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A SPHERICAL MODEL FOR HYPERBOLIC GEOMETRY

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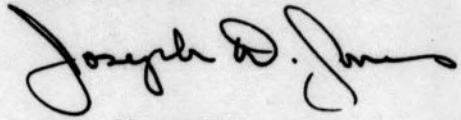
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## PREFACE

It shall be the purpose of this paper to demonstrate a model consistent with postulates of three-dimensional hyperbolic geometry. This model, to be composed of orthogonal spheres and therefore called an orthogonal-spheres model, shall be an extension into three-space of the two dimensional Poincaré model which is perhaps the most frequently-employed model of hyperbolic geometry, although the Klein model and the pseudosphere are also used.

Verification of a model entails its presentation and a demonstration of the validity of basic postulates of the geometry in question for the particular model. In order to develop the orthogonal-spheres model, some preliminary concepts of orthogonality for circles shall be introduced and extended to spheres. Although this material in itself is not directly related to the validity of the model, the concepts presented will be essential to the presentation to follow.

The postulates of Euclidean space shall be accepted and shall be used without specific reference. They shall be listed in the Appendix, along with other theorems from Euclidean geometry which shall be used from time to time.

I cannot be generous enough with an expression of gratitude to my advisor, Mr. Donald Jones. Without his assistance and encouragement, this work would not have been possible.

## I. INTRODUCTION

To contribute to the understanding of this paper, it is necessary to make some statement about notation to be used as well as various statements which shall consistently refer to a specific idea.

When we say "two points" it is implied that the points are distinct unless a statement is made to the contrary. Likewise "two lines" and "two planes" shall denote distinct lines and planes.

With regard to lines and subsets of lines, the following notation shall be employed: If we are given two points A and B, "AB" shall denote the Euclidean distance between A and B; " $\overline{AB}$ " shall denote the segment with endpoints A and B; " $\overleftrightarrow{AB}$ " shall denote the line containing A and B; and " $\overrightarrow{AB}$ " shall denote the ray with A as endpoint and B any point thereof different from A.

If C is a sphere, we shall use "I(C)" to denote the interior of the sphere; "I( $\angle$ RST)" shall denote the interior of the angle  $\angle$ RST. "C," will denote the center of the sphere C.

In general, when there is to be an extension from Euclidean space to hyperbolic space, higher case letters shall denote the former, while lower case letters shall denote the latter. For example,  $\underline{m}$  will denote the measure of a hyperbolic angle, while  $\underline{M}$  will denote the measure of the Euclidean angle associated with the given hyperbolic angle.

Euclidean lines and planes shall be referred to as "E-lines" or "E-planes", and lines and planes in hyperbolic space shall be called "H-lines" or "H-planes".

In addition, we shall make use of the following standard symbols: " $\exists$ " means "there exists"; " $\ni$ ", "such that"; " $\subset$ ", "is contained in"; " $\supset$ ", "contains"; " $\in$ ", "is an element of"; " $\perp$ ", "is perpendicular to"; " $\cong$ ", "is congruent to"; " $\sphericalangle$ ", "angle"; " $\triangle$ ", "triangle"; " $\Phi$ ", "is orthogonal to"; " $=$ ", "is equal to"; " $\cup$ ", "the union of" (with respect to sets); " $\cap$ ", "the intersection of (with respect to sets); " $>$ ", "is strictly greater than"; " $<$ ", "is strictly less than"; " $\geq$ ", "greater than or equal to"; " $\leq$ ", "less than or equal to".

In order to develop a geometry, a number of primary postulates must be assumed on the basis of which the theorems of the system may be proved. There are several equally logical successions in which these postulates may occur; we shall employ the order used by Moise [4, p. 37, ff.]. The postulates which are used for hyperbolic geometry are as follows:

### 1. Incidence Postulates

- 1) Given two points, there is exactly one line containing them.
- 2) Given three non-collinear points, there is exactly one plane containing them.
- 3) If two points lie in a plane, then the line containing them lies in the plane.
- 4) If two planes intersect, then their intersection is a line.
- 5) Every line contains at least two points. Every plane contains at least three non-collinear points. Space contains at least four non-coplanar points.

### 2. Distance Postulates

- 1) Distance is defined as a function associating with

every pair of points  $P, Q$  in space, a non-negative real number  $R$ .  $R$  is called the distance between the points.

- 2) For every pair of points  $P, Q$  the distance between  $P$  and  $Q \geq 0$ .
- 3) The distance between  $P$  and  $Q$  is 0 if and only if  $P = Q$ .
- 4) The distance between  $P$  and  $Q$  equals the distance between  $Q$  and  $P$  for every  $P$  and  $Q$  in space.
- 5) (the Ruler Postulate) Every line has a coordinate system.

3. Space-Separation Postulate: Given a plane in space. The set of all points that do not lie in the plane is the union of two sets  $H_1$  and  $H_2$  such that each of the sets is convex and such that if point  $P$  belongs to one of the sets and point  $Q$  to the other, the segment  $\overline{PQ}$  intersects the plane.

#### 4. Angle Measure Postulates

- 1) There exists a function  $m: \mathcal{A} \rightarrow \mathbb{R}^+$ , where  $\mathcal{A}$  is the set of all angles and  $\mathbb{R}^+$  is the set of positive real numbers. The real number associated with each angle in this function is called the measure of the angle.
- 2) For every angle  $A$ , the measure of  $A$  is between 0 and 180.
- 3) Let  $\overrightarrow{AB}$  be a ray on the edge of the half-plane  $H$ . For every number  $r$  between 0 and 180, there is exactly one ray  $\overrightarrow{AP}$ , with  $P$  in  $H$ , such that the measure of angle  $\sphericalangle PAB$  is  $r$ .
- 4) If  $D$  is in the interior of angle  $\sphericalangle BAC$ , then the measure of angle  $\sphericalangle BAC$  equals the measure of angle  $\sphericalangle BAD$

plus

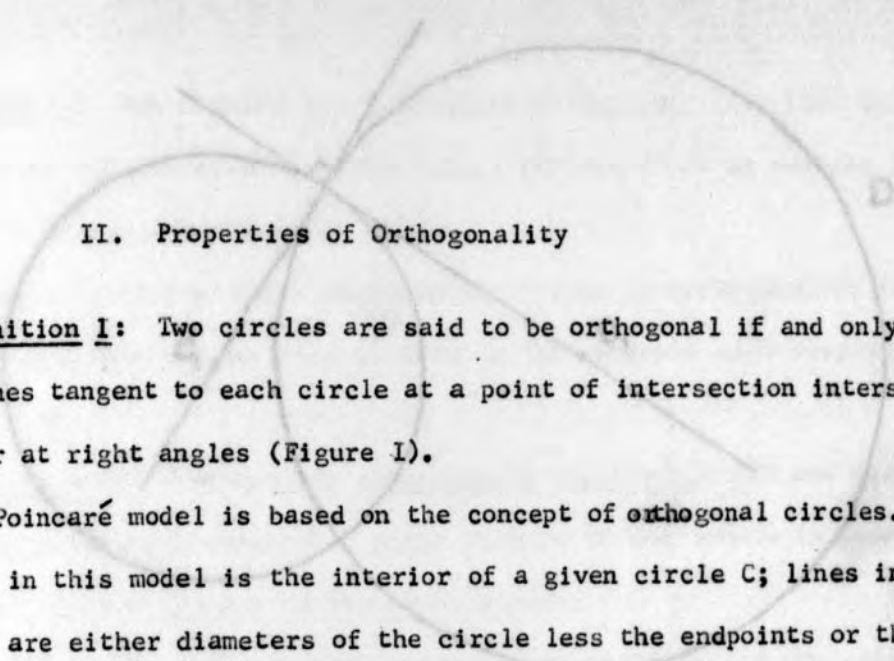
plus the measure of angle  $\angle DAC$ .

5) If two angles form a linear pair, then they are supplementary.

5. Congruence Postulate: Given a correspondence between two triangles or between a triangle and itself. If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

6. Parallel Postulate: Given a line and a point in a plane, there exist at least two lines in the plane containing the point which are parallel to the given line.





## II. Properties of Orthogonality

Definition 1: Two circles are said to be orthogonal if and only if the lines tangent to each circle at a point of intersection intersect each other at right angles (Figure I).

The Poincaré model is based on the concept of orthogonal circles. The plane in this model is the interior of a given circle  $C$ ; lines in the model are either diameters of the circle less the endpoints or the intersection of circles orthogonal to  $C$  with the interior of  $C$ . In Figure II, the interior of  $C$  is the plane of the model, while diameter  $AB - \{A, B\}$  is an example of a line as is arc  $RS$  (of circle  $D$ , orthogonal to  $C$ ).

Definition 2: Consider the circle  $C$  with center  $C$ , and radius  $r$ . Let  $P$  be a point of  $I(C)$ . The point  $P' \in \overrightarrow{CP}$  in the exterior of  $C$  such that  $C, P \cdot C, P' = r^2$  is called the inverse point of  $P$  with respect to  $C$ .

It is clear that if  $P$  lies on  $C$ , then  $P = P'$ . The inverse of a point is unique since the real number associated with  $C, P'$  is unique.

We shall state three theorems concerning orthogonal circles which will be fundamental to theorems concerning orthogonal spheres. Their proofs are given elsewhere [3, p. 98-99].

Theorem 1: The angles between two intersecting circles are the same at both points of intersection.

Theorem 2: Two circles are orthogonal if and only if the tangent to each circle at a point of intersection passes through the center of

FIGURE II - POINCARÉ MODEL

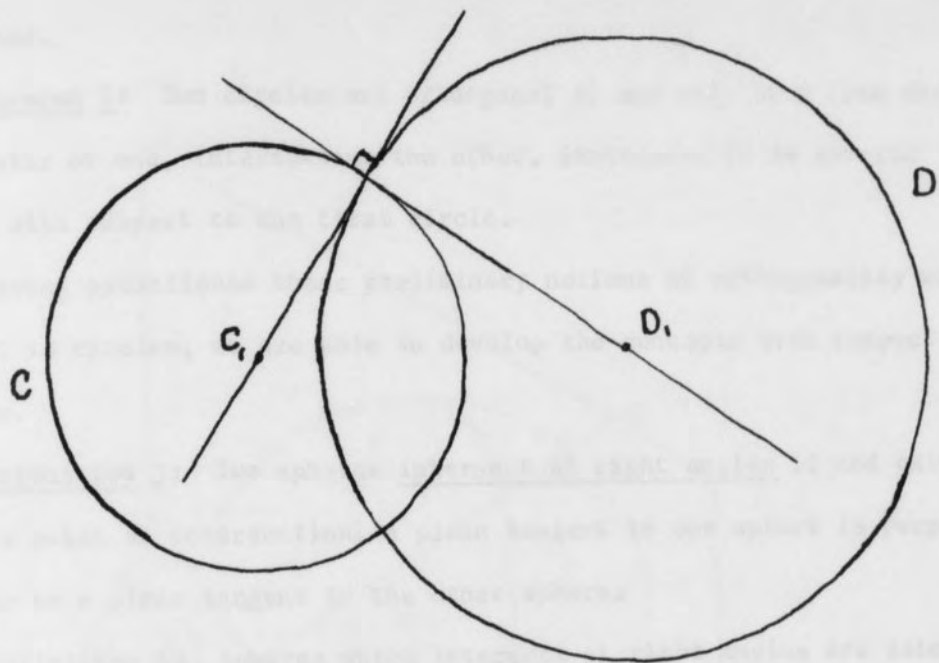


FIGURE I

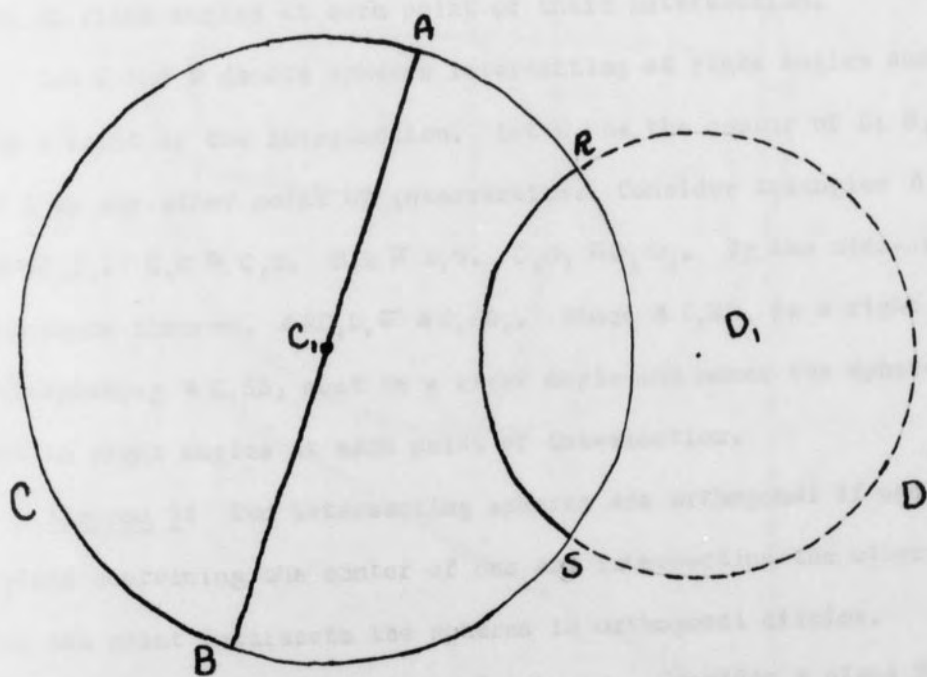


FIGURE II - POINCARÉ MODEL

the other.

Theorem 3: Two circles are orthogonal if and only if a line through the center of one, intersecting the other, intersects it in inverse points with respect to the first circle.

Having established these preliminary notions of orthogonality with respect to circles, we are able to develop the concepts with respect to spheres.

Definition 3: Two spheres intersect at right angles if and only if at a point of intersection, a plane tangent to one sphere is perpendicular to a plane tangent to the other sphere.

Definition 4: Spheres which intersect at right angles are said to be orthogonal. (Figure III)

Theorem 4: Let C and D be intersecting spheres. If C and D intersect at right angles at one point of their intersection, then they intersect at right angles at each point of their intersection.

Let C and D denote spheres intersecting at right angles and let R be a point of the intersection. Let  $C_1$  be the center of C;  $D_1$ , of D. Let S be any other point of intersection. Consider triangles  $\triangle RC_1D_1$  and  $\triangle SC_1D_1$ .  $C_1R \cong C_1S$ .  $D_1R \cong D_1S$ .  $C_1D_1 \cong C_1D_1$ . By the side-side-side congruence theorem,  $\triangle RC_1D_1 \cong \triangle SC_1D_1$ . Since  $\sphericalangle C_1RD_1$  is a right angle, corresponding  $\sphericalangle C_1SD_1$  must be a right angle and hence the spheres intersect in right angles at each point of intersection.

Theorem 5: Two intersecting spheres are orthogonal if and only if a plane containing the center of one and intersecting the other in more than one point intersects the spheres in orthogonal circles.

1) Let C and D be orthogonal spheres. Consider a plane F, containing  $C_1$ , and intersecting D in more than one point.  $C \cap F$  and  $D \cap F$

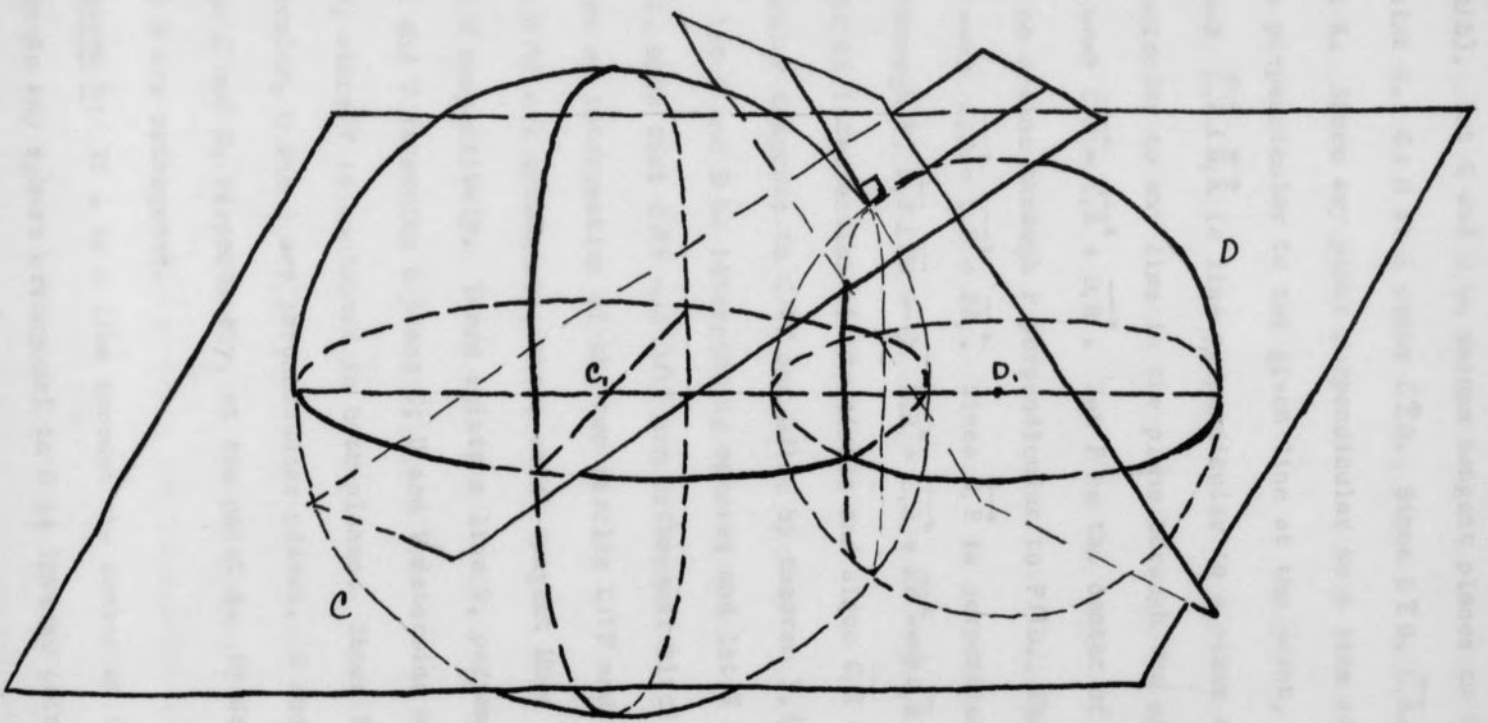


FIGURE III

are both circles since the intersection in each case contains more than one point.

$F \cap (C \cup D)$  is two intersecting circles. Let  $R$  be a point of intersection of  $F \cap (C \cup D)$ . Let  $G$  and  $H$  be unique tangent planes to  $C$  and  $D$  respectively at the point  $R$ .  $G \perp H$  at  $R$  since  $C \perp D$ . Since  $C \perp D$ ,  $\vec{C}_R \perp H$  at  $R$ ;

$\vec{D}_R \perp G$  at  $R$ . Since any plane perpendicular to a line at a point contains all lines perpendicular to the given line at the point,  $\vec{C}_R \subset G$  and

$\vec{D}_R \subset H \implies \vec{C}_R \perp \vec{D}_R$  (a line perpendicular to a plane at a given point

is perpendicular to any line in the plane through the given point).

$\vec{C}_R \perp \vec{D}_R \implies \overline{C_D}^2 = \overline{C_R}^2 + \overline{D_R}^2$ . Let  $P$  be the center of  $F \cap D$ . Then  $D_1$

must lie on a line through  $P$  perpendicular to  $F \cap D$ . Therefore

$\vec{C}_1 \perp \vec{D}_1 \implies \overline{C_D}^2 = \overline{C_P}^2 + \overline{P D_1}^2$ . Since  $\vec{D}_1 \vec{P}$  is perpendicular to any line

in  $F \cap D$  through  $P$ ,  $\vec{D}_1 \vec{P} \perp \vec{P R} \implies \overline{D_1 R}^2 = \overline{C_P}^2 + \overline{P R}^2 \implies \vec{C}_1 \vec{R} \perp \vec{P R}$ . Since  $R$

is a point of intersection of the circle and since  $\vec{C}_1 \vec{R}$  and  $\vec{P R}$  are

perpendicular tangents to  $C \cap F$  and  $D \cap F$ , by theorem 3,  $(C \cap F) \perp (D \cap F)$ .

2) Let  $C$  and  $D$  be intersecting spheres and let  $F$  be a plane containing  $C$ , such that  $C \cap F$  and  $D \cap F$  are orthogonal circles in  $F$ . Let  $R$

be a point of intersection of the two circles  $C \cap F$  and  $D \cap F$ . Since

$C \cap F$  and  $D \cap F$  are orthogonal, there exist tangent lines  $T, U \subset F$ , tangent

to  $C$  and  $D$  respectively. There exists a line  $V$ , perpendicular to  $F$

at  $R$ .  $T$  and  $V$  determine a plane  $G$ ;  $U$  and  $V$  determine a plane  $H$ ;

$H \cap G = V$ , since  $V$  is contained in both planes. Since  $T$  and  $U$  are

perpendicular,  $G$  and  $H$  are perpendicular planes.  $G$  and  $H$  are tangent

planes to  $C$  and  $D$ , respectively, at the point  $R$ . By definitions 4 and

5,  $C$  and  $D$  are orthogonal.

Theorem 6: If  $L$  is a line through the center of  $C$ , of a sphere  $C$ ,

$L$  intersects any sphere orthogonal to  $C$  in inverse points.

Let  $C$  and  $D$  be orthogonal spheres, with line  $L$  containing  $C$ , and

intersecting  $D$ . Let  $G$  be a plane containing  $L$ .  $G \cap (C \cup D)$  is two orthogonal circles (Theorem 5). Since  $L$  passes through  $C_1$ ,  $L$  intersects  $G \cap D$  in inverse points. If  $L$  intersects  $D$  in only one point, that point will be its own inverse. Since  $L$  will not intersect  $D$  except in  $G \cap D$ , the points of intersection of  $L$  with an orthogonal sphere are inverse points.

Theorem 7: Given a point  $J$ , in the exterior of sphere  $C$ . There is exactly one sphere  $J$  having  $J$ , as center and orthogonal to  $C$ .

Consider a tangent plane  $T$  from  $J$ , to  $C$ . Let  $T \cap C = \{R\}$ .  $J_1$ ,  $C_1$ , and  $R$  determine a plane  $G$ .  $T \cap G$  is  $J_1R$ , which is tangent to  $C \cap G$ . Then  $\overline{C_1R} \perp \overline{J_1R}$ . Therefore  $C \cap G$  is a circle with  $J_1$  as center and  $J_1R$  as radius. Using Theorem 5, it may seem that there exists a sphere  $J$  orthogonal to  $C$  with  $J_1$  as center and radius  $J_1R$ .

To prove that this sphere is unique, assume that there are two spheres  $K$  and  $K_2$  with  $J_1$  as center, each orthogonal to  $C$ . Consider a plane  $H$  containing  $G$  and  $J$  and therefore two points of  $K \cap C$  and two points of  $K_2 \cap C$ . Under the above assumption,  $H$  intersects  $C$  and  $K$ , as well as  $C$  and  $K_2$  in orthogonal circles (Figure IV). Let  $R$  and  $R_2$  be the points of  $K \cap C \cap H$ ;  $S$  and  $S_2$  are the points of  $K_2 \cap C \cap H$ . By definition of orthogonal circles, there exists a line through  $J_1$  tangent to  $C \cap H$  at  $R$  and a similar tangent at  $R_2$ . Also, there are tangents from  $J_1$  to  $C \cap H$  at  $S$  and  $S_2$ . However, from a point outside a circle, only two lines may be drawn tangent to the circle, leading to a contradiction of the assumption that there are two spheres orthogonal to  $C$  with  $J_1$  as center.

Theorem 8: Let  $C$  denote a sphere with center  $C_1$ . Let  $F$  denote a plane  $\exists F \cap I(C) \neq \emptyset$  and  $C_1 \notin F$ . Let  $M$  denote a circle in  $F \cap M \cap I(C) \neq \emptyset$ .



There exists a unique sphere  $D \ni MCD$  and  $D \perp C$ .

Let  $P$  denote the center of  $M$  and  $L$  denote the perpendicular to  $F$  at  $P$ . If  $L = \overleftrightarrow{C, P}$ , then consider a plane  $G$  containing  $L$ .  $G \cap M$  consists of two points  $Q$  and  $T$ .  $Q$  has a unique inverse  $Q' \in G$  with respect to  $C$  (Definition 2). The points  $Q, T$ , and  $Q'$  determine a unique circle  $NC \cap G$  orthogonal to  $G \cap C$  (Theorem 3).  $N_1$  lies on  $L$  since  $L$  is perpendicular to  $F$  at  $P$ .  $N_1$  is then the center of a unique sphere orthogonal to  $C$  (Theorem 7).

If  $L \neq \overleftrightarrow{C, P}$ , then  $C_1$  and  $L$  determine plane  $G$ . Let  $A, B \in G \cap M$  and let  $A' \in G$  denote the inverse of  $A$  with respect to  $C$ . Then  $A, B$ , and  $A'$  determine unique circle  $N$  orthogonal to  $C \cap G$ . Since  $N$  is equidistant from  $A$  and  $A'$ ,  $N_1 \in L$ .  $N_1$  determines exactly one sphere orthogonal to  $C$  (Theorem 7). Since  $N_1 \in L$ ,  $N_1$  is a distance  $N_1A$  from all points of  $M$ ; thus,  $M \subset N$ .  $N_1$  is then the center of a sphere orthogonal to  $C$  (Theorem 5) which is unique (Theorem 7).



### 3. INCIDENCE POSTULATES

Incidence Postulate 1: Given two points, there is exactly one line containing them.

### III. VERIFICATION OF MODEL

#### A. DEFINITION OF MODEL

In order to define this model, let  $C$  be a Euclidean sphere. The set of all points in the interior of  $C$  shall constitute Hyperbolic space. This space shall be denoted by  $I(C)$ , and  $E$  shall denote Euclidean space.

H-points in the model shall be the Euclidean points in  $I(C)$ .

H-lines in the model shall be of two types: An H-line of Type I shall be the intersection of a Euclidean line containing  $C$ , and  $I(C)$ .

To define an H-line of Type II, let  $D$  be a sphere orthogonal to  $C$ . An H-line of Type II is the intersection of  $I(C) \cap D$  and an E-plane  $F$  containing  $C$ .  $F \cap D \cap I(C)$  shall be denoted by  $D_f$ .

H-planes of the model shall also be of two types: An H-plane of Type I is the intersection of a Euclidean plane containing  $C$ , with  $I(C)$ . An H-plane of Type I may also be regarded as a Poincaré model. An H-plane of Type II is the intersection of a sphere  $D$ , orthogonal to  $C$ , with  $I(C)$ .  $D \cap I(C)$  shall be denoted by  $\tilde{D}$ .

Incidence Postulate 2: If two points lie in a plane, then the line containing them lies in the plane.

Consider a Type I plane containing  $A$  and  $B$ . If  $A$ ,  $B$ , and  $C$ , are

## B. INCIDENCE POSTULATES

Incidence Postulate 1: Given two points, there is exactly one line containing them.

Let  $A$  and  $B$  belong to  $I(C)$ .  $A$  and  $B$  determine a unique E-line. If  $\overleftrightarrow{AB}$  contains  $C$ , then  $\overleftrightarrow{AB} \cap I(C)$  is an H-line of Type I and from the Euclidean postulates,  $A$  and  $B$  are uniquely contained therein. If  $\overleftrightarrow{AB}$  does not contain  $C$ , then there is a unique E-plane  $F$  containing  $A$ ,  $B$  and  $C$ . This plane intersects  $C$  in a circle. In the Poincaré model, two points in the model are uniquely contained in a line  $D_f$ , an arc of a circle orthogonal to  $C \cap F$ . This line  $D_f$  is a line of Type II, uniquely containing  $A$  and  $B$ .

Incidence Postulate 2: Given three non-collinear points, there is exactly one plane containing them.

Let non-collinear points  $A$ ,  $B$ , and  $J$  belong to  $I(C)$ .  $A$ ,  $B$ , and  $J$  determine a unique E-plane  $F$ . If  $F$  contains  $C$ , then  $F \cap I(C)$  is a plane of Type I, and from the Euclidean postulates,  $A$ ,  $B$ , and  $J$  are uniquely contained therein. If  $F$  does not contain  $C$ , then  $A$ ,  $B$ , and  $J$  determine a unique circle  $MCF$ . This circle may be contained in an infinite number of spheres, all of whose centers lie on an E-line  $L$ , perpendicular to  $F$  and passing through the center of  $M$ . Exactly one of these points of  $L$  is the center of a sphere orthogonal to  $C$  (Theorem 8). The intersection of this unique sphere and  $I(C)$  defines a plane of Type II containing the three points  $A$ ,  $B$ , and  $J$ .

Incidence Postulate 3: If two points lie in a plane, then the line containing them lies in the plane.

Consider a Type I plane containing  $A$  and  $B$ . If  $A$ ,  $B$ , and  $C$ , are

collinear, then  $A, B \in$  an H-line of Type I, which lies in the plane since Type I lines are subsets of Euclidean lines. If  $A, B,$  and  $C,$  are non-collinear, then they determine a unique E-plane  $F.$   $A$  and  $B$  are uniquely contained in a line  $L$  of Type II (Incidence Postulate 1). By definition of H-lines,  $L \subset F.$  There exists a circle  $M$  (containing  $A$  and  $B$ )  $\exists M \subset F$  and  $M \perp C \cap F.$  Since the H-line containing  $A$  and  $B$  is unique,  $L = M \cap I(C)$  and is therefore contained in  $F.$  If  $A$  and  $B$  are contained in a plane of Type II,  $D_f,$  then they determine an H-line  $\overleftrightarrow{AB}.$  By definition of H-lines  $\overleftrightarrow{AB}$  must lie in  $D.$

Incidence Postulate 4: If two planes intersect, their intersection is a line.

If the intersection is of two Type I planes, the intersection is a line, since two Euclidean planes intersect in a line. Since all Type I planes contain  $C_1,$  one point of the intersection must be the center. The line of intersection must therefore be a line of Type I.

If the intersection is of a Type I plane  $F,$  and a Type II plane  $\tilde{D},$  it may be seen that  $F \cap D$  (the Euclidean sphere determining  $\tilde{D}$ ) is a circle orthogonal to  $C \cap F$  (Theorem 5).  $D$  is then an H-line of Type II by definition.

If the intersection is of two Type II planes, we consider the spheres  $B$  and  $D,$  each orthogonal to  $C.$   $B \cap D$  is a circle. Let  $R$  be a point of  $B \cap D.$  Then the E-line  $\overleftrightarrow{C_1 R}$  will intersect  $B$  in inverse points  $R$  and  $R_1';$   $\overleftrightarrow{C_1 R}$  will intersect  $D$  in inverse points  $R$  and  $R_2'. R_1' = R_2'$  since the inverse of a point is unique.  $R',$  then must be a point of  $B \cap D.$  Let  $G$  be the plane containing  $B \cap D.$   $C_1$  lies in  $G$  since  $C_1, R,$  and  $R'$  are collinear. Since  $B \cap D$  is then in  $G,$   $B \cap D \cap I(C)$  is a line of Type II (Theorem 5).

Incidence Postulate 5: Every line contains at least two points.

Every plane contains at least three non-collinear points. Space contains at least four non-coplanar points.

Since this model is a subset of Euclidean space, lines whether of Type I or II must contain at least two points. A plane of Type I, being a subset of a Euclidean plane necessarily contains at least three non-collinear points. A Euclidean sphere, forming a plane of Type II, also contains at least three non-collinear points. The interior of a Euclidean sphere contains at least four non-coplanar points and hence  $I(C)$  contains at least four non-coplanar points.

### C. DISTANCE POSTULATES

Definition 5: Let  $f: L \rightarrow \mathbb{R}$  be a one-to-one correspondence between a line  $L$  and the real numbers. If for all points,  $P, Q \in L$  we have  $PQ = |f(P) - f(Q)|$ , then  $f$  is a coordinate system for  $L$ .

Distance Postulate 1: Distance is defined as a function associating with every pair of points,  $Q, T$  in space, a non-negative real number  $r$ .  $r$  is called the distance between the points.

If  $D$  is a sphere orthogonal to  $C$  and  $G$  is a plane containing  $C$ , and intersecting  $I(D)$ ,  $D \cap C \cap G = \{R, S\}$ . For each point  $Q \in D$ , let us define  $f(Q) = \ln QR/QS$ .  $Q = Q' \implies QR = Q'R$  and  $QS = Q'S \implies QR/QS = Q'R/Q'S \implies \ln QR/QS = \ln Q'R/Q'S$ , which demonstrates that  $f$  is a function. For each real number  $S$ , there exists a positive real number  $t$  such that  $\ln t = S \implies$  a point  $Q \in D, \exists QR/QS = t \implies f$  is an "onto" mapping. If  $\ln QR/QS = \ln Q'R/Q'S$ , then  $Q = Q' \implies f$  is one-to-one.

Let us define a function  $d: I(C) \times I(C) \rightarrow \mathbb{R}^+$ . Let  $Q$  and  $T \in I(C)$ .  $Q$  and  $T$  determine a unique  $H$ -line intersecting  $C$  at  $U$  and  $V$ . We will define  $d(Q, T) = \left| \ln QU/QV - \ln TU/TV \right| = \left| \ln \frac{QU/QV}{TU/TV} \right|$ .

Distance Postulate 2: For every  $Q, T$ ,  $d(Q, T) \geq 0$ .

Since  $d(Q, T)$  is expressed as an absolute value,  $d(Q, T)$  must be positive or zero.

Distance Postulate 3:  $d(Q, T) = 0$ , if and only if  $Q = T$ .

1) Assume  $d(Q,T) = 0$

$$\text{then } \left| \ln \frac{QR/QS}{TR/TS} \right| = 0 \implies \ln \frac{QR/QS}{TR/TS} = \ln 1 \implies$$

$$\frac{QR/QS}{TR/TS} = 1 \implies Q = T$$

2) Assume  $Q = T$

$$\text{then } \left| \ln \frac{QR/QS}{TR/TS} \right| = \left| \ln 1 \right| = 0 \implies d(Q,T) = 0$$

Distance Postulate 4:  $d(Q,T) = d(T,Q)$  for every  $Q$  and  $T$  in  $S$ .

$$d(Q,T) = \left| \ln \frac{QR/QS}{TR/TS} \right| = \left| \ln QR/QS - \ln TR/TS \right| =$$

$$\left| \ln TR/TS - QR/QS \right| = \left| \ln \frac{TR/TS}{QR/QS} \right| = d(T,Q)$$

By the transitive law,  $d(Q,T) = d(T,Q)$

Distance Postulate 5: Every line has a coordinate system

(Ruler Postulate).

From the definition of the distance function, we have

$$d(Q,T) = \left| \ln QR/QS - \ln TR/TS \right| = \left| f(Q) - f(T) \right|. \text{ By definition 5,}$$

$\overleftrightarrow{QT}$  has a coordinate system.

A primary consequence of the distance postulates and especially of the Ruler Postulate is the concept of betweenness.

Definition 6: If points  $A$ ,  $B$ , and  $C$  are collinear and if

$$d(A,B) + d(B,C) = d(A,C), \text{ then } B \text{ is between } A \text{ and } C.$$

#### D. SPACE SEPARATION

Definition 7: A set  $S$  is called convex if for every two points  $P, Q \in S$ , the entire segment  $\overline{PQ}$  lies in  $S$ .

Space Separation Postulate: Given a plane in Space (H-space). The set of all points that do not lie in the plane is the union of two sets such that 1) each of the sets is convex and 2) if  $P$  belongs to one of the sets and  $Q$  to the other, then the segment  $\overline{PQ}$  intersects the plane.

In H-space, both types of planes must be considered. Let  $F$  be an H-plane of Type I.  $F \subset E$ -plane  $G$ . ( $F = G \cap I(C)$ ).

Let  $H_1$  and  $H_2$  denote the E-half-spaces determined by  $G$ . We shall make the following definitions: H-half-space  $h_1 = H_1 \cap I(C)$

$$\text{H-half-space } h_2 = H_2 \cap I(C)$$

By definition of the model as the set of points in the interior of a Euclidean sphere, neither  $h_1$  nor  $h_2$  is empty.  $I(C) = h_1 \cup h_2 \cup F$  which implies  $I(C) - F = h_1 \cup h_2$ .

To prove that each of the sets is convex, let us consider the points  $A, B \in h_1$  (the case for  $h_2$  is identical). Let  $A, B$ , and  $C_1$  be collinear. Since  $A, B \in h_1$ ,  $A, B \in H_1 \implies \overline{AB} \in H_1$ . Since a unique H-line in  $I(C)$  determines  $AB$  and since  $\overline{AB} \subset H_1$ ,  $\overline{AB}$  must belong to  $h_1$ .

If  $A, B, C_1$  are non-collinear, then  $A, B$ , and  $C_1$  determine a unique E-plane  $J$ .  $J \cap I(C)$  is a Poincaré model; therefore  $\overline{AB}$  must belong to  $J \cap h_1$ , which implies  $\overline{AB} \subset h_1$ .

Now let  $A \in h_1$  and  $B \in h_2$ . If  $A, B$ , and  $C_1$  are collinear, then  $\overleftrightarrow{AB}$  is a line of Type I, therefore a subset of an E-line. Since  $I(C)$  is an E-space and  $A \in H_1, B \in H_2$ ,  $\overline{AB}$  must intersect  $G$ . Since  $A$  and  $B$  are clearly in  $H_1$  and  $H_2$ , the point of intersection is  $C_1 \in F$ .

If  $A, B$ , and  $C_1$  are non-collinear, then  $A, B$ , and  $C_1$  determine

a unique E-plane  $J$ .  $J \cap I(C)$  is a Poincaré model, with  $F \cap J$  dividing the model into two half-planes:  $J \cap h_1$  and  $J \cap h_2$ .  $A$  and  $B$  determine a line of Type II in  $J$ . Since  $A \in h_1$ ,  $B \in h_2$ ,  $\overleftrightarrow{AB}$  must intersect  $F \cap J$  from the plane separation properties of the Poincaré model.  $\overline{AB}$  intersects  $F \cap J$  implies  $\overline{AB}$  intersects  $F$ .

Now let  $G$  be an H-plane of Type II, called  $\tilde{D}$ .  $\tilde{D} = D \cap I(C)$  where  $D$  is orthogonal to  $C$ . In H-space we shall take the following definitions:

$$\text{H-half-space } h_1 = I(C) \cap \text{Ex}(D)$$

$$\text{H-half-space } h_2 = I(C) \cap I(D)$$

Therefore  $I(C) - \tilde{D} = h_1 \cup h_2$ . Neither  $h_1$  nor  $h_2$  is empty because  $I(C)$  must contain the Euclidean points of the interior of a sphere and  $I(C) \cap I(D)$  must contain at least one point if  $D$  is orthogonal to  $C$ .

To show that each of the sets is convex, consider points  $A, B \in h_1$ . If  $A, B$ , and  $C$ , are collinear, then the Type I segment  $\overline{AB}$  is contained in  $h_1$  by virtue of its being a Euclidean line. If  $A, B$  and  $C$ , are non-collinear, then  $A, B$ , and  $C$ , determine a unique E-plane,  $F$ .  $F \cap I(C)$  is a Poincaré model.  $A$  and  $B$  determine an H-line of Type II in  $F$ . From the properties of the Poincaré model,  $\overline{AB}$  must lie in  $F \cap h_1 \implies \overline{AB} \subset h_1$ .

Now consider  $A, B \in h_2$ . If  $A, B$ , and  $C$ , are collinear, any plane containing the segment  $\overline{AB}$  of the Type I line intersects  $C$  in a Poincaré model, whose plane separation properties assure that  $\overline{AB} \subset h_2$ . If  $A, B$ , and  $C$ , are non-collinear, then  $A, B$ , and  $C$ , determine a unique plane  $F$ .  $F \cap C$  is a Poincaré model, with  $F \cap D$  separating the planes into two half-planes,  $F \cap h_1$  and  $F \cap h_2$ .  $A$  and  $B$  determine a unique H-line of Type II,  $\overline{AB}$  in  $F$ . From the plane separation properties of the Poincaré model,  $\overline{AB} \subset F \cap h_2 \implies \overline{AB} \subset h_2$ .

Now let  $A \in h_1$  and  $B \in h_2$ . If  $A, B$ , and  $C$ , are collinear, then  $\overleftrightarrow{C, B}$  is a line of Type I. Consider any E-plane  $J$  containing  $\overleftrightarrow{C, B}$ . Since



$B \in I(D)$ ,  $J$  cuts  $C$  and  $D$  in orthogonal circles.  $J \cap C$  is a Poincaré model,  $J \cap D$  separating the model into two half-planes. Since  $A \in J \cap h_1$  and  $B \in J \cap h_2$ , from plane separation properties of the Poincaré model,  $\overline{AB}$  intersects  $J \cap \tilde{D}$ ,  $\implies \overline{AB}$  intersects  $\tilde{D}$ .

If  $A$ ,  $B$ , and  $C$ , are not collinear, they determine the E-plane  $J$ .  $J \cap I(C)$  is a Poincaré model.  $A$  and  $B$  determine a unique line of Type II.  $J \cap \tilde{D}$  is a line in the model separating the Poincaré plane into two half planes,  $D \cap h_1$ , and  $J \cap h_2$ . By plane separation properties of this model,  $\overline{AB}$  must intersect  $J \cap \tilde{D}$ . Since  $\overline{AB}$  is unique in  $I(C)$ , this statement implies that  $\overline{AB}$  intersects  $\tilde{D}$ .

### E. ANGLE MEASURE

**Definition 8:** Let  $L$  be an H-line in an H-plane  $F$ . Let the H-line  $K$  intersecting  $L$  in one point  $A$  divide  $F$  into two half-planes. An H-ray  $\vec{AB}$  is defined as the point  $A$  and all points  $B \in L$  which lie in one half plane. A ray is of Type I if  $L$  is a line of Type I and is a ray of Type II if  $L$  is a line of Type II.

**Definition 9:** An H-angle is the union of two non-collinear H-rays having a common endpoint.

**Definition 10:** Let  $\vec{AB}$  denote a ray of Type II.  $A$ ,  $B$ , and  $C$ , determine an H-plane of Type I,  $F$ .  $FCE$ -plane  $G$ . Let E-line  $\vec{CA}$  divide  $G$  into two E-half-planes. Let  $LCG$  be the unique E-line tangent to arc  $\vec{AB}$  at  $A$ . Let  $\vec{B} \in L$  in the same half-plane as  $B$ . Then E-ray  $\vec{AB}$  is called the tangent ray to H-ray  $\vec{AB}$ . A tangent ray to the H-ray  $\vec{AB}$  is denoted by  $\vec{AB}$ . If  $\vec{AB}$  is a ray of Type I, the tangent ray to  $\vec{AB}$  will be the E-ray  $\vec{AB}$  containing  $\vec{AB}$ .

**Definition 11:** If  $\angle RST$  is an H-angle, then E-angle  $\angle \hat{RST}$  (where  $\vec{SR}$  is the tangent ray to  $\vec{SR}$  and  $\vec{ST}$  is the tangent ray to  $\vec{ST}$ ) shall be called the associated E-angle to  $\angle RST$ .

It may be shown that for each H-angle the associated E-angle is unique. If both rays  $\vec{SR}$  and  $\vec{ST}$  are of Type I, then  $\vec{SR}$  and  $\vec{ST}$  determine a unique E-plane  $F$ .  $\vec{SR}$ ,  $\vec{ST} \in F$  determine the E-angle  $\angle \hat{RST}$ , unique because any other rays containing  $\vec{SR}$  and  $\vec{ST}$  are subsets of  $\vec{SR}$  and  $\vec{ST}$ . If one ray is of Type I (say  $\vec{SR}$ ) and the other,  $\vec{ST}$ , is of Type II,  $\vec{SR}$  and  $\vec{ST}$  determine a unique E-plane containing the E-angle  $\angle \hat{RST}$ , as above, which is uniquely associated with H-angle  $\angle RST$ , since a tangent to a curve at a point in a plane is unique. If both rays  $\vec{SR}$  and  $\vec{ST}$

are of Type II, then each has a unique tangent ray, determining as above, the unique E-angle  $\sphericalangle \hat{RST}$ .

Measure Postulate 1: There exists a function  $m: \mathcal{A} \rightarrow \mathbb{R}^+$ , where  $\mathcal{A}$  is the set of all angles and  $\mathbb{R}^+$  is the set of all positive real numbers. This real number shall be called the measure of the angle.

We shall define  $m(\sphericalangle RST) = M(\sphericalangle \hat{RST})$  where  $M$  is the angle function for E-space. Since  $M$  uniquely associates a positive real number with each E-angle,  $m$  will associate with each H-angle a unique positive real number determined by the function  $M$  of the associated E-angle.

Measure Postulate 2: For every angle  $\sphericalangle RST$ ,  $m(\sphericalangle RST)$  is between 0 and 180.

By the unique association of every H-angle with an E-angle, ( $m(\sphericalangle RST) = M(\sphericalangle \hat{RST})$ ), and since  $0 < M(\sphericalangle \hat{RST}) < 180$ , then  $0 < m(\sphericalangle RST) < 180$ .

Measure Postulate 3: Let  $\vec{SR}$  be a ray on the edge of the H-half-plane  $H$ . For every number  $r$  between 0 and 180, there is exactly one ray  $\vec{ST}$  with  $T$  in  $H$  such that  $m(\sphericalangle RST) = r$ .

If  $\vec{SR}$  is a ray of Type I with  $S = C$ , then let  $F$  be the H-plane containing this H-half-plane. Then there exists an E-plane  $G \ni F = G \cap I(C)$ . Since  $\vec{SR} \subset F$ ,  $\vec{SR} \subset G$  and there exists a unique ray  $\vec{ST} \subset G \ni M(\sphericalangle \hat{RST}) = r$ , which implies  $\exists$  unique ray  $\vec{ST} \subset F \ni m(\sphericalangle RST) = M(\sphericalangle \hat{RST}) = r$ .

If  $\vec{SR}$  is a ray of Type I with  $S \neq C$ , then let  $F$  be the H-plane containing this H-half-plane. Then there exists an E-plane  $G \ni F = G \cap I(C)$ . Since  $\vec{SR} \subset F$ ,  $\vec{SR} \subset G$  and there exists a unique ray  $\vec{ST} \subset G \ni M(\sphericalangle \hat{RST}) = r$ , which implies  $\exists$  unique ray  $\vec{ST} \subset F \ni m(\sphericalangle RST) = M(\sphericalangle \hat{RST}) = r$ .

If  $\vec{SR}$  is a ray of Type I with  $S \neq C$ , then  $\vec{SR}$  lies on the edge of a Type I half-plane,  $H$ .  $H \subset F \subset G$  as above and there exists  $\vec{ST} \subset G \ni M(\sphericalangle \hat{RST}) = r$ . Consider the line  $LCG$  and perpendicular to  $\vec{ST}$  at  $S$ . Consider also the

E-ray  $\overrightarrow{C_1 S}$ . Let  $S'$  be the inverse point of  $S$  with respect to  $C$ . The perpendicular bisector of  $\overline{SS'}$  intersects  $L$  in a point  $D_1$ , which must be the center of a unique circle  $D$  tangent to  $\overleftrightarrow{TS}$  and orthogonal to  $C \cap G$ . Let  $T$  be any point of  $D \cap G \cap I(C)$  and note that the ray  $\overrightarrow{ST}$  is uniquely determined by this construction. Since by definition,  $\sphericalangle \hat{RST}$  is uniquely associated with  $\sphericalangle RST$ ,  $M(\sphericalangle \hat{RST}) = r \implies m(\sphericalangle RST) = r$ .

If  $\overrightarrow{SR}$  is a ray of Type II,  $\overrightarrow{SR}$  may lie on the edge of a Type I H-half-plane  $H$ .  $H \subset FCG$ , as above. Consider the tangent ray  $\overrightarrow{SR} \subset G$ . There exists a unique ray  $\overrightarrow{ST} \cap T \in G$ , where  $M(\sphericalangle \hat{RST}) = r$ . As above,  $\overrightarrow{ST}$  is a unique tangent ray to an H-ray of Type II,  $\overrightarrow{ST} \cap m(\sphericalangle RST) = r$ .

$\overrightarrow{SR}$  may also lie on the edge of a Type II H-half-plane  $h$ .  $h$  is contained in the Type II H-plane  $\tilde{D}$ . Consider E-plane  $\overleftrightarrow{C_1 RS}$  which divides  $\tilde{D}$  into two half-planes  $h_1$  and  $h_2$ . By definition of tangent ray,  $\overrightarrow{SR} \subset \overleftrightarrow{C_1 RS}$ . Now consider the E-plane  $G \perp \overleftrightarrow{C_1 RS}$  and containing  $\overleftrightarrow{SR}$ . On the  $h_1$  side of  $\overleftrightarrow{C_1 RS}$  there exists a unique ray  $\overrightarrow{ST} \subset G \cap m(\sphericalangle RST) = r$ . Now consider the plane  $\overleftrightarrow{C_1 ST}$ .  $\overleftrightarrow{C_1 ST}$  must intersect  $\tilde{D}$  in the ray  $\overrightarrow{ST}$  by definition of the tangent. Then since  $\overrightarrow{ST}$  must be the tangent ray to  $\overrightarrow{ST}$ ,  $M(\sphericalangle \hat{RST}) = r \implies m(\sphericalangle RST) = r$ .

Notation: If  $h$  is a half-plane;  $L$ , the edge of  $h$ ; and  $m$  a point of  $h$ ; " $h_m^+$ " denotes the half-plane determined by  $L$ , containing  $m$ . " $h_m^-$ " denotes the half plane determined by  $L$  not containing  $m$ .

Measure Postulate 4: If  $U$  is in the interior of  $\sphericalangle RST$ , then  $m(\sphericalangle RST) = m(\sphericalangle RSU) + m(\sphericalangle UST)$ .

We must show that if  $U$  lies in the interior of  $\sphericalangle RST$ , then  $\overrightarrow{SU}$  lies in the interior of the associated angle  $\sphericalangle \hat{RST}$ .

If  $S = C$ , then  $\overrightarrow{SR}$  and  $\overrightarrow{ST}$  are rays of Type I and  $\overrightarrow{SU}$  must also be a ray of Type I. Associated with each of these rays is an E-ray

$\vec{SR}$ ,  $\vec{ST}$ , and  $\vec{SU}$ , respectively. Since the H-rays are Euclidean lines, with  $U \in I(\angle RST)$ ,  $U \in I(\angle \hat{RST})$ ,  $\vec{SU}$  lies in  $I(\angle RST)$ . Then  $M(\angle \hat{RST}) = M(\angle \hat{RSU}) + M(\angle \hat{UST})$  which implies  $m(\angle RST) = m(\angle RSU) = m(\angle UST)$ .

If  $S \notin C_1$ , and  $\vec{SR}$  is a ray of Type I ( $\vec{SR} \subset E\text{-ray } \vec{SS}'$ ), then  $\vec{ST}$  is a ray of Type II.  $\vec{ST}$  is contained in a sphere  $D$  orthogonal to  $C$  with center  $D_1$ . (Figure V.)

$\vec{SR}$  and  $\vec{ST}$  are contained in an H-plane of Type I, F, which is contained in E-plane  $G$ .  $G \cap (C \cup D)$  is two intersecting circles,  $G \cap C$  and  $G \cap D$ . If  $U \in I(\angle RST)$  then  $(\vec{SU} - \{S\}) \subset I(\angle RST)$ .  $SU \subset B_2$ . Let  $L$  be the tangent line to  $G \cap D$  containing  $\vec{ST}$ ; likewise let  $M$  be the tangent to  $B_2$  containing  $\vec{SU}$ .  $(D - \{S\}) \subset h_S^L$ ;  $(B - \{S\}) \subset h_S^M$ .

Every ray between  $\vec{SR}$  and  $\vec{ST}$  must intersect  $G \cap D$  in a point other than  $S$ . Since  $T \in h_S^{RS}$ , every ray contained in  $I(\angle \hat{TSR})$  with endpoint  $S$  intersects  $G \cap D$ . Also every ray contained in  $I(\angle \hat{USR})$  with endpoint  $S$  intersects  $G \cap B$  as well as  $G \cap D$ . Assume that  $\vec{SU} \subset h_S^{ST}$ . Then there are some rays of  $\angle \hat{RST}$  which do not intersect  $D$  (namely those in  $I(\angle \hat{TSU})$ ) which is a contradiction of the assumption that  $\vec{SU}$  lies in  $h_S^{ST}$ . Therefore we have  $SU \subset I(\angle \hat{RST}) \implies M(\angle \hat{RST}) = M(\angle \hat{RSU}) + M(\angle \hat{UST}) \implies m(\angle RST) = m(\angle RSU) + m(\angle UST)$ .

The proofs for other angles formed by one ray of Type I and the other of Type II proceed in a similar manner.

If  $S \notin C_1$ , and  $\vec{SR}$  and  $\vec{ST}$  are rays of Type II lying in the same Type I plane  $G$ , then consider the angle  $\angle RST$ . Let  $U \in I(\angle RST)$ . Let  $\vec{SR}$ ,  $\vec{ST}$ , and  $\vec{SU}$  be tangent rays to  $\vec{SR}$ ,  $\vec{ST}$ , and  $\vec{SU}$ , respectively. Let  $\vec{SR}$  be an arc of circle  $G \cap A$ ;  $\vec{ST}$ , of circle  $G \cap D$ , and  $\vec{SU}$ , of circle  $G \cap B$ . Then all points of  $h_S^{ST}$  intersect  $G \cap D$ ; all in  $h_S^{SU}$  intersect  $B$  and all in

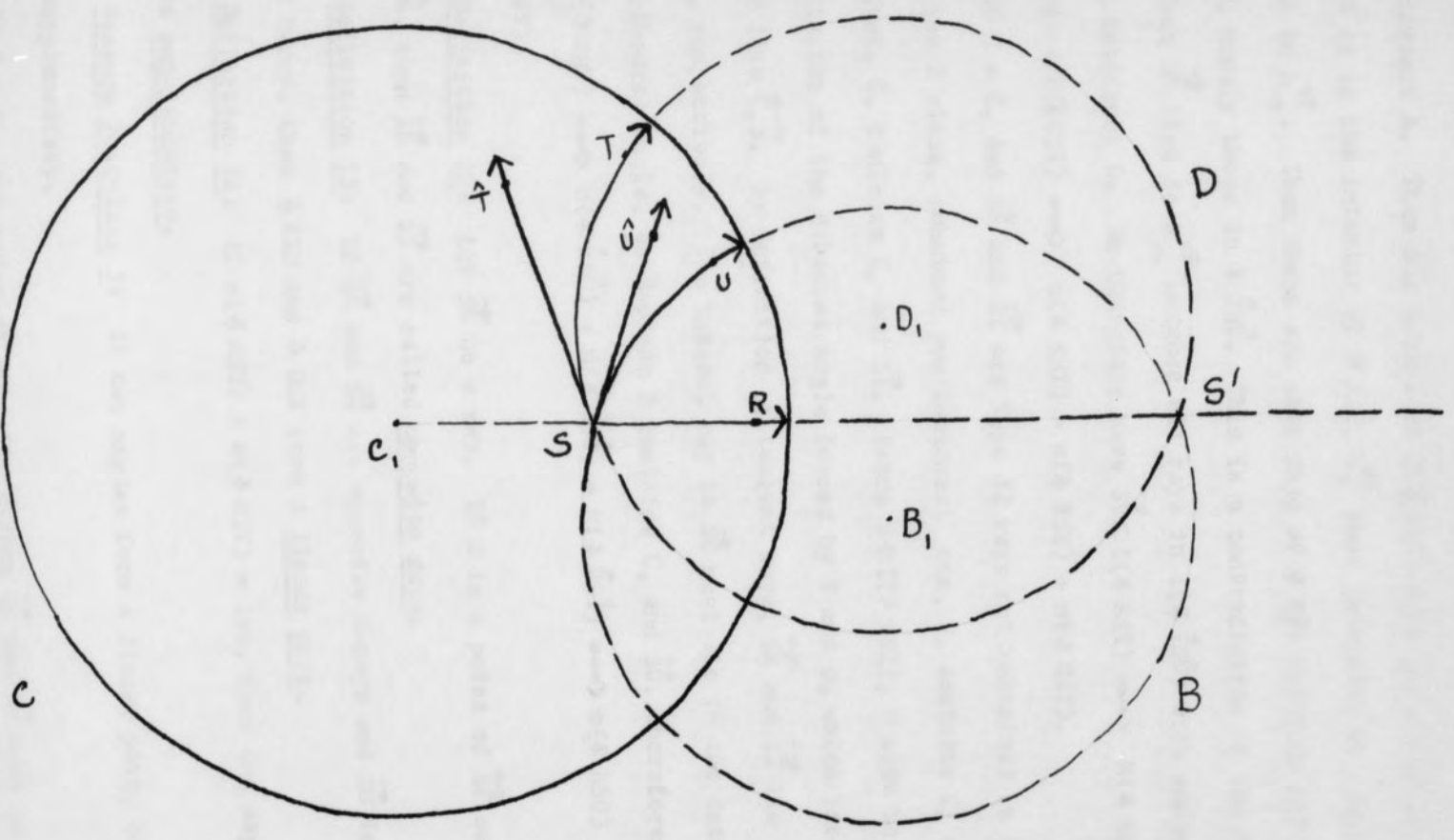


FIGURE V

$h_B^{SS'}$  intersect A. Then all E-rays of  $I(\angle RST)$  must intersect circle D. Since U is in the interior of  $\angle RST$ ,  $h_S^{SU}$  must intersect B. Assume that  $\vec{SU}$  lies in  $h_{-S}^{S\hat{T}}$ . Then there are some rays of  $\angle RST$  which do not intersect D, namely those in  $\angle \hat{T}\hat{S}\hat{U}$ . This is a contradiction of the assumption that  $\vec{SU}$  lies in  $h_{-S}^{S\hat{T}}$  because all rays in  $I(\angle \hat{T}\hat{S}\hat{U})$  with endpoint S must intersect D. We therefore have  $\vec{SU} \subset I(\angle RST) \implies m(\angle RST) = m(\angle RSU) + m(\angle UST) \implies m(\angle RST) = m(\angle RSU) + m(\angle UST)$ .

If  $S \neq C$ , and  $\vec{SR}$  and  $\vec{ST}$  are Type II rays not contained in the same Type I plane, consider two E-planes: one, F, contains C, and  $\vec{SR}$ ; the other, G, contains C, and  $\vec{ST}$ . Since  $U \in I(\angle RST)$ , U also belongs to the interior of the dihedral angle formed by F and G, which intersect in the line  $\overleftrightarrow{C, S}$ . By definition of tangent rays,  $\vec{SR}$  and  $\vec{ST}$  lie in F and G, respectively. The tangent ray to  $\vec{SU}$  must lie in the interior of this dihedral angle, as E-plane H contains C, and  $\vec{SU}$ , therefore also  $\vec{SU}$ .  $\vec{SU} \subset I(\angle RST) \implies m(\angle RSU) + m(\angle UST) = m(\angle RST) \implies m(\angle RSU) + m(\angle UST) = m(\angle RST)$ .

Definition 12: Let  $\vec{SR}$  be a ray. If T is a point of  $\overleftrightarrow{SR}$  such that  $\vec{ST} \not\subset \vec{SR}$ , then  $\vec{SR}$  and  $\vec{ST}$  are called opposite rays.

Definition 13: If  $\vec{SR}$  and  $\vec{ST}$  are opposite H-rays and  $\vec{SU}$  is any third H-ray, then  $\angle RSU$  and  $\angle UST$  form a linear pair.

Definition 14: If  $m(\angle RSU) + m(\angle UST) = 180$ , then the angles are called supplementary.

Measure Postulate 5: If two angles form a linear pair, then they are supplementary.

If  $S = C$ , the union of opposite H-rays  $\vec{SR}$  and  $\vec{ST}$  must be a line of Type I. These rays are contained in opposite E-rays  $\vec{SR}$  and  $\vec{ST}$  through  $C$ . If U is any other point of the H-space, ray  $\vec{C, U}$  is also

contained in an E-ray  $\overrightarrow{C, U}$ . Then,  $M(\hat{\angle} RS\hat{U}) + M(\hat{\angle} U\hat{S}\hat{T}) = 180$ , since  $\overrightarrow{SR}$  and  $\overrightarrow{ST}$  are opposite E-rays, which implies  $m(\angle RSU) + m(\angle UST) = 180$ .

If  $S = C$ , then opposite H-rays  $\overrightarrow{SR}$  and  $\overrightarrow{ST}$  may be contained in either a line of Type I or a line of Type II. Suppose both rays belong to a Type I line. Then, associated with each ray is a unique ray,  $\overrightarrow{SR}$  and  $\overrightarrow{ST}$  respectively. These rays lie on the E-line  $\overleftrightarrow{RT}$ . By hypothesis, we have a third ray of Type II,  $\overrightarrow{SU}$ , with which  $\overrightarrow{SU}$  is associated. Since  $M(\hat{\angle} RS\hat{U}) + M(\hat{\angle} U\hat{S}\hat{T}) = 180$ , by definition  $m(\angle RSU) + m(\angle UST) = 180$ .

Suppose both rays  $\overrightarrow{SR}$  and  $\overrightarrow{ST}$  are contained in a line of Type II. Associated with each ray is a unique tangent ray,  $\overrightarrow{SR}$  and  $\overrightarrow{ST}$ . The union of these tangent rays must be an E-line because a circle has a unique tangent at a given point. We are also given a third ray  $\overrightarrow{SU}$  which forms a linear pair with  $\overrightarrow{SR}$  and  $\overrightarrow{ST}$ . Associated with  $\overrightarrow{SU}$  is  $\overrightarrow{SU}$ . Then because  $\hat{\angle} RT$  is an E-line,  $M(\hat{\angle} RS\hat{U}) + M(\hat{\angle} U\hat{S}\hat{T}) = 180$ , which implies  $m(\angle RSU) + m(\angle UST) = 180$ .



## F. CONGRUENCE POSTULATE

Definition 15: If A, B, and C are three non-collinear points, then the set  $\overline{AB} \cup \overline{BC} \cup \overline{AC}$  is called a triangle.

Definition 16: Let  $\overline{AB}$  and  $\overline{CD}$  be segments. If  $AB = CD$ , then the segments are called congruent and we write  $\overline{AB} \cong \overline{CD}$ .

Definition 17: If  $m(\sphericalangle ABC) = m(\sphericalangle DEF)$ , then the angles are called congruent and we write  $\sphericalangle ABC \cong \sphericalangle DEF$ .

Definition 18: Given  $\triangle ABC$ ,  $\triangle DEF$  and a one-to-one correspondence  $ABC \leftrightarrow DEF$  between their vertices. If every pair of corresponding sides is congruent and every pair of corresponding angles is congruent then the correspondence is a congruence.

Side-Angle-Side Congruence Postulate: Given a correspondence between two triangles. If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is a congruence.

Given two triangles  $\triangle ABC$  and  $\triangle A'B'C'$  with the correspondence  $ABC \leftrightarrow A'B'C'$ . In addition  $\overline{AB} \cong \overline{A'B'}$ ,  $\overline{BC} \cong \overline{B'C'}$ ,  $\sphericalangle B \cong \sphericalangle B'$ .

Each triangle  $\triangle ABC$  and  $\triangle A'B'C'$  determines a unique plane  $G$  and  $G'$ , respectively. When  $G$  is a plane of Type I and  $G = G'$ , then the two triangles lie in an example of a Poincaré model in which the SAS postulate holds. [2, p. 364]; therefore  $\triangle ABC \cong \triangle A'B'C'$ .

If  $\triangle ABC \subset G$  and  $\triangle A'B'C' \subset G'$ , both  $G$  and  $G'$  being planes of Type I, then  $G$  and  $G'$  must intersect in a Type I line  $L$  through  $C$ . Suppose the dihedral angle between  $G$  and  $G'$  is  $\Theta$ . Since  $G$  and  $G'$  are subsets of  $E$ -planes, the rigid motion of  $G'$  about  $L$  of  $\Theta$  degrees, will map

$\Delta A'B'C'$  to  $\Delta A''B''C'' \in G$ , preserving distance and angle measure.

Clearly  $\Delta A'B'C' \cong \Delta A''B''C''$ . As above,  $\Delta ABC \cong \Delta A''B''C''$ . By the transitive property of the equivalence relation,  $\Delta ABC \cong \Delta A'B'C'$ .

When one triangle with the above correspondence is in a plane of Type I and the other is in a plane of Type II, we shall define a mapping from the plane of Type II onto a plane of Type I so that one of the above situations will exist. Consider the intersection of orthogonal spheres  $C$  and  $D$  with a plane  $G$ , containing centers  $C$ , and  $D$ , respectively (Figure VI). Let  $\overrightarrow{C_1D_1} \cap [D \cap G] = \{A\}$ . Let  $L$  be a line perpendicular to  $\overrightarrow{C_1D_1}$  at  $A$ .  $L \cap C = \{R, S\}$ . Let  $K$  be a line tangent to  $C \cap G$  at  $R$ . Let  $K \cap \overrightarrow{C_1D_1} = \{B_1\}$ . By this construction,  $B_1$  is the inverse of  $A$  with respect to  $C$  [3, p. 87]. By Theorem 3,  $B_1$  must be contained in  $D$ .  $B_1$  is also the center of a sphere  $B$  orthogonal to  $C$  with radius  $B_1R$ .

The mapping shall be defined as follows: Let  $F$  be the Type II plane  $\tilde{D}$  described above and let  $F' = G \cap I(C)$  where  $G$  is an E-plane perpendicular to  $\overrightarrow{C_1D_1}$  at  $C_1$ . Let  $f: (E - \{B_1\}) \rightarrow (E - \{B_1\})$  where  $f(Q) = Q'$  and  $Q'$  is the inverse of  $Q$  with respect to  $B$ .  $f$  is therefore one-to-one and onto.

We wish to show that  $f(F) = F'$ : Let  $\{T\} = \overrightarrow{B_1Q} \cap F'$  where  $Q \in F$ ,  $Q \neq A$ . Then the correspondence  $\Delta B_1QA \rightarrow \Delta B_1C_1Q'$  exists such that the three corresponding angles are equal, since the hypotenuse of  $\Delta B_1QA$  is a diameter of  $D$ . Therefore,  $\Delta B_1QA \sim \Delta B_1C_1Q'$ . Then  $B_1Q/B_1C_1 = B_1A/B_1T$  which implies  $B_1Q \cdot B_1T = B_1A \cdot B_1C_1$ . From the construction of  $B_1$ ,  $C_1$  and  $A$  are inverse points with respect to  $B$ . Therefore  $B_1A \cdot B_1C_1 = (B_1R)^2$ . Since  $Q$  and  $Q'$  are inverse points with respect to  $B$ ,  $B_1Q \cdot B_1Q' = (B_1R)^2$ . We have then  $B_1Q \cdot B_1Q' = B_1Q \cdot B_1T$ , which implies  $Q' = T$ . Since  $Q'$  lies on  $F'$ , we know that  $f(F) \subset F'$ .



To show that  $F' \subset f(F)$ , choose point  $Q' \in F'$ . Let the point  $U$  be the intersection of  $\overrightarrow{B_1 Q'}$  and  $B$ . Then we have a correspondence between  $\Delta B_1 U A$  and  $\Delta B_1 C_1 Q'$  such that corresponding angles are congruent which implies  $\Delta B_1 U A \sim \Delta B_1 C_1 Q'$ . Then  $B_1 U / B_1 C_1 = B_1 A / B_1 Q'$  which implies  $B_1 U \cdot B_1 Q' = B_1 A \cdot B_1 C_1$ .  $A$  and  $C_1$  are inverse with respect to  $B$  as above, so  $B_1 A \cdot B_1 C_1 = (B_1 R)^2$ . Again,  $B_1 Q \cdot B_1 Q' = (B_1 R)^2$ . From these two conclusions,  $B_1 Q \cdot B_1 Q' = B_1 U \cdot B_1 Q'$  which implies  $Q = U$ . Since  $Q$  lies on  $F$ , we know that  $F' \subset f(F)$ . Combining this result with the one above, it may be seen that  $f(F) = F'$ .

This mapping  $f$  preserves distance as follows: We have from Coxeter [1, p. 92], that the cross-ratio of any four points is preserved by any inversion. In this inversion with respect to  $B$ , if we take points,  $S, T, U, V \in D$  where  $\{S, V\} = D \cap C \cap G$  ( $G$  an  $E$ -plane containing  $C_1$ ) and  $T, U \in D_g$ , then  $\frac{TS}{US} / \frac{TV}{UV} = \frac{T'S'}{U'S'} / \frac{T'V'}{U'V'}$ . By the definition of distance function,  $d(T, U) = \left| \ln \frac{TS/TV}{US/UV} \right|$ . Since the ratio is preserved,  $d(T, U) = \left| \ln \frac{TS/TV}{US/UV} \right| = \left| \ln \frac{T'S'/T'V'}{U'S'/U'V'} \right| = d(T', U') \implies$  by transitive properties, distance is preserved under  $f$ .

Angle measure is also preserved under  $f$ . Consider angle  $\sphericalangle NMO$  in  $F$ , both rays of which are necessarily of Type II. (Figure VII) Consider plane  $L$  determined by  $\overrightarrow{MN}$  and  $C_1$ , and plane  $K$  determined by  $\overrightarrow{MO}$  and  $C_1$ .  $\overrightarrow{MN} \subset L$  is the unique tangent ray to  $\overrightarrow{MN}$ , as  $\overrightarrow{MO} \subset K$  is the unique tangent ray to  $\overrightarrow{MO}$ .  $m(\sphericalangle NMO) = M(\sphericalangle \widehat{NM\hat{O}})$ . Now consider the plane  $H$ , determined by  $B_1$  and  $\overrightarrow{MN}$ ; and  $J$ , determined by  $B_1$  and  $\overrightarrow{MO}$ .  $H$  intersects  $F'$  in a line  $\overleftrightarrow{M'N'}$  as  $J$  intersects  $F'$  in  $\overleftrightarrow{M'O'}$  (here  $N'$  and  $O'$  are not inverse points of any point of  $\overrightarrow{MN}$  or  $\overrightarrow{MO}$ , respectively).



[2, p. 151]. Since  $H \cap J$  contains  $B_1$ , by the above mapping  $H \cap J \cap F'$  is the point  $M'$ . By consideration of perpendiculars to  $H \cap J$  at  $M$  and  $M'$ , we are able to determine through a sequence of congruent triangles that angle  $\sphericalangle NMO \cong \sphericalangle N'M'O'$ . Thus angle measure is preserved under  $f$ .

With distance and angle measure preserved,  $\triangle A'B'C' \cong \triangle A''B''C''$  where  $\triangle A'B'C' \subset F$  and  $\triangle A''B''C'' \subset F'$ . This plane of Type I,  $F'$ , may be rotated about an axis formed by  $F' \cap G$  (the plane of  $\triangle ABC$ ) so as to map  $F'$  into  $G$  as above.

When both triangles  $\triangle ABC$  and  $\triangle A'B'C'$  are contained in planes of Type II, each may be mapped to a plane of Type I in the above manner. Then, if necessary, these Type I planes may be rotated about their line of intersection in order to bring them into coincidence as above.

## G. PARALLELISM

**Definition 19:** Two co-planar lines are said to be parallel if they do not intersect.

**Hyperbolic Parallel Postulate:** Given a line and a point in a plane, there exist at least two lines in the plane containing the point which are parallel to the given line.

Given a plane  $F$  of Type I, let us consider a line of Type I,  $L$ , in  $F$  and any point  $A \in F$  ( $A \notin L$ ). All Type II lines through  $A$  will have centers on the perpendicular bisector  $B$  of  $\overline{AA'}$ . Clearly the circle with center at the midpoint  $M$  of  $\overline{AA'}$  will not intersect  $L$ , for  $d(M, A) < d(M, C_1)$ . By choosing a point  $P \in B$  outside the circle  $\odot d(P, A) < d(P, C_1)$ , we can get an infinite number of circles with center  $P$  forming Type II lines through  $A$  parallel to  $L$ .

If  $L$  is a line of Type II, we can choose a point  $A \in F$ ,  $A \notin L$ . Then there are an infinite number of Type I lines not containing  $L$ . A Type I line  $D$  will be parallel to  $L$  if all points of  $D$  are a greater distance from the center  $L$ , of circle  $L$  than is any point of  $L$ . There are also an infinite number of Type II lines parallel to  $L$ . A Type II line  $K$  will be parallel to  $L$  if all points of  $K$  are a greater distance from  $L$ , than is any point of  $L$ .

If  $L$  lies in a plane  $G$  of Type II, there exist an infinite number of lines  $KCG \ni K$  is parallel to  $L$ . Since  $L$  is a line of Type II,  $L = G \cap F$  (an E-plane containing  $C_1$ ). Any line  $KCG$  will be parallel to  $L$  if  $KCE$ -plane  $H$  ( $C_1 \in H$ )  $\ni H \cap F \cap G = \emptyset$ .

Having established that the postulates of hyperbolic geometry are satisfied by this model, we have shown that this orthogonal-spheres model is a valid representation of hyperbolic space.

#### THEOREMS

##### THEOREM 1: THE INTERSECTION OF A PLANE AND A SPHERE

1. If a plane and a sphere intersect in more than one point, their intersection is a circle.
2. If two spheres intersect in more than one point then they intersect in a circle.
3. Three non-collinear points determine a unique circle.
4. Four non-coplanar points determine a unique sphere.
5. If two lines intersect a sphere in exactly one point, the plane determined by the lines is tangent to the sphere at that point.
6. A plane perpendicular to a given line at a given point of the line contains all lines perpendicular to the given line at the given point.
7. From a point in a plane in the exterior of a circle in the same plane, exactly two tangents may be drawn to the circle.

##### POSTULATES OF EUCLIDEAN GEOMETRY

The postulates for Euclidean geometry with the exception of the parallel postulate are the same as those listed in the introduction for hyperbolic geometry. The Euclidean parallel postulate is as follows: Given a line and a point not on the line, there is exactly one line which passes through the given point and is parallel to the line.



## V. APPENDIX

### Euclidean Theorems used in this Paper

1. If a plane and a sphere intersect in more than one point, then the intersection is a circle.
2. If two spheres intersect in more than one point then they intersect in a circle.
3. Three non-collinear points determine a unique circle.
4. Four non-coplanar points determine a unique sphere.
5. If two lines intersect a sphere in exactly one point, the plane determined by the lines is tangent to the sphere at that point.
6. A plane perpendicular to a given line at a given point of the line contains all lines perpendicular to the given line at the given point.
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