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We discuss large scale geometric properties of Cayley graphs of the integers using different infinite generating sets. We define the notion of $k$-prisms for graphs and study the large scale geometry of graphs with this property. It turns out that graphs with $k$-prisms for all $k$ cannot have property A and thus are infinite dimensional in a strong sense. We give an example of an infinite family of graphs that have $k$-prisms for all $k$ and we also give an example of a space the has property A and "almost has" $k$-prisms.

# AN OBSTRUCTION TO PROPERTY A

by

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# CHAPTER I

# INTRODUCTION

Let $G$ be a group. If we fix a generating set $S$ for $G$, then we can construct a graph $\Gamma = \Gamma(G, S)$ corresponding to $G$ and $S$ by taking the vertices of $\Gamma$ to be the elements of $G$ itself and connecting any two vertices $g$ and $h$ by an edge whenever $gs = h$ for some element $s \in S$. We can consider the graph $\Gamma$ as a metric space by setting the length of each edge to be 1 and taking the path metric on $\Gamma$; this procedure turns the algebraic object $G$ into a geometric object $\Gamma$. The drawback of this approach is that different choices of $S$ can lead to wildly different geometric objects; however, when $G$ is a finitely generated group, any two choices of finite generating sets $S$ and $S'$ will give rise to two graphs that are the same on the large scale, see Chapter II.

In this paper, we investigate the extent to which different choices of infinite generating sets $S$ can change the graph $\Gamma(G, S)$ when $G = \mathbb{Z}$. We are chiefly interested in generating sets that are closed under additive inverses and are closed under taking powers. The simplest such generating set is the collection $S_g = \{1, \pm g, \pm g^2, \pm g^3, \ldots\}$ such that $g > 1$. We denote $\Gamma(\mathbb{Z}, S_g)$ by $C_g$. Edges in the graph $C_g$ connect each vertex to infinitely many other vertices, see Figure I.1. It is not difficult to see that the graphs $C_2$ and $C_3$ are distinct, but the question of whether they are the same on the large scale remains open [Nat11] and motivates our study of the metric properties of these graphs.

In order to determine if the graphs of $C_2$ and $C_3$ are distinct on the large scale we study large scale equivalences. One way in which to show that these graphs are the same in the large scale would be to relate them via a large scale equivalence. On the other hand it would also be possible to distinguish these graphs if we could find a large scale invariant that one of the graphs possesses, but the other does not. Of main interest to us are quasi-isometries and quasi-isometry invariants. One important quasi-isometry invariant we consider is Yu's property A, see definition III.9. We show that the graphs of $C_g$ have interesting structures we name $k$-prisms. This structure allows us to show that the spaces $C_g$ do not have property A, for any $g > 1$, in fact we are able to generalize this result and say that any graph with $k$-prisms for all $k$ does not have property A.

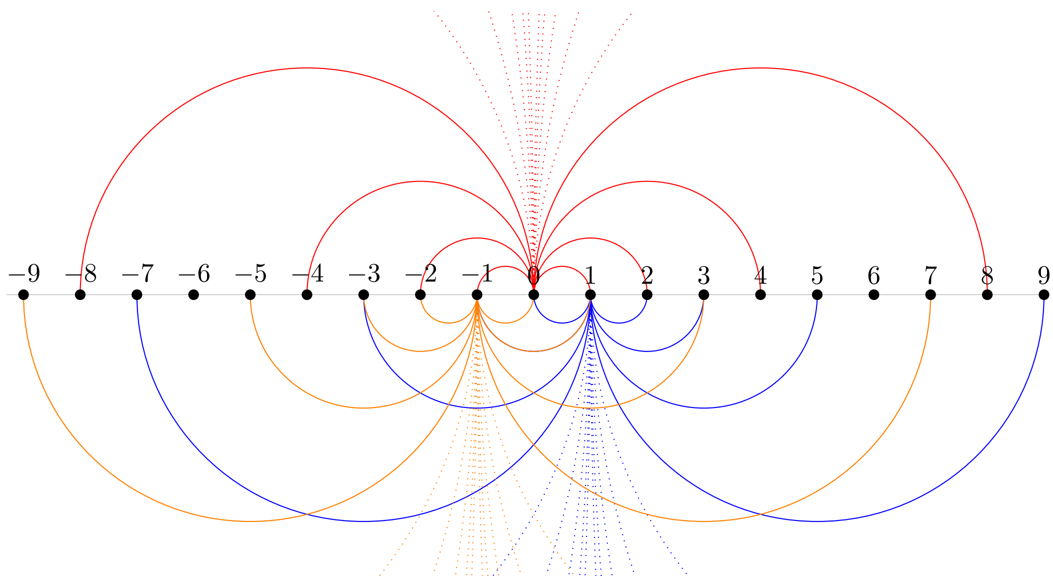


Figure I.1. Cayley Graph of $C_2$. Here we depict the edges emanating from $0, 1, -1$ in the Cayley Graph of $C_2$.

# CHAPTER II

# GEOMETRIC GROUP THEORY

We note that many of the results in this section are well known and we follow the development of [CM17] to delve into the background of geometric group theory. Geometric group theory is an area of mathematics that studies groups by considering them as geometric objects or allowing them to act on metric spaces. One way in which to view a group as a geometric object is to consider the Cayley graph of the group. Let $G$ be a group with a fixed generating set $S$. We assume that the identity is not in $S$ and that $S$ is symmetric in the sense that $s \in S$ implies $s^{-1} \in S$. We define a graph $\Gamma = \Gamma(G, S)$, called the *Cayley graph of $G$ with respect to $S$* as in the introduction: the vertices of $\Gamma$ are in one-to-one correspondence with the elements of $G$, we connect the elements $g$ and $h$ in $G$ with an edge precisely when there is an $s \in S$ such that $gs = h$. We view $G$ as a metric space by taking the edge-length metric $d_S$ on $\Gamma$. More precisely, for $g$ and $h$ in $G$, write $g^{-1}h$ as a word in the elements of $S$ with minimal length, say $g^{-1}h = s_1 s_2 \cdots s_n$ with $s_i \in S$ for all $i = 1, 2 \cdots, n$. Then $g s_1 s_2 \cdots s_n = h$, and there is a path of length $n$ between $g$ and $h$ in the Cayley graph. Thus $d_S(g, h) = n$.

Alternatively, we could define a norm $\|\cdot\|_S$ on $G$ with respect to the generating set $S$ by setting $\|g\|_S = \min\{n \colon s_1 s_2 \cdots s_n = g, s_i \in S\}$. The distance $d_S(g, h) = \|g^{-1}h\|_S$ defines a metric on $G$ called the *left-invariant word metric.*

**Example II.1.** The dihedral group of order 10 can be described as the symmetries of a regular pentagon. It can be presented in terms of generators and relations as

$$D_{2 \cdot 5} = \langle r, s \mid r^5 = 1,\ s^2 = 1,\ rs = sr^{-1} \rangle.$$

With $S = \{r, r^{-1}, s\}$, we note that $S^{-1} = S$ and thus $S$ is symmetric. We compute $d_S(rs, r^2sr^{-1}) = 2$. Indeed, using the relations of $D_{2\cdot 5}$, we find

$$d_S(rs, r^2sr^{-1}) = \|s^{-1}r^{-1}r^2sr^{-1}\|_S = \|s^{-1}rsr^{-1}\|_S = \|s^{-1}sr^{-1}r^{-1}\|_S = \|r^{-2}\|_S = 2.$$

The Cayley graph of the dihedral group of order 10 with $S = \{s, r, r^{-1}\}$ is indicated on the left-hand side of Figure II.1. Notice that vertices are connected by (undirected) edges precisely when the two vertices differ by right multiplication by an element of $\{s, r, r^{-1}\}$. In the right-hand side of that figure, we indicate a geodesic between the elements $rs = sr^4$ and $r^2sr^{-1} = sr^2$.

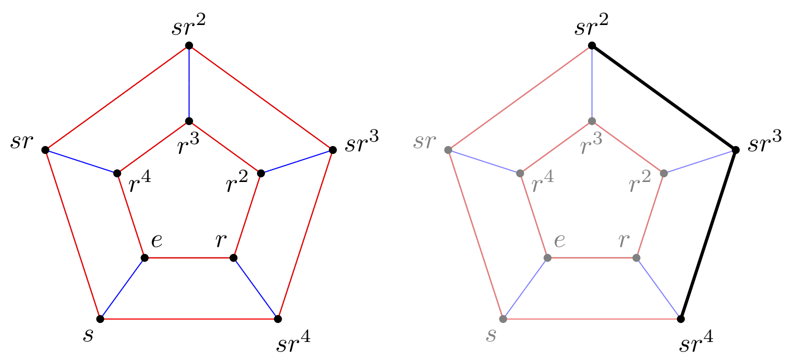


Figure II.1. Cayley Graph of the Dihedral Group of Order 10.

Other interesting types of groups to consider are free groups. First we define the *free group of rank 2*. Define a *word* in the letters $a$ and $b$ to be an arbitrary finite string made up of the symbols $a$,$a^{-1}$,$b$ and $b^{-1}$, for ease of notation we will denote $a^{-1} = A$ and $b^{-1} = B$. For instance the following are examples of words in $a$ and $b$, $aaabbbaaa$, $AABABaab$. We can also consider the empty word, which is the string containing no letters. We define multiplication by concatenating words: $(aab, Bab) \mapsto aabBab$. Based on this definition multiplication by the empty word does not change a string, so the empty word is the identity element in this group. We define a *reduced word* to be a word with the property that the symbols $a$ and $A$ or $b$ and $B$ do not appear next to each other. We can now define the *free group of rank 2* to be the set $F_2 = \{$reduced words in $a$ and $b$ and the empty word$\}$ with multiplication defined by concatenating then reducing. Similarly given an arbitrary set $S$ we can define the *free group* $F(S)$ by first taking each element $s \in S$ and creating an artificial inverse, $s^{-1}$ such that $s, s^{-1}$ do not both belong to $S$. Now like $F_2$ we define the elements of $F(S)$ to be reduced words in the elements of $S$ where the multiplication is also defined by concatenation. If $S$ is a finite set of cardinality $n$ then we denote $F(S)$ by $F_n$.

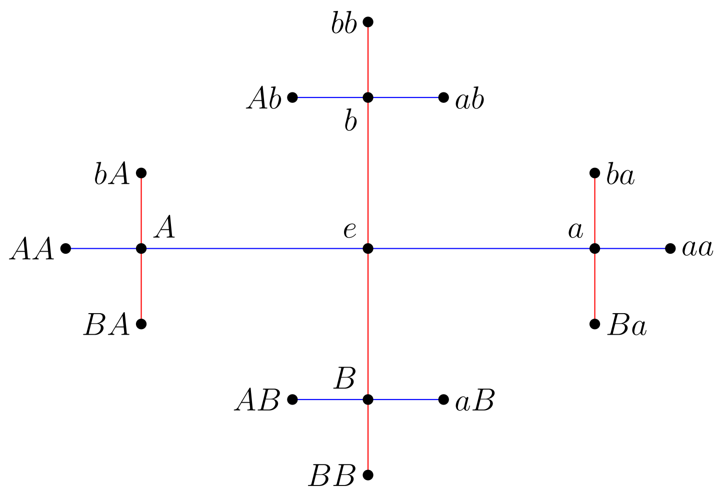


Figure II.2. Cayley Graph of $F_2$

The Cayley graph of $F_2$ can be visualized by drawing the vertex for the identity element (the empty word) first and then connecting it via an edge to the vertices $a, b, A, B$ this process is then continued so each new vertex is connected to three new vertices obtained from right multiplication of $a, b, A, B$. This idea can be extended to any $F_n$. In fact, we see that the number of edges incident to each vertex will be precisely twice the cardinality of $S$.

A natural question to consider is the extent to which different choices of generating sets give rise to different Cayley graphs. To look into this we consider the Cayley graphs of the integers with the generating sets, $\{\pm 1\}$ and $\{\pm 2, \pm 3\}$. We see from figure II.3 that $\Gamma(\mathbb{Z}, \{\pm 1\})$ is a line with each $n \in \mathbb{Z}$ connected via an edge to $n+1$.

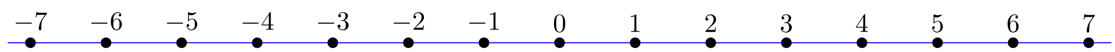


Figure II.3. $\Gamma(\mathbb{Z}, \{\pm 1\})$

The Cayley graph of the integers with the generating set $\{\pm 2, \pm 3\}$ will have the same vertex set as that of $\Gamma(\mathbb{Z}, \{\pm 1\})$. In fact the vertex set for any Cayley graph of the integers is $n \in \mathbb{Z}$. However, now instead of an edge connecting $n$ and $n+1$ there are edges connecting $n$ and $n+2$, $n$ and $n+3$, this can be seen in figure II.4.

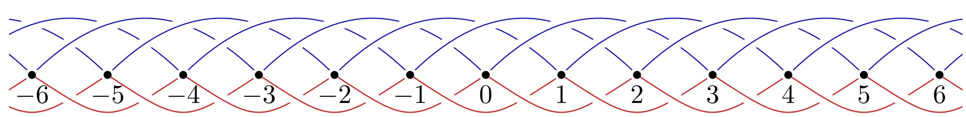


Figure II.4. $\Gamma(\mathbb{Z}, \{\pm 2, \pm 3\})$

While it is true that different choices of generating sets can give rise to vastly different metric spaces, in the case that $G$ is finitely generated, any two finite generating sets give rise to metric spaces that are large scale equivalent. For instance, the Cayley graphs of $\Gamma(\mathbb{Z}, \{\pm 1\})$ and $\Gamma(\mathbb{Z}, \{\pm 2, \pm 3\}$ "look the same on the large scale". We define two main large scale equivalences, bi-Lipschitz equivalences and quasi-isometries.

**Definition II.2.** [CM17] Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A function $f\colon X \to Y$ is called a *bi-Lipschitz embedding* if there is some constant $K \geq 1$ such that for all $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

**Definition II.3.** [CM17] A bi-Lipschitz embedding $f$ is a *bi-Lipschitz equivalence* if it is also surjective.

**Definition II.4.** [CM17] Let $(X, d)$ be a metric space and let $G$ be a group an *action by isometries* of $G$ on $X$ is an isomorphism from $G$ to the group of self isomorphisms of $X$.

**Theorem II.5.** *[CM17] Let $G$ be a finitely generated group, and let $S$ and $S'$ be two finite generating sets for $G$. Then the identity map $f \colon G \to G$ is a bi-Lipschitz equivalence from the metric space $(G, d_S)$ to the metric space $(G, d_{S'})$*

*Proof.* First we note that $G$ acts by isometries on itself with respect to any word metric $d_S$. Thus $d_S(g, h) = d_S(1, g^{-1}h)$, which is simply the word length of $g^{-1}h$ in $S$. Hence it is sufficient to show there exists a constant $K \geq 1$ such that for all $g \in G$

$$\frac{1}{K} d_S(1, g) \leq d_{S'}(1, g) \leq K d_S(1, g).$$

So we are simply comparing word lengths of $g \in G$ for two different generating sets $S$ and $S'$. Now since $S$ is finite we can define,

$$M = \max\{d_{S'}(1, s) | s \in S \cup S^{-1} \geq 1\}.$$

So for any $g \in G$ with word length $n$ in $S$ we have $s_1, s_2, \cdots, s_n \in S \cup S^{-1}$ such that

$g = s_1 s_2 \cdots s_n$. Now from the triangle inequality we obtain:

$$\begin{aligned} d_{S'}(1, g) &= d_{S'}(1, s_1 s_2 \cdots s_n) \\ &\leq d_{S'}(1, s_1) + d_{S'}(s_1, s_1 s_2 s_3 \cdots s_n) \\ &\leq d_{S'}(1, s_1) + d_{S'}(s_1, s_1 s_2) + d_{S'}(s_1 s_2, s_1 s_2 \cdots s_n) \\ &\vdots \\ &\leq d_{S'}(1, s_1) + d_{S'}(s_1, s_1 s_2) + \cdots + d_{S'}(s_1 s_2 \cdots s_{n-1}, s_1 s_2 \cdots s_n). \end{aligned}$$

We have that for $1 \leq k \leq n$,

$$d_{S'}(s_1 s_2 \cdots s_k, s_1 s_2 \cdots s_k s_{k+1}) = d_{S'}(1, (s_1 s_2 \cdots s_k)^{-1} s_1 s_2 \cdots s_k s_{k+1}) = s_{k+1},$$

and thus

$$\begin{aligned} d_{S'}(1, g) &\leq d_{S'}(1, s_1) + d_{S'}(1, s_2) + \cdots + d_{S'}(1, s_n) \\ &\leq M + M + \cdots + M \\ &\leq nM. \end{aligned}$$

Now since the word length of $g$ in $S$ is $n$ we have that for all $g \in G$,

$$d_{S'}(1, g) \leq M d_S(1, g)$$

Letting $K = M$ we have the upper bound. To obtain the lower bound one simply needs to interchange $S$ and $S'$. □

If we consider our previous example of the Cayley graph of the integers with the generating sets $S = \{\pm 1\}$ and $S' = \{\pm 2, \pm 3\}$, then Theorem II.5 tells us that the identity map is a bi-Lipschitz equivalence between the vertex set of $\Gamma(\mathbb{Z}, \{\pm 1\})$ and the vertex set of $\Gamma(\mathbb{Z}, \{\pm 2, \pm 3\})$. Next we wish to extend this idea to the edge sets of these two Cayley graphs. To this end we define the notions of *quasi-isometric embedding* and *quasi-isometry*.

**Definition II.6.** [CM17] Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A function $f\colon X \to Y$ is said to be a *quasi-isometric embedding* if there exist constants $A \geq 1$ and $B \geq 0$ such that for all $x, x' \in X$,

$$\frac{1}{A} d_Y(f(x), f(x')) - B \leq d_X(x, x') \leq A d_Y(f(x), f(x')) + B.$$

**Definition II.7.** [CM17] The metric spaces $(X, d_X)$ and $(Y, d_Y)$ are said to be *quasi-isometric* if there is a quasi-isometric embedding $f\colon X \to Y$, and there is some $K \geq 0$ such that for every $y \in Y$ there is some $x \in X$ so that $d_Y(y, f(x)) \leq K$. In this case, we describe the map $f$ as a *quasi-isometry.*

Again we come back to the question of different generating sets, and this time we ask whether or not the geometric realizations of Cayley graphs are quasi-isometric. First though we give our own proof of the following two propositions from [CM17] about quasi-isometries.

**Proposition II.8.** *The composition of quasi-isometries is a quasi-isometry.*

*Proof.* Let $(X, d_X), (Y, d_Y)$, and $(Z, d_Z)$ be metric spaces, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be quasi-isometries. Then by definition we have that there exists constants $K_1, K_2 \geq 1$ and $C_1, C_2 \geq 0$ such that

$$\frac{1}{K_1} d_X(x_1, x_2) - C_1 \leq d_Y(f(x_1), f(x_2)) \leq K_1 d_X(x_1, x_2) + C_1$$

$$\frac{1}{K_2} d_Y(y_1, y_2) - C_2 \leq d_Z(g(y_1), g(y_2)) \leq K_2 d_Y(y_1, y_2) + C_2.$$

Thus if $x_1, x_2 \in X$ are arbitrary points this implies,

$$\frac{1}{K_2} d_Y(f(x_1), f(x_2)) - C_2 \leq d_Z(g \circ f(x_1), g \circ f(x_2)) \leq K_2 d_Y(f(x_1), f(x_2)) + C_2$$

$$\frac{1}{K_2}[\frac{1}{K_1} d_x(x_1, x_2) - C_1] - C_2 \leq d_Z(g \circ f(x_1), g \circ f(x_2)) \leq K_2[K_1 d_x(x_1, x_2) + C_1] + C_2$$

then letting $K = K_2 K_1$ and $C = K_2 C_1 + C_2$ we have the desired inequalities. □

Before we give the second proposition we first need to introduce quasi-inverses.

**Definition II.9.** [CM17] Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A *quasi-inverse* of a function $f: X \to Y$ is a function $g: Y \to X$ such that, there exists a $k \geq 0$ such that for all $x \in X$ we have $d_X(g(f(x)), x) \leq k$, and for all $y \in Y$ we have $d_Y(f(g(y)), y) \leq k$.

**Proposition II.10.** *A quasi-isometric embedding $f: X \to Y$ is a quasi-isometry if and only if $f$ has a quasi-inverse.*

*Proof.* We begin with the forward implication. That is, we assume that $f: X \to Y$ is a quasi-isometry, and we wish to show that $f$ has a quasi inverse. Then by definition of quasi-isometry there exist a $D > 0$ such that for all $y \in Y$ there is an $x \in X$ such that $d_Y(f(x), y) \leq D$. Given $y \in Y$ choose an $x \in X$ such that $d_Y(f(x), y) \leq D$ and define $g: Y \to X$ by $g(y) = x$. Then for all $y \in Y$

$$d_Y(f(g(y)), y) = d_Y(f(x), y) \leq D,$$

and for all $x \in X$ by definition of quasi-isometric embedding we have

$$d_X(g(f(x)), x) \leq K_1 d_Y(f(g(f(x)), f(x)) + K_1 C_1 \leq K_1 D + K_1 C_1.$$

Taking $K = K_1 D + K_1 C_1$ the forward implication follows. Conversely, we assume that

$f$ has a quasi-inverse. Taking $D = K$ we have for all $y \in Y$ there exists an $x \in X$, $x = g(y)$ so that,

$$d_Y(f(g(y)), y) \leq D = K.$$

Therefore $f$ is a quasi isometry □

The following corollary follows trivially from Proposition II.10.

**Corollary II.11.** *[CM17] If $f\colon X \to Y$ is a quasi-isometry, and $g\colon Y \to X$ is a quasi-inverse of $f$ then $g$ is a quasi-isometry.*

**Theorem II.12.** *[CM17] Let $G$ be a finitely generated group, and let $S$ and $S'$ be two finite generating sets for $G$. Then the geometric realization of the Cayley graph $\Gamma(G, S)$ is quasi-isometric to the geometric realization of the Cayley graph $\Gamma(G, S')$.*

*Proof.* First we note that there exists a quasi-isometry from the geometric realization of any graph to its set of vertices under the path metric. This quasi-isometry is obtained by sending each point lying on an edge to the closest vertex, understanding that there are sometimes two choices. From Theorem II.5 we have that the identity map $f : G \to G$ is a bi-Lipschitz equivalence and thus clearly a quasi-isometry between the vertex sets of $\Gamma(G, S)$ and $\Gamma(G, S')$. Proposition II.10 tells us that such a quasi-isometry must have a quasi-inverse, which is also a quasi-isometry. Then considering the quasi-isometries, $f\colon \Gamma(G, S) \to G$, $g\colon G \to G$, and $h\colon G \to \Gamma(G, S')$ Proposition II.8 tells us that the composition $f \circ g \circ h\colon \Gamma(G, S) \to \Gamma(G, S')$ is the desired quasi isometry. □

**Example II.13.** The Cayley graph $\Gamma(G, S)$ is quasi-isometric to the group $G$ in the metric $d_S$ described above via the identity map, with $A = 1$, $B = 0$, and $K = 0$.

We have been working towards the fundamental theorem of geometric group theory, the *Milnor-Švarc Lemma*, but we need a few more definitions before we can state the theorem.

**Definition II.14.** [CM17] Let $(X, d)$ be a metric space. A *geodesic segment* is an isometric embedding $\gamma\colon [a, b] \to X$, where $a, b \in \mathbb{R}$ and $a \leq b$.

**Definition II.15.** [CM17] A metric space $(X, d)$ is said to be a *geodesic metric space* if for all $x_1, x_2 \in X$ there exists a geodesic segment $\gamma\colon [a, b] \to X$ such that $\gamma(a) = x_1$ and $\gamma(b) = x_2$.

**Definition II.16.** [CM17] A metric space $(X, d)$ is said to be *proper* if for all $x \in X$ and all $r > 0$, the closed ball $B(x, r)$ is a compact subset of $X$.

**Definition II.17.** [CM17] Let $(X, d)$ be a proper geodesic metric space. An action by isometries of the group $G$ on $X$ is said to be *properly discontinuous* if for each compact set $K \subseteq X$ the set $\{g \in G | gK \cap K \neq \emptyset\}$ is finite.

**Definition II.18.** [CM17] Let $(X, d)$ be a proper geodesic metric space. An action on of the group $G$ on $X$ is *cocompact* if for any base point $x_0 \in X$ there is an $R > 0$, such that for any $x \in X$ there is a $g \in G$ such that $B(gx_0, R)$ contains $x$.

**Definition II.19.** [CM17] Let $(X, d)$ be a proper geodesic metric space. An action on of the group $G$ on $X$ is *geometric* if it is a properly discontinuous and cocompact action by isometries.

Now we state the Milnor-Švarc Lemma.

**Theorem II.20.** *[CM17] Let $G$ be a group, and let $(X, d)$ be a proper geodesic metric space. Suppose that $G$ acts geometrically on $X$. Then $G$ is finitely generated and $G$ is quasi-isometric to $X$.*

*Proof.* Let $x_0 \in X$ be fixed, then since $G$ acts cocompactly on $X$ we can pick a $R > 0$ so that the $G$-translates of the closed ball $B(x_0, R)$ cover $X$. Let $S$ be the set of nontrivial $g \in G$ such that $B(gx_0, R) \cap B(x_0, R) \neq \emptyset$. Then since $G$ acts geometrically on $X$, $S$ is nonempty and finite with $S = S^{-1}$. We show that $S$ generates $G$. For ease of notation we write $B = B(x_0, R)$. Then since $(X, d)$ is proper we have that $B$ is compact ,and we can define

$$\begin{aligned} c =& \inf\{d(B, gB) | g \in G, g \neq 1, g \notin S\} \\ =& \inf\{d(x, gy) | x, y \in B, g \in G, g \neq 1, g \notin S\}. \end{aligned}$$

Note that $c > 0$. Now since the action is proper we have that for any $gB$ and $B$ disjoint and $d(B, gB) = D$ there are finitely many translates which are at most distance $D$ away from $B$. Thus $c$ is a minimum of a finite number of positive numbers and thus must be positive.

Now we want to show that $S$ generates $G$, so let $g \in G$ with $g \notin S \cup \{1\}$. Then we have that $d(x_0, gx_0) \geq 2R + c$, which implies $d(x_0, gx_0) \geq R + c$. Hence there exists a $k \geq 2$ such that

$$R + (k-1)c \leq d(x_0, gx_0) \leq R + kc.$$

Now we choose points $x_1, x_2, \cdots, x_{k+1} = gx_0$ on a geodesic segment connecting $x_0$ and $gx_0$ with $d(x_0, x_1) \leq R$ and $d(x_i, x_{i+1}) \leq c$ for $1 \leq i \leq k$. Thus we have $k + 1$ elements in $G$, $1 = g_0, g_1, \cdots, g_k = gx_0$ where $x_{i+1} \in g_i B$ for $0 \leq i \leq k$. Now if we let $s_i = g_{i-1}^{-1} g_i$ for $1 \leq i \leq k$ then,

$$d(B, s_i B) = d(g_{i-1}B, g_i B) \leq d(x_i, x_{i-1}) < c.$$

Thus $s_i \in S$ yet we have

$$s_1 s_2 \cdots s_k = (g_0^{-1} g_1)(g_1^{-1} g_2) \cdots (g_{k-1}^{-1} g_k) = g_k = g,$$

and thus $S$ generates $G$. We have shown that $G$ is finitely generated, and now we need to show that $G$ is quasi-isometric to $X$ under the word metric, $d_S$. To this end we define a map $G \to X$ by $g \mapsto gx_0$. Clearly every point in $X$ is within distance $R$ of some point $gx_0$, that is, there exists a $g \in G$ so that $d_X(gx_0, x) < r$. Now it remains to be seen that,

$$\frac{1}{K} d_S(g,h) - C \leq d(gx_0, hx_0) \leq K d_S(g,s) + C.$$

However, the equalities

$$d(gx_0, hx_0) = d(x_0, (g^{-1}h)x_0)$$
$$d_S(g,h) = d_S(1, g^{-1}h)$$

mean that it suffices to show that there are constants $K \geq 1$ and $C \geq 0$ such that for all $g \in G$,

$$\frac{1}{K} d_S(1,g) - C \leq d(x_0, gx_0) \leq K d_S(1,g) + C.$$

Now let,

$$L = \max\{d(x_0, sx_0) | s \in S\}$$

and define,

$$K = \max\{1/c, L, 2R\}$$
$$C = \max\{1/K, c\}.$$

Now let $g \in G$, then we consider three cases. If $g = 1$ then $d(x_0, gx_0) = d_S(1, g) = 0$, and thus the desired inequality follows. Next we consider $g = s \in S$; then we have $0 \leq d(x_0, sx_0) \leq 2R$ and $d_S(1, s) = 1$. Thus we have,

$$\frac{1}{K} d_s(1, g) - C = \frac{1}{K} - C \leq 0 \leq d(x_0, sx_0) \leq 2R \leq K d_S(1, g) + C.$$

Lastly we suppose that $g \notin S \cup \{1\}$. Then we have from the proof that $S$ generates $G$ that if $k$ is the largest possible integer with $R + (k-1)c \leq d(x_0, gx_0)$ then $d_s(1, g) \leq k$. combining these inequalities yields,

$$R + (d_S(1, g) - 1)c \leq d(x_0, gx_0).$$

Since $R \geq 0$ subtracting $R$ from both sides of the inequality yields,

$$cd_S(1, g) - c \leq d(x_0, gx_0) - R \leq d(x_0, gx_0).$$

The argument from Theorem II.5 yields $d(x_0, gx_0) \leq Ld_S(1, g)$. Thus we have

$$cd_S(1, g) - c \leq d(x_0, gx_0) \leq Ld_S(1, g).$$

Therefore, since $K \geq L$, $K \geq \frac{1}{c}$ and $C \geq c$ the desired inequalities follow, and thus $G$ is quasi-isometric to $X$. □

# CHAPTER III

# COARSE GEOMETRY

In the previous chapter we began to discuss the ideas of large scale equivalences, in particular quasi-isometries and bi-Lipschitz equivalences. We wish to determine whether or not $C_2$ and $C_3$ are quasi-isometric. One way to do this is to find a quasi-isometry invariant to differentiate them. To this end we begin to explore amenability, property A, large scale connectedness and asymptotic dimension, all of which we show are quasi-isometry invariants. We find that they do not distinguish $C_2$ and $C_3$. To begin, we discuss the notion of amenability through the geometric definition introduced by Følner, which concerns the relationship between the volume of a set and the area of its boundary. First we define the the $R$-boundary of a set as follows:

**Definition III.1.** [NY12] Let $G$ be a group with the word length metric, and let $A$ be a subset of $G$. The *$R$-boundary* of $A$ is the set

$$\partial_R A = \{g \in G \setminus A | d(g, A) \leq R\}.$$

**Definition III.2.** [NY12] A finitely generated group $G$ is *amenable* if for every $R > 0$ and $\varepsilon > 0$ there exists a finite set $F \subseteq G$ such that,

$$\frac{\#\partial_R F}{\#F} \leq \varepsilon.$$

**Example III.3.** [NY12] We see that any finite group is amenable. Take $F = G$ then,

$$\frac{\#\partial_R F}{\#F} = 0.$$

**Example III.4.** [NY12] The group $\mathbb{Z}$ is amenable. Given $\varepsilon > 0$ take $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ and let $R > 0$. Then set $F$ equal to the closed ball of radius $nR$ centered at 0, i.e. $F = B(0, nR)$. Then we have $\delta_R F = \{-R(n+1), -Rn - R + 1, \cdots, -Rn - 1\} \cup \{Rn + 1, Rn + 2, \cdots, R(n+1)\}$ and thus,

$$\frac{\#\partial_R F}{\#F} = \frac{2R}{2Rn+1} \leq \frac{1}{n}.$$

Above we have given two examples of groups which are amenable. An example of a group which is not amenable is the free group $\mathbb{F}_n$, [NY12].

We now introduce two nice properties of amenability: that subgroups of finitely generated amenable groups are amenable; and invariance of amenability under quasi-isometries.

**Theorem III.5.** *[NY12] Let $G$ and $H$ be finitely generated groups and, let $H$ be a subgroup of $G$. If $G$ is amenable then $H$ is also amenable.*

*Proof.* We note that amenability is independent of our choice of generating set, thus given a generating set $\Sigma_H$ of $H$ we can pick a generating set $\Sigma_G$ of $G$ so that $\Sigma_H \subseteq \Sigma_G$. Now let $R > 0$, $\varepsilon > 0$, and let $F \subseteq G$ be a finite set such that $\frac{\#\partial_R F}{\#F} \leq \varepsilon$. Consider the left cosets of $H$ in $G$, $\{H_i\}_{i \in \mathbb{N}}$ and let $F_i = F \cap H_i$ for all $i \in I$. It is clear from our choice of generating set that there is some index such that $H_i \subseteq H$ for all $i \in I$. We denote the $R$-boundary with respect to $\Sigma_G$ and $\Sigma_H$ as $\partial_R^G$ and $\partial_R^H$, respectively. Then since $H$ is a subgroup of $G$, we have

$$\#\partial_R^G F \geq \#\partial_R^H F = \sum_{i \in I} \#\partial_R^H F_i.$$

$$\varepsilon \geq \frac{\#\partial_R^G F}{\#F} \geq \sum_{i \in I} \frac{\#\partial_R^H F_i}{\#F} = \sum_{i \in I} \frac{\#F_i}{\#F} \frac{\#\partial_R^H F_i}{\#F_i}.$$

Now we note that there must be some index $i \in I$ such that $\frac{\#\partial_R^H F_i}{\#F_i} \leq \varepsilon$, since otherwise we would obtain a contradiction to the amenability of $G$ through the equality $\sum_{i\in I} \frac{\#F_i}{\#F} = 1$. Therefore since all left cosets of $H$ are isometric, $F_i$ is the desired set in $H$ given $R$ and $\varepsilon$. □

We see that amenability is a quasi-isometry invariant and give an altered proof of the one presented in [NY12].

**Theorem III.6.** *Let $G$ and $H$ be finitely generated groups such that $G$ is quasi-isometric to $H$. Then if $H$ is amenable so is $G$.*

*Proof.* Let $f : G \to H$ be a quasi-isometry with constants $A \geq 1$ and $B \geq 0$. Now since $H$ is amenable given $R > 0$ and $\varepsilon > 0$ let $F_H \subseteq H$ be a finite set which satisfies,

$$\frac{\#\partial_{AR+B} F_H}{\#F_H} \leq \varepsilon$$

and define,

$$F_G = f^{-1}(F_H) \subseteq G.$$

Where $f^{-1}$ is the quasi-inverse of $f$. We note that $F_G$ is finite and the $R$-boundary of $F_G$ is mapped inside of the $(AR + B)$ boundary of $F_H$ and thus,

$$\#\partial_R F_G \leq \#\partial_{AR+B} F_H.$$

Thus, since we have $\#F_H \leq \#F_G$,

$$\frac{\#\partial_R F_G}{\#F_G} \leq \frac{\#\partial_{AR+B} F_H}{\#F_H} \leq \varepsilon.$$

□

We now give an equivalent property for amenability in finitely generated groups. Let $F\gamma$ be the set $\{g\gamma | g \in F\}$. That is, $F\gamma$ is the right translation of $F$ by $\gamma$.

**Theorem III.7.** *[NY12] A finitely generated group is amenable if and only if for every $R > 0$ and every $\varepsilon > 0$ there is a finite set $F \subseteq G$ such that,*

$$\frac{\#(F \Delta F\gamma)}{\#F} \leq \varepsilon$$

*for all $\gamma \in G$ with $|\gamma| \leq R$.*

*Proof.* Let $R > 0$, let $\varepsilon > 0$, and let $F \subseteq G$ be a finite set. We consider sets $F \Delta F\gamma$, the symmetric difference of $F$ and $F_\gamma$, and note that,

$$\bigcup_{|\gamma| \leq R} F\gamma \setminus F = \partial_R F.$$

Thus it is clear we have

$$\#\partial_R F \leq \sum_{|\gamma| \leq R} \#(F\gamma \setminus F) \leq \sum_{|\gamma| \leq R} \#(F\gamma \Delta F). \tag{III.1}$$

Now for all $|\gamma| \leq R$

$$\#(F\gamma\Delta F) = \#(F\gamma \setminus F) + \#(F \setminus F\gamma) = \#(F \setminus F\gamma^{-1}) + \#(F \setminus F\gamma).$$

Since $|\gamma| = |\gamma^{-1}|$,

$$\begin{aligned} \sum_{|\gamma| \leq R} \#(F\gamma\Delta F) &= 2 \sum_{|\gamma| \leq R} \#(F\gamma \setminus F) \\ &\leq 2\#B(e, R) \max_{|\gamma| \leq R} \#(F\gamma \setminus F) \\ &\leq 2\#B(e, R)\#\partial_R F. \end{aligned}$$

From the above and (III.1) we obtain,

$$\frac{1}{2\#B(e,R)}\sum_{|\gamma|\leq R}\#(F\gamma\Delta F)\leq \#\partial_R F\leq \sum_{|\gamma|\leq R}\#(F\gamma\Delta F). \tag{III.2}$$

Now suppose $G$ is amenable, and let $F$ be a finite set which satisfies

$$\frac{\#\partial F}{\#F}\leq \frac{\varepsilon}{2\#B(e,R)},$$

then we have

$$\frac{\#(F\gamma\Delta F)}{\#F}\leq 2\#B(e,R)\frac{\#\partial F}{\#F}\leq \varepsilon.$$

We note that the other direction follows in a similar manner from (III.2). $\square$

A similar result can be shown for left translation by $\gamma$ by considering the map $f: G\to G$ given by $f(g)=g^{-1}$.

**Theorem III.8.** *[NY12] A finitely generated group is amenable if and only if for every $R>0$ and every $\varepsilon>0$, there is a finite set $F\subseteq G$ such that,*

$$\frac{\#(\gamma F\Delta F)}{\#F}\leq \varepsilon$$

*for all $\gamma\in G$ with $|\gamma|\leq R$.*

We now discuss a weaker notion of amenability, called property $A$.

**Definition III.9** ([Yu00])**.** A (discrete) metric space $X$ is said to have *property $A$* if for all $R>0$ and all $\varepsilon>0$, there exists a family $\{A_x\}_{x\in X}$ of finite, non-empty subsets of $X\times\mathbb{Z}_{\geq 1}$ such that

1. for all $x,y\in X$ with $d(x,y)\leq R$, we have $\frac{\#(A_x\Delta A_y)}{\#(A_x\cap A_y)}\leq\varepsilon$, and
2. there exists a $B>0$ such that for every $x\in X$, if $(y,n)\in A_x$, then $d(x,y)\leq B$.

The conditions above tell us that $A_x$ and $A_y$ are almost the same provided that $d(x,y) \leq R$ and are disjoint if $d(x,y) \geq 2B$. We see the equivalent forms of amenability given in Theorems III.7 and III.8 are similar to condition 1 from above. Recall Examples III.3 and III.4 where we showed finite groups and the integers are amenable. We now show that they also have property A.

**Example III.10.** [NY12] Given a finite group $G$, to show that $G$ has property A, let $A_x = G \times 1$. Then it is clear that

$$\frac{\#(A_x \Delta A_y)}{\#(A_x \cap A_y)} = 0,$$

and the second condition is trivially satisfied. Therefore any finite group has property $A$.

**Example III.11.** [NY12] We now show that the integers have property A. Given $\varepsilon > 0$ take $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ and let $R > 0$. Define $A_x = B(x, R(n+1)) \times \{1\}$. Now if $d(x,y) \leq R$ then $\#(A_x \Delta A_y) \leq 2R$ and $\#(A_x \cap A_y) \geq 2Rn$, thus we have,

$$\frac{\#(A_x \Delta A_y)}{\#(A_x \cap A_y)} = \frac{2R}{2Rn} < \varepsilon$$

**Definition III.12.** [NY12] Let $(X,d)$ be a metric space. Then $X$ is called *uniformly discrete* if there exists a constant $C > 0$ such that for any two distinct points $x, y \in X$ we have $d(x,y) \geq C$.

**Definition III.13.** A uniformly discrete metric space $X$ is called *locally finite* if for every $x \in X$ and every $r \geq 0$ we have $\#B(x,r) < \infty$.

**Definition III.14.** A locally finite metric space $X$ is said to have *bounded* geometry if for every $r \in \mathbb{R}$ there exists a number $N(r)$ such that for every $x \in X$ we have $\#B(x,r) \leq N(r)$.

We now show that property A is a quasi-isometry invariant.

**Theorem III.15.** *[NY12] Let $X$ and $Y$ be quasi-isometric, uniformly discrete metric spaces with bounded geometry. Then if $Y$ has property A, $X$ also has property A.*

*Proof.* Let $R > 0, \varepsilon > 0$ and let $f : X \to Y$ be a quasi-isometry with constants $A \geq 1, B \geq 0$. Now let $B_y$ be a family of sets yielding property A for $Y$ with $\frac{\varepsilon}{N}$ and $AR + B$, where $N = sup_{w \in Y} \# f^{-1}(w)$. For all $x \in X$ we define $A_x \subseteq X \times \mathbb{N}$ by,

$$A_x = \{(z, n) \in X \times \mathbb{N} | (f(z), n) \in B_{f(x)}\}.$$

It is clear from the construction that the family $A_x$ is finite and there is an $S' > 0$ so that $A_x \subseteq B(x, S') \times \mathbb{N}$. We also note that if $x, y \in X$ with $d(x, y) \leq R$ then,

$$\#(A_x \Delta A_y) \leq N \#(B_{f(x)} \Delta B_{f(y)}).$$

We also have that $d_Y(f(x), f(y)) \leq AR + B$ and thus,

$$\frac{\#(A_x \Delta A_y)}{\#(B_{f(x)} \cap B_{f(y)})} \leq N \frac{\varepsilon}{N} = \varepsilon.$$

Finally noting that,

$$\#(A_x \cap A_y) \geq \#(B_{f(x)} \cap B_{f(y)})$$

we obtain the desired inequality,

$$\frac{\#(A_x \Delta A_y)}{\#(A_x \cap A_y)} \leq \frac{\#(A_x \Delta A_y)}{\#(B_{f(x)} \cap B_{f(y)})} = \varepsilon.$$

□

**Example III.16.** We note that any tree has property A. To see this we need to show that for every $\varepsilon > 0$ and for all $R > 0$, there exists a family $\{A_x\}_{x \in T}$ of finite subsets of $T \times \mathbb{N}$ and an $r > 0$ such that

1. $\frac{\#(A_x \Delta A_y)}{\#(A_x \cap A_y)} < \varepsilon$, if $d(x, y) \leq R$, and

2. $A_x \subset B(x, r) \times \mathbb{N}$ for every $x \in T$.

Fix a geodesic ray $\gamma_0$, and suppose that for all $x \in T$, $\gamma_x$ is a ray that begins at $x$ with the property that $\gamma_0 \cap \gamma_x$ is a geodesic ray. It is easy to see that such a $\gamma_x$ exists and is unique in a tree. Now let $\varepsilon > 0$ and $R > 0$ be given, and set $r \geq \frac{R}{\varepsilon}$. Put $A_x = \gamma_x([0, r]) \times \{1\}$. Then, since $(A_x \Delta A_y) = (A_x \setminus A_y) \cup (A_y \setminus A_x)$ and both $A_x$ and $A_y$ intersect $\gamma_0$ except at possibly $R$ places, we have that $\#(A_x \Delta A_y) < 2R$. Furthermore, $\#(A_x \cap A_y) \geq \frac{2R}{\varepsilon}$, and so $\frac{\#(A_x \Delta A_y)}{\#(A_x \cap A_y)} \leq \varepsilon$. Finally, it is clear that $A_x \subset B(x, r) \times \mathbb{N}$.

Another quasi-isometry invariant that we consider is being large scale path connected. To begin with we first define what it means to be $k$-path connected.

**Definition III.17.** Let $X$ be a metric space and let $k > 0$ we say that $X$ is *$k$-path connected* if for each pair of points, $x, y \in X$ there exists a finite number of points $x_0, x_1, \cdots, x_n \in X$ such that $x_0 = x$, $x_n = y$ and $d(x_i, x_{i+1}) \leq k$ for each $i$.

**Definition III.18.** Let $X$ be a metric space, we say that $X$ is *large scale path connected* if there exists a $k > 0$ such that $X$ is $k$-path connected.

We now show that the notion of large scale path connected is a quasi isometry invariant.

**Theorem III.19.** *Let $X$ and $Y$ be metric spaces and assume that $X$ is large scale path connected and $X$ and $Y$ are quasi-isometric, then $Y$ is large scale path connected.*

*Proof.* Let $X$ and $Y$ be metric spaces and assume that $X$ is large scale path connected. To see that $Y$ is large scale path connected let $y, y'$ be two points in $Y$. Then since $X$ is quasi-isometric to $Y$ there exists points $x, x' \in X$ such that $d_Y(y, f(x)) \leq c$ and

$d_Y(y', f(x')) \leq c$ for some $c \geq 0$. Then since $X$ is large scale path connected there exists a $k > 0$ such that for all $x, x' \in X$ there exists points $x_0 = x, x_1, \cdots, x_n = x' \in X$ with $d_X(x_i, x_{i+1}) \leq k$ and

$$d_Y(f(x_i), f(x_{i+1})) \leq Ad_X(x_i, x_{i+1}) + AB \leq A(k + B)$$

for some constants $A \geq 1$ and $B \geq 0$. thus there exists points $y = y_0, f(x_0) = y_1, f(x_1) = y_2, \cdots, f(x_n) = y_{n+1}, y' = y_{n+2} \in Y$ so that $d_Y(y_i, y_{i+1}) \leq \max\{A(k + B), c)\}$. Therefore $Y$ is large scale path connected. □

It is quite easy to see that both $C_2$ and $C_3$ are large scale path connected with $k = 1$, for the same reason that the integers themselves are large scale path connected.

The last quasi-isometry invariant that we consider is that of asymptotic dimension, which is a version of covering dimension. We first need to define the $R$-multiplicity of a cover.

**Definition III.20.** [NY12] Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a cover of a metric space, $X$. Given $R > 0$ the *R-multiplicity* of $\mathcal{U}$ is the smallest integer $n$ so that for all $x \in X$ the ball $B(x, R)$ intersects at most $n$ elements of $\mathcal{U}$.

**Definition III.21.** [NY12] Let $X$ be a metric space. The *asymptotic dimension*, denoted asdim $X$, of $X$ is the smallest number $n \in \mathbb{N} \cup \{0\}$ such that for all $R > 0$ there exists a uniformly bounded cover $\mathcal{U} = \{U_i\}_{i \in I}$ with $R$-multiplicity $n + 1$.

We now show that the asymptotic dimension is a quasi-isometry invariant.

**Theorem III.22.** *[NY12] Let $X$ and $Y$ be quasi-isometric metric spaces. Then,*

$$\operatorname{asdim} X = \operatorname{asdim} Y.$$

*Proof.* Let $f : X \to Y$ be a quasi-isometry with constants $A \geq 1$ and $B \geq 0$. let $r > 0$ and $\mathcal{U} = \{U_i\}_{i\in I}$ be a cover of $Y$ satisfying Definition III.21 and set $R = Ar + B$. Now we look at the cover of $X$ given by $\mathcal{V} = \{f^{-1}(U_i)\}_{i\in I}$ and note that this cover is uniformly bounded. Thus it remains to show that the $r$-multiplicity of $\mathcal{V}$ is bounded by $1 + \operatorname{asdim} X$. We note by the construction of our cover we have that $B_X(x, r)$ intersects the same number of elements in $\mathcal{V}$ as $f(B_X(x, r))$ intersects in $\mathcal{U}$. We also have that for all $x \in X$ the image of $B_X(x, r)$ satisfies $f(B_X(x, r)) \subseteq B_Y(f(x), Ar + B) = B_Y(f(x), R)$. Now $B_Y(y, R)$ intersects at most $1 + \operatorname{asdim} Y$ we have that asdim $X \leq$ asdim $Y$. Conversely we can apply this same argument to the quasi-inverse of $f$ to obtain the inequality $\operatorname{asdim} Y \leq \operatorname{asdim} X$ and thus asdim $X =$ asdim $Y$. □

An interesting result from [BF08], tells us that a discrete metric space with bounded geometry and finite asymptotic dimension has property A. However we show in IV.11 that $C_g$ does not have property A for any $g > 1$ and thus it must be the case that $C_2$ and $C_3$ have infinite asymptotic dimension [AGV16].

# CHAPTER IV

# RESULTS

In this chapter we continue to work towards determining whether $(C_2, d_2)$ and $(C_3, d_3)$ are quasi-isometric, a question originally asked by Richard E. Schwartz [Nat11, Problem 6]. To this end we define the notion of metric spaces having $k$-prisms and by using an example of Nowak [Now07] together with the fact that $C_g$ has $k$-prisms we show that $C_g$ fails to have property A for every integer $g > 1$. This can be interpreted as saying that $C_g$ is infinite dimensional in a strong sense, since uniformly discrete metric spaces with finite asymptotic dimension have property A.

The following theorems of Nathanson [Nat11] give a method of computing length in $C_g$.

**Theorem IV.1** ([Nat11, Theorem 3])**.** *Let $g$ be an even positive integer. Every integer $n$ has a unique representation in the form*

$$n = \sum_{i=0}^{\infty} \varepsilon_i g^i$$

*such that*

1. *$\varepsilon_i \in \{0, \pm 1, \pm 2, \ldots, \pm g/2\}$ for all nonnegative integers $i$,*
2. *$\varepsilon_i \neq 0$ for only finitely many nonnegative integers $i$,*
3. *if $|\varepsilon_i| = g/2$, then $|\varepsilon_{i+1}| < g/2$ and $\varepsilon_i \varepsilon_{i+1} \geq 0$.*

*Moreover, $n$ has length $\ell_g(n) = \sum_{i=0}^{\infty} |\varepsilon_i|$.*

**Theorem IV.2** ([Nat11, Theorem 6]). *Let $g$ be an odd positive integer, $g \geq 3$. Every integer $n$ has a unique representation in the form*

$$n = \sum_{i=0}^{\infty} \varepsilon_i g^i$$

*such that*

1. $\varepsilon_i \in \{0, \pm 1, \pm 2, \ldots, \pm(g-1)/2\}$ *for all nonnegative integers $i$,*

2. $\varepsilon_i \neq 0$ *for only finitely many nonnegative integers $i$.*

*Moreover, $n$ has length $\ell_g(n) = \sum_{i=0}^{\infty} |\varepsilon_i|$.*

For any integers $n$ and $g > 2$, Theorems IV.1 and IV.2 give a unique $g$-adic expression for $n$ that realizes a geodesic path from 0 to $n$. Thus there is $N > 0$ such that $n = \sum_{i=0}^{N} \varepsilon_i g^i$, $\varepsilon_N \neq 0$, and $\ell_g(n) = \sum_{i=0}^{\infty} |\varepsilon_i|$. We call $n = \sum_{i=0}^{N} \varepsilon_i g^i$ the *minimal $g$-adic expansion*, and denote it by

$$[n]_g = [\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N].$$

**Example IV.3.** We consider the case when $g = 2$ and $n = 6$. Then we have a representation for $n = 6 = 0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2$. However this representation is not the minimal $g$-adic expansion since $\varepsilon_1 = g/2 = 1$ and $\varepsilon_2 = g/2 = 1$ as well, so the third condition is not satisfied. In this case the unique representation of $n = 6$ is $[6]_2 = [0, -1, 0, 1]$.

**Example IV.4.** Let $\{0, k\}^n$ be the set of vertices of an $n$-dimensional cube at scale $k$ endowed with the $\ell_1$-metric. The disjoint union $\coprod_{n=1}^{\infty} \{0, k\}^n$ can be metrized in such a way that it is a locally finite metric space that fails to have property A [Now07].

In order to utilize Example IV.4, we define the notion of $k$-prisms. We show that a metric space with $k$-prisms has disjoint isometric copies of $\{0, k\}^n$ for all $n$. In particular, the existence of $k$-prisms is an obstruction to having property A.

**Definition IV.5.** Let $k$ be a positive integer. We say that a metric space $X$ has *$k$-prisms* if for any finite set $F \subset X$, there exists an isometry $T$ such that

1. $T(F) \cap F = \emptyset$ and

2. $d(x, T(y)) = k + d(x, y)$ for all $x, y \in F$.

**Remark 1.** If a metric space $X$ has $k$-prisms, then $X$ has $nk$-prisms for all $n \in \mathbb{N}$. To show this we use induction on $n$. The base case is trivial since $X$ having $k$-prisms means $X$ has $1k$-prisms. Now we assume the $X$ has $(n-1)k$-prisms, and we wish to show that $X$ has $nk$-prisms. Since $X$ has $(n-1)k$-prisms for any finite subset $F$, there is an isometry $T_{n-1}$ such that $T_{n-1}(F) \cap F = \emptyset$ and $d(x, T_{n-1}(y)) = (n-1)k + d(x, y)$. Since $X$ has $k$-prisms, we can find an isometry $T_n$ taking the set $F \cup T_{n-1}(F)$ to an isometric copy so that each vertex of $F \cup T_{n-1}(F)$ is at distance $k$ from its image. Thus, if we restrict $T_n$ to the image $T_{n-1}(F)$, we see that $d(x, T_n(T_{n-1}(y))) = k + d(x, T_{n-1}(y)) = k + (n-1)k + d(x, y) = nk + d(x, y)$ and $T_n(F) \cap F = \emptyset$. Therefore $X$ has $nk$-prisms for all $n \geq 1$.

**Lemma IV.6.** *The space $C_g$ has $k$-prisms for every $k$.*

*Proof.* Let $F \neq \emptyset$ be an arbitrary finite subset of $C_g$. By Remark 1, it suffices to find an isometry $T$ such that $F \cap T(F) = \emptyset$ and $d(x, T(y)) = 1 + d(x, y)$ for all $x, y \in F$. We know from Theorem IV.1 and Theorem IV.2 that for any positive integer $g$ we have a unique representation for an integer $x$ of the form, $x = \sum_{i=0}^{\infty} \varepsilon_i g^i$,where the requirements of the $\varepsilon_i$ change depending on whether $g$ is even or odd, and $\varepsilon_i = 0$

for all but finitely many indices. Since $F$ is finite, there is some positive integer $m$ so that $\varepsilon_i = 0$ for all $i > m$ for each $x \in F$. Now, we define an isometry $T$ that takes $x = \sum_{i=0}^{m} \varepsilon_i g^i$ to $T(x) = \sum_{i=0}^{m} \varepsilon_i g^i + g^{m+2}$. We note that choosing $g^{m+1}$ is not sufficient since we cannot expect this expression to be in the canonical form. Clearly $T$ is an isometry. Now we see that

$$d(x, T(y)) = d\left(\sum_{i=0}^{m} \varepsilon_i g^i, \sum_{i=0}^{m} \delta_i g^m + g^{m+2}\right) = \sum_{i=0}^{m} |\delta_i - \varepsilon_i| + 1 = d(x, y) + 1.$$

So $d(x, T(y)) = d(x, y) + 1$, and by construction $F \cap T(F) = \emptyset$. Therefore $C_g$ has $k$-prisms for each $k$. □

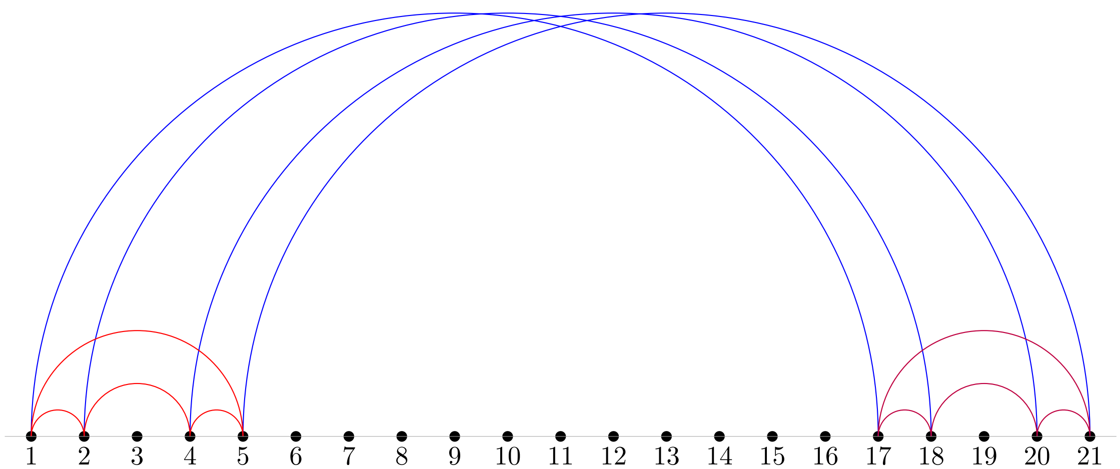


Figure IV.1. Constructing a 1-prism over $\{1, 2, 4, 5\}$ in $C_2$.

In Figure IV.1 we give an example of a 1-prism constructed by the method of this proof over a set in $C_2$.

**Lemma IV.7.** *A metric space $X$ with $k$-prisms for some $k$ contains an infinite geodesic ray.*

*Proof.* Since $X$ has $k$-prisms we can start with a single point $x \in X$. Use the isometry $T_n$ from Remark 1 to obtain an infinite geodesic ray made up of the points $\{x, T_1(x), T_2(T_1(x)), T_3(T_2(T_1(x))), ...\}$. This sequence has the property that $d(x, T_n(x)) = (n-1)k + d(x, T_1(x))$ and $d(T_i(x), T_j(x)) = ik - jk$. Thus we have an infinite geodesic ray. □

**Corollary IV.8.** *The space $C_g$ contains an infinite geodesic ray.*

**Lemma IV.9.** *If $X \neq \emptyset$ has $k$-prisms for some $k$, then $X$ contains an isometric copy of $\{0, k\}^n$. In particular, the space $C_g$ contains an isometric copy of $\{0, k\}^n$ for each $n$ and for each $k$.*

*Proof.* Let $X$ be a metric space with $k$-prisms for some $k$. In order to show that $X$ has an isometric copy of $\{0, k\}^n$, we use induction. To this end we note that if $X$ has $k$-prisms then for any finite subset $F \subset X$, there exists an isometry $T$ such that $T(F) \cap F = \emptyset$ and $d(x, T(y)) = k + d(x, y)$ for all $x, y \in F$.

Taking any $a \in X$, we apply the $k$-prism condition to find a map $T$ such that $d(a, T(a)) = k$. Clearly, $\{a, T(a)\}$ is an isometric copy of $\{0, k\}$. Now we assume that $X$ has an isometric copy, $C$, of $\{0, k\}^{n-1}$. Then $T(C)$ is also an isometric copy of $\{0, k\}^{n-1}$. We claim that $C \cup T(C)$ is an isometric copy of $\{0, k\}^n$. This follows from the fact that $d(x, T(y)) = k + d(x, y)$ for all $x, y \in C \cup T(C)$. Therefore if $X$ has $k$-prisms then $X$ has an isometric copy of $\{0, k\}^n$.

In particular, since $C_g$ has $k$-prisms for all $k$, $C_g$ has an isometric copy of $\{0, k\}^n$ for all $k$. □

**Theorem IV.10.** *Let $X$ be any discrete metric space with $k$-prisms for some $k \geq 1$, then $X$ does not have property $A$.*

*Proof.* First we note that if $Z$ has property A and $Y \subset Z$, then $Y$ has property A,[NY12]. Now given Lemma IV.6 we take $Y$ to be the graph in Example IV.4, which does not have property A. Then since $X$ has $k$-prisms for some $k$, we see that $X$ has an infinite geodesic ray and an isometric copy of $\{0,k\}^n$ for each $n$. Thus we have disjoint $k$-scale $n$-cubes for every $n$ and so $Y \subset X$. Since $Y$ does not have property A, $X$ cannot have property A. $\square$

**Corollary IV.11.** *Let $g > 1$ be an integer. Then $C_g$ fails to have Yu's property A.*

**Definition IV.12.** The direct sum of a sequence $\{G_n\}_{n=1}^{\infty}$ of groups $G_n$ is the set of all sequences $\{g_n\}_{n=1}^{\infty}$ where $g_n \in G_n$ and $g_n$ is equal to the identity element of $G_n$ for all but a finite set of indices. This is denoted $\oplus_{n=1}^{\infty} G_n$.

**Example IV.13.** We remark that the fact that $k$ is fixed in the definition of $k$-prisms is important. We describe a space $X$ with property A that contains an isometric copy of $\{0,n!\}^n$ for each $n$. This space has the property that for every finite subset $F \subset X$ there is a $k$ and an isometry $T : F \to X$ such that $d(x,T(y)) = k + d(x,y)$ for all $x,y \in F$, yet $X$ does not have $k$-prisms for any fixed $k$. Our example is $X = \oplus_{i=1}^{\infty}\mathbb{Z}$ with the metric $d(x,y) = \sum_{i=1}^{\infty} i|x_i - y_i|$, which Yamauchi showed has property A [Yam15], and thus does not have $k$-prisms. To show $X$ contains an isometric copy of $\{0,n!\}^n$ for each $n$, we define an isometry $f\colon \{0,n!\}^n \to X$ by $f(t_1,\ldots,t_n) = (\frac{t_1}{1},\frac{t_2}{2},\ldots,\frac{t_n}{n},0,0,\ldots)$. Then, since each $t_i$ is either 0 or $n!$, it follows that each $t_i$ is divisible by $i$, so $\frac{t_i}{i} \in \mathbb{Z}$. Also, for any $s$ and $t$ in $\{0,n!\}^n$,

$$d_{\ell_1}(s,t) = \sum_{i=1}^{n} |s_i - t_i| = \sum_{i=1}^{n} i\left|\frac{s_i}{i} - \frac{t_i}{i}\right| = d(f(s),f(t)).$$

Thus $X$ contains an isometric copy of $\{0,n!\}^n$.

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