SON, BYUNGJAE, Ph.D. Analysis of Classes of Singular Steady State Reaction Diffusion Equations. (2017)
Directed by Dr. Ratnasingham Shivaji. 146 pp.

We study positive radial solutions to classes of steady state reaction diffusion problems on the exterior of a ball with both Dirichlet and nonlinear boundary conditions. We study both Laplacian as well as p-Laplacian problems with reaction terms that are p-sublinear at infinity. We consider both positone and semipositone reaction terms and establish existence, multiplicity and uniqueness results. Our existence and multiplicity results are achieved by a method of sub-supersolutions and uniqueness results via a combination of maximum principles, comparison principles, energy arguments and a-priori estimates. Our results significantly enhance the literature on p-sublinear positone and semipositone problems.

Finally, we provide exact bifurcation curves for several one-dimensional problems. In the autonomous case, we extend and analyze a quadrature method, and in the nonautonomous case, we employ shooting methods. We use numerical solvers in Mathematica to generate the bifurcation curves.

ANALYSIS OF CLASSES OF SINGULAR STEADY STATE REACTION DIFFUSION EQUATIONS

by

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A Dissertation Submitted to
the Faculty of The Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

Greensboro
2017

Approved by

Committee Chair

To my father

APPROVAL PAGE

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ACKNOWLEDGMENTS

First and foremost, I would like to express my sincere gratitude to my advisor Prof. Ratnasingham Shivaji for the continuous guidance, encouragement and support during my Ph.D. study.

I would also like to thank my Ph.D. committee members Prof. Maya Chhetri, Prof. Richard Fabiano, Prof. Thomas Lewis, Prof. Jerome Goddard II (Auburn University Montgomery) and Prof. Stephen Robinson (Wake Forest University). I would especially like to thank Prof. Thomas Lewis for his invaluable advice in the computational studies.

It is also a great pleasure to thank Prof. Pavel Drábek (University of West Bohemia, Czech Republic), Dr. Eunkyung Ko (Ulsan National Institute of Science and Technology, South Korea), Prof. Eun Kyoung Lee (Pusan National University, South Korea), and Prof. Inbo Sim (University of Ulsan, South Korea) for their kind advice and inspiring discussions.

Finally, I wish to express my sincere gratitude to all faculty/staff members and graduate students in the Department of Mathematics and Statistics at the University of North Carolina at Greensboro for their support and friendship.

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INTRODUCTION

Study of nonlinear reaction diffusion equations is of great importance in various applications such as nonlinear heat generation, combustion theory, chemical reactor theory and population dynamics (see [Sem35], [Ske51], [Tur52], [Par61], [Ari69], [FK69], [Sat75], [Fif79], [KJD$^+$79], [Tam79], [ZBLM85], [OL01], [Mur03] and [CC04]). Models are of the form:

$$\begin{cases} u_t = d\Delta u + f(u); \quad x \in \Omega \\ u(x,0) = \psi_0(x); \quad x \in \Omega \\ Bu = 0; \quad x \in \partial\Omega \end{cases} \tag{1.1}$$

where $\Delta u := \operatorname{div}(\nabla u)$ is the Laplacian of u, $d > 0$ is the diffusion coefficient, $\Omega \subset \mathbb{R}^N; N \geq 1$ is a bounded domain with smooth boundary $\partial\Omega$, the reaction term $f : [0,\infty) \to \mathbb{R}$ is a continuous function, and $Bu \equiv u$ or $Bu \equiv \frac{\partial u}{\partial y} + \tilde{c}(u)u$ where $\frac{\partial u}{\partial y}$ is the outward normal derivative of u on $\partial\Omega$ and $\tilde{c} : [0,\infty) \to (0,\infty)$ is a continuous function. For above mentioned applications, u describes a temperature distribution, mass concentration or population density, and only non-negative solutions ($u \geq 0$ in $\overline{\Omega}$) are relevant.

The steady states of (1.1) (if they exist) are of great importance in understanding the dynamics of the solutions of (1.1). For the case when $Bu \equiv u$ (Dirichlet or hostile boundary condition), researchers have built a rich history for nonlinear eigenvalue problems of the form:

$$\begin{cases} -\Delta u = \lambda f(u); \quad x \in \Omega \\ u = 0; \quad x \in \partial\Omega \end{cases} \tag{1.2}$$

where $\lambda = \frac{1}{d}$ is a positive parameter.

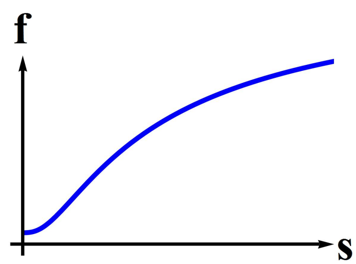

Figure 1. An Example of a Positone Function.

In the case when f is positive at the origin and monotone (see Figure 1), (1.2) is referred to in the literature as a positone problem. Classical examples arise in the theory of nonlinear heat generation (see [KC67] where the authors study the reaction term $f(s) = e^s$) and combustion theory (see [BIS81] where the authors study the reaction term $f(s) = e^{\frac{\kappa s}{\kappa + s}}$ with $\kappa > 0$). For results related to positive solutions of such positone problems, we refer the reader to [KC67], [CL70], [Lae71], [Rab71], [Ama72], [CR73], [Par74], [CR75], [Ama76], [AC78], [GNN79], [WL79], [BIS81], [Lio82], [Shi83], [CS84], [Ang85], [Dan86], [Rab86], [Shi87], [Lin91], [AP93], [Wan94], [HL97], [Du00] and [HS06].

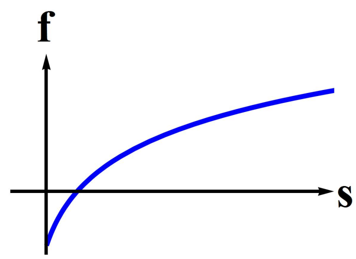

Figure 2. An Example of a Semipositone Function.

In the case when f is negative at the origin and eventually positive (see Figure 2), (1.2) is referred to as a semipositone problem, and is again of great importance in applications, for example, in population dynamics with constant yield harvesting (see [OSS02]). The study of positive solutions of semipositone problems is considerably more challenging since ranges of positive solutions must include regions where f is negative as well as where f is positive. For results related to positive solutions of such semipositone problems, we refer the reader to [BS83], [CS87], [SW87], [CS88], [BCS89], [CS89a], [CS89b], [BS91], [ANZ92], [CGS93], [AAB94], [CGS95], [CHS95], [AHS96], [CS98], [HS99], [OSS02], [HS04], [DS06], [DOS06], [CCSU07], [SY07] and [SY11]. Researchers have also studied the case when $Bu \equiv \frac{\partial u}{\partial y} + \tilde{c}(u)u$ (nonlinear boundary condition). Classical examples arise in combustion theory and population dynamics (see [CC04] and [MWUU06]). For results related to positive solutions of such positone or semipositone problems, we refer the reader to [CC06], [MWUU06], [CC07], [Sht08], [GLS10a], [GLS10b], [GLS11], [GKS14] and [GS14].

In the late 80's, researchers initiated the study of nonlinear eigenvalue problems that arise in the modeling of physical and nature phenomena involving the p-Laplacian operator. A typical model is of the form:

$$\begin{cases} -\Delta_p u = \lambda f(u); \quad x \in \Omega \\ Bu = 0; \quad x \in \partial\Omega \end{cases} \tag{1.3}$$

where $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$ is the p-Laplacian of u. For results related to positive solutions of such positone or semipositone problems, we refer the reader to [Dia85], [DEM89], [Jan93], [DD94], [HSS96], [CDG97], [DDM99], [DKT99], [EG99], [HSC01], [BK02], [CHS03], [COS03], [GR03], [HS03], [JS04], [RS04], [RSY07], [AS09], [Hai11], [KLS11], [LSS11], [DY12], [CKS13], [CDS15] and [SS16].

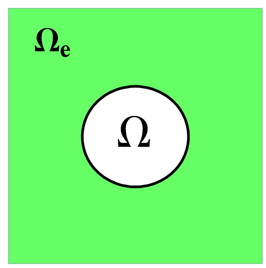

Figure 3. Exterior of a Ball.

Recently, researchers have also considered this study on the exterior of a ball (see Figure 3). However, there are only few results in this direction (see [LSY10], [CSS12], [GLSS13], [KLS13], [San13], [ACSS14], [BKLS14], [KRS15], [MSSS] and [CSSS]).

In this dissertation, the focus is to enrich this literature, namely, consider classes of nonlinear singular eigenvalue problems of the form:

$$\begin{cases} -\Delta_p u = \lambda K(|x|)\frac{f(u)}{u^\alpha}; \quad x \in \Omega_e \\ Bu = 0; \quad |x| = r_0 \\ u(x) \to 0; \quad |x| \to \infty \end{cases} \tag{1.4}$$

where $1 < p < N$, $0 \leq \alpha < 1$, $\Omega_e := \{x \in \mathbb{R}^N \mid |x| > r_0 > 0\}$, the weight function $K : [r_0, \infty) \to (0, \infty)$ is a continuous function such that $K(r) \leq \frac{1}{r^{N+\sigma}}$ for $r \gg 1$ for some $\sigma \in (0, \frac{N-p}{p-1})$, $Bu \equiv u$ or $Bu \equiv \frac{\partial u}{\partial y} + \tilde{c}(u)u$ where $\frac{\partial u}{\partial y}$ is the outward normal derivative of u on $\partial\Omega_e$ and $\tilde{c} : [0, \infty) \to (0, \infty)$ is a continuous function, and the reaction term $f : [0, \infty) \to \mathbb{R}$ is a continuous function. In particular, the main focus is the case when the reaction term f has a p-sublinear growth at infinity, namely:

(H_1) $\lim_{s\to\infty} \frac{f(s)}{s^{p-1+\alpha}} = 0$.

Existence, multiplicity and uniqueness of positive radial solutions for various classes of reaction processes are established.

1.1 Uniqueness Results for Classes of Singular Steady State Reaction Diffusion Equations with Dirichlet Boundary Conditions

Singular steady state reaction diffusion equations with Dirichlet boundary conditions of the form:

$$\begin{cases} -\Delta_p u = \lambda K(|x|)\frac{f(u)}{u^\alpha}; \quad x \in \Omega_e \\ u = 0; \quad |x| = r_0 \\ u \to 0; \quad |x| \to \infty \end{cases} \tag{1.5}$$

are studied. Here p, α, Ω_e, K and f are as before in (1.4). Note that restricting the analysis to positive radial solutions, by a Kelvin type transformation, namely the change of variables $r = |x|$ and $t = \left(\frac{r}{r_0}\right)^{\frac{N-p}{1-p}}$, (1.5) reduces to anaylzing the two point boundary value problems of the form:

$$\begin{cases} -\left(\varphi_p(u')\right)' = \lambda h(t)\frac{f(u)}{u^\alpha}; \quad t \in (0,1) \\ u(0) = 0 = u(1) \end{cases} \tag{1.6}$$

where $\varphi_p(s) := |s|^{p-2}s$ and $h(t) := \left(\frac{p-1}{N-p}\right)^p r_0^p t^{\frac{p(1-N)}{N-p}} K\left(r_0 t^{\frac{1-p}{N-p}}\right)$ (see [KLS13] and Appendix A.1). Then $h : (0,1] \to (0,\infty)$ is a continuous function and there exist $d > 0$ and $\eta \in (0,1)$ such that $h(t) \le \frac{d}{t^\eta}$ for $t \in (0,1]$. Note that since we do not assume $\sigma \ge \frac{N-p}{p-1}$, h may be singular at 0 (i.e., $\lim_{t\to 0+} h(t) = \infty$). For the analysis of positive solutions of (1.6), $p > 1$ is assumed in the rest of the dissertation since the condition $p < N$ is only needed for the Kelvin type transformation.

For the case when $f(0) > 0$ and $0 \le \alpha < 1$, the authors in [KLS13] discussed the existence of a positive solution of (1.6) for all $\lambda > 0$ and multiplicity results for a certain range of λ provided f satisfies certain additional conditions.

Our first result (Theorem 1.1) in this dissertation is to establish the uniqueness of a positive solution for $\lambda \gg 1$ when $p > 1$ and $\alpha = 0$. Assume:

(H_2) $f \in C^1(0,\infty)$ with $\limsup_{s\to 0+} sf'(s) < \infty$

(H_3) $f' > 0$ on $(0,\infty)$

(H_4) $f_* := \inf_{s\in[0,\infty)} f(s) > 0$

(H_5) there exist $q \in (0, p-1)$ and $\sigma_0 > 0$ such that $\frac{f(s)}{s^q}$ is nonincreasing on $[\sigma_0,\infty)$.

We establish:

Theorem 1.1. *Let $p > 1$, $\alpha = 0$ and $(H_1) - (H_5)$ hold. Then (1.6) has a unique positive solution $u \in C^1[0,1]$ for $\lambda \gg 1$.*

Remark. A simple example satisfying the hypotheses in Theorem 1.1 is $f(s) = s^{q^*} + 1$ and $h(t) = \frac{1}{t^\eta}$ where $0 < q^* < p - 1$ and $0 < \eta < 1$. Note that $\frac{f(s)}{s^q}$ is nonincreasing for all $s > 0$ when $q \in [q^*, p-1)$ and nondecreasing for $s \gg 1$ when $q \in (0, q^*)$.

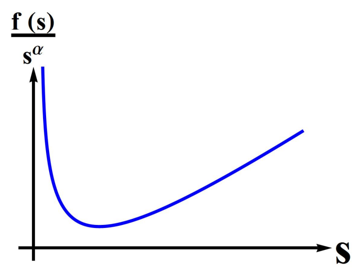

Figure 4. An Example of an Infinite Positone Function.

Our second result (Theorem 1.2) in this dissertation is to extend this uniqueness result when $p = 2$ to the case when $\alpha \neq 0$, which is referred to in the literature as an infinite positone problem (see Figure 4). Assume:

(H_6) $\alpha + \eta < 1$

(H_7) $f \in C^1(0, \infty)$ with $\limsup_{s \to 0+} sf'(s) < \alpha f(0)$

(H_8) there exists $\sigma^* > 0$ such that $\frac{f(s)}{s^\alpha}$ is nonincreasing on (σ^*, ∞).

We establish:

Theorem 1.2. *Let $p = 2$, $0 < \alpha < 1$, (H_1), (H_4) and $(H_6) - (H_8)$ hold. Then* (1.6) *has a unique positive solution $u \in C^2(0,1) \cap C^1[0,1]$ for $\lambda \gg 1$.*

Remark. A simple example satisfying the hypotheses in Theorem 1.2 is $f(s) = e^{\frac{\kappa s}{\kappa+s}}$ and $h(t) = \frac{1}{t^\eta}$ where $\kappa > 0$ and $0 < \alpha + \eta < 1$.

Remark. Theorem 1.2 can be extended to the case when the weight function h also has a singularity at $t = 1$, namely, there exist $d > 0$, $\eta > 0$ and $\eta^* > 0$ such that $h(t) \leq \frac{d}{t^\eta}$ for $t \approx 0$ and $h(t) \leq \frac{d}{(1-t)^{\eta^*}}$ for $t \approx 1$ where $\alpha + \eta < 1$ and $\alpha + \eta^* < 1$.

See [KLS11] for existence and multiplicity results of similar problems on bounded domains Ω. See also [CKS13] for a uniqueness result for $\lambda \gg 1$ when $p = 2$ and Ω is a bounded domain, namely for the following problem:

$$\begin{cases} -\Delta u = \lambda \frac{f(u)}{u^\alpha}; & x \in \Omega \\ u = 0; & x \in \partial\Omega. \end{cases}$$

Theorem 1.2 is an extension of this result to radial solutions on the exterior of a ball.

For the case when $f(0) < 0$ and $0 \leq \alpha < 1$, the existence of a positive solution of (1.6) was established for $\lambda \gg 1$ in [San13]. Further, when $p = 2$ and $\alpha = 0$, the authors in [CSS12] established the uniqueness of a positive solution for $\lambda \gg 1$.

Our third result (Theorem 1.3) in this dissertation is to extend the uniqueness result in [CSS12] to the case when $p > 1$ and $\alpha = 0$. Assume:

(H_9) $h \in C^1(0,1]$ such that h is strictly decreasing on $(0,1]$

(H_{10}) $f(0) < 0$ and $f(s) \to \infty$ as $s \to \infty$.

We establish:

Theorem 1.3. *Let $p > 1$, $\alpha = 0$, $(H_1) - (H_3)$, (H_5) and $(H_9) - (H_{10})$ hold. Then (1.6) has a unique positive solution $u \in C^1[0,1]$ for $\lambda \gg 1$.*

Remark. A simple example satisfying the hypotheses in Theorem 1.3 is $f(s) = s^{q^*} - 1$ and $h(t) = \frac{1}{t^\eta}$ where $0 < q^* < p - 1$ and $0 < \eta < 1$. Note that $\frac{f(s)}{s^q}$ is nonincreasing for $s \gg 1$ when $q \in (q^*, p-1)$.

Remark. When h is strictly decreasing, it turns out that every nonnegative solution of (1.6) is strictly positive in $(0,1)$ (see Lemma 4.1).

Remark. Let $\Omega_A := \{x \in \mathbb{R}^N \mid r_0 < |x| < R_0 \text{ where } 0 < r_0 < R_0\}$, $(H_1) - (H_3)$, (H_5) and (H_{10}) hold. If $r^{\frac{p(N-1)}{p-1}} K(r)$ is strictly increasing for $r \in [r_0, \infty)$, then the boundary value problem:

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u); \quad x \in \Omega_A \\ u = 0; \quad x \in \partial\Omega_A \end{cases} \tag{1.7}$$

has a unique positive radial solution for $\lambda \gg 1$. This follows since the change of variables $r = |x|$ and $t = \left(r^{\frac{N-p}{1-p}} - R_0^{\frac{N-p}{1-p}}\right) / \left(r_0^{\frac{N-p}{1-p}} - R_0^{\frac{N-p}{1-p}}\right)$ reduces the study of (1.7) to analyzing the boundary value problem (1.6) with in fact a nonsingular strictly decreasing h (see Appendix A.3).

1.2 Existence, Multiplicity and Uniqueness Results for Classes of Singular Steady State Reaction Diffusion Equations with Nonlinear Boundary Conditions

Singular steady state reaction diffusion equations with nonlinear boundary conditions of the form:

$$\begin{cases} -\Delta_p u = \lambda K(|x|)\frac{f(u)}{u^\alpha}; \quad x \in \Omega_e \\ \frac{\partial u}{\partial y} + \tilde{c}(u)u = 0; \quad |x| = r_0 \\ u \to 0; \quad |x| \to \infty \end{cases} \tag{1.8}$$

are studied. Here p, α, Ω_e, K, f, $\tilde{c}$ and $\frac{\partial u}{\partial y}$ are as before in (1.4). Note that restricting the analysis to positive radial solutions, (1.8) reduces to anaylzing the two point boundary value problems of the form:

$$\begin{cases} -\left(\varphi_p(u')\right)' = \lambda h(t)\frac{f(u)}{u^\alpha}; \quad t \in (0,1) \\ u'(1) + c(u(1))u(1) = 0 \\ u(0) = 0 \end{cases} \tag{1.9}$$

where h is as before in (1.6) and $c(s) := \frac{p-1}{N-p} r_0 \tilde{c}(s)$ (see [BKLS14] and Appendix A.2). For the analysis of positive solutions of (1.9), $p > 1$ is assumed in the rest of the dissertation since the condition $p < N$ is only needed for the Kelvin type transformation.

When $p = 2$ and $\alpha = 0$, the authors in [BKLS14] established the existence of a positive solution for all $\lambda > 0$, uniquness results for $\lambda \approx 0$ and $\lambda \gg 1$ and multiplicity results for a certain range of λ provided f satisfies:

(H_{11}) $f(s) > 0$ for $s \in [0, \infty)$

and certain additional conditions.

Our next five results (Theorems 1.4 - 1.8) in this dissertation are to extend the results in [BKLS14] to the case when $p = 2$ and $\alpha \neq 0$:

Theorem 1.4. *Let $p = 2$, $0 < \alpha < 1$, (H_1), (H_6) and (H_{11}) hold. Then* (1.9) *has a positive solution $u \in C^2(0, 1] \cap C^1[0, 1]$ for all $\lambda > 0$.*

Theorem 1.5. *Let $p = 2$, $0 < \alpha < 1$, (H_1), $(H_6) - (H_7)$ and (H_{11}) hold. Assume that $c(s)s$ is nondecreasing for $s \in [0, \infty)$. Then* (1.9) *has a unique positive solution $u \in C^2(0, 1] \cap C^1[0, 1]$ for $\lambda \approx 0$.*

Theorem 1.6. *Let $p = 2$, $0 < \alpha < 1$, (H_1), (H_4) and $(H_6) - (H_8)$ hold. Assume that c is bounded and $c(s)s$ is nondecreasing for $s \in [0, \infty)$. Then* (1.9) *has a unique positive solution $u \in C^2(0, 1] \cap C^1[0, 1]$ for $\lambda \gg 1$.*

Remark. A simple example satisfying the hypotheses in Theorems 1.4 - 1.6 is $f(s) = e^{\frac{\kappa s}{\kappa + s}}$, $h(t) = \frac{1}{t^\eta}$ and $c(s) = e^{\frac{s}{1+s}}$ where $\kappa > 0$ and $0 < \alpha + \eta < 1$.

Assume:

(H_{12}) $\frac{f(s)}{s^{1+\alpha}}$ is strictly decreasing for $s \in [0, \infty)$.

The following global uniqueness result is also established:

Theorem 1.7. *Let $p = 2$, $0 < \alpha < 1$, (H_1), (H_6) and $(H_{11})-(H_{12})$ hold. Assume that c is nondecreasing. Then* (1.9) *has a unique positive solution $u \in C^2(0,1] \cap C^1[0,1]$ for all $\lambda > 0$.*

Remark. A simple example satisfying the hypotheses in Theorem 1.7 is $f(s) = e^{\frac{\kappa s}{\kappa + s}}$, $h(t) = \frac{1}{t^\eta}$ and and $c(s) = e^{\frac{s}{1+s}}$ where $0 < \kappa < 4 + 4\alpha$ and $0 < \alpha + \eta < 1$. Note that $\frac{f(s)}{s^{1+\alpha}}$ is strictly decreasing for all $s > 0$.

Next existence of at least two positive solutions of (1.9) is discussed for a certain range of λ under additional assumptions on f. In order to state this result, note from [WJ93] that the boundary value problem:

$$\begin{cases} -w'' = \frac{h(t)}{w^\alpha}; \quad t \in (0,1) \\ w(1) = 2w'(1) \\ w(0) = 0 \end{cases}$$

has a positive solution $w \in C^2(0,1) \cap C^1[0,1]$. Further, it can be proven that this solution is unique. Assume:

(H_{13}) $h_* := \inf_{t \in (0,1]} h(t) > 0$

(H_{14}) there exist a_1 and a_2 such that $a_1 \in (0, \frac{a_2}{8})$ and $\frac{f(s)}{s^\alpha}$ is nondecreasing on (a_1, a_2)

(H_{15}) there exists a_3 such that $a_1 < a_3 < \frac{a_2}{8}$ and $\frac{a_1^{1+\alpha}}{f(a_1)} / \frac{a_3^{1+\alpha}}{f(a_3)} > \frac{16}{h_*} \|w\|_\infty^{1+\alpha}$.

We establish:

Theorem 1.8. *Let $p = 2$, $0 < \alpha < 1$, (H_1), (H_3), (H_6), (H_{11}) and $(H_{13}) - (H_{15})$ hold. Then (1.9) has at least two positive solutions belonging to $C^2(0,1] \cap C^1[0,1]$ for $\lambda \in (\lambda_1, \lambda_2]$ where*

$$\lambda_1 := \frac{16}{h_*} \frac{a_3^{1+\alpha}}{f(a_3)}$$

$$\lambda_2 := \min\left\{ \frac{2a_2}{h_*} \frac{a_3^{\alpha}}{f(a_3)}, \frac{1}{\|w\|_\infty^{1+\alpha}} \frac{a_1^{1+\alpha}}{f(a_1)} \right\}.$$

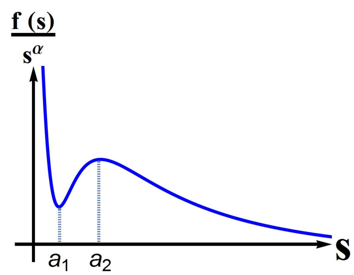

Figure 5. A Function Satisfying (H_{14}).

Remark. A simple example satisfying the hypotheses in Theorem 1.8 is $f(s) = e^{\frac{\kappa s}{\kappa + s}}$, $h(t) = \frac{1}{t^\eta}$ and $c(s) = e^{\frac{s}{1+s}}$ where $\kappa \gg 1$ and $0 < \alpha + \eta < 1$. Note that choosing $a_1 = \frac{1}{2\alpha} - \frac{1}{\kappa} + \frac{1}{2\alpha}\sqrt{1 - \frac{4\alpha}{\kappa}}$, $a_2 = \kappa^2 a_1$ and $a_3 = \kappa a_1$, it is easy to show that for $\kappa \gg 1$ $\frac{f(s)}{s^\alpha}$ is nondecreasing on (a_1, a_2) and $\frac{a_1^{1+\alpha}}{f(a_1)} / \frac{a_3^{1+\alpha}}{f(a_3)} = \frac{1}{\kappa^{1+\alpha}} e^{\frac{a_1 \kappa(\kappa-1)}{(1+a_1)(\kappa+a_1)}} \gg 1$. Hence the hypotheses of Theorem 1.8 hold for $\kappa \gg 1$.

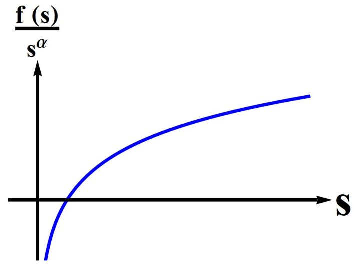

Figure 6. An Example of an Infinite Semipositone Function.

For the semipositone case, when $p = 2$ and $\alpha = 0$, the authors in [BKLS14] established the nonexistence of a positive solution for $\lambda \approx 0$ and the existence result for $\lambda \gg 1$. In [LSS16], the authors extended these results to the case when $p = 2$ and $\alpha \neq 0$, which is referred to in the literature as an infinite semipositone problem (see Figure 6).

Our next two results (Theorems 1.9 - 1.10) in this dissertation are to extend the results in [BKLS14] and [LSS16] to the case when $p > 1$. Assume:

(H_{16}) there exist $A^* > 0$ and $l_0 > 0$ such that $f(s) \geq A^* s^{l_0}$ for $s \gg 1$.

We establish:

Theorem 1.9. *Let $p > 1$, $0 \leq \alpha < 1$ and (H_1) hold. If $f(0) < 0$, then (1.9) has no positive solution for $\lambda \approx 0$.*

Theorem 1.10. *Let $p > 1$, $0 \leq \alpha < 1$, (H_1), (H_6), (H_{10}), (H_{13}) and (H_{16}) hold. Then (1.9) has a positive solution $u \in C^1[0,1]$ for $\lambda \gg 1$ such that $\|u\|_\infty \to \infty$ as $\lambda \to \infty$. Further, if (H_3) and (H_9) hold, then $u(1) > 0$ for any positive solution u, and for any given $a^\# \in (0,1)$, $\inf_{t\in[a^\#,1]} u(t) \to \infty$ as $\lambda \to \infty$.*

Further, in [KRS15], a uniqueness result for $\lambda \gg 1$ was established when $p = 2$ and $\alpha = 0$.

Our final result (Theorem 1.11) in this dissertation is to extend the uniqueness result in [KRS15] to the case when $p > 1$ and $\alpha = 0$. Assume:

(H_{17}) $c(s)$ is nondecreasing for $s \gg 1$.

We establish:

Theorem 1.11. *Let $p > 1$, $\alpha = 0$, $(H_1) - (H_3)$, (H_5), $(H_9) - (H_{10})$ and $(H_{16}) - (H_{17})$ hold. Then (1.9) has a unique positive solution $u \in C^1[0, 1]$ for $\lambda \gg 1$.*

Remark. A simple example satisfying the hypotheses in Theorems 1.9 - 1.11 is $f(s) = s^{q^*} - 1$, $h(t) = \frac{1}{t^\eta}$ and $c(s) = e^s$ where $0 < q^* < p - 1$ and $0 < \eta < 1$.

In Chapter 2, we introduce several preliminary results that will be used to establish existence and multiplicity results. Further, some regularities of positive solutions are investigated. In Chapters 3 - 4, we prove Theorems 1.1 - 1.3 that are results for the case when the Diriclet boundary condition occurs on the boundary of the ball. In Chapters 5 - 6, we prove Theorems 1.4 - 1.11 that are results for the case when a nonlinear boundary condition occurs on the boundary of the ball. In Chapters 7 - 8, we conduct computational studies providing several examples related to our results. Here we generate $(\lambda, \|u\|_\infty)$-bifurcation diagrams, and hence we obtain detailed (exact) information on the structure of positive solutions as the parameter λ changes. In Chapter 9, we discuss the conclusion of this dissertation and several open problems that we plan to study in the near future. In Appendix A, Kelvin transformations are presented. We also prove Theorems 8.1 - 8.4 in Appedix B.

CHAPTER II

PRELIMINARIES

In this chapter, we first outline degree theory and several useful properties (see [Llo78] for more details). Next, a method of sub-supersolutions for (1.9) is established which we use to show existence and multiplicity results for positive solutions. Finally, some regularity results of positive solutions for (1.6) and (1.9) are discussed.

2.1 Degree Theory

Degree Theory in Finite Dimensional Spaces: Let $\Omega \subset \mathbb{R}^N$ be open and bounded and $\phi : \overline{\Omega} \to \mathbb{R}^N$ be a continuously differentiable function. Define $J_\phi(x)$ as the Jacobian determinant of ϕ at $x \in \mathbb{R}^N$ and $Z_\phi := \{x \in \overline{\Omega} \mid J_\phi(x) = 0\}$. For $p \in \mathbb{R}^N \setminus \phi(\partial\Omega)$ and $p \notin \phi(Z_\phi)$, the degree of ϕ at p relative to Ω is defined by

$$deg(\phi, \Omega, p) := \sum_{x \in \phi^{-1}(p)} sign\ J_\phi(x).$$

For $p \in \mathbb{R}^N \setminus \phi(\partial\Omega)$ but $p \in \phi(Z_\phi)$, the degree of ϕ at p relative to Ω is defined to be

$$deg(\phi, \Omega, p) := deg(\phi, \Omega, q)$$

where q is any point in $\mathbb{R}^N \setminus \phi(\partial\Omega)$ such that $q \notin \phi(Z_\phi)$ and $|p - q| < d(p, \phi(\partial\Omega))$. Here $d(p, \phi(\partial\Omega))$ is a distance function from p to $\phi(\partial\Omega)$.

For a continuous function ϕ and $p \in \mathbb{R}^N \setminus \phi(\partial\Omega)$, the degree of ϕ at p relative to Ω is defined to be

$$deg(\phi, \Omega, p) := deg(\psi, \Omega, p)$$

where ψ is a continuously differentiable function such that $|\phi(x) - \psi(x)| < d(p, \phi(\partial\Omega))$ for $x \in \overline{\Omega}$. Then the following properties are satisfied:

Lemma 2.1. *Let I be the identity operator. If $p \in \Omega$, then $deg(I, \Omega, p) = 1$. If $p \notin \overline{\Omega}$, then $deg(I, \Omega, p) = 0$.*

Lemma 2.2. *If $deg(\phi, \Omega, p)$ is defined and non-zero, then there exists $q \in \Omega$ such that $\phi(q) = p$.*

Lemma 2.3. *(Homotopy Invariance Theorem) Let $H : [0, 1] \times \overline{\Omega} \to \mathbb{R}^N$ satisfy $H(t, \cdot) \in C(\overline{\Omega})$ for $t \in [0, 1]$ and $H(s, \cdot) \to H(t, \cdot)$ in $C(\overline{\Omega})$ as $s \to t$. Let $h_t(x) \equiv H(t, x)$. If $p \notin h_t(\partial\Omega)$ for $0 \le t \le 1$, then $deg(h_t, \Omega, p)$ is independent of $t \in [0, 1]$.*

Degree Theory in Infinite Dimensional Spaces: Let $(X, \| \cdot \|)$ be a normed linear space and $\Omega \subset X$ be open and bounded. Let $\phi := I - T$ where $T : \overline{\Omega} \to X$ is completely continuous. For $p \in X \setminus \phi(\partial\Omega)$, the degree of ϕ at p relative to Ω is defined to be

$$deg(\phi, \Omega, p) := deg(I - \hat{T}, \Omega_*, p)$$

where $\hat{T}$ is a continuous function on $\overline{\Omega}$ with finite dimensional range such that $\|T(x) - \hat{T}(x)\| < d(p, \phi(\partial\Omega))$ for $x \in \overline{\Omega}$ and $\Omega_* := \Omega \cap span\{\hat{T}(\overline{\Omega}), p\}$. Then Lemmas 2.1 - 2.3 again hold.

2.2 Methods of Sub-Supersolutions

For (1.6), several methods of sub-supersolutions were established for the case $p > 1$ to establish existence and multiplicity results (see [Cui00] and [LSY09]).

For (1.9), authors in [BKLS14] established a method of sub-supersolutions when $p = 2$ and $\alpha = 0$. In [LSS16], authors established a method of sub-supersolutions for the case $p = 2$ and $0 < \alpha < 1$. These results are extended to the case $p > 1$ by requiring subsolutions to be bounded below by a power of the distance function. In particular, by a subsolution of (1.9), we mean a function $\psi \in C^1[0,1]$ that satisfies $\psi(t) \geq Dd(t, \partial\Omega)^{\kappa^*}$ for some $D > 0$ and $\kappa^* > 0$ such that $\alpha\kappa^* + \eta < 1$ and

$$\begin{cases} -\left(\varphi_p(\psi')\right)' \leq \lambda h(t)\frac{f(\psi)}{\psi^\alpha}; \quad t \in (0,1) \\ \psi'(1) + c(\psi(1))\psi(1) \leq 0 \\ \psi(0) = 0 \end{cases} \tag{2.1}$$

where $\Omega := (0,1)$. By a supersolution of (1.9), we mean a function $\phi \in C^1[0,1]$ that satisfies $\phi(t) > 0$ for $t \in (0,1)$ and

$$\begin{cases} -\left(\varphi_p(\phi')\right)' \geq \lambda h(t)\frac{f(\phi)}{\phi^\alpha}; \quad t \in (0,1) \\ \phi'(1) + c(\phi(1))\phi(1) \geq 0 \\ \phi(0) = 0. \end{cases} \tag{2.2}$$

Now, for a subsolution ψ and a supersolution ϕ such that $\psi \le \phi$, we define the operator $T : C[0,1] \to C[0,1]$ related to (1.9) by

$$Tw(t) := \int_0^t \varphi_p^{-1}\left(\lambda \int_s^1 h(r)\frac{f(\gamma(r,w))}{\gamma(r,w)^\alpha}dr - \varphi_p(\bar{c}(w(1)))\right)ds \tag{2.3}$$

where $\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $\bar{c} : \mathbb{R} \to \mathbb{R}$ are defiend by

$$\gamma(t,s) := \begin{cases} \phi(t); & s > \phi(t) \\ s; & \psi(t) \le s \le \phi(t) \\ \psi(t); & s < \psi(t) \end{cases} \quad \text{and} \quad \bar{c}(s) := \begin{cases} c(\phi(1))\phi(1); & s > \phi(1) \\ c(s)s; & \psi(1) \le s \le \phi(1) \\ c(\psi(1))\psi(1); & s < \psi(1). \end{cases}$$

It follows that T satisfies the following properties:

Lemma 2.4. *T is completely continuous.*

Proof. Let $\{v_n\}$ be a bounded sequence in $C[0,1]$. Let $f^*(s) := \max_{0\le r\le s}|f(r)|$ and $c^*(s) := \max_{0\le r\le s}\bar{c}(r)$. Then we have

$$\begin{aligned} |(Tv_n)'(t)| &\le \varphi_p^{-1}\left(\lambda \int_t^1 h(s)\frac{|f(\gamma(s,v_n))|}{\gamma(s,v_n)^\alpha}ds + \varphi_p(\bar{c}(v_n(1)))\right) \\ &\le \varphi_p^{-1}\left(\lambda f^*(\|\phi\|_\infty)\int_0^1 \frac{h(s)}{\psi^\alpha}ds + \varphi_p(c^*(\|\phi\|_\infty))\right) \\ &\le \varphi_p^{-1}\big(\lambda D_1 f^*(\|\phi\|_\infty) + \varphi_p(c^*(\|\phi\|_\infty))\big) \end{aligned}$$

where $D_1 := \frac{1}{D^\alpha}\left(\int_0^{\frac{1}{2}} \frac{h(s)}{s^{\alpha\kappa^*}}ds + \int_{\frac{1}{2}}^1 \frac{h(s)}{(1-s)^{\alpha\kappa^*}}ds\right)$ and $\|\phi\|_\infty := \max_{t\in[0,1]}|\phi(t)|$. This implies that $\{\|(Tv_n)'\|_\infty\}$ is uniformly bounded, and hence $\{\|Tv_n\|_\infty\}$ is uniformly bounded. By the Arzela-Ascoli Theorem, $\{T(v_n)\}$ has a convergent subsequence in $C[0,1]$.

Next we show that T is continuous. Let $\{w_n\} \subset C[0,1]$ be such that $w_n \to w$ as $n \to \infty$ for some $w \in C[0,1]$. Since $\bar{c}$ is continuous, $\varphi_p(\bar{c}(w_n(1)))$ converges to $\varphi_p(\bar{c}(w(1)))$ as $n \to \infty$. We also have

$$
\begin{aligned}
&\int_t^1 h(s)\Big|\frac{f(\gamma(s,w_n))}{\gamma(s,w_n)^\alpha} - \frac{f(\gamma(s,w))}{\gamma(s,w)^\alpha}\Big|ds \\
&\le \int_t^1 h(s)\frac{|f(\gamma(s,w_n)) - f(\gamma(s,w))|}{\gamma(s,w_n)^\alpha}ds + \int_t^1 h(s)\Big|\frac{f(\gamma(s,w))}{\gamma(s,w_n)^\alpha} - \frac{f(\gamma(s,w))}{\gamma(s,w)^\alpha}\Big|ds \\
&\le D_1\|f(\gamma(\cdot,w_n)) - f(\gamma(\cdot,w))\|_\infty + f^*(\|\phi\|_\infty)\int_0^1 h(s)\Big|\frac{1}{\gamma(s,w_n)^\alpha} - \frac{1}{\gamma(s,w)^\alpha}\Big|ds.
\end{aligned}
$$

Since the last term converges to 0 as $n \to \infty$ by the Lebesgue Dominated Convergence Theorem, $\int_t^1 h(s)\frac{f(\gamma(s,w_n))}{\gamma(s,w_n)^\alpha}ds$ converges uniformly to $\int_t^1 h(s)\frac{f(\gamma(s,w))}{\gamma(s,w)^\alpha}ds$ as $n \to \infty$. For each n and for $t \in [0,1]$, we have

$$
\varphi_p(\bar{c}(w_n(1))) \le \varphi_p(c^*(\|\phi\|_\infty)) \quad \text{and} \quad \int_t^1 h(s)\frac{f(\gamma(s,w_n))}{\gamma(s,w_n)^\alpha}ds \le D_1 f^*(\|\phi\|_\infty).
$$

For $t \in [0,1]$, we also have

$$
\varphi_p(\bar{c}(w(1))) \le \varphi_p(c^*(\|\phi\|_\infty)) \quad \text{and} \quad \int_t^1 h(s)\frac{f(\gamma(s,w))}{\gamma(s,w)^\alpha}ds \le D_1 f^*(\|\phi\|_\infty).
$$

Let $D_2 := 2\big(\lambda D_1 f^*(\|\phi\|_\infty) + \varphi_p(c^*(\|\phi\|_\infty))\big)$. Since φ_p^{-1} is uniformly continuous on $[-D_2, D_2]$, we obtain that $Tw_n(t)$ converges uniformly to $Tw(t)$ as $n \to \infty$. This implies that $\|Tw_n - Tw\|_\infty \to 0$ as $n \to \infty$, so T is continuous. Hence Lemma is proven. $\square$

Now we establish:

Lemma 2.5. *Assume that there exist a subsolution ψ and a supersolution ϕ of* (1.9) *such that $\psi \le \phi$ on* $[0,1]$. *Then* (1.9) *has at least one solution $u \in C^1[0,1]$ satisfying $\psi \le u \le \phi$ on* $[0,1]$.

Proof. Note that T defined by (2.3) is bounded on $C[0,1]$ since

$$|Tw(t)| \le \int_0^t \varphi_p^{-1}\left(\lambda \int_s^1 h(r)\frac{|f(\gamma(r,w))|}{\gamma(r,w)^\alpha}dr + \varphi_p(\bar{c}(w(1)))\right) ds \le \varphi_p^{-1}(D_2).$$

This implies that there exists $D^* \gg 1$ such that $(I-T)(w) \neq 0$ for any $w \in C[0,1]$ satisfying $\|w\|_\infty = D^*$. By Lemmas 2.1 - 2.4, we have

$$deg(I-T, B_{D^*}(0), 0) = deg(I, B_{D^*}(0), 0) = 1$$

and there exists $w_0 \in C[0,1]$ such that $T(w_0) = w_0$. It is easy to show $w_0 \in C^1[0,1]$.

Now we claim that $w_0(t) \in [\psi(t), \phi(t)]$ for $t \in [0,1]$. If our claim is true, then $\gamma(t, w_0(t)) = w_0(t)$ for $t \in [0,1]$. This implies that w_0 is a positive solution of (1.9), and hence Lemma is proven. To show $w_0(t) \ge \psi(t)$ for $t \in [0,1]$, assume that there exists $t_0 \in (0,1]$ such that $\psi(t_0) > w_0(t_0)$. Then two cases are followed:

(I) there exists $(a,b) \subset (0,1)$ such that $\psi(t) > w_0(t)$ in (a,b), $\psi(a) = w_0(a)$ and $\psi(b) = w_0(b)$

(II) there exists $a \in (0,1)$ such that $\psi(t) > w_0(t)$ in $(a,1]$ and $\psi(a) = w_0(a)$.

For the case (I), there exists $\tilde{t} \in (a,b)$ such that $\psi'(\tilde{t}) = w_0'(\tilde{t})$ and $\psi'(t) > w_0'(t)$ for $t \in (a,\tilde{t})$. However, for $t \in (a,\tilde{t})$ we have

$$\varphi_p(\psi'(t)) - \varphi_p(w_0'(t)) \leq \lambda \int_t^{\tilde{t}} h(s) \left(\frac{f(\psi)}{\psi^\alpha} - \frac{f(\gamma(s,w_0))}{\gamma(s,w_0)^\alpha} \right) ds = 0.$$

Since φ_p is increasing, we obtain $\psi'(t) \leq w_0'(t)$ for $t \in (a,\tilde{t})$. This is a contradiction. For the case (II), we have $\psi'(1) - w_0'(1) \leq -c(\psi(1))\psi(1) + \bar{c}(w_0(1)) = 0$ since $\psi(1) > w_0(1)$. Thus there exists $\hat{t} \in (a,1]$ such that $\psi'(\hat{t}) = w_0'(\hat{t})$ and $\psi'(t) > w_0'(t)$ for $t \in (a,\hat{t})$. This is a contradiction by the same argument of the case (I). Hence $w_0(t) \geq \psi(t)$ for $t \in [0,1]$. By a similar argument, we can show that $w_0(t) \leq \phi(t)$ for $t \in [0,1]$. □

2.3 Regularity of Positive Solutions

In this section, we show that positive solutions of (1.6) and (1.9) have at least C^{1,α^*}-regularities for some $\alpha^* \in (0,1)$.

Lemma 2.6. *Let $\alpha = 0$. If u is a positive solution of (1.6) or (1.9), then $u \in C^{1,\alpha^*}[0,1]$ for some $\alpha^* \in (0,1)$. In particular, $u \in C^2(0,1) \cap C^{1,\alpha^*}[0,1]$ for $1 < p \leq 2$.*

Proof. Let $t_m \in (0,1)$ be such that $u(t_m) = \|u\|_\infty$. From (1.6) or (1.9), we have

$$u'(t) = \varphi_p^{-1}\left(\lambda \int_t^{t_m} h(s) f(u) ds \right).$$

If $1 < p \leq 2$, then u' is differentiable on $(0,1)$. By the Mean Value Theorem, we have

$$\begin{aligned} |u'(t_2) - u'(t_1)| &= \left|\varphi_p^{-1}\left(\lambda \int_{t_2}^{t_m} h(s)f(u)ds\right) - \varphi_p^{-1}\left(\lambda \int_{t_1}^{t_m} h(s)f(u)ds\right)\right| \\ &\leq \left|\frac{1}{p-1}\lambda^{\frac{1}{p-1}}[f^*(\|u\|_\infty)]^{\frac{1}{p-1}}\left(\int_0^1 h(s)ds\right)^{\frac{2-p}{p-1}} \int_{t_1}^{t_2} h(s)ds\right| \\ &\leq D_3\lambda^{\frac{1}{p-1}}[f^*(\|u\|_\infty)]^{\frac{1}{p-1}}\left|{t_2}^{1-\eta} - {t_1}^{1-\eta}\right| \\ &\leq D_3\lambda^{\frac{1}{p-1}}[f^*(\|u\|_\infty)]^{\frac{1}{p-1}}|t_2 - t_1|^{1-\eta} \end{aligned} \tag{2.4}$$

for any t_1 and $t_2 \in [0,1]$ where $D_3 := \frac{d}{(p-1)(1-\eta)}\left(\int_0^1 h(s)ds\right)^{\frac{2-p}{p-1}}$. Hence $u \in C^2(0,1) \cap C^{1,\alpha^*}[0,1]$ where $\alpha^* = 1 - \eta$.

For the case $p > 2$, noting that $|\varphi_p^{-1}(b) - \varphi_p^{-1}(a)| \leq 2^{\frac{p-2}{p-1}}|\varphi_p^{-1}(b-a)|$ for $a, b \in \mathbb{R}$, we have

$$\begin{aligned} |u'(t_2) - u'(t_1)| &= \left|\varphi_p^{-1}\left(\lambda \int_{t_2}^{t_m} h(s)f(u)ds\right) - \varphi_p^{-1}\left(\lambda \int_{t_1}^{t_m} h(s)f(u)ds\right)\right| \\ &\leq 2^{\frac{p-2}{p-1}}\left|\lambda \int_{t_1}^{t_2} h(s)f(u)ds\right|^{\frac{1}{p-1}} \\ &\leq D_4\lambda^{\frac{1}{p-1}}[f^*(\|u\|_\infty)]^{\frac{1}{p-1}}\left|{t_2}^{1-\eta} - {t_1}^{1-\eta}\right|^{\frac{1}{p-1}} \\ &\leq D_4\lambda^{\frac{1}{p-1}}[f^*(\|u\|_\infty)]^{\frac{1}{p-1}}|t_2 - t_1|^{\frac{1-\eta}{p-1}} \end{aligned} \tag{2.5}$$

for any t_1 and $t_2 \in [0,1]$ where $D_4 := \frac{d}{1-\eta}2^{\frac{p-2}{p-1}}$. This implies that $u \in C^{1,\alpha^*}[0,1]$ where $\alpha^* = \frac{1-\eta}{p-1}$. Hence Lemma is proven. $\square$

Lemma 2.7. *Let $0 < \alpha < 1$ and (H_6) hold. Assume that $f(s) > 0$ for $s \in (0, \infty)$. If u is a positive solution of (1.6) or (1.9), then $u \in C^{1,\alpha^*}[0,1]$ for some $\alpha^* \in (0,1)$. In particular, $u \in C^2(0,1) \cap C^{1,\alpha^*}[0,1]$ for $1 < p \leq 2$.*

Proof. Since $f(s) > 0$ for $s \in (0, \infty)$, it is easy to show that $u(t) \geq D_5 d(t, \partial\Omega)$ for some $D_5 > 0$ where $\Omega := (0,1)$. Let $t_m \in (0,1)$ be such that $u(t_m) = \|u\|_\infty$. From (1.6) or (1.9), we have

$$u'(t) = \varphi_p^{-1}\left(\lambda \int_t^{t_m} h(s)\frac{f(u)}{u^\alpha} ds\right).$$

If $1 < p \leq 2$, then u' is differentiable on $(0,1)$. By arguments similar to those in Lemma 2.6, we have

$$\begin{aligned}
|u'(t_2) - u'(t_1)| &= \left|\varphi_p^{-1}\left(\lambda \int_{t_2}^{t_m} h(s)\frac{f(u)}{u^\alpha} ds\right) - \varphi_p^{-1}\left(\lambda \int_{t_1}^{t_m} h(s)\frac{f(u)}{u^\alpha} ds\right)\right| \\
&\leq \left|\frac{D_5^{\frac{-\alpha}{p-1}}}{p-1}\lambda^{\frac{1}{p-1}}[f^*(\|u\|_\infty)]^{\frac{1}{p-1}}\left(\int_0^1 \frac{h(s)}{d(s,\partial\Omega)^\alpha} ds\right)^{\frac{2-p}{p-1}} \int_{t_1}^{t_2} \frac{h(s)}{d(s,\partial\Omega)^\alpha} ds\right| \\
&\leq D_6 \lambda^{\frac{1}{p-1}}[f^*(\|u\|_\infty)]^{\frac{1}{p-1}}|t_2 - t_1|^{1-\alpha-\eta}
\end{aligned}$$

for any t_1 and $t_2 \in [0,1]$ where $D_6 := \frac{2^{\alpha+\eta} D_5^{\frac{-\alpha}{p-1}}}{(p-1)(1-\alpha-\eta)}\left(\int_0^1 \frac{h(s)}{d(s,\partial\Omega)^\alpha} ds\right)^{\frac{2-p}{p-1}}$. Hence $u \in C^2(0,1) \cap C^{1,\alpha^*}[0,1]$ where $\alpha^* = 1 - \alpha - \eta$.

For the case $p > 2$, we also have

$$
\begin{aligned}
|u'(t_2) - u'(t_1)| &= \left|\varphi_p^{-1}\left(\lambda \int_{t_2}^{t_m} h(s)\frac{f(u)}{u^\alpha}ds\right) - \varphi_p^{-1}\left(\lambda \int_{t_1}^{t_m} h(s)\frac{f(u)}{u^\alpha}ds\right)\right| \\
&\leq 2^{\frac{p-2}{p-1}}\lambda^{\frac{1}{p-1}}\left|\int_{t_1}^{t_2} h(s)\frac{f(u)}{u^\alpha}ds\right|^{\frac{1}{p-1}} \\
&\leq 2^{\frac{p-2}{p-1}}D_5^{\frac{-\alpha}{p-1}}\lambda^{\frac{1}{p-1}}[f^*(\|u\|_\infty)]^{\frac{1}{p-1}}\left|\int_{t_1}^{t_2} \frac{h(s)}{d(s,\partial\Omega)^\alpha}ds\right|^{\frac{1}{p-1}} \\
&\leq D_7\lambda^{\frac{1}{p-1}}[f^*(\|u\|_\infty)]^{\frac{1}{p-1}}|t_2 - t_1|^{\frac{1-\alpha-\eta}{p-1}}
\end{aligned}
$$

for any t_1 and $t_2 \in [0,1]$ where $D_7 := \frac{dD_5^{\frac{-\alpha}{p-1}}}{1-\alpha-\eta}2^{\frac{2p-4+\alpha+\eta}{p-1}}$. This implies that $u \in C^{1,\alpha^*}[0,1]$ where $\alpha^* = \frac{1-\alpha-\eta}{p-1}$. Hence Lemma is proven. □

Remark. Let $1 < p \leq 2$. If u is a positive solution of (1.9) and $u(1) > 0$, then $u \in C^2(0,1] \cap C^{1,\alpha^*}[0,1]$. Further, if u is a positive solution of (1.6) or (1.9) and if h is nonsingular at 0 and $\alpha = 0$, then $u \in C^2[0,1]$.

CHAPTER III

PROOFS OF THEOREMS 1.1 - 1.2

3.1 Proof of Theorem 1.1

We first define $G : [\sigma_0, \infty) \to \mathbb{R}$ by $G(s) := \frac{s}{[f(s)]^{\frac{1}{p-1}}}$. Then G is strictly increasing and $\lim_{s\to\infty} G(s) = \infty$ by (H_1), (H_3) and (H_5). Further, G^{-1} satisfies:

Lemma 3.1. *Let (H_1), (H_3) and (H_5) hold. For each $C > 0$, there exist $L_1 > 0$ and $L_2 > 0$ (independent of λ) such that*

$$L_1 G^{-1}(\lambda^{\frac{1}{p-1}}) \leq G^{-1}(\lambda^{\frac{1}{p-1}} C) \leq L_2 G^{-1}(\lambda^{\frac{1}{p-1}})$$

for $\lambda \gg 1$ where $L_1 := \min\left\{1, C^{\frac{p-1}{p-1-q}}\right\}$ and $L_2 := \max\left\{1, C^{\frac{p-1}{p-1-q}}\right\}$.

This property was first observed in [Hai08]. We follow and extend arguments in [Hai08] to establish several results.

Lemma 3.2. *Let $p > 1$, $\alpha = 0$, (H_1) and $(H_3) - (H_5)$ hold. Let u be a positive solution of (1.6). For $\lambda \gg 1$, there exist constants C_1 and C_2 (independent of λ) with $0 < C_1 < C_2$ such that*

$$C_1 G^{-1}(\lambda^{\frac{1}{p-1}}) d(t, \partial\Omega) \leq u(t) \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}}) d(t, \partial\Omega).$$

Proof. First we show that for $\lambda \gg 1$ there exists $C_1 > 0$ such that

$$u(t) \geq C_1 G^{-1}(\lambda^{\frac{1}{p-1}}) d(t, \partial\Omega).$$

Let z be the solution of the boundary value problem of the form:

$$\begin{cases} -\left(\varphi_p(z')\right)' = f(0)h(t)\chi_{[\frac{1}{4},\frac{3}{4}]}(t); & t \in (0,1) \\ z(0) = 0 = z(1) \end{cases} \tag{3.1}$$

where $\chi_S(t) = 1$ for $t \in S$ and $\chi_S(t) = 0$ for $t \in [0,1] \setminus S$. Let $t_m \in (0,1)$ be the point such that $z(t_m) = \|z\|_\infty$. It is easy to show that $t_m \in (\frac{1}{4}, \frac{3}{4})$ and z can be written as

$$z(t) = \begin{cases} \int_0^t \varphi_p^{-1}\left(f(0)\int_s^{t_m} h(r)\chi_{[\frac{1}{4},\frac{3}{4}]}(r)dr\right) ds; & 0 \le t \le t_m \\ \int_t^1 \varphi_p^{-1}\left(f(0)\int_{t_m}^{s} h(r)\chi_{[\frac{1}{4},\frac{3}{4}]}(r)dr\right) ds; & t_m \le t \le 1. \end{cases}$$

If $t_m \in (\frac{1}{4}, \frac{1}{2}]$, then we have

$$\|z\|_\infty = \int_0^{t_m} \varphi_p^{-1}\left(f(0)\int_s^{t_m} h(r)\chi_{[\frac{1}{4},\frac{3}{4}]}(r)dr\right) ds \le \frac{1}{2}\varphi_p^{-1}\left(\overline{h}f(0)\Big(t_m - \frac{1}{4}\Big)\right)$$

and

$$z\left(\frac{3}{4}\right) = \int_{\frac{3}{4}}^{1} \varphi_p^{-1}\left(f(0)\int_{t_m}^{s} h(r)\chi_{[\frac{1}{4},\frac{3}{4}]}(r)dr\right) ds \ge \frac{1}{4}\varphi_p^{-1}\left(\underline{h}f(0)\Big(\frac{3}{4} - t_m\Big)\right)$$

where $\overline{h} := \max_{t\in[\frac{1}{4},\frac{3}{4}]} h(t)$ and $\underline{h} := \min_{t\in[\frac{1}{4},\frac{3}{4}]} h(t)$. Since $\|z\|_\infty \ge z\left(\frac{3}{4}\right)$, we have

$$\frac{1}{2}\varphi_p^{-1}\left(\overline{h}f(0)\Big(t_m - \frac{1}{4}\Big)\right) \ge \frac{1}{4}\varphi_p^{-1}\left(\underline{h}f(0)\Big(\frac{3}{4} - t_m\Big)\right).$$

This implies that $t_m \geq \frac{1}{4} + K_1$ where $K_1 := \frac{\underline{h}}{2(\varphi_p(2)\overline{h}+\underline{h})}$. If $t_m \in [\frac{1}{2}, \frac{3}{4})$ then it is clear that $t_m \geq \frac{1}{4} + K_1$. Hence $t_m \geq \frac{1}{4} + K_1$. Then we have

$$z\left(\frac{1}{4}\right) = \int_0^{\frac{1}{4}} \varphi_p^{-1}\left(f(0)\int_s^{t_m} h(r)\chi_{[\frac{1}{4},\frac{3}{4}]}(r)dr\right) ds \geq \frac{1}{4}\varphi_p^{-1}\left(\underline{h}f(0)\Big(t_m - \frac{1}{4}\Big)\right) \geq K_2$$

where $K_2 := \frac{1}{4}\varphi_p^{-1}(\underline{h}f(0)K_1)$. Then we obtain that $z(t) \geq K_* d(t, \partial\Omega)$ for some $K_* > 0$. By the comparison principle, we obtain $u(t) \geq \lambda^{\frac{1}{p-1}} z(t) \geq \lambda^{\frac{1}{p-1}} K_* d(t, \partial\Omega)$. Let K_λ (≥ 1) be the largest constant such that $u(t) \geq \lambda^{\frac{1}{p-1}} K_\lambda z(t)$. Then $u(t) \geq K_\lambda^* := \frac{\lambda^{\frac{1}{p-1}} K_\lambda K_*}{4}$ on $[\frac{1}{4}, \frac{3}{4}]$, and we have

$$\begin{aligned}
&-f(0)\left(\varphi_p(u')\right)' + \lambda f(K_\lambda^*)\left(\varphi_p(z')\right)' \\
&\qquad = \lambda f(0)h(t)f(u) - \lambda f(0)f(K_\lambda^*)h(t)\chi_{[\frac{1}{4},\frac{3}{4}]}(t) \\
&\qquad = \lambda f(0)h(t)\big(f(u) - f(K_\lambda^*)\chi_{[\frac{1}{4},\frac{3}{4}]}(t)\big) \\
&\qquad \geq 0.
\end{aligned}$$

By the comparison principle, we have $[f(0)]^{\frac{1}{p-1}} u(t) \geq \lambda^{\frac{1}{p-1}} [f(K_\lambda^*)]^{\frac{1}{p-1}} z(t)$, and hence

$$u(t) \geq \lambda^{\frac{1}{p-1}} \frac{[f(K_\lambda^*)]^{\frac{1}{p-1}}}{[f(0)]^{\frac{1}{p-1}}} z(t).$$

This implies that $K_\lambda \geq \frac{[f(K_\lambda^*)]^{\frac{1}{p-1}}}{[f(0)]^{\frac{1}{p-1}}}$. Thus for $\lambda \gg 1$ (so that $K_\lambda^* \geq \sigma_0$), we obtain

$$\frac{\lambda^{\frac{1}{p-1}} K_\lambda K_*}{4} = K_\lambda^* = G^{-1}\left(\frac{K_\lambda^*}{[f(K_\lambda^*)]^{\frac{1}{p-1}}}\right) \geq G^{-1}\left(\frac{\lambda^{\frac{1}{p-1}} K_*}{4[f(0)]^{\frac{1}{p-1}}}\right).$$

By Lemma 3.1, for $\lambda \gg 1$ there exists $C_1 > 0$ such that

$$4G^{-1}\left(\frac{\lambda^{\frac{1}{p-1}}K_*}{4[f(0)]^{\frac{1}{p-1}}}\right) \geq C_1 G^{-1}(\lambda^{\frac{1}{p-1}}).$$

Thus for $\lambda \gg 1$ we have

$$u(t) \geq \lambda^{\frac{1}{p-1}}K_\lambda z(t) \geq \frac{C_1}{K_*}G^{-1}(\lambda^{\frac{1}{p-1}})z(t) \geq C_1 G^{-1}(\lambda^{\frac{1}{p-1}})d(t,\partial\Omega).$$

Next we show that for $\lambda \gg 1$ there exists $C_2 > 0$ such that

$$u(t) \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}})d(t,\partial\Omega).$$

For $\lambda \gg 1$, we have $\|u'\|_\infty \geq \sigma_0$, and hence $\sigma_0 \leq \|u\|_1 \leq 2\lambda^{\frac{1}{p-1}}[f(\|u\|_1)]^{\frac{1}{p-1}}\varphi_p^{-1}(H)$ where $\|u\|_1 := \|u\|_\infty + \|u'\|_\infty$ and $H := \int_0^1 h(s)ds$. This implies that

$$\|u'\|_\infty \leq G^{-1}\left(\frac{\|u\|_1}{[f(\|u\|_1)]^{\frac{1}{p-1}}}\right) \leq G^{-1}\big(2\lambda^{\frac{1}{p-1}}\varphi_p^{-1}(H)\big). \tag{3.2}$$

By Lemma 3.1, for $\lambda \gg 1$ there exists $C_2 > 0$ such that

$$G^{-1}\big(2\lambda^{\frac{1}{p-1}}\varphi_p^{-1}(H)\big) \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}}). \tag{3.3}$$

Thus we have $u(t) \leq \|u'\|_\infty d(t,\partial\Omega) \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}})d(t,\partial\Omega)$ for $\lambda \gg 1$. Hence the proof is complete. $\square$

Lemma 3.3. *Let $p > 1$, $\alpha = 0$, (H_1) and $(H_3) - (H_5)$ hold. Let $\gamma_0 \leq \gamma < 1$ where $\gamma_0 := \frac{C_1}{C_2}$. Let u and v be the positive solution of* (1.6). *For $\lambda \gg 1$, there exists $\delta > 0$ (independent of λ) such that $\frac{C_1\gamma_0}{2} G^{-1}(\lambda^{\frac{1}{p-1}}) \leq |sv'(t) + (1-s)\gamma u'(t)| \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}})$ for $s \in [0,1]$ and $t \in [0,\delta] \cup [1-\delta, 1]$.*

Proof. Let $s \in [0,1]$ and $y := sv' + (1-s)\gamma u'$. From (3.2) and (3.3), we obtain that $|y(t)| \leq s\|v'\|_\infty + (1-s)\|u'\|_\infty \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}})$.

Now we show that $|y(t)| \geq \frac{C_1\gamma_0}{2} G^{-1}(\lambda^{\frac{1}{p-1}})$ for $t \in [0,\delta] \cup [1-\delta, 1]$. From (2.4) and (2.5), we obtain that $\|u\|_{1,\alpha^*} \leq \lambda^{\frac{1}{p-1}} K_3 [f(\|u\|_{1,\alpha^*})]^{\frac{1}{p-1}}$ for some $K_3 > 0$. By Lemma 3.2, we get $\|u\|_{1,\alpha^*} \geq \sigma_0$ for $\lambda \gg 1$ where $\|u\|_{1,\alpha^*} := \|u\|_1 + \|u'\|_{\alpha^*}$ and $\|u\|_{\alpha^*} := \sup\limits_{a \neq b \in (0,1)} \dfrac{|u(b) - u(a)|}{|b-a|^{\alpha^*}}$. Thus for $\lambda \gg 1$ we have

$$\|u\|_{1,\alpha^*} = G^{-1}\left(\frac{\|u\|_{1,\alpha^*}}{[f(\|u\|_{1,\alpha^*})]^{\frac{1}{p-1}}} \right) \leq G^{-1}(\lambda^{\frac{1}{p-1}} K_3).$$

Without loss of generality, we obtain $\|u\|_{1,\alpha^*} \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}})$ by Lemma 3.1. Hence $\|y\|_{\alpha^*} \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}})$. Let $\delta > 0$ be such that $\delta^{\alpha^*} \leq \frac{C_1\gamma_0}{2C_2}$. Then we have

$$|y(t) - y(0)| \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}})|t|^{\alpha^*} \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}})\delta^{\alpha^*}$$

for $t \in [0, \delta]$. By Lemma 3.2, we also have

$$y(0) = sv'(0) + (1-s)\gamma u'(0) \geq sC_1 G^{-1}(\lambda^{\frac{1}{p-1}}) + (1-s)C_1\gamma G^{-1}(\lambda^{\frac{1}{p-1}}) \geq C_1\gamma_0 G^{-1}(\lambda^{\frac{1}{p-1}}).$$

This implies that

$$|y(t)| \geq |y(0)| - |y(t) - y(0)| \geq C_1\gamma_0 G^{-1}(\lambda^{\frac{1}{p-1}}) - C_2 G^{-1}(\lambda^{\frac{1}{p-1}})\delta^{\alpha^*} \geq \frac{C_1\gamma_0}{2} G^{-1}(\lambda^{\frac{1}{p-1}}).$$

By a similar argument, we can show that $|y(t)| \geq \frac{C_1\gamma_0}{2} G^{-1}(\lambda^{\frac{1}{p-1}})$ for $t \in [1-\delta, 1]$. $\square$

Let δ and γ be as before (see Lemma 3.3) and u and v be positive solutions of (1.6). Define $a_\lambda : [0, \delta] \cup [1-\delta, 1] \to \mathbb{R}$ by

$$a_\lambda(t) := (p-1)\int_0^1 |s\tilde{v}'(t) + (1-s)\gamma\tilde{u}'(t)|^{p-2} ds$$

where $\tilde{u} := \frac{u}{G^{-1}(\lambda^{\frac{1}{p-1}})}$ and $\tilde{v} := \frac{v}{G^{-1}(\lambda^{\frac{1}{p-1}})}$. By Lemma 3.3, we obtain that $a_\lambda(t) \in [C_*, C^*]$ where

$$C_* := (p-1)\min\left\{\left(\frac{C_1\gamma_0}{2}\right)^{p-2}, C_2^{p-2}\right\} \text{ and } C^* := (p-1)\max\left\{\left(\frac{C_1\gamma_0}{2}\right)^{p-2}, C_2^{p-2}\right\}.$$

We prove:

Lemma 3.4. *Let κ_0 be the solution of the boundary value problem:*

$$-\left(a_\lambda(t){\kappa_0}'(t)\right)' = \begin{cases} 0; & t \in (0, t_\lambda] \\ h(t); & t \in (t_\lambda, \delta) \end{cases}$$

$$\kappa_0(0) = 0 = \kappa_0(\delta)$$

where $t_\lambda \to 0$ as $\lambda \to \infty$. Then there exists $\overline{M} > 0$ (independent of λ) such that $\kappa_0(t) \geq \overline{M} d(t, \partial\Omega_\delta^*)$ for $\lambda \gg 1$ where $\Omega_\delta^* := (0, \delta)$.

Proof. Let $t_m \in [0, \delta]$ be such that $\kappa_0(t_m) = \|\kappa_0\|_\infty$. It is easy to show $t_m \in (t_\lambda, \delta)$ and κ_0 can be written as

$$\kappa_0(t) = \begin{cases} \int_0^t \frac{1}{a_\lambda(s)} \int_s^{t_m} h(r)\chi_{[t_\lambda, t_m]}(r) dr ds; & 0 \leq t \leq t_m \\ \int_t^\delta \frac{1}{a_\lambda(s)} \int_{t_m}^s h(r) dr ds; & t_m \leq t \leq \delta. \end{cases}$$

For $\lambda \gg 1$, if $t_m \leq \frac{\delta}{2}$ then we have

$$\kappa_0(t_m) = \int_0^{t_m} \frac{1}{a_\lambda(s)} \int_s^{t_m} h(r)\chi_{[t_\lambda, t_m]}(r) dr ds \leq \frac{H}{C_*} t_m$$

and

$$\kappa_0\left(\frac{3\delta}{4}\right) = \int_{\frac{3\delta}{4}}^{\delta} \frac{1}{a_\lambda(s)} \int_{t_m}^{s} h(r) dr ds \geq \frac{1}{C^*} \int_{\frac{3\delta}{4}}^{\delta} \int_{\frac{\delta}{2}}^{\frac{3\delta}{4}} h(r) dr ds \geq K_4$$

where $K_4 := \frac{\delta^2 h_\#}{16C^*}$ and $h_\# := \min_{t \in [\frac{\delta}{4}, \frac{3\delta}{4}]} h(t)$. This implies that $t_m \geq K_5$ where $K_5 := \frac{C_* K_4}{H}$ (independent of λ). For $\lambda \gg 1$, if $t_m \geq \frac{\delta}{2}$ then we have

$$\kappa_0(t_m) = \int_{t_m}^{\delta} \frac{1}{a_\lambda(s)} \int_{t_m}^{s} h(r) dr ds \leq \frac{H}{C_*}(\delta - t_m)$$

and

$$\kappa_0\left(\frac{\delta}{4}\right) = \int_0^{\frac{\delta}{4}} \frac{1}{a_\lambda(s)} \int_s^{t_m} h(r)\chi_{[t_\lambda, t_m]}(r) dr ds \geq \frac{1}{C^*} \int_0^{\frac{\delta}{4}} \int_{\frac{\delta}{4}}^{\frac{\delta}{2}} h(r) dr ds \geq K_4.$$

This implies that $t_m \leq \delta - K_5$. For $\lambda \gg 1$ and $t \in (0, \frac{K_5}{2})$, we obtain

$$\kappa_0'(t) = \frac{1}{a_\lambda(t)} \int_t^{t_m} h(s)\chi_{[t_\lambda, t_m]}(s)ds \geq \frac{1}{C^*} \int_{\frac{K_5}{2}}^{K_5} h(s)ds \geq \frac{\hat{h}K_5}{2C^*}$$

and

$$\kappa_0\left(\delta - \frac{K_5}{2}\right) = \int_{\delta-\frac{K_5}{2}}^{\delta} \frac{1}{a_\lambda(s)} \int_{t_m}^{s} h(r)drds \geq \frac{1}{C^*} \int_{\delta-\frac{K_5}{2}}^{\delta} \int_{\delta-K_5}^{\delta-\frac{K_5}{2}} h(r)drds \geq \frac{\hat{h}K_5^2}{4C^*}$$

where $\hat{h} := \min_{t\in[\frac{K_5}{2}, \delta-\frac{K_5}{2}]} h(t)$. Thus $\kappa_0(t) \geq \overline{M}t$ for $\lambda \gg 1$ and $t \in [0, \frac{\delta}{2}]$ where $\overline{M} := \frac{\hat{h}K_5^2}{4C^*\delta}$. Similarly, for $\lambda \gg 1$ and $t \in (\delta - \frac{K_5}{2}, \delta)$, we deduce

$$\kappa_0'(t) = -\frac{1}{a_\lambda(t)} \int_{t_m}^{t} h(s)ds \leq -\frac{1}{C^*} \int_{\delta-K_5}^{\delta-\frac{K_5}{2}} h(s)ds \leq -\frac{\hat{h}K_5}{2C^*}$$

and

$$\kappa_0\left(\frac{K_5}{2}\right) = \int_0^{\frac{K_5}{2}} \frac{1}{a_\lambda(s)} \int_s^{t_m} h(r)\chi_{[t_\lambda, t_m]}(r)drds \geq \frac{1}{C^*} \int_0^{\frac{K_5}{2}} \int_{\frac{K_5}{2}}^{K_5} h(r)drds \geq \frac{\hat{h}K_5^2}{4C^*}.$$

Then $\kappa_0(t) \geq \overline{M}(\delta - t)$ for $\lambda \gg 1$ and $t \in [\frac{\delta}{2}, \delta]$. Hence we obtain that $\kappa_0(t) \geq \overline{M}d(t, \partial\Omega_\delta^*)$ for $\lambda \gg 1$. □

Corollary 3.5. *Let κ_1 be the solution of the boundary value problem:*

$$-\left(a_\lambda(t)\kappa_1{}'(t)\right)' = \begin{cases} h(t); & t \in (1-\delta, \tilde{t}_\lambda] \\ 0; & t \in (\tilde{t}_\lambda, 1) \end{cases}$$

$$\kappa_1(1-\delta) = 0 = \kappa_1(1)$$

where $\tilde{t}_\lambda \to 1$ as $\lambda \to \infty$. Then there exists $\widetilde{M} > 0$ (independent of λ) such that $\kappa_1(t) \geq \widetilde{M} d(t, \partial\Omega_\delta^{\#})$ for $\lambda \gg 1$ where $\Omega_\delta^{\#} := (1-\delta, 1)$.

Lemma 3.6. *Let a_λ, t_λ, δ and γ be as before (see Lemmas 3.3 - 3.4). Let κ_0^* be the solution of the boundary value problem:*

$$-\left(a_\lambda(t)\kappa_0^{*\prime}(t)\right)' = \begin{cases} h(t); & t \in (0, t_\lambda] \\ 0; & t \in (t_\lambda, \delta) \end{cases}$$

$$\kappa_0^*(0) = 0 = \kappa_0^*(\delta).$$

Then there exists $M^* > 0$ (independent of λ) such that $\kappa_0^*(t) \leq M^* t_\lambda^{1-\eta} d(t, \partial\Omega_\delta^*)$ for $\lambda \gg 1$.

Proof. Let $t_m \in [0, \delta]$ be such that $\kappa_0^*(t_m) = \|\kappa_0^*\|_\infty$. It is easy to show that $t_m \in (0, t_\lambda)$ and κ_0^* can be written as

$$\kappa_0^*(t) = \begin{cases} \int_0^t \frac{1}{a_\lambda(s)} \int_s^{t_m} h(r)drds; & 0 \leq t \leq t_m \\ \int_t^\delta \frac{1}{a_\lambda(s)} \int_{t_m}^s h(r)\chi_{[t_m, t_\lambda]}(r)drds; & t_m \leq t \leq \delta. \end{cases}$$

For $t \in [0, t_m]$, we have

$$\kappa_0^*(t) = \int_0^t \frac{1}{a_\lambda(s)} \int_s^{t_m} h(r) dr ds \leq \frac{1}{C_*} \int_0^t \int_s^{t_m} \frac{d}{r^\eta} dr ds \leq \frac{d}{C_*(1-\eta)} t_\lambda^{1-\eta} t.$$

Since κ_0^* has a maximum at t_m, we obtain that $\kappa_0^*(t) \leq M^* t_\lambda^{1-\eta} t$ for $t \in [0, \frac{\delta}{2}]$ where $M^* := \frac{d}{C_*(1-\eta)}$. For $t \in [t_m, \delta]$, we have

$$\kappa_0^*(t) = \int_t^\delta \frac{1}{a_\lambda(s)} \int_{t_m}^s h(r) \chi_{[t_m, t_\lambda]}(r) dr ds \leq \frac{1}{C^*} \int_t^\delta \int_{t_m}^{t_\lambda} \frac{d}{r^\eta} dr ds \leq M^* t_\lambda^{1-\eta} (\delta - t).$$

This implies that $\kappa_0^*(t) \leq M^* t_\lambda^{1-\eta} (\delta - t)$ for $t \in [\frac{\delta}{2}, \delta]$. Hence $\kappa_0^*(t) \leq M^* t_\lambda^{1-\eta} d(t, \partial\Omega_\delta^*)$ for $\lambda \gg 1$. $\square$

Corollary 3.7. *Let a_λ, $\tilde{t}_\lambda$, δ and γ be as before (see Lemma 3.3 and Corollary 3.5). Let κ_1^* be the solution of the boundary value problem:*

$$-\left(a_\lambda(t) {\kappa_1^*}'(t)\right)' = \begin{cases} 0; & t \in (1-\delta, \tilde{t}_\lambda] \\ h(t); & t \in (\tilde{t}_\lambda, 1) \end{cases}$$

$$\kappa_1^*(1-\delta) = 0 = \kappa_1^*(1).$$

Then there exists $M_* > 0$ (independent of λ) such that $\kappa_1^*(t) \leq M_* (1 - \tilde{t}_\lambda)^{1-\eta} d(t, \partial\Omega_\delta^\#)$ for $\lambda \gg 1$.

Now we prove Theorem 1.1. Let u and v be positive solutions of (1.6). By Lemma 3.2, we have $v \geq \gamma_0 u$ for $\lambda \gg 1$ where $\gamma_0 := \frac{C_1}{C_2}$. Let γ be the largest constant such that $v \geq \gamma u$.

We show that $\gamma \geq 1$. Assume $\gamma < 1$. Let $\tilde{u} := \frac{u}{G^{-1}(\lambda^{\frac{1}{p-1}})}$ and $\tilde{v} := \frac{v}{G^{-1}(\lambda^{\frac{1}{p-1}})}$. Since $G^{-1}(\lambda^{\frac{1}{p-1}}) = \lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}}\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)$, we obtain that

$$-\left(\varphi_p\left(\tilde{u}'\right)\right)' = h(t)\frac{f(u)}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)} \quad \text{and} \quad -\left(\varphi_p\left(\tilde{v}'\right)\right)' = h(t)\frac{f(v)}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)}.$$

By the Mean Value Theorem, we have

$$\varphi_p(\tilde{v}') - \varphi_p(\gamma\tilde{u}') = a_\lambda(t)\left(\tilde{v}' - \gamma\tilde{u}'\right)$$

for $t \in \Omega_\delta := [0,\delta] \cup [1-\delta,1]$ where $a_\lambda(t) := (p-1)\int_0^1 |s\tilde{v}'(t) + (1-s)\gamma\tilde{u}'(t)|^{p-2}ds$. This implies that

$$-\left(a_\lambda(t)\left(\tilde{v}' - \gamma\tilde{u}'\right)\right)' = h(t)\frac{f(v) - \gamma^{p-1}f(u)}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)} \geq h(t)\frac{f(\gamma u) - \gamma^{p-1}f(u)}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)}.$$

Let $I := \{t \in \Omega_\delta \mid u(t) \geq \frac{\sigma_0}{\gamma_0}\}$. It is easy to show that $I = [t_I, \delta] \cup [1-\delta, \tilde{t}_I]$ where $t_I := \min\{t \in (0,1) \mid u(t) \geq \frac{\sigma_0}{\gamma_0}\}$ and $\tilde{t}_I := \max\{t \in (0,1) \mid u(t) \geq \frac{\sigma_0}{\gamma_0}\}$. On I, it follows from (H_5) that

$$f(\gamma u) - \gamma^{p-1}f(u) \geq (\gamma^q - \gamma^{p-1})f(u) \geq m_1(1-\gamma)$$

where $m_1 := (p-1-q)\gamma_0^q f\big(\frac{\sigma_0}{\gamma_0}\big)\min\{1,\gamma_0^{p-2-q}\}$. On $\Omega_\delta \setminus I$, we have

$$
\begin{aligned}
|f(\gamma u)-\gamma^{p-1}f(u)| &\leq |f(\gamma u)-f(u)|+(1-\gamma^{p-1})f(u) \\
&\leq (1-\gamma)|uf'(\zeta)|+(1-\gamma)(p-1)\max\{1,\gamma_0^{p-2}\}f\Big(\frac{\sigma_0}{\gamma_0}\Big) \\
&\leq \frac{1-\gamma}{\gamma_0}|\zeta f'(\zeta)|+(1-\gamma)(p-1)\max\{1,\gamma_0^{p-2}\}f\Big(\frac{\sigma_0}{\gamma_0}\Big) \\
&\leq m_2(1-\gamma)
\end{aligned}
$$

where $m_2 := \frac{1}{\gamma_0}\sup_{s\in(0,\frac{\sigma_0}{\gamma_0})}|sf'(s)|+(p-1)\max\{1,\gamma_0^{p-2}\}f\big(\frac{\sigma_0}{\gamma_0}\big)$ and $\zeta(t)\in(\gamma u(t),u(t))$. Then we have

$$
-\left(\frac{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)}{1-\gamma}a_\lambda(t)(\tilde{v}'-\gamma\tilde{u}')\right)' \geq \begin{cases} m_1h(t); & t\in I \\ -m_2h(t); & t\in\Omega_\delta\setminus I. \end{cases}
$$

Let $\overline{\kappa}$ be the solution of the boundary value problem:

$$
-\left(a_\lambda(t)\overline{\kappa}'(t)\right)' = \begin{cases} m_1h(t); & t\in I \\ -m_2h(t); & t\in\Omega_\delta\setminus I \end{cases}
$$

$$
\overline{\kappa}\equiv 0 \quad \text{ on } \partial\Omega_\delta.
$$

By the comparison principle, we have

$$
\tilde{v}(t)-\gamma\tilde{u}(t) \geq \frac{1-\gamma}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)}\overline{\kappa}(t)
$$

for $t \in \Omega_\delta$. Let κ be the solution of the boundary value problem:

$$-\left(a_\lambda(t)\kappa'(t)\right)' = \begin{cases} m_1 h(t); & t \in I \\ 0; & t \in \Omega_\delta \setminus I \end{cases}$$

$$\kappa \equiv 0 \quad \text{ on } \partial\Omega_\delta.$$

We note that $t_I \to 0$ and $\tilde{t}_I \to 1$ as $\lambda \to \infty$ by Lemma 3.2. Then by Lemmas 3.4 and 3.6 and Corollaries 3.5 and 3.7, we obtain that $\kappa(t) \geq \widehat{M_1} d(t, \partial\Omega_\delta)$ and $|\kappa(t) - \overline{\kappa}(t)| \leq \widehat{M_2}|\Omega_\delta \setminus I|^{1-\eta} d(t, \partial\Omega_\delta)$ for $\lambda \gg 1$ where $\widehat{M_1} := \min\{\overline{M}, \widetilde{M}\}$, $\widehat{M_2} := \max\{M^*, M_*\}$ and $|\Omega_\delta \setminus I|$ is the length of $\Omega_\delta \setminus I$. Since $|\Omega_\delta \setminus I| \to 0$ as $\lambda \to \infty$, we have

$$\overline{\kappa}(t) \geq \kappa(t) - |\kappa(t) - \overline{\kappa}(t)| \geq \left(\widehat{M_1} - \widehat{M_2}|\Omega_\delta \setminus I|^{1-\eta}\right) d(t, \partial\Omega_\delta) \geq \frac{\widehat{M_1}}{2} d(t, \partial\Omega_\delta)$$

for $\lambda \gg 1$ and $t \in \Omega_\delta$. Since $d(t, \partial\Omega_\delta) = d(t, \partial\Omega)$ for $t \in \Omega_{\frac{\delta}{2}} := [0, \frac{\delta}{2}] \cup [1 - \frac{\delta}{2}, 1]$, we have

$$\overline{\kappa}(t) \geq \frac{\widehat{M_1}}{2} d(t, \partial\Omega)$$

for $t \in \Omega_{\frac{\delta}{2}}$. This implies that

$$\tilde{v}(t) - \gamma\tilde{u}(t) \geq \frac{1-\gamma}{f\left(G^{-1}(\lambda^{\frac{1}{p-1}})\right)} \overline{\kappa}(t) \geq \frac{\widehat{M_1}(1-\gamma)}{2f\left(G^{-1}(\lambda^{\frac{1}{p-1}})\right)} d(t, \partial\Omega)$$

for $t \in \Omega_{\frac{\delta}{2}}$. By Lemma 3.2, we obtain

$$v(t) \geq (\gamma + \epsilon_\lambda) u(t)$$

for $t \in \Omega_{\frac{\delta}{2}}$ where $\epsilon_\lambda := \frac{\widehat{M}_1(1-\gamma)}{2C_2 f\left(G^{-1}(\lambda^{\frac{1}{p-1}})\right)}$. Thus $v(t) \geq \gamma u(t) + \tilde{\epsilon}_\lambda$ for $t \in \left\{\frac{\delta}{2}, 1 - \frac{\delta}{2}\right\}$ where $\tilde{\epsilon}_\lambda := \epsilon_\lambda C_1 G^{-1}(\lambda^{\frac{1}{p-1}})\frac{\delta}{2}$. Further, we have

$$u(t) \geq C_1 G^{-1}(\lambda^{\frac{1}{p-1}})\frac{\delta}{2} \geq \frac{\sigma_0}{\gamma_0}$$

for $\lambda \gg 1$ and $t \in [0,1] \setminus \Omega_{\frac{\delta}{2}}$. By (H_5), we have

$$-\left(\varphi_p(v')\right)' = \lambda h(t) f(v) \geq \lambda h(t) f(\gamma u) \geq \lambda \gamma^{p-1} h(t) f(u)$$

on $[0,1] \setminus \Omega_{\frac{\delta}{2}}$. But

$$-\left(\varphi_p((\gamma u + \tilde{\epsilon}_\lambda)')\right)' = \lambda \gamma^{p-1} h(t) f(u)$$

on $[0,1] \setminus \Omega_{\frac{\delta}{2}}$. Thus $v \geq \gamma u + \tilde{\epsilon}_\lambda$ on $[0,1] \setminus \Omega_{\frac{\delta}{2}}$. Then

$$v \geq \gamma u + \tilde{\epsilon}_\lambda d(t, \partial\Omega) \geq \left(\gamma + \frac{C_1}{C_2}\epsilon_\lambda\right) u$$

on $[0,1] \setminus \Omega_{\frac{\delta}{2}}$. Since $C_1 < C_2$, we have $v \geq \left(\gamma + \frac{C_1}{C_2}\epsilon_\lambda\right) u$ on $[0,1]$. This is a contradiction for the maximality of γ. Hence $\gamma \geq 1$. This implies that $v \equiv u$ on $[0,1]$ for $\lambda \gg 1$.

3.2 Proof of Theorem 1.2

Lemma 3.8. *Let $p = 2$ and (H_4) hold. If u is a positive solution of (1.6), then $u \geq \delta_\lambda e$ in $(0,1)$ where $\delta_\lambda^{1+\alpha} := \lambda \frac{f_*}{\|e\|_\infty^\alpha}$ and e is the solution of*

$$\begin{cases} -e''(t) = h(t);\ t \in (0,1) \\ e(0) = 0 = e(1). \end{cases} \tag{3.4}$$

Proof. Let u be a positive solution of (1.6) for $\lambda > 0$. Assuming that $\Omega_\lambda := \{t \in (0,1) \mid u(t) < \delta_\lambda e(t)\} \neq \emptyset$, we have

$$\begin{aligned} -(u(t) - \delta_\lambda e(t))'' &= \lambda h(t)\frac{f(u)}{u^\alpha} - \delta_\lambda h(t) \\ &> \lambda h(t)\frac{f_*}{\delta_\lambda^\alpha e(t)^\alpha} - \delta_\lambda h(t) \\ &\geq \lambda h(t)\frac{f_*}{\delta_\lambda^\alpha \|e\|_\infty^\alpha} - \delta_\lambda h(t) \\ &= 0 \end{aligned}$$

in Ω_λ, and $u - \delta_\lambda e = 0$ on $\partial\Omega_\lambda$. This contradicts the maximum principle. Hence $\Omega_\lambda = \emptyset$, which proves Lemma 3.8. □

Now we prove Theorem 1.2. Let u and v be positive solutions of (1.6). Let $g(s) := \frac{f(s)}{s^\alpha}$. Then we have

$$\int_0^1 -(v-u)''(v-u)ds = \lambda \int_0^1 h(s)(g(v) - g(u))(v-u)ds. \tag{3.5}$$

Integrating by parts, we get

$$\int_0^1 -(v-u)''(v-u)ds = \int_0^1 |(v-u)'|^2 ds.$$

Further, by the Fundamental Theorem of Calculus, we have

$$\lambda \int_0^1 h(s)(g(v)-g(u))(v-u)ds = \lambda \int_0^1 h(s)\zeta(s)(v-u)^2 ds$$

where $\zeta(s) := \int_0^1 g'(u+s(v-u))dr$. From (3.5), we obtain

$$\int_0^1 |(v-u)'|^2 ds = \lambda \int_0^1 h(s)\zeta(s)(v-u)^2 ds. \tag{3.6}$$

Let $\Omega := (0,1)$ and $K_6 > 0$ be such that $e(t) \geq K_6 d(t, \partial\Omega)$. For $\lambda \gg 1$ where $\delta_\lambda > \frac{\sigma^*}{K_6}$, we define $\Omega_1 := [\frac{\sigma^*}{\delta_\lambda K_6}, 1 - \frac{\sigma^*}{\delta_\lambda K_6}]$. By Lemma 3.8, $u(t) \geq \delta_\lambda e(t) \geq \delta_\lambda K_6 d(t, \partial\Omega) \geq \sigma^*$ for $t \in \Omega_1$. Let $\Omega_2 := (0,1) \setminus \Omega_1 = (0, \sigma_\lambda) \cup (1-\sigma_\lambda, 1)$ where $\sigma_\lambda := \frac{\sigma^*}{\delta_\lambda K_6} = K_7 \lambda^{-\frac{1}{1+\alpha}}$ and $K_7 := \frac{\sigma^*}{K_6}(\frac{\|e\|_\infty^\alpha}{f_*})^{\frac{1}{1+\alpha}}$. Then we have

$$\int_0^1 |(v-u)'|^2 ds = \lambda \int_{\Omega_1} h(s)\zeta(s)(v-u)^2 ds + \lambda \int_{\Omega_2} h(s)\zeta(s)(v-u)^2 ds.$$

Since $u + s(v-u) = (1-s)u + sv \geq (1-s)\delta_\lambda e + s\delta_\lambda e = \delta_\lambda e \geq \sigma^*$ for $s \in \Omega_1$ and by (H_8), we obtain $g'(u+ s(v-u)) \leq 0$, which implies $\zeta(s) = \int_0^1 g'(u+s(v-u))dr \leq 0$ for $s \in \Omega_1$. Further, $\lim_{s\to 0+} g'(s) = -\infty$ and $g'(s) \leq 0$ for $s > \sigma^*$ by (H_7) and (H_8). Hence g' is bounded above. Let $K_8 > 0$ be such that $g'(s) \leq K_8$ for all $s > 0$. Then we have

$$\int_0^1 |(v-u)'|^2 ds \le \lambda \int_{\Omega_2} h(s)\zeta(s)(v-u)^2 ds \le \lambda K_8 \int_{\Omega_2} h(s)(v-u)^2 ds. \tag{3.7}$$

For $t \in (0, \sigma_\lambda)$, we have

$$\begin{aligned}
|v(t)-u(t)| &= \left| \int_0^t v' - u' ds \right| \\
&\le \left(\int_0^{\sigma_\lambda} (v'-u')^2 ds \right)^{\frac{1}{2}} \left(\int_0^{\sigma_\lambda} 1 \; ds \right)^{\frac{1}{2}} \\
&= \lambda^{-\frac{1}{2(1+\alpha)}} K_7^{\frac{1}{2}} \left(\int_0^{\sigma_\lambda} (v'-u')^2 ds \right)^{\frac{1}{2}} \\
&\le \lambda^{-\frac{1}{2(1+\alpha)}} K_7^{\frac{1}{2}} \left(\int_0^{1} (v'-u')^2 ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, if $t \in (1-\sigma_\lambda, 1)$ then we obtain

$$|v(t)-u(t)| \le \lambda^{-\frac{1}{2(1+\alpha)}} K_7^{\frac{1}{2}} \left(\int_{1-\sigma_\lambda}^1 (v'-u')^2 ds \right)^{\frac{1}{2}} \le \lambda^{-\frac{1}{2(1+\alpha)}} K_7^{\frac{1}{2}} \left(\int_0^1 (v'-u')^2 ds \right)^{\frac{1}{2}}.$$

Hence it follows that

$$\int_{\Omega_2} h(s)(v-u)^2 ds \le \lambda^{-\frac{1}{1+\alpha}} K_7 \int_0^1 (v'-u')^2 ds \left(\int_0^{\sigma_\lambda} h(s) ds + \int_{1-\sigma_\lambda}^1 h(s) ds \right). \tag{3.8}$$

If $\int_0^1 |(v-u)'|^2 ds \neq 0$, then from (3.7) and (3.8) we get

$$1 \leq \lambda K_7 K_8 \lambda^{-\frac{1}{1+\alpha}} \left(\int_0^{\sigma_\lambda} h(s)ds + \int_{1-\sigma_\lambda}^1 h(s)ds \right) \leq K_9 \left(\lambda^{\frac{\alpha}{1+\alpha} - \frac{1-\eta}{1+\alpha}} + \lambda^{\frac{\alpha}{1+\alpha} - \frac{1}{1+\alpha}} \right)$$

for some constant $K_9 > 0$, that is a contradiction for $\lambda \gg 1$ since $\alpha + \eta < 1$. Hence $\int_0^1 |(v-u)'|^2 ds = 0$. It follows that $v - u$ is a constant. But $v(0) - u(0) = 0 = v(1) - u(1)$. Hence $v \equiv u$.

CHAPTER IV

PROOF OF THEOREM 1.3

4.1 Proof of Theorem 1.3

Let $F(s) := \int_0^s f(r)dr$. By (H_3) and (H_{10}), there exist constants β and θ such that $0 < \beta < \theta$, $f(\beta) = 0$ and $F(\theta) = 0$. We first establish several results by extending the arguments in [CSS12]:

Lemma 4.1. *Let (H_3) and $(H_9)-(H_{10})$ hold. If u is a nonnegative solution of (1.6), then u has a unique interior maximum, say at t_m, and $u(t_m) > \theta$. Further, u is strictly positive in $(0,1)$.*

Proof. It is easy to show that $u'(t) > 0$ for $t \approx 0$ and $u'(t) < 0$ for $t \approx 1$. Let $t_m \in (0,1)$ be the first point satisfying $u'(t) = 0$. Assume $u(t_m) \leq \theta$. Define $E(t) := \lambda F(u(t))h(t) + \frac{p-1}{p}|u'(t)|^p$. Then $E(t)$ is differentiable and $E'(t) = \lambda F(u(t))h'(t)$ on $(0, t_m)$. This implies that $E(t)$ is strictly increasing on $(0, t_m)$ since $u(t) < \theta$ and by (H_9). Now integrating (1.6) from $t \in (0, t_m)$ to t_m, we obtain

$$u'(t) = \varphi_p^{-1}\left(\lambda \int_t^{t_m} h(s)f(u)ds\right) \leq \varphi_p^{-1}\left(\lambda \frac{df(\theta)}{1-\eta}\right).$$

Integrating again from 0 to $t \in (0, t_m)$, we have $u(t) \leq \lambda^{\frac{1}{p-1}} K_{10} t$ where $K_{10} := \varphi_p^{-1}\left(\frac{df(\theta)}{1-\eta}\right)$ (> 0). Since f is continuous, there exists $K_{11} > 0$ such that $|F(s)| \leq K_{11}s$ for $s \in [0, \theta]$. This implies that $\lim_{t\to 0+} E(t) \geq 0$ since $\lim_{t\to 0+} \lambda |F(u(t))|h(t) \leq \lim_{t\to 0+} \lambda^{\frac{p}{p-1}} dK_{10}K_{11}t^{1-\eta} = 0$. Since $E(t)$ is strictly increasing on $(0, t_m)$, we have

$E(t_m) > 0$. This is a contradiction since $E(t_m) = \lambda F(u(t_m))h(t_m) \leq 0$. Hence $u(t_m) > \theta$.

Now we show that u has a unique interior maximum. Assume that u has two interior maxima. Let $\tilde{t} \in (t_m, 1)$ be the first point satisfying $u'(t) = 0$. Integrating (1.6) from t_m to $\tilde{t}$, we have $\lambda \int_{t_m}^{\tilde{t}} h(s)f(u)ds = 0$. This implies that $u(\tilde{t}) < \beta < \theta < u(t_m)$. Thus there exists $\bar{t} \in (t_m, \tilde{t})$ such that $u(\bar{t}) = \theta$. Since $E(\bar{t}) = \frac{p-1}{p}|u'(\bar{t})|^p \geq 0$ and $u(t) < \theta$ on $(\bar{t}, \tilde{t})$, we obtain $E(\tilde{t}) > 0$. This is a contradiction since $E(\tilde{t}) = \lambda F(u(\tilde{t}))h(\tilde{t}) < 0$. Hence u has a unique interior maximum. □

Lemma 4.2. *Let (H_3) and $(H_9) - (H_{10})$ hold. Let u be a positive solution of* (1.6). *Let t_β and $\tilde{t}_\beta \in (0,1)$ be such that $t_\beta < \tilde{t}_\beta$ and $u(t_\beta) = u(\tilde{t}_\beta) = \beta$. Then t_β and $1 - \tilde{t}_\beta \leq O(\lambda^{-\frac{1}{p}})$.*

Proof. Let $h_* := \inf_{t\in(0,1)} h(t)$ (> 0). Let $t_{\frac{\beta}{2}} \in (0,1)$ be the first point such that $u(t_{\frac{\beta}{2}}) = \frac{\beta}{2}$. Integrating (1.6) from 0 to $t \in (0, t_{\frac{\beta}{2}})$, we obtain

$$\varphi_p(u'(t)) = \varphi_p(u'(0)) - \lambda \int_0^t h(s)f(u)ds \geq -\lambda h_* f\Big(\frac{\beta}{2}\Big)t.$$

Thus $u'(t) \geq \lambda^{\frac{1}{p-1}} K_{12} t^{\frac{1}{p-1}}$ where $K_{12} := -\varphi_p^{-1}\big(h_* f(\frac{\beta}{2})\big)$ (> 0). Integrating again from 0 to $t_{\frac{\beta}{2}}$, we have $\frac{\beta}{2} = u(t_{\frac{\beta}{2}}) - u(0) \geq \lambda^{\frac{1}{p-1}} K_{12}\big(\frac{p-1}{p}\big) t_{\frac{\beta}{2}}^{\frac{p}{p-1}}$. Thus $t_{\frac{\beta}{2}} \leq \lambda^{-\frac{1}{p}} K_{13}$ where $K_{13} := \big(\frac{\beta p}{2K_{12}(p-1)}\big)^{\frac{p-1}{p}}$. Further, by the Mean Value Theorem, there exists $t_* \in (0, t_{\frac{\beta}{2}})$ such that $\frac{\beta}{2} = u(t_{\frac{\beta}{2}}) - u(0) = u'(t_*) t_{\frac{\beta}{2}}$. Since u' is strictly increasing on $(0, t_\beta)$, we have $\lambda^{\frac{1}{p}} \frac{\beta}{2K_{13}} \leq u'(t_*) \leq u'(t)$ for $t \in [t_{\frac{\beta}{2}}, t_\beta]$. Integrating from $t_{\frac{\beta}{2}}$ to t_β, we have $t_\beta - t_{\frac{\beta}{2}} \leq \lambda^{-\frac{1}{p}} K_{13}$. This implies that $t_\beta \leq O(\lambda^{-\frac{1}{p}})$. By a similar argument, we can show that $1 - \tilde{t}_\beta \leq O(\lambda^{-\frac{1}{p}})$. □

Lemma 4.3. *Let* (H_3) *and* $(H_9)-(H_{10})$ *hold. Let* u *be a positive solution of* (1.6). *Let* $t_{\frac{\beta+\theta}{2}}$ *and* $\tilde{t}_{\frac{\beta+\theta}{2}} \in (0,1)$ *be such that* $t_{\frac{\beta+\theta}{2}} < \tilde{t}_{\frac{\beta+\theta}{2}}$ *and* $u(t_{\frac{\beta+\theta}{2}}) = u(\tilde{t}_{\frac{\beta+\theta}{2}}) = \frac{\beta+\theta}{2}$. *Then* $t_{\frac{\beta+\theta}{2}}$ *and* $1-\tilde{t}_{\frac{\beta+\theta}{2}} \leq O(\lambda^{-\frac{1}{p}})$.

Proof. Let t_θ and $\tilde{t}_\theta \in (0,1)$ be such that $t_\theta < \tilde{t}_\theta$ and $u(t_\theta) = u(\tilde{t}_\theta) = \theta$. Integrating (1.6) from $t \in (0, t_\theta)$ to t_θ, we obtain

$$u'(t) = \varphi_p^{-1}\left(\varphi_p(u'(t_\theta)) + \lambda \int_t^{t_\theta} h(s)f(u)ds\right).$$

Integrating again from t_β to $t \in (t_\beta, t_\theta)$, we have

$$u(t) = \beta + \int_{t_\beta}^{t} \varphi_p^{-1}\left(\varphi_p(u'(t_\theta)) + \lambda \int_s^{t_\theta} h(r)f(u)dr\right)ds.$$

Since $\varphi_p(u'(t_\theta)) > 0$, we have

$$\begin{aligned}
\frac{\theta-\beta}{2} &= \int_{t_\beta}^{t_{\frac{\beta+\theta}{2}}} \varphi_p^{-1}\left(\varphi_p(u'(t_\theta)) + \lambda \int_s^{t_\theta} h(r)f(u)dr\right)ds \\
&> \int_{t_\beta}^{t_{\frac{\beta+\theta}{2}}} \varphi_p^{-1}\left(\lambda \int_{t_{\frac{\beta+\theta}{2}}}^{t_\theta} h(r)f(u)dr\right)ds \\
&\geq \int_{t_\beta}^{t_{\frac{\beta+\theta}{2}}} \varphi_p^{-1}\left(\lambda h_* f\Big(\frac{\beta+\theta}{2}\Big)(t_\theta - t_{\frac{\beta+\theta}{2}})\right)ds \\
&= (t_{\frac{\beta+\theta}{2}} - t_\beta)\varphi_p^{-1}\left(\lambda h_* f\Big(\frac{\beta+\theta}{2}\Big)(t_\theta - t_{\frac{\beta+\theta}{2}})\right).
\end{aligned} \tag{4.1}$$

By the Mean Value Theorem, there exist t_1 and $t_2 \in (0,1)$ with $t_\beta < t_1 < t_{\frac{\beta+\theta}{2}} < t_2 < t_\theta$ such that $u(t_{\frac{\beta+\theta}{2}}) - u(t_\beta) = u'(t_1)(t_{\frac{\beta+\theta}{2}} - t_\beta)$ and $u(t_\theta) - u(t_{\frac{\beta+\theta}{2}}) = u'(t_2)(t_\theta - t_{\frac{\beta+\theta}{2}})$.

Since $u(t_{\frac{\beta+\theta}{2}}) - u(t_\beta) = \frac{\theta-\beta}{2} = u(t_\theta) - u(t_{\frac{\beta+\theta}{2}})$, we have $u'(t_1)(t_{\frac{\beta+\theta}{2}} - t_\beta) = u'(t_2)(t_\theta - t_{\frac{\beta+\theta}{2}})$. Since u' is strictly decreasing on (t_β, t_θ), we have $u'(t_1) > u'(t_2) > 0$. Thus $t_{\frac{\beta+\theta}{2}} - t_\beta < t_\theta - t_{\frac{\beta+\theta}{2}}$. This and (4.1) imply that $\frac{\theta-\beta}{2} > \lambda^{\frac{1}{p-1}} K_{14} (t_{\frac{\beta+\theta}{2}} - t_\beta)^{\frac{p}{p-1}}$ where $K_{14} := \varphi_p^{-1}\left(h_* f\left(\frac{\beta+\theta}{2}\right)\right)$ (> 0). Thus $t_{\frac{\beta+\theta}{2}} - t_\beta \leq \lambda^{-\frac{1}{p}} K_{15}$ where $K_{15} := \left(\frac{\theta-\beta}{2K_{14}}\right)^{\frac{p-1}{p}}$. By Lemma 4.2, we have $t_{\frac{\beta+\theta}{2}} \leq O(\lambda^{-\frac{1}{p}})$. By a similar argument, we can show that $1 - \tilde{t}_{\frac{\beta+\theta}{2}} \leq O(\lambda^{-\frac{1}{p}})$. $\square$

Lemma 4.4. *Let* (H_3) *and* $(H_9) - (H_{10})$ *hold. If* u *is a positive solution of* (1.6), *then there exists* $K^\# > 0$ *such that* $u(t) > \lambda^{\frac{1}{p-1}} K^\#$ *for* $\lambda \gg 1$ *and* $t \in [\frac{1}{4}, \frac{3}{4}]$.

Proof. First assume $t_m > \frac{3}{4}$. By Lemmas 4.2 - 4.3, for $\lambda \gg 1$ we have

$$\begin{aligned} u\left(\frac{1}{4}\right) &= \beta + \int_{t_\beta}^{\frac{1}{4}} \varphi_p^{-1}\left(\lambda \int_s^{t_m} h(r) f(u) dr\right) ds \\ &> \int_{t_\beta}^{\frac{1}{4}} \varphi_p^{-1}\left(\lambda \int_{\frac{1}{4}}^{\frac{3}{4}} h(r) f(u) dr\right) ds \\ &\geq \int_{t_\beta}^{\frac{1}{4}} \varphi_p^{-1}\left(\lambda \frac{h_*}{2} f\left(\frac{\beta+\theta}{2}\right)\right) ds \\ &\geq \lambda^{\frac{1}{p-1}} K_{16} \end{aligned}$$

where $K_{16} := \frac{1}{8}\varphi_p^{-1}\left(\frac{h_*}{2} f\left(\frac{\beta+\theta}{2}\right)\right)$ (> 0). Since u is strictly increasing on $[\frac{1}{4}, \frac{3}{4}]$, we have $u > \lambda^{\frac{1}{p-1}} K_{16}$ on $[\frac{1}{4}, \frac{3}{4}]$ for $\lambda \gg 1$. Similarly, we can show that if $t_m < \frac{1}{4}$ then there exists $K_{17} > 0$ such that $u > \lambda^{\frac{1}{p-1}} K_{17}$ on $[\frac{1}{4}, \frac{3}{4}]$ for $\lambda \gg 1$. If $t_m \in [\frac{1}{4}, \frac{3}{4}]$, we can also show that there exists $K_{18} > 0$ such that $u\left(\frac{1}{8}\right) > \lambda^{\frac{1}{p-1}} K_{18}$ and $u\left(\frac{7}{8}\right) > \lambda^{\frac{1}{p-1}} K_{18}$ for $\lambda \gg 1$. Hence for $\lambda \gg 1$ there exists $K^\# := \min\{K_{16}, K_{17}, K_{18}\} > 0$ such that $u > \lambda^{\frac{1}{p-1}} K^\#$ on $[\frac{1}{4}, \frac{3}{4}]$. $\square$

Next we consider the boundary value problem:

$$\begin{cases} -(\varphi_p(w'(t)))' = h(t)\big(A\chi_{[\frac{1}{4},\frac{3}{4}]}(t) - B\chi_{[0,\frac{1}{4})\cup(\frac{3}{4},1]}(t)\big); \quad t \in (0,1) \\ w(0) = 0 = w(1). \end{cases} \tag{4.2}$$

Note that since $h \in L^1(0,1)$, for each $A > 0$ and $B > 0$, (4.2) has a unique solution $w \in C_0^1[0,1]$ (see [MM00]). Let $A_0 := \max\left\{\frac{4(1+\varphi_p(2))BH}{h_*}, \frac{1+BH}{K_{19}h_*}, \frac{4(\varphi_p(2)+BH)}{h_*}\right\}$ where $H := \int_0^1 h(s)ds$ and $K_{19} := \frac{\varphi_p(2)h_*}{8h\left(\frac{1}{4}\right)}$. We prove:

Lemma 4.5. *Let $h_* := \inf_{t\in(0,1)} h(t)$ (> 0). Let $B > 0$ be fixed and $A > A_0$. Then a unique solution w of* (4.2) *satisfies $w(t) \geq d(t, \partial\Omega)$.*

Proof. Let $t_m \in [0,1]$ be the first point such that w has a local extremum. First we show that w has a local maximum at t_m. Assume that it has a local minimum at t_m. If $t_m \in [\frac{1}{4}, \frac{3}{4})$, then integrating (4.2) from t_m to $t \in (t_m, \frac{3}{4})$ we have $-\varphi_p(w'(t)) = A\int_{t_m}^t h(s)ds > 0$. Thus $w'(t) < 0$ for $t \in (t_m, \frac{3}{4})$. This is a contradiction since w has a local minimum at t_m. If $t_m \in [\frac{3}{4}, 1]$, then integrating (4.2) from 0 to t_m we have $\varphi_p(w'(0)) = -B\int_0^{\frac{1}{4}} h(s)ds + A\int_{\frac{1}{4}}^{\frac{3}{4}} h(s)ds - B\int_{\frac{3}{4}}^{t_m} h(s)ds \geq A\frac{h_*}{2} - 2BH > 0$. This is a contradiction since w has a local minimum at t_m and no local extremum on $(0, t_m)$. Thus $t_m \in [0, \frac{1}{4})$. Then we have

$$w'(t) = \begin{cases} \varphi_p^{-1}\left(-B\int_t^{t_m} h(s)ds\right); \quad t \in [0, t_m) \\ \varphi_p^{-1}\left(B\int_{t_m}^t h(s)ds\right); \quad t \in [t_m, \frac{1}{4}) \\ \varphi_p^{-1}\left(B\int_{t_m}^{\frac{1}{4}} h(s)ds - A\int_{\frac{1}{4}}^t h(s)ds\right); \quad t \in [\frac{1}{4}, \frac{3}{4}) \\ \varphi_p^{-1}\left(B\int_{t_m}^{\frac{1}{4}} h(s)ds - A\int_{\frac{1}{4}}^{\frac{3}{4}} h(s)ds + B\int_{\frac{3}{4}}^t h(s)ds\right); \quad t \in [\frac{3}{4}, 1]. \end{cases}$$

This implies that $w'(t) \le \varphi_p^{-1}(BH)$ on $(0, \frac{1}{2})$. Since $A > \frac{4(1+\varphi_p(2))BH}{h_*}$, we have

$$
\begin{aligned}
w'\left(\frac{1}{2}\right) &= \varphi_p^{-1}\left(B\int_{t_m}^{\frac{1}{4}} h(s)ds - A\int_{\frac{1}{4}}^{\frac{1}{2}} h(s)ds\right) \\
&\le \varphi_p^{-1}\left(BH - A\frac{h_*}{4}\right) \\
&\le -2\varphi_p^{-1}(BH).
\end{aligned}
$$

Since $w'(t)$ is strictly decreasing on $\left(\frac{1}{2}, \frac{3}{4}\right)$, we have $w'(t) \le -2\varphi_p^{-1}(BH)$ on $\left(\frac{1}{2}, \frac{3}{4}\right)$. For $t \in (\frac{3}{4}, 1)$, we obtain

$$
\begin{aligned}
w'(t) &\le \varphi_p^{-1}\left(B\int_{t_m}^{\frac{1}{4}} h(s)ds - A\int_{\frac{1}{4}}^{\frac{3}{4}} h(s)ds + B\int_{\frac{3}{4}}^{1} h(s)ds\right) \\
&\le \varphi_p^{-1}\left(-A\frac{h_*}{2} + 2BH\right) \\
&< 0.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
w(1) &= \int_0^{\frac{1}{2}} w'(s)ds + \int_{\frac{1}{2}}^{\frac{3}{4}} w'(s)ds + \int_{\frac{3}{4}}^{1} w'(s)ds \\
&< \frac{\varphi_p^{-1}(BH)}{2} - \frac{\varphi_p^{-1}(BH)}{2} \\
&= 0.
\end{aligned}
$$

This is a contradiction since $w(1) = 0$. Hence w has a local maximum at t_m.

Next we show that w has a unique interior maximum. If $t_m \in [0, \frac{1}{4})$, then $-\varphi_p(w'(t)) = -B\int_{t_m}^{t} h(s)ds < 0$ for $t \in (t_m, \frac{1}{4})$. Thus $w'(t) > 0$ on $(t_m, \frac{1}{4})$. This is a contradiction since w has a local maximum at t_m. If $t_m \in (\frac{3}{4}, 1]$, then $\varphi_p(w'(t)) = -B\int_{t}^{t_m} h(s)ds < 0$ for $t \in (\frac{3}{4}, t_m)$. Thus $w'(t) < 0$ on $(\frac{3}{4}, t_m)$. This is a contradiction. Hence $t_m \in [\frac{1}{4}, \frac{3}{4}]$. Let $\tilde{t}_m \in [t_m, 1)$ be the largest point in $(0, 1)$ such that w has a local extremum. By a similar argument, we can show that w has a local maximum at $\tilde{t}_m \in [\frac{1}{4}, \frac{3}{4}]$. Now we claim $t_m = \tilde{t}_m$. If not, there exists $\hat{t}_m \in (t_m, \tilde{t}_m)$ such that w has a local minimum at $\hat{t}_m$. Integrating (4.2) from t_m to $\hat{t}_m$, we have $A\int_{t_m}^{\hat{t}_m} h(s)ds = 0$, that is a contradiction. Thus our claim holds. Hence w has a unique interior maximum at $t_m \in [\frac{1}{4}, \frac{3}{4}]$.

Now we show that $w(t) \geq d(t, \partial\Omega)$. If $t_m \in [\frac{1}{4}, \frac{1}{2}]$ we have

$$\begin{aligned} w(t_m) &\geq w\left(\frac{3}{4}\right) \\ &= \int_{\frac{3}{4}}^{1} \varphi_p^{-1}\left(\int_{t_m}^{s} h(r)\left(A\chi_{[\frac{1}{4},\frac{3}{4}]}(r) - B\chi_{[0,\frac{1}{4})\cup(\frac{3}{4},1]}(r)\right)dr\right)ds \\ &\geq \frac{1}{4}\varphi_p^{-1}\left(A\frac{h_*}{4} - BH\right) \end{aligned}$$

and

$$\begin{aligned} w(t_m) &= \int_{0}^{t_m} \varphi_p^{-1}\left(\int_{s}^{t_m} h(r)\left(A\chi_{[\frac{1}{4},\frac{3}{4}]}(r) - B\chi_{[0,\frac{1}{4})\cup(\frac{3}{4},1]}(r)\right)dr\right)ds \\ &\leq \int_{0}^{t_m} \varphi_p^{-1}\left(A\int_{\frac{1}{4}}^{t_m} h(r)dr\right)ds \\ &\leq \frac{1}{2}\varphi_p^{-1}\left(A\Big(t_m - \frac{1}{4}\Big)h\Big(\frac{1}{4}\Big)\right). \end{aligned}$$

Hence

$$\frac{1}{4}\varphi_p^{-1}\left(A\frac{h_*}{4} - BH\right) \le \frac{1}{2}\varphi_p^{-1}\left(A\Big(t_m - \frac{1}{4}\Big)h\Big(\frac{1}{4}\Big)\right)$$

and this implies that $t_m \ge \frac{1}{4} + K_{19}$ where $K_{19} := \frac{\varphi_p(2)h_*}{8h\left(\frac{1}{4}\right)}$ since $A > \frac{4(1+\varphi_p(2))BH}{h_*} > \frac{8BH}{h_*}$. Since $A > \frac{1+BH}{K_{19}h_*}$, we also have

$$\begin{aligned}
w'(0) &= \varphi_p^{-1}\left(\int_0^{t_m} h(s)\left(A\chi_{[\frac{1}{4},\frac{3}{4}]}(s) - B\chi_{[0,\frac{1}{4})\cup(\frac{3}{4},1]}(s)\right)ds\right) \\
&\ge \varphi_p^{-1}\left(A\int_{\frac{1}{4}}^{\frac{1}{4}+K_{19}} h(s)ds - BH\right) \\
&\ge \varphi_p^{-1}\left(AK_{19}h_* - BH\right) \\
&> 1.
\end{aligned}$$

Further, $A > \frac{4(\varphi_p(2)+BH)}{h_*}$ implies that

$$\begin{aligned}
w\left(\frac{3}{4}\right) &= \int_{\frac{3}{4}}^{1} \varphi_p^{-1}\left(\int_{t_m}^{s} h(r)\left(A\chi_{[\frac{1}{4},\frac{3}{4}]}(r) - B\chi_{[0,\frac{1}{4})\cup(\frac{3}{4},1]}(r)\right)dr\right)ds \\
&\ge \frac{1}{4}\varphi_p^{-1}\left(A\frac{h_*}{4} - BH\right) \\
&\ge \frac{1}{2}.
\end{aligned}$$

Thus $w\left(\frac{1}{2}\right) \geq \frac{1}{2}$ since $t_m \in [\frac{1}{4}, \frac{1}{2}]$. Then we obtain that $w(t) \geq t$ on $[0, \frac{1}{2}]$. We also have

$$\begin{aligned}
w'(1) &= -\varphi_p^{-1}\left(\int_{t_m}^{1} h(s)\left(A\chi_{[\frac{1}{4},\frac{3}{4}]}(s) - B\chi_{[0,\frac{1}{4})\cup(\frac{3}{4},1]}(s)\right) ds\right) \\
&\leq -\varphi_p^{-1}\left(A\int_{\frac{1}{2}}^{\frac{3}{4}} h(s)ds - BH\right) \\
&\leq -\varphi_p^{-1}\left(A\frac{h_*}{4} - BH\right) \\
&< -1.
\end{aligned}$$

This implies that $w(t) \geq 1-t$ on $[\frac{1}{2}, 1]$. Thus if $t_m \in [\frac{1}{4}, \frac{1}{2}]$ then $w(t) \geq d(t, \partial\Omega)$. By a similar argument, if $t_m \in [\frac{1}{2}, \frac{3}{4}]$ we can show that $w(t) \geq d(t, \partial\Omega)$ for $A > A_0$. Hence Lemma is proven. □

Lemma 4.6. *Let* (H_3) *and* $(H_9) - (H_{10})$ *hold. Let* u *be a positive solution of* (1.6). *Then* $u(t) \geq \lambda^{\frac{1}{p-1}} d(t, \partial\Omega)$ *for* $\lambda \gg 1$.

Proof. Given $M > 0$, consider the boundary value problem:

$$\begin{cases} -\left(\varphi_p(z'(t))\right)' = h(t)\big(f(M)\chi_{[\frac{1}{4},\frac{3}{4}]}(t) + f(0)\chi_{[0,\frac{1}{4})\cup(\frac{3}{4},1]}(t)\big); \quad t \in (0,1) \\ z(0) = 0 = z(1). \end{cases} \tag{4.3}$$

By (H_3) and (H_{10}), choosing $M \gg 1$, Lemma 4.5 implies that (4.3) has a unique positive solution z such that $z(t) \geq d(t, \partial\Omega)$. Note that $u(t) \geq M$ for $\lambda \gg 1$ and

$t \in [\frac{1}{4}, \frac{3}{4}]$ by Lemma 4.4. Thus for $\lambda \gg 1$ we have

$$-\left(\varphi_p(u')\right)' + \lambda\left(\varphi_p(z')\right)' = \lambda h(t)f(u) - \lambda h(t)\big(f(M)\chi_{[\frac{1}{4},\frac{3}{4}]}(t) + f(0)\chi_{[0,\frac{1}{4})\cup(\frac{3}{4},1]}(t)\big) \geq 0.$$

By the weak comparison principle, we have $u(t) \geq \lambda^{\frac{1}{p-1}} z(t)$. Since $z(t) \geq d(t, \partial\Omega)$, we have $u(t) \geq \lambda^{\frac{1}{p-1}} d(t, \partial\Omega)$ for $\lambda \gg 1$. □

Next we establish the following result that estimates upper and lower bound functions for positive solutions. In fact, it is the same result introduced in Lemma 3.2 but for the semipositone problem:

Lemma 4.7. *Let $p > 1$, $\alpha = 0$, (H_1), (H_3), (H_5) and $(H_9) - (H_{10})$ hold. Let u be a positive solution of (1.6). For $\lambda \gg 1$, there exist constants C_1 and C_2 (independent of λ) with $0 < C_1 < C_2$ such that*

$$C_1 G^{-1}(\lambda^{\frac{1}{p-1}}) d(t, \partial\Omega) \leq u(t) \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}}) d(t, \partial\Omega).$$

Proof. First we show that for $\lambda \gg 1$ there exists $C_1 > 0$ such that

$$u(t) \geq C_1 G^{-1}(\lambda^{\frac{1}{p-1}}) d(t, \partial\Omega).$$

By Lemma 4.6, we have $u(t) \geq \lambda^{\frac{1}{p-1}} z(t) \geq \lambda^{\frac{1}{p-1}} d(t, \partial\Omega)$ for $\lambda \gg 1$. Let K_λ (≥ 1) be the largest constant such that $u(t) \geq \lambda^{\frac{1}{p-1}} K_\lambda z(t)$. Then $u(t) \geq K_\lambda^* := \frac{\lambda^{\frac{1}{p-1}} K_\lambda K_*}{4}$ on

$[\frac{1}{4}, \frac{3}{4}]$. For $\lambda \gg 1$ and $t \in [\frac{1}{4}, \frac{3}{4}]$, we have

$$\begin{aligned}
&-f(M)\left(\varphi_p(u')\right)' + \lambda f(K_\lambda^*)\left(\varphi_p(z')\right)' \\
&\qquad = \lambda f(M)h(t)f(u) - \lambda f(K_\lambda^*)h(t)\big(f(M)\chi_{[\frac{1}{4},\frac{3}{4}]}(t) + f(0)\chi_{[0,\frac{1}{4})\cup(\frac{3}{4},1]}(t)\big) \\
&\qquad = \lambda f(M)h(t)\big(f(u) - f(K_\lambda^*)\big) \\
&\qquad \geq 0
\end{aligned}$$

where M was chosen in the proof of Lemma 4.6. Since $K_\lambda \geq 1$, we have $K_\lambda^* \geq M$ for $\lambda \gg 1$. For $\lambda \gg 1$ and $t \in [0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$, we have

$$\begin{aligned}
&-f(M)\left(\varphi_p(u')\right)' + \lambda f(K_\lambda^*)\left(\varphi_p(z')\right)' \\
&\qquad = \lambda f(M)h(t)f(u) - \lambda f(K_\lambda^*)h(t)\big(f(M)\chi_{[\frac{1}{4},\frac{3}{4}]}(t) + f(0)\chi_{[0,\frac{1}{4})\cup(\frac{3}{4},1]}(t)\big) \\
&\qquad = \lambda h(t)\big(f(M)f(u) - f(0)f(K_\lambda^*)\big) \\
&\qquad \geq 0.
\end{aligned}$$

By the comparison principle, we have $[f(M)]^{\frac{1}{p-1}}u(t) \geq \lambda^{\frac{1}{p-1}}[f(K_\lambda^*)]^{\frac{1}{p-1}}z(t)$ for $\lambda \gg 1$. Hence for $\lambda \gg 1$ we have

$$u(t) \geq \lambda^{\frac{1}{p-1}} \frac{[f(K_\lambda^*)]^{\frac{1}{p-1}}}{[f(M)]^{\frac{1}{p-1}}} z(t).$$

This implies that $K_\lambda \geq \frac{[f(K_\lambda^*)]^{\frac{1}{p-1}}}{[f(M)]^{\frac{1}{p-1}}}$ for $\lambda \gg 1$. Thus we obtain

$$\frac{\lambda^{\frac{1}{p-1}} K_\lambda K_*}{4} = K_\lambda^* = G^{-1}\left(\frac{K_\lambda^*}{[f(K_\lambda^*)]^{\frac{1}{p-1}}}\right) \geq G^{-1}\left(\frac{\lambda^{\frac{1}{p-1}} K_*}{4[f(M)]^{\frac{1}{p-1}}}\right).$$

By Lemma 3.1, for $\lambda \gg 1$ there exists $C_1 > 0$ such that

$$4G^{-1}\left(\frac{\lambda^{\frac{1}{p-1}} K_*}{4[f(M)]^{\frac{1}{p-1}}}\right) \geq C_1 G^{-1}(\lambda^{\frac{1}{p-1}}).$$

Thus for $\lambda \gg 1$ we have

$$u(t) \geq \lambda^{\frac{1}{p-1}} K_\lambda z(t) \geq \frac{C_1}{K_*} G^{-1}(\lambda^{\frac{1}{p-1}}) z(t) \geq C_1 G^{-1}(\lambda^{\frac{1}{p-1}}) d(t, \partial\Omega).$$

Next we show that for $\lambda \gg 1$ there exists $C_2 > 0$ such that

$$u(t) \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}}) d(t, \partial\Omega).$$

We follow the argument in Lemma 3.2. For $\lambda \gg 1$, we have $\|u'\|_\infty \geq \sigma_0$, and hence $\sigma_0 \leq \|u\|_1 \leq 2\lambda^{\frac{1}{p-1}} [f(\|u\|_1)]^{\frac{1}{p-1}} \varphi_p^{-1}(H)$ where $H := \int_0^1 h(s)ds$. This implies that

$$\|u'\|_\infty \leq G^{-1}\left(\frac{\|u\|_1}{[f(\|u\|_1)]^{\frac{1}{p-1}}}\right) \leq G^{-1}\big(2\lambda^{\frac{1}{p-1}} \varphi_p^{-1}(H)\big).$$

By Lemma 3.1, for $\lambda \gg 1$ there exists $C_2 > 0$ such that

$$G^{-1}\big(2\lambda^{\frac{1}{p-1}} \varphi_p^{-1}(H)\big) \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}}).$$

Thus we have $u(t) \le \|u'\|_\infty d(t,\partial\Omega) \le C_2 G^{-1}(\lambda^{\frac{1}{p-1}}) d(t,\partial\Omega)$ for $\lambda \gg 1$. Hence the proof is complete. $\square$

Using Lemma 4.7 and the ideas in the proof of Lemma 3.3, one can establish:

Lemma 4.8. *Let $p > 1$, $\alpha = 0$, (H_1), (H_3), (H_5) and $(H_9) - (H_{10})$ hold. Let u and v be the positive solution of* (1.6)*. Let $\gamma_0 \le \gamma < 1$ where $\gamma_0 := \frac{C_1}{C_2}$. For $\lambda \gg 1$, there exists $\delta > 0$ (independent of λ) such that $\frac{C_1\gamma_0}{2} G^{-1}(\lambda^{\frac{1}{p-1}}) \le |sv'(t) + (1-s)\gamma u'(t)| \le C_2 G^{-1}(\lambda^{\frac{1}{p-1}})$ for $s \in [0,1]$ and $t \in [0,\delta]$.*

Now we prove Theorem 1.3 by arguments similar to those in the proof of Theorem 1.1. Let u and v be positive solutions of (1.6). By Lemma 4.7, we have $v \ge \gamma_0 u$ for $\lambda \gg 1$ where $\gamma_0 := \frac{C_1}{C_2}$. Let γ be the largest constant such that $v \ge \gamma u$.

We show that $\gamma \ge 1$. Assume $\gamma < 1$. Let $\tilde{u} := \frac{u}{G^{-1}(\lambda^{\frac{1}{p-1}})}$ and $\tilde{v} := \frac{v}{G^{-1}(\lambda^{\frac{1}{p-1}})}$. Since $G^{-1}(\lambda^{\frac{1}{p-1}}) = \lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}}\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)$, we obtain that

$$-\left(\varphi_p\left(\tilde{u}'\right)\right)' = h(t)\frac{f(u)}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)} \quad \text{and} \quad -\left(\varphi_p\left(\tilde{v}'\right)\right)' = h(t)\frac{f(v)}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)}.$$

By the Mean Value Theorem, we have

$$\varphi_p(\tilde{v}') - \varphi_p(\gamma\tilde{u}') = a_\lambda(t)\left(\tilde{v}' - \gamma\tilde{u}'\right)$$

for $t \in \Omega_\delta := [0,\delta] \cup [1-\delta,1]$ where $a_\lambda(t) := (p-1)\int_0^1 |s\tilde{v}'(t) + (1-s)\gamma\tilde{u}'(t)|^{p-2} ds$. This implies that

$$-\left(a_\lambda(t)\left(\tilde{v}' - \gamma\tilde{u}'\right)\right)' = h(t)\frac{f(v) - \gamma^{p-1} f(u)}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)} \ge h(t)\frac{f(\gamma u) - \gamma^{p-1} f(u)}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)}.$$

By (H_5) and (H_{10}), without loss of generality, we assume $\sigma_0 \gg 1$ so that $\frac{\sigma_0}{\gamma_0} > \frac{\beta+\theta}{2}$ and $f\big(\frac{\sigma_0}{\gamma_0}\big) \geq |f(0)|$ hold. Let $I := \{t \in \Omega_\delta \mid u(t) \geq \frac{\sigma_0}{\gamma_0}\}$. It is easy to show that $I = [t_I, \delta] \cup [1-\delta, \tilde{t}_I]$ where $t_I := \min\{t \in (0,1) \mid u(t) \geq \frac{\sigma_0}{\gamma_0}\}$ and $\tilde{t}_I := \max\{t \in (0,1) \mid u(t) \geq \frac{\sigma_0}{\gamma_0}\}$. On I, it follows from (H_5) that

$$f(\gamma u) - \gamma^{p-1} f(u) \geq (\gamma^q - \gamma^{p-1}) f(u) \geq m_1(1-\gamma)$$

where $m_1 := (p-1-q)\gamma_0^q f\big(\frac{\sigma_0}{\gamma_0}\big) \min\{1, \gamma_0^{p-2-q}\}$. On $\Omega_\delta \setminus I$, we have

$$\begin{aligned}
|f(\gamma u) - \gamma^{p-1} f(u)| &\leq |f(\gamma u) - f(u)| + (1-\gamma^{p-1}) f(u) \\
&\leq (1-\gamma)|u f'(\zeta)| + (1-\gamma)(p-1)\max\{1, \gamma_0^{p-2}\} f\Big(\frac{\sigma_0}{\gamma_0}\Big) \\
&\leq \frac{1-\gamma}{\gamma_0}|\zeta f'(\zeta)| + (1-\gamma)(p-1)\max\{1, \gamma_0^{p-2}\} f\Big(\frac{\sigma_0}{\gamma_0}\Big) \\
&\leq m_2(1-\gamma)
\end{aligned}$$

where $m_2 := \frac{1}{\gamma_0}\sup_{s\in(0,\frac{\sigma_0}{\gamma_0})} |sf'(s)| + (p-1)\max\{1, \gamma_0^{p-2}\} f\big(\frac{\sigma_0}{\gamma_0}\big)$ and $\zeta(t) \in (\gamma u(t), u(t))$. Then we have

$$-\left(\frac{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)}{1-\gamma} a_\lambda(t)(\tilde{v}' - \gamma \tilde{u}')\right)' \geq \begin{cases} m_1 h(t); & t \in I \\ -m_2 h(t); & t \in \Omega_\delta \setminus I. \end{cases}$$

Let $\overline{\kappa}$ be the solution of the boundary value problem:

$$-\left(a_\lambda(t)\overline{\kappa}'(t)\right)' = \begin{cases} m_1 h(t); & t \in I \\ -m_2 h(t); & t \in \Omega_\delta \setminus I \end{cases}$$

$$\overline{\kappa} \equiv 0 \quad \text{ on } \partial\Omega_\delta.$$

By the comparison principle, we have

$$\tilde{v}(t) - \gamma\tilde{u}(t) \geq \frac{1-\gamma}{f\left(G^{-1}(\lambda^{\frac{1}{p-1}})\right)}\overline{\kappa}(t)$$

for $t \in \Omega_\delta$. Let κ be the solution of the boundary value problem:

$$-\left(a_\lambda(t)\kappa'(t)\right)' = \begin{cases} m_1 h(t); & t \in I \\ 0; & t \in \Omega_\delta \setminus I \end{cases}$$

$$\kappa \equiv 0 \quad \text{ on } \partial\Omega_\delta.$$

We note that $t_I \to 0$ and $\tilde{t}_I \to 1$ as $\lambda \to \infty$ by Lemma 4.7. Then by Lemmas 3.4 and 3.6 and Corollaries 3.5 and 3.7, we obtain that $\kappa(t) \geq \widehat{M}_1 d(t, \partial\Omega_\delta)$ and $|\kappa(t) - \overline{\kappa}(t)| \leq \widehat{M}_2 |\Omega_\delta \setminus I|^{1-\eta} d(t, \partial\Omega_\delta)$ for $\lambda \gg 1$ where $\widehat{M}_1 := \min\{\overline{M}, \widetilde{M}\}$, $\widehat{M}_2 := \max\{M^*, M_*\}$ and $|\Omega_\delta \setminus I|$ is the length of $\Omega_\delta \setminus I$. Since $|\Omega_\delta \setminus I| \to 0$ as $\lambda \to \infty$, we have

$$\overline{\kappa}(t) \geq \kappa(t) - |\kappa(t) - \overline{\kappa}(t)| \geq \left(\widehat{M}_1 - \widehat{M}_2|\Omega_\delta \setminus I|^{1-\eta}\right)d(t, \partial\Omega_\delta) \geq \frac{\widehat{M}_1}{2} d(t, \partial\Omega_\delta)$$

for $\lambda \gg 1$ and $t \in \Omega_\delta$. Since $d(t, \partial\Omega_\delta) = d(t, \partial\Omega)$ for $t \in \Omega_{\frac{\delta}{2}} := [0, \frac{\delta}{2}] \cup [1 - \frac{\delta}{2}, 1]$, we have

$$\overline{\kappa}(t) \geq \frac{\widehat{M}_1}{2} d(t, \partial\Omega)$$

for $t \in \Omega_{\frac{\delta}{2}}$. This implies that

$$\tilde{v}(t) - \gamma\tilde{u}(t) \geq \frac{1-\gamma}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)} \overline{\kappa}(t) \geq \frac{\widehat{M}_1(1-\gamma)}{2f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)} d(t, \partial\Omega)$$

for $t \in \Omega_{\frac{\delta}{2}}$. By Lemma 4.7, we obtain

$$v(t) \geq (\gamma + \epsilon_\lambda) u(t)$$

for $t \in \Omega_{\frac{\delta}{2}}$ where $\epsilon_\lambda := \frac{\widehat{M}_1(1-\gamma)}{2C_2 f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)}$. Thus $v(t) \geq \gamma u(t) + \tilde{\epsilon}_\lambda$ for $t \in \left\{\frac{\delta}{2}, 1 - \frac{\delta}{2}\right\}$ where $\tilde{\epsilon}_\lambda := \epsilon_\lambda C_1 G^{-1}(\lambda^{\frac{1}{p-1}})\frac{\delta}{2}$. Further, we have

$$u(t) \geq C_1 G^{-1}(\lambda^{\frac{1}{p-1}}) \frac{\delta}{2} \geq \frac{\sigma_0}{\gamma_0}$$

for $\lambda \gg 1$ and $t \in [0, 1] \setminus \Omega_{\frac{\delta}{2}}$. By (H_5), we have

$$-\left(\varphi_p(v')\right)' = \lambda h(t) f(v) \geq \lambda h(t) f(\gamma u) \geq \lambda \gamma^{p-1} h(t) f(u)$$

on $[0,1]\setminus\Omega_{\frac{\delta}{2}}$. But

$$-\left(\varphi_p((\gamma u+\tilde{\epsilon}_\lambda)')\right)' = \lambda\gamma^{p-1}h(t)f(u)$$

on $[0,1]\setminus\Omega_{\frac{\delta}{2}}$. Thus $v \geq \gamma u+\tilde{\epsilon}_\lambda$ on $[0,1]\setminus\Omega_{\frac{\delta}{2}}$. Then

$$v \geq \gamma u+\tilde{\epsilon}_\lambda d(t,\partial\Omega) \geq \Big(\gamma+\frac{C_1}{C_2}\epsilon_\lambda\Big)u$$

on $[0,1]\setminus\Omega_{\frac{\delta}{2}}$. Since $C_1 < C_2$, we have $v \geq \big(\gamma+\frac{C_1}{C_2}\epsilon_\lambda\big)u$ on $[0,1]$. This is a contradiction for the maximality of γ. Hence $\gamma \geq 1$. This implies that $v \equiv u$ on $[0,1]$ for $\lambda \gg 1$.

CHAPTER V

PROOFS OF THEOREMS 1.4 - 1.8

5.1 Proof of Theorem 1.4

Let $f^*(s) := \max_{0\le r\le s} f(r)$. Then $f^*(s)$ is nondecreasing and $\frac{f^*(s)}{s^{1+\alpha}} \to 0$ as $s\to\infty$. Thus there exists $M_\lambda \gg 1$ such that $\frac{f^*(M_\lambda\|w\|_\infty)}{(M_\lambda\|w\|_\infty)^{1+\alpha}} \le \frac{1}{\lambda\|w\|_\infty^{1+\alpha}}$. Let $\phi_1 := M_\lambda w$. Then we have

$$-\phi_1'' = M_\lambda \frac{h(t)}{w^\alpha} \ge \lambda h(t)\frac{f^*(M_\lambda\|w\|_\infty)}{(M_\lambda w)^\alpha} \ge \lambda h(t)\frac{f(M_\lambda w)}{(M_\lambda w)^\alpha} = \lambda h(t)\frac{f(\phi_1)}{\phi_1^\alpha}.$$

We also have $\phi_1'(1) + c(\phi_1(1))\phi_1(1) > 0$ since $w'(1) > 0$ and $w(1) > 0$. Thus ϕ_1 is a positive supersolution of (1.9).

Next we construct a positive subsolution ψ_1 of (1.9). Let $e \in C^2(0,1)\cap C^1[0,1]$ be the unique positive solution of (3.4). By (H_{11}), there exists $m_\lambda \approx 0$ such that $m_\lambda^{1+\alpha}\|e\|_\infty^\alpha \le \lambda \min_{s\in[0,\|e\|_\infty]} f(s)$. Let $\psi_1 := m_\lambda e$. Then we have

$$-\psi_1'' = m_\lambda h(t) \le \lambda h(t)\frac{f(m_\lambda e)}{(m_\lambda e)^\alpha} = \lambda h(t)\frac{f(\psi_1)}{\psi_1^\alpha}.$$

We also have $\psi_1'(1) + c(\psi_1(1))\psi_1(1) < 0$ since $e'(1) < 0$ and $e(1) = 0$. It is easy to show that $\psi_1(t) \ge M_1 d(t,\partial\Omega)$ for some $M_1 > 0$. Thus ψ_1 is a positive subsolution of (1.9).

We can again choose m_λ such that $\psi_1 \le \phi_1$. Hence there exists a positive solution of (1.9) for each $\lambda > 0$ by Lemma 2.5.

5.2 Proof of Theorem 1.5

We first prove the following Lemma that we need to establish Theorem 1.5.

Lemma 5.1. *Let* (H_1), (H_6) *and* (H_{11}) *hold. There exists* $\hat{\lambda} > 0$ *such that if* u *is a positive solution of* (1.9) *for* $\lambda \in (0, \hat{\lambda})$, *then* $\|u\|_\infty < 1$.

Proof. Let t_m be such that $u(t_m) = \|u\|_\infty$. Integrating (1.9) from t to t_m, we have

$$u'(t) = \lambda \int_t^{t_m} h(s) \frac{f(u)}{u^\alpha} ds.$$

Integrating again from 0 to t_m, we have

$$u(t_m) = \lambda \int_0^{t_m} \int_s^{t_m} h(r) \frac{f(u)}{u^\alpha} drds.$$

Note that u is concave by (H_{11}). Thus $u(t) \geq \|u\|_\infty t$ for $t \in [0, t_m]$. Then we obtain

$$\begin{aligned}
\|u\|_\infty &\leq \lambda \int_0^{t_m} \int_s^{t_m} h(r) \frac{f^*(\|u\|_\infty)}{\|u\|_\infty^\alpha r^\alpha} drds \\
&\leq \lambda \frac{f^*(\|u\|_\infty)}{\|u\|_\infty^\alpha} \int_0^{t_m} \int_s^{t_m} \frac{d}{r^{\alpha+\eta}} drds \\
&\leq \lambda \frac{f^*(\|u\|_\infty)}{\|u\|_\infty^\alpha} \int_0^1 \frac{d}{s^{\alpha+\eta}} ds.
\end{aligned}$$

Since $\alpha + \eta < 1$, there exists $M_2 > 0$ such that $\int_0^1 \frac{d}{s^{\alpha+\eta}} ds \leq M_2$. Thus we have

$$1 \leq \lambda M_2 \frac{f^*(\|u\|_\infty)}{\|u\|_\infty^{1+\alpha}}.$$

Since $\frac{f^*(s)}{s^{1+\alpha}} \to 0$ as $s \to \infty$, there exists $M_3 > 0$ such that $\frac{f^*(s)}{s^{1+\alpha}} \leq M_3$ for $s \geq 1$. Let $\hat{\lambda} := \frac{1}{M_2 M_3}$. If $\|u\|_\infty \geq 1$, then $1 \leq \lambda M_2 M_3 = \frac{\lambda}{\hat{\lambda}}$. This is a contradiction for $\lambda \in (0, \hat{\lambda})$. Thus $\|u\|_\infty < 1$ for $\lambda \in (0, \hat{\lambda})$. $\square$

Now we prove Theorem 1.5. Assume that there exist distinct positive solutions u and v of (1.9). Let $t_1 \in (0,1]$ be such that $\|v-u\|_\infty = v(t_1) - u(t_1)$. If $t_1 \in (0,1)$, then $v'(t_1) - u'(t_1) = 0$. If $t_1 = 1$, then we have

$$v'(t_1) - u'(t_1) = v'(1) - u'(1) = -c(v(1))v(1) + c(u(1))u(1).$$

Since $\|v-u\|_\infty = v(t_1) - u(t_1)$ and $c(s)s$ is nondecreasing for $s \in [0,\infty)$, we have $v'(t_1) - u'(t_1) = 0$. Choose $t_2 \in [0, t_1)$ such that $v(t_2) - u(t_2) = 0$ and $v(t) - u(t) > 0$ for $t \in (t_2, t_1]$. Then we have

$$\begin{aligned} v(t) - u(t) &= \lambda \int_{t_2}^{t} \int_{s}^{t_1} h(r)\left(g(v) - g(u)\right) drds \\ &= \lambda \int_{t_2}^{t} \int_{s}^{t_1} h(r) g'(\theta(r))(v-u) drds \end{aligned}$$

where $g(s) := \frac{f(s)}{s^\alpha}$ and $\theta(t) \in [\min\{u(t), v(t)\}, \max\{u(t), v(t)\}]$. By (H_7), it follows that $g'(s) < 0$ for $s \approx 0$. Hence there exists $M_4 > 0$ such that $g'(s) \leq M_4$ for $s \in (0,1]$. Thus we have $\|v-u\|_\infty \leq \lambda M_4 \|v-u\|_\infty \int_0^1 h(s)ds$ for $\lambda < \hat{\lambda}$ by Lemma 5.1, and consequently $1 \leq \lambda M_4 \int_0^1 h(s)ds$. This is a contradiction for $\lambda \approx 0$. Hence (1.9) has a unique positive solution for $\lambda \approx 0$.

5.3 Proof of Theorem 1.6

We prove the following Lemma to establish Theorem 1.6.

Lemma 5.2. *Let* (H_4) *hold and* c *be bounded. If* u *is a positive solution of* (1.9)*, then there exists* $\delta_\lambda > 0$ *such that* $u \geq \delta_\lambda l$ *in* $(0,1)$*, where* $\delta_\lambda^{1+\alpha} = \frac{\lambda}{\|l\|_\infty^\alpha}$ *and* $l \in C^2(0,1) \cap C^1[0,1]$ *is the unique positive solution of the problem (see [BKLS14]):*

$$\begin{cases} -l''(t) = f_* h(t), \quad t \in (0,1) \\ l'(1) + c^* l(1) = 0 \\ l(0) = 0 \end{cases}$$

where $c^* := \sup_{s \in [0,\infty)} c(s)$.

Proof. Let u be a positive solution of (1.9). Suppose $u(t) < \delta_\lambda l(t)$ for some $t \in (0,1]$. If $u(1) - \delta_\lambda l(1) \geq 0$, then there exists an interval $(a,b) \subset (0,1)$ such that $u(t) - \delta_\lambda l(t) < 0$ in (a,b) and $u(a) - \delta_\lambda l(a) = 0 = u(b) - \delta_\lambda l(b)$. In (a,b), we have

$$\begin{aligned} -(u - \delta_\lambda l)'' &= \lambda h(t) \frac{f(u)}{u^\alpha} - \delta_\lambda f_* h(t) \\ &\geq \lambda h(t) \frac{f_*}{\delta_\lambda^\alpha l^\alpha} - \delta_\lambda f_* h(t) \\ &\geq \lambda h(t) \frac{f_*}{\delta_\lambda^\alpha \|l\|_\infty^\alpha} - \delta_\lambda f_* h(t) \\ &= 0. \end{aligned}$$

By concavity, $u(t) - \delta_\lambda l(t) = 0$ on (a,b). This is a contradiction. Thus $u(1) - \delta_\lambda l(1) < 0$. Then there exists $a \in [0,1)$ such that $u(t) - \delta_\lambda l(t) < 0$ in $(a,1]$ and $u(a) - \delta_\lambda l(a) = 0$.

By a similar argument, $u - \delta_\lambda l$ is concave on $(a, 1)$. However, we have

$$u'(1) - \delta_\lambda l'(1) = -c(u(1))u(1) + c^*\delta_\lambda l(1) > \delta_\lambda l(1)(-c(u(1)) + c^*) \geq 0.$$

This is a contradiction. Thus Lemma is proven. $\square$

Now we prove Theorem 1.6. Let u and v be positive solutions of (1.9) with $u \not\equiv v$ on $[0, 1]$. Without loss of generality, there exists $t_3 \in (0, 1]$ such that $v'(t_3) - u'(t_3) = 0$, $v(t_3) - u(t_3) > 0$ and $v(t_3) - u(t_3) \geq v(t) - u(t)$ for $t \in [t_3, 1]$ since $c(s)s$ is nondecreasing for $s \in [0, \infty)$. Then

$$\int_0^{t_3} -(v-u)''(v-u)ds = \lambda \int_0^{t_3} h(s)\left(g(v) - g(u)\right)(v-u)ds.$$

By integration by parts, we have

$$\int_0^{t_3} -(v-u)''(v-u)ds = \int_0^{t_3} |v' - u'|^2 ds.$$

By the Fundamental Theorem of Calculus, we have

$$\begin{aligned}
&\lambda \int_0^{t_3} h(s)\left(g(v) - g(u)\right)(v-u)ds \\
&= \lambda \int_0^{t_3} h(s)\left(\int_0^1 g'(u + r(v-u))(v-u)dr\right)(v-u)ds \\
&= \lambda \int_0^{t_3} h(s)\zeta(s)|v-u|^2 ds
\end{aligned}$$

where $\zeta(t) := \int_0^1 g'(u + r(v-u))dr$. Thus we have

$$\int_0^{t_3} |v' - u'|^2 ds = \lambda \int_0^{t_3} h(s)\zeta(s)|v-u|^2 ds. \tag{5.1}$$

Let $b_\lambda := \frac{\sigma^*}{\delta_\lambda l(1)} = \lambda^{-\frac{1}{1+\alpha}} M_5$ where $M_5 := \frac{\sigma^* \|l\|_\infty^{\frac{\alpha}{1+\alpha}}}{l(1)}$.

We first obtain a contradiction in the case when $t_3 \leq b_\lambda$. By (H_7) and (H_8), we can choose $M_6 > 0$ such that $g'(s) \leq M_6$ for $s \in (0, \infty)$. From (5.1), we obtain

$$\int_0^{t_3} |v' - u'|^2 ds \leq \lambda M_6 \int_0^{t_3} h(s)|v-u|^2 ds.$$

By Hölder's inequality, we have

$$|v(t) - u(t)| \leq t^{\frac{1}{2}} \left(\int_0^t |v' - u'|^2 ds \right)^{\frac{1}{2}}. \tag{5.2}$$

This implies that

$$\begin{aligned}
\int_0^{t_3} |v' - u'|^2 ds &\leq \lambda M_6 \int_0^{t_3} h(s) \left(s \int_0^s |v' - u'|^2 dr \right) ds \\
&\leq \lambda b_\lambda M_6 \left(\int_0^{t_3} h(s) ds \right) \left(\int_0^{t_3} |v' - u'|^2 ds \right) \\
&\leq \lambda^{\frac{\alpha}{1+\alpha}} M_5 M_6 \left(\int_0^{b_\lambda} h(s) ds \right) \left(\int_0^{t_3} |v' - u'|^2 ds \right).
\end{aligned}$$

If $\int_0^{t_3} |v' - u'|^2 ds \neq 0$, then

$$1 \leq \lambda^{\frac{\alpha}{1+\alpha}} M_5 M_6 \int_0^{b_\lambda} h(s) ds. \tag{5.3}$$

Since $h(t) \leq \frac{d}{t^\eta}$ for $t \in (0, 1]$, we have

$$\int_0^{b_\lambda} h(s) ds \leq \int_0^{b_\lambda} \frac{d}{s^\eta} ds = \frac{d}{1-\eta} b_\lambda^{1-\eta} = \lambda^{-\frac{1-\eta}{1+\alpha}} \frac{d M_5^{1-\eta}}{1-\eta}.$$

Thus we have $1 \leq \lambda^{\frac{\alpha+\eta-1}{1+\alpha}} \frac{d M_5^{2-\eta} M_6}{1-\eta}$. This is a contradiction for $\lambda \gg 1$ by (H_6). Hence $\int_0^{t_3} |v' - u'|^2 ds = 0$. It follows that $v - u$ is a constant on $[0, t_3]$. Since $v(0) = u(0)$, we have $v(t) = u(t)$ for $t \in [0, t_3]$. This is a contradiction because $v(t_3) - u(t_3) > 0$. Thus $t_3 > b_\lambda$.

Then again from (5.1), we have

$$\int_0^{t_3} |v' - u'|^2 ds = \lambda \int_0^{b_\lambda} h(s)\zeta(s)|v - u|^2 ds + \lambda \int_{b_\lambda}^{t_3} h(s)\zeta(s)|v - u|^2 ds.$$

Since l is concave, $l(t) \geq l(1)t$ for $t \in (0, 1)$. For $t \in (b_\lambda, 1)$, we have

$$u(t) + s(v(t) - u(t)) = (1 - s)u(t) + sv(t) \geq \delta_\lambda l(t) \geq \delta_\lambda l(1)t \geq \sigma^*.$$

By (H_8), $\zeta(t) \leq 0$ for $t \in (b_\lambda, 1)$. Thus we have $\lambda \int_{b_\lambda}^{t_3} h(s)\zeta(s)|v - u|^2 ds \leq 0$ and

$$\int_0^{t_3} |v' - u'|^2 ds \leq \lambda \int_0^{b_\lambda} h(s)\zeta(s)|v - u|^2 ds.$$

Since $g'(s) \leq M_6$ for $s \in (0, \infty)$ and from (5.2), we have

$$\int_0^{t_3} |v' - u'|^2 ds \leq \lambda^{\frac{\alpha}{1+\alpha}} M_5 M_6 \left(\int_0^{b_\lambda} h(s) ds \right) \left(\int_0^{t_3} |v' - u'|^2 ds \right).$$

If $\int_0^{t_3} |v' - u'|^2 ds \neq 0$, then we again obtain (5.3), and by the same argument as before, we have a contradiction. Hence (1.9) has a unique positive solution for $\lambda \gg 1$.

5.4 Proof of Theorem 1.7

Let u and v be positive solutions of (1.9) such that $u \not\equiv v$ on $[0, 1]$. Without loss of generality, let $t_4 \in [0, 1)$ be such that $v(t_4) - u(t_4) = 0$, $v(t) - u(t) \geq 0$ on $[t_4, 1]$ and $v(t) - u(t) > 0$ for some $(a, b) \subset (t_4, 1]$. For $t \in (a, b)$, we have

$$-(uv'' - vu'') = \lambda h(t) \left(u \frac{f(v)}{v^\alpha} - v \frac{f(u)}{u^\alpha} \right) = \lambda h(t) \frac{f(u)}{u^\alpha} \frac{f(v)}{v^\alpha} \left(\frac{u^{1+\alpha}}{f(u)} - \frac{v^{1+\alpha}}{f(v)} \right) < 0.$$

Thus $\int_{t_4}^1 -(uv'' - vu'') ds < 0$. We note that $v'(t_4) \geq u'(t_4)$ since $v(t_4) = u(t_4)$. Since c is nondecreasing, we obtain

$$\begin{aligned} \int_{t_4}^1 -(uv'' - vu'') ds &= -u(1)v'(1) + v(1)u'(1) + u(t_4)v'(t_4) - u'(t_4)v(t_4) \\ &\geq u(1)v(1) \left(c(v(1)) - c(u(1)) \right) \\ &\geq 0. \end{aligned}$$

This is a contradiction. Therefore $u \equiv v$ on $[0, 1]$. Hence (1.9) has a unique positive solution for all $\lambda > 0$.

5.5 Proof of Theorem 1.8

Let $\phi_2 := \frac{a_1}{\|w\|_\infty} w$. For $\lambda \leq \frac{1}{\|w\|_\infty^{1+\alpha}} \frac{a_1^{1+\alpha}}{f(a_1)}$, we have

$$-\phi_2'' = \frac{a_1}{\|w\|_\infty} \frac{h(t)}{w^\alpha} \geq \lambda h(t) \frac{f(a_1)}{\left(\frac{a_1}{\|w\|_\infty} w\right)^\alpha} \geq \lambda h(t) \frac{f(\phi_2)}{\phi_2^\alpha}.$$

We also have $\phi_2'(1) + c(\phi_2(1))\phi_2(1) > 0$ since $w'(1) > 0$ and $w(1) > 0$. Thus ϕ_2 is a positive supersolution of (1.9) with $\|\phi_2\|_\infty = a_1$.

Next we construct a positive subsolution ψ_2 of (1.9) for $\frac{16}{h_*} \frac{a_3^{1+\alpha}}{f(a_3)} < \lambda \leq \frac{2a_2}{h_*} \frac{a_3^\alpha}{f(a_3)}$. Let $a^* \in (0, a_1]$ be such that $\frac{f(a^*)}{a^{*\alpha}} := \inf_{0<r\leq a_1} \frac{f(r)}{r^\alpha}$. Define $\tilde{g} \in C[0,\infty)$ such that $\tilde{g}$ is nondecreasing on $[0, a_1]$, $\tilde{g}(s) \leq \frac{f(s)}{s^\alpha}$ for $s > 0$ and

$$\tilde{g}(s) = \begin{cases} \frac{f(a^*)}{a^{*\alpha}}; \quad s \leq a^* \\ \frac{f(s)}{s^\alpha}; \quad s \geq a_1. \end{cases}$$

For $0 < \epsilon < \frac{1}{2}$ and δ, $\eta > 1$, define $\rho(t) : [0,1] \to [0,1]$ by

$$\rho(t) := \begin{cases} 1 - (1 - (\frac{t}{\epsilon})^\eta)^\delta; \quad 0 \leq t \leq \epsilon \\ 1; \quad \epsilon < t \leq \frac{1}{2} \end{cases}$$

and $\rho(t) = \rho(1-t)$ for $t \in [\frac{1}{2}, 1]$. Then

$$\rho'(t) = \begin{cases} \frac{\delta\eta}{\epsilon} (\frac{t}{\epsilon})^{\eta-1} \left(1 - (\frac{t}{\epsilon})^\eta\right)^{\delta-1}; \quad 0 \leq t \leq \epsilon \\ 0; \quad \epsilon < t \leq \frac{1}{2}. \end{cases}$$

Let $v(t) := a_3\rho(t)$. Define $\tilde{v}$ as the positive solution of

$$\begin{cases} -\tilde{v}'' = \lambda h_* \tilde{g}(v); \quad t \in (0,1) \\ \tilde{v}(0) = 0 = \tilde{v}(1). \end{cases}$$

Define ψ_2 as the positive solution of

$$\begin{cases} -\psi_2'' = \lambda h_* \tilde{g}(v); \quad t \in (0,1) \\ \psi_2'(1) + c(\psi_2(1))\psi_2(1) = 0 \\ \psi_2(0) = 0. \end{cases}$$

By the maximum principle, it is easy to show that $\psi_2 \geq \tilde{v}$. Now we claim that for $\lambda \in (\frac{16}{h_*}\frac{a_3^{1+\alpha}}{f(a_3)}, \frac{2a_2}{h_*}\frac{a_3^{\alpha}}{f(a_3)}]$ and for $t \in (0,1]$, we have

$$\tilde{v}(t) > v(t) \tag{5.4}$$

and

$$\|\psi_2\|_\infty \leq a_2. \tag{5.5}$$

If our claim is true, then ψ_2 is a positive subsolution of (1.9) since

$$-\psi_2'' = \lambda h_* \tilde{g}(v) \leq \lambda h_* \tilde{g}(\tilde{v}) \leq \lambda h_* \tilde{g}(\psi_2) \leq \lambda h(t) \frac{f(\psi_2)}{\psi_2^\alpha}$$

and $\psi_2(t) \geq \lambda M_7 d(t, \partial\Omega)$ for some $M_7 > 0$. It suffices to show that $\tilde{v}'(t) > v'(t)$ for $t \in (0, \frac{1}{2}]$ in order to show (5.4) since $\tilde{v}(0) = 0 = v(0)$. It is obvious on $[\epsilon, \frac{1}{2}]$ since $\tilde{v}'(t) > 0 = v'(t)$. For $0 < t < \epsilon$, we have

$$\tilde{v}'(t) = \lambda \int_t^{\frac{1}{2}} h_* \tilde{g}(v) ds \geq \lambda \int_\epsilon^{\frac{1}{2}} h_* \tilde{g}(a_3) ds = \lambda h_* \tilde{g}(a_3) \Big(\frac{1}{2} - \epsilon \Big) = \lambda h_* \frac{f(a_3)}{a_3^\alpha} \Big(\frac{1}{2} - \epsilon \Big).$$

Noting that $|v'(t)| \leq a_3 \frac{\delta\eta}{\epsilon}$ on $(0, \epsilon)$, it is easy to see that $\tilde{v}'(t) > v'(t)$ on $(0, \epsilon)$ provided

$$\lambda h_* \frac{f(a_3)}{a_3^\alpha} \Big(\frac{1}{2} - \epsilon \Big) > a_3 \frac{\delta\eta}{\epsilon}.$$

Equivalently, if

$$\lambda > \frac{\delta\eta}{h_* \epsilon \left(\frac{1}{2} - \epsilon \right)} \frac{a_3^{1+\alpha}}{f(a_3)}. \tag{5.6}$$

Note that

$$\inf_{\epsilon \in (0, \frac{1}{2})} \frac{\delta\eta}{h_* \epsilon \left(\frac{1}{2} - \epsilon \right)} \frac{a_3^{1+\alpha}}{f(a_3)} = \delta\eta \frac{16}{h_*} \frac{a_3^{1+\alpha}}{f(a_3)}.$$

This is achived at $\epsilon = \frac{1}{4}$. Returning to the definition of the function ρ, we choose $\epsilon = \frac{1}{4}$. Since $\lambda > \frac{16}{h_*} \frac{a_3^{1+\alpha}}{f(a_3)}$, we can choose δ (> 1) and η (> 1) such that $\lambda > \delta\eta \frac{16}{h_*} \frac{a_3^{1+\alpha}}{f(a_3)}$. Hence (5.6) holds and consequently (5.4) is satisfied for $\lambda > \frac{16}{h_*} \frac{a_3^{1+\alpha}}{f(a_3)}$. Next we show

(5.5). For $t \in (0,1)$, we have

$$\begin{aligned}
\psi_2(t) &= \lambda h_* \int_0^t \int_s^1 \tilde{g}(v) dr ds - c(\psi_2(1))\psi_2(1)t \\
&\leq \lambda h_* \int_0^1 \int_s^1 \tilde{g}(a_3) dr ds \\
&\leq \lambda h_* \tilde{g}(a_3) \int_0^1 (1-s)\, ds \\
&= \frac{\lambda h_*}{2} \frac{f(a_3)}{a_3^\alpha}.
\end{aligned}$$

Thus $\|\psi_2\|_\infty \leq \frac{\lambda h_*}{2}\frac{f(a_3)}{a_3^\alpha}$. Since $\lambda \leq \frac{2a_2}{h_*}\frac{a_3^\alpha}{f(a_3)}$, we have that $\|\psi_2\|_\infty \leq a_2$. Thus ψ_2 is a positive subsolution of (1.9) with $a_3 < \|\psi_2\|_\infty \leq a_2$.

From the proof of Theorem 1.4, we have a sufficiently small positive subsolution ψ_1 such that $\psi_1 \leq \phi_2$ and a sufficiently large positive supersolution ϕ_1 such that $\psi_2 \leq \phi_1$. There exist two positive solutions u_1 and u_2 such that $\psi_1 \leq u_1 \leq \phi_2$ and $\psi_2 \leq u_2 \leq \phi_1$. Clearly, $u_1 \not\equiv u_2$ since $\|\psi_2\|_\infty > \|\phi_2\|_\infty$. Hence there exist at least two positive solutions of (1.9) for $\lambda \in (\lambda_*, \lambda^*]$.

CHAPTER VI

PROOFS OF THEOREMS 1.9 - 1.11

6.1 Proof of Theorem 1.9

Since $f(0) < 0$ and (H_1), there exists $M_8 > 0$ such that $\frac{f(s)}{s^\alpha} \leq M_8 s^{p-1}$ for $s \in (0, \infty)$. Assume that u is a positive solution of (1.9). Then we have

$$\begin{aligned} u(t) &= \int_0^t \varphi_p^{-1}\left(\lambda \int_s^1 h(r)\frac{f(u)}{u^\alpha}dr - \varphi_p(c(u(1))u(1))\right) ds \\ &\leq \int_0^t \varphi_p^{-1}\left(\lambda M_8 \int_s^1 h(r)u^{p-1}dr\right) ds \\ &\leq \varphi_p^{-1}\left(\lambda M_8 H \|u\|_\infty^{p-1}\right) \end{aligned}$$

where $H := \int_0^1 h(s)ds$. Thus we obtain that $1 \leq \varphi_p^{-1}(\lambda M_8 H)$. This is a contradiction for $\lambda \approx 0$. Hence there is no positive solution of (1.9) for $\lambda \approx 0$.

6.2 Proof of Theorem 1.10

We first construct a subsolution of (1.9). Let $\lambda_1 > 0$ be the first eigenvalue and $e \in C^1[0,1]$ be the corresponding eigenvector of the boundary value problem:

$$\begin{cases} -(\varphi_p(e'))' = \lambda \varphi_p(e); \quad t \in (0,1) \\ e(0) = 0 = e(1) \end{cases}$$

where $\|e\|_\infty = 1$ and $e > 0$ on $(0,1)$. Then there exist $M_9 > 0$ and $M_{10} > 0$ such that $M_9 d(t,\partial\Omega) \le e(t) \le M_{10} d(t,\partial\Omega)$. Let $\kappa = \frac{p-\eta}{p-1+\alpha}$. There exist $m > 0$, $\epsilon > 0$ and $\mu > 0$ such that $\lambda_1 |e(t)|^p - (\kappa-1)(p-1)|e'(t)|^p \le -m$ for $t \in (0,\epsilon) \cup (1-\epsilon, 1)$ and $e(t) \ge \mu$ in $[\epsilon, 1-\epsilon]$. Choose $A^* > 0$ and $l_0 \in (0,\alpha)$ such that $f(s) \ge A^* s^{l_0}$ for $s \gg 1$ from (H_{16}). Let $\psi := \lambda^{\tilde{\sigma}} e^{\kappa}$ where $\tilde{\sigma} \in (\frac{1}{p-1+\alpha}, \frac{1}{p-1+\alpha-l_0})$. Then we have

$$\begin{aligned} -(\varphi_p(\psi'))' &= \lambda^{\tilde{\sigma}(p-1)} \kappa^{p-1} \left(\lambda_1 e^{\kappa(p-1)} - (\kappa-1)(p-1) e^{\kappa(p-1)-p} |e'|^p \right) \\ &= \frac{\lambda^{\tilde{\sigma}(p-1)} \kappa^{p-1}}{e^{p-\kappa(p-1)}} \left(\lambda_1 e^p - (\kappa-1)(p-1)|e'|^p \right). \end{aligned}$$

Since $\tilde{\sigma} > \frac{1}{p-1+\alpha}$, we obtain $-\frac{dM_{10}^{\eta}}{m\kappa^{p-1}} \min_{s\in[0,\infty)} f(s) \le \lambda^{\tilde{\sigma}(p-1+\alpha)-1}$ for $\lambda \gg 1$. For $\lambda \gg 1$ and $t \in (0,\epsilon)$, we have

$$-(\varphi_p(\psi'))' \le \frac{-\lambda^{\tilde{\sigma}(p-1)} \kappa^{p-1} m}{e^{\alpha\kappa+\eta}} \le \frac{-\lambda^{\tilde{\sigma}(p-1)} \kappa^{p-1} m}{(M_{10} t)^{\eta} e^{\alpha\kappa}} \le \lambda \frac{d \min_{s\in[0,\infty)} f(s)}{t^{\eta} (\lambda^{\tilde{\sigma}} e^{\kappa})^{\alpha}} \le \lambda h(t) \frac{f(\psi)}{\psi^{\alpha}}.$$

By a similar argument, we have

$$-(\varphi_p(\psi'))' \le \frac{-\lambda^{\tilde{\sigma}(p-1)} \kappa^{p-1} m}{e^{\alpha\kappa+\eta}} \le \frac{-\lambda^{\tilde{\sigma}(p-1)} \kappa^{p-1} m}{M_{10}^{\eta} (1-t)^{\eta} e^{\alpha\kappa}} \le \lambda \frac{d \min_{s\in[0,\infty)} f(s)}{(1-t)^{\eta} (\lambda^{\tilde{\sigma}} e^{\kappa})^{\alpha}} \le \lambda h(t) \frac{f(\psi)}{\psi^{\alpha}}$$

for $\lambda \gg 1$ and $t \in (1-\epsilon, 1)$. Since $\tilde{\sigma} < \frac{1}{p-1+\alpha-l_0}$, we have $\frac{\lambda_1 \kappa^{p-1}}{h_* A^* \mu^{\kappa l_0}} \le \lambda^{1-\tilde{\sigma}(p-1+\alpha-l_0)}$ for $\lambda \gg 1$. We also have $A^*(\lambda^{\tilde{\sigma}} e(t)^{\kappa})^{l_0} \le f(\lambda^{\tilde{\sigma}} e(t)^{\kappa})$ for $\lambda \gg 1$ and $t \in [\epsilon, 1-\epsilon]$. Then we have

$$-(\varphi_p(\psi'))' \le \frac{\lambda^{\tilde{\sigma}(p-1)} \kappa^{p-1} \lambda_1 e^{p-\eta}}{e^{\alpha\kappa}} \le \frac{\lambda^{\tilde{\sigma}(p-1)} \kappa^{p-1} \lambda_1}{e^{\alpha\kappa}} \le \lambda \frac{h_* A^* (\lambda^{\tilde{\sigma}} \mu^{\kappa})^{l_0}}{(\lambda^{\tilde{\sigma}} e^{\kappa})^{\alpha}} \le \lambda h(t) \frac{f(\psi)}{\psi^{\alpha}}$$

for $\lambda \gg 1$ and $t \in [\epsilon, 1-\epsilon]$. We also have $\psi'(1) + c(\psi(1))\psi(1) = 0$ since $e(1) = 0$. Further, we obtain $\psi(t) \geq \lambda^{\tilde{\sigma}} M_9^{\kappa} d(t, \partial\Omega)^{\kappa}$ such that $\alpha\kappa + \eta < 1$ since $e(t) \geq d(t, \partial\Omega)$ and $\kappa = \frac{p-\eta}{p-1+\alpha}$. Hence ψ is a subsolution of (1.9) for $\lambda \gg 1$.

Next we construct a supersolution of (1.9). Let $\tilde{z} \in C^1[0,1]$ be the solution of the boundary value problem:

$$\begin{cases} -(\varphi_p(z'(t))' = \frac{h(t)}{t^{\alpha}}; \quad t \in (0,1) \\ \\ z(0) = 0 = z'(1). \end{cases}$$

Then there exists $M_{11} > 0$ such that $\tilde{z}(t) \geq M_{11}t$ for $t \in [0,1]$. Since $f^*(s) = \max_{0 \leq r \leq s} |f(r)|$, $f^*(s)$ is nondecreasing and $\frac{f^*(s)}{s^{p-1+\alpha}} \to 0$ as $s \to \infty$. Thus there exists $M_{\lambda}^* \gg 1$ such that $\psi(t) \leq M_{\lambda}^* \tilde{z}(t)$ for $t \in [0,1]$ and $\frac{f^*(M_{\lambda}^* \|\tilde{z}\|_{\infty})}{(M_{\lambda}^* \|\tilde{z}\|_{\infty})^{p-1+\alpha}} \leq \frac{M_{11}^{\alpha}}{\lambda \|\tilde{z}\|_{\infty}^{p-1+\alpha}}$. Let $\phi := M_{\lambda}^* \tilde{z}$. Then we have

$$-(\varphi_p(\phi'))' = M_{\lambda}^{*p-1} \frac{h(t)}{t^{\alpha}} \geq \lambda h(t) \frac{f^*(M_{\lambda}^* \|\tilde{z}\|_{\infty})}{(M_{\lambda}^* \tilde{z})^{\alpha}} \geq \lambda h(t) \frac{f(M_{\lambda}^* \tilde{z})}{(M_{\lambda}^* \tilde{z})^{\alpha}} = \lambda h(t) \frac{f(\phi)}{\phi^{\alpha}}.$$

We also have $\phi'(1) + c(\phi(1))\phi(1) > 0$ since $\tilde{z}(1) > 0$ and $\tilde{z}(1) = 0$. Hence ϕ is a supersolution of (1.9) such that $\psi(t) \leq \phi(t)$ for $t \in [0,1]$.

By Lemma 2.5, there exists a positive solution u of (1.9) for $\lambda \gg 1$ such that $\psi \leq u \leq \phi$. Further, $\|u\|_{\infty} \to \infty$ as $\lambda \to \infty$ since $\|\psi\|_{\infty} \to \infty$ as $\lambda \to \infty$.

Next we show that $u(1) > 0$ for any positive solution u and $\inf_{t \in [a^{\#}, 1]} u(t) \to \infty$ as $\lambda \to \infty$ for any constant $a^{\#} \in (0,1)$. Let $t_m \in (0,1)$ be the first point satisfying $u'(t) = 0$. Let (H_3) and (H_{10}) hold. Then there exist unique constants $\beta > 0$ and $\theta > 0$ such that $f(\beta) = 0$ and $F_{\alpha}(\theta) = 0$ where $F_{\alpha}(s) := \int_0^s \frac{f(r)}{r^{\alpha}} dr$. We also define

$E_\alpha(t) := \lambda F_\alpha(u(t))h(t) + \frac{p-1}{p}|u'(t)|^p$. We first establish the following Lemmas by arguments similar to those in Lemmas 4.1 - 4.3.

Lemma 6.1. *Let* (H_3) *and* $(H_9) - (H_{10})$ *hold. If* u *is a positive solution of* (1.9), *then* u *has a unique interior maximum, say at* t_m, *and* $u(t_m) > \theta$.

Proof. To show $u(t_m) > \theta$, we assume $u(t_m) \leq \theta$. Integrating (1.9) from $t \in (0, t_m)$ to t_m, we obtain

$$u'(t) = \varphi_p^{-1}\left(\lambda \int_t^{t_m} h(s)\frac{f(u)}{u^\alpha}ds\right) \leq \varphi_p^{-1}\left(\lambda \frac{dg_\theta}{1-\eta}\right)$$

where $g_\theta := \sup_{s\in(0,\theta)} \frac{f(s)}{s^\alpha}$ (> 0). Integrating again from 0 to $t \in (0, t_m)$, we have $u(t) \leq \lambda^{\frac{1}{p-1}} M_{12} t$ where $M_{12} := \varphi_p^{-1}\left(\frac{dg_\theta}{1-\eta}\right)$. Let $M_{13} > 0$ be such that $|F_\alpha(s)| \leq M_{13} s^{1-\alpha}$ for $s \in [0, \theta]$. Then we obtain

$$\lim_{t\to 0+} \lambda |F_\alpha(u(t))| h(t) \leq \lim_{t\to 0+} \lambda^{\frac{p-\alpha}{p-1}} d M_{12}^{1-\alpha} M_{13} t^{1-\alpha-\eta} = 0.$$

This implies that $\lim_{t\to 0+} E_\alpha(t) \geq 0$. We note that $E_\alpha(t)$ is differentiable and $E'_\alpha(t) = \lambda F_\alpha(u(t))h'(t)$ on $(0, t_m)$. Then $E_\alpha(t_m) > 0$ by (H_9). This is a contradiction since $E_\alpha(t_m) = \lambda F_\alpha(u(t_m))h(t_m) \leq 0$. Hence $u(t_m) > \theta$.

Now we assume that u has two interior maxima. Let $\tilde{t} \in (t_m, 1)$ be the first point satisfying $u'(t) = 0$. Integrating (1.9) from t_m to $\tilde{t}$, we have $\int_{t_m}^{\tilde{t}} h(s)\frac{f(u)}{u^\alpha}ds = 0$. This implies that $u(\tilde{t}) < \beta < \theta < u(t_m)$. Let $\bar{t} \in (t_m, \tilde{t})$ be such that $u(\bar{t}) = \theta$. Then $E_\alpha(\bar{t}) = \frac{p-1}{p}|u'(\bar{t})|^p > 0$. We again note that $E_\alpha(t)$ is differentiable on $(t_m, \tilde{t})$ and $E'_\alpha(t) = \lambda F_\alpha(u(t))h'(t)$ on $(t_m, \tilde{t})$. This implies that $E_\alpha(t)$ is strictly increasing on $(\bar{t}, \tilde{t})$ since $u(t) < \theta$ on $(\bar{t}, \tilde{t})$. Thus $E_\alpha(\tilde{t}) > 0$. This is a contradiction since $E_\alpha(\tilde{t}) = \lambda F_\alpha(u(\tilde{t}))h(\tilde{t}) < 0$. Hence u has a unique interior maximum. $\square$

Lemma 6.2. *Let (H_3) and $(H_9) - (H_{10})$ hold. Let u be a positive solution of* (1.9). *Let $t_\beta \in (0,1)$ be the first point such that $u(t_\beta) = \beta$. Then $t_\beta \leq O(\lambda^{-\frac{1}{p}})$.*

Proof. Let $t_{\frac{\beta}{2}} \in (0,1)$ be the first point such that $u(t_{\frac{\beta}{2}}) = \frac{\beta}{2}$. Integrating (1.9) from 0 to $t \in (0, t_{\frac{\beta}{2}})$, we obtain

$$\varphi_p(u'(t)) = \varphi_p(u'(0)) - \lambda \int_0^t h(s)\frac{f(u)}{u^\alpha} ds \geq -\lambda h_* g_{\frac{\beta}{2}} t$$

where $g_{\frac{\beta}{2}} := \sup_{s\in(0,\frac{\beta}{2})} \frac{f(s)}{s^\alpha}$. Thus $u'(t) \geq \lambda^{\frac{1}{p-1}} M_{14} t^{\frac{1}{p-1}}$ where $M_{14} := -\varphi_p^{-1}\big(h_* g_{\frac{\beta}{2}}\big)$. Integrating again from 0 to $t_{\frac{\beta}{2}}$, we have $\frac{\beta}{2} = u(t_{\frac{\beta}{2}}) - u(0) \geq \lambda^{\frac{1}{p-1}} M_{14} \big(\frac{p-1}{p}\big) t_{\frac{\beta}{2}}^{\frac{p}{p-1}}$. Thus $t_{\frac{\beta}{2}} \leq \lambda^{-\frac{1}{p}} M_{15}$ where $M_{15} := \big(\frac{\beta p}{2M_{14}(p-1)}\big)^{\frac{p-1}{p}}$. Further, by the Mean Value Theorem, there exists $t_* \in (0, t_{\frac{\beta}{2}})$ such that $\frac{\beta}{2} = u(t_{\frac{\beta}{2}}) - u(0) = u'(t_*) t_{\frac{\beta}{2}}$. Since u' is strictly increasing on $(0, t_\beta)$, we have $\lambda^{\frac{1}{p}} \frac{\beta}{2M_{15}} \leq u'(t_*) < u'(t)$ for $t \in [t_{\frac{\beta}{2}}, t_\beta]$. Integrating from $t_{\frac{\beta}{2}}$ to t_β, we have $t_\beta - t_{\frac{\beta}{2}} < \lambda^{-\frac{1}{p}} M_{15}$. Hence $t_\beta \leq O(\lambda^{-\frac{1}{p}})$. $\square$

Lemma 6.3. *Let (H_3) and $(H_9) - (H_{10})$ hold. Let u be a positive solution of* (1.9). *Let $t_{\frac{\beta+\theta}{2}} \in (0,1)$ be the first point such that $u(t_{\frac{\beta+\theta}{2}}) = \frac{\beta+\theta}{2}$. Then $t_{\frac{\beta+\theta}{2}} \leq O(\lambda^{-\frac{1}{p}})$.*

Proof. Let $t_\theta \in (0,1)$ be the first point such that $u(t_\theta) = \theta$. Integrating (1.9) from $t \in (0, t_\theta)$ to t_θ, we obtain $u'(t) = \varphi_p^{-1}\Big(\varphi_p(u'(t_\theta)) + \lambda \int_t^{t_\theta} h(s)\frac{f(u)}{u^\alpha} ds\Big)$. Integrating again from t_β to $t \in (t_\beta, t_\theta)$, we have

$$u(t) = \beta + \int_{t_\beta}^t \varphi_p^{-1}\left(\varphi_p(u'(t_\theta)) + \lambda \int_s^{t_\theta} h(r)\frac{f(u)}{u^\alpha} dr\right) ds.$$

Since $\varphi_p(u'(t_\theta)) > 0$, we have

$$\begin{aligned}
\frac{\theta-\beta}{2} &= \int_{t_\beta}^{t_{\frac{\beta+\theta}{2}}} \varphi_p^{-1}\left(\varphi_p(u'(t_\theta)) + \lambda\int_s^{t_\theta} h(r)\frac{f(u)}{u^\alpha}dr\right)ds \\
&> \int_{t_\beta}^{t_{\frac{\beta+\theta}{2}}} \varphi_p^{-1}\left(\lambda\int_{t_{\frac{\beta+\theta}{2}}}^{t_\theta} h(r)\frac{f(u)}{u^\alpha}dr\right)ds \qquad (6.1)\\
&\geq (t_{\frac{\beta+\theta}{2}} - t_\beta)\varphi_p^{-1}\left(\lambda h_* M_{16}(t_\theta - t_{\frac{\beta+\theta}{2}})\right)
\end{aligned}$$

where $M_{16} := \frac{1}{\theta^\alpha} f\big(\frac{\beta+\theta}{2}\big)$. By the Mean Value Theorem, there exist t_1 and $t_2 \in (0,1)$ with $t_\beta < t_1 < t_{\frac{\beta+\theta}{2}} < t_2 < t_\theta$ such that $u(t_{\frac{\beta+\theta}{2}}) - u(t_\beta) = u'(t_1)(t_{\frac{\beta+\theta}{2}} - t_\beta)$ and $u(t_\theta) - u(t_{\frac{\beta+\theta}{2}}) = u'(t_2)(t_\theta - t_{\frac{\beta+\theta}{2}})$. Since $u(t_{\frac{\beta+\theta}{2}}) - u(t_\beta) = \frac{\theta-\beta}{2} = u(t_\theta) - u(t_{\frac{\beta+\theta}{2}})$, we have $u'(t_1)(t_{\frac{\beta+\theta}{2}} - t_\beta) = u'(t_2)(t_\theta - t_{\frac{\beta+\theta}{2}})$. Since u' is strictly decreasing on (t_β, t_θ), we have $u'(t_1) > u'(t_2) > 0$, so $t_{\frac{\beta+\theta}{2}} - t_\beta < t_\theta - t_{\frac{\beta+\theta}{2}}$. This and (6.1) imply that $\frac{\theta-\beta}{2} > \lambda^{\frac{1}{p-1}} M_{17}(t_{\frac{\beta+\theta}{2}} - t_\beta)^{\frac{p}{p-1}}$ where $M_{17} := \varphi_p^{-1}(\underline{h}M_{16})$. Thus $t_{\frac{\beta+\theta}{2}} - t_\beta < \lambda^{-\frac{1}{p}} M_{18}$ where $M_{18} := \big(\frac{\theta-\beta}{2M_{17}}\big)^{\frac{p-1}{p}}$. Hence we have $t_{\frac{\beta+\theta}{2}} \leq O(\lambda^{-\frac{1}{p}})$. □

Now we show $u(1) > 0$. Assume $u(1) = 0$. By Lemma 6.1, there exists $\tilde{t}_\theta \in (t_m, 1)$ such that $u(\tilde{t}_\theta) = \theta$ and $u'(\tilde{t}_\theta) < 0$. Then $E_\alpha(\tilde{t}_\theta) = \frac{p-1}{p}|u'(\tilde{t}_\theta)|^p > 0$. Hence $E_\alpha(1) > 0$ since $E_\alpha(t)$ is strictly increasing for $t \in (\tilde{t}_\theta, 1)$. However, $u(1) = 0$ implies that $u'(1) = 0$, therefore $E(1) = 0$. This is a contradiction. Hence $u(1) > 0$.

We also show $u(1) \geq \frac{\beta+\theta}{2}$ for $\lambda \gg 1$. Assume $u(1) < \frac{\beta+\theta}{2}$. By the above argument, we obtain $E_\alpha(1) = \lambda F_\alpha(u(1))h(1) + \frac{p-1}{p}|u'(1)|^p > 0$. Then we have

$$c(u(1))u(1) = -u'(1) > \Big(-\lambda\frac{p}{p-1}F_\alpha(u(1))h(1)\Big)^{\frac{1}{p}}.$$

This implies that $u(1) \approx 0$ for $\lambda \gg 1$. Thus $\frac{F_\alpha(u(1))}{u(1)} \approx -\infty$ for $\lambda \gg 1$, and hence $\left(-\lambda \frac{p}{p-1} \frac{F_\alpha(u(1))}{u(1)} h(1)\right)^{\frac{1}{p}} \gg 1$ for $\lambda \gg 1$. Then $c(u(1))u(1)^{\frac{p-1}{p}} \gg 1$ for $\lambda \gg 1$. This is a contradiction since $u(1) \approx 0$ for $\lambda \gg 1$. Hence $u(1) \geq \frac{\beta+\theta}{2}$ for $\lambda \gg 1$.

Next we show that $u(1) \to \infty$ as $\lambda \to \infty$. Let $a^\# \in (0,1)$ be any constant, where $a^\#$ is independent of λ. First we assume $t_m > \frac{a^\#+1}{2}$. By Lemma 6.3, for $\lambda \gg 1$ we have

$$\begin{aligned} u\left(a^\#\right) &= \beta + \int_{t_\beta}^{a^\#} \varphi_p^{-1}\left(\lambda \int_s^{t_m} h(r)\frac{f(u)}{u^\alpha} dr\right) ds \\ &\geq \int_{\frac{a^\#}{2}}^{a^\#} \varphi_p^{-1}\left(\lambda \int_{a^\#}^{\frac{a^\#+1}{2}} h(r)\frac{f(u)}{u^\alpha} dr\right) ds \\ &\geq \int_{\frac{a^\#}{2}}^{a^\#} \varphi_p^{-1}\left(\lambda \frac{1-a^\#}{2} \frac{h_*}{\|u\|_\infty^\alpha} f\Big(\frac{\beta+\theta}{2}\Big)\right) ds \\ &\geq \lambda^{\frac{1}{p-1}} \frac{M_{19}}{\|u\|_\infty^{\frac{\alpha}{p-1}}} \end{aligned} \tag{6.2}$$

where $M_{19} := \frac{a^\#}{2}\varphi_p^{-1}\left(\frac{1-a^\#}{2} h_* f\left(\frac{\beta+\theta}{2}\right)\right)$ (> 0). Thus $\|u\|_\infty^{\frac{p-1+\alpha}{p-1}} \geq \lambda^{\frac{1}{p-1}} M_{19}$. By the Mean Value Theorem, we have

$$u(t_m) - u(1) = -u'(\tilde{t})(1-t_m) \leq -u'(1) = c(u(1))u(1) \tag{6.3}$$

where $\tilde{t} \in (t_m, 1)$. This implies that $(c(u(1))+1)u(1) \geq \|u\|_\infty \geq \lambda^{\frac{1}{p-1+\alpha}} M_{19}^{\frac{p-1}{p-1+\alpha}}$ if $t_m > \frac{a^\#+1}{2}$. Hence $u(1) \to \infty$ as $\lambda \to \infty$. Next we assume $t_m \leq \frac{a^\#+1}{2}$. Since

$u(1) \geq \frac{\beta+\theta}{2}$, we have

$$c(u(1))u(1) = -u'(1) = \varphi_p^{-1}\left(\lambda \int_{t_m}^{1} h(s)\frac{f(u)}{u^\alpha} ds\right) \geq \lambda^{\frac{1}{p-1}} \frac{M_{20}}{\|u\|_\infty^{\frac{\alpha}{p-1}}} \tag{6.4}$$

where $M_{20} := \varphi_p^{-1}\left(\frac{1-a^\#}{2} h_* f\left(\frac{\beta+\theta}{2}\right)\right)$. By (6.3), we have

$$\lambda^{\frac{1}{p-1}} M_{20} \leq c(u(1))u(1)\|u\|_\infty^{\frac{\alpha}{p-1}} \leq (c(u(1)+1)u(1))^{\frac{p-1+\alpha}{p-1}}.$$

Thus $(c(u(1))+1)u(1) \geq \lambda^{\frac{1}{p-1+\alpha}} M_{20}^{\frac{p-1}{p-1+\alpha}}$ if $t_m \leq \frac{a^\#+1}{2}$. Hence $u(1) \to \infty$ as $\lambda \to \infty$.

Finally we show that $\inf_{t\in[a^\#,1]} u(t) \to \infty$ as $\lambda \to \infty$. Let $\lambda \gg 1$ be such that $t_\beta < \frac{a^\#}{4}$ and $t_{\frac{\beta+\theta}{2}} < \frac{a^\#}{2}$. If $t_m > a^\#$, then we have

$$\begin{aligned} u\left(\frac{a^\#}{2}\right) &= \beta + \int_{t_\beta}^{\frac{a^\#}{2}} \varphi_p^{-1}\left(\lambda \int_s^{t_m} h(r)\frac{f(u)}{u^\alpha}\right) ds \\ &\geq \int_{\frac{a^\#}{4}}^{\frac{a^\#}{2}} \varphi_p^{-1}\left(\lambda \int_{\frac{a^\#}{2}}^{a^\#} h(r)\frac{f(u)}{u^\alpha}\right) ds \\ &\geq \lambda^{\frac{1}{p-1}} \frac{M_{21}}{u(a^\#)^{\frac{\alpha}{p-1}}} \end{aligned}$$

where $M_{21} := \frac{a^\#}{4}\varphi_p^{-1}\left(\frac{a^\#}{2} h_* f\left(\frac{\beta+\theta}{2}\right)\right)$. Since u is increasing on $[\frac{a^\#}{2}, a^\#]$, we have $u(a^\#) \geq u(\frac{a^\#}{2}) \geq \lambda^{\frac{1}{p-1}} \frac{M_{21}}{u(a^\#)^{\frac{\alpha}{p-1}}}$. Thus $u(a^\#) \geq \lambda^{\frac{1}{p-1+\alpha}} M_{21}^{\frac{p-1}{p-1+\alpha}}$. If $t_m \leq a^\#$, then $u(a^\#) \geq u(1)$ since u is decreasing on $[a^\#, 1]$. Therefore $u(a^\#) \geq \min\{\lambda^{\frac{1}{p-1+\alpha}} M_{21}^{\frac{p-1}{p-1+\alpha}}, u(1)\}$. Since u is concave on $[a^\#, 1]$, we obtain that $\inf_{t\in[a^\#,1]} u(t) \geq \min\{u(a^\#), u(1)\} \geq \min\{\lambda^{\frac{1}{p-1+\alpha}} M_{21}^{\frac{p-1}{p-1+\alpha}}, u(1)\}$. Hence $\inf_{t\in[a^\#,1]} u(t) \to \infty$ as $\lambda \to \infty$.

6.3 Proof of Theorem 1.11

We first establish the following Lemmas to establish Theorem 1.11.

Lemma 6.4. *Let $p > 1$, $\alpha = 0$, (H_1), (H_3), (H_5) and $(H_9) - (H_{10})$ hold. For $\lambda \gg 1$, there exist constants C_1 and C_2 (independent of λ) with $0 < C_1 < C_2$ such that*

$$C_1 G^{-1}(\lambda^{\frac{1}{p-1}}) d(t, \partial\Omega) \leq u(t) \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}}) t.$$

Lemma 6.5. *Let $p > 1$, $\alpha = 0$, (H_1), (H_3), (H_5) and $(H_9) - (H_{10})$ hold. Let u and v be the positive solution of* (1.9). *Let $\gamma_0 \leq \gamma < 1$ where $\gamma_0 := \frac{C_1}{C_2}$. For $\lambda \gg 1$, there exists $\delta > 0$ (independent of λ) such that $\frac{C_1\gamma_0}{2} G^{-1}(\lambda^{\frac{1}{p-1}}) \leq |sv'(t) + (1-s)\gamma u'(t)| \leq C_2 G^{-1}(\lambda^{\frac{1}{p-1}})$ for $s \in [0,1]$ and $t \in [0, \delta]$.*

The proofs of Lemmas 6.4 - 6.5 are done by arguments similar to those in Lemmas 4.7 - 4.8.

Now we prove Theorem 1.11 by arguments similar to those in the proof of Theorem 1.3. Let u and v be positive solutions of (1.9). By Lemma 6.4, $v \geq \gamma_0 u$ on $[0, \frac{1}{2}]$ for $\lambda \gg 1$ where $\gamma_0 := \frac{C_1}{C_2}$. Let γ $(= \gamma_\lambda)$ be the largest constant such that $v \geq \gamma u$ on $[0, 1]$.

First we show $\gamma \geq \gamma_0$ when $\lambda \gg 1$. Assume $\gamma < \gamma_0$ for $\lambda \gg 1$. Then $v(t) > \gamma u(t)$ for $t \in (0, \frac{1}{2}]$, and there exists $t_5 \in (\frac{1}{2}, 1]$ such that $v(t_5) - \gamma u(t_5) = 0$ since γ is the largest constant such that $v \geq \gamma u$ on $[0, 1]$. Let $t_6 \in (\frac{1}{2}, 1]$ be the first point such that $v(t_6) - \gamma u(t_6) = 0$. If $t_6 = 1$, then $v'(1) - \gamma u'(1) = -c(v(1))v(1) + \gamma c(u(1))u(1) \geq 0$ by (H_{17}). This implies that $v'(1) - \gamma u'(1) = 0$. Let $t_7 \in (0, 1)$ be the largest point such that $v(t_7) - \gamma u(t_7) > 0$ and $v'(t_7) - \gamma u'(t_7) = 0$. We can also choose $t_{\sigma_0} \in (0, 1)$ such that $\gamma u(t_{\sigma_0}) = \sigma_0$ since $v(1) = \gamma u(1)$ and $v(1) \gg 1$ for $\lambda \gg 1$.

Let $t_8 := \max\{t_7, t_{\sigma_0}, \frac{1}{2}\}$. By (H_3) and (H_5), we have

$$\begin{aligned} 0 &\geq -\varphi_p(v'(1)) + \varphi_p(\gamma u'(1)) - (-\varphi_p(v'(t_8)) + \varphi_p(\gamma u'(t_8))) \\ &= \lambda \int_{t_8}^{1} h(s) \left(f(v) - \gamma^{p-1} f(u) \right) ds \\ &\geq \lambda \int_{t_8}^{1} h(s) \left(f(\gamma u) - \gamma^{p-1} f(u) \right) ds \\ &\geq \lambda(\gamma^q - \gamma^{p-1}) \int_{t_8}^{1} h(s) f(u) ds. \end{aligned}$$

This is a contradiction since $\int_{t_8}^1 h(s)f(u)ds > 0$. Hence $t_6 \in (\frac{1}{2}, 1)$. Then $v(t_6) - \gamma u(t_6) = 0$ and $v'(t_6) - \gamma u'(t_6) = 0$. By the above argument, we again get a contradiction. Hence $\gamma \geq \gamma_0$.

Now we show that $\gamma \geq 1$. Assume $\gamma < 1$. Let $\tilde{u} = \frac{u}{G^{-1}(\lambda^{\frac{1}{p-1}})}$ and $\tilde{v} = \frac{v}{G^{-1}(\lambda^{\frac{1}{p-1}})}$. Since $G^{-1}(\lambda^{\frac{1}{p-1}}) = \lambda^{\frac{1}{p-1}} f^{\frac{1}{p-1}}\left(G^{-1}(\lambda^{\frac{1}{p-1}})\right)$, we have

$$-\left(\varphi_p\left(\tilde{u}'\right)\right)' = h(t)\frac{f(u)}{f\left(G^{-1}(\lambda^{\frac{1}{p-1}})\right)} \quad \text{and} \quad -\left(\varphi_p\left(\tilde{v}'\right)\right)' = h(t)\frac{f(v)}{f\left(G^{-1}(\lambda^{\frac{1}{p-1}})\right)}.$$

By the Mean Value Theorem, we obtain $\varphi_p(\tilde{v}') - \varphi_p(\gamma\tilde{u}') = a_\lambda(t)\left(\tilde{v}' - \gamma\tilde{u}'\right)$ for $t \in \Omega_\delta^*$ where $\Omega_\delta^* := (0, \delta)$ and $a_\lambda(t) := (p-1)\int_0^1 |s\tilde{v}'(t) + (1-s)\gamma\tilde{u}'(t)|^{p-2} ds$. Then we have

$$-\left(a_\lambda(t)\left(\tilde{v}' - \gamma\tilde{u}'\right)\right)' = h(t)\frac{f(v) - \gamma^{p-1}f(u)}{f\left(G^{-1}(\lambda^{\frac{1}{p-1}})\right)} \geq h(t)\frac{f(\gamma u) - \gamma^{p-1}f(u)}{f\left(G^{-1}(\lambda^{\frac{1}{p-1}})\right)}.$$

Without loss of generality, we assume $\sigma_0 \gg 1$ so that $\frac{\sigma_0}{\gamma_0} > \frac{\beta+\theta}{2}$ and $f\left(\frac{\sigma_0}{\gamma_0}\right) \geq |f(0)|$ hold. Let $I := \{t \in \Omega_\delta^* \mid u(t) \geq \frac{\sigma_0}{\gamma_0}\}$ and $J := \Omega_\delta^* \setminus I$. Then $I = [t_I, \delta)$ and $J = (0, t_I)$

where $t_I := \min\{t \in (0,1) \mid u(t) \geq \frac{\sigma_0}{\gamma_0}\}$. On I, it follows from (H_5) that

$$f(\gamma u) - \gamma^{p-1} f(u) \geq (\gamma^q - \gamma^{p-1}) f(u) \geq m_1(1-\gamma)$$

where $m_1 := (p-1-q)\gamma_0^q f\big(\frac{\sigma_0}{\gamma_0}\big)\min\{1, \gamma_0^{p-2-q}\}$. On J, we have

$$\begin{aligned}
|f(\gamma u) - \gamma^{p-1} f(u)| &\leq |f(\gamma u) - f(u)| + (1-\gamma^{p-1})|f(u)| \\
&\leq (1-\gamma) u f'(\zeta) + (1-\gamma)(p-1)\max\{1, \gamma_0^{p-2}\} f\Big(\frac{\sigma_0}{\gamma_0}\Big) \\
&\leq \frac{1-\gamma}{\gamma_0}\zeta f'(\zeta) + (1-\gamma)(p-1)\max\{1, \gamma_0^{p-2}\} f\Big(\frac{\sigma_0}{\gamma_0}\Big) \\
&\leq m_2(1-\gamma)
\end{aligned}$$

where $\zeta(t) \in (\gamma u(t), u(t))$ and $m_2 := \frac{1}{\gamma_0}\sup_{s\in(0,\frac{\sigma_0}{\gamma_0})} s f'(s) + (p-1)\max\{1, \gamma_0^{p-2}\} f\big(\frac{\sigma_0}{\gamma_0}\big)$. Thus we have

$$-\left(\frac{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)}{1-\gamma} a_\lambda(t)(\tilde{v}' - \gamma \tilde{u}')\right)' \geq \begin{cases} m_1 h(t); & t \in I \\ -m_2 h(t); & t \in J. \end{cases}$$

Let $\overline{\kappa}$ be the solution of the boundary value problem:

$$-\left(a_\lambda(t)\overline{\kappa}'(t)\right)' = \begin{cases} m_1 h(t); & t \in I \\ -m_2 h(t); & t \in J \end{cases}$$

$$\overline{\kappa} \equiv 0 \quad \text{ on } \partial\Omega_\delta^*.$$

By the comparison principle, we have

$$\tilde{v}(t) - \gamma \tilde{u}(t) \geq \frac{1-\gamma}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)} \overline{\kappa}(t)$$

for $t \in \Omega_\delta^*$. Let κ be the solution of the boundary value problem:

$$-\left(a_\lambda(t)\kappa'(t)\right)' = \begin{cases} m_1 h(t); & t \in I \\ 0; & t \in J \end{cases}$$

$$\kappa \equiv 0 \quad \text{ on } \partial\Omega_\delta^*.$$

We note that $t_I \to 0$ and $|J| \to 0$ as $\lambda \to \infty$ by Lemma 6.4 where $|J|$ is the length of J. Then by Lemmas 3.4 and 3.6, we obtain that $\kappa(t) \geq \overline{M} d(t, \partial\Omega_\delta^*)$ and $|\kappa(t) - \overline{\kappa}(t)| \leq M^* |J|^{1-\eta} d(t, \partial\Omega_\delta^*)$ for $\lambda \gg 1$. Thus we have

$$\overline{\kappa}(t) \geq \kappa(t) - |\kappa(t) - \overline{\kappa}(t)| \geq \big(\overline{M} - M^*|J|^{1-\eta}\big) d(t, \partial\Omega_\delta^*) \geq \frac{\overline{M}}{2} d(t, \partial\Omega_\delta^*)$$

for $\lambda \gg 1$ and $t \in \Omega_\delta^*$. Since $d(t, \partial\Omega_\delta^*) = t$ for $t \in \Omega_{\frac{\delta}{2}}^* := [0, \frac{\delta}{2}]$, we have

$$\overline{\kappa}(t) \geq \frac{\overline{M}}{2} d(t, \partial\Omega)$$

for $t \in \Omega_{\frac{\delta}{2}}^*$. This implies that

$$\tilde{v}(t) - \gamma \tilde{u}(t) \geq \frac{1-\gamma}{f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)} \overline{\kappa}(t) \geq \frac{\overline{M}(1-\gamma)}{2f\big(G^{-1}(\lambda^{\frac{1}{p-1}})\big)} t$$

for $t \in \Omega^*_{\frac{\delta}{2}}$. Then we obtain

$$v(t) \geq (\gamma + \epsilon_\lambda) u(t)$$

for $t \in \Omega^*_{\frac{\delta}{2}}$ by Lemma 6.4 where $\epsilon_\lambda := \frac{\overline{M}(1-\gamma)}{2C_2 f\left(G^{-1}(\lambda^{\frac{1}{p-1}})\right)}$. Then we have $v\left(\frac{\delta}{2}\right) \geq \gamma u\left(\frac{\delta}{2}\right) + \tilde{\epsilon}_\lambda$ where $\tilde{\epsilon}_\lambda := \epsilon_\lambda C_1 G^{-1}(\lambda^{\frac{1}{p-1}})\frac{\delta}{2}$. Now we claim that $v(1) > \gamma u(1)$. If not, $v(1) = \gamma u(1)$. By (H_{17}), we have $v'(1) - \gamma u'(1) = -c(v(1))v(1) + \gamma c(u(1))u(1) \geq 0$. This implies that $v'(1) = \gamma u'(1)$ since $v \geq \gamma u$. Let $t^* \in (0,1)$ be such that $v(t) - \gamma u(t) \geq 0$ and $v'(t) - \gamma u'(t) \leq 0$ for $t \in [t^*, 1]$, $v(t^*) - \gamma u(t^*) > 0$ and $v'(t^*) - \gamma u'(t^*) = 0$. If $t^* \geq t_I$, then by (H_5) we have

$$\begin{aligned}
0 &= -\varphi_p(v'(1)) + \varphi_p(\gamma u'(1)) \\
&= \lambda \int_{t^*}^1 h(s)\left(f(v) - \gamma^{p-1} f(u)\right) ds \\
&\geq \lambda \int_{t^*}^1 h(s)\left(f(\gamma u) - \gamma^{p-1} f(u)\right) ds \\
&\geq \lambda(\gamma^q - \gamma^{p-1}) \int_{t^*}^1 h(s) f(u) ds.
\end{aligned}$$

This implies that $\int_{t^*}^1 h(s) f(u) ds = 0$. This is a contradiction. Thus $t^* < t_I$. Then we have

$$0 \leq -\varphi_p(v'(t_I)) + \varphi_p(\gamma u'(t_I)) = \lambda \int_{t^*}^{t_I} h(s)\left(f(v) - \gamma^{p-1} f(u)\right) ds.$$

Thus we obtain

$$\begin{aligned}
0 &= -\varphi_p(v'(1)) + \varphi_p(\gamma u'(1)) \\
&= \lambda \int_{t^*}^{t_I} h(s)\big(f(v) - \gamma^{p-1} f(u)\big)\, ds + \lambda \int_{t_I}^{1} h(s)\big(f(v) - \gamma^{p-1} f(u)\big)\, ds \\
&\geq \lambda(\gamma^q - \gamma^{p-1}) \int_{t_I}^{1} h(s) f(u) ds.
\end{aligned}$$

This implies that $\int_{t_I}^{1} h(s)f(u)ds = 0$. This is a contradiction. Hence $v(1) > \gamma v(1)$. Let $\hat{\epsilon}_\lambda := v(1) - \gamma u(1)$. On $(\frac{\delta}{2}, 1]$, by (H_5) we have

$$-\left(\varphi_p(v')\right)' = \lambda h(t) f(v) \geq \lambda h(t) f(\gamma u) \geq \lambda \gamma^{p-1} h(t) f(u).$$

But

$$-\left(\varphi_p((\gamma u + \epsilon_\lambda^*)')\right)' = \lambda \gamma^{p-1} h(t) f(u)$$

where $\epsilon_\lambda^* := \min\{\tilde{\epsilon}_\lambda, \hat{\epsilon}_\lambda\}$. Thus $v \geq \gamma u + \epsilon_\lambda^*$ on $(\frac{\delta}{2}, 1]$ by the comparison principle. By Lemma 6.4, we have

$$v \geq \gamma u + \epsilon_\lambda^* \geq \Big(\gamma + \frac{\epsilon_\lambda^*}{C_2 G^{-1}(\lambda^{\frac{1}{p-1}})}\Big) u$$

on $(\frac{\delta}{2}, 1]$. Thus $v \geq \big(\gamma + \epsilon_\lambda^{\#}\big) u$ on $[0, 1]$ where $\epsilon_\lambda^{\#} := \min\big\{\epsilon_\lambda, \frac{\epsilon_\lambda^*}{C_2 G^{-1}(\lambda^{\frac{1}{p-1}})}\big\}$. This is a contradiction for the maximality of γ. Hence $\gamma \geq 1$. This implies that $v \equiv u$ on $[0, 1]$.

CHAPTER VII

COMPUTATIONAL RESULTS FOR BOUNDARY VALUE PROBLEMS WITH DIRICHLET BOUNDARY CONDITIONS

7.1 Autonomous Problems with Dirichlet Boundary Conditions

Autonomous boundary value problems of the form:

$$\begin{cases} -u'' = \lambda \tilde{f}(u); \quad t \in (0,1) \\ u(0) = 0 = u(1) \end{cases} \tag{7.1}$$

are studied in this section. Here $\tilde{f} : (0, \infty) \to \mathbb{R}$ is a continuously differentiable function that is integrable on $(0, \epsilon)$ for some $\epsilon > 0$ and satisfies one of the additional hypotheses (P) or (S):

(P) $\tilde{f}(s) > 0$ for all $s > 0$

(S) there exist unique $\beta > 0$ and $\theta > 0$ such that $\tilde{f}(s) < 0$ for $s \in (0, \beta)$, $\tilde{f}(s) > 0$ for $s \in (\beta, \infty)$ and $F(\theta) = 0$ where $F(s) := \int_0^s \tilde{f}(r)dr$.

Since the equation (7.1) is autonomous, all solutions are symmetric about $t = \frac{1}{2}$. For the case (P), positive solutions must be concave on $(0, 1)$ (see Figure 7). For the case (S), positive solutions u must be convex on regions where $u(t) < \beta$ (near $t = 0$ and $t = 1$) and be concave on regions where $u(t) > \beta$ (see Figure 8).

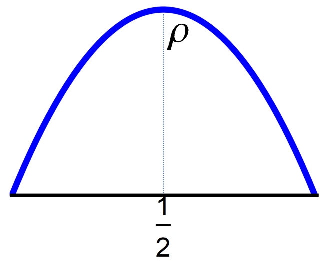

Figure 7. Positive Solutions of (7.1) in the Case (P).

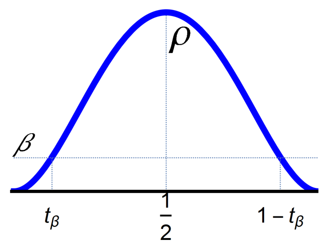

Figure 8. Positive Solutions of (7.1) in the Case (S).

In fact, by the quadrature method as described in [Lae71], solutions are determined by

$$\sqrt{2\lambda}t = \int_0^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}; \quad t \in \left(0, \frac{1}{2}\right) \tag{7.2}$$

with $u(t) = u(1-t)$ for $t \in \left(\frac{1}{2}, 1\right)$ and $u(\frac{1}{2}) = \|u\|_\infty = \rho$. Evaluating (7.2) at $t = \frac{1}{2}$, it follows that a $(\lambda, \|u\|_\infty)$-bifurcation curve is given by

$$\sqrt{\lambda} = \sqrt{2} \int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}. \tag{7.3}$$

Further, (7.1) has a positive solution $u \in C^2(0,1) \cap C^1[0,1]$ of (7.1) with $\|u\|_\infty = \rho$ for a given (λ, ρ) satisfying (7.3) (see [Lae71]).

Now we provide $(\lambda, \|u\|_\infty)$-bifurcation diagrams for several examples of (7.1). The general procedure to generate bifurcation curves is that we first choose several appropriate values for ρ and evaluate corresponding λ from (7.3) using Mathematica. In the following examples, more detailed cases of the behavior of $\tilde{f}$ are considered:

$(P1)$ (P) and $\tilde{f}(0) > 0$

$(P2)$ (P) and $\lim_{s\to 0+} \tilde{f}(s) = \infty$

$(S1)$ (S) and $\tilde{f}(0) < 0$

$(S2)$ (S) and $\lim_{s\to 0+} \tilde{f}(s) = -\infty$.

First, we consider the following example related to $(P1)$:

$$\tilde{f}(s) = e^{\frac{5s}{5+s}}. \tag{7.4}$$

This is a particular example of (1.6) with $\alpha = 0$ and $f(s) = e^{\frac{5s}{5+s}}$. Since f satisfies the hypotheses of Theorem 1.1 with $h \equiv 1$, the uniqueness result holds for $\lambda \gg 1$. Computationally, we seek a value λ^* such that the corresponding value of ρ is unique for all $\lambda > \lambda^*$. According to Figure 9, we have the estimate $\lambda^* \approx 4.65$. We can also see that a unique ρ exists for $\lambda < 3.51$ and three values of ρ exist for $\lambda \in (3.51, 4.65)$. Further, positive solutions satisfy the following properties: as λ increases in $(0, 3.51) \cup (4.65, \infty)$, $\|u\|_\infty$ and $u'(0)$ increase (see Figures 11 - 12).

Second, we consider the following example related to $(P2)$:

$$\tilde{f}(s) = \frac{e^{\frac{13\sqrt{s}}{13+\sqrt{s}}}}{\sqrt{s}}. \tag{7.5}$$

This is a particular example of (1.6) with $\alpha = \frac{1}{2}$ and $f(s) = e^{\frac{13\sqrt{s}}{13+\sqrt{s}}}$. Since f satisfies the hypotheses of Theorem 1.2 with $h \equiv 1$, the uniqueness result holds for $\lambda \gg 1$. Computationally, we seek a value λ^* such that the corresponding value of ρ is unique for all $\lambda > \lambda^*$. According to Figure 13, we have the estimate $\lambda^* \approx 35.49$. We can also see that a unique ρ exists for $\lambda < 29.85$ and three values of ρ exist for $\lambda \in (29.85, 35.49)$. Further, positive solutions satisfy the following properties: as λ increases in $(0, 29.85) \cup (35.49, \infty)$, $\|u\|_\infty$ and $u'(0)$ increase (see Figures 15 - 16).

Next, we consider the following example related to $(S1)$:

$$\tilde{f}(s) = \sqrt{s+1} - 2. \tag{7.6}$$

This is a particular example of (1.6) with $\alpha = 0$ and $f(s) = \sqrt{s+1} - 2$. It is easy to show that $\beta = 3$, $\theta \approx 6.46$ and f satisfies the hypotheses of Theorem 1.3. However, $h \equiv 1$ does not satisfy (H_9). Hence the result of Theorem 1.3 does not hold. Nevertheless, computationally we can seek a value λ^* such that the corresponding value of ρ is unique for all $\lambda > \lambda^*$. According to Figure 17, we have the estimate $\lambda^* \approx 141.70$. We can also see that two values of ρ exist for $\lambda \in (75.57, 141.70)$ and no such ρ exists for $\lambda < 75.57$. Further, positive solutions satisfy the following properties: as λ increases for $\lambda \gg 1$, $\|u\|_\infty$ and $u'(0)$ increase, and as λ increases along the lower bifurcation branch, $\|u\|_\infty \to 6.46 \approx \theta$ and $u'(0) \to 0$ (see Figures 19 - 20).

Finally, we consider the following example related to $(S2)$:

$$\tilde{f}(s) = \frac{s-2}{\sqrt{s}}. \tag{7.7}$$

This is a particular example of (1.6) with $\alpha = \frac{1}{2}$ and $f(s) = s - 2$. It is easy to show that $\beta = 2$ and $\theta \approx 6$. No uniqueness result is established for the case when $f(0) < 0$ and $\alpha \neq 0$, but computationally we can still seek a value λ^* such that the corresponding value of ρ is unique for all $\lambda > \lambda^*$. According to Figure 21, we have the estimate $\lambda^* \approx 42.73$. We can also see that two values of ρ exist for $\lambda \in (38.35, 42.73)$ and no such ρ exists for $\lambda < 38.35$ in the bifurcation curves of (7.7). Further, positive solutions satisfy the following properties: as λ increases for $\lambda > 42.73$, $\|u\|_\infty$ and $u'(0)$ increase, and as λ increases along the lower bifurcation branch, $\|u\|_\infty \to \theta \approx 6$ and $u'(0) \to 0$ (see Figures 23 - 24).

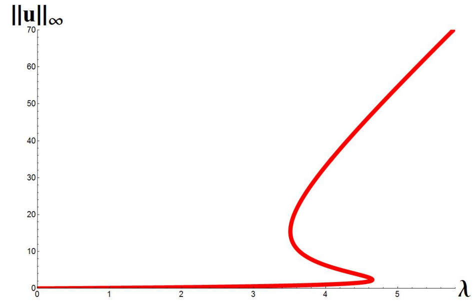

Figure 9. Bifurcation Curve for (7.4).

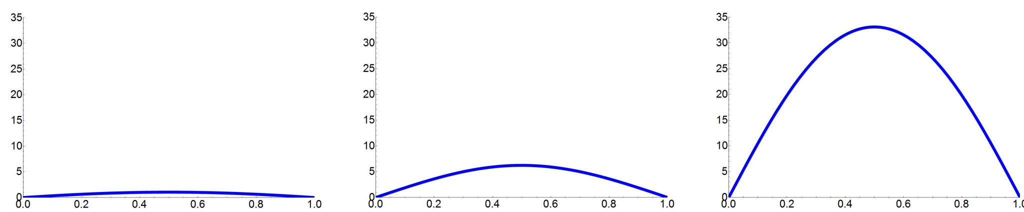

Figure 10. Three Positive Solutions of (7.4) at $\lambda \approx 4.00$.

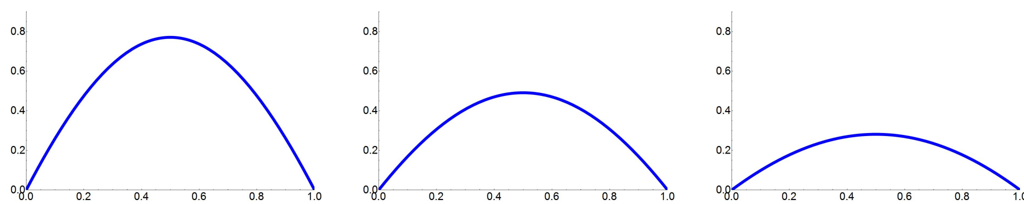

Figure 11. Positive Solutions of (7.4) at $\lambda \approx 3.50$, $\lambda \approx 2.69$ and $\lambda \approx 1.80$. Along the lower bifurcation branch, $\|u\|_\infty$ and $u'(0)$ decrease as λ approaches 0.

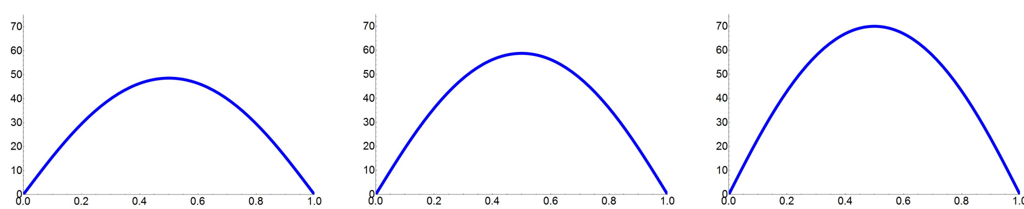

Figure 12. Positive Solutions of (7.4) at $\lambda \approx 4.70$, $\lambda \approx 5.20$ and $\lambda \approx 5.77$. Along the upper bifurcation branch, $\|u\|_\infty$ and $u'(0)$ increase as λ approaches ∞.

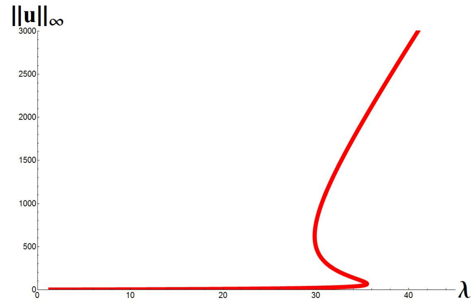

Figure 13. Bifurcation Curve for (7.5).

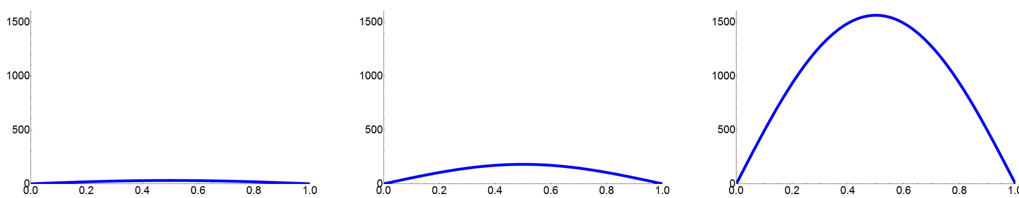

Figure 14. Three Positive Solutions of (7.5) at $\lambda \approx 33.03$.

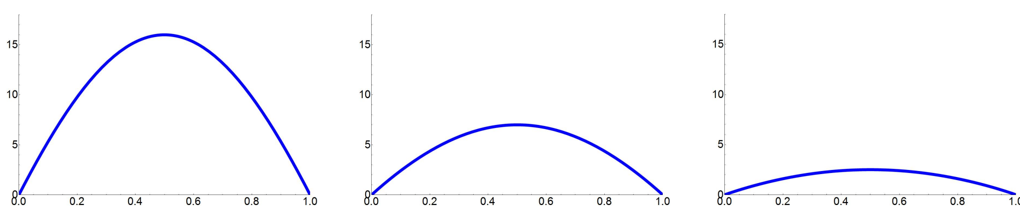

Figure 15. Positive Solutions of (7.5) at $\lambda \approx 27.19$, $\lambda \approx 17.62$ and $\lambda \approx 7.80$. Along the lower bifurcation branch, $\|u\|_\infty$ and $u'(0)$ decrease as λ approaches 0.

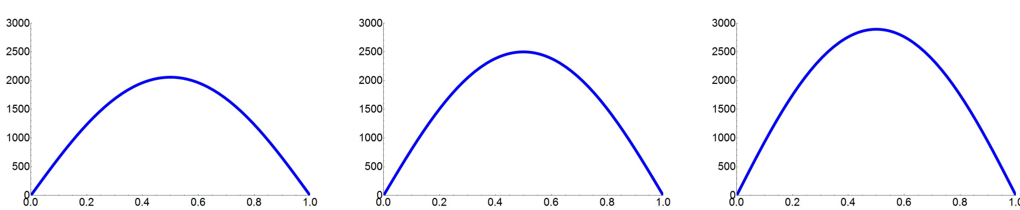

Figure 16. Positive Solutions of (7.5) at $\lambda \approx 35.62$, $\lambda \approx 38.09$ and $\lambda \approx 40.37$. Along the upper bifurcation branch, $\|u\|_\infty$ and $u'(0)$ increase as λ approaches ∞.

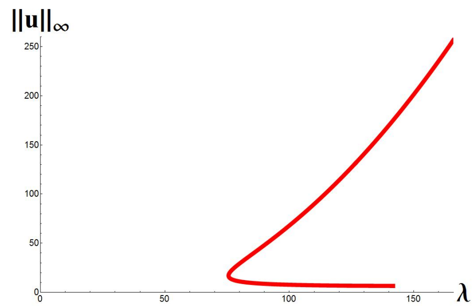

Figure 17. Bifurcation Curve for (7.6).

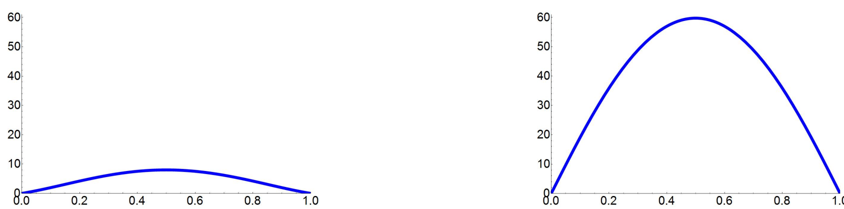

Figure 18. Two Positive Solutions of (7.6) at $\lambda \approx 96.00$.

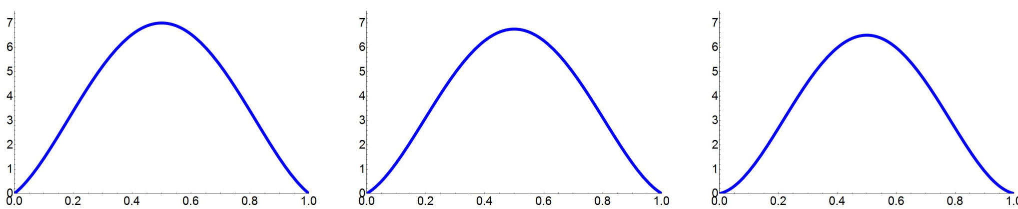

Figure 19. Positive Solutions of (7.6) at $\lambda \approx 113.94$, $\lambda \approx 123.09$ and $\lambda \approx 141.70$. Along the lower bifurcation branch, $\|u\|_\infty$ and $u'(0)$ decrease as λ increases.

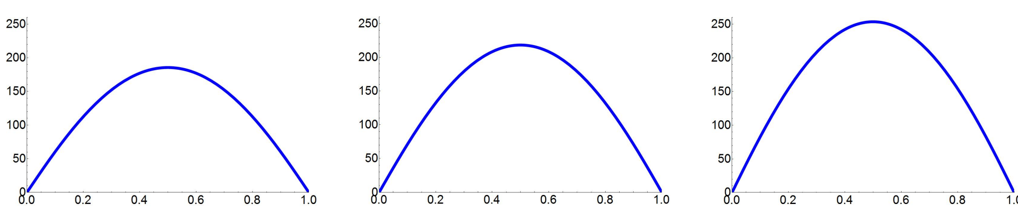

Figure 20. Positive Solutions of (7.6) at $\lambda \approx 145.06$, $\lambda \approx 155.05$ and $\lambda \approx 165.04$. Along the upper bifurcation branch, $\|u\|_\infty$ and $u'(0)$ increase as λ increases.

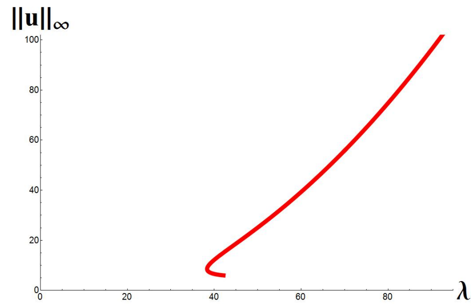

Figure 21. Bifurcation Curve for (7.7).

Figure 22. Two Positive Solutions of (7.7) at $\lambda \approx 40.14$.

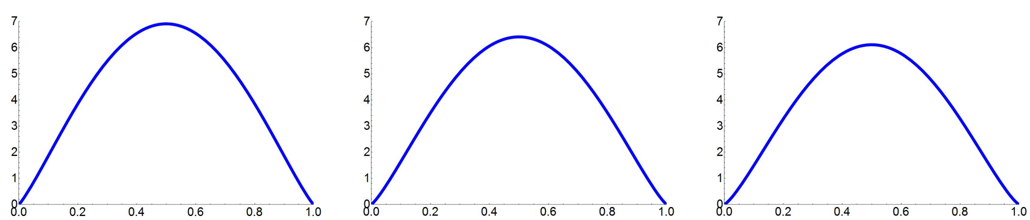

Figure 23. Positive Solutions of (7.7) at $\lambda \approx 39.34$, $\lambda \approx 40.43$ and $\lambda \approx 41.56$. Along the lower bifurcation branch, $\|u\|_\infty$ and $u'(0)$ decrease as λ increases.

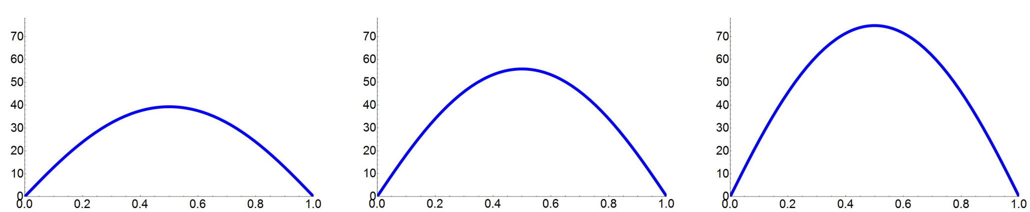

Figure 24. Positive Solutions of (7.7) at $\lambda \approx 60.00$, $\lambda \approx 69.99$ and $\lambda \approx 80.02$. Along the upper bifurcation branch, $\|u\|_\infty$ and $u'(0)$ increase as λ increases.

7.2 Nonautonomous Problems with Dirichlet Boundary Conditions

In this section, nonautonomous boundary value problems (1.6) (when h is nonconstant) are studied. When dealing with (1.6), depending on the sign of $f(0)$, positive solutions must be one of the two forms represented in Figure 25.

Figure 25. Positive Solutions of (1.6).

Since the equation is nonautonomous, the quadrature method does not apply. Hence shooting methods are used to generate $(\lambda, \|u\|_\infty)$-bifurcation curves. The general procedure is that we first guess ranges of λ and $u'(1)$ using Matlab and/or Mathematica. Typically, to find an initial solution corresponding to a single $(\lambda, u'(1))$ pair, a finite difference discretization of the PDE and Fsolve are used in Matlab to directly approximate the solution for a given value of λ. Using the finite difference solution, we can approximate $u'(1)$. This point serves as a starting point for generating the full bifurcation curve using Mathematica. To this end, we apply the shooting method to the initial value problem:

$$\begin{cases} -u'' = \lambda h(t) f(u); \quad t \in (0,1) \\ u(1) = 0, \quad u'(1) = -q \end{cases} \tag{7.8}$$

with $u(0) = 0$ and $u(t) > 0$ for $t \in (0, 1)$ for the case when $\alpha = 0$, or

$$\begin{cases} -u'' = \lambda h(t) \frac{f(u)}{(u+\epsilon)^\alpha}; \quad t \in (0,1) \\ u(1) = 0, \quad u'(1) = -q \end{cases} \tag{7.9}$$

with $u(0) = 0$ and $u(t) > 0$ for $t \in (0, 1)$ where $\epsilon \approx 0$ for the case when $0 < \alpha < 1$. The initial value problems are discretized using NDSolve, an ODE solver in Mathematica. If successful, $\rho = \|u\|_\infty$ can be evaluated. Then we again numerically solve the initial value problems (7.8) or (7.9) to estimate values for λ and ρ after slightly changing q, exploiting the continuity of the curve. The shooting method can generate all pairs of the points (λ, ρ) since it is based on the bisection method and performs an exhaustive search, yielding the $(\lambda, \|u\|_\infty)$-bifurcation curve.

Now we provide several $(\lambda, \|u\|_\infty)$-bifurcation diagrams for examples corresponding to (1.6). First, we consider the following example related to $(P1)$:

$$h(t) = \frac{1}{\sqrt[3]{t}}, \quad f(s) = e^{\frac{6s}{6+s}} \quad \text{and} \quad \alpha = 0. \tag{7.10}$$

Since h and f satisfy the hypotheses of Theorem 1.1, the uniqueness result holds for $\lambda \gg 1$. Computationally, we seek a value λ^* such that the corresponding value of ρ is unique for all $\lambda > \lambda^*$. According to Figure 26, we have the estimate $\lambda^* \approx 3.30$. We can also see that a unique ρ exists for $\lambda < 1.51$ and three values of ρ exist for $\lambda \in (1.51, 3.30)$. Further, positive solutions satisfy the following properties: as λ increases in $(0, 1.51) \cup (3.30, \infty)$, $\|u\|_\infty$ and $u'(0)$ increase (see Figures 28 - 29).

Second, we consider the following example related to $(P2)$:

$$h(t) = \frac{1}{\sqrt[3]{t}}, \quad f(s) = e^{\frac{6s}{6+s}} \quad \text{and} \quad \alpha = \frac{1}{3}. \tag{7.11}$$

Since h and f satisfy the hypotheses of Theorem 1.2, the uniqueness result holds for $\lambda \gg 1$. Computationally, choosing $\epsilon = 0.005$ in (7.9), we seek a value λ^* such that the corresponding value of ρ is unique for all $\lambda > \lambda^*$. According to Figure 30, we have the estimate $\lambda^* \approx 4.28$. We can also see that a unique ρ exists for $\lambda < 3.81$ and three values of ρ exist for $\lambda \in (3.81, 4.28)$. Further, positive solutions satisfy the following properties: as λ increases in $(0, 3.81) \cup (4.28, \infty)$, $\|u\|_\infty$ and $u'(0)$ increase (see Figures 32 - 33).

Next, we consider the following example related to $(S1)$:

$$h(t) = \frac{1}{\sqrt[3]{t}}, \quad f(s) = \sqrt{s+1} - 2 \quad \text{and} \quad \alpha = 0. \tag{7.12}$$

It is easy to show that h and f satisfy the hypotheses of Theorem 1.3 with $\beta = 3$, $\theta \approx 6.46$. Hence the uniqueness result of Theorem 1.3 holds for $\lambda \gg 1$. Computationally, we seek a value λ^* such that the corresponding value of ρ is unique for all $\lambda > \lambda^*$. According to Figure 34, we have the estimate $\lambda^* \approx 79.94$. We can also see that two values of ρ exist for $\lambda \in (57.59, 79.94)$ and no such ρ exists for $\lambda < 57.59$. Further, positive solutions satisfy the following properties: as λ increases for $\lambda \gg 1$, $\|u\|_\infty$ and $u'(0)$ increase, and as λ increases along the below bifurcation branch, $\|u\|_\infty$ and $u'(0)$ decrease (see Figures 36 - 37).

Finally, we consider the following example related to $(S2)$:

$$h(t) = \frac{1}{\sqrt[3]{t}}, \quad f(s) = s - 1 \quad \text{and} \quad \alpha = \frac{1}{3}. \tag{7.13}$$

No uniqueness result is established for the case when $f(0) < 0$ and $\alpha \neq 0$, but computationally, choosing $\epsilon = 0.005$ in (7.9), we can seek a value λ^* such that the corresponding value of ρ is unique for all $\lambda > \lambda^*$. According to Figure 38, we have the estimate $\lambda^* \approx 17.71$. We can also see that two values of ρ exist for $\lambda \in (16.39, 17.71)$ and no such ρ exists for $\lambda < 16.39$. Further, positive solutions satisfy the following properties: as λ increases for $\lambda \gg 1$, $\|u\|_\infty$ and $u'(0)$ increase, and as λ increases along the below bifurcation branch, $\|u\|_\infty$ and $u'(0)$ decrease (see Figures 40 - 41).

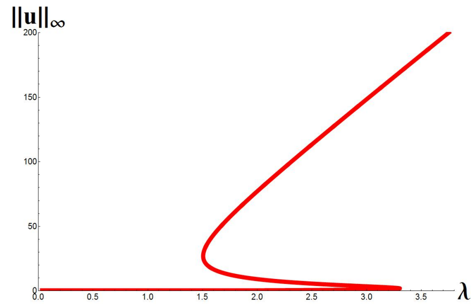

Figure 26. Bifurcation Curve for (7.10).

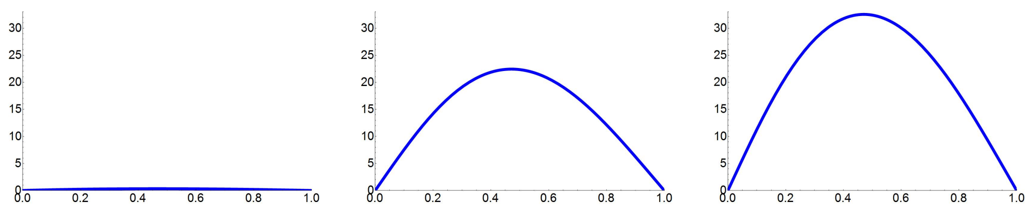

Figure 27. Three Positive Solutions of (7.10) at $\lambda \approx 1.52$.

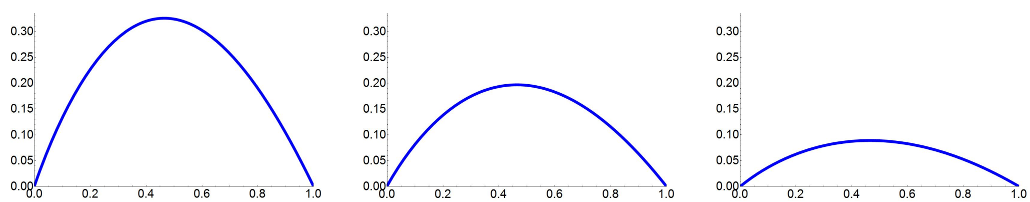

Figure 28. Positive Solutions of (7.10) at $\lambda \approx 1.50$, $\lambda \approx 1.00$ and $\lambda \approx 0.50$. Along the lower bifurcation branch, $\|u\|_\infty$ and $u'(0)$ decrease as λ approaches 0.

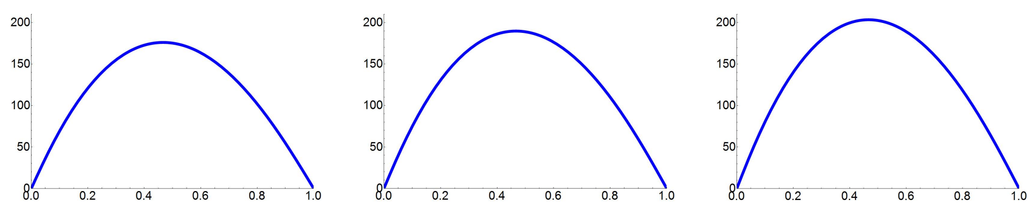

Figure 29. Positive Solutions of (7.10) at $\lambda \approx 3.40$, $\lambda \approx 3.60$ and $\lambda \approx 3.80$. Along the upper bifurcation branch, $\|u\|_\infty$ and $u'(0)$ increase as λ approaches ∞.

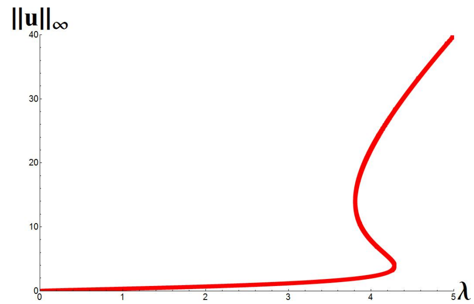

Figure 30. Bifurcation Curve for (7.11).

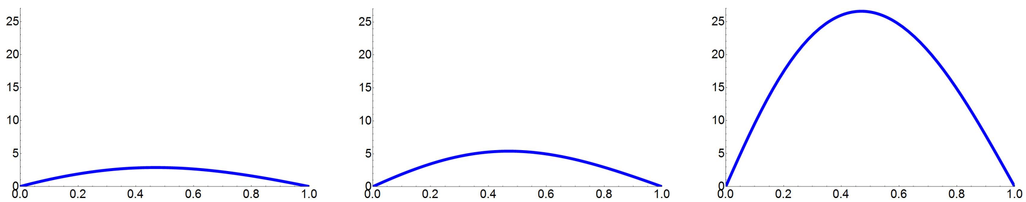

Figure 31. Three Positive Solutions of (7.11) at $\lambda \approx 4.20$.

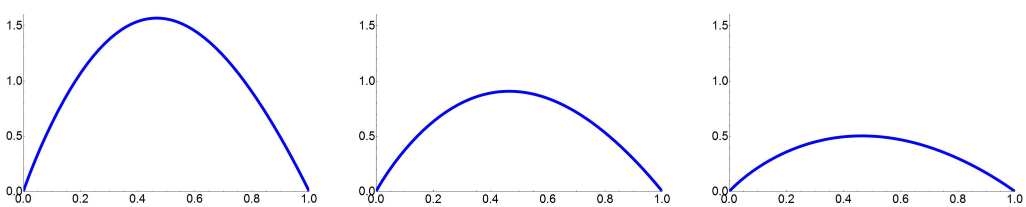

Figure 32. Positive Solutions of (7.11) at $\lambda \approx 3.50$, $\lambda \approx 2.50$ and $\lambda \approx 1.50$. Along the lower bifurcation branch, $\|u\|_\infty$ and $u'(0)$ decrease as λ approaches 0.

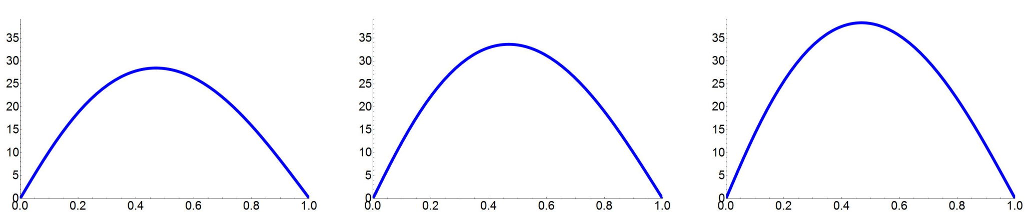

Figure 33. Positive Solutions of (7.11) at $\lambda \approx 4.30$, $\lambda \approx 4.60$ and $\lambda \approx 4.90$. Along the upper bifurcation branch, $\|u\|_\infty$ and $u'(0)$ increase as λ approaches ∞.

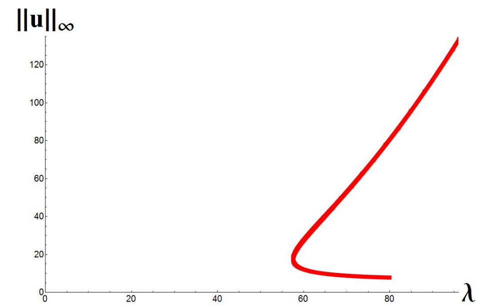

Figure 34. Bifurcation Curve for (7.12).

Figure 35. Two Positive Solutions of (7.12) at $\lambda \approx 60.00$.

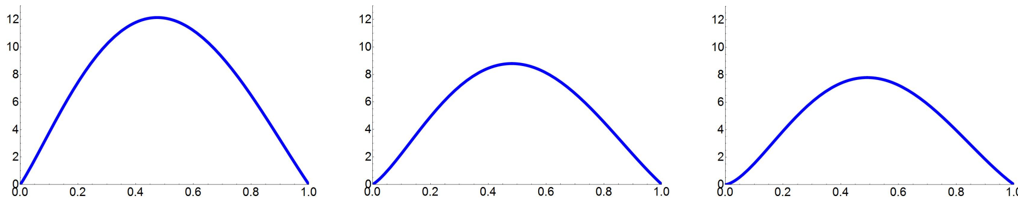

Figure 36. Positive Solutions of (7.12) at $\lambda \approx 59.95$, $\lambda \approx 69.95$ and $\lambda \approx 79.94$. Along the lower bifurcation branch, $\|u\|_\infty$ and $u'(0)$ decrease as λ increases.

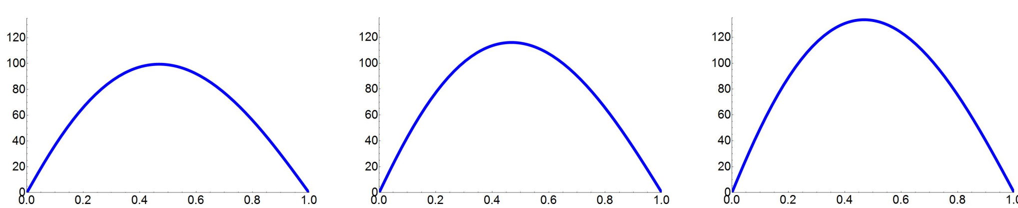

Figure 37. Positive Solutions of (7.12) at $\lambda \approx 86.15$, $\lambda \approx 91.15$ and $\lambda \approx 96.15$. Along the upper bifurcation branch, $\|u\|_\infty$ and $u'(0)$ increase as λ increases.

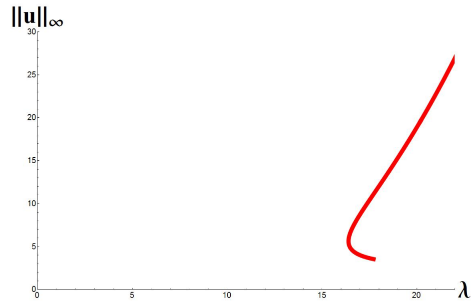

Figure 38. Bifurcation Curve for (7.13).

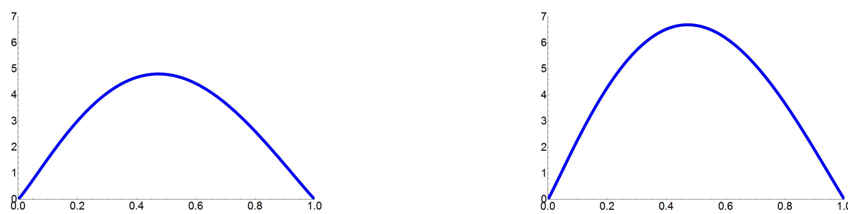

Figure 39. Two Positive Solutions of (7.13) at $\lambda \approx 16.50$.

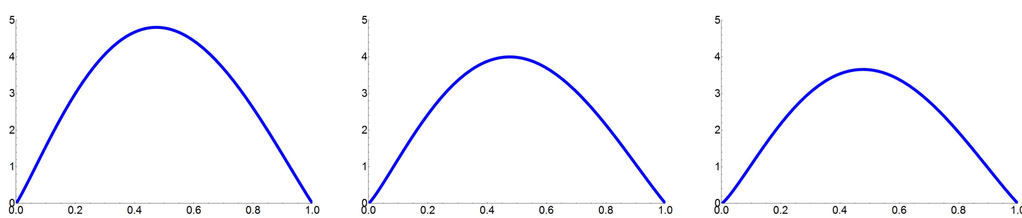

Figure 40. Positive Solutions of (7.13) at $\lambda \approx 16.50$, $\lambda \approx 17.01$ and $\lambda \approx 17.49$. Along the lower bifurcation branch, $\|u\|_\infty$ and $u'(0)$ decrease as λ increases.

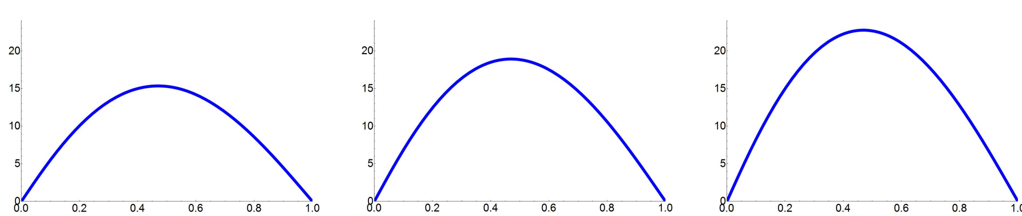

Figure 41. Positive Solutions of (7.13) at $\lambda \approx 19.00$, $\lambda \approx 20.00$ and $\lambda \approx 21.00$. Along the upper bifurcation branch, $\|u\|_\infty$ and $u'(0)$ increase as λ increases.

CHAPTER VIII

COMPUTATIONAL RESULTS FOR BOUNDARY VALUE PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

8.1 Autonomous Problems with Nonlinear Boundary Conditions

One of the goals in this section is to extend the quadrature method for generating $(\lambda, \|u\|_\infty)$-bifurcation curves for positive solutions to autonomous problems with nonlinear boundary conditions.

Consider autonomous boundary value problems of the form:

$$\begin{cases} -u'' = \lambda \tilde{f}(u); \quad t \in (0,1) \\ u'(1) + c(u(1))u(1) = 0 \\ u(0) = 0. \end{cases} \tag{8.1}$$

Here $\tilde{f}$ is as before in (7.1) and $c : [0, \infty) \to (0, \infty)$ is a continuous function.

For the case (P) introduced in Chapter 7, positive solutions have positive values at $t = 1$, so solutions must be concave on $(0, 1)$ (see Figure 42). For the case (S) introduced in Chapter 7, positive solutions u have nonnegative function values at $t = 1$. Each solution of (S) is convex on regions where $u(t) < \beta$ (near $t = 0$ and possibly near $t = 1$) and is concave on regions $u(t) > \beta$ (see Figure 43). In order to preserve the unique challenges posed by the presence of the nonlinear boundary condition, we only consider solutions where $u(1) > 0$ in this section, which implies that $u'(1) < 0$.

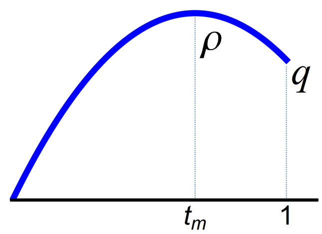

Figure 42. Positive Solutions of (8.1) in the Case (P).

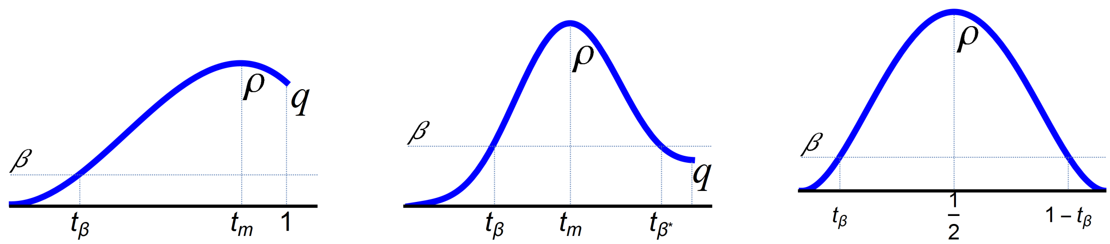

Figure 43. Positive Solutions of (8.1) in the Case (S).

We note that any positive solution of (8.1) has a unique interior maximum at some $t_m \in (0, 1)$ and must be symmetric about t_m (see the proof of Lemma B.2).

Of particular interest is the shape of the bifurcation curves. Laetsch studied such problems in [Lae71] with Dirichlet boundary conditions using a quadrature method (or time map analysis). The ideas of Laetsch have been been adapted to problems with a number of different boundary conditions, for example Neumann (see [MS93]), mixed (see [AMS99]), and nonlinear boundary conditions (see [GPS]). In particular, in [GPS], the authors study a certain example of c arising in population dynamics involving density dependent dispersal on the boundary. We expand the ideas in [GPS] for certain classes of c where $\tilde{f}$ satisfies (P) or (S). In particular, we provide more detailed analysis of the quadrature method for such two-point boundary value problems involving nonlinear boundary conditions. Namely, we establish:

Theorem 8.1. *For $\tilde{f}$ satisfying either (P) or (S), there exists a positive solution $u \in C^2(0,1) \cap C^1[0,1]$ of (8.1) with $\|u\|_\infty = \rho$, $u(1) = q$ and $0 < q < \rho$ if and only if the equations*

$$\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{c(q)q}{\sqrt{F(\rho) - F(q)}} = 0 \tag{8.2}$$

and

$$\sqrt{2\lambda} = \frac{c(q)q}{\sqrt{F(\rho) - F(q)}} \tag{8.3}$$

hold. Further, for (λ, ρ, q) satisfying (8.2) and (8.3), (8.1) has a positive solution u given by

$$\sqrt{2\lambda}t = \int_0^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}; \quad t \in [0, t_m)$$

$$\sqrt{2\lambda}(1 - t) = \int_q^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}; \quad t \in (t_m, 1]$$

$$u(t_m) = \rho \quad \textit{and} \quad u(1) = q$$

where t_m satisfies

$$t_m = \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \Bigg/ \left(\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \right).$$

A series of Theorems related to $(\lambda, \|u\|_\infty)$-bifurcation diagrams are also established.

Theorem 8.2. *If $\tilde{f}$ satisfies (P), then for every $\rho > 0$ there exists $q > 0$ such that (8.2) is satisfied. If $\tilde{f}$ satisfies (S), then for every $\rho \geq \theta$ there exists $q > 0$ such that (8.2) is satisfied.*

Theorem 8.3. *If $\tilde{f}$ satisfies either $(P1)$ or $(P2)$ and $s + c(s)s$ is continuously differentiable and nondecreasing for $s > 0$, then for each fixed $\rho > 0$ there exists a unique $q > 0$ such that (8.2) is satisfied.*

Theorem 8.4. *If $\tilde{f}$ satisfies either $(S1)$ or $(S2)$, $c(s)s$ is continuously differentiable and either*

(S3) $\frac{s+c(s)s}{\sqrt{-F(s)}}$ *is nondecreasing for $s \in (0, \beta)$ and $s + c(s)s$ is nondecreasing for $s > 0$,*
or

(S4) $(f(s)c(s)s)' > 2f(s)$ *for $s \in (0, \beta)$ and $c(s)s$ is nondecreasing for $s > 0$*

is satisfied, then for each fixed $\rho \geq \theta$ there exists a unique $q > 0$ such that (8.2) is satisfied.

See Appendix B.1 - B.4 for the proofs of Theorems 8.1 - 8.4, respectively.

Now we provide $(\lambda, \|u\|_\infty)$-bifurcation diagrams for several examples of (8.1). The general procedure to generate bifurcation curves is that for several appropriate values for ρ, we first find corresponding q from (8.2) and evaluate λ from (8.3) using Mathematica.

First, we consider the following example related to $(P1)$:

$$\tilde{f}(s) = e^{\frac{6s}{6+s}} \quad \text{and} \quad c(s) = e^{\frac{s}{1+s}}. \tag{8.4}$$

Clearly, $\tilde{f}$ satisfies $(P1)$, and $s + c(s)s$ is nondecreasing for $s > 0$. Hence the results of Theorems 8.2 - 8.3 apply. This implies that for $\rho > 0$ there exists a unique $\lambda > 0$ satisfying (8.2) and (8.3). The bifurcation diagram is shown in Figure 44. This is also a particular example of (1.9) with $\alpha = 0$ and $f(s) = e^{\frac{6s}{6+s}}$. Note that the existence and uniqueness results of Theorems 1.4 - 1.6 hold when $\alpha = 0$ (see [BKLS14]). Computationally, we seek values λ_* and λ^* such that the corresponding value of ρ is unique for all $\lambda < \lambda_*$ or $\lambda > \lambda^*$. According to Figure 44, we have the estimates $\lambda_* \approx 1.14$ and $\lambda^* \approx 2.27$. We can also see that three values of ρ exist for $\lambda \in (1.14, 2.27)$. Further, positive solutions satisfy the following properties: as λ increases in $(0, 1.14) \cup (2.27, \infty)$, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase and t_m decreases (see Figures 46 - 47).

Second, we consider the following example related to $(P2)$:

$$\tilde{f}(s) = \frac{e^{\frac{13\sqrt{s}}{13+\sqrt{s}}}}{\sqrt{s}} \quad \text{and} \quad c(s) = e^{\frac{s}{1+s}}. \tag{8.5}$$

Clearly, $\tilde{f}$ satisfies $(P2)$, and $s + c(s)s$ is nondecreasing for $s > 0$. Hence the results of Theorems 8.2 - 8.3 apply. This implies that for $\rho > 0$ there exists a unique $\lambda > 0$ satisfying (8.2) and (8.3). The bifurcation diagram is shown in Figure 48. This is also a particular example of (1.9) with $\alpha = \frac{1}{2}$ and $f(s) = e^{\frac{13\sqrt{s}}{13+\sqrt{s}}}$. Since f and c satisfy the hypotheses of Theorems 1.4 - 1.6 with $h \equiv 1$, the existence and uniqueness results hold. Computationally, we seek values λ_* and λ^* such that the corresponding value of ρ is unique for all $\lambda < \lambda_*$ or $\lambda > \lambda^*$. According to Figure 48, we have the estimates $\lambda_* \approx 17.59$ and $\lambda^* \approx 20.88$. We can also see that three values of ρ exist for $\lambda \in (17.59, 20.88)$. Further, positive solutions satisfy the following properties: as

λ increases in $(0, 17.59) \cup (20.88, \infty)$, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase and t_m decreases (see Figures 50 - 51).

Next, we consider the following example related to $(S1)$:

$$\tilde{f}(s) = \sqrt{s+1} - 2 \quad \text{and} \quad c(s) = \frac{s+2}{s+1}. \tag{8.6}$$

Clearly, $\tilde{f}$ satisfies $(S1)$ and $(S4)$ with $\beta = 3$ and $\theta \approx 6.46$, and $c(s)s$ is nondecreasing for $s > 0$. Hence the results of Theorems 8.2 and 8.4 apply. This implies that for each $\rho > 6.46$ there exists a unique $\lambda > 0$ satisfying (8.2) and (8.3). The bifurcation diagram is shown in Figure 52. This is also a particular example of (1.9) with $\alpha = 0$ and $f(s) = \sqrt{s+1} - 2$. Since f satisfies the hypotheses of Theorems 1.9 - 1.10 with $h \equiv 1$, the existence result for $\lambda \gg 1$ and the nonexistence result for $\lambda \approx 0$ both hold. Computationally, we seek a value λ^* such that the corresponding value of ρ exists for all $\lambda > \lambda^*$ and no corresponding value of ρ exists for all $\lambda < \lambda^*$. According to Figure 52, we have the estimate $\lambda^* \approx 31.98$. We can also see that two values of ρ exist for $\lambda \in (31.98, 52.91)$ and a unique ρ exists for $\lambda > 52.91$ in the Figure. Further, positive solutions satisfy the following properties: as λ increases for $\lambda > 52.91$, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase and t_m and t_β decrease, and as λ increases along the lower bifurcation branch, $\|u\|_\infty \to 6.46 \approx \theta$ and $u'(0) \to 0$ (see Figures 54 - 55). For any positive solution u, $u(1) > 3 = \beta$, so u is convex on $(0, t_\beta)$ and concave on $(t_\beta, 1)$ where t_β is the point where $u(t_\beta) = \beta$.

Finally, we consider the following example related to $(S2)$:

$$\tilde{f}(s) = \frac{s-1}{\sqrt{s}} \quad \text{and} \quad c(s) = \frac{s+2}{s+1}. \tag{8.7}$$

Clearly, $\tilde{f}$ satisfies $(S2)$ and $(S4)$ with $\beta = 1$ and $\theta \approx 3$, and $c(s)s$ is nondecreasing for $s > 0$. Hence the results of Theorems 8.2 and 8.4 apply. This implies that for $\rho > 3$ there exists a unique $\lambda > 0$ satisfying (8.2) and (8.3). The bifurcation diagram is shown in Figure 56. This is also a particular example of (1.9) with $\alpha = \frac{1}{2}$ and $f(s) = s - 1$. Since f satisfies the hypotheses of Theorems 1.9 - 1.10 with $h \equiv 1$, the existence result for $\lambda \gg 1$ and the nonexistence result for $\lambda \approx 0$ both hold. Computationally, we seek a value λ^* such that the corresponding value of ρ exists for all $\lambda > \lambda^*$ and no corresponding value of ρ exists for all $\lambda < \lambda^*$. According to Figure 56, we have the estimate $\lambda^* \approx 11.95$. We can also see that two values of ρ exist for $\lambda \in (11.95, 12.78)$ and a unique ρ exists for $\lambda > 12.78$ in the Figure. Further, positive solutions satisfy the following properties: as λ increases for $\lambda > 12.78$, $\|u\|_\infty$, $u(1)$, $u'(0)$ and t_m increase and t_β decreases, and as λ increases along the lower bifurcation branch, $\|u\|_\infty \to 3 \approx \theta$ and $u'(0) \to 0$ (see Figures 58 - 59). For any positive solution u, $u(1) > 1 = \beta$, so u is convex on $(0, t_\beta)$ and concave on $(t_\beta, 1)$.

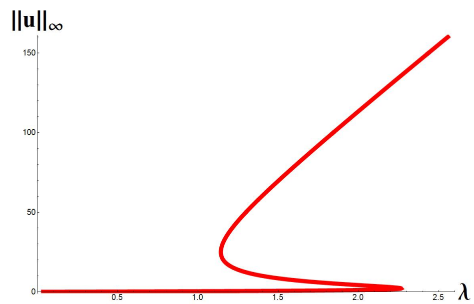

Figure 44. Bifurcation Curve for (8.4).

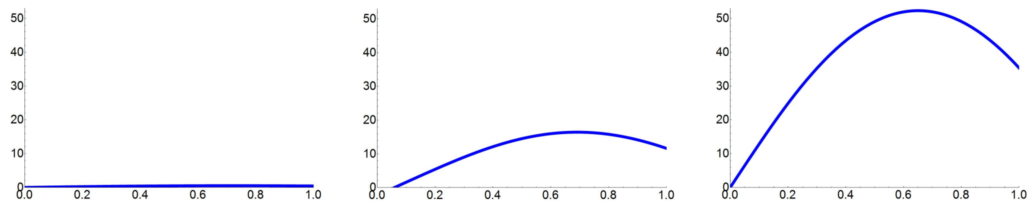

Figure 45. Three Positive Solutions of (8.4) at $\lambda \approx 1.32$.

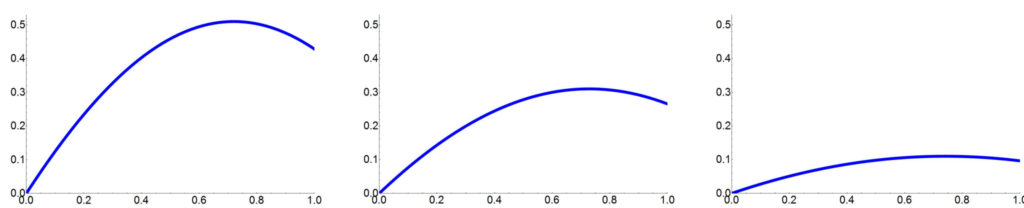

Figure 46. Positive Solutions of (8.4) at $\lambda \approx 1.32$, $\lambda \approx 0.91$ and $\lambda \approx 0.37$. Along the lower bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ decrease as λ approaches 0.

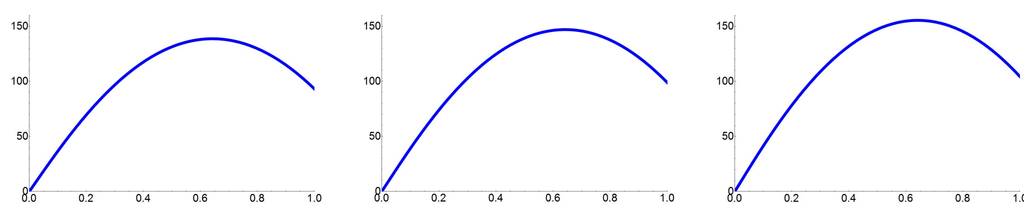

Figure 47. Positive Solutions of (8.4) at $\lambda \approx 2.30$, $\lambda \approx 2.40$ and $\lambda \approx 2.50$. Along the upper bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase as λ approaches ∞.

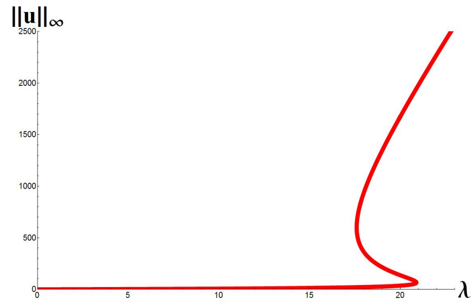

Figure 48. Bifurcation Curve for (8.5).

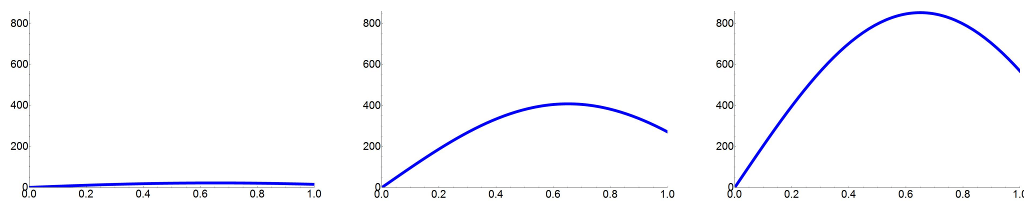

Figure 49. Three Positive Solutions of (8.5) at $\lambda \approx 17.83$.

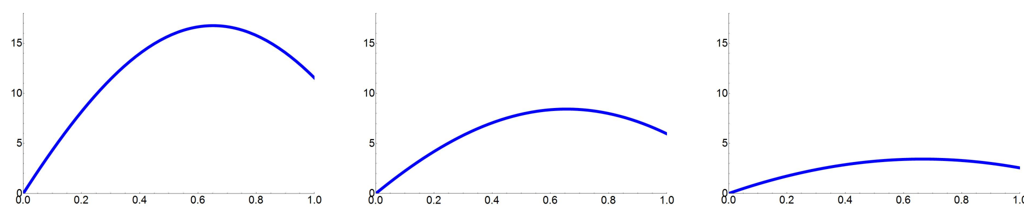

Figure 50. Positive Solutions of (8.5) at $\lambda \approx 16.29$, $\lambda \approx 11.53$ and $\lambda \approx 5.83$. Along the lower bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ decrease as λ approaches 0.

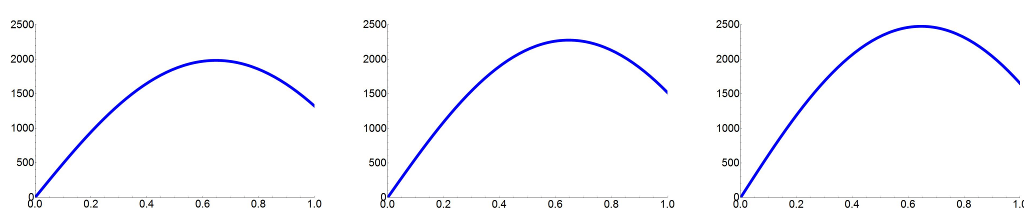

Figure 51. Positive Solutions of (8.5) at $\lambda \approx 21.00$, $\lambda \approx 22.00$ and $\lambda \approx 22.70$. Along the upper bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase as λ approaches ∞.

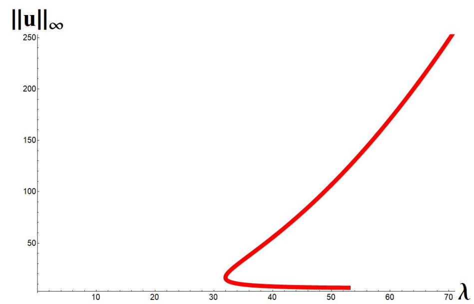

Figure 52. Bifurcation Curve for (8.6).

Figure 53. Two Positive Solutions of (8.6) at $\lambda \approx 33.01$.

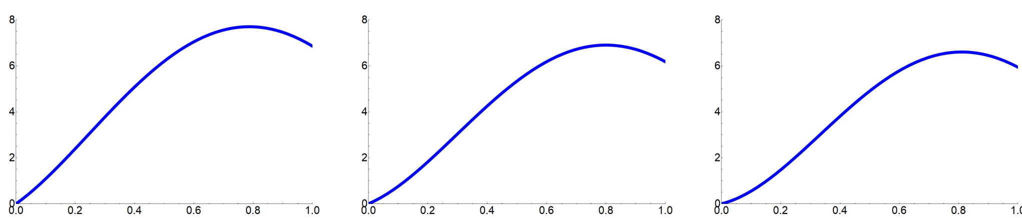

Figure 54. Positive Solutions of (8.6) at $\lambda \approx 40.28$, $\lambda \approx 45.80$ and $\lambda \approx 50.10$. Along the lower bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ decrease as λ increases.

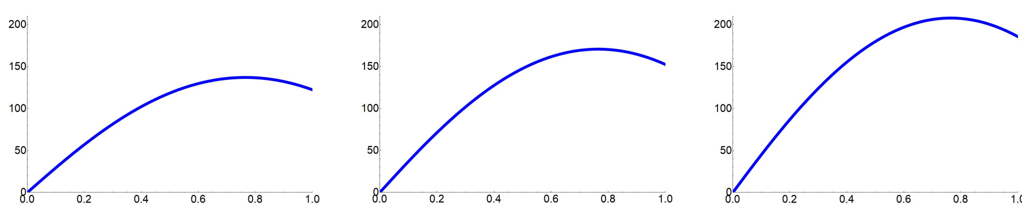

Figure 55. Positive Solutions of (8.6) at $\lambda \approx 55.00$, $\lambda \approx 60.00$ and $\lambda \approx 65.00$. Along the upper bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase as λ approaches ∞.

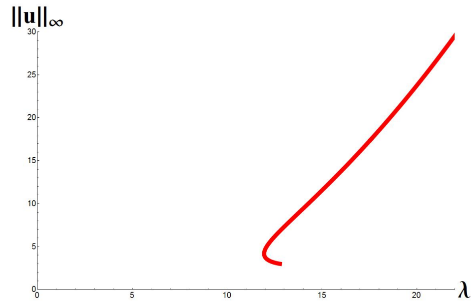

Figure 56. Bifurcation Curve for (8.7).

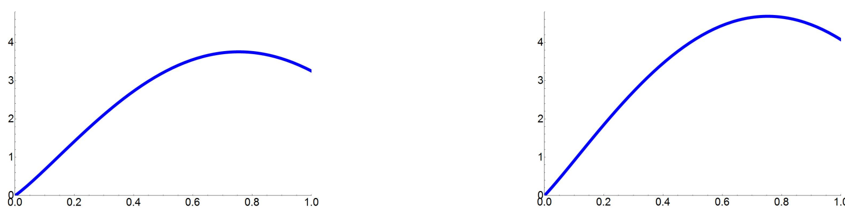

Figure 57. Two Positive Solutions of (8.7) at $\lambda \approx 12.00$.

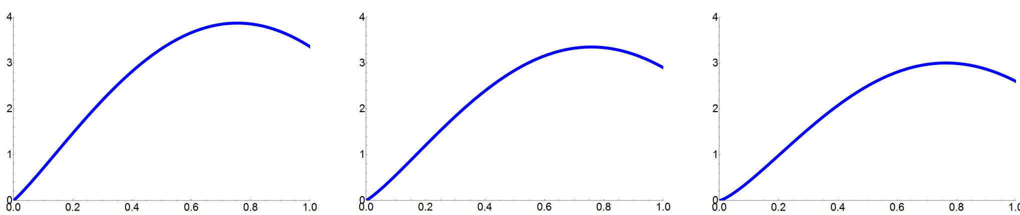

Figure 58. Positive Solutions of (8.7) at $\lambda \approx 11.97$, $\lambda \approx 12.23$ and $\lambda \approx 12.78$. Along the lower bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ decrease as λ increases.

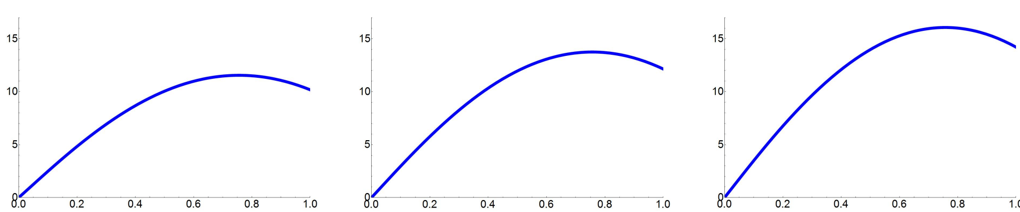

Figure 59. Positive Solutions of (8.7) at $\lambda \approx 15.00$, $\lambda \approx 16.00$ and $\lambda \approx 17.00$. Along the upper bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase as λ approaches ∞.

8.2 Nonautonomous Problems with Nonlinear Boundary Conditions

In this section, nonautonomous boundary value problems (1.9) (when h is nonconstant) are studied. When dealing with (1.9), depending on the sign of $f(0)$, positive solutions must be one of the forms in Figures 42 - 43. Since the equation is nonautonomous, the quadrature method again does not apply. As in Section 7.2, shooting methods are used to generate $(\lambda, \|u\|_\infty)$-bifurcation curves. The general procedure is similar to the method introduced in Section 7.2. We first guess ranges of λ and $u'(1)$ using Matlab and/or Mathematica. Typically, to find an initial solution corresponding to a single $(\lambda, u'(1))$ pair, a finite difference discretization of the PDE and Fsolve are used in Matlab to directly approximate the solution for a given value of λ. Using the finite difference solution, we can approximate $u'(1)$. This point serves as a starting point for generating the full bifurcation curve using Mathematica. To this end, we apply the shooting method to the initial value problem:

$$\begin{cases} -u'' = \lambda h(t)\frac{f(u)}{u^\alpha}; \quad t \in (0,1) \\ u(1) = q, \quad u'(1) = -c(q)q \end{cases} \tag{8.8}$$

with $u(0) = 0$ and $u(t) > 0$ for $t \in (0,1)$. The initial value problems are discretized using NDSolve, an ODE solver in Mathematica. If successful, $\rho = \|u\|_\infty$ can be evaluated. Then we again numerically solve the initial value problem (8.8) to estimate values for λ and ρ after slightly changing q, exploiting the continuity of the curve. The shooting method can generate all pairs of the points (λ, ρ) in the bifurcation curve since it is based on the bisection method and performs an exhaustive search.

Now we provide several $(\lambda, \|u\|_\infty)$-bifurcation diagrams for examples corresponding to (1.9). First, we consider the following example related to $(P1)$:

$$h(t) = \frac{1}{\sqrt[3]{t}}, \quad f(s) = e^{\frac{6s}{6+s}}, \quad \alpha = 0 \quad \text{and} \quad c(s) = e^{\frac{s}{1+s}}. \tag{8.9}$$

Note that the existence and uniqueness results of Theorems 1.4 - 1.6 hold when $\alpha = 0$ (see [BKLS14]). Computationally, we seek values λ_* and λ^* such that the corresponding value of ρ is unique for all $\lambda < \lambda_*$ or $\lambda > \lambda^*$. According to Figure 60, we have the estimates $\lambda_* \approx 0.94$ and $\lambda^* \approx 1.86$. We can also see that three values of ρ exist for $\lambda \in (0.94, 1.86)$. Further, positive solutions satisfy the following properties: as λ increases in $(0, 0.94) \cup (1.86, \infty)$, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase and t_m decreases (see Figures 62 - 63).

Second, we consider the following example related to $(P2)$:

$$h(t) = \frac{1}{\sqrt[3]{t}}, \quad f(s) = e^{\frac{6s}{6+s}}, \quad \alpha = \frac{1}{3} \quad \text{and} \quad c(s) = e^{\frac{s}{1+s}}. \tag{8.10}$$

Since h, f and c satisfy the hypotheses of Theorems 1.4 - 1.6, the existence and uniqueness results hold. Computationally, we seek values λ_* and λ^* such that the corresponding value of ρ is unique for all $\lambda < \lambda_*$ or $\lambda > \lambda^*$. According to Figure 64, we have the estimates $\lambda_* \approx 2.34$ and $\lambda^* \approx 2.52$. We can also see that three values of ρ exist for $\lambda \in (2.34, 2.52)$. Further, positive solutions satisfy the following properties: as λ increases in $(0, 2.34) \cup (2.52, \infty)$, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase and t_m decreases (see Figures 66 - 67).

Next, we consider the following example related to $(S1)$:

$$h(t) = \frac{1}{\sqrt[3]{t}}, \quad f(s) = \sqrt{s+1} - 2, \quad \alpha = 0 \quad \text{and} \quad c(s) = e^{\frac{s}{1+s}}. \tag{8.11}$$

Since h, f and c satisfy the hypotheses of Theorems 1.9 - 1.11, the existence and uniqueness results for $\lambda \gg 1$ and the nonexistence result for $\lambda \approx 0$ hold. Computationally, we seek values λ_* and λ^* such that the corresponding value of ρ exists for all $\lambda > \lambda_*$, the corresponding value of ρ is unique for all $\lambda > \lambda^*$ and no corresponding value of ρ exists for all $\lambda < \lambda_*$. According to Figure 68, we have the estimates $\lambda_* \approx 35.51$ and $\lambda^* \approx 46.68$. We can also see that two values of ρ exist for $\lambda \in (35.51, 46.68)$. Further, positive solutions satisfy the following properties: as λ increases for $\lambda > 46.68$, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase, and as λ increases along the lower bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ decrease (see Figures 70 - 71).

Finally, we consider the following example related to $(S2)$:

$$h(t) = \frac{1}{\sqrt[3]{t}}, \quad f(s) = s - 1, \quad \alpha = \frac{1}{3} \quad \text{and} \quad c(s) = e^{s}. \tag{8.12}$$

Since h and f satisfy the hypotheses of Theorems 1.9 - 1.10, the existence result for $\lambda \gg 1$ and the nonexistence result for $\lambda \approx 0$ hold. Computationally, we seek a value λ^* such that the corresponding value of ρ exists for all $\lambda > \lambda^*$ and no corresponding value of ρ exists for all $\lambda < \lambda^*$. According to Figure 72, we have the estimate $\lambda^* \approx 12.80$. We can also see that two values of ρ exist for $\lambda \in (12.80, 12.92)$ and a unique ρ exists for $\lambda > 12.92$. Further, positive solutions satisfy the following properties: as λ

increases for $\lambda > 12.92$, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase, and as λ increases along the lower bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ decrease (see Figures 74 - 75).

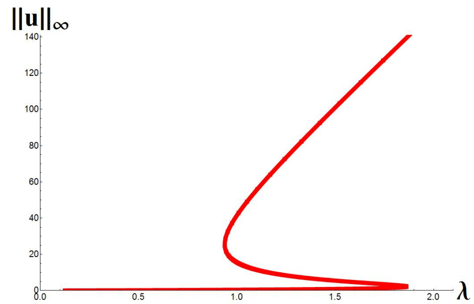

Figure 60. Bifurcation Curve for (8.9).

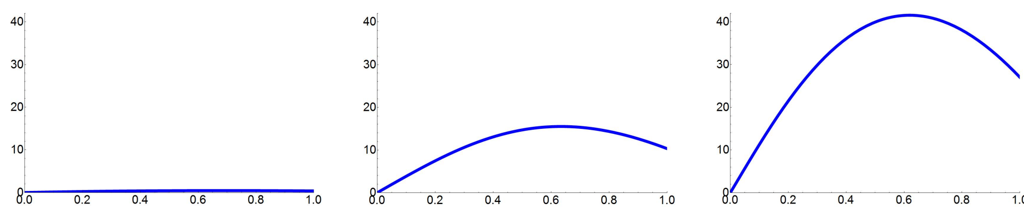

Figure 61. Three Positive Solutions of (8.9) at $\lambda \approx 1.00$.

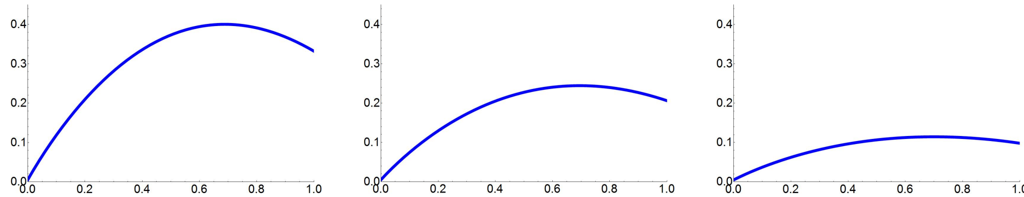

Figure 62. Positive Solutions of (8.9) at $\lambda \approx 0.90$, $\lambda \approx 0.60$ and $\lambda \approx 0.30$. Along the lower bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ decrease as λ approaches 0.

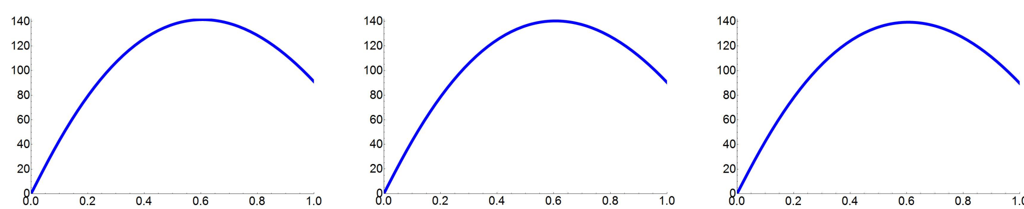

Figure 63. Positive Solutions of (8.9) at $\lambda \approx 1.86$, $\lambda \approx 1.87$ and $\lambda \approx 1.88$. Along the upper bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase as λ approaches ∞.

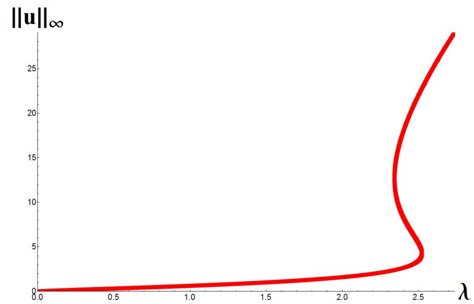

Figure 64. Bifurcation Curve for (8.10).

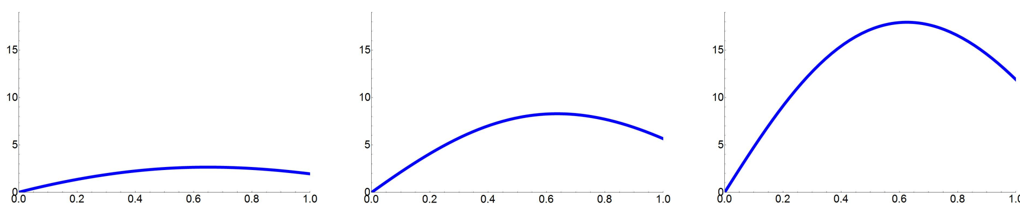

Figure 65. Three Positive Solutions of (8.10) at $\lambda \approx 2.40$.

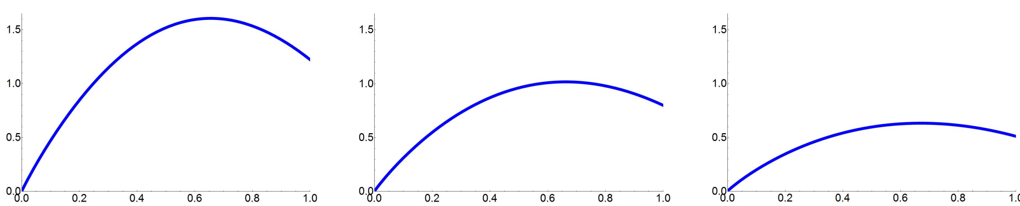

Figure 66. Positive Solutions of (8.10) at $\lambda \approx 2.00$, $\lambda \approx 1.50$ and $\lambda \approx 1.00$. Along the lower bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ decrease as λ approaches 0.

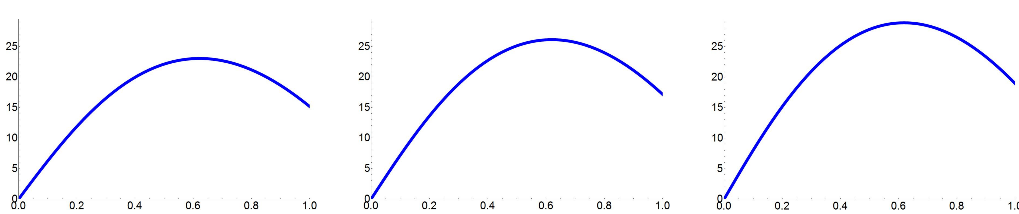

Figure 67. Positive Solutions of (8.10) at $\lambda \approx 2.53$, $\lambda \approx 2.63$ and $\lambda \approx 2.73$. Along the upper bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase as λ approaches ∞.

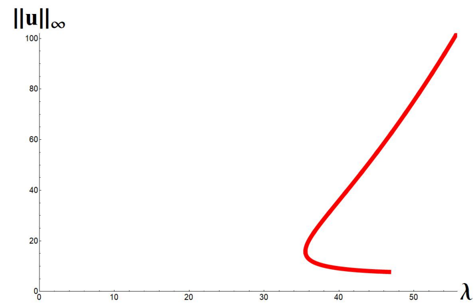

Figure 68. Bifurcation Curve for (8.11).

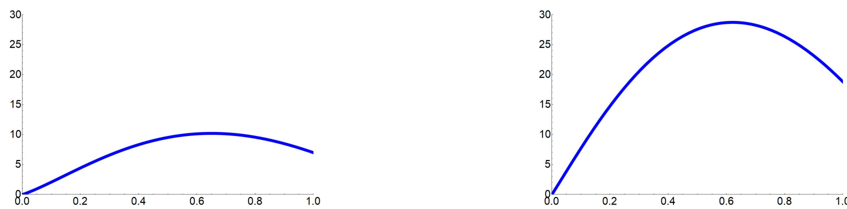

Figure 69. Two Positive Solutions of (8.11) at $\lambda \approx 38.01$.

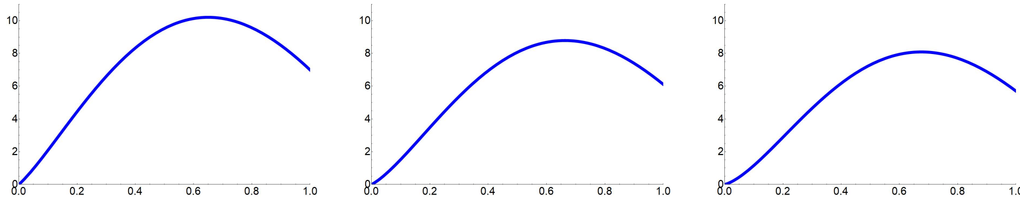

Figure 70. Positive Solutions of (8.11) at $\lambda \approx 38.01$, $\lambda \approx 41.04$ and $\lambda \approx 44.06$. Along the lower bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ decrease as λ increases.

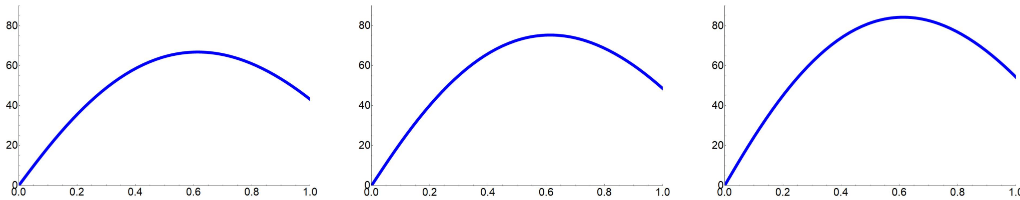

Figure 71. Positive Solutions of (8.11) at $\lambda \approx 48.00$, $\lambda \approx 50.00$ and $\lambda \approx 52.00$. Along the upper bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase as λ approaches ∞.

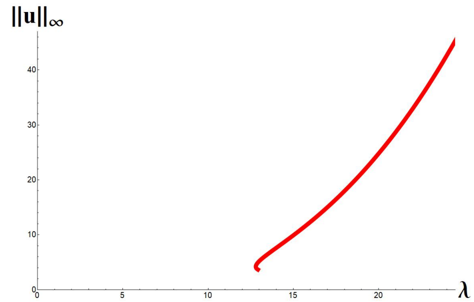

Figure 72. Bifurcation Curve for (8.12).

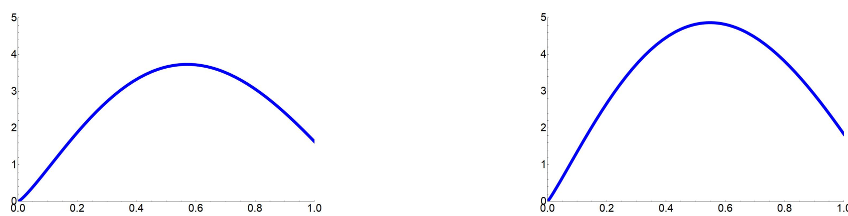

Figure 73. Two Positive Solutions of (8.12) at $\lambda \approx 12.90$.

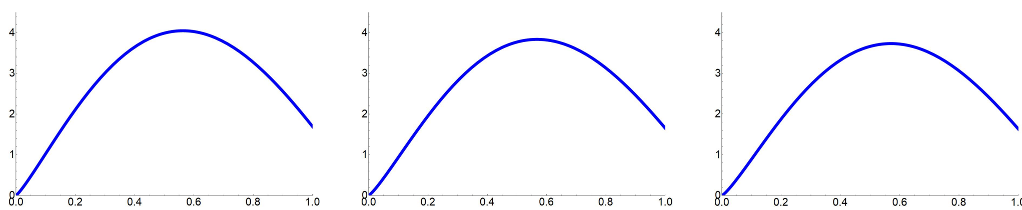

Figure 74. Positive Solutions of (8.12) at $\lambda \approx 12.81$, $\lambda \approx 12.85$ and $\lambda \approx 12.90$. Along the lower bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ decrease as λ increases.

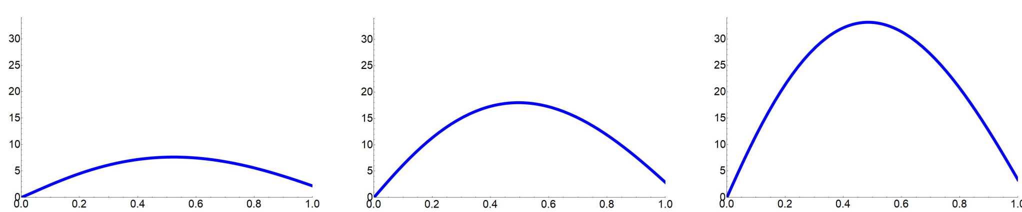

Figure 75. Positive Solutions of (8.12) at $\lambda \approx 14.00$, $\lambda \approx 18.00$ and $\lambda \approx 22.04$. Along the upper bifurcation branch, $\|u\|_\infty$, $u(1)$ and $u'(0)$ increase as λ approaches ∞.

CHAPTER IX

CONCLUSION AND FUTURE DIRECTIONS

9.1 Conclusion

In this dissertation, we studied positive radial solutions to steady state reaction diffusion equations on the exterior of a ball with Dirichlet boundary conditions and nonlinear boundary conditions. In particular, we established existence, multiplicity and uniqueness results for various classes of reaction processes, including positone and semipositone problems in both singular and nonsingular cases. Some of these contributions have already been published or have been accepted for publication in [KLSS14], [LSS16] and [SSS17]. Others are included in [SS] and [LSSS], which are under preparation. We also performed numerical studies generating $(\lambda, \|u\|_\infty)$-bifurcation curves of positive solutions for several examples related to our results yielding more detailed information on the structure of positive solutions as the parameter λ changes.

9.2 Future Directions

We plan to study the following open questions concerning uniqueness for semipositone problems:

(1) Extend Theorem 1.3 to the case $\alpha \neq 0$.

(2) Extend Theorem 1.3 to systems.

(3) Extend Theorem 1.11 to the case $\alpha \neq 0$.

(4) Extend Theorem 1.11 to systems.

We also plan to study non-radial positive solutions, first on the exterior of a ball and next on the exterior of any bounded domain. Recently, a few results have been reported in this direction (see [DS14], [CDS16] and [CDS]). However, many questions remain open. In particular, I plan to focus on uniqueness results for semipositone problems.

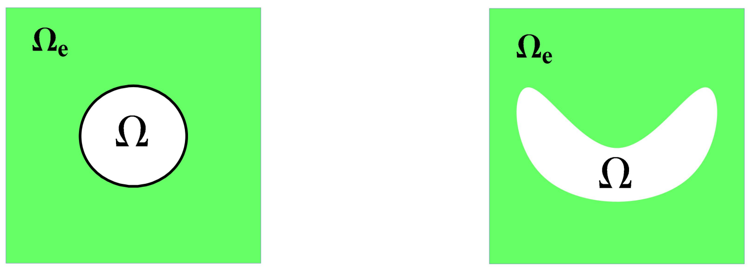

Figure 76. Radial and Non-radial Exterior Domains.

Finally, we plan to extend the quadrature method and the shooting method analysis for autonomous and nonautonomous problems involving the p-Laplacian operator.

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APPENDIX A

KELVIN TRANSFORMATIONS

A.1 Kelvin Transformation for Boundary Value Problems with Dirichlet Boundary Conditions

Consider boundary value problems of the form:

$$\begin{cases} -\Delta_p u = \lambda K(|x|)\frac{f(u)}{u^\alpha}; \quad x \in \Omega_e \\ u = 0; \quad |x| = r_0 \\ u \to 0; \quad |x| \to \infty \end{cases} \tag{A.1}$$

where $1 < p < N$, $0 \leq \alpha < 1$, $\Omega_e := \{x \in \mathbb{R}^N \mid |x| > r_0 > 0\}$, $f : [0, \infty) \to \mathbb{R}$ is a continuous function and $K : [r_0, \infty) \to (0, \infty)$ is a continuous function such that $K(r) \leq \frac{1}{r^{N+\sigma}}$ for $r \gg 1$ for some $\sigma \in (0, \frac{N-p}{p-1})$. Setting $r = |x|$ and $v(r) = u(x)$, we obtain

$$\Delta_p u(x) = \frac{1}{r^{N-1}} \left(r^{N-1}|v'(r)|^{p-2}v'(r)\right)'.$$

Then (A.1) reduces to the boundary value problem of the form:

$$\begin{cases} -\left(r^{N-1}|v'(r)|^{p-2}v'(r)\right)' = \lambda r^{N-1}K(r)\frac{f(v)}{v^\alpha}; \quad r \in [r_0, \infty) \\ v(r_0) = 0 \\ v(r) \to 0; \quad r \to \infty. \end{cases} \tag{A.2}$$

Setting again $t = \left(\frac{r}{r_0}\right)^{\frac{N-p}{1-p}}$ and $w(t) = v(r)$, we obtain

$$\left(r^{N-1}|v'(r)|^{p-2}v'(r)\right)' = \frac{1}{r_0^{p-N+1}}\left(\frac{N-p}{p-1}\right)^p t^{\frac{N-1}{N-p}}\left(|w'(t)|^{p-2}w'(t)\right)'$$

and

$$r^{N-1}K(r)\frac{f(v)}{v^\alpha} = r_0^{N-1}t^{\frac{(1-p)(N-1)}{N-p}}K\left(r_0 t^{\frac{1-p}{N-p}}\right)\frac{f(w)}{w^\alpha}.$$

Thus (A.2) reduces to the two point boundary value problem of the form:

$$\begin{cases} -\left(\varphi_p(w')\right)' = \lambda h(t)\frac{f(w)}{w^\alpha}; \quad t \in (0,1) \\ w(0) = 0 = w(1) \end{cases} \tag{A.3}$$

where $\varphi_p(s) := |s|^{p-2}s$ and $h(t) := \left(\frac{p-1}{N-p}\right)^p r_0^p t^{\frac{p(1-N)}{N-p}} K\left(r_0 t^{\frac{1-p}{N-p}}\right)$.

A.2 Kelvin Transformation for Boundary Value Problems with Nonlinear Boundary Conditions

Now we consider the following boundary value problems:

$$\begin{cases} -\Delta_p u = \lambda K(|x|)\frac{f(u)}{u^\alpha}; \quad x \in \Omega_e \\ \frac{\partial u}{\partial y} + \tilde{c}(u)u = 0; \quad |x| = r_0 \\ u \to 0; \quad |x| \to \infty \end{cases} \tag{A.4}$$

where p, α, Ω_e, f and K are as before, $\frac{\partial u}{\partial y}$ is the outward normal derivative of u on $\partial\Omega_e$ and $\tilde{c} : [0, \infty) \to (0, \infty)$ is a continuous function. Changing variables $r = |x|$ and $t = \left(\frac{r}{r_0}\right)^{\frac{N-p}{1-p}}$, we obtain

$$
\begin{aligned}
\frac{\partial u}{\partial y}(x) + \tilde{c}(u(x))u(x) &= -v'(r_0) + \tilde{c}(v(r_0))v(r_0) \\
&= \frac{N-p}{r_0(p-1)} w'(1) + \tilde{c}(w(1))w(1)
\end{aligned}
$$

for all x satisfying $|x| = r_0$. Then, by arguments similar to those in Section 10.1, (A.4) reduces to the two point boundary value problem of the form:

$$
\begin{cases}
-\left(\varphi_p(w')\right)' = \lambda h(t)\frac{f(w)}{w^\alpha}; \quad t \in (0,1) \\
w'(1) + c(w(1))w(1) = 0 \\
w(0) = 0
\end{cases}
\tag{A.5}
$$

where p, α, f and h are as before and $c(s) := \frac{p-1}{N-p} r_0 \tilde{c}(s)$.

A.3 Kelvin Transformation for Boundary Value Problems on an Annulus

Next we consider the boundary value problem of the form:

$$
\begin{cases}
-\Delta_p u = \lambda K(|x|) f(u); \quad x \in \Omega_A \\
u = 0; \quad x \in \partial\Omega_A
\end{cases}
\tag{A.6}
$$

where p, f and K are as before and $\Omega_A := \{x \in \mathbb{R}^N \mid r_0 < |x| < R_0\}$. By arguments similar to those in Section 10.1, setting $r = |x|$ and $v(r) = u(x)$, (A.6) reduces to the boundary value problem (A.2). Setting again $t = \big(r^{\frac{N-p}{1-p}} - R_0^{\frac{N-p}{1-p}}\big)/\big(r_0^{\frac{N-p}{1-p}} - R_0^{\frac{N-p}{1-p}}\big)$, we obtain

$$\big(r^{N-1}|v'(r)|^{p-2}v'(r)\big)' = C_1\big(C_2t + R_0^{\frac{N-p}{1-p}}\big)^{\frac{N-1}{N-p}}\left(|w'(t)|^{p-2}w'(t)\right)'$$

and

$$r^{N-1}K(r)\frac{f(v)}{v^\alpha} = \big(C_2t + R_0^{\frac{N-p}{1-p}}\big)^{\frac{(N-1)(1-p)}{N-p}}K\Big(\big(C_2t + R_0^{\frac{N-p}{1-p}}\big)^{\frac{1-p}{N-p}}\Big)\frac{f(w)}{w^\alpha}$$

where $C_1 := \big(\frac{N-p}{C_2(p-1)}\big)^p$ and $C_2 := (r_0^{\frac{N-p}{1-p}} - R_0^{\frac{N-p}{1-p}})$. Thus (A.2) reduces to the two point boundary value problem of the form:

$$\begin{cases} -\left(\varphi_p(w')\right)' = \lambda h(t)\frac{f(w)}{w^\alpha}; \quad t \in (0,1) \\ w(0) = 0 = w(1) \end{cases} \tag{A.7}$$

where p and f are as before and $h(t) := \frac{1}{C_1}\big(C_2t + R_0^{\frac{N-p}{1-p}}\big)^{\frac{-p(N-1)}{N-p}}K\Big(\big(C_2t + R_0^{\frac{N-p}{1-p}}\big)^{\frac{1-p}{N-p}}\Big)$.

APPENDIX B

PROOFS OF THEOREMS 8.1 - 8.4

B.1 Proof of Theorem 8.1

First we establish the following two Lemmas needed to prove the result.

Lemma B.1. *If $\tilde{f}$ satisfies (S) and $\rho < \theta$, then there is no $\lambda > 0$ for which* (8.1) *has a positive solution u satisfying $\|u\|_\infty = \rho$.*

Proof. Assume to the contrary that u is a positive solution to (8.1) for some $\lambda > 0$ such that $\|u\|_\infty = \rho < \theta$. Note that $u'(1) < 0$ since we are only interested in the case where $u(1) > 0$. Hence there exists $t_m \in (0, 1)$ such that $u'(t_m) = 0$ and $u(t_m) = \rho$. Multiplying the differential equation by u' and integrating, we obtain

$$(u'(t))^2 = 2\lambda[F(\rho) - F(u(t))] \tag{B.1}$$

for $t \in (0, t_m)$. But this implies that $(u'(0))^2 = 2\lambda F(\rho) < 0$, a contradiction. Hence no such solution can exist. □

Lemma B.2. *Any positive solution u of* (8.1) *has a unique interior maximum at some $t_m \in (0, 1)$, is strictly increasing on $(0, t_m)$, is strictly decreasing on $(t_m, 1)$ and is symmetric about t_m.*

Proof. Let $t_m \in (0, 1)$ be such that $\|u\|_\infty = u(t_m) = \rho$. Suppose there exists another local maximum. Then there must be a local minimum at some $t_m^* \in (0, 1)$, at which $u''(t_m^*) \geq 0$. This implies that $u(t_m^*) \leq \beta$. Let $E(t) := \lambda F(u(t)) + \frac{1}{2}(u'(t))^2$ for $t \in (0, 1)$. A simple calculation will show that $E'(t) = 0$, and hence $E(t)$ is constant on $[0,1]$. But $E(t_m) = \lambda F(\rho) \geq 0$ while $E(t_m^*) = \lambda F(u(t_m^*)) < 0$, and hence we have a contradiction. Therefore t_m is the unique critical point. From (B.1), we easily see that

$$u'(t) = \begin{cases} \sqrt{2\lambda\left[F(\rho) - F(u(t))\right]} > 0; \quad t \in (0, t_m) \\ -\sqrt{2\lambda\left[F(\rho) - F(u(t))\right]} < 0; \quad t \in (t_m, 1). \end{cases} \tag{B.2}$$

Further, note that both $w_1(t) = u(t_m + t)$ and $w_2(t) = u(t_m - t)$ satisfy

$$\begin{cases} -w''(t) = \lambda f(w(t)); \quad t \in (0,1) \\ w'(0) = 0 \\ w(0) = \rho. \end{cases}$$

Hence, by Picard's Theorem, we have $w_1(t) = w_2(t)$ which implies that u is symmetric about t_m. $\square$

We now begin the proof of Theorem 8.1 by showing first that if $u \in C^2(0,1) \cap C^1[0,1]$ is a positive solution to (8.1) with $\|u\|_\infty = u(t_m) = \rho$ and $u(1) = q$, then λ, ρ and q must satisfy (8.2) and (8.3). We note here that the improper integral in (8.2) is convergent since $f(\rho) > 0$. Integrating (B.2), we obtain

$$t\sqrt{2\lambda} = \int_0^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}} \tag{B.3}$$

for $t \in (0, t_m)$ and

$$(1-t)\sqrt{2\lambda} = \int_q^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}} \tag{B.4}$$

for $t \in (t_m, 1)$. Setting $t = t_m$, we obtain

$$t_m\sqrt{2\lambda} = \int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} \tag{B.5}$$

and

$$(1-t_m)\sqrt{2\lambda} = \int_q^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}. \tag{B.6}$$

Adding (B.5) and (B.6), we obtain

$$\sqrt{2\lambda} = \int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}$$

and hence from (B.5) we obtain

$$t_m = \int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} \Bigg/ \left(\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} \right). \tag{B.7}$$

Further, using the boundary conditions and (B.2), we obtain

$$c(q)q = -u'(1) = \sqrt{2\lambda\,[F(\rho) - F(q)]}.$$

Hence (8.2) and (8.3) are satisfied.

Next, if λ, ρ and q satisfy (8.2) and (8.3), let t_m be defined by (B.7) and define $u : [0,1] \to [0,\rho]$ via (B.3) and (B.4) for $t \in (0,t_m) \cup (t_m,1)$ with $u(0) = 0$, $u(t_m) = \rho$ and $u(1) = q$. Note that u is well-defined on $(0,t_m)$ since both $\int_0^u \frac{ds}{\sqrt{F(\rho)-F(s)}}$ and $t\sqrt{2\lambda}$ increase from 0 to $\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}$ as u increases from 0 to ρ and t increases from 0 to t_m, respectively. Also, u is well-defined on $(t_m,1)$ since both $\int_q^u \frac{ds}{\sqrt{F(\rho)-F(s)}}$ and $(1-t)\sqrt{2\lambda}$ decrease from $\int_q^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}$ to 0 as u decreases from ρ to q and t increases from t_m to 1, respectively. Now we define $H : (0,t_m) \times (0,\rho) \to \mathbb{R}$ by

$$H(\ell, v) := \int_0^v \frac{ds}{\sqrt{F(\rho) - F(s)}} - \ell\sqrt{2\lambda}.$$

Clearly H is C^1, $H(t,u(t)) = 0$ for $t \in (0,t_m)$ and

$$H_v \mid_{(t,u(t))} = \frac{1}{\sqrt{F(\rho) - F(u(t))}} \neq 0.$$

Hence, by the Implicit Function Theorem, u is C^1 on $(0,t_m)$. Similarly, u is C^1 on $(t_m,1)$. From (B.3) and (B.4), we have

$$u'(t) = \begin{cases} \sqrt{2\lambda\,[F(\rho) - F(u(t))]}; & t \in (0,t_m) \\ -\sqrt{2\lambda\,[F(\rho) - F(u(t))]}; & t \in (t_m,1). \end{cases} \tag{B.8}$$

Differentiating (B.8) again, we have

$$-u''(t) = \lambda\tilde{f}(u(t))$$

for $t \in (0, t_m) \cup (t_m, 1)$. But $u(t_m) = \rho$ and $\tilde{f}$ is continuous, and hence $u \in C^2(0,1) \cap C^1[0,1]$. Further, (B.8) implies that $-u'(1) = \sqrt{2\lambda\,[F(\rho) - F(q)]}$, and hence by (8.3) we have $u'(1) + c(u(1))u(1) = 0$. Thus u is a solution of (8.1).

B.2 Proof of Theorem 8.2

Define

$$J(\rho, q) := \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{c(q)q}{\sqrt{F(\rho) - F(q)}}.$$

Note that if (P) is satisfied, then for every $\rho > 0$ there exists $q > 0$ such that $J(\rho, q) = 0$ since

$$J(\rho, 0) = 2\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} > 0 \qquad \text{and} \qquad \lim_{q \to \rho} J(\rho, q) = -\infty.$$

Hence ρ and q satisfy (8.2). Similarly, if (S) is satisfied then the claim holds for all $\rho > \theta$. For $\rho = \theta$, we again have $\lim_{q \to \theta} J(\theta, q) = -\infty$ and observe that

$$\begin{aligned}
\lim_{q \to 0} J(\theta, q) &= 2\int_0^\theta \frac{ds}{\sqrt{-F(s)}} - \lim_{q \to 0^+} \frac{c(q)q}{\sqrt{-F(q)}} \\
&= 2\int_0^\theta \frac{ds}{\sqrt{-F(s)}} - \lim_{q \to 0^+} \frac{c(q)q}{\sqrt{-q\tilde{f}(z)}} \\
&= 2\int_0^\theta \frac{ds}{\sqrt{-F(s)}} \\
&> 0
\end{aligned}$$

for some $z \in (0, q)$. Hence there exists $q > 0$ satisfying (8.2) for all $\rho \geq \theta$.

B.3 Proof of Theorem 8.3

Let $\rho > 0$ be fixed. The existence of $q > 0$ satisfying (8.2) follows from Theorem 8.2. As for the uniqueness of q, a straightforward calculation will show

$$J_q(\rho, q) = -\frac{2[1 + (c(q)q)'](F(\rho) - F(q)) + \tilde{f}(q)c(q)q}{2\left(F(\rho) - F(q)\right)^{\frac{3}{2}}}. \tag{B.9}$$

Since $\tilde{f}(q) > 0$ and $1 + (c(s)s)' = (s + c(s)s)' > 0$ by assumption, $J_q(\rho, q) < 0$ for all $q > 0$. Hence there cannot be two values of q such that $J(\rho, q) = 0$.

B.4 Proof of Theorem 8.4

Let $\rho \geq \theta$ be fixed. The existence of $q > 0$ satisfying (8.2) again follows from Theorem 8.2. If $(S3)$ holds, then we have

$$\left(\ln\left(\frac{s + c(s)s}{\sqrt{F(\rho) - F(s)}}\right)\right)' \geq 0$$

for $s \in (0, \beta)$. A straightforward calculation will show that this implies

$$\frac{1 + (c(s)s)'}{s + c(s)s} \geq \frac{-\tilde{f}(s)}{2(F(\rho) - F(s))}. \tag{B.10}$$

Further, we observe from (B.10) that

$$\frac{1 + (c(s)s)'}{c(s)s} \geq \frac{1 + (c(s)s)'}{s + c(s)s} \geq \frac{-f(s)}{2(-F(s))} \geq \frac{-f(s)}{2(F(\rho) - F(s))} \tag{B.11}$$

for $s \in (0, \beta)$. Hence, using (B.11), we conclude that

$$2[1 + (c(s)s)'](F(\rho) - F(s)) + \tilde{f}(s)c(s)s > 0 \tag{B.12}$$

for $s \in (0, \beta)$. It is easy to show that the inequality (B.12) also holds for $s \in [\beta, \rho)$. Therefore we have $J_q(\rho, q) < 0$ for all $q > 0$, and hence the result follows from (B.9).

If $(S4)$ holds, then let

$$g(s) := 2(F(\rho) - F(s)) + \tilde{f}(s)c(s)s$$

and observe that g is continuous on $[0, \rho]$, $g(0) = 2F(\rho) \geq 0$ and $g'(s) > 0$ for $s \in (0, \beta)$ by $(S4)$. This implies that $g(s) > 0$ for $s \in (0, \beta]$. Note that $(c(s)s)' \geq 0$ implies that $1 + (c(s)s)' \geq 1$. Therefore $J_q(\rho, q) < 0$ for $q \in (0, \beta]$. For $q \in (\beta, \rho)$, we also have $J_q(\rho, q) < 0$ since $f(s) > 0$ for $s \in (\beta, \rho)$. Therefore $J_q(\rho, q) < 0$ for all $q > 0$, and hence the result follows from (B.9).