X, and let F be the filter generated by φ. Then $F \to x_0$ iff $\varphi \to x_0$.

PROOF: (i) Suppose $F \to x_0$; then $\forall$ nbd U of x_0, $U \in F$. This means $\exists\, d \in D$, such that $\varphi(T_d) \subset U$, which says $\varphi \to x_0$.

(ii) Suppose $\varphi \to x_0$; then $\forall$ nbd U of x_0, $\exists\, d \in D$ such that $\varphi(T_d) \subset U$. This says $U \in F$ and $F \to x_0$.

THEOREM 17: Let F be a filter on a space X. Then $F \to x_0$ iff each net based on F also converges to x_0.

PROOF: (i) Suppose $F \to x_0$; then $\forall$ nbd U of x_0, $U \in F$. If $V \in F$ such that $U \supset V$, then if $\varphi: F \to X$ is any net based on F, $\varphi(V) \in V \subset U$, so $\varphi \to x_0$.

(ii) Now suppose that $F \not\to x_0$. Then $\exists$ nbd U of x_0 such that $U \notin F$. Note that for each $A \in F$, $A - U \neq \emptyset$, since if $A - U = \emptyset$, we would have $A \subset U$ and by definition of a filter, $U \in F$. So, the choice function $\varphi: F \to X$ defined by $\varphi(A) \in A - U$ is a net based on F which is never in U, and hence could not converge to x_0.

The above two theorems show that convergence based on nets is equivalent to convergence based on filters. All of the previous theorems in this paper could be proved by replacing nets with filters. The concept of a subnet of a net has its analog in the idea of a finer filter; i.e. if F is a filter on X, then F' is finer than F iff $F \subset F'$. The concept of ultranet is analogous to what is known as an ultrafilter; i.e., a filter F on a set X is said to be an ultrafilter iff given any filter F' finer than F, $F' = F$.

We may also discuss convergence in terms of filterbases whose definition is motivated by our above discussion of a basis for a filter on a space.

DEFINITION 19: Let X be a topological space. Then a filterbase in X is a family of non-empty subsets of X, $F = \{A_\alpha \mid \alpha \in A\}$, such that $\forall \alpha, \beta \in A$, $\exists \lambda \in A$ such that $A_\lambda \subset A_\alpha \cap A_\beta$.

Analogous to a subnet and a finer filter is the notion of a subordinate filterbase.

DEFINITION 20: If $\mathcal{F} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ and $\mathcal{F}' = \{B_\beta \mid \beta \in \mathcal{B}\}$ are two filterbases on a space X, we say $\mathcal{F}$ is subordinate to $\mathcal{F}$ iff $\forall A_\alpha \in \mathcal{F}$, $\exists B_\beta \in \mathcal{F}'$ such that $B_\beta \subset A_\alpha$.

Similarly, ultranets and ultrafilters have corresponding to them the concept of a maximal filterbase.

DEFINITION 21: A filterbase $\mathcal{F}$ on a space X is said to be maximal iff for each filterbase $\mathcal{F}'$ on X subordinate to $\mathcal{F}$, it is true that $\mathcal{F}$ is subordinate to $\mathcal{F}'$.

As with nets and filters, we may consider filterbases determined by nets and nets based on filterbases.

First let $\varphi: D \to X$ be a net in a topological space X. Then the family $\mathcal{F} = \{\varphi(T_d) \mid d \in D\}$ of the images of all terminal sets of D is a filterbase in X, as we have seen in our discussion of a basis for a filter, and is called the filterbase determined by the net φ.

Second, if $\mathcal{F} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ is a filterbase in a space X, we may construct a net based on $\mathcal{F}$ as follows:
Let $D = \{(a_\alpha, A_\alpha) \mid a_\alpha \in A_\alpha \text{ and } A_\alpha \in \mathcal{F}\}$. This is possible since each element of $\mathcal{F}$ is non-empty. Then direct D by saying $(a_\alpha, A_\alpha) \prec (a_\beta, A_\beta)$ iff $A_\alpha \supset A_\beta$. The relation $\prec$ directs D, since $\mathcal{F}$ is a filterbase. Now we define the map $\varphi: D \to X$ by $\varphi(a_\alpha, A_\alpha) = a_\alpha$, and we have a net based on the filterbase $\mathcal{F}$.

We need to define the convergence of a filterbase and prove that the concept of convergence based on nets is equivalent to that based on filterbases.

DEFINITION 22: Let $\mathcal{F} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a filterbase on a space X. Then $\mathcal{F}$ converges to $x_0 (\mathcal{F} \to x_0)$ iff for each nbd U of x_0, there is an $A_\beta \in \mathcal{F}$ such that $A_\beta \subset U$. A filterbase $\mathcal{F}$ accumulates at $x_0 (\mathcal{F} \succ\!\circ\, x_0)$ iff $\forall$ nbd U of x_0 and $\forall\, A_\alpha \in \mathcal{F}$, $U \cap A_\alpha \neq \emptyset$.

THEOREM 18: Let $\varphi: D \to X$ be a net in a topological space X, and let $\mathcal{F} = \{\varphi(T_d) \mid d \in D\}$ be the filterbase determined by φ. Then $\varphi \to x_0$ iff $\mathcal{F} \to x_0$.

PROOF: The proof is simply Definitions 6 and 22.

THEOREM 19: Let $\mathcal{F} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a filterbase. Then $\mathcal{F} \to x_0$ iff each net based on $\mathcal{F}$ converges to x_0.

PROOF: (i) Let $\mathcal{F} = \{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a filterbase, let $D = \{(a_\alpha, A_\alpha) \mid a_\alpha \in A_\alpha,\ A_\alpha \in \mathcal{F}\}$, and let $\varphi: D \to X$ be a net based on $\mathcal{F}$ as defined above. If $\mathcal{F} \to x_0$, then $\forall$ nbd U of x_0, $\exists\ A_\beta \in \mathcal{F}$ such that $A_\beta \subset U$. Thus, $\forall\ (a_\lambda, A_\lambda) \in D$ such that $(a_\beta, A_\beta) \prec (a_\lambda, A_\lambda)$, we have $\varphi(a_\lambda, A_\lambda) = a_\lambda \in A_\lambda \subset A_\beta \subset U$, so that $\varphi \to x_0$.

(ii) Suppose $\mathcal{F} \not\to x_0$. Then there is a nbd U of x_0 such that $\forall\, A_\alpha \in \mathcal{F}$, $A_\alpha - U \neq \emptyset$. Then define $D = \{(a_\alpha, A_\alpha) \mid a_\alpha \in A_\alpha - U,\ A_\alpha \in \mathcal{F}\}$ and $\varphi: D \to X$ by $\varphi(a_\alpha, A_\alpha) = a_\alpha$. Then φ is a net, based on the filterbase $\mathcal{F}$, which does not converge to x_0, so the theorem is proved.

Thus convergence in general topological spaces may be expressed adequately in terms of nets, filters or filterbases; the theorems in Chapter III of this paper have parallel statements using both filters and filterbases. However, neither of the latter two ideas is as natural a generalization of sequences as are nets. In fact, it is only through the nets based on filters and filterbases that we see the similarity of those concepts to the idea of a sequence.

SUMMARY

In this thesis it was pointed out that sequences are inadequate to describe the concept of convergence in a general topological space. A sequence was then generalized to a net, which was shown adequate to formulate the concept of convergence in any topological space. It was shown that the notion of closure in a general topological space, and hence all topological concepts, could be defined using nets. Cauchy nets were introduced and a Cauchy criterion for nets of real numbers was proved. Finally, it was shown that convergence in terms of nets was equivalent to convergence in terms of the alternative concepts of filters and filterbases.

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