Symmetry in Atonal Music

By

Sergei L. Miles

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Approved by:

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Vicky Klima, Ph.D., Thesis Director

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William Cook, Ph.D., Departmental Honors Director

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Eric Marland, Ph.D., Chair, Mathematical Sciences

Abstract

Atonal music includes each of the twelve pitch-classes repeated equally within a composition. The composer then gives no preference to a particular subset of the twelve pitch-classes and avoids key-structure in the music, which is a significant part of the structure in traditional tonal music. A particular piece of atonal music is often written to favor a group of symmetric permutations of a given 12-tone row reached via the operations transposition, inversion, and retrograde. Here, we investigate symmetry in twelve-tone rows and apply these ideas to *n*-tone rows for microtonal systems. In terms of algebra, the goal is to count the unique groupings of permutations, or orbits, which can be reached via combinations of the three possible operations that preserve symmetry.

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Introduction: Tonal vs. Atonal Music

Traditional Western-European style music contains twelve possible notes. These twelve notes, or pitch-classes, are {C, $C^{\#}/D^{b}$, D, $D^{\#}/E^{b}$, E, F, $F^{\#}/G^{b}$, G, $G^{\#}/A^{b}$, A, $A^{\#}/B^{b}$, B}. In moving from one note to the next, we take a step of one semitone. Additionally, an entire jump of twelve semitones at a time increases a tone by one octave. Stepping by an octave occurs on a piano when jumping between successive C keys. As a note raises one octave the pitch of the note doubles in frequency, the speed of the sound vibration. Notes a whole number of octaves apart sound similar to the ear, and so we give them the same name forming a pitch-class. Finally, traditional tonal music is equal tempered meaning each step of a semitone increases the frequency by equal amounts. In the case of a twelve note system each step increases the pitch by a factor of $2^{1/12}$ which is further discussed in AsKew, Kennedy, and Klima's work [1]. Labeling the pitch-class C as 0, $C^{\#}/D^{b}$ as 1, D as 2 and so on, the twelve pitch-classes may be represented algebraically by the group $\mathbb{Z}_{12}$, the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} together with the operation of addition modulo 12. However, in Western tonal music a given composition would typically emphasize repetitions of a particular subset of seven notes as dictated by the key of the music. A piece could be written according to the key of C-major which would feature only the white keys on a piano. In this instance, as shown in figure 1 to the left, the subset of notes {0, 2, 4, 5, 7, 9, 11} would be favored and arranged in the music along with the occasional sharp or flat.

[0 2 4 5 7 9 11 0]

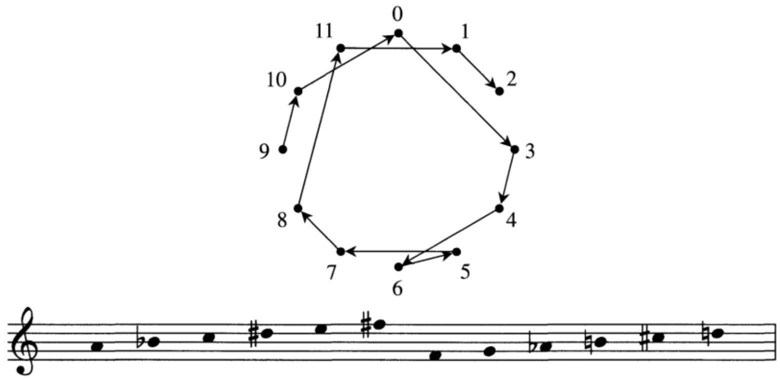
Figure 1: C-Major Scale

Atonal music by contrast does not showcase a favored key structure. Instead atonal music strive to maintain an even distribution for the occurrence count of each pitch class within a composition. As an example Hunter and Hippel [5] consider Schoenberg's Serenade opus 24 movement 5 which was composed to feature symmetries of the prime row, shown in figure 3. The prime row represents a specific permutation or ordering of the twelve pitch-classes. We may also think of this permutation as a bijection, *p*: Positions → Pitch Classes. Here we number the positions zero through eleven. For example, in figure 3 we see A = 9 occurs in position 0 so then $p(0) = 9$. Similarly, since C = 0 occurs in position 2, $p(2) = 0$, etc. Therefore the prime row permutation, *p*, determines in what order the twelve notes are to be played. In this manner, the particular permutation *p* may be represented in the 2-row array notation format shown in figure 2.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 9 & 10 & 0 & 3 & 4 & 6 & 5 & 7 & 8 & 11 & 1 & 2 \end{bmatrix}$$

Figure 2: 2-Row Array Notation for the prime row *p*

Here the first row represents the positions, 0 through 11, and the second row showcases the outputted order of the notes. Alternatively, we can represent the prime row *p* using cycle notation, $p = (0\ 9\ 11\ 2)(1\ 10)(5\ 6)$. The cycle notation indicates position 0 maps to the 9^{th} pitch-class, position 9 maps to the 11^{th} pitch class, position 1 maps to the 10^{th} pitch class, position 2 wraps back around to the 0^{th} pitch class etc. Equivalently in one-row notation we have $p = [9\ 10\ 0\ 3\ 4\ 6\ 5\ 7\ 8\ 11\ 1\ 2]$. This indicates that the pitch class A will be played first, followed by $A^{\#}/B^{b}$, then C etc. Finally, this prime row *p* may be visualized geometrically as in figure 3.



[9 10 0 3 4 6 5 7 8 11 1 2]

Figure 3: Prime Row *p* from Schoenberg’s Serenade [5]

The notes 0 through 11 go around clockwise to create a musical clock. The arrows represent the progression of notes in the prime row p = [9 10 0 3 4 6 5 7 8 11 1 2]. Starting from 9, 10 follows, then 0, and so on. As presented in figure 3 the dodecagon shape is equivalent to the treble clef representation of the prime row as it would appear on sheet music. Schoenberg’s Serenade then contained and sometimes favored permutations that resulted in shapes symmetric via operations of transposition, inversion, and retrograde to the one shown in figure 3. In this manner, permutations that result in geometric shapes symmetric to the one created in figure 3 will be a part of the same collection of elements which we define as $\text{RowClass}_{12}(p)$. For a more detailed history of atonal music see Haimo [3] and Headlam’s [4] writings on Schoenberg and Alban Berg’s atonal musical compositions.

Introduction: The Problem

Since atonal music is built around reorderings of the same twelve notes, what we really care about are the possible permutations of twelve elements (i.e., the algebraic group S_{12}). As explained by Fraleigh [2], S_n is of size $n!$ where in our case n indicates the number of notes in our microtonal system. For the twelve note system, the goal is then to examine the RowClass constructed from a given element of S_{12}. In their work Hunter and Hippel [5] describe the musical actions that result in symmetries of a given RowClass. These include transposition, inversion, and retrograde as possible operations. They then counted the number of unique RowClasses, or geometric shapes, which could be constructed via the above mentioned operations. In this paper we review their work and duplicate the results with the aim of extending their calculations to generic n-note microtonal systems. For an alternative approach consider Reiner's [6] work on enumeration in music theory.

Row Operations: Transposition

First, we will examine the transposition operation. The name transposition comes from the musical term transpose which means to uniformly increase or decrease the pitch of all of the notes. In other words, to apply a transposition by one semitone we take note 0 and raise it to note 1 i.e. C to $C^{\#}$, etc. Algebraically, each note is simply being raised by a single semitone. Continuing with Schoenberg's prime row as an example:

Position	=	[	0	1	2	3	4	5	6	7	8	9	10	11	]
p	=	[	9	10	0	3	4	6	5	7	8	11	1	2	]
T^1p	=	[	10	11	1	4	5	7	6	8	9	0	2	3	]

Here T^1 denotes the operation of transposing once as a left action on the prime row p. Observe T^1p is a composition of functions; p occurs first, mapping positions to notes then T is applied, and the note passed through is increased (mod 12). Also notice there are twelve possible transpositions as transposing the twelfth time will result in the original prime row, for example, 9 + 12 mod 12 = 9. Alternatively, this same operation may be observed geometrically in figure 4.

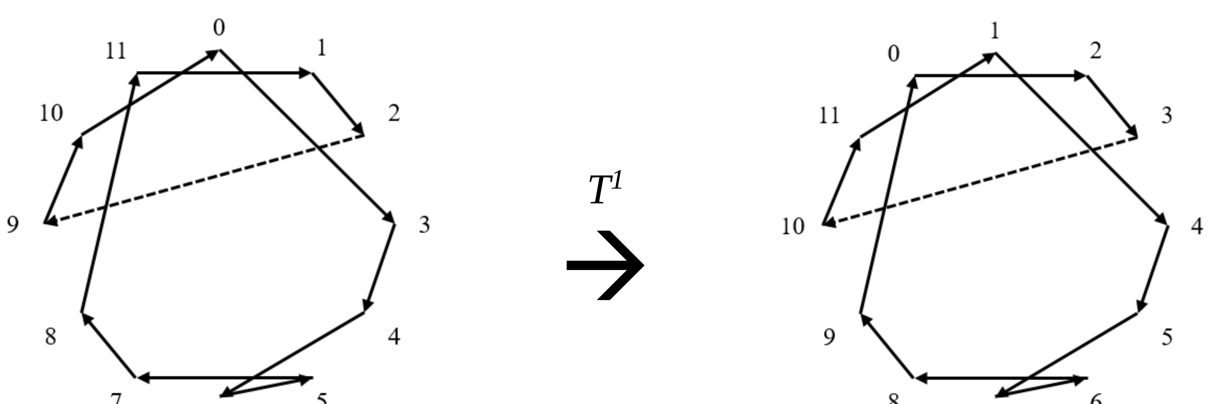


Figure 4: Transposition by 1 represented geometrically

T^1p then is represented as [10 11 1 4 5 7 6 8 9 0 2 3] which is equivalent to the dodecagon on the right in figure 4. Here the shape of the permutation as drawn by the arrows has not changed; however, the labels have rotated by $({}^{360}/_{12})^0$ in the counter clockwise direction. The two permutations, p and T^1p, are considered to be symmetric and belong to the same RowClass.

Finally, observe from the geometric picture that the musical clock has been rotated once counterclockwise. In general this is true as we continue to apply transpositions until the twelfth time when the clock has wrapped around full circle. Thus transpositions of n-note rows act as rotations in the Dihedral Groups D_n (see [2] for details). However, Dihedral groups concern the symmetries of a non-directed figure and so the directions of the arrows are ignored (i.e. remains fixed) when a rotation is applied.

Row Operations: Inversion

Next, we will examine the second row operation, inversion. As an introduction, first consider the more formal definition of a Dihedral group given in [2]:

$$D_n = \{\ x, y \mid x^n = 1, y^2 = 1, (xy)^2 = 1\ \}.$$

Here n denotes the number of nodes, x denotes a rotation, and y denotes a reflection. We have observed that transposition acts a rotation in D_n, inversion behaves similarly as a reflection.

Inversion is easier to describe geometrically rather than algebraically. First place an axis of rotation on the original row diagram then reflect the note labels across this line. Let ℓ be the line of reflection passing through the note in the first position of our prime row, p. We take reflection across ℓ as our initial operation y in the Dihedral group. The other reflections can then be reached by first transposing then applying this reflection. Figure 5 illustrates inversion as applied to the prime row from Schoenberg's Serenade.

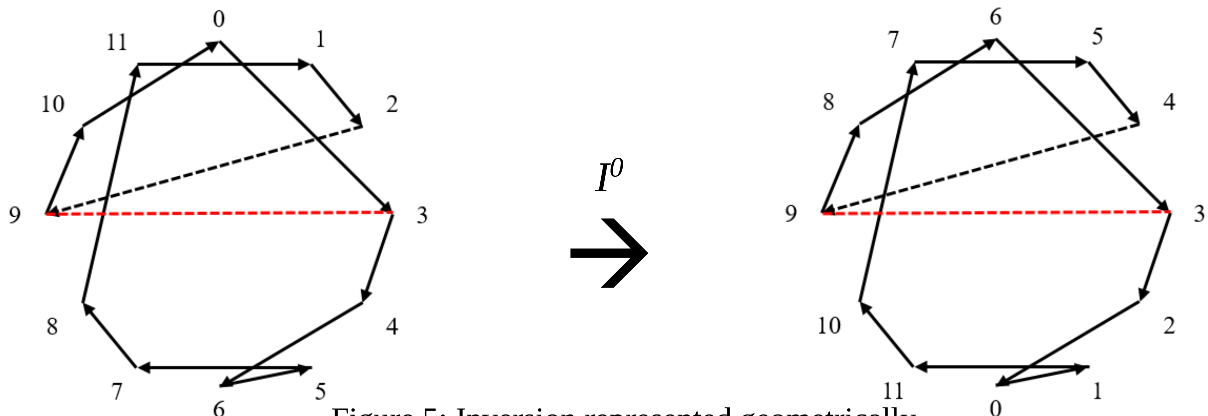


Figure 5: Inversion represented geometrically

Here I^0 denotes inversion as an operation being applied to the prime row *p*. Then the notes are reflected across the axis of reflection and so note 10 swaps with 8, note 11 with note 7, note 0 with note 6, note 1 with note 5, and note 2 with note 4. Notice 9 and 3 remain in their same position given that they lie on the line of reflection.

We can represent this inversion algebraically as follows

position	=	[	0	1	2	3	4	5	6	7	8	9	10	11	]
p	=	[	9	10	0	3	4	6	5	7	8	11	1	2	]
I^0p	=	[	9	8	6	3	2	0	1	11	10	7	5	4	]

In general a prime row may be written as $[P_0, P_1, \ldots, P_{11}]$ and when we apply I^0 we may write the resulting row as $[P_0, 2P_0 - P_1, \ldots, 2P_0 - P_{11}] \bmod 12$ [5]. Note that applying an inversion again will result in the same original prime row *p*. In this case the inversion operation is more clearly seen as being equivalent to a reflection operation.

Row Operations: Retrograde

The retrograde operation simply reverses the arrows in our geometric representation of the prime row as seen in figure 6.

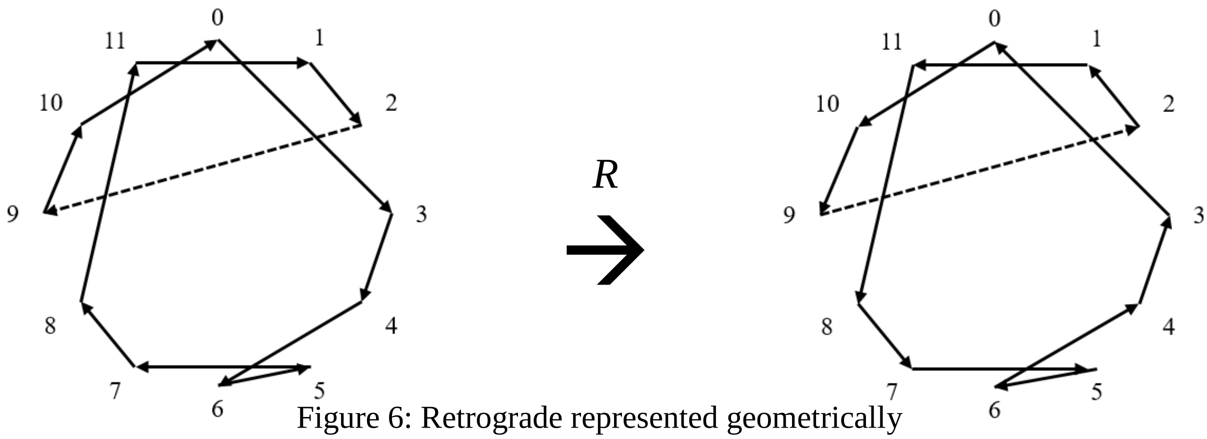


Figure 6: Retrograde represented geometrically

Because this operation directly deals with the arrows of our object, algebraically we must first apply retrograde to reverse the initial position row prior to applying the permutation for the prime row p. That is retrograde occurs as a right action on p. Retrograde may then be represented in cycle notation as $R = (0\ \ 11)(1\ \ 10)(2\ \ 9)(3\ \ 8)(4\ \ 7)(5\ \ 6)$ which is equivalent to $R = [11\ \ 10\ \ 9\ \ 8\ \ 7\ \ 6\ \ 5\ \ 4\ \ 3\ \ 2\ \ 1\ \ 0]$ in one row notation. So therefore,

position	=	[	0	1	2	3	4	5	6	7	8	9	10	11	]
R	=	[	11	10	9	8	7	6	5	4	3	2	1	0	]
pR	=	[	2	1	11	8	7	5	6	4	3	0	10	9	]

Notice when pR is compared against the original p, pR is just p in the reverse direction. Again, the original shape created from the arrows has been maintained and the permutations are considered to be symmetric. Note that the Dihedral groups contain rigid nodes of non-directed regular n-gons. Here similarities with the Dihedral end; however, what correlations we have seen will be useful as we continue forward. More specifically, as we work towards calculating all of the permutations that belong to the $\text{RowClass}_{12}(p)$ we will be calculating the double coset *DpR* with Dihedral operations and retrograde operation occurring as left and right actions respectively on the original prime row *p*.

Row Operations: Non-symmetric Permutations

As an example of a permutation that would not be symmetric to our prime row *p*, and so would not be an element of $\text{RowClass}_{12}(p)$, consider the row *q* = [0 1 2 3 4 5 6 7 8 9 10 11]. Geometrically, we can plainly see that these two permutations are not symmetric, as illustrated in figure 7.

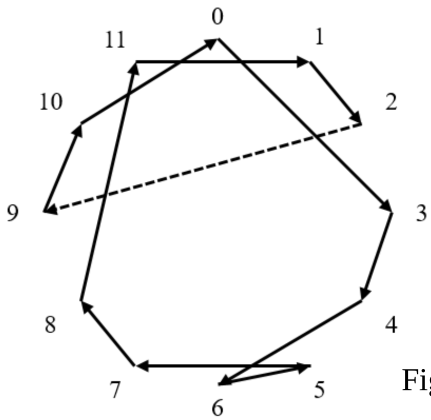
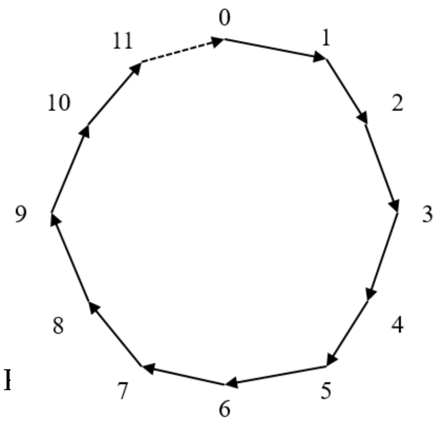


Figure 7: Non-Symmetric I



Additionally as a proof by exhaustion, this may be seen algebraically as we can calculate all of the possible operations' results on the row *q* which is done in figure 8.

Transpositions	**Inversions**
T^0q = [0 1 2 3 4 5 6 7 8 9 10 11]	IT^0q = [0 11 10 9 8 7 6 5 4 3 2 1]
T^1q = [1 2 3 4 5 6 7 8 9 10 11 0]	IT^1q = [1 0 11 10 9 8 7 6 5 4 3 2]
T^2q = [2 3 4 5 6 7 8 9 10 11 0 1]	IT^2q = [2 1 0 11 10 9 8 7 6 5 4 3]
T^3q = [3 4 5 6 7 8 9 10 11 0 1 2]	IT^3q = [3 2 1 0 11 10 9 8 7 6 5 4]
T^4q = [4 5 6 7 8 9 10 11 0 1 2 3]	IT^4q = [4 3 2 1 0 11 10 9 8 7 6 5]
T^5q = [5 6 7 8 9 10 11 0 1 2 3 4]	IT^5q = [5 4 3 2 1 0 11 10 9 8 7 6]
T^6q = [6 7 8 9 10 11 0 1 2 3 4 5]	IT^6q = [6 5 4 3 2 1 0 11 10 9 8 7]
T^7q = [7 8 9 10 11 0 1 2 3 4 5 6]	IT^7q = [7 6 5 4 3 2 1 0 11 10 9 8]
T^8q = [8 9 10 11 0 1 2 3 4 5 6 7]	IT^8q = [8 7 6 5 4 3 2 1 0 11 10 9]
T^9q = [9 10 11 0 1 2 3 4 5 6 7 8]	IT^9q = [9 8 7 6 5 4 3 2 1 0 11 10]
$T^{10}q$ = [10 11 0 1 2 3 4 5 6 7 8 9]	$IT^{10}q$ = [10 9 8 7 6 5 4 3 2 1 0 11]
$T^{11}q$ = [11 0 1 2 3 4 5 6 7 8 9 10]	$IT^{11}q$ = [11 10 9 8 7 6 5 4 3 2 1 0]

Retrograde

qR = [11 10 9 8 7 6 5 4 3 2 1 0] = $IT^{11}q$

Figure 8: $\text{RowClass}_{12}(q)$

Here we have generated all of the elements of $\text{RowClass}_{12}(q)$, none of which equal p = [9 10 0 3 4 6 5 7 8 11 1 2]. Therefore, q and p are not symmetric. However, the figure 8 calculations also showcase another curiosity; we have overlap in results from our operations. As shown above, $qR = IT^{11}q$. This occurred because for some RowClasses the retrograde operation will not be required as retrograde will be equivalent to some operation offered by the Dihedral group. Additionally, because the retrograde of q is captured each of the possible retrograde combinations will also be absorbed by some operation in the Dihedral group. We will see additional examples of this phenomenon in chapters 3 and 4.

RowClass Structure: RowClass Cardinality

In this section we work to determine the cardinality of $\mathrm{RowClass}_{12}(\alpha)$. Let $\alpha \in S_{12}$, where the chosen α results in overlap between the retrograde operation and the Dihedral operations then $|\mathrm{RowClass}_{12}(\alpha)| = 24$ [5]. We saw an example in figure 7 using the prime row $\alpha = q = [0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7\ \ 8\ \ 9\ \ 10\ \ 11]$. The general case will follow a similar format, shown in figure 9.

Transpositions	**Inversions**
$T^0\alpha$	$IT^0\alpha$
$T^1\alpha$	$IT^1\alpha$
$T^2\alpha$	$IT^2\alpha$
$T^3\alpha$	$IT^3\alpha$
$T^4\alpha$	$IT^4\alpha$
$T^5\alpha$	$IT^5\alpha$
$T^6\alpha$	$IT^6\alpha$
$T^7\alpha$	$IT^7\alpha$
$T^8\alpha$	$IT^8\alpha$
$T^9\alpha$	$IT^9\alpha$
$T^{10}\alpha$	$IT^{10}\alpha$
$T^{11}\alpha$	$IT^{11}\alpha$

Figure 9: $\mathrm{RowClass}_{12}(\alpha)$ with Overlap

Here, just as it happened for q, αR will be equivalent to one of the above listed operations. In this case, where retrograde is equivalent to some operation in the Dihedral group, the number of symmetric permutations will be equal to the size of the Dihedral group which, depending on n, is $2n$ or 24 when working in a traditional 12-note microtonal system.

Finally we consider when a given α does not yield overlap between the retrograde operation and the Dihedral operations. In this case, for each of the Dihedral operations retrograde may be applied in addition as seen in figure 10.

Transpositions	**Inversions**
$T^{0}\alpha$	$IT^{0}\alpha$
$T^{1}\alpha$	$IT^{1}\alpha$
$T^{2}\alpha$	$IT^{2}\alpha$
$T^{3}\alpha$	$IT^{3}\alpha$
$T^{4}\alpha$	$IT^{4}\alpha$
$T^{5}\alpha$	$IT^{5}\alpha$
$T^{6}\alpha$	$IT^{6}\alpha$
$T^{7}\alpha$	$IT^{7}\alpha$
$T^{8}\alpha$	$IT^{8}\alpha$
$T^{9}\alpha$	$IT^{9}\alpha$
$T^{10}\alpha$	$IT^{10}\alpha$
$T^{11}\alpha$	$IT^{11}\alpha$
Retrograde	
$T^{0}\alpha R$	$IT^{0}\alpha R$
$T^{1}\alpha R$	$IT^{1}\alpha R$
$T^{2}\alpha R$	$IT^{2}\alpha R$
$T^{3}\alpha R$	$IT^{3}\alpha R$
$T^{4}\alpha R$	$IT^{4}\alpha R$
$T^{5}\alpha R$	$IT^{5}\alpha R$
$T^{6}\alpha R$	$IT^{6}\alpha R$
$T^{7}\alpha R$	$IT^{7}\alpha R$
$T^{8}\alpha R$	$IT^{8}\alpha R$
$T^{9}\alpha R$	$IT^{9}\alpha R$
$T^{10}\alpha R$	$IT^{10}\alpha R$
$T^{11}\alpha R$	$IT^{11}\alpha R$

Figure 10: $\text{RowClass}_{12}(\alpha)$ with no Overlap

With retrograde applied we double the number of unique possible operations and $|\text{RowClass}_{12}(\alpha)| = 48$ [5]. In general when $\alpha R \neq d\alpha$ for any $d \in \text{D}_n$ we have $|\text{RowClass}_{12}(\alpha)| = 2 * |\text{D}_n| = 4n$.

RowClass Structure: Unique RowClasses

Now that we can count the number of permutations contained within a given RowClass, we work to calculate the number of unique RowClasses. In other words, we will determine the number of unique geometric shapes within a 12-note system that are not symmetric to one another. Using notation introduced by Hunter and Hippel [5], we define the set of all unique RowClasses as follows:

$$T = \{ D\alpha R \mid \alpha \in S_{12} \}.$$

For the $D\alpha R$ construction we have two possibilities: 1) $D\alpha \cap D\alpha R = \emptyset$ or 2) $D\alpha = D\alpha R$. In other words, we either have no overlap between the Dihedral operations with the retrograde operation or there is complete overlap. For ease of calculation let us begin by supposing $D\alpha = D\alpha R$ and count the number of RowClasses in the case that there is complete overlap. It follows algebraically:

$$D\alpha = D\alpha R,$$

$$D = D\alpha R\alpha^{-1}.$$

So we are looking for the instances when $\alpha R\alpha^{-1} \in D$. The construction $\alpha R\alpha^{-1}$ means R is conjugated with α; Remember R = (0 11)(1 10)(2 9)(3 8)(4 7)(5 6) and as Fraleigh explains in [2] conjugation preserves cycle structure. Hence conjugates of R must contain exactly six 2-cycles where a 2-cycle is simply a switching of two notes i.e. (0 1). First, we will count the number of distinct conjugates of R, then we can determine how many α's yield a specific conjugate. To count the distinct conjugates we need to know how many distinct conjugates of six 2-cycles exist. For instance, the cycle c = (0 11)(1 10)(2 9)(3 8)(4 7)(5 6) and cycle d = (11 0)(1 10)(2 9)(3 8)(4 7)(5 6) are the same because (0 11) and (11 0) are the two ways of notating the single permutation that switches 0 and 11. We then have two equivalent results and this can happen six times for each possible 2-cycle. At this point, we know we have no more than 12! / 2^6 distinct conjugates. Additionally, when g = (1 10)(0 11)(2 9)(3 8)(4 7)(5 6) then $c = g$. This is because swapping two 2-cycles does not change the interpretation. We have 6! ways of writing the same conjugate via re-orderings, giving

$$\frac{12!}{2^6 * 6!} \text{ distinct conjugates of } R.$$

We now have 12! permutations being pigeon holed into $\frac{12!}{2^6 * 6!}$ distinct conjugates. Therefore 2^6 * 6! α's map R to each of its conjugates.

Next, we find the distinct conjugates of R within the Dihedral group. For this it will again be helpful to think about the cycle structure of retrograde where nodes are interchanged. In this respect, reflections would seem like promising conjugates of R; however, because twelve is even only half of the reflections will pass through two nodes and so leave them fixed. We then see only half of the reflections appear as conjugates of R. Among the rotations only the rotation by 180°, or T^6, will result in nodes flipping with one another in the fashion we desire. Then 2^6 * 6! α's conjugate R to each of the 7 possible Dihedral operations.

Therefore, within a 12-note system we now know that there are 2^6 * 6! * 7 α's for which $D\alpha = D\alpha R$ as αR is equivalent to some Dihedral operation. However, we are counting the number of RowClasses for which this happens and so we should divide by the number of α's within each RowClass. As calculated in the last section, we know a single RowClass with complete overlap contains 24 permutations. Therefore, within the 12-note system we have $\frac{2^6*6!*7}{24}$ unique RowClasses containing complete overlap between *R* and *D*.

Lastly, we have $12! - 2^6 * 6! * 7$ elements of S_{12} left to consider. These are exactly the permutations, α, for when $\alpha R \notin D$. We take this as the number of α's for which retrograde is needed to generate the RowClass and divide by 48, the number of α's that would be contained within each RowClass. Putting our counts together leads to the count of the distinct RowClasses given in Theorem 1.

<u>Theorem 1 [5]:</u> Let $T = \{ D\alpha R \mid \alpha \in S_{12} \}$ where D is the dihedral group associated with a regular 12-gon and $R = (0\ \ 11)\ (1\ \ 10)\ (2\ \ 9)\ (3\ \ 8)\ (4\ \ 7)\ (5\ \ 6)$ then

$$|T| = \frac{2^6*6!*7}{24} + \frac{12!-2^6*6!*7}{48} = 13{,}440 + 9{,}972{,}480 = 9{,}985{,}920.$$

Examples: 3-note Microtonal System

As a concrete example, consider the 3-note microtonal system. Here all 3-tone rows are elements of

$$S_3 = \{ (0), (0\ \ 1), (0\ \ 2), (1\ \ 2), (0\ \ 1\ \ 2), (0\ \ 2\ \ 1) \}.$$

Because S_3 is so small we can simply calculate the elements in each RowClass as seen below:

RowClass$_3$(α_1) for α_1 = [0 1 2]

$T^0\alpha$	=	[0 1 2]	=	(0)
$T^1\alpha$	=	[1 2 0]	=	(0 1 2)
$T^2\alpha$	=	[2 0 1]	=	(0 2 1)
$I\alpha$	=	[0 2 1]	=	(1 2)
$IT^1\alpha$	=	[1 0 2]	=	(0 1)
$IT^2\alpha$	=	[2 1 0]	=	(0 2)

Figure 11: RowClass$_3$(α)

As seen in figure 11 all six permutations in S_3 are elements of $D_3\alpha$, and here including retrograde will not add additional rows. More specifically αR = [2 1 0] = $IT^2\alpha$ and so overlaps with a Dihedral operation.

Therefore a 3-note microtonal system leads to only one RowClass represented geometrically in figure 12.

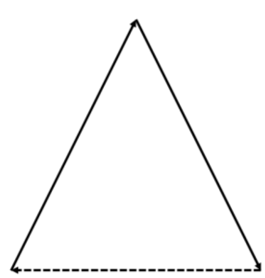


Figure 12: RowClass$_3$(α) geometrically

This result is not surprising. If we take a 3-node equilateral shape and draw two straight lines, the only possible shape is a triangle. Here the dotted line maps the last note in the 3-tone row back to the first.

We can also conclude we have only one RowClass of 3-tone rows using a counting argument mirroring our 12-tone argument. We start by observing that in the 3-note system retrograde is given by $R = (0\ \ 2)$. As before we take 3!, divide by the number of equivalent conjugates of R, and take the denominator $2^1 * 1!$ to indicate the number of α's that conjugate with R to return R.

Next we consider conjugates of R in D_3, that is we look for elements of D_3 with the same cycle structure as found in R. All of the reflections in D_3 maintain this structure. However, because $n = 3$ is odd we have no rotation by 180°. Therefore, we have $2^1 * 1! * 3$ α's that have complete overlap between D and R. Mirroring our construction of Theorem 1 we then have a Theorem 2.

Theorem 2: Let $T = \{ D\alpha R \mid \alpha \in S_3 \}$ where D is the dihedral group associated with a regular 3-gon and $R = (0\ \ 2)$ then

$$|T| = \frac{2^1 * 1! * 3}{6} + \frac{3! - 2^1 * 1! * 3}{12} = 1 + 0 = 1 .$$

Our algebraic result agrees with our direct computations. We have only one unique RowClass in a 3-note microtonal system.

Examples: 4-note Microtonal System

Four-tone rows are elements of the group S_4, a group of order 4! = 24. Here we will first consider a combinatorial argument so as to get an idea of how many RowClasses we should expect to find using direct calculations. Again, we start by considering *R* in cycle notation as it would appear in a 4-note microtonal system. Now $R = (0\ \ 3)\ (1\ \ 2)$. As before we take 4!, divide by the number of equivalent conjugates, and take the denominator $2^2 * 2!$ to indicate the number of α's that conjugate with *R* to return *R*.

Next we consider the conjugates of *R* in D_4, that is we look for elements of D_4 with the same cycle structure as *R*. In the same manner as our 12-note system, only half of the reflections have this cycle-structure, giving 2 possible reflections. Additionally, because 4 is even D_4 contains the rotation by 180°, or $T^2 = (0\ \ 2)\ (1\ \ 3)$. Therefore, we have $2^2 * 2! * 3$ α's that will have complete overlap between *D* and *R*. Mirroring our presentation of Theorem 1 we have

<u>Theorem 3:</u> Let $T = \{\ D\alpha R \mid \alpha \in S_4\ \}$ where D is the dihedral group associated with a regular 14-gon and $R = (0\ \ 3)\ (1\ \ 2)$ then

$$|T| = \frac{2^2 * 2! * 3}{8} + \frac{4! - 2^2 * 2! * 3}{16} = 3 + 0 = 3\,.$$

Now we know to expect 3 possible unique RowClasses in the 4-note microtonal system. More than that, we also know to expect that all three RowClasses can be generated without retrograde due to the 0 term in Theorem 3. Direct calculations confirm this in figure 13.

RowClass$_4$(α_1) for α_1 = [0 1 2 3]

α_1	= [0 1 2 3] =	(0)
$T^1\alpha_1$	= [1 2 3 0] =	(0 1 2 3)
$T^2\alpha_1$	= [2 3 0 1] =	(0 2)(1 3)
$T^3\alpha_1$	= [3 0 1 2] =	(0 3 2 1)
$I\alpha_1$	= [0 3 2 1] =	(1 3)
$IT^1\alpha_1$	= [1 0 3 2] =	(0 1)(2 3)
$IT^2\alpha_1$	= [2 1 0 3] =	(0 2)
$IT^3\alpha_1$	= [3 2 1 0] =	(0 3)(1 2)

RowClass$_4$(α_2) for α_2 = [0 1 3 2]

α_2	= [0 1 3 2] =	(2 3)
$T^1\alpha_2$	= [1 2 0 3] =	(0 1 2)
$T^2\alpha_2$	= [2 3 1 0] =	(0 2 1 3)
$T^3\alpha_2$	= [3 0 2 1] =	(0 3 1)
$I\alpha_2$	= [0 3 1 2] =	(1 3 2)
$IT^1\alpha_2$	= [1 0 2 3] =	(0 1)
$IT^2\alpha_2$	= [2 1 3 0] =	(0 2 3)
$IT^3\alpha_2$	= [3 2 0 1] =	(0 3 1 2)

RowClass$_4$(α_3) for α_3 = [3 1 2 0]

α_3	= [3 1 2 0] =	(0 3)
$T^1\alpha_3$	= [0 2 3 1] =	(1 2 3)
$T^2\alpha_3$	= [1 3 0 2] =	(0 1 3 2)
$T^3\alpha_3$	= [2 0 1 3] =	(0 2 1)
$I\alpha_3$	= [3 1 0 2] =	(0 3 2)
$IT^1\alpha_3$	= [0 2 1 3] =	(1 2)
$IT^2\alpha_3$	= [1 3 2 0] =	(0 1 3)
$IT^3\alpha_3$	= [2 0 3 1] =	(0 2 3 1)

Figure 13: RowClass$_4$(α_1), RowClass$_4$(α_2), and RowClass$_4$(α_3)

As seen in figure 13, all 24 permutations are represented, and there is no crossover between these three unique sets. Geometrically, they create three unique shapes that are not symmetric to one another as displayed in figure 14.

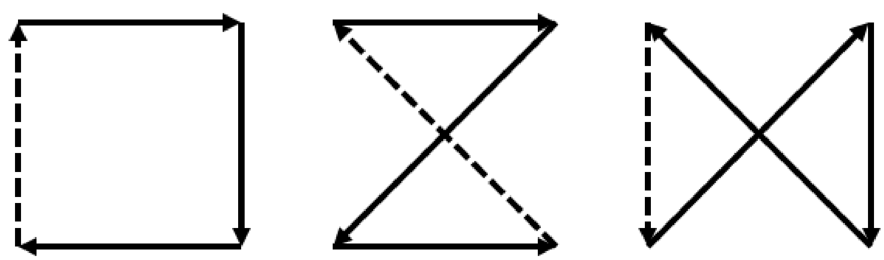


Figure 14: RowClass$_4$(α_1), RowClass$_4$(α_2), and RowClass$_4$(α_3) geometrically left to right

Examples: 5-note Microtonal System

As a final example we will examine the five-note microtonal system. Not only is this another odd *n*-note system, but five is the smallest *n* for which retrograde will be needed in order to generate all of our RowClasses. We then continue as before thinking of S_5, a group of order 5! = 120, as the collection of all 5-note rows.

Again, we first consider a combinatorial argument to get an idea of how many RowClasses we should expect to find by direct calculations. We start by considering *R* in cycle notation as it would appear in a 5-note microtonal system. Now *R* = (0 4) (1 3). Following our previous work we take 5!, divide by the number of equivalent conjugates of *R*, and take the denominator $2^2 * 2!$ to indicate the number of α's that conjugate with *R* to return *R*.

Next, we consider the conjugates of *R* in D_5, that is we look for elements of D_5 with the same cycle structure. As in our 3-note system, each of the five reflections in D_5 have this structure. But, because this is an odd Dihedral we do not have the rotation by 180°. Therefore, we have $2^2 * 2! * 5$ α's that will have complete overlap between *D* and *R*. Mirroring our presentation of Theorem 1 we have

Theorem 4: Let $T = \{ D\alpha R \mid \alpha \in S_5 \}$ where D is the dihedral group associated with a regular 5-gon and $R = (0\ \ 4)\ (1\ \ 3)$ then

$$|T| = \frac{2^2 * 2! * 5}{10} + \frac{5! - 2^2 * 2! * 5}{20} = 4 + 4 = 8\,.$$

Now we know to expect 8 possible unique RowClasses in the 5-note microtonal system. More than that, we also know to expect four of them to not require retrograde and to expect four of them to require retrograde in order to generate the sets.

Again, we verify our combinatorial results using direct calculations. Figures 15 and 16 give the four RowClasses that do not require retrograde to be generated.

RowClass$_5$(α_1) for α_1 = [0 1 2 3 4]				
α_1	=	[0 1 2 3 4]	=	(0)
$T^1\alpha_1$	=	[1 2 3 4 0]	=	(0 1 2 3 4)
$T^2\alpha_1$	=	[2 3 4 0 1]	=	(0 2 4 1 3)
$T^3\alpha_1$	=	[3 4 0 1 2]	=	(0 3 1 4 2)
$T^4\alpha_1$	=	[4 0 1 2 3]	=	(0 4 3 2 1)
$I\alpha_1$	=	[0 4 3 2 1]	=	(1 4)(2 3)
$IT^1\alpha_1$	=	[1 0 4 3 2]	=	(0 1)(2 4)
$IT^2\alpha_1$	=	[2 1 0 4 3]	=	(0 2)(3 4)
$IT^3\alpha_1$	=	[3 2 1 0 4]	=	(0 3)(1 2)
$IT^4\alpha_1$	=	[4 3 2 1 0]	=	(0 4)(1 3)

RowClass$_5$(α_2) for α_2 = [1 3 0 2 4]				
α_2	=	[1 3 0 2 4]	=	(0 1 3 2)
$T^1\alpha_2$	=	[2 4 1 3 0]	=	(0 2 1 4)
$T^2\alpha_2$	=	[3 0 2 4 1]	=	(0 3 4 1)
$T^3\alpha_2$	=	[4 1 3 0 2]	=	(0 4 2 3)
$T^4\alpha_2$	=	[0 2 4 1 3]	=	(1 2 4 3)
$I\alpha_2$	=	[1 4 2 0 3]	=	(0 1 4 3)
$IT^1\alpha_2$	=	[2 0 3 1 4]	=	(0 2 3 1)
$IT^2\alpha_2$	=	[3 1 4 2 0]	=	(0 3 2 4)
$IT^3\alpha_2$	=	[4 2 0 3 1]	=	(0 4 1 2)
$IT^4\alpha_2$	=	[0 3 1 4 2]	=	(1 3 4 2)

Figure 15: RowClass$_5$(α_1) and RowClass$_5$(α_2)

RowClass$_5$(α_3) for α_3 = [1 2 0 3 4]				
α_3	=	[1 2 0 3 4]	=	(0 1 2)
$T^1\alpha_3$	=	[2 3 1 4 0]	=	(0 2 1 3 4)
$T^2\alpha_3$	=	[3 4 2 0 1]	=	(0 3)(1 4)
$T^3\alpha_3$	=	[4 0 3 1 2]	=	(0 4 2 3 1)
$T^4\alpha_3$	=	[0 1 4 2 3]	=	(2 4 3)
$I\alpha_3$	=	[1 0 2 4 3]	=	(0 1)(3 4)
$IT^1\alpha_3$	=	[2 1 3 0 4]	=	(0 2 3)
$IT^2\alpha_3$	=	[3 2 4 1 0]	=	(0 3 1 2 4)
$IT^3\alpha_3$	=	[4 3 0 2 1]	=	(0 4 1 3 2)
$IT^4\alpha_3$	=	[0 4 1 3 2]	=	(1 4 2)

RowClass$_5$(α_4) for α_4 = [4 1 2 3 0]				
α_4	=	[4 1 2 3 0]	=	(0 4)
$T^1\alpha_4$	=	[0 2 3 4 1]	=	(1 2 3 4)
$T^2\alpha_4$	=	[1 3 4 0 2]	=	(0 1 3)(2 4)
$T^3\alpha_4$	=	[2 4 0 1 3]	=	(0 2)(1 4 3)
$T^4\alpha_4$	=	[3 0 1 2 4]	=	(0 3 2 1)
$I\alpha_4$	=	[4 2 1 0 3]	=	(0 4 3)(1 2)
$IT^1\alpha_4$	=	[0 3 2 1 4]	=	(1 3)
$IT^2\alpha_4$	=	[1 4 3 2 0]	=	(0 1 4)(2 3)
$IT^3\alpha_4$	=	[2 0 4 3 1]	=	(0 2 4 1)
$IT^4\alpha_4$	=	[3 1 0 4 2]	=	(0 3 4 2)

Figure 16: RowClass$_5$(α_3) and RowClass$_5$(α_4)

Figures 15 and 16 represent 40 permutations, and there is no crossover between these four unique sets. We also know these four RowClasses do not require retrograde as $\alpha_1 R$ = [4

3 2 1 0] = $IT^4\alpha_1$, α_2R = [4 2 0 3 1] = $IT^3\alpha_2$, α_3R = [4 3 0 2 1] = $IT^3\alpha_3$, and α_4R = [0 3 2 1 4] = $IT^1\alpha_4$. Geometrically, the RowClasses in figures 15 and 16 create four unique shapes that are not symmetric to one another as displayed in figure 17.

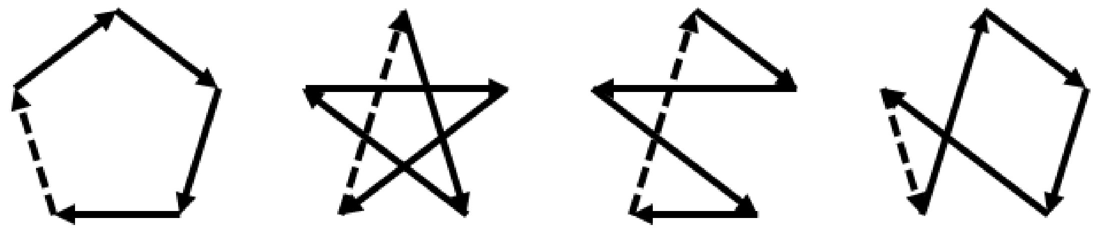


Figure 17: RowClass$_5$(α_1), RowClass$_5$(α_2), RowClass$_5$(α_3), and RowClass$_5$(α_4) geometrically left to right

Lastly, we will show the four remaining RowClasses within the 5-note system that do require retrograde. Because each of these RowClasses have 20 elements we will abbreviate their elements as seen in figures 18 and 19.

RowClass$_5$(α_5) for α_5 = [1 0 2 3 4]				
α_5	=	[1 0 2 3 4]	=	(0 1)
...		…		…
$T^4\alpha_5$	=	[0 4 1 2 3]	=	(1 4 3 2)
$I\alpha_5$	=	[1 2 0 4 3]	=	(0 1 2)(3 4)
...		…		…
$IT^4\alpha_5$	=	[0 1 4 3 2]	=	(2 4)
α_5R	=	[4 3 2 0 1]	=	(0 4 1 3)
...		…		…
$T^4\alpha_5R$	=	[3 2 1 4 0]	=	(0 3 4)(1 2)
$I\alpha_5R$	=	[4 0 1 3 2]	=	(0 4 2 1)
...		…		…
$IT^4\alpha_5R$	=	[3 4 0 2 1]	=	(0 3 2)(1 4)

RowClass$_5$(α_6) for α_6 = [3 1 2 0 4]				
α_6	=	[3 1 2 0 4]	=	(0 3)
...		…		…
$T^4\alpha_6$	=	[2 0 1 4 3]	=	(0 2 1)(3 4)
$I\alpha_6$	=	[3 0 4 1 2]	=	(0 3 1)(2 4)
...		…		…
$IT^4\alpha_6$	=	[2 4 3 0 1]	=	(0 2 3)(1 4)
A_6R	=	[4 0 2 1 3]	=	(0 4 3 1)
...		…		…
$T^4\alpha_6R$	=	[3 4 1 0 2]	=	(0 3)(1 4 2)
$I\alpha_6R$	=	[4 3 1 2 0]	=	(0 4)(1 3 2)
...		…		…
$IT^4\alpha_6R$	=	[3 2 0 1 4]	=	(0 3 1 2)

Figure 18: RowClass$_5$(α_5) and RowClass$_5$(α_6)

RowClass$_5$(α_7) for α_7 = [2 0 1 3 4]

α_7	=	[2 0 1 3 4]	=	(0 2 1)
...		…		…
$T^4\alpha_7$	=	[1 4 0 2 3]	=	(0 1 4 3 2)
$I\alpha_7$	=	[2 4 3 1 0]	=	(0 2 3 1 4)
...		…		…
$IT^4\alpha_7$	=	[1 3 2 0 4]	=	(0 1 3)
α_7R	=	[4 3 1 0 2]	=	(0 4 2 1 3)
...		…		…
$T^4\alpha_7R$	=	[3 2 0 4 1]	=	(0 3 4 1 2)
$I\alpha_7R$	=	[4 0 2 3 1]	=	(0 4 1)
...		…		…
$IT^4\alpha_7R$	=	[3 4 1 2 0]	=	(0 3 2 1 4)

RowClass$_5$(α_8) for α_8 = [3 0 2 1 4]

α_8	=	[3 0 2 1 4]	=	(0 3 1)
...		…		…
$T^4\alpha_8$	=	[2 4 1 0 3]	=	(0 2 1 4 3)
$I\alpha_8$	=	[3 1 4 0 2]	=	(0 3)(2 4)
...		…		…
$IT^4\alpha_8$	=	[2 0 3 4 1]	=	(0 2 3 4 1)
α_8R	=	[4 1 2 0 3]	=	(0 4 3)
...		…		…
$T^4\alpha_8R$	=	[3 0 1 4 2]	=	(0 3 4 2 1)
$I\alpha_8R$	=	[4 2 1 3 0]	=	(0 4)(1 2)
...		…		…
$IT^4\alpha_8R$	=	[3 1 0 2 4]	=	(0 3 2)

Figure 19: RowClass$_5$(α_7) and RowClass$_5$(α_8)

Figures 18 and 19 represent all 80 remaining permutations, and there is no crossover between these four unique sets. Geometrically, the RowClasses in figures 18 and 19 create four unique shapes that are not symmetric to one another as displayed in figure 20.

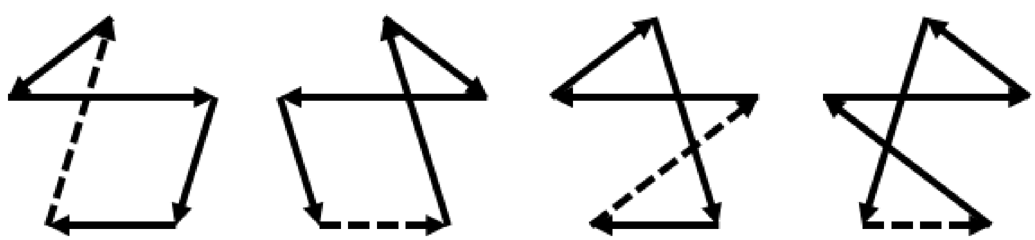


Figure 20: RowClass$_5$(α_5), RowClass$_5$(α_6), RowClass$_5$(α_7), and RowClass$_5$(α_8) geometrically left to right

Generalization: Equation for Even *n*-note Microtonal Systems ($n \geq 4$)

We model our calculations for even *n*-note microtonal systems after the combinatorial approach taken in the 4 and 12 note systems. We know all of the possible permutations will be elements of S_n and have *n*! orderings. Additionally, we need the general cycle notation format for retrograde in the even case: $R = (0\ \ n\text{-}1)\ (1\ \ n\text{-}2)\ldots(\frac{n-2}{2}\ \ \frac{n}{2})$. As we can see from all previous examples, the first position swaps with the last position, the second position swaps with the next to last position, and so on until the two middle positions are swapped.

As before we begin by considering the case where $D\alpha = D\alpha R$ to determine the number of unique RowClasses with complete overlap between *R* and *D*. We start by determining the distinct conjugates of *R*. As seen from the general cycle notation of *R* there are *n*/2 2-cycles. For each of these 2-cycles flipping the elements result in the same 2-cycle (i.e., $(0\ \ n\text{-}1) = (n\text{-}1\ \ 0)$) and so we get $2^{n/2}$ equivalent conjugate results. Additionally, if the 2-cycles are rearranged then we still have equivalent results (i.e., $(0\ \ n\text{-}1)\ (1\ \ n\text{-}2) = (1\ \ n\text{-}2)\ (0\ \ n\text{-}1)$). So because there are *n*/2 2-cycles, there are (*n*/2)! reorderings of a given conjugate result. Then it follows that we have $\frac{n!}{2^{n/2} * \frac{n}{2}!}$ distinct conjugates of *R*.

We have *n*! permutations being pigeon holed into $\frac{n!}{2^{n/2} * \frac{n}{2}!}$ possible conjugates. However, as we are counting for the case where retrograde completely overlaps with the Dihedral group, we more specifically care about the number of *α*'s that preserve *R* when conjugated. This equals the number of *α*'s that give results in a single pigeon hole, or $2^{n/2} * \frac{n}{2}!$.

Next, we also have to account for how many distinct conjugates of *R* are in the Dihedral group. For this it will again be helpful to think about the cycle structure of retrograde and take the number of reflections with this cycle structure. For even *n*, half of the reflections in D_n will pass through two nodes and so hold them fixed. The other half of the reflections have the same cycle structure of *R*. Among the rotations we again have the rotation by 180°, or $T^{n/2}$. Thus yielding a total of $2^{n/2} * \left({}^{n}/_{2}\right)! * \left({}^{n}/_{2} + 1\right)$ α's that have complete overlap with *R* since $2^{n/2} * \frac{n}{2}!$ α's conjugate to each of the $\frac{n}{2} + 1$ possible Dihedral operations.

However, we are trying to count the number of RowClasses for which this happens and so we should divide by the number of α's that will belong to each RowClass. In this case we only have Dihedral operations, and so we know there will be $2n$ permutations contained in a single RowClass with complete overlap. Therefore within an even *n*-note microtonal system we have $\frac{2^{n/2} * ({}^{n}/_{2})! * ({}^{n}/_{2}+1)}{2n}$ unique RowClasses containing complete overlap between *R* and *D*.

Finally, to account for the case where we have no overlap between *R* and *D* we simply remove from the $n!$ elements of S_n all possible α's that have already been counted to get $n! - 2^{n/2} * \left({}^{n}/_{2}\right)! * \left({}^{n}/_{2} + 1\right)$ α's for which retrograde is needed to generate the RowClass. We then divide by the number of α's that would be contained within each RowClass which would be $4n$, double the Dihedral operations. Thus $\frac{n! - 2^{n/2} * ({}^{n}/_{2})! * ({}^{n}/_{2}+1)}{4n}$ unique RowClasses require *R* for generation within an even *n*-note microtonal system. Combining these two counts then leads to Theorem 5.

Theorem 5: Let $T = \{ D\alpha R \mid \alpha \in S_n \}$ where D is the dihedral group associated with a regular n-gon for even n and $R = (0\ \ n\text{-}1)\ (1\ \ n\text{-}2)\ldots(\frac{n-2}{2}\ \ \frac{n}{2})$ then

$$|T| = \frac{2^{n/2} * \left({}^{n}/_{2}\right)! * \left({}^{n}/_{2} + 1\right)}{2n} + \frac{n! - 2^{n/2} * \left({}^{n}/_{2}\right)! * \left({}^{n}/_{2} + 1\right)}{4n}.$$

Generalization: Equation for Odd n-note Microtonal Systems ($n \geq 3$)

We model our calculations for the odd n-note microtonal systems after the combinatorial approach taken in the three and five note systems. In general, we know all of the possible permutations will be elements of S_n and have n! possibilities. Additionally, we will need the general cycle notation format for retrograde in the odd case: R = $(0\ \ n\text{-}1)\ (1\ \ n\text{-}2)\ldots(\frac{n-3}{2}\ \ \frac{n+1}{2})$. As we can see from all previous examples, the first position swaps with the last position, the second position swaps with the next to last position, and so on until the two positions on either side of the middle position are swapped leaving the middle position fixed.

As before, we begin by considering the case where $D\alpha = D\alpha R$ to determine the number of RowClasses with complete overlap between R and D. We first determine the number of distinct conjugates of R. As seen from the general cycle notation of R, retrograde contains (n-1)/2 2-cycles. For each of these 2-cycles flipping the elements result in the same 2-cycle (i.e., $(0\ \ n\text{-}1) = (n\text{-}1\ \ 0)$). This may happen for each 2-cycle and so this results in $2^{(n-1)/2}$ equivalent conjugate results. Additionally, if the 2-cycles are rearranged we still have equivalent results (i.e., $(0\ \ n\text{-}1)\ (1\ \ n\text{-}2) = (1\ \ n\text{-}2)\ (0\ \ n\text{-}1)$). Then, because there are (n-1)/2 2-cycles, there are ((n-1)/2)! reorderings of a given conjugate result. It follows that we have $\frac{n!}{2^{(n-1)/2} * \frac{n-1}{2}!}$ distinct conjugates of R.

We have n! permutations being pigeon holed into $\frac{n!}{2^{(n-1)/2} * \frac{n-1}{2}!}$ possible conjugates. However, as we are counting for the case where retrograde overlaps with the Dihedral group, we more specifically care about the number of α's that preserve R when conjugated which equals the number of a's that give equivalent results in a single pigeon hole, or $2^{(n-1)/2} * \frac{n-1}{2}!$.

Next, we also have to account for how many distinct conjugates of R are in the Dihedral group. For this it will again be helpful to think about the cycle structure of retrograde and take the number of reflections with this cycle structure. For odd n, all of the reflections hold one node constant while swapping the rest. Thus, we have all n reflections since they maintain the cycle structure of R. We do not have the rotation by 180^{o} as we are working with an odd n. Then $2^{(n-1)/2} * \frac{n-1}{2}!$ α's conjugate to each of the n possible Dihedral conjugates of R and we have $2^{(n-1)/2} * \frac{n-1}{2}! * n$ α's that have complete overlap with R.

However, we are trying to count the number of RowClasses for which this happens and so we should divide by the number of α's that will belong to each RowClass. In this case we will only have the Dihedral operations, and so we know there will be 2n operations or permutations contained in a single RowClass with overlap. Therefore, within an odd n-note microtonal system, $\frac{2^{(n-1)/2} * (n-1/_2)! * n}{2n}$ unique RowClasses contain complete overlap between R and D.

Finally, to account for the case where we have no overlap between R and D we simply remove from the n! elements of S_n all possible α's that have already been counted to get $n! - 2^{(n-1)/2} * (n - 1/_2)! * n$ α's for which retrograde is needed to generate the RowClass and divide by the number of α's that would be contained within each RowClass which would be 4n, double the Dihedral operations. Thus within an odd n-note microtonal system $\frac{2^{(n-1)/2} * (n-1/_2)! * n}{2n}$ unique RowClasses require R for generation. Combining these two counts leads to Theorem 6.

<u>Theorem 6:</u> Let T = { $D\alpha R$ | $\alpha \in S_n$ } where D is the dihedral group associated with a regular n-gon for odd n and R = (0 n-1) (1 n-2)…($\frac{n-3}{2}$ $\frac{n+1}{2}$) then

$$|T| = \frac{2^{(n-1)/2} * (n - 1/_2)! * n}{2n} + \frac{n! - 2^{(n-1)/2} * (n - 1/_2)! * n}{4n}.$$

Generalization: Trivial Cases, n = 1 & 2

The symmetric groups S_1 and S_2 are so small (of orders 1 and 2 respectively) that our interpretations regarding Dihedral operations break down. Direct calculations, as seen in figures 21 and 22, quickly show that we have a single RowClass in both situations.

n=1: RowClass$_1$(α) for α = [0]

α	=	[0]	=	(0)

Figure 21: RowClass$_1$(α)

n=2: RowClass$_2$(α) for α = [0 1]

α	=	[0 1]	=	(0)
$T^1\alpha$	=	[1 0]	=	(0 1)

Figure 22: RowClass$_2$(α)

Future Work: Inclusion of Cyclic Shift

In our work, we have only considered three symmetric row operations: transposition, inversion, and retrograde. However, within atonal music a rare fourth row operation may be applied, though infrequently, in order to maintain symmetric rows. This fourth and final row operation is the cyclic shift.

For the sake of simplicity, cyclic shift was not included in this research as it would have increased the complexity of the problem. If cyclic shift were to be included as an additional operation this would create more possible combinations of operations, or a larger RowClass cardinality. A larger RowClass cardinality would then in turn mean we would have fewer unique RowClasses as each RowClass could potentially contain additional permutations. For example, take the 4-note microtonal system. Without getting into the details of the cyclic shift operations, $RowClass_4(\alpha_2)$ and $RowClass_4(\alpha_3)$ would have been symmetric. Then for the 4-note system there would have only been two unique RowClasses.

If cyclic shift were to be included, the possible RowClass cardinalities would have to be reconsidered and thus the number of unique RowClasses would need adjustment. These questions are worth examining in future work.

Future Work: Significance for Smallest Orbits

As a final thought, work could also be done in examining the unique RowClasses that are smallest by cardinality. Our RowClasses are orbits in Group theory. In our case of the 12-note system, we have $DS_{12}R$ where D acts as a left action and R acts as a right action on the group S_{12}. The RowClasses then perfectly partition S_{12} without duplicating permutations found in the other RowClasses. This is the same behavior found of orbits in Group theory.

So for these smallest orbits we already know that when we have complete overlap between D and R then a given RowClass's cardinality will equal 24, in the case of the 12-note system. Given this additional structure of overlap, these smaller RowClasses are unique and may have other patterns not yet understood. Furthermore, this added structure may have some connections or resemblance to tonal music given that tonal music is inherently more structured.

While it is certainly true that there may not be any connection between tonal music and these smallest orbits, this is still a question work examining. In this respect, giving additional attention to these smallest orbits is another direction for possible future work.

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