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We study positive radial solutions for classes of steady state reaction diffusion problems on the exterior of a ball with both Dirichlet and nonlinear boundary conditions. We consider p-Laplacian problems ($p > 1$) with reaction terms which are superlinear at infinity and semipositone.

In the case $p = 2$, using variational methods, we establish the existence of a solution, and via detailed analysis of the Green's function, we prove the positivity of the solution. In the case $p \neq 2$, we again use variational methods to establish the existence of a solution, but the positivity of the solution is achieved via sophisticated a priori estimates. In the case $p \neq 2$, the Green's function analysis is no longer available. Our results significantly enhance the literature on superlinear semipositone problems.

Finally, we provide algorithms for the numerical generation of exact bifurcation curves for one-dimensional problems. In the autonomous case, we extend and analyze a quadrature method, and using nonlinear solvers in Mathematica, generate bifurcation curves. In the nonautonomous case, we employ shooting methods in Mathematica to generate bifurcation curves.

ANALYSIS OF CLASSES OF SUPERLINEAR SEMIPOSITONE PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

by

Quinn A. Morris

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Approved by

Committee Chair

To all my teachers.

APPROVAL PAGE

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CHAPTER I

INTRODUCTION

Consider the nonlinear reaction-diffusion problem

$$\begin{cases} u_t = d\Delta_p u + f(u); & x \in \Omega, t > 0, \\ u(x,0) = \psi_0(x); & x \in \Omega, \\ Bu = 0; & x \in \partial\Omega, t > 0, \end{cases} \tag{1.1}$$

where $d > 0$ is the diffusion coefficient, $\Delta_p z = \text{div}(|\nabla z|^{p-2}\nabla z)$ with $p > 1$, $f : [0,\infty) \to \mathbb{R}$ is a continuous function, $\Omega \subset \mathbb{R}^N$, and B is a generic, possibly nonlinear, operator to be specified later. Such problems arise in the study of nonlinear heat generation, combustion theory, chemical reactor theory and population dynamics. For such applications, only non-negative solutions ($u \geq 0$ in $\overline{\Omega}$) are relevant. The study of steady states (if they exist) for (1.1) is of great importance in understanding the dynamics of the solutions of (1.1), and researchers (since 1967, see [KC67]) have focused on the study of nonlinear eigenvalue problems of the form:

$$\begin{cases} -\Delta_p u = \lambda f(u); & x \in \Omega, \\ Bu = 0; & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda = \frac{1}{d}$ is a positive parameter.

In the case that f is positive and monotone, (1.2) is referred to in the literature as a “positone" problem (see Figure 1). Classical examples arise in the theory of nonlinear heat generation (see [KC67] where the authors consider the reaction term

$f(u) = e^u$) and combustion theory (see [BIS81] where the authors consider the reaction $f(u) = e^{\frac{\alpha u}{\alpha + u}}$; $\alpha > 0$). For a rich history of results related to positive solutions of such positone problems in the case $p = 2$, we refer the reader to [KC67], [Ama76], [Rab71], [CL70], [CR73], [CR75], [Par61], [Sat75], [Par74], [Tam79], [Ari69], [Ama72], [WL79], [KJD$^+$79], [Lae71], and [AC77]. In particular, the celebrated SIAM Review paper of P. L. Lions in 1982 (see [Lio82]) provides an excellent overview of results for positone problems, as well as a list of open problems at the time.

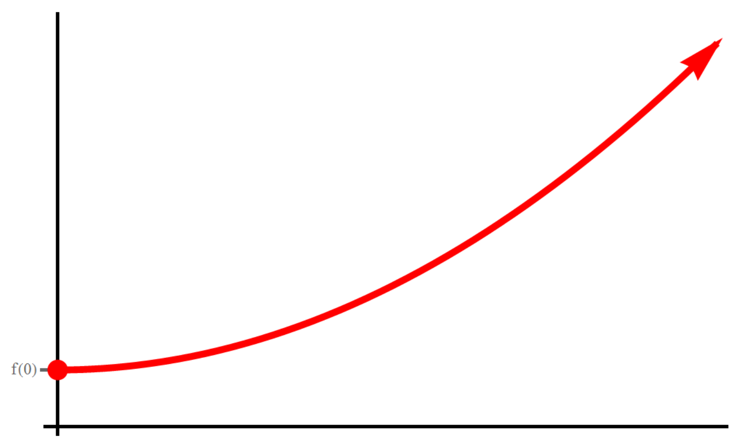


Figure 1. An Example of a Positone f

Of particular interest in this dissertation is one such problem, the case where f satisfies,

(F1) $f(0) < 0$, f is monotone and eventually positive.

which is referred to in the literature as a semipositone problem (see Figure 2). The study of positive solutions to semipostone problems is considerably more challenging, since the range of a solution must include regions where f is negative as well as where f is positive. Such problems were discussed in [Lio82], and it was noted that they posed signficant challenges, a fact later confirmed by H. Berestycki, L.A. Caffarelli, and L. Nirenberg in 1996 (see [BCN96]). The study of semipositone problems was first formally introduced by Castro and Shivaji in 1988 (see [CS88]) in the case of Dirichlet boundary conditions, where several challenging differences were noted in their study when compared to the study of positone problems. See also [BS83], where Brown and Shivaji, in 1982, first encountered the difficulty with semipositone problems in a study of perturbed bifurcation theory. Castro and Shivaji's initial work in [CS88] has lead to a plethora of work in recent years, particularly in the case $Bu = u$ (Dirichlet boundary conditions) and Ω is a bounded domain in $\mathbb{R}^N$. See [ANZ92], [AAB94], [CG09], [CG13], [CHS03], [HSC01], [JS04], [BCS89], [CS89a], [CS89b], [BS91], [CGS93], [CGS95], [CHS95], [AHS96], [CS98], [HS99], [OSS02], [HS03], [HS04], [DOS06], [CCSU07], [SY07], [SY11], [CSS12], and [SSS13] for results for problems with Dirichlet boundary conditions on bounded domains. An important application of semipositone problems arises in the study of population dynamics with constant yield harvesting, as was illustrated by Oruganti, Shi, and Shivaji in 2002 (see [OSS02]).

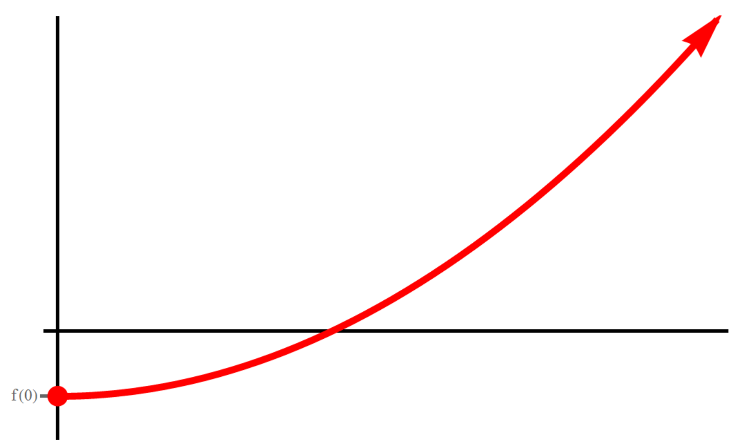


Figure 2. An Example of a Semipositone f

The focus of this dissertation is to enrich the literature on a class of semipositone problems, namely, when the reaction term f is p-superlinear at infinity ($\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \infty$). In the case $p = 2$, Castro and Shivaji in 1989 made significant breakthroughs in the study of such problems when Ω is a ball in $\mathbb{R}^N$, $N > 1$, by first proving that non-negative solutions are in fact positive and, hence, radially symmetric (see [CS89b]), and then establishing an optimal existence result for $\lambda \approx 0$ (see [CS89a]). Also, in 1989, Brown, Castro, and Shivaji established a non-existence result for $\lambda >> 1$ (see [BCS89]). Further, in 1993, Ali, Castro, and Shivaji established the uniqueness of this positive solution for $\lambda \approx 0$ under additional assumptions on f (see [ACS93]). The existence result was extended to a general bounded domain by Allegretto, Nistri, and Zecca in 1992 (see [ANZ92]); Ambrosetti, Arcoya, and Buffoni in 1994 (see [AAB94]); and Sumallee in 1988 (see [Uns88]). Further, non-existence for $\lambda >> 1$ was extended to a general bounded domain by Allegretto, Nistri, and Zecca in 1992 (see [ANZ92]).

See [ACS93], [ANZ92], [AAB94], [AHS96], [AZ94], [BCS89], [BS91], [CCSU07], [CDS15], [CdFL16], [CS88], [CS89a], [CS89b], [CG09], [CG13], [DOS06], [HSC01], [Hai14], [HSS98], [JS04], [SW87], [Uns88] for results on p-superlinear, semipositone problems in both $p = 2$ and $p > 1$ cases where Ω is a general bounded domain. To date, the extension of the uniqueness result to general bounded domains remains open.

A more challenging problem is to consider such superlinear, semipositone problems on unbounded domains. In this dissertation, we will consider the problems

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u); & x \in \Omega_e, \\ u = 0; & |x| = r_0, \\ u \to 0; & |x| \to \infty, \end{cases} \tag{1.3}$$

and

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u); & x \in \Omega_e, \\ \frac{\partial u}{\partial \eta} + \tilde{c}(u) u = 0; & |x| = r_0, \\ u \to 0; & |x| \to \infty, \end{cases} \tag{1.4}$$

where $\lambda > 0$ is a parameter, $\Delta_p z = \text{div}(|\nabla z|^{p-2} \nabla z)$ with $p > 1$,

$$\Omega_e = \left\{ x \in \mathbb{R}^N \mid |x| > r_0, r_0 > 0, N > p \right\},$$

$K \in C\left([r_0, \infty), (0, \infty)\right)$ satisfies $K(r) \leq \frac{1}{r^{N+\mu}}$; $\mu > 0$ for $r >> 1$ and K is eventually decreasing, $\frac{\partial}{\partial \eta}$ is the outward normal derivative, and $\tilde{c} \in C\left([0, \infty), (0, \infty)\right)$.

We assume the reaction term f satisfies the additional condition

(F2) there exist $A, B \in (0, \infty)$ and $q \in (p-1, \infty)$ such that for $s > 0$ sufficiently large, $As^q \leq f(s) \leq Bs^q$.

In addition to (F1) and (F2), we make the following technical assumptions:

(AR1) there exists $\theta > p$ such that for s sufficiently large, $sf(s) > \theta F(s)$, where

$$F(s) = \int_0^s f(z)\, dz,$$

and

(AR2) for θ satisfying (AR1),

$$c(s)s^p < \theta \int_0^s c(z)\phi_p(z)\, dz$$

for s sufficiently large, where $\phi_p(z) = |z|^{p-2}z$.

Remark. If we take $f(s) = s^q - 1$ with $q > p-1$ and $c(s) = s^\delta + 1$ with $0 < \delta < q+1-p$, then for any $\theta \in [\delta + p, q+1]$, it is easy to show that (AR1) and (AR2) are satisfied (in fact) for all $s > 0$.

In an effort to find positive, radial solutions of (1.3) and (1.4), we apply the change of variables $\zeta = |x|$ and $t = \left(\frac{\zeta}{r_0}\right)^{\frac{p-N}{p-1}}$ to transform (1.3) and (1.4) to the boundary value problems

$$\begin{cases} -\left(\phi_p(u')\right)' = \lambda h(t) f(u);\ t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{1.5}$$

and

$$\begin{cases} -\left(\phi_p(u')\right)' = \lambda h(t) f(u);\ t \in (0,1), \\ u(0) = 0, \\ \phi_p\left(u'(1)\right) + c(u(1))\phi_p\left(u(1)\right) = 0, \end{cases} \tag{1.6}$$

respectively, where $\phi_p(s) = |s|^{p-2}s$,

$$h(t) = \left(\frac{p-1}{N-p} r_0\right)^p t^{-\frac{p(N-1)}{N-p}} K\left(r_0 t^{\frac{1-p}{N-p}}\right),$$

and $c(s) = \left(\frac{r_0(p-1)}{N-p}\tilde{c}(s)\right)^{p-1}$. Due to our earlier assumptions on K, the weight function $h \in C(0,1] \cap L^1(0,1)$ and $\inf_{x\in[0,1]} h(x) > 0$. See Appendix A.1 for full details of the transformation.

Remark. We may consider the related problems on the annulus, namely

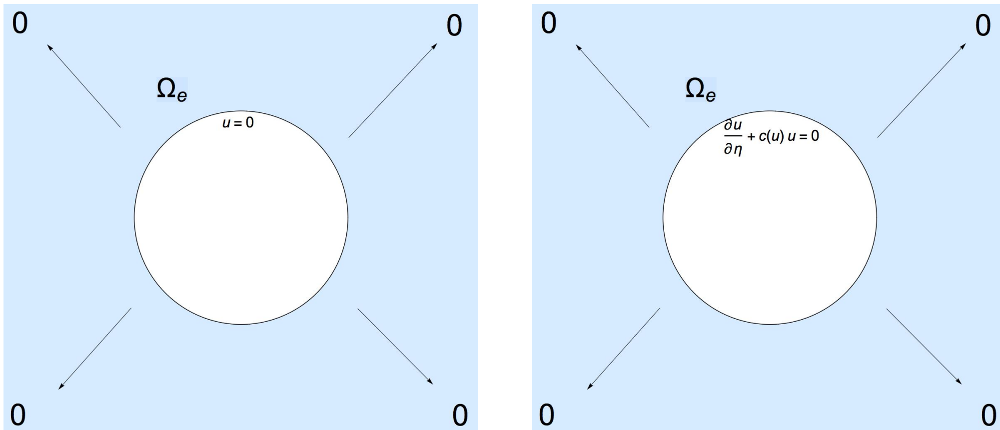
$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u); & x \in \Omega_a, \\ u(x) = 0; & |x| = R_1, \\ u(x) = 0; & |x| = R_2, \end{cases} \tag{1.7}$$

and

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u); & x \in \Omega_a, \\ u(x) = 0; & |x| = R_1, \\ \frac{\partial u}{\partial \eta} + \tilde{c}(u)u = 0; & |x| = R_2, \end{cases} \tag{1.8}$$

where $\Omega_a = \left\{x \in \mathbb{R}^N | R_1 < |x| < R_2; R_2 > R_1 > 0, N > p\right\}$ and $K \in C([R_1, R_2], (0, \infty))$. Applying a change of variables from [GLS10] (see Appendix A.2), we may transform the problems into (1.5) and (1.6), respectively, where now $h \in C[0,1]$.

In the case $p = 2$ with Dirichlet boundary conditions, Abebe, Chhetri, Sankar, and Shivaji proved existence of a positive, radial solution on the exterior of a ball in $\mathbb{R}^N$ for $\lambda \approx 0$ in [ACSS14] using degree theory. The more general $p > 1$ case and the case of nonlinear boundary conditions both remained untreated until now.



(a) Dirichlet Boundary Condition as in (1.3)

(b) Nonlinear Boundary Condition as in (1.4)

Figure 3. Exterior Domains with Boundary Conditions

We consider the semilinear ($p = 2$) cases of (1.3) and (1.4), where $\Delta_2 = \Delta$ is the usual Laplace operator. In this case, (1.5) and (1.6) become

$$\begin{cases} -u'' = \lambda h(t) f(u); \ \ t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{1.9}$$

and

$$\begin{cases} -u'' = \lambda h(t) f(u); \ \ t \in (0,1), \\ u(0) = 0, \\ u'(1) + c(u(1))u(1) = 0, \end{cases} \tag{1.10}$$

where $h(t) = \frac{r_0^2}{(2-N)^2} t^{\frac{-2(N-1)}{N-2}} K\left(r_0 t^{\frac{1}{2-N}}\right)$ and $c(s) = \frac{r_0}{N-2}\tilde{c}(s)$.

We establish the following result for (1.9) and (1.10), respectively.

Theorem 1.1. *Let* $p = 2$ *and* f *satisfy (F1), (F2), and (AR1). There exists* $\overline{\lambda} > 0$ *so that* (1.9) *has a positive solution,* $u_\lambda \in C^2(0,1) \cap C^1[0,1]$*, for all* $\lambda \in (0, \overline{\lambda})$*.*

Theorem 1.2. *Let* $p = 2$ *and* f *satisfy (F1), (F2), and (AR1), and let* c *satisfy (AR2) with* $p = 2$*. There exists* $\tilde{\lambda} > 0$ *so that* (1.10) *has a positive solution,* $u_\lambda \in C^2(0,1) \cap C^1[0,1]$*, for all* $\lambda \in (0, \tilde{\lambda})$*.*

We prove Theorem 1.1 and Theorem 1.2 by employing variational methods, namely the Mountain Pass Theorem, to establish the existence of a solution. In particular, conditions (AR1) and (AR2) are Ambrosetti-Rabinowitz-type conditions which are used to establish that the functionals associated to (1.9) and (1.10) satisfy the Palais-Smale compactness condition. Since $f(0) < 0$, it is challenging to establish that the mountain pass solution is in fact positive. Crucial a priori estimates of the solutions when $\lambda \approx 0$ and estimates of the Green's function near the boundary will be used to overcome this difficulty. We note that positive solutions to (1.9) and (1.10) give rise to positive radial solutions of (1.3) and (1.4), respectively.

We also treat the quasilinear ($p \neq 2$) cases of (1.3) and (1.4), where Δ_p is the p-Laplace operator. We establish the following results for (1.5) and (1.6), respectively.

Theorem 1.3. *Let* f *satisfy (F1), (F2), and (AR1). There exists* $\hat{\lambda} > 0$ *so that* (1.5) *has a positive solution,* $u_\lambda \in C^1[0,1]$*, for all* $\lambda \in (0, \hat{\lambda})$*.*

Theorem 1.4. *Let* f *satisfy (F1), (F2), and (AR1), and let* c *satisfy (AR2). There exists* $\check{\lambda} > 0$ *so that* (1.6) *has a positive solution,* $u_\lambda \in C^1[0,1]$*, for all* $\lambda \in (0, \check{\lambda})$*.*

Remark. Note that solutions to (1.9), (1.10), (1.5), and (1.6), by reversing the earlier change of variables, guarantee positive, radial solutions to (1.3), (1.4), (1.7), and (1.8), respectively, in the both the $p = 2$ and $p \neq 2$ cases.

While the variational framework developed in the semilinear case can be adapted with only minor modifications, we face a more difficult challenge in proving positivity of the solutions for the quasilinear case, since the Green's function is no longer available. Though some recent work had been done in this direction on bounded domains without singular weights (see [CdFL16] and [CDS15]), we address (1.3) and (1.4) for the first time on exterior domains, which leads to the presence of singular weight functions. Again obtaining crucial a priori estimates of the solutions, we show by contradiction that, for $\lambda \approx 0$, the mountain pass solution must be positive.

Finally, we establish numerical and computational schemes for generating bifurcation curves of positive solutions to problems (1.9) and (1.10). Of particular interest in the study of (1.9) and (1.10) is the shape of bifurcation curves of positive solutions. Laetsch studied the autonomous case ($h(t) = 1$ for all $t \in (0, 1)$) of (1.9) in [Lae71] using a quadrature method (or time map analysis) and here, we establish such a method for the autonomous case of (1.10). We also employ shooting methods and nonlinear solvers to plot bifurcation curves in the nonautonomous cases of (1.9) and (1.10).

In order to prove Theorems 1.1–1.4, we employ variational methods, and in particular the Mountain Pass Theorem. In Chapter 2, we introduce the necessary background material. In Chapter 3, we deal with the case $p = 2$, proving Theorems 1.1 and 1.2. In Chapter 4, we deal with the more general $p \neq 2$ case, proving Theorems 1.3 and 1.4. In Chapters 5 and 6, we provide the numerical and computational methods for generating exact bifurcation curves in the autonomous and nonautonomous cases, respectively, as well as their application to some superlinear problems.

CHAPTER II

PRELIMINARIES

In order to establish the existence of solutions to (1.5), (1.6), (1.9), and (1.10), we will employ variational methods. The fundamental idea behind variational methods is to reformulate the search for solutions of an equation as a search for critical points of an appropriate functional. To this end, we employ several Banach spaces, $C[0,1]$, $C^1[0,1]$, $L^s(0,1)$, and $W_0^{1,p}(0,1)$, whose definitions we now recall.

Definition 2.1. Given $1 \leq s < \infty$ and $1 \leq p < \infty$, we define the following function spaces as,

$$\begin{aligned}
C[0,1] &:= \{u : [0,1] \to \mathbb{R} \mid u \text{ is continuous on } [0,1]\} \\
C^1[0,1] &:= \{u : [0,1] \to \mathbb{R} \mid u \in C[0,1] \text{ and } u' \in C[0,1]\} \\
L^s(0,1) &:= \left\{u : [0,1] \to \mathbb{R} \mid u \text{ is measurable and } \int_0^1 |u(r)|^s \, dr < \infty\right\} \\
W^{1,p}(0,1) &:= \{u : [0,1] \to \mathbb{R} \mid u \in L^p(0,1) \text{ and } u' \in L^p(0,1)\} \\
W_0^{1,p}(0,1) &:= \big\{u : [0,1] \to \mathbb{R} \mid u \in W^{1,p}(0,1) \text{ and } u(0) = 0 = u(1)\big\}
\end{aligned}$$

To give the idea of the variational method, consider the case $p = 2$ with Dirichlet boundary conditions. We define a weak solution to (1.9) as follows.

Definition 2.2. A function $u_0 \in W_0^{1,2}(0,1)$ is said to be a *weak solution* to (1.9) if

$$\int_0^1 u_0' v' \, dx - \lambda \int_0^1 f(u_0) v \, dx = 0 \quad \forall v \in W_0^{1,2}(0,1).$$

To find a weak solution directly, we seek a functional $J : W_0^{1,2}(0,1) \to \mathbb{R}$ such that $J'(u)$ at any $u \in W_0^{1,2}(0,1)$ satisfies

$$< J'(u), v > = \int_0^1 u'v' \, dx - \lambda \int_0^1 f(u)v \, dx \quad \forall v \in W_0^{1,2}(0,1).$$

Critical points of such a functional are clearly weak solutions of (1.9).

Now, we may turn our focus to finding critical points of a functional J. In order to do this, we recall the celebrated Mountain Pass Theorem which is stated below.

Theorem 2.3 (Mountain Pass Theorem (see [AR73])). *Let X be a Banach space, and let $J \in C^1(X;\mathbb{R})$ satisfy:*

(PS) any sequence $\{u_n\} \subset X$ such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence,

(MP1) $J(0) = 0$,

(MP2) there exist $\alpha, R > 0$ such that $J(u) \geq \alpha \ \forall \|u\|_X = R$, and

(MP3) there exists $v \in X$ such that $\|v\|_X > R$ and $J(v) < 0$.

Further, let

$$\Gamma := \{\gamma \in C([0,1];X) \mid \gamma(0) = 0, \gamma(1) = v\},$$

and

$$\hat{c} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).$$

Then $\hat{c}$ is a critical value of the functional J.

The condition (PS) is the well-known Palais-Smale condition developed by R. Palais and S. Smale (see [Pal63], [Pal66], [PS64], and [Sma64]) which is sufficient to prove the existence of $\hat{c}$. The condition ensures an appropriate sense of compactness in the functional J by ensuring that the set $\{u \in X | J(u) = c \text{ and } J'(u) = 0\}$ is compact for each $c \in \mathbb{R}$. The other three conditions (MP1)–(MP3) ensure that the functional has the correct geometry. See Figure 4 for a visualization of the Mountain Pass Theorem.

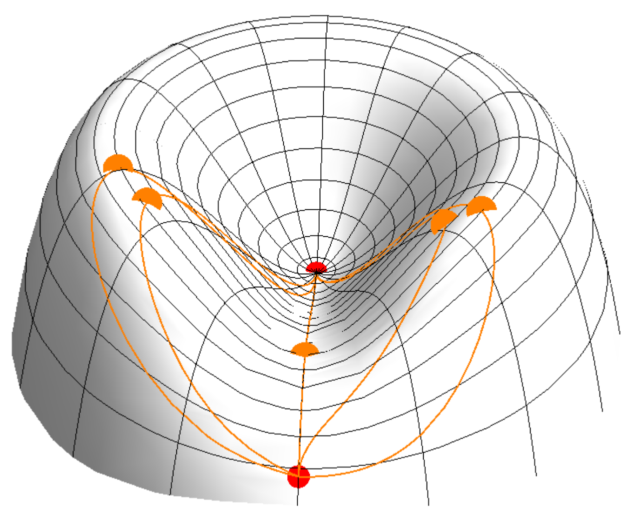


Figure 4. A Visualization of the Mountain Pass Theorem. The red points correspond to 0 and v, and each orange curve represents an element of Γ. The orange points are the maximum value of each curve, and by taking the infimum of all such orange points, we find a critical value.

It is well known (see [Ada75]) that each of the spaces $C[0,1]$, $C^1[0,1]$, $L^s(0,1)$, $W^{1,p}(0,1)$, and $W_0^{1,p}(0,1)$ are Banach spaces when paired with the norms

$$\begin{aligned}
\|u\|_\infty &:= \max_{[0,1]} |u(t)|, \\
\|u\|_{C^1} &:= \|u\|_\infty + \|u'\|_\infty, \\
\|u\|_s &:= \left(\int_0^1 |u(r)|^s \, dr \right)^{\frac{1}{s}}, \\
\|u\|_{1,p} &:= \left(\|u\|_p^p + \|u'\|_p^p \right)^{\frac{1}{p}}, \text{ and} \\
\|u\|_{1,p,0} &:= \|u'\|_p,
\end{aligned}$$

respectively.

In the special cases $s = 2$ or $p = 2$, we recall that the spaces $L^2(0,1)$, $W^{1,2}(0,1)$, and $W_0^{1,2}(0,1)$ are all Hilbert spaces. We also recall that $W^{1,p}(0,1)$ is compactly embedded in $C[0,1]$, which implies the existence of a constant $k > 0$ such that $\|u\|_\infty \leq k\|u\|_{1,p}$ for every $u \in W^{1,p}(0,1)$ (see [Ada75]).

We will also be interested in a particular subspace of $W^{1,p}(0,1)$, namely the subset

$$\widetilde{W}^{1,p}(0,1) = \left\{ u \in W^{1,p}(0,1) \mid u(0) = 0 \right\}.$$

The subspace is well defined due to the compact embedding of $W^{1,p}(0,1)$ into $C[0,1]$, and, further, we may show that the norms $\|\cdot\|_{1,p,0}$ and $\|\cdot\|_{1,p}$ are equivalent on $\widetilde{W}^{1,p}(0,1)$.

Proposition 2.4. *Let $\|u\|_{1,p}$ and $\|u\|_{1,p,0}$ be defined on $\widetilde{W}^{1,p}(0,1)$. Then $\|\cdot\|_{1,p,0}$ is equivalent to $\|\cdot\|_{1,p}$ on $\widetilde{W}^{1,p}(0,1)$.*

Proof. Let $u \in \widetilde{W}^{1,p}(0,1)$. Then clearly, $\|u\|_{1,p,0} \leq \|u\|_{1,p}$. Further, applying Jensen's inequality, we have

$$\begin{aligned}\int_0^1 |u(x)|^p \, dx &= \int_0^1 \left| \int_0^x u'(s) \, ds \right|^p \, dx \\ &\leq \int_0^1 \left(\int_0^x |u'(s)| \, ds \right)^p \, dx \\ &\leq \int_0^1 \left(\int_0^1 |u'(s)| \, ds \right)^p \, dx \\ &\leq \int_0^1 \int_0^1 |u'(s)|^p \, ds \, dx \\ &= \int_0^1 |u'(s)|^p \, ds,\end{aligned}$$

which implies that

$$\|u\|_{1,p} = \left(\int_0^1 |u|^p \, dx + \int_0^1 |u'|^p \, dx \right)^{\frac{1}{p}} \leq \left(2 \int_0^1 |u'|^p \, dx \right)^{\frac{1}{p}} = 2^{\frac{1}{p}} \|u\|_{1,p,0}.$$

Hence, $\| \cdot \|_{1,p,0}$ is equivalent to $\| \cdot \|_{1,p}$ on $\widetilde{W}^{1,p}(0,1)$. □

We also recall the concept of the (S^+) condition (see [Bro70]). The proof of the following proposition can be found in [GP04].

Proposition 2.5 ((S^+) Property)**.** *Let* $\Psi : W^{1,p}(0,1) \to [0,\infty)$ *be defined by* $\Psi(u) = \frac{1}{p}\int_0^1 |u'|^p \, dx$. *Then* Ψ' *exists,*

$$\langle \Psi'(u), v \rangle = \int_0^1 |u'|^{p-2} u'v' \, dx \quad \forall v \in W^{1,p}(0,1),$$

and if u_n *converges weakly to* u *and* $\limsup_{n\to\infty} \langle \Psi'(u_n), u_n - u \rangle \leq 0$, *then* $u_n \to u$ *strongly in* $W^{1,p}(0,1)$.

While we will use the Mountain Pass Theorem to prove the existence of weak solutions to (1.5) and (1.6), we show that these solutions have higher regularity. In particular, by solution to (1.5) or (1.6), we mean $u \in C^1[0,1]$ and $\phi_p(u') \in W^{1,1}(0,1)$ satisfying equation (1.5) or (1.6), respectively. In the case $p = 2$ (i.e., (1.9) and (1.10)), we further mean that $u \in C^2(0,1) \cap C^1[0,1]$.

Finally, we state the following proposition, which provides alternative forms of the growth condition (F2) and Ambrosetti-Rabinowitz type conditions (AR1) and (AR2).

Proposition 2.6. *Given the extension of $f(s) := f(0)$; $s < 0$, (F2) implies that there exists constants $\tilde{A}, \tilde{B} > 0$ so that*

$$f(s) \geq A|s|^q - \tilde{A} \quad \forall s \geq 0,$$

and

$$f(s) \leq B|s|^q + \tilde{B} \quad \forall s \in \mathbb{R}.$$

Furthermore, there exist constants $A_1, B_1, \tilde{A}_1, \tilde{B}_1 > 0$ such that

$$F(s) \geq A_1|s|^{q+1} - \tilde{A}_1 \quad \forall s \geq 0,$$

and

$$F(s) \leq B_1|s|^{q+1} + \tilde{B}_1 \quad \forall s \in \mathbb{R}.$$

Similarly, if f satisfies (AR1), then there exists a constant $\tilde{\theta} > 0$ such that

$$sf(s) > \theta F(s) - \tilde{\theta} \quad \forall s \geq 0.$$

Finally, (AR2) combined with the extension $c(s) \coloneqq c(-s)$; $s < 0$ implies that there exists $\tilde{\theta}_1 \in \mathbb{R}$ such that

$$\tilde{\theta}_1 < \theta \int_0^s c(z)\phi_p(z)\, dz - c(s)|s|^p$$

for all $s \in \mathbb{R}$ since $c(z)\phi_p(z) = -c(-z)\phi_p(-z)$, and hence

$$\int_0^s c(z)\phi_p(z)\, dz = \int_0^{-s} c(z)\phi_p(z)\, dz,$$

for all $s < 0$.

The proposition follows directly from (F2), (AR1), and (AR2).

CHAPTER III

THE SEMILINEAR CASE

3.1 Proof of Theorem 1.1

We will work primarily with three spaces, $W_0^{1,2}(0,1)$, $C[0,1]$, and $L^p(0,1)$ for $p = 1, 2$ with the standard norms on each space. For ease of notation in this chapter, we set $H = W_0^{1,2}(0,1)$, $\Omega = (0,1)$, and let $\|\cdot\|_H = \|\cdot\|_{1,2,0}$.

Let $J : H \to \mathbb{R}$ be defined by

$$J(u) = \frac{1}{2}\int_0^1 (u')^2 \, dx - \lambda \int_0^1 hF(u) \, dx. \tag{3.1}$$

The second term in the definition of J is well-defined, since $H \hookrightarrow C[0,1]$ and

$$\left|\lambda \int_0^1 hF(u) \, dx\right| \leq \lambda \|h\|_1 \max_{-M_1 \leq s \leq M_1} |F(s)| \quad \text{where } M_1 = \|u\|_\infty.$$

Since f is a C^1 map, we find that the map J is continuous, differentiable and

$$J'(u)(v) = \int_0^1 u'v' \, dx - \lambda \int_0^1 hf(u)v \, dx \quad \forall v \in H.$$

Next we will show that J is a C^1 map. Define

$$L_u(v) := \int_0^1 hf(u)v \, dx \quad \forall v \in H.$$

If $\|u_1 - u_2\|_H < \epsilon$, then

$$\begin{aligned} |L_{u_1}(v) - L_{u_2}(v)| &= |\int_0^1 h(x)(f(u_1) - f(u_2))v \, dx| \\ &\leq \int_0^1 |h(x)||f'(\eta)||u_1 - u_2||v| \, dx, \end{aligned}$$

where $\eta(x)$ is such that $\min\{u_1(x), u_2(x)\} < \eta(x) < \max\{u_1(x), u_2(x)\}$ for any fixed x. Since $u_1, u_2 \in H$ and f' is continuous we have

$$|L_{u_1}(v) - L_{u_2}(v)| \leq C\epsilon \|h\|_1 \|v\|_H,$$

for some $C > 0$, and hence J is C^1. The critical points of the functional J are weak solutions of (1.9).

We will first establish the existence of a solution for (1.9) using the Mountain Pass Theorem and then prove that the solution thus obtained is positive.

3.1.1 *Existence of a Mountain Pass Solution*

We wish to apply the standard Mountain Pass Theorem.

3.1.1.1 *J Satisfies (PS)*

Lemma 3.1. *The map J satisfies the Palais-Smale condition.*

Proof. First, we wish to show that any sequence, $\{u_n\} \subset H$, satisfying the hypotheses of (PS) must be bounded. Assume to the contrary that $\{u_n\}$ is such that $J'(u_n) \to 0$, there exists some $M > 0$ such that $|J(u_n)| < M$ for all $n \geq 1$, and $\|u_n\|_H \to \infty$. Then consider the quantity $\frac{\theta J(u_n) - \langle J'(u_n), u_n \rangle}{\|u_n\|_H}$, where $\theta > 2$ is chosen as in (AR1). Taking a limit as $n \to \infty$, we see that $\lim_{n\to\infty} \frac{\theta J(u_n) - \langle J'(u_n), u_n \rangle}{\|u_n\|_H} = 0$, since $J(u_n)$ is bounded and

$J'(u_n) \to 0$. Also we can write

$$\theta J(u_n) - \langle J'(u_n), u_n \rangle = \left(\frac{\theta}{2} - 1\right) \int_0^1 (u_n')^2 \, dx - \lambda \int_0^1 h(x) \left(\theta F(u_n) - f(u_n)u_n\right) \, dx.$$

Note that when $u_n \geq 0$, $\theta F(u_n) - f(u_n)u_n \leq \tilde{\theta}$ and when $u_n < 0$, $\theta F(u_n) - f(u_n)u_n = (\theta - 1)f(0)u_n$. Hence

$$\begin{aligned}\theta J(u_n) - \langle J'(u_n), u_n \rangle &\geq \left(\frac{\theta}{2} - 1\right) \int_0^1 (u_n')^2 \, dx - \lambda\tilde{\theta}\|h\|_1 \\ &\quad - \lambda(\theta - 1)|f(0)|\|u_n\|_\infty \|h\|_1 \\ &\geq \left(\frac{\theta}{2} - 1\right) \|u_n\|_H^2 - \lambda\tilde{\theta}\|h\|_1 \\ &\quad - \lambda k(\theta - 1)|f(0)|\|u_n\|_H \|h\|_1,\end{aligned}$$

where $k > 0$ satisfies $\|z\|_\infty \leq k\|z\|_H$ for all $z \in H$. Dividing both sides by $\|u_n\|_H$ and taking a limit as $n \to \infty$, we get a contradiction. Hence, $\{u_n\}$ is bounded in H. Since it is bounded in H, there exists a subsequence, call it again $\{u_n\}$, which converges weakly in H and strongly in $C[0, 1]$.

Now, since $J'(u_n) \to 0$, it is easy to show that $\{u_n\}$ is Cauchy in H and therefore, converges strongly in H. Hence, the Palais-Smale compactness condition holds. □

3.1.1.2 Geometry of J

First, note that $J(0) = 0$. Now for any $v \in H$ such that $\|v\|_H = 1$, $v(x) > 0$ for all $x \in (0,1)$ and any parameter $s > 0$, we have

$$\begin{aligned} J(sv) &= \frac{s^2}{2} - \frac{\lambda}{2}\int_0^1 h(x)F(sv)\,dx \\ &\leq \frac{s^2}{2} + s\left(\frac{\lambda\tilde{A}}{2}\int_0^1 h(x)v\,dx\right) - s^{q+1}\left(\frac{\lambda A}{2(q+1)}\int_0^1 h(x)v^{q+1}\,dx\right), \end{aligned}$$

since $F(\tilde{s}) \geq \frac{A}{q+1}(\tilde{s})^{q+1} - \tilde{A}\tilde{s}$ for all $\tilde{s} > 0$.

Now, letting $s \to \infty$, we note that $\lim_{s\to\infty} J(sv) = -\infty$ since $q > 1$. Choose $s^* >> 1$ such that $J(s^*v) < 0$.

Now, in order to apply the Mountain Pass Theorem we need a lemma which will show that there exists an $r > 0$ and an $\alpha > 0$ such that $J(u) > \alpha$ for all $\|u\| = r$. Later, however, we will also need information on how J grows when $r \to 0^+$ in order to show that the mountain pass solution is positive. We prove:

Lemma 3.2. *There exists $\overline{\lambda} > 0$ such that if $\lambda \in (0, \overline{\lambda})$, then for any $u \in H$ such that $\|u\|_H = \lambda^{\frac{-1}{q-1}}$,*

$$J(u) \geq \frac{1}{4}\lambda^{-\frac{2}{q-1}}.$$

Proof. Let $\|u\|_H = r$, where $r = \lambda^{\frac{-1}{q-1}}$. Now, rewriting $J(u)$ as

$$J(u) = \frac{1}{2}r^2 - \lambda\int_{\Omega_1} hF(u)\,dx - \lambda\int_{\Omega_2} hF(u)\,dx - \lambda\int_{\Omega_3} hF(u)\,dx \qquad (3.2)$$

where $\Omega_1 := \{x \in (0,1) : u(x) < 0\}$, $\Omega_2 := \{x \in (0,1) : 0 \leq u(x) \leq \tilde{\beta}\}$, and $\Omega_3 := \{x \in (0,1) : \tilde{\beta} < u(x)\}$, where $\tilde{\beta} > 0$ is the unique value where $F(\tilde{\beta}) = 0$, we obtain,

$$J(u) \geq \frac{1}{2}r^2 - \lambda \int_{\Omega_1} hf(0)u\, dx - \lambda \int_{\Omega_3} hF(u)\, dx. \tag{3.3}$$

Hence, applying (AR1) to (3.3), we obtain,

$$\begin{aligned} J(u) &\geq \frac{1}{2}r^2 + \lambda f(0)\|h\|_1 k\|u\|_H - \frac{\lambda B}{q+1}\|h\|_1 k^{q+1}\|u\|_H^{q+1} - \lambda\tilde{B}\|h\|_1 k\|u\|_H \\ &= \lambda^{\frac{-2}{q-1}}\left(\frac{1}{2} - \|h\|_1 k(|f(0)| + \tilde{B})\lambda^{\frac{q}{q-1}} - \frac{B}{q+1}\|h\|_1 k^{q+1}\lambda^{\frac{2q+2}{q-1}}\right). \end{aligned}$$

Hence, for λ sufficiently small, $J(u) \geq \frac{1}{4}\lambda^{\frac{-2}{q-1}}$. □

Hence, the hypotheses of the Mountain Pass Theorem have been satisfied, and we have the existence of at least one weak solution u_λ of (1.9).

3.1.2 Positivity of Solution u_λ for $\lambda \approx 0$.

We first establish an upper bound on $\|u_\lambda\|_\infty$.

Lemma 3.3. *There exists $\hat{\lambda} \in (0, \overline{\lambda})$ and $c_4 > 0$, independent of λ, such that for $\lambda \in (0, \hat{\lambda})$, $\|u_\lambda\|_\infty \leq c_4\lambda^{-\frac{1}{q-1}}$.*

Proof. Let v_1 denote the eigenfunction corresponding to the principal eigenvalue, λ_1, of $-u''$ with Dirichlet boundary conditions with $v_1 > 0$ and $\|v_1\|_H = 1$. Then, for

$s \geq 0$,

$$\begin{aligned} J(sv_1) &= \frac{1}{2}s^2 - \lambda \int_0^1 hF(sv_1)\, dx \\ &\leq \frac{1}{2}s^2 - \lambda \int_0^1 h \left(A_1 (sv_1)^{q+1} - \tilde{A}_1 \right)\, dx \\ &= \frac{1}{2}s^2 - \frac{\lambda A s^{q+1}}{q+1} \int_0^1 h v_1^{q+1}\, dx + \lambda \tilde{A}_1 \int_0^1 h\, dx \\ &\leq \frac{1}{2}s^2 - \frac{\lambda A c_1 s^{q+1}}{q+1} + \lambda \tilde{A}_1 \|h\|_1 \\ &= p(s) \ \ \text{(say)}, \end{aligned}$$

where $c_1 = \int_0^1 hv_1^{q+1}\, dx$.

Now $p(s)$ is maximized when $s = (\lambda A c_1)^{\frac{-1}{q-1}}$, and hence for $\lambda \approx 0$,

$$J(sv_1) \leq \left(\frac{1}{2} - \frac{1}{q+1} \right) (Ac_1)^{\frac{-2}{q-1}} \lambda^{\frac{-2}{q-1}} + \lambda \tilde{A}_1 \|h\|_1 \leq c_2 \lambda^{\frac{-2}{q-1}},$$

for some $c_2 > 0$ independent of λ. Hence, $J(u_\lambda) \leq c_2 \lambda^{\frac{-2}{q-1}}$, for $\lambda \approx 0$.

Now, recalling that $\theta F(s) - f(s)s \le \tilde{\theta}$ for $s \ge 0$,

$$
\begin{aligned}
\|u_\lambda\|_H^2 &= 2J(u_\lambda) + 2\lambda \int_{\Omega_1} hF(u_\lambda)\,dx + 2\lambda \int_{\Omega_1^c} hF(u_\lambda)\,dx \\
&\le 2c_2\lambda^{\frac{-2}{q-1}} + 2\lambda \int_{\Omega_1} hu_\lambda f(0)\,dx + 2\lambda \int_0^1 h\left(\frac{u_\lambda f(u_\lambda)}{\theta} + \frac{\tilde{\theta}}{\theta}\right)\,dx \\
&\qquad - 2\lambda \int_{\Omega_1} h\left(\frac{u_\lambda f(0)}{\theta} + \frac{\tilde{\theta}}{\theta}\right)\,dx \\
&= 2c_2\lambda^{\frac{-2}{q-1}} + 2\lambda\left(1 - \frac{1}{\theta}\right)\int_{\Omega_1} hu_\lambda f(0)\,dx - 2\lambda \int_{\Omega_1} h\frac{\tilde{\theta}}{\theta}\,dx \\
&\qquad + \frac{2\lambda}{\theta}\int_0^1 hu_\lambda f(u_\lambda)\,dx + 2\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 \\
&\le 3c_2\lambda^{\frac{-2}{q-1}} + 2\lambda k|f(0)|\|h\|_1\|u_\lambda\|_H + \frac{2}{\theta}\|u_\lambda\|_H^2,
\end{aligned}
$$

for $\lambda > 0$ small.

Now, this implies that $a\|u_\lambda\|_H^2 + b\lambda\|u_\lambda\|_H - 3c_2\lambda^{\frac{-2}{q-1}} < 0$, for $a = 1 - \frac{2}{\theta} > 0$ and $b = -2k|f(0)|\|h\|_1 < 0$. So, $\|u_\lambda\|_H$ must be less than the largest root of the quadratic $as^2 + b\lambda s - 3c_2\lambda^{\frac{-2}{q-1}}$. In other words, $\|u_\lambda\|_H \le \frac{-b\lambda + \sqrt{b^2\lambda^2 + 12ac_2\lambda^{\frac{-2}{q-1}}}}{2a} \le c_3\lambda^{\frac{-1}{q-1}}$ for some constant c_3, for $\lambda > 0$ small. Hence, $\|u_\lambda\|_\infty \le c_4\lambda^{\frac{-1}{q-1}}$ where $c_4 = kc_3$. □

3.1.2.1 Proof of Main Result

First we note that

$$
\begin{aligned}
\lambda \int_0^1 hf(u_\lambda)u_\lambda\,dx &= 2J(u_\lambda) + 2\lambda \int_0^1 hF(u)\,dx \\
&\ge \frac{1}{2}\lambda^{-\frac{2}{q-1}} + 2\lambda F(\beta)\int_0^1 h\,dx \qquad (3.4)\\
&\ge \frac{1}{4}\lambda^{-\frac{2}{q-1}}
\end{aligned}
$$

for $\lambda > 0$ sufficiently small. Now, choose $\gamma > 0$ such that $\hat{B}\|h\|_1\gamma^{q+1} = \frac{1}{16}$, where $\hat{B} = \max\{B, \tilde{B}\}$, and define $\Omega_\lambda := \{x | u_\lambda(x) \geq \gamma\lambda^{-\frac{1}{q-1}}\}$. Then for λ sufficiently small, $u_\lambda(x)$ will be sufficiently large on Ω_λ and hence $f(u_\lambda(x)) < Bu_\lambda(x)^q$ on Ω_λ. Then

$$\begin{aligned}\lambda \int_0^1 hf(u_\lambda)u_\lambda \, dx &= \lambda \int_{\Omega_\lambda} hf(u_\lambda)u_\lambda \, dx + \lambda \int_{\Omega_\lambda^c} hf(u_\lambda)u_\lambda \, dx \\ &\leq \lambda \int_{\Omega_\lambda} hBu_\lambda^{q+1} \, dx + \lambda \int_{\Omega_\lambda^c} h \left(B\,|u_\lambda|^q + \tilde{B} \right) |u_\lambda| \, dx, \qquad (3.5)\end{aligned}$$

since $f(s) \leq B|s|^q + \tilde{B}$ for all $s \in \mathbb{R}$. Now, recalling that on Ω_λ^c, $u_\lambda(x) \leq \gamma\lambda^{-\frac{1}{q-1}}$ and, by Lemma 3.3, on Ω_λ, $u_\lambda(x) \leq c_4\lambda^{-\frac{1}{q-1}}$, from (3.4) and (3.5) for $\lambda \approx 0$ we have,

$$\begin{aligned}\frac{1}{4}\lambda^{-\frac{2}{q-1}} &\leq B|\Omega_\lambda|\|h\|_1 c_4^{q+1}\lambda^{-\frac{2}{q-1}} + B(1 - |\Omega_\lambda|)\|h\|_1\gamma^{q+1}\lambda^{-\frac{2}{q-1}} \\ &\qquad + \tilde{B}(1 - |\Omega_\lambda|)\|h\|_1\gamma\lambda^{\frac{-1}{q-1}} \\ &\leq \hat{B}\|h\|_1\lambda^{-\frac{2}{q-1}} \left(|\Omega_\lambda| c_4^{q+1} + \gamma^{q+1} + \gamma\lambda^{\frac{1}{q-1}} \right) \\ &\leq \hat{B}\|h\|_1\lambda^{-\frac{2}{q-1}} \left(|\Omega_\lambda| c_4^{q+1} + 2\gamma^{q+1} \right)\end{aligned}$$

Hence, by the definition of γ, we may conclude that $|\Omega_\lambda| \geq \frac{1}{8\hat{B}\|h\|_1 c_4^{q+1}} = K$, (say). Let $N_\epsilon := [0, \epsilon) \cup (1 - \epsilon, 1]$ for $\epsilon \in (0, \frac{1}{2})$, where ϵ is chosen sufficiently small such that $|N_\epsilon| \leq \frac{K}{2}$. Letting $K_\lambda := \Omega_\lambda - N_\epsilon$, we also have that $|K_\lambda| \geq \frac{K}{2}$. Recall that the Green's function for the second derivative operator with Dirichlet boundary conditions is given by

$$G(x, \xi) = \begin{cases} (1 - x)\xi; & 0 \leq \xi \leq x \leq 1, \\ (1 - \xi)x; & 0 \leq x \leq \xi \leq 1. \end{cases} \qquad (3.6)$$

Define $d(\xi) = \min\{\xi, 1-\xi\}$ and $\hat{h} = \inf_{t\in(0,1]} h(t)$. Then for $x \in K_\lambda$ and $\xi \in N_\epsilon$, we have that $G(x,\xi) \geq \epsilon d(\xi)$. So, for any ξ such that $d(\xi) < \epsilon$, for $\lambda \approx 0$, we have

$$\begin{aligned} u_\lambda(\xi) &= \lambda \int_0^1 G(x,\xi) h f(u_\lambda)\, dx \\ &\geq \lambda \int_{K_\lambda} G(x,\xi) h A (u_\lambda)^q \, dx + \lambda f(0) \int_0^1 G(x,\xi) h \, dx \\ &\geq A\lambda^{-\frac{1}{q-1}} \hat{h} \int_{K_\lambda} \epsilon d(\xi) \gamma^q \, dx + \lambda f(0) \|h\|_1 \\ &\geq A\lambda^{-\frac{1}{q-1}} \hat{h} \epsilon d(\xi) \gamma^q \frac{K}{2} + \lambda f(0) \|h\|_1 \\ &\geq c_5 d(\xi) \lambda^{-\frac{1}{q-1}}, \end{aligned} \tag{3.7}$$

for some $c_5 > 0$.

We define w_λ and z_λ such that

$$\begin{cases} -w_\lambda'' = \lambda h f^+(u_\lambda); \quad x \in (0,1), \\ w_\lambda(0) = 0 = w_\lambda(1), \end{cases}$$

and

$$\begin{cases} -z_\lambda'' = \lambda h f^-(u_\lambda); \quad x \in (0,1), \\ z_\lambda(0) = 0 = z_\lambda(1) \end{cases}$$

where $f^+(s) = \max\{f(s), 0\}$ and $f^-(s) = \min\{f(s), 0\}$.

Clearly, $u_\lambda = w_\lambda + z_\lambda$, and also $z_\lambda(\xi) = \lambda \int_0^1 G(x,\xi) h(x) f^-(u_\lambda(x))\, dx$ ≤ 0 since $f^-(u_\lambda(x)) \leq 0$ and $G(x,\xi), h(x) \geq 0$. Furthermore, since $f^-(u_\lambda(x)) \geq f(0)$, we see that $z_\lambda(\xi) = \lambda \int_0^1 h G(x,\xi) f^-(u_\lambda(x))\, dx \geq \lambda f(0) \|h\|_1$. So $\lambda f(0) \|h\|_1$

$\leq z_\lambda(\xi) \leq 0$. Also, for ξ such that $d(\xi) = \epsilon$, we have $w_\lambda(\xi) = u_\lambda(\xi) - z_\lambda(\xi) \geq u_\lambda(\xi) \geq c_5 \epsilon \lambda^{-\frac{1}{q-1}}$. Hence, by the maximum principle, we have $w_\lambda(\xi) \geq c_5 \epsilon \lambda^{-\frac{1}{q-1}}$ for all $\xi \in \Omega - N_\epsilon$. Therefore, $u_\lambda(\xi) = w_\lambda(\xi) + z_\lambda(\xi) \geq c_5 \epsilon \lambda^{-\frac{1}{q-1}} + \lambda f(0) \|h\|_1$, and hence, for $\lambda > 0$ small enough, $u_\lambda > 0$ on $\Omega - N_\epsilon$. This, combined with the earlier proof that $u_\lambda(\xi) \geq c_5 d(\xi) \lambda^{-\frac{1}{q-1}}$ for all $\xi \in N_\epsilon$, completes the proof of the theorem.

3.2 Proof of Theorem 1.2

Here we establish the existence result for $\lambda \approx 0$ for the boundary value problem (1.10) which involves a nonlinear boundary condition at $x = 1$.

3.2.1 Variational Formulation

For ease of notation, in this chapter we take $\tilde{H} = \widetilde{W}^{1,p}(0,1)$ and take $\|\cdot\|_H$ as before (see Proposition 2.4 for justification). We extend the function c by letting $c(s) = c(-s)$ for $s < 0$, and define $E : \tilde{H} \to \mathbb{R}$ by

$$E(u) = J(u) + g(u(1)). \tag{3.8}$$

where

$$g(s) = \int_0^s c(z) z \, dz,$$

and $J(u)$ is defined as before.

Now, we wish to establish a regularity result to show that critical points of E are solutions to (1.10).

Definition 3.4. We say $u \in \tilde{H}$ is a critical point of E if

$$\int_0^1 u'\varphi'\,dx - \lambda \int_0^1 h(x)f(u)\varphi\,dx + g'(u(1))\varphi(1) = 0 \quad \forall \varphi \in \tilde{H}.$$

Lemma 3.5. *If u is a critical point of E, then u satisfies* (1.10) *almost everywhere in* $(0,1)$ *and the boundary conditions in the classical sense. Additionally, if we know that $h(x)$ is locally Hölder continuous in $(0,1)$, then the solution $u \in C^2(0,1) \cap C[0,1]$ and the equation is satisfied in the classical sense.*

Proof. Clearly $u(0) = 0$ since the critical point $u \in \tilde{H}$. If $\varphi \in C_c^\infty(0,1)$, then we have

$$\int_0^1 u'\varphi' - \lambda \int_0^1 h(x)f(u)\varphi = 0. \tag{3.9}$$

In other words, u is a weak solution of $-u'' = \lambda h(x)f(u)$. But from the assumptions on h and since $u \in \tilde{H} \subset C[0,1]$ we have $\lambda h(x)f(u(x)) \in L^\infty_{loc}((0,1])$. By standard elliptic regularity, $u \in W^{2,2}_{loc}(0,1)$, and from the definition of the weak second derivative, we have $-\int_0^1 u''\varphi\,dx = \int_0^1 u'\varphi'\,dx$ for all $\varphi \in C_c^\infty(0,1)$. Now, from (3.9), we have

$$-\int_0^1 u''\varphi\,dx - \lambda \int_0^1 h(x)f(u)\varphi\,dx = 0 \quad \forall \varphi \in C_c^\infty(0,1).$$

Since we now know that $u'' + \lambda h(x)f(u) \in L^1_{loc}(0,1)$ from the previous expression, then we have that,

$$-u'' = \lambda h(x)f(u) \text{ a.e in } (0,1). \tag{3.10}$$

Since $u \in \tilde{H}$, we have both u and u' in $L^1(0,1)$. Now from the previous representation, $u'' = -\lambda h(x) f(u) \in L^1(0,1)$. Thus we have improved regularity, with $u \in W^{2,1}(0,1) \cap W^{2,2}_{loc}(0,1)$.

Now we know that $u \in W^{2,1}(0,1)$. So $u' \in W^{1,1}(0,1)$, and hence u' is an absolutely continuous function in $[0,1]$. Therefore, it now makes sense to talk about the pointwise value $u'(1)$.

Finally, we will show that $u'(1) + c(u(1))u(1) = 0$. Let

$$\mathcal{C} = \{\varphi \in C^\infty(0,1) \cap C^1[0,1] \mid \text{support}(\varphi) \subset\subset (0,1], \varphi(1) \neq 0\}.$$

Clearly $\mathcal{C} \subset \tilde{H}$. From Definition 3.4 applied for an arbitrary $\varphi \in \mathcal{C}$ we have,

$$\int_0^1 u'\varphi' \, dx - \lambda \int_0^1 h(x) f(u) \varphi \, dx + g'(u(1))\varphi(1) = 0.$$

Using the integration by parts formula on $W^{2,1}(0,1)$, we have

$$\int_0^1 u'\varphi' \, dx = -\int_0^1 u''\varphi \, dx + \varphi u']_0^1,$$

and hence,

$$-\int_0^1 (u''\varphi + \lambda h(x) f(u)) \, \varphi \, dx + \varphi(1)u'(1) + g'(u(1))\varphi(1) = 0 \quad \forall \varphi \in \mathcal{C}.$$

Using (3.9) we have $u'(1) + g'(u(1)) = 0$, i.e u satisfies the boundary condition at $x = 1$.

□

3.2.2 Existence of a Mountain Pass Solution

3.2.2.1 E is C^1

Recall that $E(u) = J(u) + g(u(1))$. Since J has already been shown to be a C^1 functional, we need only show that $g(u(1))$ is C^1 to conclude that E is C^1.

Fix $u \in \tilde{H}$ and consider the functional $H(u) := g(u(1))$. For any $v \in \tilde{H}$, $\langle H'(u), v\rangle = g'(u(1))v(1)$. It is clear that the fuction $g(s)$, as previously defined, is differentiable. Further, since pointwise evaluation is a continuous operation, we may conclude that the derivative is also continuous. Hence, $E(u)$ is a C^1 functional.

3.2.2.2 E Satisfies (PS)

Again, we first wish to show that any sequence $\{u_n\}$ satisfying the hypotheses of (PS) must be bounded. Assume to the contrary that $\{u_n\}$ is such that $E'(u_n) \to 0$, there exists some $M > 0$ such that $|E(u_n)| < M$ for all $n \geq 1$, and $\|u_n\|_H \to \infty$. Then consider the quantity $\frac{\theta E(u_n) - \langle E'(u_n), u_n\rangle}{\|u_n\|_H}$ where $\theta > 2$ is chosen as in (AR1).

Taking a limit as $n \to \infty$, we see that $\lim_{n\to\infty} \frac{\theta E(u_n) - \langle E'(u_n), u_n\rangle}{\|u_n\|_H} = 0$ since $J(u_n)$ is bounded and $J'(u_n) \to 0$. However,

$$\begin{aligned}
\theta E(u_n) - \langle E'(u_n), u_n\rangle &= (\theta J(u_n) - \langle J'(u_n), u_n\rangle) \\
&\qquad + \left(\theta g(u_n(1)) - c(u_n(1))\,(u_n(1))^2\right) \\
&\geq c\|u_n\|_H^2 - \lambda\tilde{\theta}\|h\|_1 \\
&\qquad + \left(\theta g(u_n(1)) - c(u_n(1))\,(u_n(1))^2\right) \\
&\geq c\|u_n\|_H^2 - \lambda\tilde{\theta}\|h\|_1 \\
&\qquad - \lambda k(\theta - 1)|f(0)|\|u_n\|_H\|h\|_1 + \tilde{\theta}_1,
\end{aligned}$$

for some $c > 0$. Dividing both sides by $\|u_n\|_H$ and taking a limit as $n \to \infty$, we get a contradiction. Hence, $\{u_n\}$ is bounded in $\tilde{H}$. Since it is bounded in $\tilde{H}$, there exists a subsequence, call it again $\{u_n\}$, which converges weakly in $\tilde{H}$ and strongly in $C[0,1]$.

Since $\{u_n\}$ converges strongly in $C[0,1]$, for any $\epsilon > 0$, there exists an $N_1 > 0$ such that for all $n, m > N_1$, $\|u_n - u_m\|_\infty < \epsilon$. Further, since $E'(u_n) \to 0$, for any $\epsilon > 0$, there exists an $N_2 > 0$ such that for all $n, m > N_2$, $\|E'(u_n) - E'(u_m)\|_* < \epsilon$, where $\|\cdot\|_*$ is the associated operator norm. Furthermore, since u_n converges in $C[0,1]$, there exists an $\tilde{M} > 0$ so that $|f(u_n) - f(u_m)| \le \tilde{M}$. Hence, we may choose $N = \max\{N_1, N_2\}$, and, for all $n, m > N$,

$$
\begin{aligned}
\|u_n - u_m\|_H^2 = \langle E'(u_n) - E'(u_m), u_n - u_m\rangle \\
&+ \lambda \int_0^1 h(x)(f(u_n) - f(u_m))(u_n - u_m)\, dx \\
&\quad - c\,(u_n(1) - u_m(1)) \cdot (u_n(1) - u_m(1))^2 \\
&\le \|J'(u_n) - J'(u_m)\|_* \|u_n - u_m\|_{L^2} \\
&\quad + \lambda \|h\|_1 \|f(u_n) - f(u_m)\|_\infty \|u_n - u_m\|_\infty \\
&\le \epsilon^2 + \lambda \|h\|_1 \|f(u_n) - f(u_m)\|_\infty \epsilon.
\end{aligned}
$$

Hence $\{u_n\}$ is a Cauchy sequence in $\tilde{H}$, and therefore $\{u_n\}$ converges strongly in $\tilde{H}$. This proves the Palais-Smale Compactness Condition.

3.2.2.3 Geometry of E

Again, we wish to show that the function E satisfies the appropriate geometric conditions of the Mountain Pass Theorem. It is again clear that $E(0) = 0$. We again take v_1 to be the principle eigenfunction of the operator $-u''$ with Dirichlet boundary conditions such that $\|v_1\|_H = 1$ and $v(x) > 0$ for all $x \in (0,1)$, and note that since

$H \subset \tilde{H}$, then $v_1 \in \tilde{H}$. Then we note that $E(sv_1) = J(sv_1) + g(v_1(1)) = J(sv_1)$ and hence, as before, $E(sv_1) \to -\infty$ as $s \to \infty$. Thus, we may choose $s^* >> 1$ sufficiently large so that $E(s^* v_1) < 0$.

Finally, we establish a lemma similar to Lemma 3.2.

Lemma 3.6. *There exists $\overline{\lambda} > 0$ such that if $\lambda \in (0, \overline{\lambda})$, then for any $u_\lambda \in \tilde{H}$ such that $\|u_\lambda\|_H = \lambda^{\frac{-1}{q-1}}$, $E(u_\lambda) \geq \frac{1}{4}\lambda^{-\frac{2}{q-1}}$.*

Proof. Recall again that $E(u_\lambda) = J(u_\lambda) + g\left(u_\lambda(1)\right)$. But recall that as in Lemma 3.2, $J(u_\lambda) \geq \frac{1}{4}\lambda^{-\frac{2}{q-1}}$. Since $g(s) \geq 0$ by definition, $E(u_\lambda) \geq J(u_\lambda) \geq \frac{1}{4}\lambda^{-\frac{2}{q-1}}$. □

Hence, E satisfies the hypotheses of the Mountain Pass Theorem, and therefore, there exists a solution u_λ to (1.10).

3.2.3 Positivity of Solution u_λ for $\lambda \approx 0$.

We again need a lemma similar to Lemma 3.3 in order to establish the result.

Lemma 3.7. *There exists $\hat{\lambda} \in (0, \overline{\lambda})$ and $c_6 > 0$, independent of λ, such that for $\lambda \in (0, \hat{\lambda})$, $\|u_\lambda\|_\infty \leq c_6 \lambda^{-\frac{1}{q-1}}$.*

Proof. Let v_1 be as before. We note that for any $s > 0$, since $v_1(1) = 0$, for $\lambda \approx 0$, $E(sv_1) = J(sv_1) + g(sv_1(1)) = J(sv_1) \leq c_2 \lambda^{\frac{-2}{q-1}}$ as before. Hence, $E(u_\lambda) \leq c_2 \lambda^{\frac{-2}{q-1}}$ for $\lambda \approx 0$.

Now, by Proposition 2.6,

$$\begin{aligned}
\|u_\lambda\|_H^2 &= 2E(u_\lambda) + 2\lambda \int_{\Omega_1} hF(u_\lambda)\,dx + 2\lambda \int_{\Omega_1^c} hF(u_\lambda)\,dx - 2g(u_\lambda(1)) \\
&\leq 2c_2\lambda^{\frac{-2}{q-1}} + 2\lambda \int_{\Omega_1} hu_\lambda f(0)\,dx + 2\lambda \int_0^1 h\left(\frac{u_\lambda f(u_\lambda)}{\theta} + \frac{\tilde{\theta}}{\theta}\right)\,dx \\
&\qquad\qquad - 2\lambda \int_{\Omega_1} h\left(\frac{u_\lambda f(0)}{\theta} + \frac{\tilde{\theta}}{\theta}\right)\,dx - 2g(u_\lambda(1)) \\
&= 2c_2\lambda^{\frac{-2}{q-1}} + 2\lambda\left(1 - \frac{1}{\theta}\right) \int_{\Omega_1} hu_\lambda f(0)\,dx - 2\lambda \int_{\Omega_1} h\frac{\tilde{\theta}}{\theta}\,dx \\
&\qquad + \frac{2\lambda}{\theta} \int_0^1 hu_\lambda f(u_\lambda)\,dx + 2\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 - 2g(u_\lambda(1)) \\
&\leq 2c_2\lambda^{\frac{-2}{q-1}} + 2\lambda|f(0)|\|h\|_1\|u_\lambda\|_\infty + \frac{2}{\theta}\|u_\lambda\|_H^2 + \frac{2}{\theta}c(u_\lambda(1))(u_\lambda(1))^2 \\
&\qquad + 2\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 - 2g(u_\lambda(1)) \\
&= 2c_2\lambda^{\frac{-2}{q-1}} + 2\lambda|f(0)|\|h\|_1\|u_\lambda\|_\infty + \frac{2}{\theta}\|u_\lambda\|_H^2 + 2\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 \\
&\qquad + \frac{2}{\theta}\left(c(u_\lambda(1))(u_\lambda(1))^2 - \theta g(u_\lambda(1))\right) \\
&\leq 2c_2\lambda^{\frac{-2}{q-1}} + 2\lambda|f(0)|\|h\|_1\|u_\lambda\|_\infty + \frac{2}{\theta}\|u_\lambda\|_H^2 + 2\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 - 2\frac{\tilde{\theta}_1}{\theta} \\
&\leq 3c_2\lambda^{\frac{-2}{q-1}} + 2\lambda|f(0)|\|h\|_1\|u_\lambda\|_\infty + \frac{2}{\theta}\|u_\lambda\|_H^2,
\end{aligned}$$

for $\lambda > 0$ small.

Now, similar to the Dirichlet case, this implies that $a\|u_\lambda\|_H^2 + b\lambda\|u_\lambda\|_H - 3c_2\lambda^{\frac{-2}{q-1}} < 0$ for $a = 1 - \frac{2}{\theta} > 0$ and $b = -2k|f(0)|\|h\|_1 < 0$. So, $\|u_\lambda\|_H$ must be less than the largest root of the quadratic $as^2 + b\lambda s - 3c_2\lambda^{\frac{-2}{q-1}}$. In other words, $\|u_\lambda\|_H \leq \frac{-b\lambda + \sqrt{b^2\lambda^2 + 12ac_2\lambda^{\frac{-2}{q-1}}}}{2a} \leq c_5\lambda^{\frac{-1}{q-1}}$, for some constant $c_5 > 0$, for $\lambda > 0$ small. Hence $\|u_\lambda\|_\infty \leq c_6\lambda^{\frac{-1}{q-1}}$ where $c_6 = kc_5$. □

3.2.3.1 Proof of Main Result

First, we note that

$$\begin{aligned}
\lambda \int_0^1 hf(u_\lambda)u_\lambda \, dx &= \|u_\lambda\|_H^2 + c(u_\lambda(1))\,(u_\lambda(1))^2 \\
&= 2J(u_\lambda) + 2\lambda \int_0^1 hF(u_\lambda)\, dx + c(u_\lambda(1))\,(u_\lambda(1))^2 \\
&\geq \frac{c_1^2}{2}\lambda^{-\frac{2}{q-1}} + 2\lambda F(\beta)\int_0^1 h\, dx \\
&\geq \frac{c_1^2}{4}\lambda^{-\frac{2}{q-1}}, \qquad (3.11)
\end{aligned}$$

for $\lambda > 0$ sufficiently small. Now, choose $\gamma > 0$ such that $\hat{B}\|h\|_1\gamma^{q+1} = \frac{c_1^2}{16}$, where $\hat{B} = \max\{B, \tilde{B}\}$, and define $\Omega_\lambda := \{x | u_\lambda(x) \geq \gamma\lambda^{-\frac{1}{q-1}}\}$ as before. Then for λ sufficiently small, $u_\lambda(x)$ will be sufficiently large on Ω_λ and hence $f(u_\lambda(x)) \leq Bu_\lambda(x)^q$ on Ω_λ. Hence,

$$\begin{aligned}
\lambda \int_0^1 hf(u_\lambda)u_\lambda \, dx &= \lambda \int_{\Omega_\lambda} hf(u_\lambda)u_\lambda \, dx + \lambda \int_{\Omega_\lambda^c} hf(u_\lambda)u_\lambda \, dx \\
&\leq \lambda \int_{\Omega_\lambda} hBu_\lambda^{q+1} \, dx + \lambda \int_{\Omega_\lambda^c} h\left(B\,|u_\lambda|^q + \tilde{B}\right)|u_\lambda| \; dx. \qquad (3.12)
\end{aligned}$$

Now, recalling that on Ω_λ^c, $u_\lambda(x) \leq \gamma\lambda^{-\frac{1}{q-1}}$ and, by Lemma 3.7, on Ω_λ, $u_\lambda(x) \leq c_4\lambda^{-\frac{1}{q-1}}$, then combining (3.11) and (3.12), for $\lambda \approx 0$ we have,

$$\begin{aligned}
\frac{c_1^2}{4}\lambda^{-\frac{2}{q-1}} &\leq B|\Omega_\lambda|\|h\|_1 c_4^{q+1}\lambda^{-\frac{2}{q-1}} + B(1-|\Omega_\lambda|)\|h\|_1\gamma^{q+1}\lambda^{-\frac{2}{q-1}} \\
&\qquad + \tilde{B}(1-|\Omega_\lambda|)\|h\|_1\gamma\lambda^{\frac{-1}{q-1}} \\
&\leq \hat{B}\|h\|_1\lambda^{-\frac{2}{q-1}}\left(|\Omega_\lambda|c_4^{q+1} + \gamma^{q+1} + \gamma\lambda^{\frac{1}{q-1}}\right) \\
&\leq \hat{B}\|h\|_1\lambda^{-\frac{2}{q-1}}\left(|\Omega_\lambda|c_4^{q+1} + 2\gamma^{q+1}\right).
\end{aligned}$$

Hence, by the definition of γ, we may conclude that $|\Omega_\lambda| \geq \frac{1}{8\hat{B}\|h\|_1 c_4^{q+1}} \equiv K$. Defining N_ϵ and K_λ as before, we again see that $|K_\lambda| \geq \frac{K}{2}$. Now, we define the new Green's function, $\tilde{G}(x,\epsilon)$ to be

$$\tilde{G}(x,\xi) = \begin{cases} \xi; & 0 \leq \xi \leq x \leq 1, \\ x; & 0 \leq x \leq \xi \leq 1, \end{cases}$$

so that

$$u_\lambda(\xi) = \lambda \int_0^1 \tilde{G}(x,\xi)h(x)f(u_\lambda(x))\,dx - c(u_\lambda(1))u_\lambda(1)\xi. \tag{3.13}$$

Using the boundary condition $u'_\lambda(1) + c(u_\lambda(1))u_\lambda(1) = 0$, we may rewrite (3.13) as

$$u_\lambda(\xi) = \lambda \int_0^1 \tilde{G}(x,\xi)h(x)f(u_\lambda(x))\,dx + u'_\lambda(1)\xi. \tag{3.14}$$

Further, since $u'_\lambda(1) = u_\lambda(1) - \int_0^1 xu''(x)\,dx = u_\lambda(1) - \lambda\int_0^1 xh(x)f(u_\lambda(x))\,dx$, by substituting we obtain,

$$u_\lambda(\xi) = \lambda \int_0^1 G(x,\xi)h(x)f(u_\lambda(x))\,dx + u_\lambda(1)\xi, \tag{3.15}$$

where G is as in (3.6).

Now, proceeding as before, we recall that by (3.7),

$$u_\lambda(\epsilon) \geq c_5 d(\xi)\lambda^{\frac{-1}{q-1}} + u_\lambda(1)\xi,$$

for $\xi \in N_\epsilon$. Hence, if $u_\lambda(1)$ were nonnegative for $\lambda > 0$ sufficiently small, then we could conclude that $u_\lambda(\xi) \geq c_5 d(\xi)\lambda^{\frac{-1}{q-1}}$ for all $\xi \in N_\epsilon$.

Assume, to the contrary, that $u_\lambda(1) < 0$. Then by (3.13) and the fact that $c(s) > 0$ for $s < 0$,

$$\begin{aligned} u_\lambda(1) &= \lambda \int_0^1 xh(x)f(u_\lambda(x))\,dx - c(u_\lambda(1))u_\lambda(1) \\ &\geq \lambda \int_0^1 xh(x)f(u_\lambda(x))\,dx \\ &\geq \lambda \int_{K_\lambda} xh(x)f(u_\lambda(x))\,dx + \lambda f(0)\|h\|_1 \\ &\geq A\hat{h}\epsilon\gamma^q\lambda^{\frac{-1}{q-1}} + \lambda f(0)\|h\|_1 \\ &> 0 \end{aligned}$$

for $\lambda > 0$ sufficiently small. Hence, we have a contradiction, and $u_\lambda(1) \geq 0$. Therefore, $u_\lambda(\xi) \geq c_5 d(\xi)\lambda^{\frac{-1}{q-1}}$ for all $\xi \in N_\epsilon$.

Now, let w_λ and z_λ be defined as

$$\begin{cases} -w_\lambda'' = \lambda h(x)f^+(u_\lambda); \quad x \in (0,1), \\ w_\lambda(0) = 0, \\ w_\lambda'(1) + c(u_\lambda(1))w_\lambda(1) = 0, \end{cases} \tag{3.16}$$

and

$$\begin{cases} -z_\lambda'' = \lambda h(x)f^-(u_\lambda); \quad x \in (0,1), \\ z_\lambda(0) = 0, \\ z_\lambda'(1) + c(u_\lambda(1))z_\lambda(1) = 0. \end{cases} \tag{3.17}$$

Then clearly $u_\lambda = w_\lambda + z_\lambda$. Further, note that since $z''_\lambda(x) \geq 0$ and $z'_\lambda(1) = -c(u_\lambda(1))z_\lambda(1)$, then $z_\lambda(x) < 0$ for all $x \in (0,1]$.

Also, $z_\lambda(\xi) = \lambda \int_0^1 \tilde{G}(x,\xi)h(x)f^-(u_\lambda(x))\ dx - c(u_\lambda(1))z_\lambda(1)\xi \geq \lambda f(0)\|h\|_1$. So $\lambda f(0)\|h\|_1 \leq z_\lambda(\xi) \leq 0$. Further, for ξ such that $d(\xi) = \epsilon$, we have $w_\lambda(\xi) = u_\lambda(\xi) - z_\lambda(\xi) \geq u_\lambda(\xi) \geq c_5\epsilon\lambda^{\frac{-1}{q-1}}$. Hence, by the maximum principle, we have that $w_\lambda(\xi) \geq c_5\epsilon\lambda^{\frac{-1}{q-1}}$ for all $\xi \in (0,1) - N_\epsilon$. Therefore, $u_\lambda(\xi) = w_\lambda(\xi) + z_\lambda(\xi) \geq c_5\epsilon\lambda^{\frac{-1}{q-1}} + \lambda f(0)\|h\|_1$, and hence for λ sufficiently small, we have that $u_\lambda(\xi) > 0$ on $(0,1) - N_\epsilon$. This, combined with the estimate of $u_\lambda(\xi)$ on N_ϵ, completes the proof.

CHAPTER IV

THE QUASILINEAR CASE

4.1 Proof of Theorem 1.3

For ease of notation in this chapter, we take $W = W_0^{1,p}(0,1)$ and let $\|\cdot\|_W = \|\cdot\|_{1,p,0}$.

Let $J : W \to \mathbb{R}$ be defined by

$$J(u) = \frac{1}{p}\int_0^1 (u')^p \, dx - \lambda \int_0^1 hF(u)\, dx. \tag{4.1}$$

The second term in the definition of J is well defined, since $W \hookrightarrow C[0,1]$ and

$$\left|\lambda \int_0^1 hF(u)\, dx\right| \leq \lambda \|h\|_1 \max_{-M_1 \leq s \leq M_1} |F(s)| \quad \text{where } M_1 = \|u\|_\infty.$$

Further, the map J is continuously differentiable and

$$\langle J'(u), v\rangle = \int_0^1 |u'|^{p-2}\, u'v'\, dx - \lambda \int_0^1 hf(u)v\, dx \quad \forall v \in W.$$

Clearly, the first term of J' is well defined. The second term is well defined since $W \hookrightarrow C[0,1]$ and the extended function $f \in C(\mathbb{R})$. Indeed, to show that J' is a continuous map, let us show that $L_u(v) := \int_0^1 hf(u)v\, dx$ is continuous for any $v \in W$.

Let $\epsilon > 0$ be given. Since the extended function f is continuous, there exists $\delta_1 > 0$ so that for every $t_1, t_2 \in \mathbb{R}$ such that $|t_2 - t_1| < \delta_1$, $|f(t_2) - f(t_1)| < \frac{\epsilon}{k\|h\|_1}$.

Choose $\delta = \frac{\delta_1}{k}$ so that when $\|u_1 - u_2\|_W < \delta$, we have $\|u_1 - u_2\|_\infty < \delta_1$. Then for any fixed $v \in W$ with $\|v\|_W \leq 1$,

$$\begin{aligned} |L_{u_1}(v) - L_{u_2}(v)| &= |\int_0^1 h(f(u_1) - f(u_2))v\, dx| \\ &\leq \int_0^1 h|f(u_1) - f(u_2)|\|v\|_\infty\, dx \\ &\leq k\int_0^1 h|f(u_1) - f(u_2)|\, dx \\ &\leq k\int_0^1 h\frac{\epsilon}{k\|h\|_1}\, dx \\ &= \epsilon, \end{aligned}$$

for all u_1, u_2 with $\|u_1 - u_2\|_W < \delta$. Hence,

$$\|L_{u_1} - L_{u_2}\| = \sup_{\|v\|_W \leq 1} \{|L_{u_1}(v) - L_{u_2}(v)|\} \leq \epsilon.$$

Therefore, J is C^1.

We will first establish the existence of a solution for (1.5) using the Mountain Pass Theorem and then prove that the solution thus obtained is positive.

Lemma 4.1. *The critical point $u \in W$ of* (4.1) *is a solution of* (1.5).

Proof. If u is a critical point of (4.1), then

$$\int_0^1 \phi_p(u'(s))v'(s)\, ds = \lambda\int_0^1 h(s)f(u(s))v(s)\, ds \quad \forall v \in C_0^\infty[0,1].$$

Using integration by parts, we then have,

$$\int_0^1 \left((\phi_p(u'(s))' + \lambda h(s)f(u(s))\right) v(s)\, ds = 0 \quad \forall v \in C_0^\infty[0,1].$$

Hence, $(\phi_p(u'(x)))' = -\lambda h(x)f(u(x))$ almost everywhere in (0,1). But since f is continuous, $u \in C[0,1]$, and $h \in C(0,1)$, then $(\phi_p(u'(x)))' = -\lambda h(x)f(u(x))$ holds for every $x \in (0,1)$. Furthermore, since $h \in L^1(0,1)$, f is continuous, and $u \in C[0,1]$, we have that $(\phi_p(u'))' \in L^1(0,1)$, i.e., $\phi_p(u') \in W^{1,1}(0,1)$.

Let $x_0 \in (0,1)$ so that $u'(x_0) = 0$. Then,

$$u'(x) = \phi_p^{-1}\left(-\lambda \int_{x_0}^{x} h(s)f(u(s))\, ds\right).$$

Since h is continuous on $(0,1]$ and f is continuous on $[0,\infty)$, $-\lambda \int_{x_0}^{x} h(s)f(u(s))\, ds$ is continuous for all $x \in (0,1]$. Since ϕ_p^{-1} is also continuous, we find that u' is continuous on $(0,1]$.

For $x = 0$, we have

$$\begin{aligned}\lim_{x\to 0^+} u'(x) &= \lim_{x\to 0^+} \phi_p^{-1}\left(-\lambda \int_{x_0}^{x} h(s)f(u(s))\, ds\right)\\ &= \phi_p^{-1}\left(-\lambda \int_{x_0}^{0} h(s)f(u(s))\, ds\right),\end{aligned}$$

exists since ϕ_p^{-1} is a continuous function and $h \in L^1(0,1)$. Hence, $u \in C^1[0,1]$. □

4.1.1 Existence of a Mountain Pass Solution

In the following theorem, we establish the existence of a Mountain Pass solution.

Theorem 4.2. *For $\lambda \approx 0$, the hypotheses of the Mountain Pass Theorem are satisfied, and there exists a solution u_λ to* (1.5).

In order to prove Theorem 4.2, we first prove several lemmas. Throughout the calculations to follow, we let $r = \frac{1}{q+1-p}$.

Lemma 4.3. *The map J satisfies the Palais-Smale condition.*

Proof. First, we wish to show that any sequence, $\{u_n\}$ satisfying the hypotheses of (PS) must be bounded. Assume to the contrary that $\{u_n\}$ is a sequence such that $J'(u_n) \to 0$, there exists some $M > 0$ such that $|J(u_n)| < M$ for all $n \geq 1$, and $\|u_n\|_W \to \infty$. Then consider the quantity

$$\frac{\theta J(u_n) - \langle J'(u_n), u_n \rangle}{\|u_n\|_W},$$

where $\theta > p$ is chosen as in (AR1). Taking a limit as $n \to \infty$, we see that

$$\lim_{n\to\infty} \frac{\theta J(u_n) - \langle J'(u_n), u_n \rangle}{\|u_n\|_W} = 0,$$

since $J(u_n)$ is bounded and $J'(u_n) \to 0$. Also we can write

$$\begin{aligned}\theta J(u_n) - \langle J'(u_n), u_n \rangle = \left(\frac{\theta}{p} - 1\right) \int_0^1 (u_n')^p \, dx \\ - \lambda \int_0^1 h\left(\theta F(u_n) - f(u_n)u_n\right) \, dx.\end{aligned}$$

Note that when $u_n \geq 0$, $\theta F(u_n) - f(u_n)u_n \leq \tilde{\theta}$, and when $u_n < 0$,

$$\begin{aligned}\theta F(u_n) - f(u_n)u_n &= \theta u_n f(0) - f(0)u_n \\ &= (\theta - 1)f(0)u_n.\end{aligned}$$

Hence

$$\theta J(u_n) - \langle J'(u_n), u_n\rangle \geq \left(\frac{\theta}{p} - 1\right) \int_0^1 (u_n')^p \, dx - \lambda\tilde{\theta}\|h\|_1$$
$$- \lambda(\theta - 1)|f(0)|\|u_n\|_\infty \|h\|_1$$
$$\geq \left(\frac{\theta}{p} - 1\right) \|u_n\|_W^p - \lambda\tilde{\theta}\|h\|_1 - \lambda k(\theta - 1)|f(0)|\|u_n\|_W \|h\|_1.$$

Dividing both sides by $\|u_n\|_W$ and taking a limit as $n \to \infty$, we get a contradiction. Hence, $\{u_n\}$ is bounded in W and therefore there exists a subsequence, call it again $\{u_n\}$, which converges weakly in W and strongly in $C[0,1]$.

Since $u_n \to u$ strongly in $C[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 hf(u_n)(u_n - u)\, dx \to 0.$$

Furthermore, since $\{u_n\}$ is a Palais-Smale sequence, $J'(u_n) \to 0$. Therefore, since $u_n - u$ is bounded in W, we obtain

$$\lim_{n\to\infty} \langle J'(u_n), u_n - u\rangle \to 0.$$

Hence,

$$\langle J'(u_n), u_n - u\rangle + \lambda \int_0^1 hf(u_n)(u_n - u)\, dx = \langle \Psi'(u_n), u_n - u\rangle \to 0,$$

where Ψ is as in Proposition 2.5. Therefore, by the (S^+) property, $u_n \to u$ strongly in W, and so J satisfies (PS). □

Lemma 4.4. *There exists $\bar{\lambda} > 0$ and $u \in W$ such that if $\lambda \in (0, \bar{\lambda})$, then $J(u) < 0$.*

Proof. Let $v_1 \in W$ such that $\|v_1\|_W = 1$, $v_1(x) > 0$ for all $x \in (0,1)$ (which implies that $v_1 \in L^{q+1}(0,1)$), and $c_1 = \left(\frac{2}{pA_1\hat{h}\|v_1\|_{q+1}^{q+1}}\right)^r$. Then for $s = c_1\lambda^{-r}$,

$$\begin{aligned} J(sv_1) &= \frac{1}{p}\int_0^1 ((sv_1)')^p \, dx - \lambda \int_0^1 hF(sv_1)\, dx \\ &\le \frac{s^p}{p} - \lambda \int_0^1 h(A_1 s^{q+1} v_1^{q+1} - \tilde{A}_1)\, dx \\ &\le \frac{s^p}{p} - \lambda A_1 s^{q+1}\hat{h}\|v_1\|_{q+1}^{q+1} + \lambda\tilde{A}_1\|h\|_1 \qquad (4.2) \\ &= c_1^p\left(\frac{\lambda^{-rp}}{p} - \lambda\hat{h}A_1 c_1^{q+1-p}\lambda^{-r(q+1)}\|v_1\|_{q+1}^{q+1}\right) + \lambda\tilde{A}_1\|h\|_1. \end{aligned}$$

Now, substituting in our choice of c_1, we have

$$\begin{aligned} J(sv_1) &\le c_1^p\left(\frac{\lambda^{-rp}}{p} - \frac{2}{p}\lambda^{1-r(q+1)}\right) + \lambda\tilde{A}_1\|h\|_1 \\ &= c_1^p\lambda^{-rp}\left(\frac{1}{p} - \frac{2}{p}\lambda^{1-r(q+1-p)}\right) + \lambda\tilde{A}_1\|h\|_1 \\ &= -c_1^p\lambda^{-rp}\frac{1}{p} + \lambda\tilde{A}_1\|h\|_1 \\ &= \lambda^{-rp}\left(\frac{-c_1^p}{p} + \lambda^{1+rp}\tilde{A}_1\|h\|_1\right). \end{aligned}$$

Hence, choosing $\bar{\lambda} < \left(p\|h\|_1\tilde{A}_1 c_1^{-p}\right)^{\frac{-1}{1+rp}}$, we see that for all $\lambda \in (0,\bar{\lambda})$, there exists s^* (for example $s^* = c_1\left(\frac{\bar{\lambda}}{2}\right)^{-r}$) so that $J(u) < 0$ for $u = s^*v_1$. □

Lemma 4.5. *There exist $\tau \in (0, c_1)$ and $\tilde{\lambda} > 0$ such that if $\|u\|_W = \tau\lambda^{-r}$, then $J(u) \ge c_2(\tau\lambda^{-r})^p$ for all $\lambda \in (0,\tilde{\lambda})$, where $c_2 = \frac{1}{4p}$.*

Proof. Let $\|u\|_W = \tau\lambda^{-r}$, where $\tau > 0$ is to be chosen later. Then

$$
\begin{aligned}
J(u) &= \frac{(\tau\lambda^{-r})^p}{p} - \lambda \int_0^1 hF(u)\,dx \\
&\geq \frac{(\tau\lambda^{-r})^p}{p} - \lambda B_1 \int_0^1 h|u|^{q+1}\,dx - \lambda\tilde{B}_1\|h\|_1 \\
&\geq \frac{(\tau\lambda^{-r})^p}{p} - \lambda B_1\|h\|_1\|u\|_\infty^{q+1} - \lambda\tilde{B}_1\|h\|_1 \\
&\geq \frac{(\tau\lambda^{-r})^p}{p} - \lambda k^{q+1}B_1\|h\|_1\|u\|_W^{q+1} - \lambda\tilde{B}_1\|h\|_1 \\
&= \frac{(\tau\lambda^{-r})^p}{p} - \lambda k^{q+1}B_1\|h\|_1\left(\tau\lambda^{-r}\right)^{q+1} - \lambda\tilde{B}_1\|h\|_1 \\
&\geq \lambda^{-rp}\left(\frac{\tau^p}{2p} - \lambda^{1+rp}\tilde{B}_1\|h\|_1\right),
\end{aligned}
$$

where $\tau < \min\left\{\left(\frac{1}{2pB_1\|h\|_1k^{q+1}}\right)^{\frac{1}{r}}, c_1\right\}$ has now been chosen. Taking

$$
\tilde{\lambda} = \tau^{\frac{p}{1+rp}}\left(4p\tilde{B}_1\|h\|_1\right)^{-\frac{1}{1+rp}},
$$

we have $J(u) \geq c_2\tau^p\lambda^{-rp}$ for all $\lambda \in (0, \tilde{\lambda})$ which proves the claim.

□

4.1.1.1 Proof of Theorem 4.2

We have already established that $J \in C^1(W;\mathbb{R})$. Observe that $J(0) = 0$ and by Lemmas 4.3, 4.4, and 4.5, for $\lambda < \min\{\overline{\lambda}, \tilde{\lambda}\}$, we have satisfied hypotheses (PS), (MP1)-(MP3) of the Mountain Pass Theorem (where we note that the choice $\tau < c_1$ in Lemma 4.5 is sufficient to ensure $\|v\|_W > R$ in hypothesis (MP2)). Hence, there exists a solution u_λ to (1.5).

Remark. To show the simple existence of a mountain pass solution (not necessarily positive) to (1.5), we may choose $\|u\|_W$ sufficiently small and quickly get the desired result. However this solution likely has negative values, and therefore does not make sense in the context of the problems (1.3), since $f(s)$ is only defined for $s \geq 0$.

4.1.2 Positivity of Solution

Let u_λ be as in Theorem 4.2 as the mountain pass solution to (1.5). We first establish two a priori bounds on u_λ which are necessary for establishing positivity.

Lemma 4.6. *Let u_λ be as in Theorem 4.2. Then there exist an $M_0 > 0$ and $\hat{\lambda} > 0$ such that,*

$$M_0 \lambda^{-r} \leq \|u_\lambda\|_\infty,$$

for all $\lambda \in (0, \hat{\lambda})$.

Proof. Recall that $J(u_\lambda) \geq c_2 \tau^p \lambda^{-rp}$ for $\lambda \in (0, \tilde{\lambda})$, $0 > \hat{F} := \inf_{s \in \mathbb{R}} F(s) > -\infty$, and $f(s)s \leq \hat{B}\left(|s|^{q+1} + |s|\right)$ for all $s \in \mathbb{R}$, where $\hat{B} = \max\{B, \tilde{B}\}$. Letting

$$\hat{\lambda} = \min\left\{ \left(\frac{(p-1)c_2\tau^p}{p|\hat{F}|\|h\|_1} \right)^{\frac{1}{1+rp}}, (2\hat{B}\|h\|_1 c_2^{-1} \tau^{-p})^{-\frac{1}{1+rp}}, \tilde{\lambda} \right\},$$

we have

$$
\begin{aligned}
\lambda \int_0^1 hf(u_\lambda)u_\lambda \, dx &= \int_0^1 |u_\lambda'|^p \, dx \\
&= pJ(u_\lambda) + p\lambda \int_0^1 hF(u_\lambda) \, dx \\
&\geq pc_2\tau^p\lambda^{-rp} - p|\hat{F}|\|h\|_1\lambda \\
&\geq c_2\tau^p\lambda^{-rp}, \qquad (4.3)
\end{aligned}
$$

for $\lambda \in (0, \hat{\lambda})$. We further note that

$$
\begin{aligned}
c_2\tau^p\lambda^{-rp} &\leq \lambda \int_0^1 hf(u_\lambda)u_\lambda \, dx \\
&\leq \hat{B}\lambda \int_0^1 h\left(|u_\lambda|^{q+1} + |u_\lambda|\right) \, dx \\
&\leq \hat{B}\lambda \int_0^1 h\left(\|u_\lambda\|_\infty^{q+1} + \|u_\lambda\|_\infty\right) \, dx \\
&\leq \hat{B}\lambda\|h\|_1(\|u_\lambda\|_\infty^{q+1} + \|u_\lambda\|_\infty),
\end{aligned}
$$

so that for $\lambda < \hat{\lambda} \leq (2\hat{B}\|h\|_1 c_2^{-1}\tau^{-p})^{-\frac{1}{1+rp}}$, $\|u_\lambda\|_\infty \geq 1$. We also have that

$$
\begin{aligned}
\lambda \int_0^1 hf(u_\lambda)u_\lambda \, dx &\leq \hat{B}\lambda \int_0^1 h\left(|u_\lambda|^{q+1} + |u_\lambda|\right) \, dx \\
&\leq \hat{B}\lambda \int_0^1 h\left(\|u_\lambda\|_\infty^{q+1} + \|u_\lambda\|_\infty\right) \, dx \\
&\leq 2\hat{B}\lambda\|h\|_1\|u_\lambda\|_\infty^{q+1}, \qquad (4.4)
\end{aligned}
$$

since $\|u_\lambda\|_\infty \geq 1$. Combining (4.3) and (4.4) and taking $M_0 = \left(\frac{c_2\tau^p}{2\hat{B}\|h\|_1}\right)^{\frac{1}{q+1}}$, the claim is proven. $\square$

Lemma 4.7. *Let u_λ be as in Theorem 4.2. Then there exist $c_3 > 0$ and $\lambda^* > 0$ such that,*

$$\|u_\lambda\|_W^p \leq c_3 \lambda^{-rp}$$

for all $\lambda \in (0, \lambda^*)$.

Proof. Let $\Omega^+ = \{x \in [0,1] \mid u_\lambda(x) \geq 0\}$ and $\Omega^- = [0,1] \backslash \Omega^+$. Since u_λ is a critical point of J and using Proposition 2.6,

$$\begin{aligned}
\|u_\lambda\|_W^p &= pJ(u_\lambda) + p\lambda \int_{\Omega^-} hF(u_\lambda)\, dx + p\lambda \int_0^1 hF(u_\lambda)\, dx - p\lambda \int_{\Omega^-} hF(u_\lambda)\, dx \\
&\leq pJ(u_\lambda) + p\lambda \int_{\Omega^-} hu_\lambda f(0)\, dx + p\lambda \int_0^1 h \left(\frac{u_\lambda f(u_\lambda)}{\theta} + \frac{\tilde{\theta}}{\theta} \right) dx \\
&\qquad - p\lambda \int_{\Omega^-} h \left(\frac{u_\lambda f(0)}{\theta} + \frac{\tilde{\theta}}{\theta} \right) dx \\
&= pJ(u_\lambda) + p\lambda \left(1 - \frac{1}{\theta}\right) \int_{\Omega^-} hu_\lambda f(0)\, dx - p\lambda \int_{\Omega^-} h\frac{\tilde{\theta}}{\theta}\, dx \\
&\qquad + \frac{p\lambda}{\theta} \int_0^1 hu_\lambda f(u_\lambda)\, dx + p\lambda \frac{\tilde{\theta}}{\theta} \|h\|_1 \\
&\leq pJ(u_\lambda) + p\lambda k |f(0)| \|h\|_1 \|u_\lambda\|_W + \frac{p}{\theta} \|u_\lambda\|_W^p + p\lambda \frac{\tilde{\theta}}{\theta} \|h\|_1. \qquad (4.5)
\end{aligned}$$

On the other hand, by the mountain pass characterization of u_λ,

$$\begin{aligned}
J(u_\lambda) &\leq \max_{s \geq 0} \{J(sv_1)\} \\
&\leq \max_{s \geq 0} \left\{ \frac{s^p}{p} - \lambda A_1 s^{q+1} \hat{h} \|v_1\|_{q+1}^{q+1} + \lambda \tilde{A}_1 \|h\|_1 \right\}, \qquad (4.6)
\end{aligned}$$

as in (4.2). Let

$$p(s) := \frac{s^p}{p} - \lambda A_1 s^{q+1}\hat{h}\|v_1\|_{q+1}^{q+1} + \lambda\tilde{A}_1\|h\|_1,$$

so that by solving $p'(s) = 0$, we find that $p(s)$ is maximized when $s = \bar{K}\lambda^{-r}$ where

$$\bar{K} = \left(A_1(q+1)\hat{h}\|v_1\|_{q+1}^{q+1}\right)^{-r}.$$

Hence, if $\lambda \leq 1$, then $\lambda^{-rp} \geq \lambda$, and therefore,

$$\begin{aligned}
pJ(u_\lambda) + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 &\leq \bar{K}^p\lambda^{-rp} - p\lambda A_1\hat{h}\bar{K}^{q+1}\lambda^{-r(q+1)}\|v_1\|_{q+1}^{q+1} + \lambda p\left(\tilde{A}_1 + \frac{\tilde{\theta}}{\theta}\right)\|h\|_1 \\
&\leq \bar{K}^p\lambda^{-rp} - pA_1\hat{h}\bar{K}^{q+1}\lambda^{-rp}\|v_1\|_{q+1}^{q+1} + \lambda^{-rp}p\left(\tilde{A}_1 + \frac{\tilde{\theta}}{\theta}\right)\|h\|_1 \\
&\leq \left(\bar{K}^p - pA_1\hat{h}\bar{K}^{q+1}\|v_1\|_{q+1}^{q+1} + p\left(\tilde{A}_1 + \frac{\tilde{\theta}}{\theta}\right)\|h\|_1\right)\lambda^{-rp} \\
&= \tilde{c}_3\lambda^{-rp}, \qquad (4.7)
\end{aligned}$$

where $\tilde{c}_3 = \bar{K}^p - pA_1\hat{h}\bar{K}^{q+1}\|v_1\|_{q+1}^{q+1} + p\left(\tilde{A}_1 + \frac{\tilde{\theta}}{\theta}\right)\|h\|_1$.

By Lemma 4.6, if $\lambda < \min\left\{\hat{\lambda}, \left(\frac{k}{M_0}\right)^{-\frac{1}{r}}\right\}$, then

$$\|u_\lambda\|_W \geq \frac{1}{k}\|u_\lambda\|_\infty \geq \frac{M_0}{k}\lambda^{-r} \geq 1.$$

From (4.5) and (4.7), we have that

$$a\|u_\lambda\|_W^p \leq b\lambda\|u_\lambda\|_W + \tilde{c}_3\lambda^{-rp},$$

for $a = 1 - \frac{p}{\theta} > 0$ and $b = pk|f(0)|\|h\|_1 > 0$. Since $\|u_\lambda\|_W \geq 1$,

$$a\|u_\lambda\|_W^p \leq b\lambda\|u_\lambda\|_W^p + \tilde{c}_3\lambda^{-rp}.$$

Hence if $\lambda \leq \frac{a}{2b} = \frac{\theta-p}{2\theta pk|f(0)|\|h\|_1}$, then

$$(a - b\lambda)\|u_\lambda\|_W^p \leq \tilde{c}_3\lambda^{-rp},$$

implies that

$$\frac{1}{2}a\|u_\lambda\|_W^p \leq \tilde{c}_3\lambda^{-rp}.$$

The lemma is proven taking $c_3 = \frac{2\tilde{c}_3}{a}$ and $\lambda^* = \min\left\{1, \hat{\lambda}, \left(\frac{k}{M_0}\right)^{-\frac{1}{r}}, \frac{\theta-p}{2\theta pk|f(0)|\|h\|_1}\right\}$. □

4.1.2.1 Proof of Theorem 1.3

We prove the theorem by contradiction. Suppose there exists a sequence

$$\{(\lambda_j, u_{\lambda_j})\}_{j=1}^{\infty} \subset (0,1) \times C^1[0,1]$$

of mountain pass solutions to (1.5) as in Theorem 4.2, such that $\lambda_j \to 0$, and $m\left(\left\{x \in (0,1)|u_{\lambda_j}(x) \leq 0\right\}\right) > 0$. Let $w_j = \frac{u_{\lambda_j}}{\|u_{\lambda_j}\|_\infty}$. Then we have,

$$-\left(\phi_p\left(w_j'\right)\right)' = \lambda_j h \frac{f(u_{\lambda_j})}{\|u_{\lambda_j}\|_\infty^{p-1}}.$$

By (F2) and Lemmas 4.6 and 4.7,

$$\begin{aligned}\left|\lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_\infty^{1-p}\right| &\leq \lambda_j \hat{B}\left(\|u_{\lambda_j}\|_\infty^{q+1-p} + \|u_{\lambda_j}\|_\infty^{1-p}\right)\\ &\leq \lambda_j \hat{B}\left(k^{\frac{1}{r}}\|u_{\lambda_j}\|_W^{\frac{1}{r}} + M_0^{1-p}\lambda^{-r(1-p)}\right)\\ &\leq \lambda_j \hat{B} c_4\left(\lambda_j^{-1} + \lambda_j^{-r(1-p)}\right),\end{aligned} \tag{4.8}$$

where $c_4 = \max\{(c_3 k)^{\frac{1}{r}}, M_0^{1-p}\}$. Hence, we observe from (4.8) that

$$\begin{aligned}\left|\lambda_j f(u_{\lambda_j})\|u_{\lambda_j}\|_\infty^{1-p}\right| &\leq \hat{B}c_4 + \hat{B}c_4\lambda_j^{-r(1-p)+1}\\ &\leq \hat{B}c_4 + \hat{B}c_4\lambda_j^{\overline{\frac{q}{q+1-p}}}\\ &\leq 2\hat{B}c_4,\end{aligned} \tag{4.9}$$

for λ_j sufficiently small. Hence $\lambda_j f(u_{\lambda_j}(x))\|u_{\lambda_j}\|_\infty^{1-p}$ converges to a limit, $z_1(x)$, for every $x \in [0,1]$. Furthermore, since $\lambda_j\|u_{\lambda_j}\|_\infty^{1-p} \to 0$ as $j \to \infty$ and f is bounded from below,

$$\begin{aligned}z_1(x) &= \lim_{j\to\infty} \lambda_j f(u_{\lambda_j}(x))\|u_{\lambda_j}\|_\infty^{1-p}\\ &\geq \lim_{j\to\infty} -\lambda_j|f(0)|\|u_{\lambda_j}\|_\infty^{1-p}\\ &= 0.\end{aligned}$$

Therefore,

$$\lambda_j h(x) f(u_{\lambda_j}(x))\|u_{\lambda_j}\|_\infty^{1-p} \to h(x)z_1(x) =: z(x) \quad \forall x \in (0,1],$$

and $z(x) \geq 0$ for all $x \in (0,1]$.

Let $x_j \in (0,1)$ be a maximum of $w_j(x)$. Then,

$$\begin{aligned}\phi_p(w_j'(x)) &= \int_x^{x_j} -\left(\phi_p(w_j'(s)\right)' \, ds \\ &= \int_x^{x_j} \lambda_j h(s) f(u_{\lambda_j}(s)) \|u_{\lambda_j}\|_\infty^{1-p} \, ds.\end{aligned}$$

By (4.9), this implies that $|\phi_p(w_j'(x))| \leq 2\hat{B}c_4\|h\|_1$ for all $x \in [0,1]$, and therefore $|w_j'(x)| \leq \left(2\hat{B}c_4\|h\|_1\right)^{\frac{1}{p-1}}$ for all $x \in [0,1]$. By the Arzela-Ascoli Theorem, this implies that there exists $w \in C[0,1]$ so that $w_j \to w$ in $C[0,1]$.

Meanwhile, again by (4.9), we have that $\left|\lambda_j h(x) f(u_{\lambda_j}(x))\|u_{\lambda_j}\|_\infty^{1-p}\right| \leq 2\hat{B}c_4 h(x)$ for all $x \in (0,1]$. Since $h \in L^1(0,1)$, by the Lebesgue Dominated Convergence Theorem, we may choose a subsequence u_{λ_j} with $x_j \to x_0$ so that

$$\int_x^{x_j} \lambda_j h(s) f(u_{\lambda_j}(s)) \|u_{\lambda_j}\|_\infty^{1-p} \, ds \to \int_x^{x_0} h(s) z_1(s) \, ds = \int_x^{x_0} z(s) \, ds.$$

Hence,

$$\phi_p^{-1}\left(\int_x^{x_j} \lambda_j h(s) f(u_{\lambda_j}(s)) \|u_{\lambda_j}\|_\infty^{1-p} \, ds\right) \to \phi_p^{-1}\left(\int_x^{x_0} z(s) \, ds\right),$$

and therefore

$$\begin{aligned}&\int_0^t \phi_p^{-1}\left(\int_x^{x_j} \lambda_j h(s) f(u_{\lambda_j}(s)) \; \|u_{\lambda_j}\|_\infty^{1-p} \, ds\right) \, dx \\ &\qquad\qquad \to \int_0^t \phi_p^{-1}\left(\int_x^{x_0} z(s) \, ds\right) \, dx.\end{aligned}$$

Furthermore, observe that $w_j(t) \to \int_0^t \phi_p^{-1}\left(\int_x^{x_0} z(s)\, ds\right)\, dx = w(t)$, and consequently

$$\begin{aligned} w_j'(t) &= \phi_p^{-1}\left(\int_t^{x_j} \lambda_j h(s) f(u_{\lambda_j}(s)) \|u_{\lambda_j}\|_\infty^{1-p}\, ds\right) \\ &\to \phi_p^{-1}\left(\int_t^{x_0} z(s)\, ds\right) \\ &= w'(t), \end{aligned}$$

for all $t \in [0,1]$.

Hence, $-\left(\phi_p\left(w'\right)\right)' = z \geq 0$ with $w(0) = 0 = w(1)$. Since $\|w_j\|_\infty = 1$, $w \not\equiv 0$ and since w is concave, $w > 0$ in $(0,1)$, $w'(0) > 0$, and $w'(1) < 0$. Since $w_j \to w$ in $C^1[0,1]$, then $w_j(x) > 0$ for all $x \in (0,1)$ for j sufficiently large. Hence, $u_{\lambda_j}(x) > 0$ for all $x \in (0,1)$ for j sufficiently large, which implies that $m\left(\{x \in (0,1); u_{\lambda_j}(x) \leq 0\}\right) = 0$ for all j sufficiently large, a contradiction. Therefore, there exists some $\check{\lambda}$ such that (1.5) has a positive solution for all $\lambda \in (0, \check{\lambda})$.

Remark. By Lemmas 4.6 and 4.7, we have

$$\|w_j\|_W \leq \frac{c_3}{M_0},$$

where we note that c_3 and M_0 are independent of λ, and therefore independent of j. In the case that $h \in C[0,1]$ (that is, $\mu \geq \frac{N-p}{p-1}$), Proposition 3.7 in [dFGU09] implies that the sequence $\{w_j\}_{j=1}^\infty$ is uniformly bounded in $C_0^{1,\beta}[0,1]$ for some $\beta \in (0,1)$. We could then conclude that $w \in C^{1,\beta^*}[0,1]$ for some $\beta^* \in (0,\beta)$. This makes the proof simpler when $h \in C[0,1]$.

4.2 Proof of Theorem 1.4

We begin by establishing the appropriate variational formulation of the problem. Let $\widetilde{W} := \widetilde{W}^{1,p}(0,1)$ and take $\|\cdot\|_{\widetilde{W}} = \|\cdot\|_{1,p,0}$. Let E be defined on $\widetilde{W}$ as

$$E(u) = J(u) + g(u(1)), \tag{4.10}$$

where

$$g(s) = \int_0^s c(z)\phi_p(z)\,dz,$$

and $J(u)$ is as in (4.1). Once again, the compact embedding of $W^{1,p}(0,1)$ into $C[0,1]$ implies that E is well defined.

Since we have already established that J is a C^1 functional, we need only to show that $H(u) := g(u(1))$ is C^1. Fix $u \in \widetilde{W}$ so that for any $v \in \widetilde{W}$, $\langle H'(u), v\rangle = g'(u(1))v(1)$. It is clear that the function $g(s)$ as previously defined is continuously differentiable, and further, since pointwise evaluation is a continuous operation, we may conclude that the $H'(u)$ is a continuous functional on $\widetilde{W}$. Hence, $E(u)$ is a C^1 functional as it is the sum of two C^1 functionals.

By Proposition 2.4, we may continue our analysis using

$$\|u\|_{\widetilde{W}} = \|u\|_{1,p,0}.$$

Lemma 4.8. *The critical point $u \in \widetilde{W}$ of* (4.10) *is a solution of* (1.6).

Proof. If u is a critical point of (4.10), then

$$\int_0^1 \phi_p(u'(s))v'(s)\,ds + g'(u(1))v(1) = \lambda \int_0^1 h(s)f(u(s))v(s)\,ds \quad \forall v \in C_0^\infty[0,1].$$

Using integration by parts and the fact that $v(1) = 0$, we have that

$$\int_0^1 ((\phi_p(u'(s))' + \lambda h(s)f(u(s)))\,v(s)\,ds = 0 \quad \forall v \in C_0^\infty[0,1].$$

As in the proof of Lemma 4.1, we have that $(\phi_p(u'(x)))' = -\lambda h(x)f(u(x))$ for all $x \in (0,1)$, $\phi_p(u') \in W^{1,1}(0,1)$, and $u \in C^1[0,1]$.

Clearly, $u(0) = 0$ since $u \in \widetilde{W}$. Let $\tilde{C} = \{v \in C^\infty[0,1] \mid v(0) = 0\}$. Then since $\tilde{C} \subset \widetilde{W}$ and u is a critical point of (4.10),

$$\int_0^1 \phi(u'(s))v'(s)\,ds + g'(u(1))v(1) = \lambda \int_0^1 h(s)f(u(s))v(s)\,ds \quad \forall v \in \tilde{C}.$$

Hence, using integration by parts,

$$\begin{aligned}\phi_p(u'(1))v(1) - &\int_0^1 (\phi_p(u'(s)))'\,v(s)\,ds + g'(u(1))v(1) \\ &= \lambda \int_0^1 h(s)f(u(s))v(s)\,ds \quad \forall v \in \tilde{C},\end{aligned}$$

which implies that for all $v \in \tilde{C}$,

$$\begin{aligned}(\phi_p(u'(1)) + c(u(1))\phi_p(u(1)))\,v(1) &= \phi_p(u'(1))v(1) + g'(u(1))v(1) \\ &= \int_0^1 \left((\phi_p(u'(s)))' + \lambda h(s)f(u(s))\right)v(s)\,ds \\ &= 0\end{aligned}$$

since $(\phi_p(u'(x)))' + \lambda h(x) f(u(x)) = 0$ almost everywhere in $(0,1)$. Since $v(1)$ is arbitrary, we may conclude that $\phi_p(u'(1)) + c(u(1))\phi_p(u(1)) = 0$, and therefore the boundary conditions are satisfied. □

4.2.1 Existence of a Mountain Pass Solution

Again, our goal will be to establish the existence of a mountain pass solution.

Theorem 4.9. *For $\lambda \approx 0$, the hypotheses of the Mountain Pass Theorem are satisfied, and there exists a solution u_λ to* (1.6)*.*

We again establish several lemmas which will help to prove the theorem.

Lemma 4.10. *The map E satisfies the Palais-Smale condition.*

Proof. As before, we first wish to show that any sequence, $\{u_n\}$ satisfying the hypotheses of (PS) must be bounded. Assume to the contrary that $\{u_n\}$ is a sequence such that $E'(u_n) \to 0$, there exists some $M > 0$ such that $|E(u_n)| < M$ for all $n \geq 1$, and $\|u_n\|_{\widetilde{W}} \to \infty$. Then, choosing $\theta > p$ satisfying (AR1) and (AR2), we note that

$$\lim_{n\to\infty} \frac{\theta E(u_n) - \langle E'(u_n), u_n\rangle}{\|u_n\|_{\widetilde{W}}} = 0.$$

Also note that

$$\begin{aligned} \theta E(u_n) - \langle E'(u_n), u_n\rangle &= (\theta J(u_n) - \langle J'(u_n), u_n\rangle) \\ &\qquad + (\theta g(u_n(1)) - c(u_n(1))(u_n(1))^p) \\ &\geq \left(\frac{\theta}{p} - 1\right) \|u_n\|_{\widetilde{W}}^p - \lambda\tilde{\theta}\|h\|_1 \\ &\qquad - \lambda k(\theta - 1)|f(0)|\|u_n\|_{\widetilde{W}}\|h\|_1 + \tilde{\theta}_1, \end{aligned}$$

by combining the earlier estimate on $\theta J(u_n) - \langle J'(u_n), u_n \rangle$ with (AR2). But this implies,

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{\theta E(u_n) - \langle E'(u_n), u_n \rangle}{\|u_n\|_{\widetilde{W}}} \\
&\geq \lim_{n\to\infty} \frac{\left(\frac{\theta}{p} - 1\right) \|u_n\|_{\widetilde{W}}^p - \lambda\tilde{\theta}\|h\|_1 - \lambda k(\theta - 1)|f(0)|\|u_n\|_{\widetilde{W}}\|h\|_1 + \tilde{\theta}_1}{\|u_n\|_{\widetilde{W}}} \\
&= \infty,
\end{aligned}$$

a contradiction. Hence, $\{u_n\}$ is bounded in $\widetilde{W}$, and therefore contains a subsequence which converges weakly in $\widetilde{W}$ and strongly in $C[0, 1]$.

Since $u_n \to u$ strongly in $C[0, 1]$, then

$$\lim_{n\to\infty} \int_0^1 hf(u_n)(u_n - u)\, dx \to 0.$$

Furthermore, since $\{u_n\}$ is a Palais-Smale sequence, $E'(u_n) \to 0$. Therefore, since $u_n - u$ is bounded in $\widetilde{W}$, we obtain

$$\lim_{n\to\infty} \langle E'(u_n), u_n - u \rangle \to 0.$$

Finally, we note that

$$c(u_n(1))\phi_p(u_n(1))(u_n(1) - u(1)) \to 0$$

since $u_n \to u$ strongly in $C[0,1]$ implies pointwise convergence and c, ϕ_p are both continuous functions. Hence,

$$\begin{aligned}\langle E'(u_n), u_n - u\rangle + \lambda \int_0^1 & hf(u_n)(u_n - u)\, dx \\ & - c(u_n(1)) \cdot \phi_p(u_n(1)) \cdot (u_n(1) - u(1)) \\ &= \langle \Psi'(u_n), u_n - u\rangle \\ &\to 0.\end{aligned}$$

Therefore, by the (S^+) property, $u_n \to u$ strongly in $\widetilde{W}$, and so E satisfies (PS). □

The following two lemmas are analogous to Lemmas 4.4 and 4.5 presented in the Dirichlet case, and rely heavily on the estimates there.

Lemma 4.11. *Let u and $\overline{\lambda} > 0$ be as in Lemma 4.4. Then for $\lambda \in (0, \overline{\lambda})$, $E(u) < 0$.*

Proof. Choose $v_1 \in W \subset \widetilde{W}$ as in the proof of Lemma 4.4. Then $E(sv_1) = J(sv_1) + g(sv_1(1)) = J(sv_1)$ since $v_1(1) = 0$ and $g(0) = 0$. The conclusion follows from Lemma 4.4. □

Lemma 4.12. *Let $\tau \in (0, c_1)$ and $c_2, \tilde{\lambda} > 0$ be as in Lemma 4.5. Then if $\|u\|_{\widetilde{W}} = \tau\lambda^{-r}$, $E(u) \geq c_2(\tau\lambda^{-r})^p$ for all $\lambda \in (0, \tilde{\lambda})$.*

Proof. Since $g(s) \geq 0$ for all $s \in \mathbb{R}$, we have that $E(u) \geq J(u)$. From Lemma 4.5, $J(u) \geq c_2(\tau\lambda^{-r})^p$ for all $\lambda \in (0, \tilde{\lambda})$. This completes the proof. □

4.2.1.1 Proof of Theorem 4.9

Again, $E \in C^1(W, \mathbb{R})$, $E(0) = 0$ and by Lemmas 4.10, 4.11, and 4.12, for $\lambda < \min\{\overline{\lambda}, \tilde{\lambda}\}$, we have satisfied hypotheses (PS) and (MP1)-(MP3) of the Mountain Pass Theorem. Hence, there exists a solution u_λ to (1.6).

4.2.2 Positivity of Solution

To utilize the argument as in the proof of Theorem 1.3, we need two lemmas as before.

Lemma 4.13. *Let u_λ be as in Theorem 4.9. For $M_0 > 0$ and $\hat{\lambda} > 0$ as in Lemma 4.6,*

$$M_0 \lambda^{-r} \leq \|u_\lambda\|_\infty,$$

for all $\lambda \in (0, \hat{\lambda})$.

Proof. Using the same notation as in the proof of Lemma 4.6, since u_λ is a solution to (1.6), we have that

$$\begin{aligned} \lambda \int_0^1 h f(u_\lambda) u_\lambda \, dx &= \int_0^1 |u_\lambda'|^p \, dx + c(u_\lambda(1))\phi_p(u_\lambda(1))u_\lambda(1) \\ &= pJ(u_\lambda) + p\lambda \int_0^1 hF(u_\lambda) \, dx + c(u_\lambda(1)) \, |u_\lambda(1)|^p \\ &\geq pc_2\lambda^{-rp} - p|\hat{F}| \|h\|_1 \lambda \\ &\geq c_2 \lambda^{-rp}, \end{aligned} \tag{4.11}$$

for $\lambda \in (0, \hat{\lambda})$. The conclusion follows from the argument in the proof of Lemma 4.6. □

Lemma 4.14. *Let u_λ be as in Theorem 4.9. There exist $C_3 > 0$ and $\Lambda^* > 0$ such that,*

$$\|u_\lambda\|_{\widetilde{W}}^p \leq C_3 \lambda^{-rp}$$

for all $\lambda \in (0, \Lambda^*)$.

Proof. Since u_λ is a critical point of E and using Proposition 2.6,

$$\begin{aligned}
\|u_\lambda\|_{\widetilde{W}}^p &= pE(u_\lambda) + p\lambda \int_{\Omega^-} hF(u_\lambda)\,dx + p\lambda \int_{\Omega^+} hF(u_\lambda)\,dx - pg(u_\lambda(1)) \\
&\le pE(u_\lambda) + p\lambda \int_{\Omega^-} hu_\lambda f(0)\,dx + p\lambda \int_0^1 h\left(\frac{u_\lambda f(u_\lambda)}{\theta} + \frac{\tilde{\theta}}{\theta}\right)\,dx \\
&\qquad - p\lambda \int_{\Omega^-} h\left(\frac{u_\lambda f(0)}{\theta} + \frac{\tilde{\theta}}{\theta}\right)\,dx - pg(u_\lambda(1)) \\
&= pE(u_\lambda) + p\lambda\left(1 - \frac{1}{\theta}\right)\int_{\Omega^-} hu_\lambda f(0)\,dx - p\lambda \int_{\Omega^-} h\frac{\tilde{\theta}}{\theta}\,dx \\
&\qquad + \frac{p\lambda}{\theta}\int_0^1 hu_\lambda f(u_\lambda)\,dx + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 - pg(u_\lambda(1)) \\
&\le pE(u_\lambda) + p\lambda k|f(0)|\|h\|_1\|u_\lambda\|_{\widetilde{W}} \\
&\qquad + \frac{p}{\theta}\|u_\lambda\|_{\widetilde{W}}^p + \frac{p}{\theta}c(u_\lambda(1))\phi_p\left(u_\lambda(1)\right)u_\lambda(1) + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 - pg(u_\lambda(1)) \\
&= pE(u_\lambda) + p\lambda k|f(0)|\|h\|_1\|u_\lambda\|_{\widetilde{W}} + \frac{p}{\theta}\|u\|_{\widetilde{W}}^p + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 \\
&\qquad + \frac{p}{\theta}\left(c(u_\lambda(1))\,|u_\lambda(1)|^p - \theta g(u_\lambda(1))\right) \\
&\le pE(u_\lambda) + p\lambda k|f(0)|\|h\|_1\|u_\lambda\|_{\widetilde{W}} + \frac{p}{\theta}\|u_\lambda\|_{\widetilde{W}}^p + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 - p\frac{\tilde{\theta}_1}{\theta}.
\end{aligned}$$

Finally, if we choose $\lambda \le \left(\frac{|\tilde{\theta}_1|}{M_0}\right)^{-\frac{1}{rp}}$, then $-\tilde{\theta}_1 \le M_0\lambda^{-rp}$, so that

$$\begin{aligned}
\|u_\lambda\|_{\widetilde{W}}^p &\le pE(u_\lambda) + p\lambda k|f(0)|\|h\|_1\|u_\lambda\|_{\widetilde{W}} + \frac{p}{\theta}\|u_\lambda\|_{\widetilde{W}}^p + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 - p\frac{\tilde{\theta}_1}{\theta} \\
&\le pE(u_\lambda) + p\lambda k|f(0)|\|h\|_1\|u_\lambda\|_{\widetilde{W}} + \frac{p}{\theta}\|u_\lambda\|_{\widetilde{W}}^p \\
&\qquad + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 + p\frac{M_0\lambda^{-rp}}{\theta}.
\end{aligned} \tag{4.12}$$

By the mountain pass characterization of u_λ,

$$
\begin{aligned}
E(u_\lambda) &\leq \max_{s\geq 0}\{E(sv_1)\} \\
&= \max_{s\geq 0}\{J(sv_1)\} \\
&\leq \max_{s\geq 0}\left\{\frac{s^p}{p} - \lambda A_1 s^{q+1}\hat{h}\|v_1\|_{q+1}^{q+1} + \lambda\tilde{A}_1\|h\|_1\right\},
\end{aligned}
\tag{4.13}
$$

by (4.2).

Now, note that the inequality (4.13) is identical to the inequality (4.6), except that the functional J has now been replaced by the functional E. Hence, we may conclude from (4.13) that

$$
pE(u_\lambda) + p\lambda\frac{\tilde{\theta}}{\theta}\|h\|_1 \leq \tilde{c}_3\lambda^{-rp}, \tag{4.14}
$$

where $\tilde{c}_3 = \bar{K}^p - pA_1\hat{h}\bar{K}^{q+1}\|v_1\|_{q+1}^{q+1} + p\left(\tilde{A}_1 + \frac{\tilde{\theta}}{\theta}\right)\|h\|_1$ as in Lemma 4.7.

Hence, following the proof of Lemma 4.7, we may combine (4.12) and (4.14) to observe that

$$
a\|u_\lambda\|_{\widetilde{W}}^p \leq b\lambda\|u_\lambda\|_{\widetilde{W}} + \tilde{C}_3\lambda^{-rp}
$$

for $a = 1 - \frac{p}{\theta} > 0$, $b = pk|f(0)|\|h\|_1 > 0$, and $\tilde{C}_3 = \tilde{c}_3 + \frac{pM_0}{\theta}$. Now, choosing $\lambda \leq \frac{a}{2b}$ and taking $C_3 = \frac{2\tilde{C}_3}{a}$, we may follow the proof of Lemma 4.7 to conclude that

$$
\|u_\lambda\|_{\widetilde{W}}^p \leq C_3\lambda^{-rp}
$$

for all $\lambda \in (0, \Lambda^*)$, where $\Lambda^* = \min\left\{1, \hat{\lambda}, \left(\frac{|\tilde{\theta}_1|}{M_0}\right)^{-\frac{1}{rp}}, \frac{\theta - p}{2\theta pk|f(0)|\|h\|_1}\right\}$. □

4.2.2.1 Proof of Theorem 1.4

We again prove the theorem by contradiction. Suppose there exists a sequence $\{(\lambda_j, u_{\lambda_j})\}_{j=1}^{\infty} \subset (0,1) \times C^1[0,1]$ of mountain pass solutions to (1.6) as in Theorem 4.9, such that $\lambda_j \to 0$ and $m\left(\{x \in (0,1) | u_{\lambda_j}(x) \le 0\}\right) > 0$.

Let $w_j = \frac{u_{\lambda_j}}{\|u_{\lambda_j}\|_\infty}$. Then

$$\begin{cases} -\left(\phi_p(w_j')\right)' = \lambda h \frac{f(u_{\lambda_j})}{\|u_{\lambda_j}\|_\infty^{p-1}}; \quad x \in (0,1), \\ w_j(0) = 0, \\ \phi_p(w_j'(1)) + c(u_{\lambda_j}(1))\phi_p(w_j(1)) = 0, \end{cases} \tag{4.15}$$

and as in the proof of Theorem 1.3, $w_j \to w$ strongly in $C^1[0,1]$ with w satisfying,

$$\begin{cases} -\left(\phi_p(w')\right)' = z; \quad x \in (0,1), \\ w(0) = 0, \\ \phi_p(w'(1)) + c(L)\phi_p(w(1)) = 0, \end{cases} \tag{4.16}$$

where $L = \lim_{j\to\infty} u_{\lambda_j}(1)$.

Since $\|w_j\|_\infty = 1$, $w \not\equiv 0$. Furthermore, since $z \ge 0$ and $c(L) > 0$, w is concave and satisfies the nonlinear boundary condition at $x = 1$ so that $w'(0) > 0$, $w'(1) < 0$, $w(1) > 0$, and $w > 0$ in $(0,1)$. The conclusion follows from the same argument as in the proof of Theorem 1.3.

CHAPTER V

COMPUTATIONALLY GENERATED BIFURCATION CURVES FOR AUTONOMOUS PROBLEMS

Throughout this chapter, we are interested in problems (1.9) and (1.10) in the case $h(t) = 1$ for all $t \in (0, 1)$ (i.e., the autonomous case). Specifically, we consider two-point boundary value problems of the form,

$$\begin{cases} -u''(t) = \lambda f(u(t)); \quad t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{5.1}$$

and

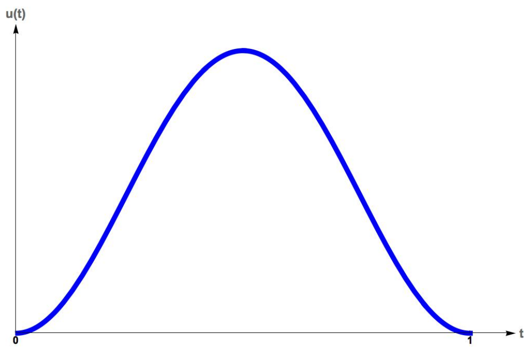
$$\begin{cases} -u''(t) = \lambda f(u(t)); \quad t \in (0,1), \\ u(0) = 0, \\ u'(1) = -c(u(1))u(1), \end{cases} \tag{5.2}$$

where $f : [0, \infty) \to \mathbb{R}$ is a continuously differentiable function and $c : [0, \infty) \to (0, \infty)$ is a continuous function. Here, we study positive solutions of (5.1) and (5.2) when the function f satisfies the hypothesis,

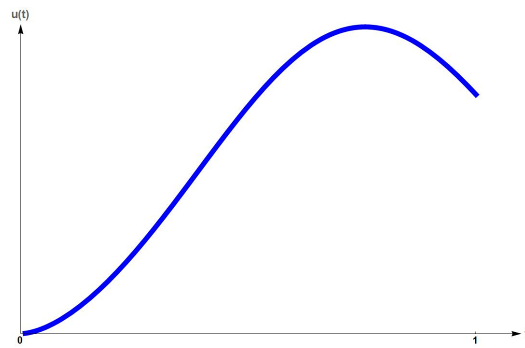
(S) there exist unique $\beta, \theta > 0$ such that $f(s) < 0$ for $s \in [0, \beta)$, $f(s) > 0$ for $s \in (\beta, \infty)$, and $F(\theta) = 0$.

We note that any solution of (5.1) or (5.2) must be symmetric about any point $t_0 \in (0, 1)$ where $u'(t_0) = 0$ (see the proof of Lemma 5.6). Further, in the case of (5.2), since we will be only interested in the case where $u(1) > 0$, we must have

$u'(1) < 0$. Now, when (S) is satisfied, solutions are convex near $t = 0$ and $t = 1$ in the Dirichlet case, and convex near $t = 0$ (and possibly near $t = 1$) in the case of nonlinear boundary conditions, and concave otherwise. See Figure 5 for examples.



(a) Shape of Solution to (5.1)



(b) Shape of Solution to (5.2)

Figure 5. Possible Solution Shapes for Semipositone Problems

Of particular interest in this chapter is the shape of the corresponding bifurcation curves. Laetsch studied bifurcation curves of (5.1) in [Lae71] using a quadrature method (or time map analysis). He established the following relationship between the parameter λ and $\|u\|_\infty$.

Theorem 5.1 (see [Lae71])**.** *There exists a positive solution $u \in C^2[0,1]$ of (5.1) with $\|u\|_\infty = \rho$ if and only if*

$$\lambda = 2\left(\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}}\right)^2. \tag{5.3}$$

Further, for a (λ, ρ) satisfying (5.3), (5.1) has a positive solution u given by $u\left(\frac{1}{2}\right) = \rho$,

$$t\sqrt{2\lambda} = \int_0^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}; \quad t \in \left(0, \frac{1}{2}\right),$$

and $u(t) = u(1-t)$ *for all* $t \in \left(\frac{1}{2}, 1\right)$.

The ideas of Laetsch have been been adapted to problems with a number of different boundary conditions, for example Neumann (see [MS93]), mixed (see [AMS99]), and nonlinear boundary conditions (see [GPS17]). In particular, in [GPS17], the authors study a certain example of c arising in population dynamics involving density dependent dispersal on the boundary. Here, we expand the ideas in [GPS17] for general classes of c when f satisfies (S). In particular, we provide more detailed analysis of the quadrature method for such two-point boundary value problems involving nonlinear boundary conditions. Namely, we establish:

Theorem 5.2. *For* f *satisfying* (S), *there exists a positive solution* $u \in C^2(0,1) \cap C^1[0,1]$ *of* (5.2) *with* $\|u\|_\infty = \rho$, $u(1) = q$, *and* $0 < q < \rho$ *if and only if*

$$\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{c(q)q}{\sqrt{F(\rho) - F(q)}} = 0, \tag{5.4}$$

and

$$\sqrt{2\lambda} = \frac{c(q)q}{\sqrt{F(\rho) - F(q)}}, \tag{5.5}$$

hold. Further, for a (λ, ρ, q) *satisfying* (5.4) *and* (5.5), (5.2) *has a positive solution given by*

$$t\sqrt{2\lambda} = \int_0^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}; \quad t \in [0, t_0),$$

$$(1-t)\sqrt{2\lambda} = \int_q^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}; \quad t \in (t_0, 1],$$

$u(t_0) = \rho$ *and* $u(1) = q$, *where* t_0 *satisfies*

$$t_0 = \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \; \Big/ \; \left(\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \right).$$

Theorem 5.3. *If* f *satisfies (S), then for every* $\rho \geq \theta$, *there exists* $q > 0$ *so that* (5.4) *is satisfied.*

Theorem 5.4. *If* f *satisfies (S),* $c(s)s$ *is continuously differentiable, and either*

(S1) $\frac{s+c(s)s}{\sqrt{-F(s)}}$ *is nondecreasing for* $s \in (0, \beta)$ *and* $s + c(s)s$ *is nondecreasing for all* $s > 0$, *or*

(S2) $(f(s)c(s)s)' > 2f(s)$ *for* $s \in (0, \beta)$ *and* $c(s)s$ *is nondecreasing for all* $s > 0$,

is satisfied, then for each fixed $\rho \geq \theta$, *there exists a unique* $q > 0$ *so that* (5.4) *is satisfied.*

In Section 5.1, we prove Theorems 5.2-5.4. In Sections 5.2 and 5.3, we provide Mathematica-generated plots of the bifurcation curves for some specific superlinear semipositone problems with Dirichlet and nonlinear boundary conditions, respectively, and highlight interesting behavior of solutions in Section 5.4. Finally, in Section 5.5, we present an interesting example and its bifurcation diagram where the hypotheses of Theorem 5.4 are violated, and for fixed ρ in a certain range, there exist multiple values of q satisfying (5.4).

5.1 Proofs of Theorems 5.2-5.4

5.1.1 Proof of Theorem 5.2

First we establish the following two lemmas needed to prove our results.

Lemma 5.5. *If f satisfies (S) and $\rho < \theta$, then there is no $\lambda > 0$ for which* (5.2) *has a positive solution, u, satisfying $\|u\|_\infty = \rho$.*

Proof. Assume to the contrary that u is a positive solution to (5.2) for some $\lambda > 0$ such that $\|u\|_\infty = \rho < \theta$. Note that $u'(1) < 0$ since we are only interested in the case where $u(1) > 0$. Hence, there exists $t_0 \in (0,1)$ such that $u'(t_0) = 0$ and $u(t_0) = \rho$. Now, multiplying the differential equation by u', we obtain:

$$-\left[\frac{(u'(t))^2}{2}\right]' = \lambda\left(F(u(t))\right)'.$$

Further, integrating we obtain

$$(u'(t))^2 = 2\lambda\left[F(\rho) - F(u(t))\right]; \quad t \in (0, t_0). \tag{5.6}$$

But this implies that $(u'(0))^2 = 2\lambda F(\rho) < 0$, a contradiction. Hence, no such solution can exist. □

Lemma 5.6. *Any positive solution u of* (5.2) *has a unique interior maximum at some $t_0 \in (0,1)$, is strictly increasing on $(0, t_0)$, is strictly decreasing on $(t_0, 1)$, and is symmetric about t_0.*

Proof. Let $t_0 \in (0,1)$ be such that $\|u\|_\infty = u(t_0) = \rho$. Suppose there exists another local maximum. Then there must be a local minimum at some $t_1 \in (0,1)$, at which $u''(t_1) \geq 0$, which implies that $u(t_1) \leq \beta$. Let $E(t) = \lambda F(u(t)) + \frac{1}{2}(u'(t))^2$ for $t \in (0,1)$. A simple calculation will show that $E'(t) = 0$, and hence $E(t)$ is constant on $[0,1]$. But $E(t_0) = \lambda F(\rho) \geq 0$ while $E(t_1) = \lambda F(u(t_1)) < 0$, and hence we have a contradiction. Therefore, t_0 is the unique critical point and from (5.6), we easily see that

$$u'(t) = \begin{cases} \sqrt{2\lambda\left[F(\rho) - F(u(t))\right]} > 0; & t \in (0, t_0), \\ -\sqrt{2\lambda\left[F(\rho) - F(u(t))\right]} < 0; & t \in (t_0, 1). \end{cases} \tag{5.7}$$

Further, note that both $w_1(t) = u(t_0 + t)$ and $w_2(t) = u(t_0 - t)$ satisfy

$$\begin{cases} -w''(t) = \lambda f(w(t)); \quad t \in (0,1), \\ w(0) = \rho, \\ w'(0) = 0. \end{cases}$$

Hence, by Picard's Theorem, we have $w_1(t) = w_2(t)$ which implies that u is symmetric about t_0. □

We now begin the proof of Theorem 5.2 by showing first that if $u \in C^2(0,1] \cap C^1[0,1]$ is a positive solution to (5.2) with $\|u\|_\infty = u(t_0) = \rho$ and $u(1) = q$, then λ, ρ, and q must satisfy (5.4) and (5.5). We note here that the improper integral in (5.4) is convergent since $f(\rho) > 0$.

Integrating (5.7), we obtain

$$t\sqrt{2\lambda} = \int_0^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}; \quad t \in (0, t_0), \tag{5.8}$$

and

$$(1-t)\sqrt{2\lambda} = \int_q^{u(t)} \frac{ds}{\sqrt{F(\rho) - F(s)}}; \quad t \in (t_0, 1). \tag{5.9}$$

Setting $t = t_0$, we obtain

$$t_0\sqrt{2\lambda} = \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \tag{5.10}$$

and

$$(1 - t_0)\sqrt{2\lambda} = \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}}. \tag{5.11}$$

Adding (5.10) and (5.11), we obtain

$$\sqrt{2\lambda} = \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}},$$

and hence from (5.10) we obtain

$$t_0 = \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \quad \Big/ \quad \left(\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}}\right). \tag{5.12}$$

Further, using the boundary conditions and (5.7), we obtain

$$u'(1) = c(q)q = \sqrt{2\lambda\,[F(\rho) - F(q)]}.$$

Hence (5.4) and (5.5) are satisfied.

Next, if λ, ρ, and q satisfy (5.4) and (5.5), let t_0 be defined by (5.12), and define $u : [0,1] \to [0,\rho]$ via (5.8) and (5.9) for $t \in (0,t_0) \cup (t_0,1)$ with $u(0) = 0$, $u(t_0) = \rho$, $u(1) = q$. Note that u is well defined on $(0,t_0)$ since both

$$\int_0^u \frac{ds}{\sqrt{F(\rho) - F(s)}}$$

and $t\sqrt{2\lambda}$ increase from 0 to

$$\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}$$

as u increases from 0 to ρ and t increases from 0 to t_0, respectively. Also, u is well defined on $(t_0, 1)$ since both

$$\int_q^u \frac{ds}{\sqrt{F(\rho)-F(s)}},$$

and $(1-t)\sqrt{2\lambda}$ decrease from

$$\int_q^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}},$$

to 0 as u decreases from ρ to q and t increases from t_0 to 1, respectively. Now, define $H : (0, t_0) \times (0, \rho) \to \mathbb{R}$ by

$$H(\ell, v) = \int_0^v \frac{ds}{\sqrt{F(\rho)-F(s)}} - \ell\sqrt{2\lambda}.$$

Clearly H is C^1, $H(t, u(t)) = 0$ for $t \in (0, t_0)$, and

$$H_v \mid_{(t,u(t))} = \frac{1}{\sqrt{F(\rho)-F(u(t))}} \neq 0.$$

Hence, by the Implicit Function Theorem, u is C^1 on $(0, t_0)$. Similarly, u is C^1 on $(t_0, 1)$, and from (5.8)-(5.9), we get

$$u'(t) = \begin{cases} \sqrt{2\lambda\left[F(\rho) - F(u(t))\right]}; & t \in (0, t_0), \\ -\sqrt{2\lambda\left[F(\rho) - F(u(t))\right]}; & t \in (t_0, 1). \end{cases} \tag{5.13}$$

Differentiating (5.13) again, we get

$$-u''(t) = \lambda f(u(t)); \quad t \in (0, t_0) \cup (t_0, 1).$$

But $u(t_0) = \rho$ and f is continuous, and hence $u \in C^2(0,1) \cap C^1[0,1]$. Further, (5.13) implies that $-u'(1) = \sqrt{2\lambda\left[F(\rho) - F(q)\right]}$, and hence by (5.5) we have $u'(1) + c(u(1))u(1) = 0$. Thus u is a solution of (5.2).

5.1.2 Proof of Theorem 5.3

Define

$$J(\rho, q) := \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{c(q)q}{\sqrt{F(\rho) - F(q)}},$$

and note that if (S) is satisfied, then for every fixed $\rho > \theta$, there exists a $q > 0$ so that $J(\rho, q) = 0$ since

$$J(\rho, 0) = 2\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} > 0 \text{ and } \lim_{q \to \rho^-} J(\rho, q) = -\infty.$$

Hence, ρ, q satisfy (5.4). For $\rho = \theta$, we again have

$$\lim_{q\to\theta} J(\theta, q) = -\infty$$

and observe that

$$\begin{aligned}
\lim_{q\to 0^+} J(\theta, q) &= 2\int_0^\theta \frac{ds}{\sqrt{-F(s)}} - \lim_{q\to 0^+} \frac{c(q)q}{\sqrt{-F(q)}} \\
&= 2\int_0^\theta \frac{ds}{\sqrt{-F(s)}} - \lim_{q\to 0^+} \frac{c(q)q}{\sqrt{-qf(z)}} \\
&= 2\int_0^\theta \frac{ds}{\sqrt{-F(s)}} \\
&> 0
\end{aligned}$$

for some $z \in (0, q)$. Hence, there exists $q > 0$ satisfying (5.4) for all $\rho \geq \theta$.

5.1.3 Proof of Theorem 5.4

Let $\rho \geq \theta$ be fixed. The existence of $q > 0$ follows from Theorem 5.3. As for the uniqueness of q, a straightforward calculation will show

$$J_q(\rho, q) = -\frac{2[1 + (c(q)q)'](F(\rho) - F(q)) + f(q)c(q)q}{2\,(F(\rho) - F(q))^{\frac{3}{2}}}. \tag{5.14}$$

If (S1) holds, then for $s \in (0, \beta)$,

$$\left(\ln\left(\frac{s + c(s)s}{\sqrt{-F(s)}}\right)\right)' \geq 0.$$

A straightforward calculation will show that this implies that

$$\frac{1+(c(s)s)'}{s+c(s)s} \geq \frac{-f(s)}{2(-F(s))}, \tag{5.15}$$

and we further observe from (5.15) that, for all $s \in (0, \beta)$,

$$\frac{1+(c(s)s)'}{c(s)s} \geq \frac{1+(c(s)s)'}{s+c(s)s} \geq \frac{-f(s)}{2(-F(s))} \geq \frac{-f(s)}{2(F(\rho)-F(s))}. \tag{5.16}$$

Hence, using (5.16), we conclude that

$$2[1+(c(s)s)'](F(\rho)-F(s)) + f(s)c(s)s > 0 \tag{5.17}$$

for $s \in (0, \beta)$. Since $f(s) \geq 0$ for all $s \in [\beta, \infty)$, it is easy to see that the inequality (5.17) also holds for $s \in [\beta, \rho)$. Therefore, by (5.14), we have $J_q(\rho, q) < 0$ for all $q > 0$, and the result follows.

If (S2) holds, then let

$$g(s) = 2(F(\rho) - F(s)) + f(s)c(s)s,$$

and observe that g is continuous on $[0, \rho]$, $g(0) = 2F(\rho) \geq 0$, and $g'(s) > 0$ for $s \in (0, \beta)$ by (S4). Hence, $g(s) > 0$ on $(0, \beta]$. Now, $(c(s)s)' \geq 0$ implies $1+(c(s)s)' \geq 1$, and therefore, $J_q(\rho, q) < 0$ for $q \in (0, \beta]$. For $q \in (\beta, \rho)$, since $f(s) > 0$ for all $s \in (\beta, \rho)$, it easily follows that $J_q(\rho, q) < 0$ for all $q > 0$ from (5.14), and the result follows.

5.2 Bifurcation Diagrams for Dirichlet Problems

In this section, we will provide two examples of bifurcation curves for problems with Dirichlet boundary conditions which are numerically generated in Mathematica. The general procedure is outlined in Algorithm 5.7

Algorithm 5.7 (Quadrature Method for Dirichlet Boundary Conditions)**.** *This is a numerical method for generating bifurcation curves for* (5.1)*.*

Input: List of N values of ρ

Output: List of N corresponding λ values

(1) Create empty list of points `pts` $= \{\}$.

(2) for $i = 1 : N$.

   (a) Evaluate (5.3) given $\rho = \rho(\texttt{i})$ to find $\lambda(\texttt{i})$.

   (b) Append $\{\lambda(i), \rho(i)\}$ to the list `pts`.

(3) Plot the ordered pairs in `pts`.

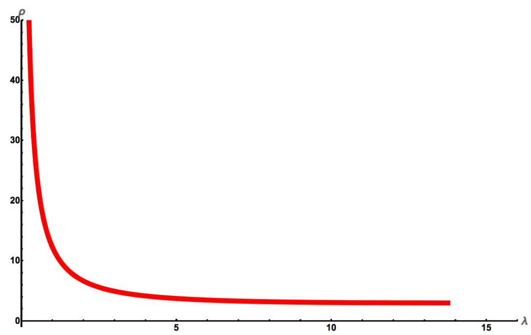
We apply this algorithm to the problems,

$$\begin{cases} -u''(t) = \lambda((u(t))^2 - 3), \quad t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{5.18}$$

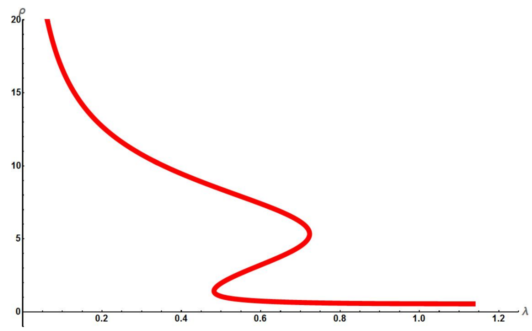
and

$$\begin{cases} -u''(t) = \lambda((u(t))^3 - 10(u(t))^2 + 40u(t) - 10), \quad t \in (0,1), \\ u(0) = 0 = u(1). \end{cases} \tag{5.19}$$

Note that the reaction terms in both (5.18) and (5.19) are semipositone and superlinear. Bifurcation diagrams for these problems are shown in Figure 6.



(a) Bifurcation Curve for (5.18)



(b) Bifurcation Curve for (5.19)

Figure 6. Bifurcation Diagrams for Two Different Autonomous Semipositone Problems with Dirichlet Boundary Conditions. Plots of solutions corresponding to selected (λ, ρ) pairs can be found in Section 5.4.

It is well known that the shape of bifurcation curves depends on characteristics of the nonlinearity f (see [Lio82]). The nonlinearities in (5.18) and (5.19) are both superlinear at infinity, and indeed we observe that $\|u\|_\infty \to \infty$ as $\lambda \to 0^+$. Furthermore, the nonlinearity in (5.19) gives rise to what is referred to in the literature as a reverse S-shaped bifurcation curve. See [CS88] for early work on reverse S-shaped bifurcation curves.

5.3 Bifurcation Diagrams for Problems with Nonlinear Boundary Conditions

In this section, we provide two examples of bifurcation diagrams for problems with nonlinear boundary conditions which are numerically generated in Mathematica. The general procedure is outlined in Algorithm 5.8.

Algorithm 5.8 (Quadrature Method for Nonlinear Boundary Conditions)**.** *This is a numerical method for generating bifurcation curves for* (5.2)*.*

Input: List of N values of ρ

Output: List of N corresponding λ values

(1) Create empty list of points `pts` $= \{\}$.

(2) for $i = 1 : N$.

  (a) Use `FindRoot` to solve (5.4) for `q(i)` given $\rho = \rho($`i`$)$.

  (b) Evaluate (5.5) given $\rho = \rho($`i`$)$ and $q =$ `q(i)` to find $\lambda($`i`$)$.

  (c) Append $\{\lambda(i), \rho(i)\}$ to the list `pts`.

(3) Plot the ordered pairs in `pts`.

We apply this algorithm to the problems

$$\begin{cases} -u''(t) = \lambda((u(t))^2 - 3), \quad t \in (0,1), \\ u(0) = 0, \\ u'(1) = -e^{\frac{u(1)}{1+u(1)}} u(1), \end{cases} \tag{5.20}$$

and

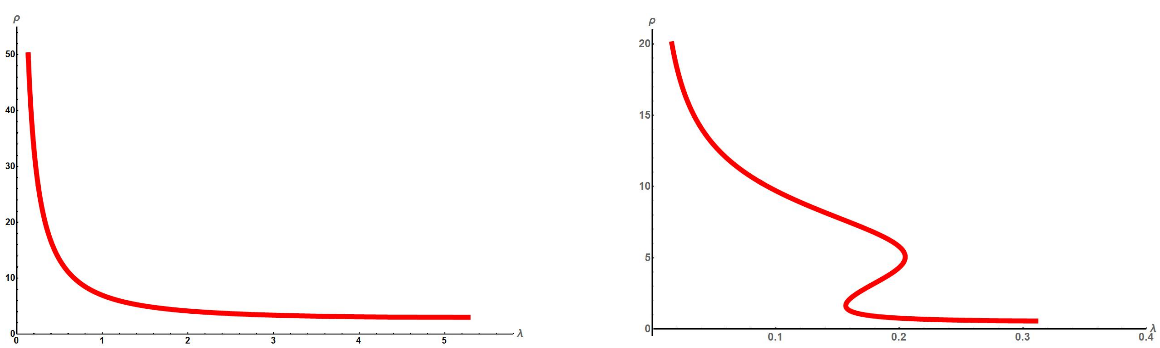
$$\begin{cases} -u''(t) = \lambda((u(t))^3 - 10(u(t))^2 + 40u(t) - 10), \quad t \in (0,1), \\ u(0) = 0, \\ u'(1) = -\frac{1}{1+u(1)} u(1). \end{cases} \tag{5.21}$$

Note again that the reaction terms in (5.20) and (5.21) are both semipositone and superlinear, and the functions f and c satisfy (S2). Hence, the results of Theorem 5.4 apply. Bifurcation diagrams for these problems are shown in Figure 7.

As before, the nonlinearities in the differential equations are both superlinear at infinity, and we again observe that $\|u\|_\infty \to \infty$ as $\lambda \to 0^+$. Furthermore, the bifurcation diagram for (5.21) remains reverse S-shaped despite the addition of the nonlinear boundary condition.

5.4 Behavior of Solutions

We observe from Figures 6 and 7 that the bifurcation diagrams for (5.18), (5.19), (5.20), and (5.21) end at some maximal value of λ, say λ^*, for which each problem has a solution. Indeed, the exact end point of the each bifurcation curve is the point (λ^*, θ).



(a) Bifurcation Curve for (5.20)

(b) Bifurcation Curve for (5.21)

Figure 7. Bifurcation Diagrams for Two Autonomous Semipositone Problems with Nonlinear Boundary Conditions. Plots of solutions corresponding to selected (λ, ρ) pairs can be found in Section 5.4.

It is known in the Dirichlet case that the solution to (5.18) and (5.19) with $\lambda = \lambda^*$ is such that $u'(0) = 0 = u'(1)$ (see [CS88]). See Figure 8 for a plot of the solution to (5.18) in the case $\lambda \approx \lambda^*$.

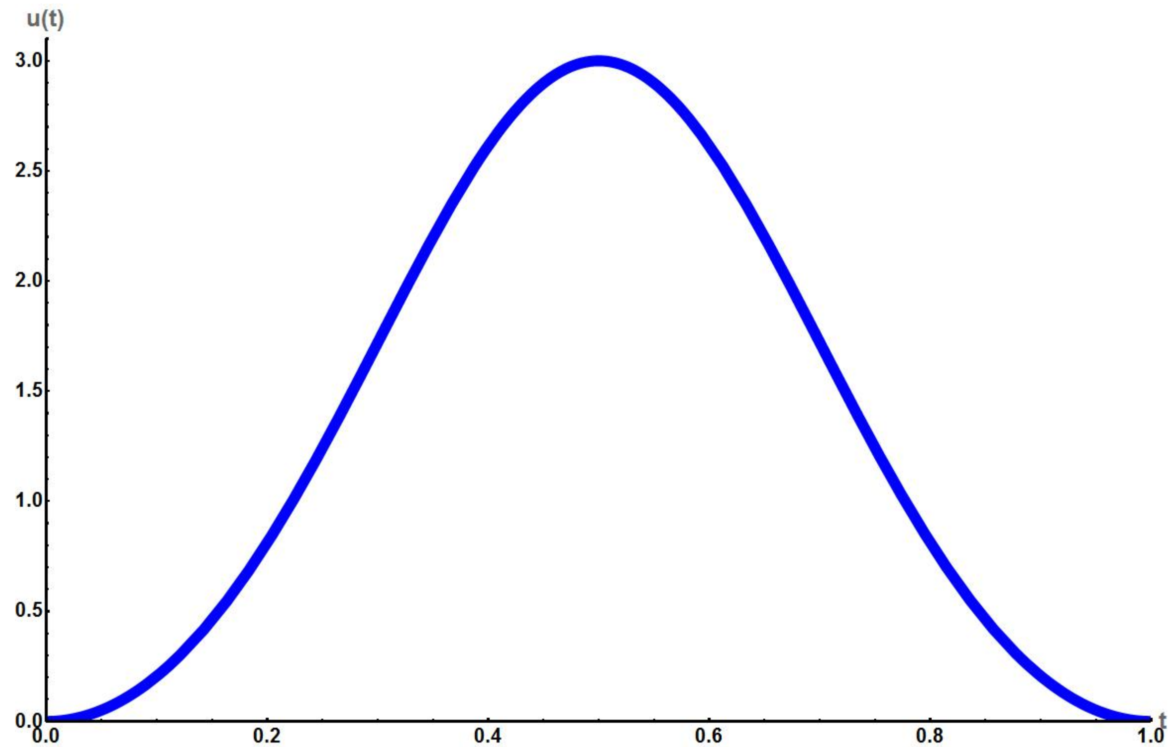
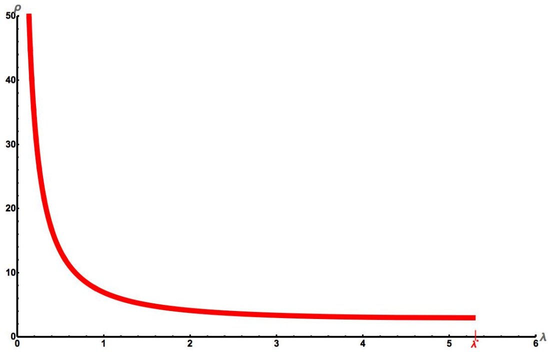
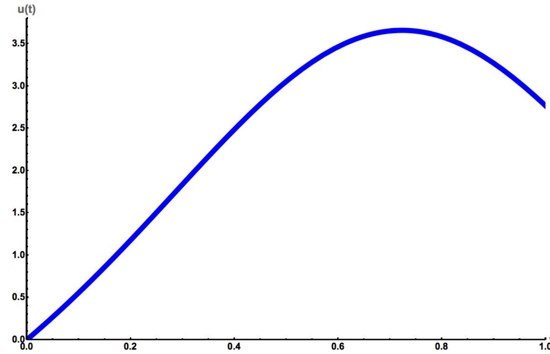


Figure 8. Solution Plot for (5.18) with $\lambda = \lambda^*$. We find that $\lambda^* \approx 13.7504$ using `FindRoot`. Solution for $\lambda = 13.7504$ obtained using `NDSolve` command in Mathematica using conditions $u(0) = 0$ and $u\left(\frac{1}{2}\right) = 3$ (since $\theta = 3$ for $f(u) = u^2 - 3$). The derivates $u'(0) \approx u'(1) \approx 1.54019 \times 10^{-3}$.

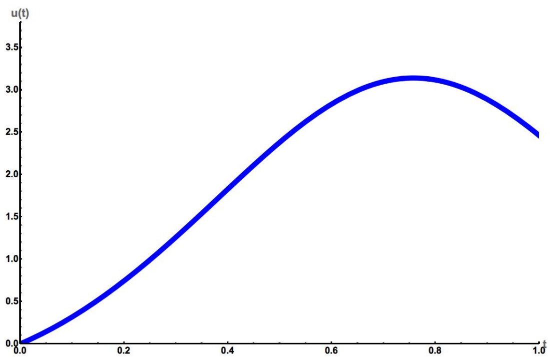
We see from (5.6) that if $\|u\|_\infty = \theta$, then any solution of (5.2) must also satisfy $u'(0) = 0$. In Figures 9 and 10, we illustrate for problems (5.20) and (5.21), respectively, that as $\lambda \to \lambda^*$, $\|u\|_\infty \to \theta$ and $u'(0) \to 0$.



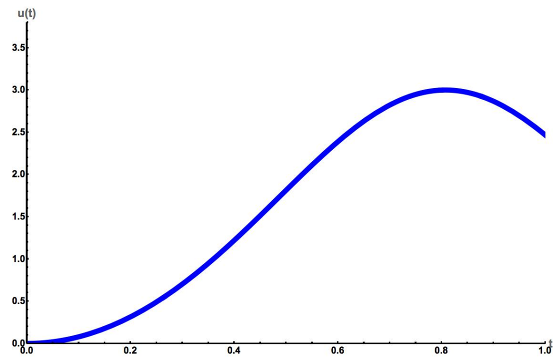
(a) Bifurcation Curve for (5.20). The curve ends at $\lambda^* \approx 5.27171$.



(b) Solution with $\lambda = 2.5152$. Here, $\|u\|_\infty \approx 3.658$ and $u'(0) \approx 5.18379$.

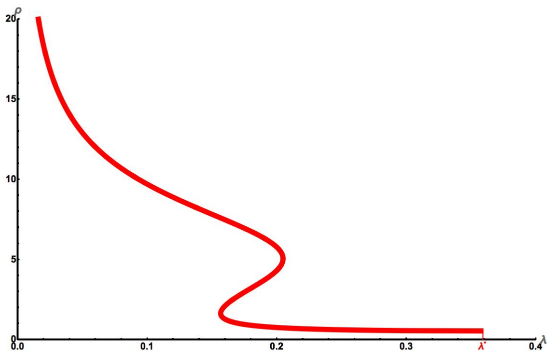


(c) Solution with $\lambda = 3.77645$. Here, $\|u\|_\infty \approx 3.141$ and $u'(0) \approx 2.61673$.

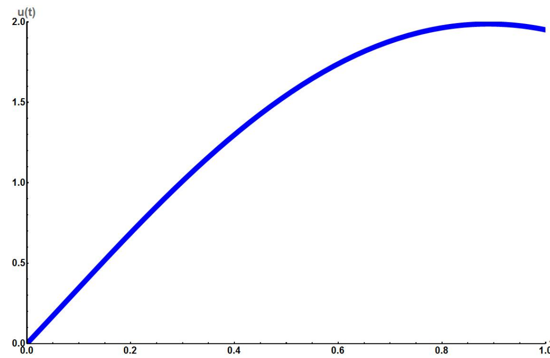


(d) Solution with $\lambda = 5.27171$. Here, $\|u\|_\infty \approx 3$ and $u'(0) \approx 0$.

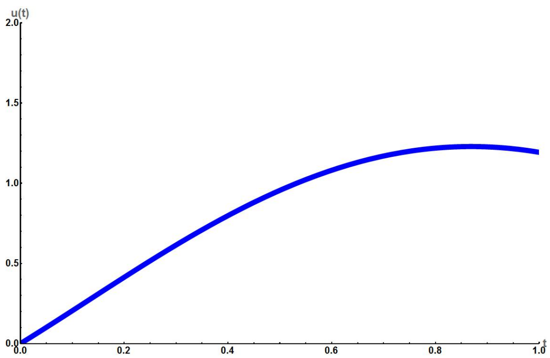
Figure 9. Bifurcation Curve and Solution Plots for (5.20). Here, we show plots of solutions for varying values of λ converging to $\lambda^* \approx 5.27171$. Note that as $\lambda \to \lambda^*$, the solutions are such that $\|u\|_\infty \to \theta = 3$ and $u'(0) \to 0$. Solutions obtained using `NDSolve` command in Mathematica with conditions $u(1) = q$ and $u'(1) = -c(q)q$, where q is found by using the `FindRoot` command to solve (5.5) for ρ given λ and then using `FindRoot` again to solve (5.4) for q given ρ.



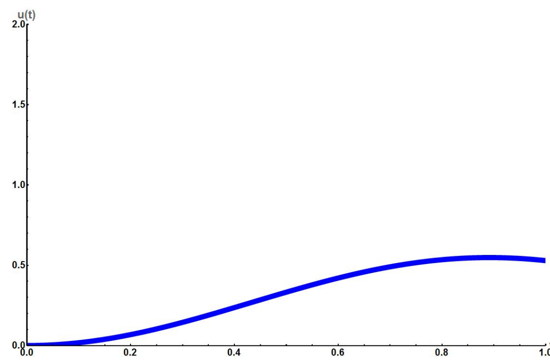
(a) Bifurcation Curve for (5.21). The curve ends at $\lambda^* \approx 0.357438$.



(b) Solution with $\lambda = 0.158524$. Here, $\|u\|_\infty \approx 1.98744$ and $u'(0) \approx 3.41839$.



(c) Solution with $\lambda = 0.161253$. Here, $\|u\|_\infty \approx 1.22881$ and $u'(0) \approx 1.99141$.



(d) Solution with $\lambda = 0.357438$. Here, $\|u\|_\infty \approx 0.547992$ and $u'(0) \approx 3.08611 \times 10^{-8}$.

Figure 10. Bifurcation Curve and Solution Plots for (5.21). Here, we show plots of solutions for varying values of λ converging to $\lambda^* \approx 0.357438$. Note that as $\lambda \to \lambda^*$, the solutions are such that $\|u\|_\infty \to \theta = 0.547992$ and $u'(0) \to 0$. Solutions obtained using `NDSolve` command in Mathematica with conditions $u(1) = q$ and $u'(1) = -c(q)q$, where q is found by using the `FindRoot` command to solve (5.5) for ρ given λ and then using `FindRoot` again to solve (5.4) for q given ρ.

We also note that, due to the reverse S-shape of the bifurcation curves for (5.19) and (5.21), there exist ranges of λ for each problem where three solutions exist. For example, taking $\lambda = 0.6$ in (5.19), we observe from Figure 6b that there are three distinct solutions with distinct norms. It remains an open problem to establish such a result analytically in higher dimension. In Figure 11, we provide plots of these solution curves.

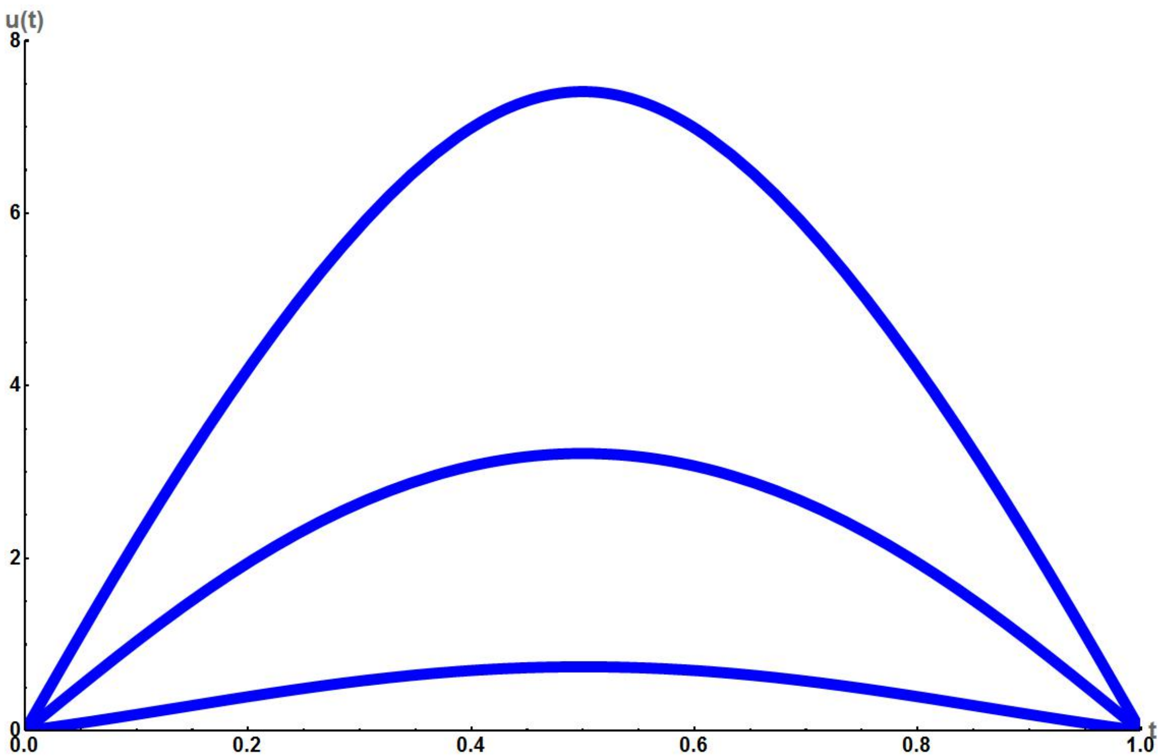


Figure 11. Solution Plot for (5.19) with $\lambda = 0.6$. The `FindRoot` command was used to find the three distinct values, $\rho_1 \approx 0.742067$, $\rho_2 \approx 3.21472$, and $\rho_3 \approx 7.41075$. Solution obtained using `NDSolve` command in Mathematica using conditions $u(0) = 0$ and $u'(0) = \sqrt{2\lambda F(\rho_i)}$ for $i = 1, 2, 3$. See [CS88] for justification of this boundary condition.

Similarly, taking $\lambda = 0.18$ in (5.21), we observe from Figure 7b that there are again three distinct solutions with distinct maximum values. In Figure 12, we provide plots of these solution curves.

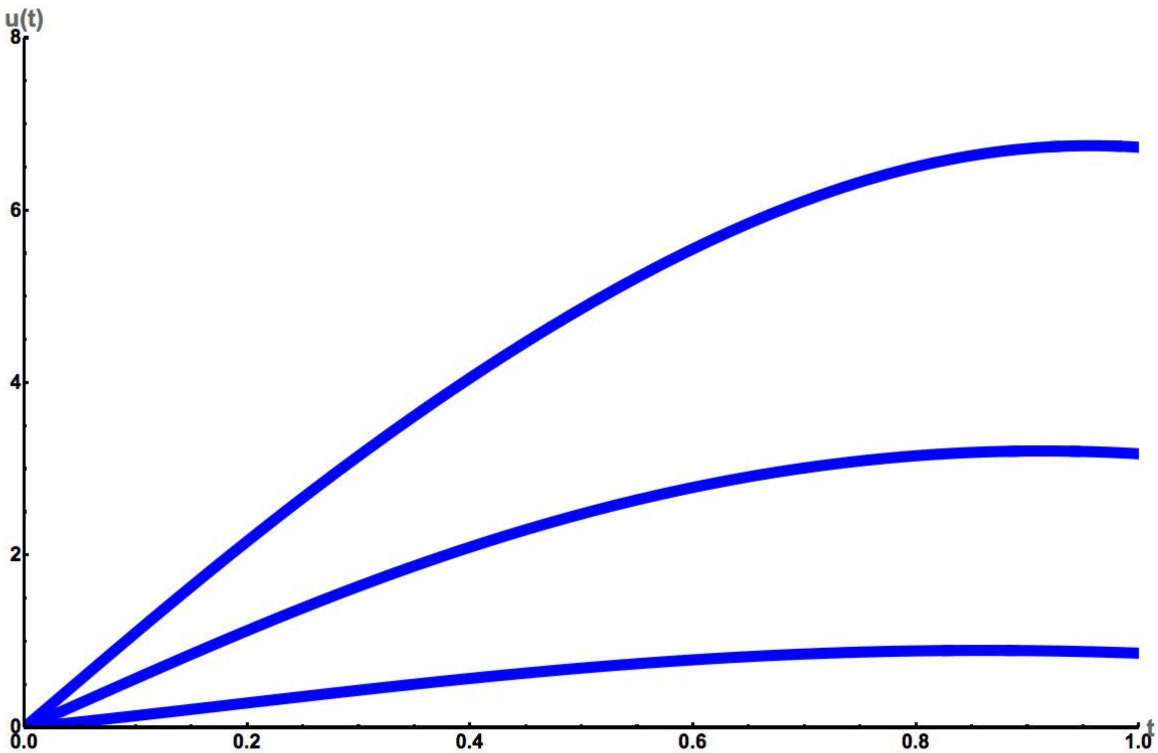


Figure 12. Solution Plot for (5.21) with $\lambda = 0.18$. The `FindRoot` command was used to find the three distinct pairs, $(\rho_1, q_1) \approx (0.897735, 0.864852)$, $(\rho_2, q_2) \approx (3.211253, 3.178000)$, and $(\rho_3, q_3) \approx (6.753183, 6.734341)$ satisfying (5.4) and (5.5) for $\lambda = 0.18$. Solution obtained using `NDSolve` command in Mathematica using conditions $u(1) = q_i$ and $u'(1) = -c(q_i)q_i$ for $i = 1, 2, 3$.

5.5 Multiplicity Generated by $s + c(s)s$ Oscillation

In the case that $(s^* + c(s^*)s^*)' < 0$ for some $s^* \in [0, \infty)$, Theorem 5.4 does not apply. In this case, it is possible that for some fixed $\rho \geq \theta$, there are multiple values of $q > 0$ so that (5.4) is satisfied. Below, we provide such an example.

Consider

$$\begin{cases} -u''(t) = \lambda\left((u(t))^2 - 3\right), t \in (0,1), \\ u(0) = 0, \\ u'(1) = -\left(\frac{1}{2}(u(1) - 10)^2 + 1\right)u(1), \end{cases} \tag{5.22}$$

and note that although $\frac{s+c(s)s}{\sqrt{-F(s)}}$ is nondecreasing on $(0, \sqrt{3})$, $s + c(s)s$ is decreasing on the interval

$$\left(\frac{20 - 2\sqrt{22}}{3}, \frac{20 + 2\sqrt{22}}{3}\right).$$

Applying Algorithm 5.8 to (5.22) , we now need to consider the possibility that for a fixed $\rho \geq \theta$, there may exist multiple q values so that (5.4) is satisfied. In Figure 13, we provide the numerically generated bifurcation curve. Observe that the oscillation of $s + c(s)s$ has introduced multiple solutions to (5.22) with the same norm. For example, if we take $\rho = 20$, then there are three values of λ for which (5.22) has a solution with $\|u\|_\infty = \rho$. See Figure 14 for plots of such solutions.

In particular, if we track q values as we plot the bifurcation diagram, we observe numerical evidence of some correspondence to changes in the sign of $(s + c(s)s)'$. See Figure 15, where (λ, ρ) pairs are visually associated with $(q, q + c(q)q)$ pairs.

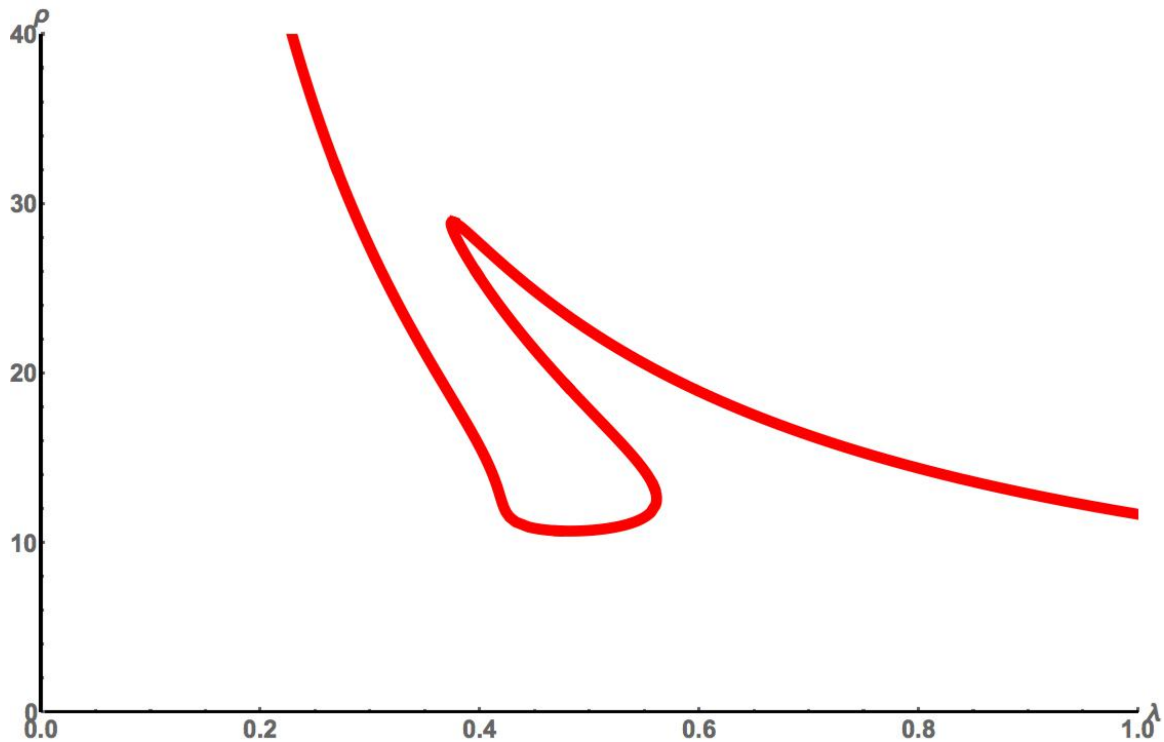


Figure 13. Bifurcation Curve for (5.22)

Many problems related to the existence, uniqueness, and exact multiplicity of solutions to (5.2) remain open. Our aim in this chapter has been to provide a quadrature method framework for addressing such problems, proofs of some results related to solutions of (5.4), and numerically generated bifurcation curves, which motivate further inquiry.

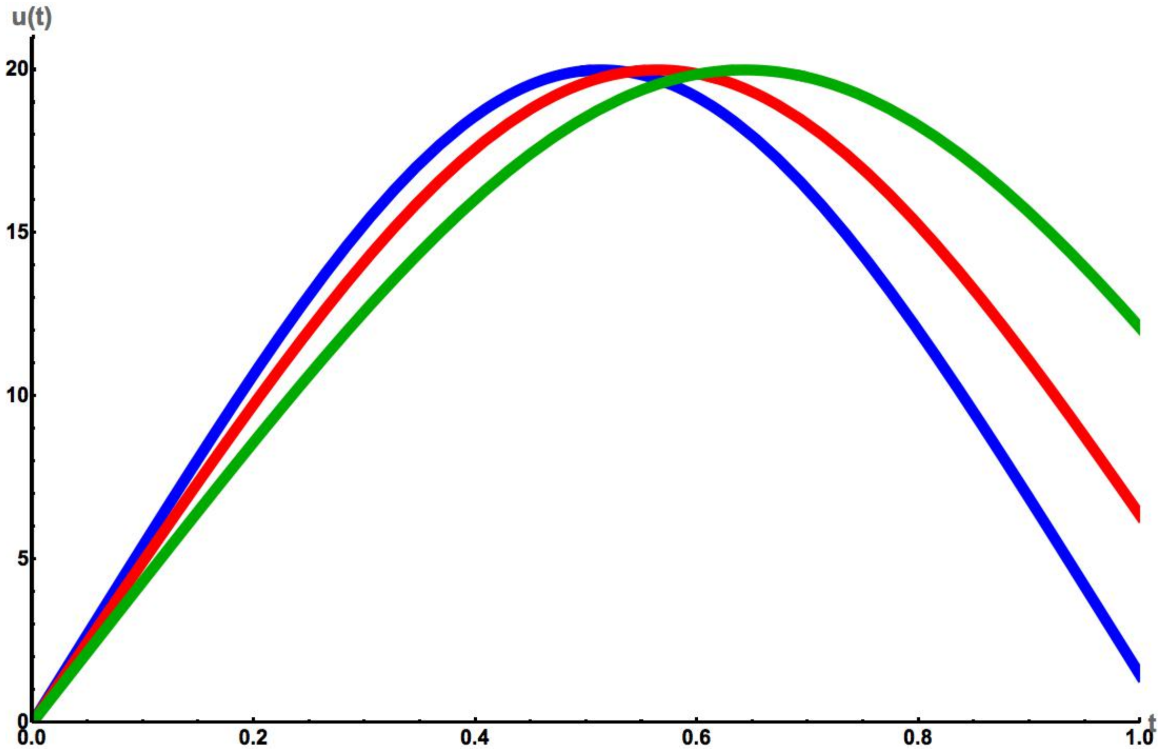
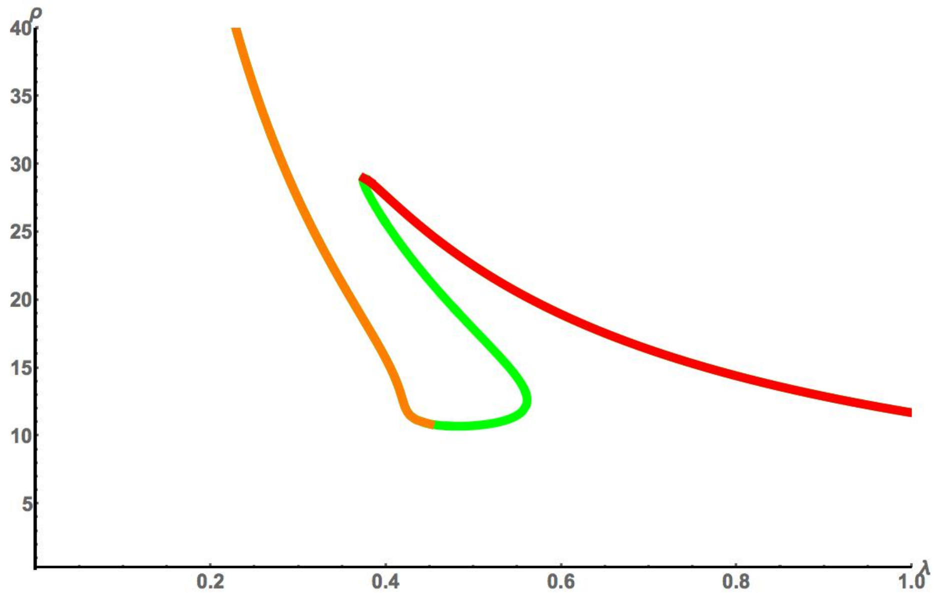
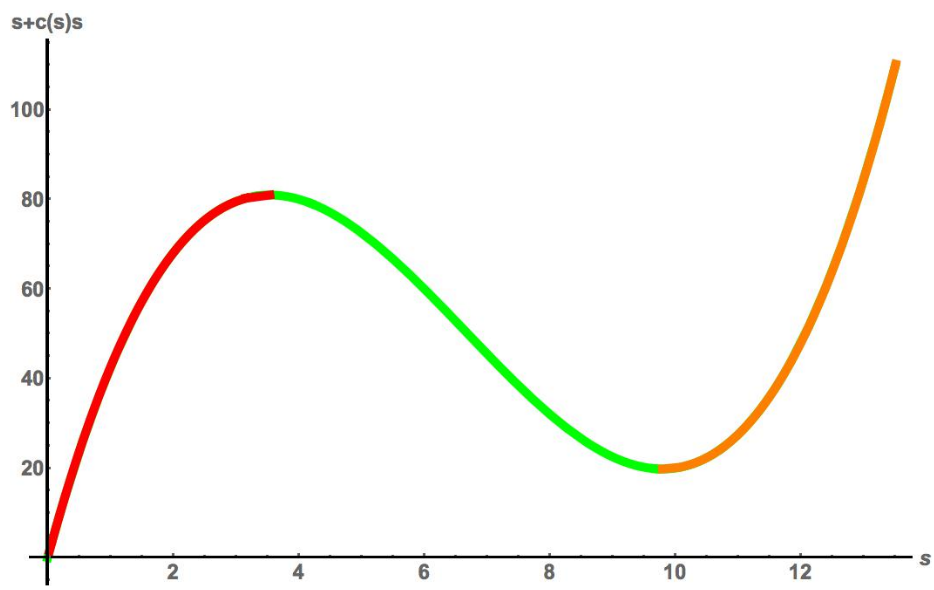


Figure 14. Solution Plots for (5.22) with $\|u\|_\infty = 20$. The `FindRoot` command was used to find the three distinct values, $q_1 = 1.44725$, $q_2 = 6.33969$, and $q_3 = 12.0901$ so that (5.4) is satisfied for $\rho = 20$. Then (5.5) is evaluated for $\rho = 20$ and each q_i for $i = 1, 2, 3$ to generate $\lambda_1 = 0.566512$, $\lambda_2 = 0.468819$, $\lambda_3 = 0.360811$. Solutions for (5.22) with each λ_i are obtained using `NDSolve` command in Mathematica using conditions $u(1) = q_i$ and $u'(1) = -c(q_i)q_i$ for $i = 1, 2, 3$.



(a) Bifurcation Curve for (5.22)



(b) Graph of $s + c(s)s$

Figure 15. Correspondence Between Shape of the Bifurcation Diagram and Shape of $s + c(s)s$

CHAPTER VI

COMPUTATIONALLY GENERATED BIFURCATION CURVES FOR NONAUTONOMOUS PROBLEMS

In this chapter, we consider problems of the form (1.9) and (1.10), where $h(t) \in C(0,1] \cap L^1(0,1)$ satisfies the more general condition $h(t) > 0$ for all $t \in (0,1)$. As this problem is no longer autonomous, the theory developed in Chapter V no longer applies, and more traditional numerical schemes for solving ordinary differential equations must be implemented. In particular, we will implement shooting methods to numerically generate bifurcation curves for problem (1.9) and (1.10).

In the case of problem (1.9), we consider the related initial value problem

$$\begin{cases} -v''(t) = \lambda h(t) f(v(t)); \quad t \in (0,1), \\ v(1) = 0, \\ v'(1) = -\alpha, \end{cases} \tag{6.1}$$

which has a unique solution, say $v(t,\lambda,\alpha)$, guaranteed by Picard's Theorem. For this problem, we take a fixed $\alpha^* > 0$ and search for $\lambda^* > 0$ so that $v(0,\lambda^*,\alpha^*) = 0$. If such a λ^* can be found, then $v(t,\lambda^*,\alpha^*)$ is a solution to (1.9) with $\lambda = \lambda^*$.

Similarly, in the case of problem (1.10), we consider the related initial value problem

$$\begin{cases} -w''(t) = \lambda h(t) f(w(t)); \quad t \in (0,1), \\ w(1) = q, \\ w'(1) = -c(q)q, \end{cases} \tag{6.2}$$

which has a unique solution, say $w(t, \lambda, q)$, guaranteed by Picard's Theorem. For a fixed $q^* > 0$, we search for $\lambda^* > 0$ so that $w(0, \lambda^*, q^*) = 0$. If such a λ^* can be found, then $w(t, \lambda^*, q^*)$ is a solution to (1.10) with $\lambda = \lambda^*$.

Remark. In setting up the shooting method for problems (1.9) and (1.10), we have chosen initial conditions at $t = 1$ for problems (6.1) and (6.2). The choice of $t = 1$ (as opposed to $t = 0$) is made due to the fact that h may be singular at $t = 0$.

6.1 Bifurcation Diagrams for Dirichlet Problems

We now provide two examples of bifurcation curves for nonautonomous problems with Dirichlet boundary conditions which are numerically generated in Mathematica. The general procedure is outlined in Algorithm 6.1.

Algorithm 6.1 (Shooting Method for Dirichlet Boundary Conditions)**.** *This is a numerical method for generating bifurcation curves for* (1.9).

Input: List of N values of α

Output: List of N corresponding (λ, ρ) pairs.

(1) Define $V(\lambda, \alpha) := v(0, \lambda, \alpha)$.

(2) for $i = 1 : N$.

(a) For $\alpha^* = \alpha(\mathtt{i})$, use `FindRoot` to find λ^* such that $V(\lambda^*, \alpha^*) = 0$. Set $\lambda(\mathtt{i}) = \lambda^*$.

(b) Use `NDSolve` to numerically solve (6.1) with $\alpha = \alpha(\mathtt{i})$ and $\lambda = \lambda(\mathtt{i})$. Set $\rho(\mathtt{i}) = \max_{t\in(0,1)} v(t)$.

(c) Append $\{\lambda(\mathtt{i}), \rho(\mathtt{i})\}$ to the list `pts`.

(3) Plot the ordered pairs in `pts`.

We apply Algorithm 6.1 to the problems,

$$\begin{cases} -u''(t) = \lambda t^{-\frac{1}{3}}((u(t))^2 - 3); \quad t \in (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{6.3}$$

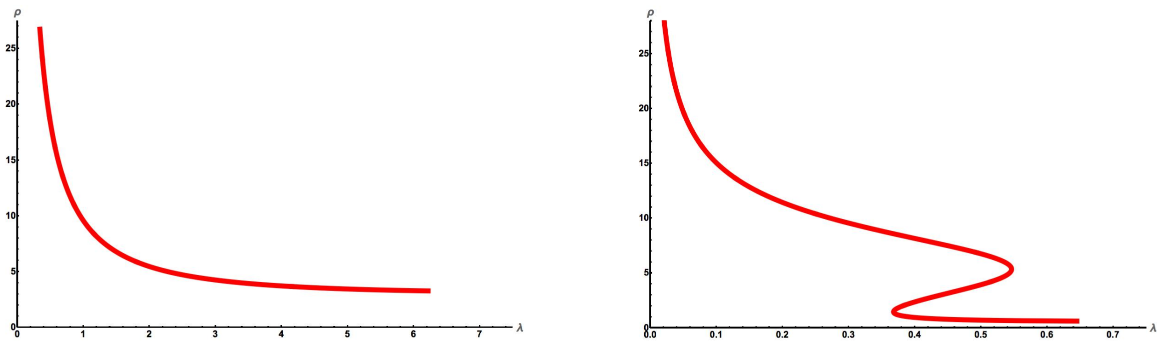
and

$$\begin{cases} -u''(t) = \lambda t^{-\frac{1}{3}}((u(t))^3 - 10(u(t))^2 + 40u(t) - 10); \quad t \in (0,1), \\ u(0) = 0 = u(1). \end{cases} \tag{6.4}$$

We have chosen the same nonlinear functions f and c in (6.3) and (6.4) as were used in the autonomous cases, (5.18) and (5.19) respectively, but have here added the singular weight function $h(t) = t^{-\frac{1}{3}}$. Bifurcation diagrams for these problems are shown in Figure 16.

6.2 Bifurcation Diagrams for Problems with Nonlinear Boundary Conditions

In order to generate bifurcation curves for problem (1.10), we implement Algorithm 6.2.



(a) Bifurcation Curve for (6.3) (b) Bifurcation Curve for (6.4)

Figure 16. Bifurcation Diagrams for Two Nonautonomous Semipositone Problems with Dirichlet Boundary Conditions

Algorithm 6.2 (Shooting Method for Nonlinear Boundary Conditions)**.** *This is a numerical method for generating bifurcation curves for* (1.10)*.*

Input: List of N values of q

Output: List of N corresponding (λ, ρ) pairs

(1) Define $W(\lambda, q) := w(0, \lambda, q)$.

(2) for $i = 1 : N$.

 (a) For $q^* = \mathtt{q}(\mathtt{i})$, use `FindRoot` to find λ^* such that $W(\lambda^*, q^*) = 0$. Set $\lambda(\mathtt{i}) = \lambda^*$.

 (b) Use `NDSolve` to numerically solve (6.2) with $q = \mathtt{q}(\mathtt{i})$ and $\lambda = \lambda(\mathtt{i})$. Set $\rho(\mathtt{i}) = \max\limits_{t \in (0,1)} w(t)$.

 (c) Append $\{\lambda(\mathtt{i}), \rho(\mathtt{i})\}$ to the list `pts`.

(3) Plot the ordered pairs in `pts`.

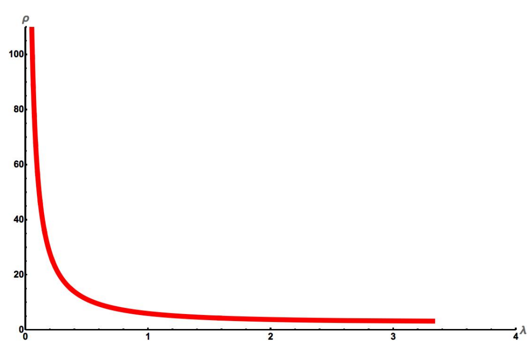
We apply Algorithm 6.2 to the problems,

$$\begin{cases} -u''(t) = \lambda t^{-\frac{1}{3}}((u(t))^2 - 3); \quad t \in (0,1), \\ u(0) = 0, \\ u'(1) = -e^{\frac{u(1)}{1+u(1)}} u(1), \end{cases} \tag{6.5}$$

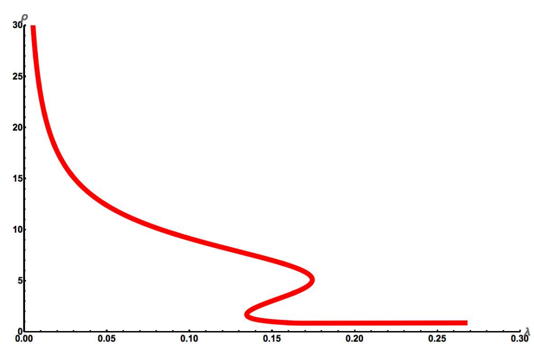
and

$$\begin{cases} -u''(t) = \lambda t^{-\frac{1}{3}}((u(t))^3 - 10(u(t))^2 + 40u(t) - 10); \quad t \in (0,1), \\ u(0) = 0, \\ u'(1) = -\frac{1}{1+u(1)} u(1). \end{cases} \tag{6.6}$$

We have chosen the same nonlinear functions f and c in (6.5) and (6.6) as were used in the autonomous cases, (5.20) and (5.21) respectively, but have here added the singular weight function $h(t) = t^{-\frac{1}{3}}$. Bifurcation diagrams for these problems are shown in Figure 17.



(a) Bifurcation Curve for (6.5)



(b) Bifurcation Curve for (6.6)

Figure 17. Bifurcation Diagrams for Two Nonautonomous Semipositone Problems with Nonlinear Boundary Conditions

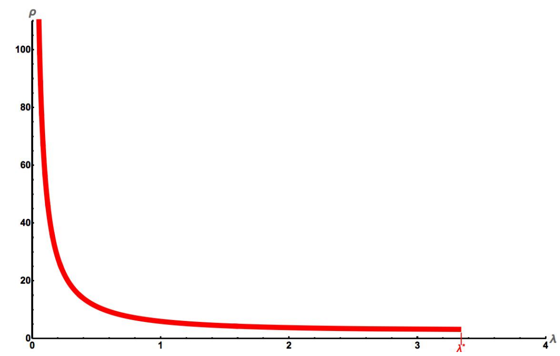
6.3 Behavior of Solutions

As in the autonomous case, we observe from Figures 16 and 17 that the bifurcation diagrams for (6.3), (6.4), (6.5), and (6.6) end at some maximal value of λ, say λ^*.

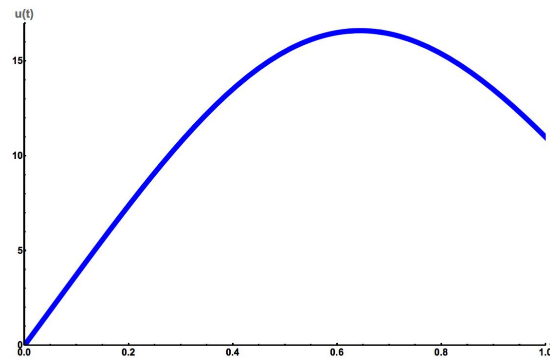
It is known that when h is a decreasing function, nonnegative solutions of (1.9) have a unique interior maximum, say at t_0, with $u(t_0) = \|u\|_\infty > \theta$ (see [CSS12]). The case where h is increasing on some portion of the domain remains open.

In Figures 18 and 19, we illustrate the behavior of solutions as $\lambda \to \lambda^*$ for problems (6.5) and (6.6), respectively.

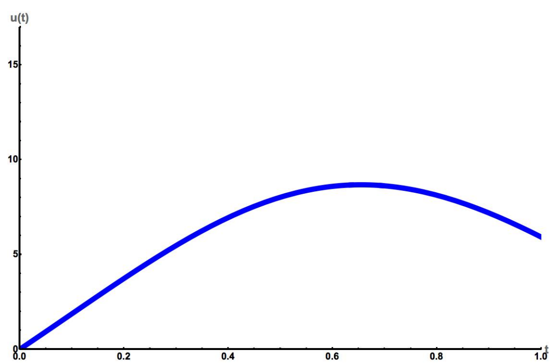
We also note that, due to the reverse S-shape of the bifurcation curves for (6.4) and (6.6), there exist ranges of λ for each problem where at least three solutions exist. For example, taking $\lambda = 0.45$ in (6.4), we observe from Figure 16b that there are three distinct solutions with distinct maximum values. In Figure 20, we provide plots of these solutions curves. Similarly, taking $\lambda = 0.16$ in (6.6), we observe from Figure 17b that there are again three distinct solutions with distinct maximum values. In Figure 21, we provide plots of these solution curves.



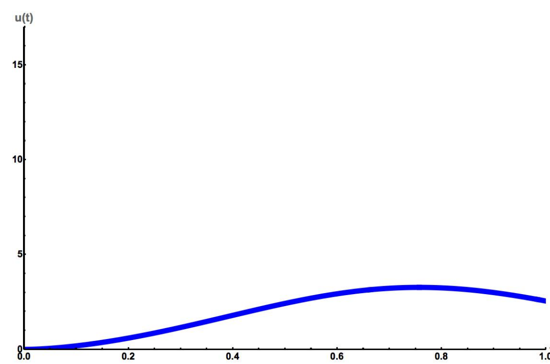
(a) Bifurcation Curve for (6.5). The curve ends at $(\lambda^*, \rho) \approx (3.32009, 3.27078)$.



(b) Solution with $\lambda = 0.333696$. Here, $\|u\|_\infty \approx 16.6106$ and $u'(0) \approx 37.2779$.

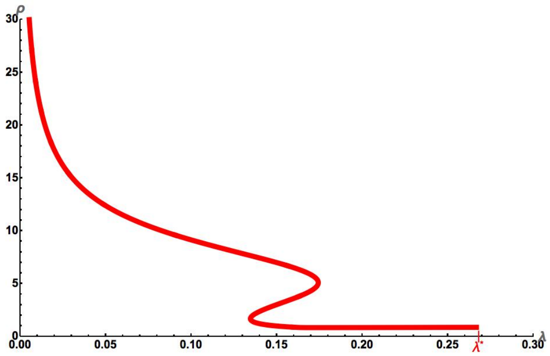


(c) Solution with $\lambda = 0.655625$. Here, $\|u\|_\infty \approx 8.6796$ and $u'(0) \approx 18.4024$.

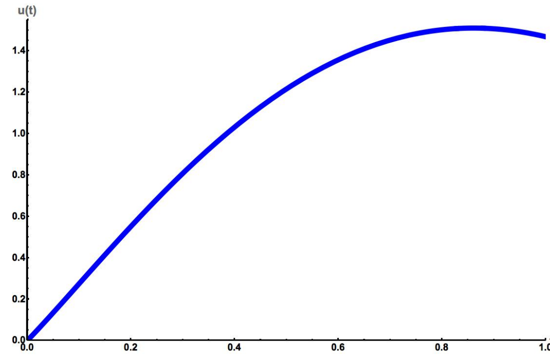


(d) Solution with $\lambda = 3.32009$. Here, $\|u\|_\infty \approx 3.27078$ and $u'(0) \approx 0.0504337$.

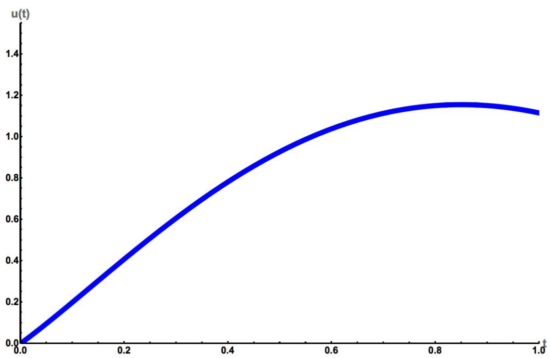
Figure 18. Bifurcation Curve and Solution Plots for (6.5). Here, we show plots of solutions for varying values of λ converging to $\lambda^* \approx 3.32009$. Note that as $\lambda \to \lambda^*$, the solutions are such that $\|u\|_\infty \not\to \theta = 3$, as they did in the autonomous case, however $u'(0) \to 0$. Solutions obtained using `NDSolve` command in Mathematica with conditions $u(1) = q$ and $u'(1) = -c(q)q$, where q is found using the procedure outlined in Algorithm 6.2.



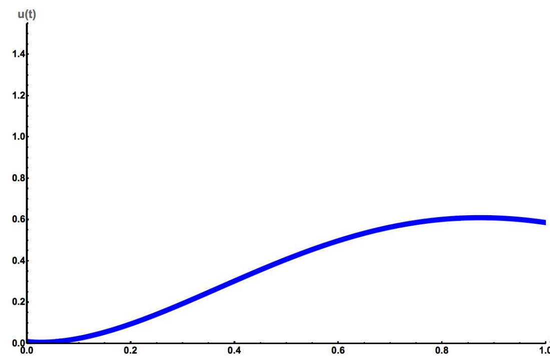
(a) Bifurcation Curve for (6.6). The curve ends at $(\lambda^*, \rho) \approx (0.266674, 0.872975)$.



(b) Solution with $\lambda = 0.135132$. Here, $\|u\|_\infty \approx 1.51014$ and $u'(0) \approx 2.55737$.



(c) Solution with $\lambda = 0.141907$. Here, $\|u\|_\infty \approx 1.15532$ and $u'(0) \approx 1.76858$.



(d) Solution with $\lambda = 0.266674$. Here, $\|u\|_\infty \approx 0.872975$ and $u'(0) \approx 0.345048$.

Figure 19. Bifurcation Curve and Solution Plots for (6.6). Here, we show plots of solutions for varying values of λ converging to $\lambda^* \approx 0.266674$. Note that as $\lambda \to \lambda^*$, the solutions are such that $\|u\|_\infty \not\to \theta = 0.547992$. It is difficult, in this case, to conclude whether $u'(0) \to 0$. Solutions obtained using the `NDSolve` command in Mathematica with conditions $u(1) = q$ and $u'(1) = -c(q)q$, where q is found using the procedure outlined in Algorithm 6.2.

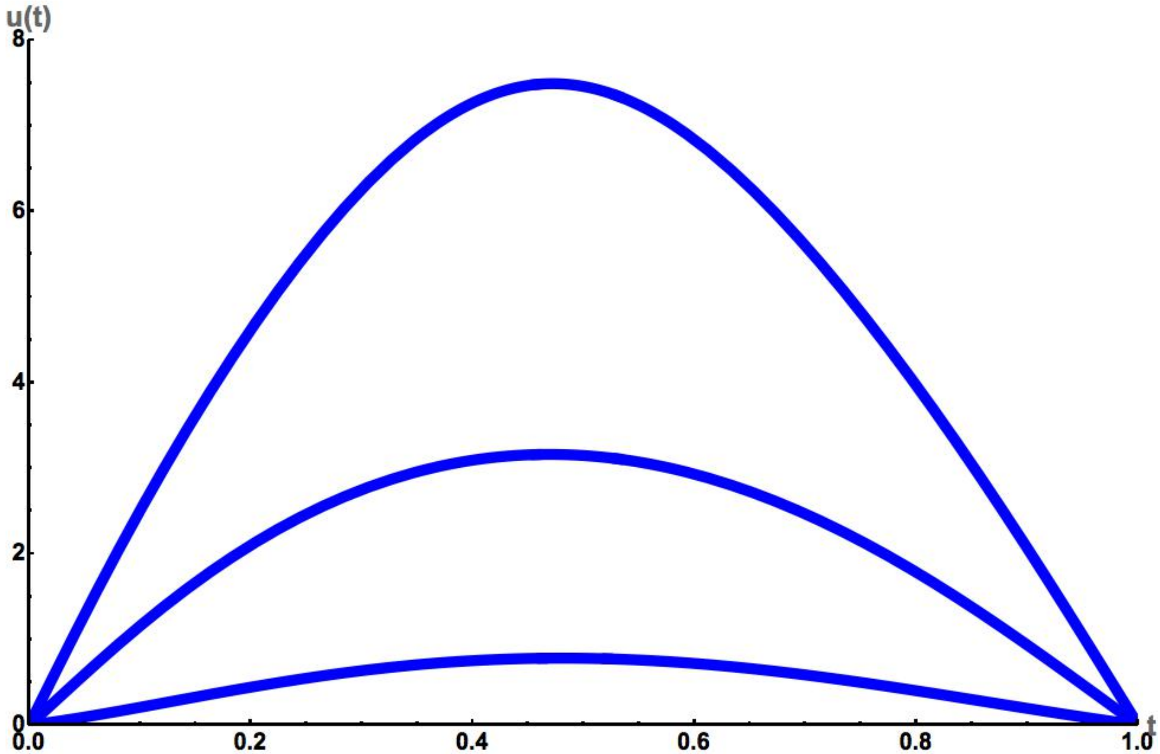


Figure 20. Solution Plots for (6.4) with $\lambda = 0.45$. The `FindRoot` command was used to find the three distinct values, α_1, α_2, and α_3 so that $V(\lambda, \alpha_i) = 0$ for $i = 1, 2, 3$. Solution obtained using the `NDSolve` command in Mathematica with conditions $u(1) = 0$ and $u'(1) = \alpha_i$ for $i = 1, 2, 3$. The maximum of the solutions are $\rho_1 \approx 0.780876$, $\rho_2 \approx 3.16105$, and $\rho_3 \approx 7.49173$, occuring at $t_1 \approx 0.479972$, $t_2 \approx 0.469016$, and $t_3 \approx 0.472155$, respectively. In the nonautonomous case, solutions need not be symmetric, and hence the location of the maximum may change.

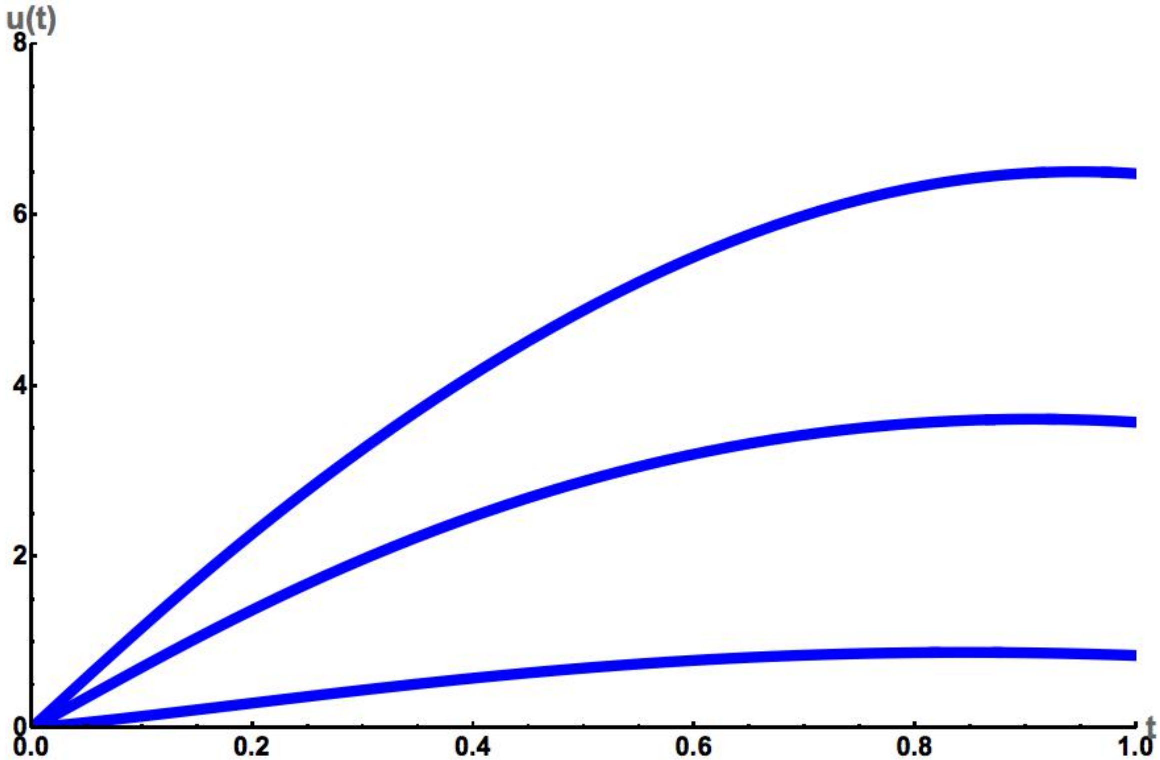


Figure 21. Solution Plots for (6.6) with $\lambda = 0.16$. The `FindRoot` command was used to find the three distinct values, $q_1 \approx 0.846099$, $q_2 \approx 3.575090$, and $q_3 \approx 6.484586$ satisfying $W(\lambda, q_i) = 0$ for $i = 1, 2, 3$. Solution obtained using the `NDSolve` command in Mathematica with conditions $u(1) = q_i$ and $u'(1) = -c(q_i)q_i$ for $i = 1, 2, 3$. The maximum of the solutions are $\rho_1 \approx 0.882413$, $\rho_2 \approx 3.6121$, and $\rho_3 \approx 6.50745$, occuring at $t_1 \approx 0.844304$, $t_2 \approx 0.905882$, and $t_3 \approx 0.947446$, respectively.

CHAPTER VII

CONCLUSION AND FUTURE DIRECTIONS

7.1 Conclusion

In this dissertation, we have established the existence of positive radial solutions for classes of superlinear, semipositone Laplacian and p-Laplacian problems with singular weights and both Dirichlet and nonlinear boundary conditions for small values of the parameter λ. In particular, we have exhibited methods for overcoming the difficulties posed by the semipositone nature of the reaction terms, the presence of singular weights, and nonlinear boundary conditions. These contributions have been published or accepted for publication in [DMS16] and [MSS16].

Further, we provided a detailed analysis of the quadrature method for autonomous ordinary differential equations with nonlinear boundary conditions, and provided algorithms which are suitably versatile to allow implementation in many programs. We have also provided algorithms for generating bifurcation curves for nonautonomous problems via shooting methods. Finally, we have obtained (computationally) exact bifurcation diagrams for several one-dimensional problems with both Dirichlet and nonlinear boundary conditions.

7.2 Future Directions

7.2.1 Existence of Non-radial Solutions

While Theorems 1.1-1.4 prove the existence of a positive radial solution on the exterior of a radial domain, these results may be extended to the non-radial cases by again employing variational methods. There are a number of natural generalizations, the first being simply to consider non-radial solutions to (1.3) and (1.4). Beyond that, one may also consider solutions on the exterior of a non-radial domain, or solutions on the exterior of a ball with non-radial weight K. While a mountain pass solution may be tractable in the correct variational setting (by separate analyses of the solution on both the interior and exterior of a sufficiently large ball), showing the positivity of the solution in these cases poses challenges which cannot be addressed by our current methods. See Figure 22 for examples.

7.2.2 Uniqueness

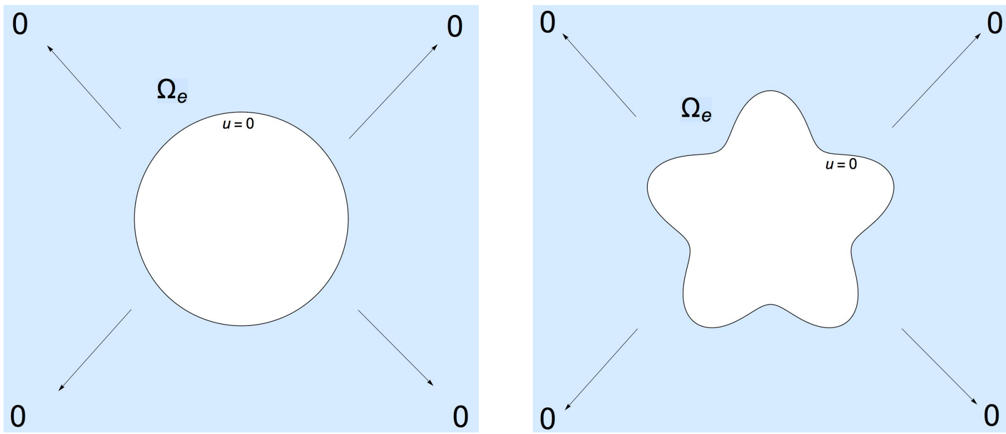
In addition to extending existence results, the question of the uniqueness of solutions to semipostione superlinear problems is wide open. The only uniqueness result for such superlinear semipositone problems that is available in the literature is [ACS93], where they study radial solutions in the ball via bifurcation theory and implicit function theorem arguments. All other cases, even in the case of general bounded domains, remain open, and no results are available in the case of unbounded domains.

7.2.3 Infinite Semipositone Problems

A natural extension of (1.3) and (1.4) is to consider $f(u) = \frac{g(u)}{u^{\alpha}}$ with g being superlinear and semipositone, and $\alpha > 0$ small. In this case, $\lim\limits_{s\to 0} f(s) = -\infty$, which will pose significant challenges in the analysis.

7.2.4 Numerical Methods

Our computational results in both the autonomous and nonautonomous cases treat only the $p = 2$ case. More work is needed to develop numerical methods to treat the cases when $p \neq 2$, including the development of a quadrature method for such problems, as well as adapting shooting methods to treat such problems.



(a) One May Consider Non-radial Solutions on a Radial Exterior Domain.

(b) One May Also Consider Solutions on a Non-radial Exterior Domain.

Figure 22. Extensions of (1.3) and (1.4) to Non-radial Cases

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APPENDIX A

KELVIN TRANSFORMATIONS

A.1 Kelvin Transformation on the Exterior of a Ball

We first consider the problem

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u), & x \in \Omega_e, \\ u = 0, & |x| = r_0, \\ u \to 0, & |x| \to \infty, \end{cases} \tag{A.1}$$

where $\lambda > 0$ is a parameter, $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2} \nabla z)$ with $p > 1$,

$$\Omega_e = \left\{ x \in \mathbb{R}^N \mid |x| > r_0, r_0 > 0, N > p \right\},$$

and $K \in C\left([r_0, \infty), (0, \infty)\right)$ satisfies $K(r) \leq \frac{1}{r^{N+\mu}}$; $\mu > 0$ for $r >> 1$. Let $r = |x|$ and $v(r) = u(x)$. Then,

$$-\Delta_p u(x) = r^{1-N} \left(r^{N-1} |v'(r)|^{p-2} v'(r) \right)'.$$

Substituting into (A.1), we see

$$\begin{cases} -\left(r^{N-1} |v'(r)|^{p-2} v'(r) \right)' = \lambda r^{N-1} K(r) f(v(r)), & r_0 < r < \infty, \\ v(r_0) = 0, & \\ v(r) \to 0, & r \to \infty. \end{cases} \tag{A.2}$$

If we now let $t = \left(\frac{r}{r_0}\right)^{\frac{p-N}{p-1}}$ and $z(t) = v(r)$, then we note that since

$$v'(r) = z'(t)\frac{p-N}{p-1}\left(\frac{r}{r_0}\right)^{\frac{1-N}{p-1}} = \frac{p-N}{p-1}t^{\frac{1-N}{p-N}}z'(t), \tag{A.3}$$

we have

$$-\left(r^{N-1}|v'(r)|^{p-2}v'(r)\right)' = \left(\frac{N-p}{p-1}\right)^p \left(\frac{1}{r_0}\right)^{p-N+1} t^{\frac{1-N}{p-N}} \left(|z'(t)|^{p-2}\, z'(t)\right)'.$$

Hence, substituting back into (A.2) we observe that

$$\begin{aligned}\left(|z'(t)|^{p-2}\, z'(t)\right)' &= \left(\frac{p-1}{N-p}\right)^p r_0^{p-N+1} t^{\frac{N-1}{p-N}} \lambda \left(r_0 t^{\frac{p-1}{p-N}}\right)^{N-1} K\left(r_0 t^{\frac{p-1}{p-N}}\right) f(z(t)) \\ &= \lambda \left(\frac{p-1}{N-p}\right)^p r_0^p t^{\frac{p(1-N)}{N-p}} K\left(r_0 t^{\frac{p-1}{p-N}}\right) f(z(t)).\end{aligned}$$

Therefore, the problem (A.1) is reduced to

$$\begin{cases} -(\phi_p(z'(t)))' = \lambda h(t) f(z(t)), \quad t \in (0,1), \\ \qquad z(0) = 0 = z(1), \end{cases} \tag{A.4}$$

where $h(t) = \left(\frac{p-1}{N-p}r_0\right)^p t^{-\frac{p(N-1)}{N-p}} K\left(r_0 t^{\frac{1-p}{p-N}}\right)$.

We may apply the same transformation to

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u), & x \in \Omega_e, \\ \frac{\partial u}{\partial \eta} + \tilde{c}(u)u = 0, & |x| = r_0, \\ \qquad u \to 0, & |x| \to \infty, \end{cases} \tag{A.5}$$

and observe that the differential equation transforms as before. Additionally,

$$0 = \frac{\partial u}{\partial \eta} + \tilde{c}(u(x))u(x) = -\frac{1}{r_0}v'(r_0) + \tilde{c}(v(r_0))v(r_0), \quad |x| = r_0,$$

and hence, by (A.3),

$$\begin{aligned} 0 &= \phi_p\left(\frac{N-p}{r_0(p-1)}z'(1)\right) + \phi_p\left(\tilde{c}(z(1))z(1)\right) \\ &= \left(\frac{N-p}{r_0(p-1)}\right)^{p-1}\phi_p(z'(1)) + (\tilde{c}(z(1))^{p-1}\phi_p(z(1)). \end{aligned}$$

Dividing through by $\left(\frac{N-p}{r_0(p-1)}\right)^{p-1}$ gives

$$\phi_p(z'(1)) + c(z(1))\phi_p(z(1)) = 0,$$

with $c(s) = \left(\frac{r_0(p-1)}{N-p}\tilde{c}(s)\right)^{p-1}$. Hence, (A.5) has been transformed to

$$\begin{cases} -\left(\phi_p(z')\right)' = \lambda h(t)f(z), \ t \in (0,1), \\ u(0) = 0, \\ \phi_p\left(z'(1)\right) + c(z(1))\phi_p\left(z(1)\right) = 0. \end{cases} \tag{A.6}$$

A.2 Kelvin Transformation on an Annulus

In the case of an annular domain, we first consider the problem,

$$\begin{cases} -\Delta_p u = \lambda K(|x|)f(u); & x \in \Omega_a, \\ u(x) = 0; & |x| = R_1, \\ u(x) = 0; & |x| = R_2. \end{cases} \tag{A.7}$$

As in Section A.1, making the change of variables $r = |x|$ and taking $v(r) = u(x)$ yields

$$\begin{cases} -\left(r^{N-1}|v'(r)|^{p-2}v'(r)\right)' = \lambda r^{N-1}K(r)f(v(r)), \quad R_1 < r < R_2, \\ v(R_1) = 0, \\ v(R_2) = 0. \end{cases} \tag{A.8}$$

Now, making the change of variables $s = -\int_r^{R_2} \tau^{\frac{1-N}{p-1}}\, d\tau$, letting $m = -\int_{R_1}^{R_2} \tau^{\frac{1-N}{p-1}}\, d\tau$, and taking $w(s) = v(r)$ yields,

$$\begin{cases} -\left(|w'(s)|^{p-2}w'(s)\right)' = \lambda \tilde{h}(s)f(w(s)), \quad m < s < 0, \\ w(m) = 0, \\ w(0) = 0, \end{cases} \tag{A.9}$$

where

$$\tilde{h}(s) = \left(R_2^{\frac{p-N}{p-1}} - \frac{N-p}{p-1}s\right)^{\frac{2(N-1)(p-1)}{p-N}} K\left(\left(R_2^{\frac{p-N}{p-1}} - \frac{N-p}{p-1}s\right)^{\frac{p-1}{p-N}}\right).$$

Finally, making the change of variables $t = \frac{m-s}{m}$ and taking $z(t) = w(s)$, we see

$$\begin{cases} -(\phi_p(z(t)))' = \lambda h(t)f(z(t)), \quad 0 < t < 1, \\ z(0) = 0, \\ z(1) = 0, \end{cases} \tag{A.10}$$

where

$$h(t) = m^{p-1}(h_1(t))^{2(N-1)} K(h_1(t))$$

with

$$h_1(t) = \left(R_2^{\frac{p-N}{p-1}} - \frac{m(1-t)(N-p)}{p-1} \right)^{\frac{p-1}{p-N}}.$$

We observe that $h \in C[0,1]$ as long as $K \in C[R_1, R_2]$.

We next apply the same transformation to

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u); & x \in \Omega_a, \\ u(x) = 0; & |x| = R_1, \\ \frac{\partial u}{\partial \eta} + \tilde{c}(u)u = 0; & |x| = R_2, \end{cases} \tag{A.11}$$

and observe that the differential equation transforms as before. Additionally,

$$0 = \frac{\partial u}{\partial \eta} + \tilde{c}(u(x))u(x) = \frac{1}{R_2} v'(R_2) + \tilde{c}(v(R_2))v(R_2), \quad |x| = R_2.$$

Note, however, that

$$v'(R_2) = R_2^{\frac{1-N}{p-1}} w'(0) = -\frac{1}{m} R_2^{\frac{1-N}{p-1}} z'(1).$$

Hence, we have

$$0 = \frac{1}{R_2} v'(R_2) + \tilde{c}(v(R_2))v(R_2) = -\frac{1}{m} R_2^{\frac{2-N-p}{p-1}} z'(1) + \tilde{c}(z(1))z(1).$$

But this is equivalent to

$$\phi_p(z'(1)) + c(z(1))\phi_p(z(1)) = 0,$$

where $c(z(1)) = -mR_2^{N+p-2}\left(\tilde{c}(z(1))\right)^{p-1}$, and hence the problem (A.11) is transformed into

$$\begin{cases} -\left(\phi_p(z')\right)' = \lambda h(t) f(z),\ t \in (0,1), \\ z(0) = 0, \\ \phi_p\left(z'(1)\right) + c(z(1))\phi_p\left(z(1)\right) = 0. \end{cases} \tag{A.12}$$