ON DENSITY OF RATIO SETS OF POWERS OF PRIMES

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Introduction

Denote by $\mathbb{R}^+$ and $\mathbb{N}$ the set of all positive real numbers and the natural numbers, respectively. Let $P = \{p_1, \dots, p_n, \dots\}$ be the set of all primes enumerated in increasing order. Denote by $R(A, B) = \{\frac{a}{b}; a \in A, b \in B\}$ the ratio set of $A, B \subset \mathbb{R}^+$ and put $R(A) = R(A, A)$ for $A \subset \mathbb{R}^+$ (cf. [3],[4],[5]). Note that $R(A, B) \neq R(B, A)$ in general, however $R(A, B)$ is dense in $\mathbb{R}^+$ if and only if $R(B, A)$ is dense in $\mathbb{R}^+$.

If we consider the sets of powers of prime numbers $K_\alpha = \{p^\alpha; p \in P\}$ for $\alpha > 0$ and $K = \{p^p; p \in P\}$ it can be shown that they are sparser on the real line than P. More precisely, denote by $K_\alpha(x), K(x)$ and $\pi(x)$, respectively, the number of elements of K_α, K and P, respectively, not exceeding $x \in \mathbb{N}$. Then clearly $K_\alpha(x) = \pi(x^{1/\alpha})$, hence if $\alpha > 1$, then

$$0 \leq \frac{K_\alpha(x)}{\pi(x)} = \frac{\pi(x^{1/\alpha})}{\pi(x)} \sim \alpha x^{\frac{1}{\alpha}-1} \to 0 \ (x \to \infty)$$

by the Prime Number Theorem (cf.[2],p.152). Further let $\alpha > 0$. Choose $\beta > \alpha$ and let x be sufficiently large so that $p_n > \beta$, where $n = K(x)$. Then $K(x) \leq K_\beta(x)$ and hence

$$0 \leq \frac{K(x)}{K_\alpha(x)} \leq \frac{K_\beta(x)}{K_\alpha(x)} = \frac{\pi(x^{1/\beta})}{\pi(x^{1/\alpha})} \sim \frac{\beta}{\alpha} x^{\frac{1}{\beta}-\frac{1}{\alpha}} \to 0 \ (x \to \infty)$$

by the Prime Number Theorem.

On the other hand it is known that $R(P)$ is dense in $\mathbb{R}^+$ (see [2],p.155 and [1]), thus in light of the previous considerations it is of interest to investigate what is the density of sets $R(A, B)$ if we choose for A, B the sets K_α and K, respectively.

It is the purpose of this note to show that the sets $R(K_\alpha, K_\beta)$ and $R(K_\alpha, K)$, respectively are still dense in $\mathbb{R}^+$, while $R(K)$ already consists of isolated points in $\mathbb{R}^+$ as do the larger sets $\mathcal{A}_0 = \{(\frac{m}{n})^m; m, n \in \mathbb{N}\}$ and $\mathcal{A}_1 = \{(\frac{m}{n})^n; m, n \in \mathbb{N}\}$, respectively.

We will say that the set A of values of the sequence $\{a_n\}_{n=1}^\infty$ of positive real numbers is *lacunary* if $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$. The symbol X^d will stand for the set of all accumulation points of $X \subset \mathbb{R}^+$.

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Main Results

An argument similar to that of [2] justifying the density of $R(P)$ in $\mathbb{R}^+$ yields the following

Proposition 1. *The set* $R(K_\alpha, K_\beta)$ *is dense in* $\mathbb{R}^+$ *for every* $\alpha, \beta > 0$.

Proof. Choose $0 < a < b$ arbitrarily. It is not hard to show by the Prime Number Theorem ([2],p.152) that $\frac{\pi((bx)^{1/\alpha})}{\pi((ax)^{1/\alpha})} \to (\frac{b}{a})^{1/\alpha}$ as $x \to \infty$. Further $(\frac{b}{a})^{1/\alpha} > 1$, and consequently there exists an $x_0 > 0$ such that $\pi((bx)^{1/\alpha}) - \pi((ax)^{1/\alpha}) > 0$ for all $x \geq x_0$.

It means that there exist primes p, q such that

$$(aq^\beta)^{1/\alpha} < p < (bq^\beta)^{1/\alpha}, \text{ i.e. } \frac{p^\alpha}{q^\beta} \in (a, b) \cap R(K_\alpha, K_\beta).$$

Hence $R(K_\alpha, K_\beta)$ is dense in $\mathbb{R}^+$. □

Further we have

Proposition 2. *The set* $R(K, K_\alpha)$ *is dense in* $\mathbb{R}^+$ *for all* $\alpha > 0$.

Proof. Choose $0 < a < b$ arbitrarily. It is known that $\frac{p_{n+1}}{p_n} \to 1$ as $n \to \infty$ (see [2],p.153), thus $(\frac{p_{n+1}}{p_n})^\alpha \to 1$ as $n \to \infty$.

Therefore we can find $m \in \mathbb{N}$ such that for each $n \geq m$

$$\frac{p_{n+1}^\alpha}{p_n^\alpha} < \frac{b}{a}. \tag{1}$$

Pick $p_0 \in P$ for which $p_m^\alpha a < p_0^{p_0}$ and put

$$q_0 = \max\{p \in P; p \geq p_m \text{ and } p^\alpha a < p_0^{p_0}\}.$$

Then clearly $q_0 \in P$, say $q_0 = p_s$ $(s \geq m)$, and by (1) we get

$$q_0^\alpha a = p_s^\alpha a < p_0^{p_0} \leq p_{s+1}^\alpha a < p_s^\alpha b = q_0^\alpha b, \text{ hence } \frac{p_0^{p_0}}{q_0^\alpha} \in (a, b) \cap R(K, K_\alpha),$$

which proves the density of $R(K, K_\alpha)$ in $\mathbb{R}^+$. □

The following proposition demonstrates that $R(K)$ cannot be dense:

Proposition 3. *Let the sequence* $0 < a_1 < a_2 < \dots$ *be lacunary. Denote*

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = c \text{ where } c \in (1, +\infty].$$

Then $R(A)$ *is not dense in* $\mathbb{R}^+$*, moreover*

$$R(A)^d \cap (\frac{1}{c}, c) = \emptyset. \tag{2}$$

Proof. Let $t \in R(A)^d$. Clearly $t \neq 1$. Suppose $t > 1$. Then there exist increasing sequences $\{m_k\}_{k=1}^\infty$ and $\{n_k\}_{k=1}^\infty$ of natural numbers such that $m_k > n_k$ $(k \in \mathbb{N})$ and $\frac{a_{m_k}}{a_{n_k}} \to t$ $(k \to \infty)$. The sequence $\{a_n\}_{n=1}^\infty$ is increasing thus

$$t \geq \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = c.$$

If $t < 1$ we analogously get that

$$t \leq \limsup_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{c}.$$

This justifies (2). □

Corollary. *Each point of $R(K)$ is isolated in $\mathbb{R}^+$.*

Proof. It suffices to observe that K is lacunary, moreover $\lim_{n\to\infty} \frac{p_{n+1}^{p_{n+1}}}{p_n^{p_n}} = +\infty$. Therefore in view of (2) we get $R(K)^d \cap (0, +\infty) = \emptyset$. □

Finally we have

Proposition 4. *The sets $\mathcal{A}_0$ and $\mathcal{A}_1$ consist of isolated points in $\mathbb{R}^+$.*

Proof. Let $t \in \mathcal{A}_0^d$. We can easily see that $t \neq 1$. Suppose $t > 1$ and denote by $t_k = (\frac{m_k}{n_k})^{m_k} \in \mathcal{A}_0$ a sequence converging to t such that $t_k > 1$ for all $k \in \mathbb{N}$. There are two possibilities:

i) there exists an $s \in \mathbb{N}$ such that $m_k = n_k + s$ for infinitely many indices k. Then we can find a subsequence $\{t_{k_l}\}_{l=1}^\infty$ of $\{t_k\}_{k=1}^\infty$ for which

$$t_{k_l} = (\frac{n_{k_l} + s}{n_{k_l}})^{n_{k_l}+s} = (1 + \frac{s}{n_{k_l}})^{n_{k_l}} (1 + \frac{s}{n_{k_l}})^s \to e^s, \text{ as } l \to \infty.$$

Consequently $t = e^s$.

ii) for all $s \in \mathbb{N}$ there are only finitely many k's such that $m_k = n_k + s$. Then there exists an increasing sequence $\{k_s\}_{s=1}^\infty$ of natural numbers such that $m_{k_s} > n_{k_s} + s$ for each $s \in \mathbb{N}$. In view of the well-known Bernoulli's inequality we get

$$t_{k_s} = (\frac{m_{k_s}}{n_{k_s}})^{m_{k_s}} > (1 + \frac{s}{n_{k_s}})^{n_{k_s}} \geq 1 + s \text{ for every } s \in \mathbb{N},$$

accordingly $t = +\infty$.

Considering the case $t < 1$ we can similarly get that $e^{-s} \in \mathcal{A}_0^d$ for all $s \in \mathbb{N}$ and $0 \in \mathcal{A}_0^d$. It is now clear that

$$\mathcal{A}_0^d = \{0, +\infty\} \cup \{e^s; s = \pm 1, \pm 2, \dots\},$$

thus $\mathcal{A}_0 \cap \mathcal{A}_0^d = \emptyset$ since the elements of $\mathcal{A}_0$ are positive rationals.

A similar argument applies to $\mathcal{A}_1$. □

Remark. In connection with Proposition 1 and the Corollary Prof. Šalát has posed the following problem:

Characterize sequences $\{\alpha_n\}_{n=1}^\infty$ of positive reals for which the ratio set of $\{p_1^{\alpha_1}, p_2^{\alpha_2}, \dots\}$ is dense in $\mathbb{R}^+$.

References

[1] D.Hobby and D.M.Silberger, *Quotients of Primes*, Amer. Math. Monthly **100** (1993), 50-52.
[2] W.Sierpiński, *Elementary Theory of Numbers*, PWN, Warszawa, 1964.
[3] J.Smítal, *Remarks on ratio sets of sets of natural numbers*, Acta F.R.N. Univ. Comen.-Math. **25** (1971), 93-99.
[4] T.Šalát, *On ratio sets of sets of natural numbers*, Acta Arith. **15** (1969), 273-278.
[5] T.Šalát, *Quotientbasen und (R)-dichte Mengen*, Acta Arith. **19** (1971), 63-78.