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Definition: Let f be a function from $[a,b]$ into the real numbers. Then f is said to be locally variable on $[a,b]$ provided there is a positive integer N such that if $\{s_p\}_0^n$ is an increasing sequence with $s_o = a$ and $s_n = b$, then f is of bounded variation on all but at most N of the intervals $[s_{p-1}, s_p]$ for $0 < p \leq n$.

Theorem: Let f be a continuous function from $[a,b]$ to the real numbers that is locally variable. If there exists a number M such that if $\{s_p\}_0^{2n}$ is a Stieltjes subdivision of $[a,b]$, then

$$\left| \sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) \right| < M, \text{ then } \int_a^b f \, df \text{ exists.}$$

A STIELTJES INTEGRAL EXISTENCE THEOREM

by

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TABLE OF CONTENTS

INTRODUCTION

This paper is the result of inquiry into questions that arose concerning Stieltjes integrals. During a course in real analysis at the University of North Carolina at Greensboro, the students were asked to find an example of a continuous function f from [0,1] to the real numbers such that $\int_0^1 f\, df$ did not exist. The function f such that $f(x) = \begin{cases} x \sin \frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ was selected since it is a function not of bounded variation. It was observed that the set of sums over all Stieltjes subdivisions of [0,1] was bounded for the function f with respect to itself. Thus the question remained, does $\int_0^1 f\, df$ exist? It is shown in the paper that $\int_0^1 f\, df$ does exist.

In looking for a general condition weaker than bounded variation under which Stieltjes integrals exist, three questions were asked. First, if f is continuous does $\int_a^b f\, df$ exist? A counter example is given in the paper. Second, does $\int_a^b f\, df$ exist if and only if f is continuous and f^2 is of bounded variation? A counter example for this question is given in the paper. Last, if f is continuous and the set of sums over all Stieltjes subdivisions of [a,b] is bounded does $\int_a^b f\, df$ exist? By the addition of a condition called locally variable as a restriction on the function, the last question can be answered affirmatively.

CHAPTER I

Notation, Definitions and Some Properties of Stieltjes Integrals

Notation: The symbol $\{s_p\}_a^b = s$ means that a and b are nonnegative integers and s is a sequence whose domain is the set to which the integer p belongs only in case $a \leq p \leq b$.

Definition 1: A Stieltjes subdivision of the interval [a,b] is a nondecreasing finite sequence $s = \{s_p\}_0^{2m}$ such that $s_0 = a$ and $s_{2m} = b$. If $\{s_p\}_0^{2m}$ is a Stieltjes subdivision of [a,b] the norm of s, denoted $\|s\|$, is defined by $\|s\| = \sup \{s_{2p} - s_{2p-2} | 1 \leq p \leq m\}$. If $\{s_p\}_0^{2m}$ is a Stieltjes subdivision of [a,b] then the even part of s is the set $\{s_{2p} | 0 \leq p \leq m\}$ and the odd part of s is the set $\{s_{2p-1} | 1 \leq p \leq m\}$.

Definition 2: A refinement of a Stieltjes subdivision $\{s_p\}_0^{2n}$ of [a,b] is a Stieltjes subdivision $\{t_p\}_0^{2m}$ of [a,b] such that the even part of $\{s_p\}_0^{2n}$ is a subsequence of the even part of $\{t_p\}_0^{2m}$.

Definition 3: The function f from [a,b] to the real numbers is said to be of bounded variation only in case there is a number $V < \infty$ such that if $\{s_p\}_0^{2m}$ is a Stieltjes subdivision of [a,b] then $\sum_{p=1}^{m} |f(s_{2p}) - f(s_{2p-2})| < V$. The total variation of f is the smallest number V such that if $\{s_p\}_0^{2m}$ is a Stieltje subdivision of [a,b] then $\sum_{p=1}^{m} |f(s_{2p}) - f(s_{2p-2})| \leq V$.

Definition 4: Let a and b be real numbers with $a \leq b$. Let f and g be functions from [a,b] to the real numbers. The Stieltjes integral from a to b of f with respect to g denoted $\int_a^b f\,dg$ is a number z such that if ϵ is a positive number there is a Stieltjes subdivision $\{s_p\}_0^{2n}$ of [a,b] such that if $\{t_p\}_0^{2m}$ is a refinement of $\{s_p\}_0^{2n}$ then

$$\left| \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - z \right| < \epsilon.$$

The following three theorems are easily proved and hence are stated without proof.

Theorem 1: Let $\{s_p\}_0^{2n}$ be a Stieltjes subdivision of [a,b] and let $\{t_p\}_0^{2m}$ be a refinement of $\{s_p\}_0^{2n}$. If $\{r_p\}_0^{2q}$ is a refinement of $\{t_p\}_0^{2m}$ then $\{r_p\}_0^{2q}$ is a refinement of $\{s_p\}_0^{2n}$.

Theorem 2: Let $\{s_p\}_0^{2n}$ and $\{t_p\}_0^{2m}$ be Stieltjes subdivisions of [a,b]. There exists a Stieltjes subdivision $\{r_p\}_0^{2q}$ that is a common refinement of $\{s_p\}_0^{2n}$ and $\{t_p\}_0^{2m}$.

Theorem 3: Let f and g be functions from [a,b] to the real numbers. If $\int_a^b f\,dg$ exists then $\int_a^b f\,dg$ is unique.

Theorem 4: If $\int_a^b f\,dg$ exists and c is a real number then $c\int_a^b f\,dg = \int_a^b c\,f\,dg = \int_a^b f\,d(cg)$.

Proof: Let ϵ be a positive number. Either $c = 0$ or $c \neq 0$. If $c = 0$ then let $\{s_p\}_0^{2n}$ be a Stieltjes subdivision of [a,b] such that if $\{t_p\}_0^{2m}$ is a refinement of $\{s_p\}_0^{2n}$ then

$$\left| \int_a^b f\,dg - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \epsilon.$$ Let $\{t_p\}_0^{2m}$ be a refinement of $\{s_p\}_0^{2n}$. Now

$$\left| c \int_a^b f\,dg - c \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right|$$

$$= |c| \left| \int_a^b f\,dg - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < |c| \cdot \epsilon = 0 < \epsilon.$$

Then $\left| c \int_a^b f\,dg - \sum_{p=1}^{m} c\, f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| =$

$\left| c \int_a^b f\,dg - c \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \epsilon$ and by theorem 3

$c \int_a^b f\,dg = \int_a^b c\, f\,dg.$ Also

$$\left| c \int_a^b f\,dg - \sum_{p=1}^{m} f(t_{2p-1})(cg(t_{2p}) - cg(t_{2p-2})) \right|$$

$$= \left| c \int_a^b f\,dg - \sum_{p=1}^{m} f(t_{2p-1})\, c\, (g(t_{2p}) - g(t_{2p-2})) \right|$$

$= \left| c \int_a^b f\,dg - c \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \epsilon$ and by

theorem 3, $c \int_a^b f\,dg = \int_a^b f\,d(cg)$. If $c \neq 0$ then let $\{s_p\}_0^{2n}$ be a

Stieltjes subdivision of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{s_p\}_0^{2n}$ then $\left| \int_a^b f\,dg - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \frac{\epsilon}{|c|}$.

Let $\{t_p\}_0^{2m}$ be a refinement of $\{s_p\}_0^{2n}$. Now

$$\left| c\int_a^b f\,dg - c\sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right|$$

$$= |c| \left| \int_a^b f\,dg - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < |c| \cdot \frac{\epsilon}{|c|} = \epsilon.$$

Then $\left| c\int_a^b f\,dg - \sum_{p=1}^{m} c\, f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right|$

$= \left| c\int_a^b f\,dg - c\sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \epsilon$ and by theorem

3 $c\int_a^b f\,dg = \int_a^b c\,f\,dg$. Also

$$\left| c\int_a^b f\,dg - \sum_{p=1}^{m} f(t_{2p-1})(cg(t_{2p}) - cg(t_{2p-2})) \right|$$

$$= \left| c\int_a^b f\,dg - \sum_{p=1}^{m} f(t_{2p-1})\, c\,(g(t_{2p}) - g(t_{2p-2})) \right|$$

$= \left| c\int_a^b f\,dg - c\sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \epsilon$ and by theorem

3, $c\int_a^b f\,dg = \int_a^b f\,d(cg)$.

<u>Theorem 5:</u> If $\int_a^b f\,dh$ and $\int_a^b g\,dh$ exist then $\int_a^b (f+g)\,dh =$

$\int_a^b f\,dh + \int_a^b g\,dh$.

Proof: Let ϵ be a positive number. Let $\{s_p\}_0^{2n}$ be a Stieltjes subdivision of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{s_p\}_0^{2n}$ then $\left| \int_a^b f\,dh - \sum_{p=1}^{m} f(t_{2p-1})(h(t_{2p}) - h(t_{2p-2})) \right| < \epsilon/2$. Let $\{r_p\}_0^{2k}$ be a Stieltjes subdivision of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{r_p\}_0^{2k}$ then

$\left| \int_a^b g\,dh - \sum_{p=1}^{m} g(t_{2p-1})(h(t_{2p}) - h(t_{2p-2})) \right| < \epsilon/2$. Let $\{t_p\}_0^{2m}$ be a common refinement of $\{s_p\}_0^{2n}$ and $\{r_p\}_0^{2k}$. Let $\{z_p\}_0^{2q}$ be a refinement of $\{t_p\}_0^{2m}$ then

$$\left| \int_a^b f\,dh + \int_a^b g\,dh - \sum_{p=1}^{q} (f+g)(z_{2p-1})(h(z_{2p}) - h(z_{2p-2})) \right| =$$

$$\left| \int_a^b f\,dh + \int_a^b g\,dh - \sum_{p=1}^{q} (f(z_{2p-1}) + g(z_{2p-1}))(h(z_{2p}) - h(z_{2p-2})) \right| =$$

$$\left| \int_a^b f\,dh + \int_a^b g\,dh - \sum_{p=1}^{q} f(z_{2p-1})(h(z_{2p}) - h(z_{2p-2})) - \sum_{p=1}^{q} g(z_{2p-1})(h(z_{2p}) - h(z_{2p-2})) \right| \le$$

$$\left| \int_a^b f\,dh - \sum_{p=1}^{q} f(z_{2p-1})(h(z_{2p}) - h(z_{2p-2})) \right| +$$

$$\left| \int_a^b g\,dh - \sum_{p=1}^{q} g(z_{2p-1})(h(z_{2p}) - h(z_{2p-2})) \right| < \epsilon/2 + \epsilon/2 = \epsilon.$$

By theorem 3, $\int_a^b f\,dh + \int_a^b g\,dh = \int_a^b (f+g)\,dh$.

<u>Theorem 6:</u> If $\int_a^b f\,dh$ and $\int_a^b f\,dg$ exist then $\int_a^b f\,d(h+g)$

$$= \int_a^b f\,dh + \int_a^b f\,dg.$$

<u>Proof:</u> Let ϵ be a positive number. Let $\{s_p\}_0^{2n}$ be a Stieltjes subdivision of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{s_p\}_0^{2n}$ then $\left| \int_a^b f\,dh - \sum_{p=1}^{m} f(t_{2p-1})(h(t_{2p}) - h(t_{2p-2})) \right| < \epsilon/2$. Let $\{r_p\}_0^{2k}$ be a Stieltjes subdivision of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{r_p\}_0^{2k}$ then

$$\left| \int_a^b f\,dg - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \epsilon/2.$$ Let $\{t_p\}_0^{2m}$ be a common refinement of $\{s_p\}_0^{2n}$ and $\{r_p\}_0^{2k}$. Let $\{z_p\}_0^{2q}$ be a refinement of $\{t_p\}_0^{2m}$.

$$\left| \int_a^b f\,dh + \int_a^b f\,dg - \sum_{p=1}^{q} f(z_{2p-1})[(h+g)(z_{2p}) - (h+g)(z_{2p-2})] \right| =$$

$$\left| \int_a^b f\,dh + \int_a^b f\,dg - \sum_{p=1}^{q} f(z_{2p-1})[(h(z_{2p}) - h(z_{2p-2})) + \right.$$

$$\left. (g(z_{2p}) - g(z_{2p-2}))] \right| =$$

$$\left| \int_a^b f\,dh + \int_a^b f\,dg - \sum_{p=1}^{q} f(z_{2p-1})(h(z_{2p}) - h(z_{2p-2})) - \right.$$

$$\left. \sum_{p=1}^{q} f(z_{2p-1})(g(z_{2p}) - g(z_{2p-2})) \right| \le \left| \int_a^b f\,dh - \right.$$

$$\sum_{p=1}^{q} f(z_{2p-1})(h(z_{2p}) - h(z_{2p-2})) \mid +$$

$\left| \int_a^b f\, dg - \sum_{p=1}^{q} f(z_{2p-1})(g(z_{2p}) - g(z_{2p-2})) \right| < \epsilon/2 + \epsilon/2 = \epsilon$. By

theorem 3, $\int_a^b f\, dh + \int_a^b f\, dg = \int_a^b f\, d(h + g)$.

<u>Theorem 7:</u> If $\int_a^b f\, dg$ exists then $\int_a^b g\, df$ exists and

$$\int_a^b f\, dg + \int_a^b g\, df = f(b)g(b) - f(a)g(a)$$

<u>Proof:</u> Let ϵ be a positive number. Let $\{s_p\}_0^{2n}$ be a Stieltjes subdivision of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{s_p\}_0^{2n}$

then $\left| \int_a^b f\, dg - \sum_{p=1}^{m} f(s_{2p-1})(g(s_{2p}) - g(s_{2p-2})) \right| < \epsilon$. Let $\{t_p\}_0^{2m}$

be a refinement of $\{s_p\}_0^{2n}$ such that if $0 \leq p \leq 2n$ then

$$\begin{cases} t_{2p} = t_{2p+1} = t_{2p+2} = s_p & \text{for } p \text{ even} \\ t_{2p+1} = s_p & \text{for } p \text{ odd} \end{cases} .$$

Let $\{r_p\}_0^{2k}$ be a refinement of $\{t_p\}_0^{2m}$. Then

$$\left| f(b)g(b) - f(a)g(a) - \int_a^b f\, dg - \sum_{p=1}^{k} g(r_{2p-1})(f(r_{2p}) - f(r_{2p-2})) \right| =$$

$$| f(r_{2k})g(r_{2k-1}) - f(r_0)g(r_1) - \int_a^b f\,dg - \sum_{p=1}^{k} g(r_{2p-1})f(r_{2p}) +$$

$$\sum_{p=1}^{k} g(r_{2p-1})f(r_{2p-2}) | =$$

$$| - \int_a^b f\,dg - \sum_{p=1}^{k-1} g(r_{2p-1})f(r_{2p}) + \sum_{p=2}^{k} g(r_{2p-1})f(r_{2p-2}) | =$$

$$| \sum_{p=1}^{k-1} g(r_{2p+1})f(r_{2p}) - \sum_{p=1}^{k-1} g(r_{2p-1})f(r_{2p}) - \int_a^b f\,dg | =$$

$| \sum_{p=1}^{k-1} f(r_{2p})(g(r_{2p+1}) - g(r_{2p-1})) - \int_a^b f\,dg |$. If p is an even integer and $0 \leq p \leq 2n$ then there is an integer j such that $p \leq j \leq k-1$ and $s_p = t_{2p} = t_{2p+1} = t_{2p+2} = r_{2j} = r_{2j+1} = r_{2j+2}$. Thus $a = s_0 = t_1 = r_1$ and $b = s_{2n} = t_{4n+1} = r_{2k-1}$. Let $v_{p-1} = r_p$ for $1 \leq p \leq 2k-1$ then $\{v_p\}_0^{2k-2}$ is a refinement of $\{s_p\}_0^{2n}$ and

$$| \sum_{p=1}^{k-1} f(r_{2p})(g(r_{2p+1}) - g(r_{2p-1})) - \int_a^b f\,dg | =$$

$| \sum_{p=1}^{k-1} f(v_{2p-1})(g(v_{2p}) - g(v_{2p-2})) - \int_a^b f\,dg | < \epsilon$. Thus

$$| f(b)g(b) - f(a)g(a) - \int_a^b f\,dg - \sum_{p=1}^{k} g(r_{2p-1})(f(r_{2p}) - f(r_{2p-2})) | < \epsilon,$$

hence by theorem 3, $\int_a^b g\,df$ exists, and $\int_a^b g\,df =$

$$f(b)g(b) - f(a)g(a) - \int_a^b f\,dg.$$

Corollary 1: If $\int_a^b f\,df$ exists then $\int_a^b f\,df = \frac{(f(b))^2-(f(a))^2}{2}$.

Proof: By theorem 7, $\int_a^b f\,df + \int_a^b f\,df = f(b)f(b) - f(a)f(a)$

and therefore $\int_a^b f\,df = \frac{(f(b))^2-(f(a))^2}{2}$.

Corollary 2: If $\int_a^b f\,df$ exists and $f(a) = f(b)$ then $\int_a^b f\,df = 0$.

Proof: By corollary 1, $\int_a^b f\,df = \frac{(f(b)^2- f(a))^2}{2} = \frac{0}{2} = 0$.

Let f and g be functions from $[a,b]$ to the real numbers such that if $\epsilon > 0$ then there is a Stieltjes subdivision $\{s_p\}_0^{2n}$ of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{s_p\}_0^{2n}$ then

$$\left|\sum_{p=1}^{n} f(s_{2p-1})(g(s_{2p}) - g(s_{2p-2})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2}))\right| < \epsilon.$$

Let $S = \{s_p\}_1^{\infty}$ be the sequence defined by s_1 = Stieltjes subdivision $\{(s_1)p\}_0^{2n}$ of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of s_1 then

$$\left|\sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \sum_{p=1}^{n} f({s_1}_{2p-1})(g({s_1}_{2p}) - g({s_1}_{2p-2}))\right| < 1.$$

s_{i+1} = Stieltjes subdivision $\{(s_{i+1})_p\}_0^{2n}$ of $[a,b]$ such that s_{i+1} refines s_i and if $\{t_p\}_0^{2m}$ is a refinement of s_{i+1} then

$$\left|\sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \sum_{p=1}^{n} f({s_{i+1}}_{2p-1})(g({s_{i+1}}_{2p}) - g({s_{i+1}}_{2p-2}))\right| < \frac{1}{i+1}.$$

Let $V = \{V_k\}_1^\infty$ be the sequence defined by

$$V_k = \sum_{p=1}^{n} f(s_{k_{2p-1}})(g(s_{k_{2p}}) - g(s_{k_{2p-2}}))$$

In the following three lemmas f, g, S, and V are as above.

<u>Lemma 1:</u> Let ϵ be a positive number and let § be a positive number such that § $< \epsilon$. Let $\{s_p\}_0^{2n}$ be the Stieltjes subdivision of [a,b] such that if $\{t_p\}_0^{2m}$ is a refinement of $\{s_p\}_0^{2n}$ then

$$\left| \sum_{p=1}^{n} f(s_{2p-1})(g(s_{2p}) - g(s_{2p-2})) - \right.$$

$$\left. \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \epsilon.$$ There exist a refinement $\{\bar{s}_p\}_0^{2q}$ of $\{s_p\}_0^{2n}$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{\bar{s}_p\}_0^{2q}$ then

$$\left| \sum_{p=1}^{q} f(\bar{s}_{2p-1})(g(\bar{s}_{2p}) - g(\bar{s}_{2p-2})) - \right.$$

$$\left. \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < §.$$

<u>Proof:</u> Let $\{r_p\}_0^{2k}$ be a Stieltje subdivision of [a,b] such that if $\{t_p\}_0^{2m}$ is a refinement of $\{r_p\}_0^{2k}$ then

$$\left| \sum_{p=1}^{k} f(r_{2p-1})(g(r_{2p}) - g(r_{2p-2})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - \right.$$

$g(t_{2p-2}))| < \S/2$. Let $\{\bar{s}_p\}_0^{2q}$ be a common refinement of $\{r_p\}_0^{2k}$ and $\{s_p\}_0^{2n}$. Then $\{\bar{s}_p\}_0^{2q}$ is a refinement of $\{s_p\}_0^{2n}$ and if $\{t_p\}_0^{2m}$ is a refinement of $\{\bar{s}_p\}_0^{2q}$ it is also a refinement of $\{r_p\}_0^{2k}$ and

$$\left| \sum_{p=1}^{q} f(\bar{s}_{2p-1})(g(\bar{s}_{2p}) - g(\bar{s}_{2p-2})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| \leq$$

$$\left| \sum_{p=1}^{q} f(\bar{s}_{2p-1})(g(\bar{s}_{2p}) - g(\bar{s}_{2p-2})) - \sum_{p=1}^{k} f(r_{2p-1})(g(r_{2p}) - g(r_{2p-2})) \right| +$$

$$\left| \sum_{p=1}^{k} f(r_{2p-1})(g(r_{2p}) - g(r_{2p-2})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \frac{\S}{2} + \frac{\S}{2} = \S.$$

Thus $\{\bar{s}_p\}_0^{2q}$ is a refinement of $\{s_p\}_0^{2n}$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{\bar{s}_p\}_0^{2q}$ then $\left| \sum_{p=1}^{q} f(\bar{s}_{2p-1})(g(\bar{s}_{2p}) - g(\bar{s}_{2p-2})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \S$.

<u>Lemma 2:</u> The sequence V converges.

<u>Proof:</u> Let ϵ be a positive number. There exist a positive integer N such that $\frac{1}{N} < \epsilon/2 < \epsilon$. Let c and d be positive integers such that $c \geq N$ and $d \geq N$. There is a Stieltjes subdivision s_c of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of s_c then

$$\left| \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \sum_{p=1}^{n} f(s_{c_{2p-1}})(g(s_{c_{2p}}) - g(s_{c_{2p-2}})) \right| < \frac{1}{c}.$$

There is a Stieltjes subdivision s_d of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of s_d then

$$\left| \sum_{p=1}^{q} f(s_{d_{2p-1}})(g(s_{d_{2p}}) - g(s_{d_{2p-2}})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \frac{1}{d}.$$

Let $\{t_p\}_0^{2m}$ be a common refinement of s_c and s_d then

$| V_d - V_c | =$

$$\left| \sum_{p=1}^{q} f(s_{d_{2p-1}})(g(s_{d_{2p}}) - g(s_{d_{2p-2}})) - \sum_{p=1}^{n} f(s_{c_{2p-1}})(g(s_{c_{2p}}) - g(s_{c_{2p-2}})) \right| =$$

$$\left| \sum_{p=1}^{q} f(s_{d_{2p-1}})(g(s_{d_{2p}}) - g(s_{d_{2p-2}})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) + \right.$$

$$\left. \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \sum_{p=1}^{n} f(s_{c_{2p-1}})(g(s_{c_{2p}}) - g(s_{c_{2p-2}})) \right| \leq$$

$$\left| \sum_{p=1}^{q} f(s_{d_{2p-1}})(g(s_{d_{2p}}) - g(s_{d_{2p-2}})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - \right.$$

$g(t_{2p-2}))\ | +$

$$\left| \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \sum_{p=1}^{n} f(s_{c_{2p-1}})(g(s_{c_{2p}}) - g(s_{c_{2p-2}})) \right| < \frac{1}{d} + \frac{1}{c} \leq \frac{1}{N} + \frac{1}{N} < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus if $\epsilon > 0$ there exist a positive integer N such that if c and d are integers and $N < c, d$ then $| V_d - V_d | =$

$$\left| \sum_{p=1}^{q} f(s_{d_{2p-1}})(g(s_{d_{2p}}) - g(s_{d_{2p-2}})) - \sum_{p=1}^{n} f(s_{c_{2p-1}})(g(s_{c_{2p}}) - g(s_{c_{2p-2}})) \right| < \epsilon.$$

Thus V is a Cauchy sequence and V converges.

<u>Lemma 3:</u> $\lim_{k \to \infty} V_k = \int_a^b f\, dg$

<u>Proof:</u> Let ϵ be a positive number. Since V converges let $\lim_{k \to \infty} V_k = Z$. There exists a positive integer N such that if k is an integer and $N \leq k$ then $| Z - V_k | < \epsilon/2$. Let k be an integer such that $N \leq k$ and $\frac{1}{k} < \epsilon/2$. Let $\{t_p\}_0^{2m}$ be a refinement of $s_k = \{s_{k_p}\}_0^{2n}$ then

$$\left| \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \sum_{p=1}^{n} f(s_{k_{2p-1}})(g(s_{k_{2p}}) - g(s_{k_{2p-2}})) \right| =$$

$\left| \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2}) - V_k \right| < \frac{1}{k}$. Then

$$\left| Z - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| = \left| Z - \sum_{p=1}^{n} f(s_{k_{2p-1}})(g(s_{k_{2p}}) - g(s_{k_{2p-2}})) + \right.$$

$$\left. \sum_{p=1}^{n} f(s_{k_{2p-1}})(g(s_{k_{2p}}) - g(s_{k_{2p-2}})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| \leq$$

$$\left| Z - \sum_{p=1}^{n} f(s_{k_{2p-1}})(g(s_{k_{2p}}) - g(s_{k_{2p-2}})) \right| + \left| \sum_{p=1}^{n} f(s_{k_{2p-1}})(g(s_{k_{2p}}) - g(s_{k_{2p-2}})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right|$$

$$= \left| Z - V_k \right| + \left| V_k - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \epsilon/2 + \frac{1}{k} <$$

$\epsilon/2 + \epsilon/2 = \epsilon$. Thus if $\epsilon > 0$, Z is a number and s_k is a Stieltjes subdivision of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of s_k then $\left| Z - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \epsilon$. Therefore $Z = \int_a^b f\,dg$ and $\lim_{k \to \infty} V_k = \int_a^b f\,dg$.

The preceding lemmas and definitions may be combined and stated as follows.

Theorem 8: Let f and g be functions from $[a,b]$ to the real numbers. If ϵ is a positive number and there exists a Stieltjes subdivision $\{s_p\}_0^{2n}$ of $[a,b]$ such that if $\{t_p\}_0^{2m}$ refines $\{s_p\}_0^{2n}$ then $\left| \sum_{p=1}^{n} f(s_{2p-1})(g(s_{2p}) - g(s_{2p-2})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right| < \epsilon$, then $\int_a^b f\,dg$ exists.

Theorem 9: Let f and g be functions from $[a,b]$ to the real numbers. If f is continuous and g is of bounded variation on $[a,b]$ then $\int_a^b f\,dg$ exists.

Proof: Let ϵ be a positive number. Let H be the total variation of g. By the uniform continuity of f, let $\S$ be a positive number such that if $x, y \in [a,b]$ and $| x - y | < \S$ then $| f(x) - f(y) | < \frac{\epsilon}{1+H}$. Let $\{s_p\}_0^{2n}$ be a Stieltjes subdivision of $[a,b]$ such that $\|s\| < \S$. Let $\{t_p\}_0^{2m}$ be a refinement of $\{s_p\}_0^{2n}$. Since each t_{2p-1}, $1 \leq p \leq m$, is contained in some $[s_{2k-2}, s_{2k}]$, then there is a number z_{2p-1} such that $f(s_{2k-1}) + z_{2p-1} = f(t_{2p-1})$ and $| z_{2p-1} | < \frac{\epsilon}{1+H}$. Then $\left| \sum_{p=1}^{n} f(s_{2p-1})(g(s_{2p}) - g(s_{2p-2})) - \right.$

$\left. \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) \right|$

$= \left| \sum_{p=1}^{n} f(s_{2p-1})(g(s_{2p}) - g(s_{2p-2})) - \sum_{p=1}^{m} (f(s_{2k-1}) + z_{2p-1})(g(t_{2p}) - \right.$

$g(t_{2p-2}))\,|$ and since the even part of $\{s_p\}_0^{2n}$ is contained in the even part of $\{t_p\}_0^{2m}$ the above equals $|\sum_{p=1}^{n} f(s_{2p-1})(g(s_{2p}) -$

$$g(s_{2p-2})) - \sum_{p=1}^{n} f(s_{2p-1})(g(s_{2p}) - g(s_{2p-2})) - \sum_{p=1}^{m} z_{2p-1}(g(t_{2p})-g(t_{2p-2}))|$$

$$= |\sum_{p=1}^{m} z_{2p-1}(g(t_{2p}) - g(t_{2p-2}))| \leq \sum_{p=1}^{m} |z_{2p-1}||g(t_{2p}) - g(t_{2p-2})| <$$

$$\sum_{p=1}^{m} \frac{\epsilon}{1+H} |\, g(t_{2p}) - g(t_{2p-2})| = \frac{\epsilon}{1+H} \sum_{p=1}^{m} |g(t_{2p}) - g(t_{2p-2})| \leq \frac{\epsilon}{1+H}\cdot$$

$H < \epsilon$. Thus if $\epsilon > 0$ there is a Stieltjes subdivision $\{s_p\}_0^{2n}$ of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{s_p\}_0^{2n}$ then

$$|\sum_{p=1}^{n} f(s_{2p-1})(g(s_{2p}) - g(s_{2p-2})) - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2}))| < \epsilon$$

and by theorem 8, $\int_a^b f\,dg$ exists.

<u>Theorem 10:</u> If $a \leq c \leq b$ and $\int_a^b f\,dg$ exists then $\int_a^b f\,dg =$

$$\int_a^c f\,dg + \int_c^b f\,dg.$$

<u>Proof:</u> Let ϵ be a positive number. Since $\int_a^b f\,dg$ exists let $\{s_p\}_0^{2n}$ be a Stieltjes subdivision of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{s_p\}_0^{2n}$ then $|\int_a^b f\,dg - \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2}))| < \epsilon/2$. Let $\{t_p\}_0^{2m}$ be a refinement of $\{s_p\}_0^{2n}$ such that

$t_{2k} = c$. Let $\{r_p\}_0^{2q}$ be a refinement of $\{t_p\}_0^{2m}$ such that $r_{2j} = c$ and $\{r_p\}_0^{2q}$ is identical to $\{t_p\}_0^{2m}$ on $[c,b]$. Then $\epsilon = \epsilon/2 + \epsilon/2 >$

$$| \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \int_a^b f\,dg | + | \int_a^b f\,dg - \sum_{p=1}^{q} f(r_{2p-1})(g(r_{2p}) - g(r_{2p-2}))|$$

$$\geq | \sum_{p=1}^{m} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \sum_{p=1}^{q} f(r_{2p-1})(g(r_{2p}) - g(r_{2p-2}))| =$$

$$| \sum_{p=1}^{k} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \sum_{p=1}^{j} f(r_{2p-1})(g(r_{2p}) - g(r_{2p-2}))|$$ thus

$\{t_p\}_0^{2k}$ is a Stieltjes subdivision of $[a,c]$ such that if $\{r_p\}_0^{2j}$ is a refinement of $\{t_p\}_0^{2k}$ then $| \sum_{p=1}^{k} f(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \sum_{p=1}^{j} f(r_{2p-1})(g(r_{2p}) - g(r_{2p-2}))| < \epsilon$. Therefore by theorem 8, $\int_a^c f\,dg$ exists. By similar method $\int_c^b f\,dg$ exists. Let $\{u_p\}_0^{2k}$ be a Stieltjes subdivision of $[a,c]$ such that if $\{W_p\}_0^{2j}$ is a refinement of $\{u_p\}_0^{2k}$ then $| \sum_{p=1}^{j} f(W_{2p-1})(g(W_{2p}) - g(W_{2p-2})) - \int_a^c f\,dg | < \epsilon/2$. Let $\{V_p\}_0^{2i}$ be a Stieltjes subdivision of $[c,b]$ such that if $\{W_p\}_0^{2q}$ is a refinement of $\{V_p\}_0^{2i}$ then $| \sum_{p=1}^{q} f(W_{2p-1})(g(W_{2p}) - g(W_{2p-2})) - \int_c^b f\,dg | < \epsilon/2$. Let $\{z_p\}_0^{2(i+k)}$ be the Stieltjes subdivision of $[a,b]$ such that $z_p = u_p$ for $0 \leq p \leq 2k$ and $z_p = V_{p-2k}$ for $2k \leq p \leq 2(i+k)$. Let $\{W_p\}_0^{2m}$ be a refinement of $\{z_p\}_0^{2(i+k)}$ then there is an integer $d \leq m$ such that $W_{2d} = z_{2i}$.

Then $\epsilon = \epsilon/2 + \epsilon/2 > | \sum_{p=1}^{d} f(W_{2p-1})(g(W_{2p}) - g(W_{2p-2})) - \int_a^c f\,dg | +$

$| \sum_{p=d+1}^{m} f(W_{2p-1})(g(W_{2p}) - g(W_{2p-2})) - \int_c^b f\,dg | \geq | \sum_{p=1}^{m} f(W_{2p-1})(g(W_{2p}) -$

$g(W_{2p-2})) - (\int_a^c f\,dg + \int_c^b f\,dg)|$. Thus by theorem 3,

$$\int_a^c f\,dg + \int_c^b f\,dg = \int_a^b f\,dg.$$

<u>Theorem 11:</u> If f is a function from $[a,b]$ to the real numbers and $\int_a^b f\,d\,f$ exists then f is continuous.

Proof: Suppose f is not continuous on $[a,b]$. Let $c \in [a,b]$ such that f is not continuous at c. Either the discontinuity at c is on the right or the left. Let the discontinuity be on the right. Let ϵ be a positive number such that if § is a positive number there is an $x \in [a,b]$ and $| x - c | <$ § such that $|f(x) - f(c)| > \epsilon$. Let $\gamma = \frac{\epsilon^2}{2}$. Let $\{s_p\}_0^{2n}$ be the Stieltjes subdivision of $[a,b]$ such that if $\{t_p\}_0^{2m}$ is a refinement of $\{s_p\}_0^{2n}$ then

$| \sum_{p=1}^{m} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})) - \int_a^b f\,d\,f | < \gamma$. Let k be a positive integer. Let $\{t_p\}_0^{2m}$ be a refinement of $\{s_p\}_0^{2n}$ such that

$t_{2k} = c$ and $t_{2k+1} \neq c$. Let $d \in [t_{2k}, t_{2k+1}]$ such that $| f(c) - f(d)| > \epsilon$. Let $\{r_p\}_0^{2m+2}$ be a refinement of $\{t_p\}_0^{2m}$ such that $r_{2k+1} = r_{2k+2} = d$, $r_p = t_p$ for $0 \leq p \leq 2k$, and $r_{p+2} = t_p$ for $2k+1 \leq p \leq 2m$. Let $\{u_p\}_0^{2m+2}$ be a refinement of $\{t_p\}_0^{2m}$ such that $u_{2k+1} = t_{2k}$, $u_{2k+2} = d$, $u_p = t_p$ for $0 \leq p \leq 2k$, and $u_{p+2} = t_p$ for $2k+1 \leq p \leq 2m$.

Then $\left| \sum_{p=1}^{m+1} f(r_{2p-1})(f(r_{2p}) - f(r_{2p-2})) - \sum_{p=1}^{m+1} f(u_{2p-1})(f(u_{2p}) - f(u_{2p-2}))\right| =$

$$| f(r_{2k+1})(f(r_{2k+2}) - f(r_{2k}) - f(u_{2k+1})(f(u_{2k+2}) - f(u_{2k}))| =$$

$$| f(d)(f(d) - f(c)) - f(c)(f(d) - f(c))| = |(f(d) - f(c))(f(d) - f(c))| = | f(d) - f(c)|^2 > \epsilon^2.$$

Since $\{r_p\}_0^{2m+2}$ and $\{u_p\}_0^{2m+2}$ are refinements of $\{s_p\}_0^{2n}$ then

$$\epsilon^2 = \frac{2}{2}\epsilon^2 = 2\,\Upsilon > \left| \sum_{p=1}^{m+1} f(r_{2p-1})(f(r_{2p}) - f(r_{2p-2})) - \int_a^b f\, d\, f \right| +$$

$$\left| \int_a^b f\, d\, f - \sum_{p=1}^{m+1} f(u_{2p-1})(f(u_{2p}) - f(u_{2p-2}))\right| \geq$$

$$\left| \sum_{p=1}^{m+1} f(r_{2p-1})(f(r_{2p}) - f(r_{2p-2})) - \int_a^b f\, d\, f + \int_a^b f\, d\, f - \right.$$

$$\left. \sum_{p=1}^{m+1} f(u_{2p-1})(f(u_{2p}) - f(u_{2p-2}))\right|$$

$$= \left| \sum_{p=1}^{m+1} f(r_{2p-1})(f(r_{2p}) - f(r_{2p-2})) - \sum_{p=1}^{m+1} f(u_{2p-1})(f(u_{2p}) - f(u_{2p-2})) \right|,$$

a contradiction. Thus f is continuous. A similar argument holds for discontinuity on the left.

CHAPTER II

A Stieltjes Integral Existence Theorem for Some Functions Not of Bounded Variation

Example 1: Let $g\colon [0,1] \to$ reals be defined by $g(x) = \begin{cases} \sqrt{x} \sin \frac{\pi}{x} \text{ if } x \neq 0. \\ 0 \text{ if } x = 0. \end{cases}$ Then g is continuous on $[0,1]$ but $\int_0^1 g\, dg$ does not exist.

Proof: The function g is the product of continuous functions for $x \neq 0$; thus if g is continuous at zero then it is continuous on $[0,1]$. Let ϵ be a positive number. Let $§ = \epsilon^2$ and if $x \in [0,1]$ such that $|x-0| < § = \epsilon^2$ then

$$\left|\sqrt{x} \sin \frac{\pi}{x} - 0\right| = \left|\sqrt{x} \sin \frac{\pi}{x}\right| = |\sqrt{x}|\left|\sin \frac{\pi}{x}\right| \leq |\sqrt{x}| = \sqrt{x} < \sqrt{§} = \sqrt{\epsilon^2}$$

$= \epsilon$ and g is continuous at zero.

Let $\{s_p\}_0^\infty$ be the sequence defined by $s_0 = 1$, $s_{2p} = \frac{2}{2p+1}$, $s_{2p-1} = s_{2p-2}$, where $0 < p$.

$$\sum_{p=1}^{\infty} g(s_{2p-1})(g(s_{2p-2}) - g(s_{2p})) = \sum_{p=1}^{\infty} g^2(s_{2p-2}) - g(s_{2p-2})g(s_{2p}) =$$

$$\sum_{p=2}^{\infty} \left(\sqrt{s_{2p-2}} \sin \frac{\pi}{s_{2p-2}}\right)^2 - \sqrt{s_{2p-2}} \sin \frac{\pi}{s_{2p-2}} \sqrt{s_{2p}} \sin \frac{\pi}{s_{2p}} =$$

$$\sum_{p=2}^{\infty} s_{2p-2} \sin^2 \frac{\pi}{s_{2p-2}} - \sqrt{s_{2p}s_{2p-2}} \sin \frac{\pi}{s_{2p-2}} \sin \frac{\pi}{s_{2p}} = \sum_{p=2}^{\infty} \frac{2}{2p-1} +$$

$$\sqrt{\frac{2}{2p+1} \cdot \frac{2}{2p-1}} > \sum_{p=2}^{\infty} \frac{1}{2p-1} = \sum_{p=1}^{\infty} \frac{1}{2p+1}.$$ By theorem 3.27 of Rudin,

<u>Principles of Mathematical Analysis</u>, $\sum_{p=1}^{\infty} \frac{1}{2p+1}$ converges if and only if $\sum_{p=0}^{\infty} \frac{2^p}{2^{p+1}+1}$ converges. $\lim_{p\to\infty} \frac{2^p}{2^{p+1}+1} = \lim_{p\to\infty} \frac{1}{2+\frac{1}{2}p} = \frac{1}{2}$ thus $\sum_{p=1}^{\infty} \frac{1}{2p+1}$ diverges and $\sum_{p=1}^{\infty} g(s_{2p-1})(g(s_{2p-2}) - g(s_{2p}))$ diverges by comparison. Therefore if $\{r_p\}_0^{2n}$ is a Stieltjes subdivision of $[0,1]$ with q the smallest integer such that $r_{2q} \neq 0$ then there is a refinement $\{t_p\}_0^{2m}$ of $\{r_p\}_0^{2n}$ such that if M is a positive number there is a positive integer k such that $t_{2k} = r_{2q}$, $\{t_p\}_1^{2k-1}$ is identical to $2k-2$ values of $\{s_p\}_0^{\infty}$ on $(0, r_{2q})$ and $\sum_{p=1}^{k} g(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \sum_{p=1}^{q} g(r_{2p-1})(g(r_{2p}) - g(r_{2p})) > M$. Thus if $\epsilon > 0$ and $\{r_p\}_0^{2n}$ is a Stieltjes subdivision of $[0,1]$ there is a refinement $\{t_p\}_0^{2m}$ of $\{r_p\}_0^{2n}$ such that $\left| \sum_{p=1}^{m} g(t_{2p-1})(g(t_{2p}) - g(t_{2p-2})) - \sum_{p=1}^{n} g(r_{2p-1})(g(r_{2p}) - g(r_{2p-2})) \right| > \epsilon$ and $\int_0^1 g\, dg$ does not exist.

<u>Example 2:</u> Let f be a function from $[0,1]$ to the real numbers defined by $f(x) = \begin{cases} x \sin \frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

The function f is continuous at 0.

<u>Proof:</u> Let ϵ be a positive number. Let $\delta = \epsilon$. Let $x \in [0,1]$ such that $|0-x| < \delta$ then $|f(0) - f(x)| =$

$|0 - x \sin \frac{\pi}{x}| = |x \sin \frac{\pi}{x}| = |x| |\sin \frac{\pi}{x}| < |x| \cdot 1 = |x| < \delta = \epsilon$. Thus if $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in [0,1]$ and $|0-x| < \delta$ then $|f(0) - f(x)| < \epsilon$ and f is continuous at 0.

The function f is not of bounded variation.

<u>Proof:</u> Suppose f is of bounded variation on $[0,1]$ then there is a number V such that if $\left\{t_p\right\}_0^{2n}$ is a Stieltjes subdivision of $[0,1]$ then $\sum_{p=1}^{n} |f(t_{2p-2}) - f(t_{2p})| < V$. Let $\left\{t_p\right\}_0^{2n}$ be a Stieltjes subdivision of $[0,1]$ such that n is even, $t_0 = 0$, $t_{2n} = t_{2n-1} = 1$, and $t_{2p} = t_{2p-1} = \frac{2}{n-p+2}$ for $1 \leq p < n$. Then $\sum_{p=1}^{n} |f(t_{2p}) - f(t_{2p-2})|$

$= \sum_{p=0}^{n-1} |f(t_{2n-2p}) - f(t_{2n-2p-2})| = |f(t_{2n}) - f(t_{2n-2})| + |f(t_{2n-2}) -$

$f(t_{2n-4})| + \dots + |f(t_2) - f(t_0)| = |f(1) - f(\frac{2}{3})| + |f(\frac{2}{3}) - f(\frac{2}{4})| +$

$\dots + |f(\frac{2}{n-1}) - f(0)| =$

$|0 - (-\frac{2}{3})| + |(-\frac{2}{3}) - 0| + |0 - (\frac{2}{5})| + |(\frac{2}{5}) - 0| + \dots + |(+\frac{2}{n-1}) - 0| =$

$\frac{4}{3} + \frac{4}{5} + \frac{4}{7} + \dots + \frac{4}{n-1} = 4(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{n-1})$. Then

$\sum_{p=1}^{n} |f(t_{2p}) - f(t_{2p-2})| = 4 \sum_{k=1}^{(n/2)-1} \frac{1}{2k+1}$, the first $(n/2)-1$ terms of

$\sum_{k=1}^{\infty} \frac{1}{2k+1}$. By theorem 3.27 of Rudin, <u>Principles of Mathematical Analysis</u>, $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ converges if and only if $\sum_{k=0}^{\infty} \frac{2^k}{2^{k+1}+1}$ converges.

$\lim_{k\to\infty} \frac{2^k}{2^{k+1}+1} = \lim_{k\to\infty} \frac{1}{2+\frac{1}{2}k} = \frac{1}{2}$. Thus $\sum_{k=0}^{\infty} \frac{2^k}{2^{k+1}+1}$ diverges as does

$\sum_{k=1}^{\infty} \frac{1}{2k+1}$. Since the partial sums of $\sum_{k=1}^{\infty} \frac{1}{2k+1}$ form a monotonic

increasing sequence that does not converge then the sequence is unbounded. Thus there is an integer m such that if $\{s_p\}_0^{2m}$ is a Stieltjes subdivision of $[0,1]$ then $\sum_{p=1}^{m} |f(s_{2p}) - f(s_{2p-2})| > V$, a contradiction.

<u>Definition 5:</u> Let f be a function from $[a,b]$ into the real numbers. Then f is said to be locally variable on $[a,b]$ provided there is positive integer N such that if $\{s_p\}_0^n$ is an increasing sequence with $s_o = a$ and $s_n = b$, then f is of bounded variation on all but at most N of the intervals $[s_{p-1}, s_p]$ for $0 < p \leq n$.

<u>Example 3:</u> Let i be a natural number. Let $h_i: [0,1] \to$ real numbers be defined by $h_i(x) = xi^2 + xi - i$. Let $f: [0,1] \to$ real numbers be defined by $f(x) = \begin{cases} x \sin \frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Let $g_i: [\frac{1}{i+1}, \frac{1}{i}]$

$\rightarrow$ real numbers be defined by $g_i(x) = \frac{f(h_i(x))}{i}$. Let $g: [0,1] \rightarrow$ real numbers be defined by $g(x) = \begin{cases} g_i(x) \text{ if } \frac{1}{i+1} \leq x \leq \frac{1}{i}. \\ 0 \text{ if } x = 0 \end{cases}$ Then g is continuous but is not locally variable.

Proof: Let i be a natural number. Then $g(\frac{1}{i}) = g_i(\frac{1}{i}) = \frac{f(h_i(\frac{1}{i}))}{i} = \frac{f(1)}{i} = 0$ and $g(\frac{1}{i+1}) = g_i(\frac{1}{i+1}) = \frac{f(h_i(\frac{1}{i+1}))}{i} = \frac{f(0)}{i} = 0$. The function g_i is continuous in $[\frac{1}{i+1}, \frac{1}{i}]$ by the composition of continuous functions and thus g is continuous on $(0,1]$. Let ϵ be a positive number. Let n be a natural number such that $\frac{1}{n} < \epsilon$. Let $x \epsilon [0,1]$ such that $|0-x| < \frac{1}{n}$ then $|g(0) - g(x)| = |0-g(x)| = |g(x)| \leq |g_i(x)| = |\frac{f(h_i(x))}{i}| \leq |\frac{1}{i}| \leq \frac{1}{n} < \epsilon$. Thus if $\epsilon > 0$ there is a $\S > 0$ such that if $|0-x| < \S$ then $|g(0) - g(x)| < \epsilon$ and g is continuous at 0. The function h_i is monotonically increasing on $[\frac{1}{i+1}, \frac{1}{i}]$ with $h_i(\frac{1}{i+1}) = 0$ and $h_i(\frac{1}{i}) = 1$ thus $f(h_i(x))$ on $[\frac{1}{i+1}, \frac{1}{i}]$ takes on all the values that f takes on $[0,1]$. Then by example 2, $f(h_i(x))$ is not of bounded variation on $[\frac{1}{i+1}, \frac{1}{i}]$. Since there are an infinite number of such intervals then g is not locally variable on $[0,1]$.

Theorem 12: Let f be a function from $[a,b]$ to the real numbers. If $\{s_p\}_0^{2n}$ is a Stieltjes subdivision of $[a,b]$ then there exists a Stieltjes subdivision, $\{r_p\}_0^{2n}$, of $[a,b]$ such that

$$\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) + \sum_{p=1}^{m} f(r_{2p-1})(f(r_{2p}) - f(r_{2p-2})) =$$

$$f^2(b) - f^2(a).$$

Proof: Let $\{s_p\}_0^{2n}$ be a Stieltjes subdivision of $[a,b]$. Let $\{t_p\}_0^{4n}$ be a refinement of $\{s_p\}_0^{2n}$ such that $t_{4p-3} = t_{4p-2} = t_{4p-1} = s_{2p-1}$ and $t_{4p} = s_{2p}$. Let $\{r_p\}_0^{4n}$ be a Stieltjes subdivision of $[a,b]$ such that $r_{2p} = t_{2p}$, $r_{4p-3} = t_{4p-4}$, and $r_{4p-1} = t_{4p}$. Let $m = 2n$. Then $\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) +$

$$\sum_{p=1}^{m} f(r_{2p-1})(f(r_{2p}) - f(r_{2p-2})) =$$

$$\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-1}) + f(s_{2p-1}) - f(s_{2p-2})) +$$

$$\sum_{p=1}^{2n} f(r_{2p-1})(f(r_{2p})- f(r_{2p-2})) =$$

$$\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-1})) + f(s_{2p-1})(f(s_{2p-1}) - f(s_{2p-2})) +$$

$$\sum_{p=1}^{n} f(r_{4p-3})(f(r_{4p-2}) - f(r_{4p-4})) + f(r_{4p-1})(f(r_{4p}) - f(r_{4p-2})).$$ By substitution the above is equal

$$\sum_{p=1}^{n} f(t_{4p-2})(f(t_{4p}) - f(t_{4p-2})) + f(t_{4p-2})(f(t_{4p-2}) - f(t_{4p-4})) +$$

$$\sum_{p=1}^{n} f(t_{4p-4})(f(t_{4p-2}) - f(t_{4p-4})) + f(t_{4p})(f(t_{4p}) - f(t_{4p-2})) =$$

$$\sum_{p=1}^{n} (f(t_{4p}) + f(t_{4p-2}))(f(t_{4p}) - f(t_{4p-2})) + (f(t_{4p-2}) +$$

$$f(t_{4p-4}))(f(t_{4p-2}) - f(t_{4p-4})) =$$

$$\sum_{p=1}^{n} f^2(t_{4p}) - f^2(t_{4p-2}) + f^2(t_{4p-2}) - f^2(t_{4p-4}) =$$

$$\sum_{p=1}^{n} f^2(t_{4p}) - f^2(t_{4p-4}) = f^2(t_{4n}) - f^2(t_0) = f^2(b) - f^2(a).$$ Thus

$\left\{r_p\right\}_0^{2m}$ is a Stieltjes subdivision of $[a,b]$ such that

$$\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) + \sum_{p=1}^{m} f(r_{2p-1})(f(r_{2p}) - f(r_{2p-2})) =$$

$$f^2(b) - f^2(a).$$

<u>Theorem 13:</u> Let f be a continuous function from $[a,b]$ to the real numbers such that if $\left\{s_p\right\}_0^{2n}$ is a Stieltjes subdivision of $[a,b]$ then f is not of bounded variation on at most one of the intervals $[s_{2p-2}, s_{2p}]$. If there exists a number M such that

$$\left| \sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))\right| < M$$ then $\int_a^b f\, d\, f$ exists.

<u>Proof:</u> Since $\int_a^b f\, df$ exists if f is of bounded variation consider f is not of bounded variation. Let ϵ be a positive number. Let $\bar{f} = \max \left\{|f(x)| \,|\, x \in [a,b]\right\}$. Let $\S$ be a positive number such that $\bar{f} \cdot \S < \epsilon/10$. Since f is uniformly continuous on

$[a,b]$ then let γ be a positive number such that if $x, y \in [a,b]$ and $|x-y| < \gamma$ then $|f(x) - f(y)| < \S$. Since f^2 is uniformly continuous on $[a,b]$ then let λ be a positive number such that if $x, y \in [a,b]$ and $|x-y| < \lambda$ then $|f^2(x)-f^2(y)| < \epsilon/10$. Let $\sigma = \min\left\{\gamma, \lambda\right\}$. Let N be the smallest number such that if $\left\{s_p\right\}_0^{2n}$ is a Stieltjes subdivision of $[a,b]$ and $\|s\| < \sigma$ then

$|\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))| \leq N$. Let $\left\{s_p\right\}_0^{2n}$ be a Stieltjes subdivision of $[a,b]$ such that $\|s\| < \sigma$ and $N - \epsilon/2 <$

$|\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))|$. Let c be an integer such that

$1 \leq c \leq n$. Let f not be of bounded variation on $[s_{2c-2}, s_{2c}]$. Then

$N - \epsilon/2 < |\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))| = |\sum_{p=1}^{c-1} f(s_{2p-1})(f(s_{2p}) -$

$f(s_{2p-2})) + f(s_{2c-1})(f(s_{2c}) - f(s_{2c-2})) + \sum_{p=c+1}^{n} f(s_{2p-1})(f(s_{2p}) -$

$f(s_{2p-2}))| \leq | f(s_{2c-1})(f(s_{2c}) - f(s_{2c-2}))| + | \sum_{p=1}^{c-1} f(s_{2p-1})(f(s_{2p}) -$

$f(s_{2p-2})) + \sum_{p=c+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))| = |f(s_{2c-1})||f(s_{2c}) -$

$f(s_{2c-2})| + | \sum_{p=1}^{c-1} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) + \sum_{p=c+1}^{n} f(s_{2p-1})(f(s_{2p}) -$

$f(s_{2p-2}))| < \bar{f} \cdot \S + |\sum_{p=1}^{c-1} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) +$

$\sum_{p=c+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))| < \epsilon/10 + |\sum_{p=1}^{c-1} f(s_{2p-1})(f(s_{2p}) -$

$f(s_{2p-2})) + \sum_{p=c+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))|$. Thus $N - \epsilon/2 < \epsilon/10 +$

$|\sum_{p=1}^{c-1} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) + \sum_{p=c+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))|$ or

$N - \frac{3\epsilon}{5} < |\sum_{p=1}^{c-1} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) + \sum_{p=c+1}^{n} f(s_{2p-1})(f(s_{2p}) -$

$f(s_{2p-2}))|$.

Suppose there exists, $\{z_p\}_0^{2k}$, a Stieltjes subdivision of

$[s_{2c-2}, s_{2c}]$ such that $|\sum_{p=1}^{k} f(z_{2p-1})(f(z_{2p}) - f(z_{2p-2}))| > \frac{4}{5}\epsilon$.
By Theorem 12 there exists a Stieltjes subdivision $\{t_p\}_0^{2m}$ of

$[s_{2c-2}, s_{2c}]$ such that $\sum_{p=1}^{k} f(z_{2p-1})(f(z_{2p}) - f(z_{2p-2})) +$

$\sum_{p=1}^{m} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})) = f^2(s_{2c}) - f^2(s_{2c-2})$. Thus

$|\sum_{p=1}^{k} f(z_{2p-1})(f(z_{2p}) - f(z_{2p-2})) + \sum_{p=1}^{m} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2}))| =$

$|f^2(s_{2c}) - f^2(s_{2c-2})|$ and $|\sum_{p=1}^{k} f(z_{2p-1})(f(z_{2p}) - f(z_{2p-2}))| -$

$$\left|\sum_{p=1}^{m} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2}))\right| \leq \left|\sum_{p=1}^{k} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) + \right.$$

$$\left.\sum_{p=1}^{m} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2}))\right| = \left| f^2(s_{2c}) - f^2(s_{2c-2})\right| < \frac{\epsilon}{10}$$ since

$|s_{2c} - s_{2c-2}| < \sigma$. Thus $\frac{4\epsilon}{5} < \left|\sum_{p=1}^{k} f(z_{2p-1})(f(z_{2p}) - f(z_{2p-1})\right| <$

$\left|\sum_{p=1}^{m} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2}))\right| + \epsilon/10$ and $\frac{3\epsilon}{5} <$

$\left|\sum_{p=1}^{m} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2}))\right|$. Observe that values of

$\sum_{p=1}^{k} f(z_{2p-1})(f(z_{2p}) - f(z_{2p-2}))$ and $\sum_{p=1}^{m} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2}))$ must be opposite in sign since the absolute value of each exceeds $\epsilon/10$ and the absolute value of their sum is less than $\epsilon/10$. Let

$\sum_{p=1}^{k} f(z_{2p-1})(f(z_{2p}) - f(z_{2p-2}))$ be positive.

Either $N - \frac{3\epsilon}{5} < \sum_{p=1}^{c-1} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) +$

$\sum_{p=c+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))$ or $N - \frac{3\epsilon}{5} < -$

$(\sum_{p=1}^{c-1} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) + \sum_{p=c+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))$. In the former case let $\{r_p\}_0^{2j}$ be a Stieltjes subdivision of $[a,b]$ such that $\{r_p\}_0^{2j}$ is identical to $\{s_p\}_0^{2n}$ on $[a, s_{2c-2}]$, identical to

$\{z_p\}_0^{2k}$, on $[s_{2c-2}, s_{2c}]$ and identical to $\{s_p\}_0^{2n}$ on $[s_{2c}, b]$. Then

$$|\sum_{p=1}^{j} f(r_{2p-1})(f(r_{2p}) - f(r_{2p-2}))| = \sum_{p=1}^{c-1} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) +$$

$$\sum_{p=c+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) + \sum_{p=1}^{k} f(z_{2p-1})(f(z_{2p}) - f(z_{2p-2})) >$$

$N - \frac{3\epsilon}{5} + \frac{3\epsilon}{5} = N$ a contradiction. In the latter case let $\{r_p\}_0^{2j}$ be a Stieltjes subdivision of $[a,b]$ such that $\{r_p\}_0^{2j}$ is identical to $\{s_p\}_0^{2n}$ on $[a, s_{2c-2}]$, identical to $\{t_p\}_0^{2m}$ on $[s_{2c-2}, s_{2c}]$ and identical to $\{s_p\}_0^{2n}$ on $[s_{2c}, b]$. Then $|\sum_{p=1}^{j} f(r_{2p-1})(f(r_{2p}) -$

$$f(r_{2p-2}))| = -\left(\sum_{p=1}^{c-1} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) + \sum_{p=c+1}^{n} f(s_{2p-1})(f(s_{2p}) -\right.$$

$$\left.f(s_{2p-2}))\right) - \sum_{p=1}^{m} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})) > N - 3\epsilon/5 + 3\epsilon/5 = N$$ a

contradiction. Thus if $\{z_p\}_0^{2k}$ is a Stieltjes subdivision of

$[s_{2c-2}, s_{2c}]$ then $|\sum_{p=1}^{k} f(z_{2p-1})(f(z_{2p}) - f(z_{2p-2}))| \leq 4\epsilon/5$. The other case is similar.

Since f is of bounded variation on $[a, s_{2c-2}]$ then $\int_a^{s_{2c-2}}$ $f\,df$ exists. Let $\{\bar{u}_p\}_0^{2i}$ be a Stieltjes subdivision of $[a, s_{2c-2}]$ such that if $\{u_p\}_0^{2w}$ is a refinement of $\{\bar{u}_p\}_0^{2i}$ then $|\int_a^{s_{2c-2}} f\,df -$

$\sum_{p=1}^{i} f(u_{2p-1})(f(u_{2p}) - f(u_{2p-2}))| < \epsilon/20$. Since f is of bounded variation on $[s_{2c}, b]$ then $\int_{s_{2c}}^{b} f\,df$ exists. Let $\left\{\bar{v}_p\right\}_0^{2h}$ be a Stieltjes subdivision of $[s_{2c}, b]$ such that if $\left\{v_p\right\}_0^{2d}$ is a refinement of $\left\{\bar{v}_p\right\}_0^{2l}$ then $\left| \int_{s_{2c}}^{b} f\,df - \sum_{p=1}^{d} f(v_{2p-1})(f(v_{2p}) - f(v_{2p-2}))\right| < \epsilon/20$. Let $\left\{\bar{h}_p\right\}_0^{2e}$ be a Stieltjes subdivision of $[a,b]$ such that $\left\{\bar{h}_p\right\}_0^{2e}$ is identical to $\left\{\bar{u}_p\right\}_0^{2i}$ on $[a, s_{2c-2}]$, $\bar{h}_{2i+1} = \bar{h}_{2i+2} = s_{2c-2}$, $\bar{h}_{2i+3} = s_{2c}$, and identical to $\left\{\bar{v}_p\right\}_0^{2l}$ on $[s_{2c}, b]$. Let $\left\{h_p\right\}_0^{2g}$ be a refinement of $\left\{\bar{h}_p\right\}_0^{2e}$ such that j, k are integers and $h_{2j} = s_{2c-2}$, $h_{2k} = s_{2c}$. Then

$$\left| \int_a^{s_{2c-2}} f\,df + \int_{s_{2c}}^{b} f\,df - \sum_{p=1}^{g} f(h_{2p-1})(f(h_{2p}) - f(h_{2p-2}))\right| = \left| \int_a^{s_{2c-2}} f\,df + \int_{s_{2c}}^{b} f\,df - \sum_{p=1}^{j} f(h_{2p-1})(f(h_{2p}) - f(h_{2p-2})) - \sum_{p=j+1}^{k} f(h_{2p-1})(f(h_{2p}) - f(h_{2p-2})) - \sum_{p=k+1}^{g} f(h_{2p-1})(f(h_{2p}) - f(h_{2p-2}))\right| \leq \left| \int_a^{s_{2c-2}} f\,df - \sum_{p=1}^{j} f(h_{2p-1})(f(h_{2p}) - f(h_{2p-2}))\right| + \left| \int_{s_{2c}}^{b} f\,df - \sum_{p=k+1}^{g} f(h_{2p-1})(f(h_{2p}) - f(h_{2p-2}))\right| + \left| \sum_{p=j+1}^{k} f(h_{2p-1})(f(h_{2p}) - f(h_{2p-2}))\right| < \epsilon/20 + \epsilon/20 + 4\epsilon/5 = \frac{9\epsilon}{10}.$$

$$\left| \frac{f^2(b) - f^2(a)}{2} - \left(\int_a^{s_{2c-2}} f\,df + \int_{s_{2c}}^{b} f\,df\right)\right| = \left| \frac{f^2(b) - f^2(a)}{2} - \right.$$

$$\frac{(f^2(s_{2c-2}) - f^2(a)}{2} + \frac{f^2(b) - f^2(s_{2c}))}{2}| =$$

$$|\frac{f^2(b) - f^2(a) - f^2(s_{2c-2}) + f^2(a) - f^2(b) + f^2(s_{2c})}{2}| =$$

$$|\frac{f^2(s_{2c}) - f^2(s_{2c-2})}{2}| < \frac{\epsilon/10}{2} = \epsilon/20.$$

Thus $|\frac{f^2(b) - f^2(a)}{2} - \sum_{p=1}^{g} f(h_{2p-1})(f(h_{2p}) - f(h_{2p-2}))| =$

$|\frac{f^2(b) - f^2(a)}{2} - (\int_a^{s_{2c-2}} f\,df + \int_{s_{2c}}^{b} f\,df) + (\int_a^{s_{2c-2}} f\,df +$

$\int_{s_{2c}}^{b} f\,df) - \sum_{p=1}^{g} f(h_{2p-1})(f(h_{2p}) - f(h_{2p-2}))| \leq |\frac{f^2(b) - f^2(a)}{2} -$

$(\int_a^{s_{2c-2}} f\,df + \int_{s_{2c}}^{b} f\,df)| + |(\int_a^{s_{2c-2}} f\,df + \int_{s_{2c}}^{b} f\,df) -$

$\sum_{p=1}^{g} f(h_{2p-1})(f(h_{2p}) - f(h_{2p-2}))| < \epsilon/20 + 9\epsilon/10 < \epsilon.$ Therefore if ϵ is a positive number then there exists a Stieltjes subdivision, $\{\bar{h}_p\}_0^{2e}$ such that if $\{h_p\}_0^{2g}$ is a refinement of $\{\bar{h}_p\}_0^{2e}$ then

$|\frac{f^2(b) - f^2(a)}{2} - \sum_{p=1}^{g} f(h_{2p-1})(f(h_{2p}) - f(h_{2p-2}))| < \epsilon$ and $\int_a^b f\,df$ exists.

Theorem 14: Let f be a locally variable continuous function from $[a,b]$ to the real numbers. If there exists a number M such that if $\{s_p\}_0^{2n}$ is a Stieltjes subdivision of $[a,b]$ then

$|\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))| < M$ then $\int_a^b f\,df$ exists.

Proof: Let N be the positive integer such that if $\{s_p\}_0^{2n}$ is a Stieltjes subdivision of $[a,b]$ then f is of bounded variation on all but at most N intervals $[s_{2p-2}, s_{2p}]$ for $0 < p \leq n$. Let $\{s_p\}_0^{2n}$ be a Stieltjes subdivision of $[a,b]$ such that f is not of bounded variation on N intervals $[s_{2p-2}, s_{2p}]$ for $0 < p \leq n$. Let p be a positive integer such that $0 < p \leq n$. Either f is of bounded variation on $[s_{2p-2}, s_{2p}]$ or it is not. If f is of bounded variation on $[s_{p-2}, s_{2p}]$ then $\int_{s_{2p-2}}^{s_{2p}} f\,df$ exists by theorem 9. If f is not of bounded variation on $[s_{2p-2}, s_{2p}]$ then $\int_{s_{2p-2}}^{s_{2p}} f\,df$ exists by theorem 13. By theorem 10, $\sum_{p=1}^{n} [\int_{s_{2p-2}}^{s_{2p}} f\,df] = \int_a^b f\,df.$

<u>Lemma 4:</u> Let f be a continuous function from $[a,b]$ to the real numbers. Let $\{s_p\}_0^{2n}$ be a Stieltjes subdivision of $[a,b]$. If there exists a positive integer k such that $1 < k \leq n$ and either $f(s_{2k}) - f(s_{2k-2}) \geq 0$ and $f(s_{2k-2}) - f(s_{2k-4}) \geq 0$ or $f(s_{2k}) - f(s_{2k-2}) \leq 0$ and $f(s_{2k-2}) - f(s_{2k-4}) \leq 0$ then there is a Stieltjes subdivision $\{t_p\}_0^{2n-2}$ such that $\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) \leq$

$\sum_{p=1}^{n-1} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})).$

<u>Proof:</u> <u>Case I.</u> Let k be a positive integer such that

$1 < k \leq n$, $f(s_{2k}) - f(s_{2k-2}) \geq 0$ and $f(s_{2k-2}) - f(s_{2k-4}) \geq 0$. Let $\{t_p\}_0^{2n-2}$ be a Stieltjes subdivision of $[a,b]$ such that $t_p = s_p$ for $0 \leq p \leq 2k-4$, $t_{2k-3} = c \in [s_{2k-4}, s_{2k}]$ such that $f(c) = \sup \{f(x) \mid x \in [s_{2k-4}, s_{2k}]\}$, and $t_p = s_{2p+2}$ for $2k - 2 \leq p \leq 2n - 2$. Then

$$\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) = \sum_{p=1}^{k-2} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) +$$

$$f(s_{2k-3})(f(s_{2k-2}) - f(s_{2k-4})) + f(s_{2k-1})(f(s_{2k}) - f(s_{2k-2})) +$$

$$\sum_{p=k+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) \leq \sum_{p=1}^{k-2} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) +$$

$$f(c)(f(s_{2k-2}) - f(s_{2k-4})) + f(c)(f(s_{2k}) - f(s_{2k-2})) + \sum_{p=k+1}^{n}$$

$$f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) = \sum_{p=1}^{k-2} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2}))$$

$$+ f(c)(f(s_{2k}) - f(s_{2k-4})) + \sum_{p=k+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) =$$

$$\sum_{p=1}^{k-2} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})) + f(t_{2k-3})(f(t_{2k-2}) - f(t_{2k-4})) +$$

$$\sum_{p=k}^{n-1} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})) = \sum_{p=1}^{n-1} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})).$$

Thus $\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) \leq \sum_{p=1}^{n-1} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2}))$.

Case II. Let k be a positive integer such that $1 < k \leq n$, $f(s_{2k}) - f(s_{2k-2}) \leq 0$, and $f(s_{2k-2}) - f(s_{2k-4}) \leq 0$. Let $\{t_p\}_0^{2n-2}$ be a Stieltjes subdivision of $[a,b]$ such that $t_p = s_p$ for

$0 \leq p \leq 2k - 4$, $t_{2k-3} = d \in [s_{2k-4}, s_{2k}]$ such that $f(d) = \inf \{f(x) | x \in [s_{2k-4}, s_{2k}]\}$, and $t_p = s_{p+2}$ for $2k - 2 \leq p \leq 2n-2$. Then

$$\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) = \sum_{p=1}^{k-2} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) +$$

$$f(s_{2k-3})(f(s_{2k-2}) - f(s_{2k-4})) + f(s_{2k-1})(f(s_{2k}) - f(s_{2k-2})) +$$

$$\sum_{p=k+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) \leq \sum_{p=1}^{k-2} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) +$$

$$f(d)(f(s_{2k-2}) - f(s_{2k-4})) + f(d)(f(s_{2k}) - f(s_{2k-2})) +$$

$$\sum_{p=k+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) = \sum_{p=1}^{k-2} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) +$$

$$f(d)(f(s_{2k}) - f(s_{2k-4})) + \sum_{p=k+1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) =$$

$$\sum_{p=1}^{k-2} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})) + f(t_{2k-3})(f(t_{2k-2}) - f(t_{2k-4})) +$$

$$\sum_{p=k}^{n-1} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})) = \sum_{p=1}^{n-1} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})).$$ Thus

$$\sum_{p=1}^{n} f(s_{2p-1})(f(s_{2p}) - f(s_{2p-2})) \leq \sum_{p=1}^{n-1} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})).$$

Let f be the function from $[0,1]$ to the real numbers defined by $f(x) = \begin{cases} x \sin \frac{\pi}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$. The function f was earlier shown to be continuous and not of bounded variation in example 2.

In the remainder of the paper f is as above.

<u>Lemma 5</u>: If p is a positive integer there is one and only one number $x \in [\frac{1}{p+1}, \frac{1}{p}]$ such that $f'(x) = 0$, $f(x)$ is a maximum of f on $[\frac{1}{p+1}, \frac{1}{p}]$ if p is even and $f(x)$ is as minimum of f on $[\frac{1}{p+1}, \frac{1}{p}]$ if p is odd.

<u>Proof</u>: f is differentiable on $(0,1]$ and $f'(x) = \sin \frac{\pi}{x} - \frac{\pi}{x} \cos \frac{\pi}{x}$. $f'(x) = 0$ when $\sin \frac{\pi}{x} = \frac{\pi}{x} \cos \frac{\pi}{x}$. Since $\cos \frac{\pi}{x}$ cannot be zero here then $f'(x) = 0$ when $\tan \frac{\pi}{x} = \frac{\pi}{x}$. Let p be a positive integer. Since the tangent function takes on all real values on the interval $[p\pi, (p+1)\pi]$ then there is an x such that $\frac{1}{p+1} \leq x \leq \frac{1}{p}$, $\frac{\pi}{\frac{1}{p}} \leq \frac{\pi}{x} \leq \frac{\pi}{\frac{1}{p+1}}$ or $p\pi \leq \frac{\pi}{x} (p+1)\pi$ and $\tan \frac{\pi}{x} = \frac{\pi}{x}$. Since $\frac{\pi}{x}$ is positive for $x \in (0,1]$ then $(\frac{1}{p+1/2}, \frac{1}{p})$ only need be considered since $\tan \frac{\pi}{x}$ is nonpositive elsewhere on $[\frac{1}{p+1}, \frac{1}{p}]$ that it is defined. Suppose there exist x_1 and x_2 such that $p < x_1 < x_2 < p + 1/2$, $\tan \frac{\pi}{x_1} = \frac{\pi}{x_1}$ and $\tan \frac{\pi}{x_2} = \frac{\pi}{x_2}$. Define function g from $(\frac{1}{p+1/2}, \frac{1}{p})$ to the real numbers by $g(x) = \frac{\pi}{x} - \tan \frac{\pi}{x}$. The function g is differentiable. Since $g(x_1) = g(x_2) = 0$ there is a number $c \in (\frac{1}{p+1/2}, \frac{1}{p})$ such that $g'(c) = \frac{g(x_1) - g(x_2)}{x_1 - x_2} = 0$. Consider $g'(x) = \frac{\pi}{x^2}(\sec^2 \frac{\pi}{x} - 1)$ and $\sec^2 \frac{\pi}{x} \neq 1$ on $(\frac{1}{p+1/2}, \frac{1}{p})$ thus $g'(x) \neq 0$, a contradiction. Thus there exists at most one number

$x \epsilon[\frac{1}{p+1}, \frac{1}{p}]$ such that $\tan \frac{\pi}{x} = \frac{\pi}{x}$ or $f'(x) = 0$. $f'(\frac{1}{p}) = \sin p\pi - p\pi \cos p\pi = \begin{cases} -p\pi & \text{if } p \text{ is even} \\ p\pi & \text{if } p \text{ is odd.} \end{cases}$ Thus if p is a positive integer there is one and only one number $x \epsilon[\frac{1}{p+1}, \frac{1}{p}]$ such that $f'(x) = 0$, $f(x)$ is a maximum of f on $[\frac{1}{p+1}, \frac{1}{p}]$ if p is even and $f(x)$ is a minimum of f on $[\frac{1}{p+1}, \frac{1}{p}]$ if p is odd.

Lemma 6: Let $c, d \epsilon(0,1]$ such that $c < d$. $f'(c) = f'(d) = 0$. Then $|f(c)| < |f(d)|$.

Proof: By lemma 5, $f(c)$ and $f(d)$ are local maximums or local minimums. Let p_1 be a positive integer such that $c \epsilon[\frac{1}{p_1+1}, \frac{1}{p_1}]$. Let p_2 be a positive integer such that $d \epsilon[\frac{1}{p_2+1}, \frac{1}{p_2}]$. Then $p_1 > p_2$. In $[\frac{1}{p_1+1}, \frac{1}{p_1}]$, $|x \sin \frac{\pi}{x}|$ is bounded by $\frac{1}{p_1}$. There is an $x \epsilon[\frac{1}{p_2+1}, \frac{1}{p_2}]$ such that $|\sin \frac{\pi}{x}| = 1$ and $|f(x)| = x$. Then $|f(x)| = x > \frac{1}{p_2+1} \geq \frac{1}{p_1}$. Thus $|f(c)| < \frac{1}{p_1}$, $\frac{1}{p_1} < |f(x)|$, $|f(x)| \leq |f(d)|$ and $|f(c)| < \frac{1}{p_1} < |f(x)| \leq |f(d)|$ or $|f(c)| < |f(d)|$.

Define $\{s_p\}_0^\infty$ by $s_0 = 1$, $s_{2p} = x$ such that $f'(x) = 0$ on $[\frac{1}{p+1}, \frac{1}{p}]$, and $s_{2p-1} = s_{2p-2}$. $\sum_{p=1}^{\infty} f(s_{2p-1})(f(s_{2p-2}) - f(s_{2p})) =$

$$\sum_{p=1}^{\infty} f(s_{2p-2})(f(s_{2p-2}) - f(s_{2p})) = \sum_{p=1}^{\infty} f^2(s_{2p-2}) - f(s_{2p-2})f(s_{2p}) =$$

$$\sum_{p=1}^{\infty} [(s_{2p-2})^2 \sin^2 \frac{\pi}{s_{2p-2}} - s_{2p-2}\, s_{2p} \sin \frac{\pi}{s_{2p-2}} \sin \frac{\pi}{s_{2p}}] =$$

$$0 + \sum_{p=2}^{\infty} [(s_{2p-2})^2 \sin^2 \frac{\pi}{s_{2p-2}} - s_{2p-2}\, s_{2p} \sin \frac{\pi}{s_{2p-2}} \sin \frac{\pi}{s_{2p}}] <$$

$\sum_{p=2}^{\infty} [(\frac{1}{p-1})^2 + (\frac{1}{p-1})^2] = 2 \sum_{p=2}^{\infty} \frac{1}{(p-1)}^2 = 2 \sum_{p=1}^{\infty} \frac{1}{p^2}$. By theorem 3.28 of Rudin, <u>Principles</u> <u>of</u> <u>Mathematical</u> <u>Analysis</u> $\sum_{p=1}^{\infty} \frac{1}{p^2}$ converges and thus $\sum_{p=1}^{\infty} f(s_{2p-1})(f(s_{2p-2}) - f(s_{2p}))$ converges by comparison. In lemma 7, $\{s_p\}_0^{\infty}$ is as defined above.

<u>Lemma 7:</u> Let $M = \sum_{p=1}^{\infty} f(s_{2p-1})(f(s_{2p-2}) - f(s_{2p}))$. Let $\{t_p\}_0^{2n}$ be a Stieltjes subidvision of $[0,1]$ then $|\sum_{p=1}^{n} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2}))| < M + 1$.

<u>Proof:</u> By theorem 12 there is a Stieltjes subdivision $\{\bar{t}_p\}_0^{2m}$ of $[0,1]$ such that $\sum_{p=1}^{n} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})) +$ $\sum_{p=1}^{m} f(\bar{t}_{2p-1})(f(\bar{t}_{2p}) - f(\bar{t}_{2p-2})) = f^2(1) - f^2(0) = 0$. Thus let

$\sum_{p=1}^{n} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})) > 0$. By lemma 4, let $\{r_p\}_0^{2q}$ be a Stieltjes subdivision of $[0,1]$ such that if k is an integer and $1 < k \leq q$ then $f(r_{2k}) - f(r_{2k-2})$ and $f(r_{2k-2}) - f(r_{2k-4})$ are opposite in sign and $\sum_{p=1}^{q} f(r_{2p-1})(f(r_{2p}) - f(r_{2p-2})) \geq$

$\sum_{p=1}^{n} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2}))$. $\sum_{p=1}^{q} f(r_{2p-1})(f(r_{2p}) - f(r_{2p-2})) =$

$\sum_{p=0}^{q-1} f(r_{2q-2p-1})(f(r_{2q-2p}) - f(r_{2q-2p-2})) = \sum_{p=1}^{q} f(r_{2q-2p+1})(f(r_{2q-2p+2}) -$

$f(r_{2q-2p}))$. Suppose there exists an integer j such that $1 < j \leq q$ and $f(s_{2j-1})(f(s_{2j-2}) - f(s_{2j})) < |f(r_{2q-2j+1})(f(r_{2q-2j+2}) -$

$f(r_{2q-2j})|$. Then either $|f(s_{2j-1})| < |f(r_{2q-2j+1})|$ or $|f(s_{2j-2}) -$

$f(s_{2j})| < |f(r_{2q-2j+2}) - f(r_{2q-2j})|$. In both cases $s_{2j-2} < r_{2q-2j+2}$ by lemma 6 and $[r_{2q-2j+2}, 1] \subset [s_{2j-2}, 1]$. Then there is a partition $\{v_p\}_0^j$ of $[s_{2j-2}, 1]$ consisting of j intervals such that if i is an integer and $0 < i < j$ then $f(v_i) - f(v_{i-1})$ and $f(v_{i+1}) - f(v_i)$ are opposite in sign. By the definition of $\{s_p\}_0^\infty$, $[s_{2j-2}, 1]$ may be partitioned into at most $j-1$ intervals with this property, a contradiction. Thus if j is an integer and $1 < k \leq q$ then $|f(r_{2q-2j+1})(f(r_{2q-2j+2}) - f(r_{2q-2j}))| \leq f(s_{2j-1})(f(s_{2j-2}) - f(s_{2j}))$.

Then by comparison $\sum_{p=1}^{n} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2})) \leq \sum_{p=1}^{q} f(r_{2q-2p+1})$

$(f(r_{2q-2p+2}) - f(r_{2q-2p})) \leq \sum_{p=1}^{\infty} f(s_{2p-1})(f(s_{2p-2}) - f(s_{2p})) +$

$f(r_{2q-1})(f(r_{2q}) - f(r_{2q-2})) < M + 1.$

<u>Lemma 8:</u> $\int_0^1 f\, df$ exists

<u>Proof:</u> By lemma 7 there is a positive number K such that if

$\{t_p\}_0^{2n}$ is a Stieltjes subdivision $[0,1]$ then $|\sum_{p=1}^{n} f(t_{2p-1})(f(t_{2p}) - f(t_{2p-2}))| < K$. Let $c \in (0,1]$. The function f is of bounded variation on $[c,1]$. Thus if j is an integer and $1 \leq j \leq n$ then f is not of bounded variation on at most one interval $[t_{2j-2}, t_{2j}]$ and f is locally variable. Then by theorem 14 $\int_0^1 f\, df$ exists.

<u>Example 4:</u> Let g be a function from $[0,1]$ to the real numbers

defined by $g(x) = 2 + f(x)$. $\int_0^1 g\, d\, g$ exists and g^2 is not of bounded variation.

<u>Proof:</u> Since $\int_0^1 f\, df$ exists and the constant function 2 is

continuous and of bounded variation then $\int_0^1 f\, d\, 2$ exists and $\int_0^1 f\, df +$ $\int_0^1 f\, d\, 2 = \int_0^1 f\, d\,(2+f)$ by theorem 6. Then $\int_0^1 (2+f)\, df$ exists by theorem 7, 2+f is continuous and $\int_0^1 (2+f)\, d\, 2$ exists, thus $\int_0^1 (2+f)\, df$

$\int_0^1 (2+f)\, d\,2 = \int_0^1 (2+f)\, d(2+f) = \int_0^1 g\, dg$ exists. $g^2 = (2+f)^2 = 4+4f+f^2$.

Let L be a positive number. Let $\left\{t_p\right\}_0^{2n}$ be a Stieltjes subdivision of $[0,1]$ such that $\sum_{p=1}^{n} |f(t_{2p}) - f(t_{2p-2})| > L$. Then $\sum_{p=1}^{n} |g^2(t_{2p}) - g^2(t_{2p-2})| = \sum_{p=1}^{n} |4+4f(t_{2p}) + f^2(t_{2p}) - 4 - 4f(t_{2p-2}) - f^2(t_{2p-2})| = \sum_{p=1}^{n} |4f(t_{2p}) - 4f(t_{2p-2}) + f^2(t_{2p}) - f^2(t_{2p-2})| = \sum_{p=1}^{n} |f(t_{2p}) - f(t_{2p-2})||\,4 + f(t_{2p}) + f(t_{2p-2})| \geq \sum_{p=1}^{n} |f(t_{2p}) - f(t_{2p-2})|(|4| - |f(t_{2p})| - |f(t_{2p-2})|) > \sum_{p=1}^{n} |f(t_{2p}) - f(t_{2p-2})|(4-1-1) = 2\sum_{p=1}^{n} |f(t_{2p}) - f(t_{2p-2})| > \sum_{p=1}^{n} |f(t_{2p}) - f(t_{2p-2})| > L$. Thus if L is a positive number there is a Stieltje subdivision $\left\{t_p\right\}_0^{2n}$ such that $\sum_{p=1}^{n} |g^2(t_{2p}) - g^2(t_{2p-2})| > L$ and g^2 is not of bounded variation.

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