A STUDY OF THE UPPER DOMATIC NUMBER OF A GRAPH

A Thesis
by
NICHOLAS PHILLIPS

Submitted to the Graduate School
at Appalachian State University
in partial fulfillment of the requirements for the degree of
MASTER OF COMPUTER SCIENCE

May 2017
Department of Computer Science

A STUDY OF THE UPPER DOMATIC NUMBER OF A GRAPH

A Thesis
by
NICHOLAS PHILLIPS
May 2017

APPROVED BY:

Alice McRae, Ph.D
Chairperson, Thesis Committee

Raghuveer Mohan, Ph.D
Member, Thesis Committee

Rahman Tashakkori, Ph.D
Member, Thesis Committee

James Wilkes, Ph.D
Chairperson, Department of Computer Science

Max C. Poole, Ph.D
Dean, Cratis D. Williams School of Graduate Studies

Abstract

A STUDY OF THE UPPER DOMATIC NUMBER OF A GRAPH

Nicholas Phillips
B.S., Appalachian State University
M.S., Appalachian State University

Chairperson: Alice McRae

Given a graph G we can *partition* the vertices of G into k disjoint sets represented as $\pi = \{V_1, V_2, \ldots, V_k\}$. We say a set A of vertices *dominates* another set of vertices, B, if for every vertex $b \in B$ there exists some vertex $a \in A$ adjacent to b. The *upper domatic number* of a graph G is written $D(G)$ and defined as the maximum integer k such that G can be partitioned into k sets where for every pair of sets $V_i, V_j \in \pi$ either V_i dominates V_j or V_j dominates V_i or both. In this thesis we introduce the upper domatic number of a graph and provide various results on the properties of the upper domatic number, notably that $D(G) \leq \Delta(G)$, as well as relating it to other well-studied graph properties such as the achromatic, pseudoachromatic, and transitive numbers.

Acknowledgments

First, I wish to thank my advisor, Dr. Alice McRae, for her guidance through every step of the researching and writing process. The wealth of knowledge and support Dr. McRae shared with me was invaluable to the creation of this thesis. Second, I would like to thank my committee members, Dr. Mohan and Dr. Tashakkori, for the wisdom, advice, and time they have provided me during my education.

In addition, I wish to thank Dr. Stephen Hedetniemi, Professor Emeritus at Clemson University, for proposing the upper domatic number and allowing Dr. McRae and myself to explore the idea.

Finally, I wish to thank my wonderful parents and their unwavering support in my pursuit of higher learning.

Dedication

This thesis is dedicated to my loving, marvelous, and ceaselessly inspiring partner, Madie.

Contents

List of Figures

Chapter 1: Introduction

1.1 An Overview

Around the middle of the 19th century, one of the most famous problems in graph theory was first posed: Is it true that any map drawn on a plane (i.e., a planar graph [1]) can be colored with only four colors in such a way that no two countries sharing a common border have the same color? This problem became known far and wide as the Four-Color Problem. Alfred Kempe in 1879 provided the first recorded proof of the four color theorem, but an error was found eleven years later [14]. His work was not in vein though, it was used as the basis for the theorem stating that the problem could definitely be solved with five colors. The Four-Color Problem, however, would go on unsolved for over 70 years before in the 1970s the Four-Color Problem became the Four-Color Theorem after several mathematicians constructed a proof by computer. [14]

This problem opened up an entire new realm of study in graph theory based on coloring the vertices of a graph [2]. A *graph* $G = (V, E)$ is comprised of a set of *vertices* and *edges* between vertices, denoted as $V(G)$ and $E(G)$ respectively. Maps can be easily transformed into graphs by representing countries with vertices and if two countries share a common border then the two corresponding vertices share an edge. The quality required in the Four-Color Problem that adjacent countries, or vertices, not use the same color is formalized as a *proper coloring* in graph theory. Another way to phrase the problem would be to ask do all planar graphs have a proper coloring using only four colors? One of the early fruits in the new study of coloring was the *chromatic number*, a number which encapsulated the idea of the Four-Color Problem. The chromatic number is but one of many graph parameters,

[1] A *planar graph* is a graph that can be drawn on a plane without any edges crossing over one another,

[2] Graphs can be colored by edges, vertices, or both. Here we will focus only on vertex graph colorings.

or properites, that are commonly studied and represent some quality about the graph. For example, the chromatic number is a form of measure of a quality called *independence* that we will come back to shortly.

The idea of coloring can be abstracted into the idea of *partitioning*, the process of separating the vertices in a graph into disjoint sets. We can think of each of these sets as being one particular color. We can define the chromatic number of a graph G as the smallest integer k for which the vertex set of G can be partitioned into k independent sets. An *independent set* of vertices is a set in which no two vertices are adjacent to one another. Graph theorists began investigating different ways of partitioning a graph based on different properties like independence or *domination* following the Four-Color Problem. A *dominating set* of vertices is a set where every vertex in a graph G that is not in the dominating set is adjacent to at least one vertex from the dominating set. In other words, a set D of vertices from G is a dominating set if and only if for any vertex $v \notin D$ there exists some vertex $u \in D$ such that $vu \in E(G)$[3]. In a similar way to how the chromatic number relates independence and partitioning, there is a graph parameter that relates domination and partitioning. The *domatic number* of a graph, G, is the maximum integer k such that $V(G)$ can be partitioned into k dominating sets. The term 'domatic' comes from the words 'dominating' and 'chromatic' [1].

The process of finding and studying new graph parameters is frequently based on making changes to requirements of previously studied parameters. First there was the chromatic number. Adding in a requirement for the relationships between the different color classes brought about study of the Grundy number [2] and the achromatic number [8]. Removing the independence property from the achromatic number introduces the pseudoachromatic number [15]. Lowering the requirement for the relationships between different color classes for the Grundy number creates the partial Grundy number [5]. Instead of partitioning based on independence with the the chromatic number, partitioning on domination introduced the domatic number [3]. Then changing the relationships between the color classes introduced the transitive number [11]. Lowering the domination requirement for the domatic number to only requiring domination in at least one direction instead of both between color classes

[3]Edges can more formally be written as (v, u) but we will use the shorthand vu in this thesis

creates the upper domatic number.

An understanding of the history of research around graph parameters serves to illuminate the importance and relevance of the upper domatic number. In order to better understand a problem, it often helps to view the problem under a different lens, thus tweaking the requirements of a known parameter allows the parameter to be seen from a new perspective. The numerous known and studied graph parameters all serve to further the body of knowledge not just about graphs but also about complexity and NP-completeness. By teasing out the relationships between different requirements, we are better able to understand what makes a problem more or less difficult or when a problem becomes NP-hard. The upper domatic number is a new perspective from which we can better understand other graph parameters, graphs as a whole, and how the domination property relates to complexity. For example, we will later see that the achromatic number and the upper domatic number do not have a clear inequality relationship thus suggesting that there is some similarity in the two parameters despite being basing partitions of different requirements.

1.2 Vocabulary

We will assume no knowledge of graph theory terminology here beyond the terms previously defined in this thesis. Therefore the following definitions provide a sufficient knowledge base from which any reader can follow along with the results reported here. Additionally an alphabetized list of all graph theory terminology used can be found in Appendix A.

The *order* and *size* of a graph, G, specify the number of vertices and edges respectively. We say that two vertices, v and u, are *adjacent* if $vu \in E(G)$. A *subgraph*, typically denoted as H, is a set of vertices from a graph G with all edges from G that are between any pair of vertices in H.

The *degree* of a vertex v, written as $deg(v)$, is the number of vertices adjacent to v. The maximum degree of all vertices in G is written as $\Delta(G)$ and the minimum degree is $\delta(G)$. The *open neighborhood* of a vertex v, denoted $N(v)$, is the set of vertices adjacent to v. The *closed neighborhood* of a vertex v, denoted as $N[v]$, is the set $N(v) + \{v\}$. A *leaf* is a vertex of degree one.

A *star* S_n is a graph containing only a single vertex with n leaves.

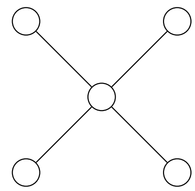

Figure 1.1: The S_4 graph

A *complete graph* K_n is a graph of order n where there is an edge between every pair of vertices. A *clique* is subgraph of a graph that is complete, i.e., an induced K_n subgraph of a graph G. A *k-regular graph* is a graph G where $\Delta(G) = \delta(G) = k$, i.e., all vertices have the same degree.

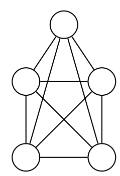

Figure 1.2: The K_5 graph which is also an example of a 4-regular graph

A *path graph* P_n is a graph of order n with exactly two leaves and $n - 2$ vertices of degree 2.

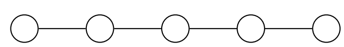

Figure 1.3: The P_5 graph

A *cycle graph* C_n is a graph of order and size n and every vertex is of degree 2. In other words, a cycle graph is an unbroken chain of adjacent vertices that start and end at the same vertex.

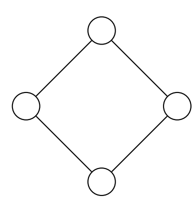

Figure 1.4: The C_4 graph

A *k-coloring* is a coloring of the vertices of a graph G using k colors. We can also think of this as a *partition* of G into k disjoint sets of vertices, written as $\pi = \{V_1, V_2, \ldots, V_k\}$.

A *proper k-coloring* is a k-coloring where no two adjacent vertices are the same color. An *independent set* of vertices is a set in which no two vertices are adjacent to one another. The *chromatic number* of a graph G, written $\chi(G)$, is the smallest positive integer k for which G has a proper k-coloring. We can also think of the chromatic number as the smallest positive integer k such that G can be partitioned into k independent sets.

A *complete k-coloring* is a proper k-coloring such that for every pair of distinct colors, there exists two adjacent vertices assigned these two colors; i.e., there exists at least one edge between every pair of color classes. The *achromatic number* of a graph G, denoted $\alpha(G)$, is the largest positive integer k such that G has a complete k-coloring. The *pseudoachromatic number* of a graph G, written as $\psi(G)$, is the largest positive integer k for which G has a k-coloring where there exists at least one edge between every distinct pair of color classes but the coloring does not have to be proper.

A *dominating set* of vertices is a set where every vertex in a graph G that is not in the dominating set, is adjacent to at least one vertex from the dominating set. The *domatic number* of a graph G, denoted $d(G)$, is the maximum integer k such that $V(G)$ can be partitioned into k dominating sets. A *d-partition* of a graph G is a partition π that achieves $d(G)$.

We say a set of vertices A *dominates* another set of vertices B, written as $A \to B$, if for every vertex $b \in B$ there is some vertex $a \in A$ such that a and b are adjacent. The *upper domatic number* of a graph G, denoted $D(G)$, is the maximum integer k such that $V(G)$ can be partitioned into k sets where for every $V_i, V_j \in \pi$, where π is a partition of $V(G)$, either $V_i \to V_j$ or $V_j \to V_i$ or both. A *D-partition* of a graph G is a partition π that achieves $D(G)$.

Another important graph property we will look at is the *transitive number* of a graph, denoted $Tr(G)$. The *transitive number* of a graph G is the largest positive integer k such that G can be partitioned into k sets where for every pair of sets, $V_i, V_j \in \pi$ if $i < j$ then $V_i \to V_j$. A *transitive partition* is a partition π that achieves $Tr(G)$.

One way to visualize the domination relationships between sets of vertices in a partition is with a *domination digraph* [7]. A *digraph*, or directed graph, is a graph where edges have direction, called arcs, and every arc has a start and end vertex. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a partition of the vertices of a graph, G, into k sets. From this partition a digraph $D(\pi)$ can be constructed where the sets in π are represented by the vertices in $D(\pi)$, and there is an arc from V_i to V_j if $V_i \to V_j$. This digraph is the *domination digraph* of the partition π of a graph G.

For example, consider the graph G shown in Figure 1.5.

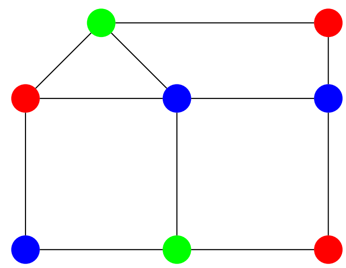

Figure 1.5: A graph G with $\pi = \{R, B, G\}$

Then the resulting domination digraph is shown in Figure 1.6.

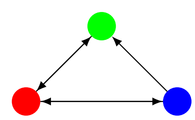

Figure 1.6: A domination digraph of the color partitions of G from Figure 1.5

A domatic partition of a graph will yield a domination digraph where every possible edge is included. On the other hand, an upper domatic partition will create a domination digraph where there is some edge between every pair of vertices. The domination digraph shown in Figure 1.6 is an example upper domatic partition. The transitive partition of a graph will yield a particular kind of domination digraph like the one shown in Figure 1.7.

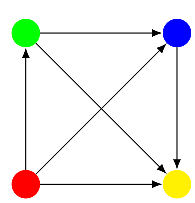

Figure 1.7: A domination digraph showing a transitive partition

In Chapter 2 we will provide a brief review of the vast literature on coloring and partitioning and a small sampling of results for the domatic number. Following that, Chapter 3 contains numerous results for the upper domatic number broken into several different categories. Chapter 4 provides results relating the upper domatic number to other graph parameters in the form of inequalities. Next, Chapter 5 lists some open problems and Chapter 6 lists references.

Chapter 2: Literature Review

A brief review of the vast literature on graph properties is presented here. The review is split into two sections, one outlining the areas of interest frequently studied in graph property research and the second reviewing the domatic number in particular.

2.1 General Review

There is a very deep, rich literature of graph theory research exploring the nature of numerous graph parameters. In particular, colorings of graphs are incredibly well studied. There are several common areas of interest for researchers to study for graph parameters such as bounds for classes of graphs, Nordhaus-Gaddum inequalities, and relating graph parameters to one another with inequality chains.

One of the most obvious questions a researcher can ask when studying a graph parameter is: What are the bounds for this parameter for different classes of graphs? Gerard Chang [1] determined bounds and values for different graphs obtained from small graphs by performing graph operations such as union, join, and Cartesian product. Another paper on the domatic number presents bounds for different classes of regular graphs such as random r-regular graphs and 3-regular random graphs [4]. Several French professors published a paper in a similar vein of study, but with an inverted approach [6]. They studied r-regular graphs that have a Grundy number of $r + 1$ and determined classes of graphs that had these qualities instead of looking at specific classes of graphs and determining the Grundy number for those classes. A little further around the world, a couple of researchers in India published a paper on the pseudoachromatic number of a graph in which they presented bounds for the parameter for a variety of classes of graphs [15].

In 1956, two researchers, Nordhaus and Gaddum [13], gave lower and upper bounds for both the sum of the chromatic number of a graph and the chromatic number of the complement along with the product of these numbers. Results of this type were soon studied for numerous graph parameters and became known as Nordhaus-Gaddum inequalities. In 1968, Frank Harary and Stephen Hedetniemi published a study on the achromatic number [8] with Nordhaus-Gaddum results for the achromatic number. Later, in 1993 Harary [9] provided numerous Nordhaus-Gaddum inequalities for a few different domination-related parameters.

The matter of relating graph parameters to one another is another area of particular interest for research. A paper on iterated colorings [10] focuses on relating the found values to other graph parameters and investigating those relationships. When Hedetniemi and Cockayne introduced the domatic number [3] they made sure to include several results relating the new parameter to other previously studied parameters. Again when Hedetniemi and others wrote on the partial grundy number [5], they included a study of the number's relationships with other graph parameters.

Clearly, answering questions about the bounds of a graph parameter, determining Nordhaus-Gaddum inequalities for a graph parameter, and relating graph parameters to one another are important steps in the study of a new parameter. With the introduction of the upper domatic number, it would be prudent to begin research on this new parameter with these general areas. Hence, these areas of research, will be the focus for this study of the upper domatic number.

2.2 Brief Review of the Domatic Number

The domatic number was first formalized by Cockayne and Hedetniemi in 1977 [3]. Perhaps the most important result of this paper was the connection drawn between the theory of domination and the theory of colorings. A simple upper bound for the domatic number was also presented as follows:

Proposition 2.1 (Cockayne, Hedetniemi)**.** *For any graph G, $d(G) \leq \delta(G) + 1$.*

This result is derived from the basic observation that in a domatic partition, a vertex, v, in some set V_i can be dominated by at most $deg(v)$ other sets. From this, we gain the term *domatically full* used to describe a graph G where $d(G) = \delta(G) + 1$.

Later, in 1990 Chang found an entire class of graphs that were always domatically full [12].

Theorem 2.1 (Chang)**.** *$d(G) = \delta(G) + 1$ for any interval graph G.*

And again in 1991 Chang found that all 2-dimensional graphs, with two exceptions, are domatically full [1].

Theorem 2.2 (Chang)**.** *For all $n_1 \geq n_2 \geq 2, d(P_{n_1} \times P_{n_2}) = 3$, unless $n_1 = n_2 = 2$ or $n_1 = 4$ and $n_2 = 2$.*

A few years after the original paper defining the domatic number, Zelinka provided a lower bound in 1983 [16].

Proposition 2.2 (Zelinka)**.** *For any graph G of order n, $d(G) \geq \lfloor \frac{n}{n-\delta(G)} \rfloor$.*

Cockayne and Hedetniemi additionally provided a Nordhaus-Gaddum result for the domatic number in the original paper [3].

Theorem 2.3 (Cockayne, Hedetniemi)**.** *For any graph G of order n, $d(G) + d(\overline{G}) \leq n + 1$, with equality if and only if $G = K_n$ or $\overline{K_n}$.*

Chapter 3: Results for the Upper Domatic Number

In this chapter, we will provide numerous results for the upper domatic number. These results are broken into subsections based on the content of the results.

3.1 Classes and General Results

Define a class of k-regular graphs, denoted NK_n, of even order n where NK_n is identical to a complete graph minus a perfect matching. A *matching* is an independent edge set, in other words, no two edges in the matching share a common vertex. A *perfect matching* is a matching in which every vertex is incident to exactly one edge of the matching.

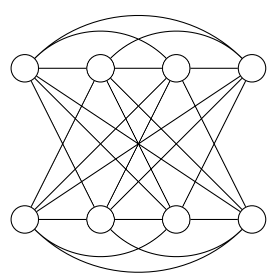

Figure 3.1: The NK_8 graph

Theorem 3.1. $D(NK_n) = \lfloor \frac{3n}{4} \rfloor$.

Proof. Let G be an arbitrary NK_n graph. Let S be a set of half of the vertices in $V(G)$ such that S is a clique of order $\frac{n}{2}$ and let S' be the remaining vertices. It is clear that S' is also a clique of order $\frac{n}{2}$, and that for any vertex $v \in S$, there exists a vertex $u \in S'$ such that $N[v] = V(G) - \{u\}$. Now, assign each vertex in S a unique color, creating $\frac{n}{2}$ singleton color classes. Every one of these colors dominates and is dominated by all of the colors currently assigned. The vertices in S' each have one color that is not adjacent. Therefore, the vertices in S' must be assigned colors in pairs. If $\frac{n}{2}$ is even, then pick any two vertices in S' and assign them a new color, and repeat until all of the vertices have been colored. Otherwise pick any two vertices in S' and assign them a new color and repeat until only 3 vertices remain. Assign all 3 vertices a new color. Therefore, we get $\lfloor \frac{n}{4} \rfloor$ colors from S'. In total, there are $\frac{n}{2}$ colors from the vertices in set S and $\lfloor \frac{n}{4} \rfloor$ colors from the vertices in set S'. Thus, $D(NK_n) \geq \lfloor \frac{3n}{4} \rfloor$.

Next we must show that $D(NK_n) \leq \lfloor \frac{3n}{4} \rfloor$. Observe that if two vertices v and u are in singleton color classes, then there must be an edge from v to u in the graph. Therefore in this graph with maximum clique size $\frac{n}{2}$, there can be no more than $\frac{n}{2}$ singleton sets in π. Therefore, $D(NK_n) \leq \frac{n}{2} + \frac{\frac{n}{2}}{2}$. Hence, $D(NK_n) = \lfloor \frac{3n}{4} \rfloor$. $\square$

We would like to next show a relationship between $D(G)$ and the maximum degree of the graph $\Delta(G)$. To do this, we will need to first present an algorithm that will be necessary in proving the relationship between $D(G)$ and $\Delta(G)$. Therefore, we will provide an algorithm that constructs a graph, G', from a graph G such that $\Delta(G') \leq \Delta(G)$. Let $\pi = \{V_1, V_2, \ldots, V_m\}$ be a D-partition of G such that $|V_1| \geq |V_2| \geq \ldots \geq |V_m|$.

Algorithm 1

Input: Graph G, D-partition of G $\pi = \{V_1, V_2, \dots, V_m\}$

Output: graph G' with $\Delta(G') \leq \Delta(G)$, π is both a D-partition and a Transitive partition for G'

1: initialize graph G' such that $G' = G$
2: remove any edges between two vertices belonging to the same set in π
3: **for** $(z = m; z > 1; z--)$ **do**
4:     **for** $(i = z - 1; i \geq 1; i--)$ **do**
5:         **while** V_i does not dominate V_z **do**
6:             find a vertex $x \in V_z$ that is not dominated by V_i
7:             find a vertex $y \in V_z$ that has multiple neighbors in V_i
8:             let ay, by, be edges between y and V_i
9:             **if** $deg(x) < \Delta(G)$ **then**
10:                 remove edge ay and add edge ax
11:             **else if** x has multiple neighbors in some other set V_p **then**
12:                 let cx be an edge between x and V_p
13:                 remove edges ay, cx and add edges ax, cy (if edge cy does not already exist)
14:             **else**
15:                 let V_k be a set in π such that there exists an edge fx where $f \in V_k$ and y has no neighbors in V_k
16:                 remove edges fx, ay and add edges ax, fy
17:             **end if**
18:         **end while**
19:     **end for**
20: **end for**

Proof of Algorithm 1 Correctness. We will show the following:

1. $\Delta(G') \leq \Delta(G)$.
2. π is a D-partition throughout the algorithm and is a transitive partition at algorithm termination.
3. Algorithm 1 terminates.

Note: we will use the term "line" here to refer to a specfic line of pseudocode in the algorithm above.

1. We can easily show that $\Delta(G') \leq \Delta(G)$ throughout the entire algorithm and at termination. Initially, $\Delta(G') = \Delta(G)$ after line 1 completes. On line 2 of the algorithm, some edges may be removed from G' therefore $\Delta(G') \leq \Delta(G)$. At line 10, vertex x will have its degree increased by one but only if $deg(x) \leq \Delta(G)$, vertex y will have it's degree decreased by one, and vertex a will have the same degree. This will maintain the relationship $\Delta(G') \leq \Delta(G)$. In lines 13 and 16, all vertices involved lose an edge and gain an edge so the degrees of each vertex is unchanged. Hence, $\Delta(G')$ never increases higher than $\Delta(G)$ but it may be lower.

2. The D-partition π will remain a D-partition for G' throughout the execution of the algorithm since no domination relationships are ever removed during execution. We will look at lines 2, 10, 13, and 16 since these are the only lines that remove edges from the graph. Line 2 does not remove any domination relationships since only edges between vertices in the same set are removed. At line 10, since π is a D-partition, either $V_i \to V_z$ or $V_z \to V_i$, but we know V_i does not dominate V_z thus $V_z \to V_i$. After line 10, the vertex $a \in V_i$ is still dominated by V_z so the domination relationship is unchanged. On line 13, either $V_p \to V_z$ or $V_z \to V_p$. If $V_p \to V_z$ then vertex $x \in V_z$ is still dominated by V_p since it has neighbors in V_p other than vertex c. If instead $V_z \to V_p$ then that remains the same since vertex $c \in V_p$ is still dominated by vertex $y \in V_z$. At line 16, set V_k does not dominate set V_z since vertex y has no neighbors in V_k. If $V_z \to V_k$ then that still holds true since vertex $f \in V_k$ is dominated by vertex $y \in V_z$ instead of by vertex $x \in V_z$. Hence, throughout execution of the while loop no domination relationships are ever removed, only new ones are added. Therefore, π remains a D-partition throughout execution of the algorithm. After the algorithm terminates, for every $V_i, V_j \in \pi$, where $i < j$, $V_i \to V_j$ therefore π is also a transitive partition.

3. In order to show that Algorithm 1 terminates, we must show that the while loop on line 4 terminates successfully. We can safely say that line 5 will always find such a vertex, otherwise the while loop condition would not be met and the statement would not be reached. As has already been shown, $V_z \to V_i$ and this relationship is not changed during the while loop. Therefore, for any vertex $v \in V_i$, there exists a vertex $u \in V_z$ such that $u \in N(v)$. But since $|V_i| \geq |V_z|$ and because there is a vertex $x \in V_z$ with no neighbor in V_i, there must be at least one vertex $y \in V_z$ with multiple neighbors in V_i. After the nested if-else tree beginning on line 9 completes, the vertices x and y will both have neighbors in V_i. Therefore, during every iteration of the while loop at least one more vertex becomes dominated by V_i than in the previous iteration of the while loop. Now we must show that at least one branch of the if-else tree beginning on line 9 will always be applicable. If the first two conditions fail, then we know that $deg(x) = \Delta(G)$ and that x has at most one neighbor in any given set. Since we know vertex y has at least two neighbors in set V_i and $deg(y) \leq deg(x)$, there must be at least one set V_k that vertex y has no neighbors in but vertex x does. Therefore, at least one branch of the if-else tree will always be applicable. $\square$

Theorem 3.2. $D(G) \leq \Delta(G) + 1$.

Proof of 3.2. Let G be an arbitrary, undirected graph of order n. Let $\pi = \{V_1, V_2, \ldots, V_m\}$ be a D-partition of G ordered such that $|V_1| \geq |V_2| \geq \ldots \geq |V_m|$. Let G' be a graph constructed from G using Algorithm 1. The construction assures that $\Delta(G') \leq \Delta(G)$. Because π is a D-partition, $D(G) = |\pi|$. With respect to G', for every $i, j, 1 \leq i \leq j \leq m$, $V_i \to V_j$ for all $V_i, V_j, \in \pi$. Thus, π is a transitive partition of G' and therefore $|\pi| \leq Tr(G')$. As Hedetniemi proved [11], $Tr(G') \leq \Delta(G')+1$. Hence, $D(G) = |\pi| \leq Tr(G') \leq \Delta(G')+1 \leq \Delta(G) + 1$, or more simply, $D(G) \leq \Delta(G) + 1$. $\square$

Corollary 3.2.1. *For any P_n where $n \geq 4$, $D(P_n) = 3$.*

Proof. For any vertex, $v \in P_n$, $deg(x) \leq 2$ so $D(P_n) \leq \Delta(P_n) + 1 = 3$. Since $n \geq 4$ we can let $a, b, c \in V(P_n)$ such that $ab, bc \in E(P_n)$. If we assign vertex a color 1, vertex b color 2, and vertex c color 3, then we just need an edge between colors 1 and 3. Let $d \in V(P_n)$. If $ad \in E(P_n)$ then assign vertex d color 3. Thus color 3 dominates colors 1 and 2 while colors 1 and 2 dominate each other. If $cd \in E(P_n)$ then assign vertex d color 1. Thus color 1 dominates colors 2 and 3 while colors 2 and 3 dominate each other. If $n \geq 4$ then we can simply assign all remaining vertices the same color as vertex d. Hence, $D(P_n) = 3$ if $n \geq 4$. $\square$

Corollary 3.2.2. *For any C_n where $n \geq 3$, $D(C_n) = 3$.*

Proof. For any vertex, $v \in C_n$, $deg(x) = 2$, so $D(C_n) \leq \Delta(C_n) + 1 = 2 + 1 = 3$. Therefore, we know that $D(C_n) \leq 3$ so we must simply show that $D(C_n) \geq 3$. Since $n \geq 3$ we can let $a, b, c \in V(C_n)$ such that $ab, bc \in E(C_n)$. If we assign vertex a color 1, vertex b color 2, and vertex c color 3, then we just need an edge between colors 1 and 3. If $n = 3$ then there exists $ac \in E(C_n)$. Otherwise, if $n > 3$ then the remaining vertices can all simply be assigned color 1. Therefore color 1 dominates all colors, and colors 2 and 3 are both dominated by and dominate each other. Thus, $D(C_n) \geq 3$. Hence, $D(C_n) = 3$ if $n \geq 3$. □

Theorem 3.3. *For any star, S_n, $D(S_n) = 2$.*

Proof. Let $c \in S_n$ be the center vertex that has n leaves, call them $v_1, v_2, \ldots, v_n$. For any leaf, v_i, $N(v_i) = \{c\}$. Without loss of generality, we can assign vertex c the color 1 and pick a leaf, v_i, and assign it the color 2. If we assign another leaf, v_j, any color but 1 or 2 then there can be no domination relationship between that color and color 2. Hence, $D(S_n) = 2$. □

Theorem 3.4. *If there exists a D-partition of a graph G with a sink set, then there exists a D-partition of G with a sink set of cardinality 1.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_n\}$ be a D-partition of a graph G with a sink set. Let $U \in \pi$ be the sink set. Because U is a sink set, we know that every element in U is dominated by every other set in π, or, more simply, every element $u \in U$ has at least one neighbor from every other set in π. Therefore, we can move all but one element from U into any other set in π, say set X, without removing any domination relationships. The one element left in U is still dominated by some element from every other set in π. The moved elements are still adjacent to at least one element from every set. Thus if set X is dominated by some subset of π each of the sets in this subset will dominate the new elements so the domination relationships are maintained. Note that if set X dominated some subset of π it still dominates each set from this subset. □

Theorem 3.5. *If there exists a D-partition of a graph G such that there exists a set V in the partition that is dominated by all but one other set, U, then there is also a D-partition of G such that set V is a sink set.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_n\}$ be a D-partition of a graph G. Let the set $V \in \pi$ be the set that is dominated by all but one other set, say set U. Since set V is not dominated by set U in a D-partition, then set V dominates set U. Note that there exists at least one vertex $v \in V$ that does not have a neighbor in set U. If we move all such vertices from set V to set U then set V becomes a sink set. Now every element in V has a neighbor in every other set, thus set V is dominated by every other set. The moved elements now in U are still adjacent to at least one element from every set other than set V. Thus if set U is dominated by some subset of π each of the sets in this subset will dominate the new elements so the relationships are maintained. Note also if set U dominated some subset of π it still dominates each set from this subset. □

3.2 Subgraphs and Joins

3.2.1 Subgraphs

Theorem 3.6. *For a graph, G, $D(G) \geq \mu$ where μ is the size of the largest clique subgraph in G.*

Proof. Let G be an arbitrary graph of order n such that the largest clique of the graph is of order μ. Let $V(\mu) = \{v_1, v_2, \ldots, v_\mu\}$ be the set of vertices in the largest clique of G. Assign each vertex in $V(\mu)$ a unique color. The rest of the graph can be colored any one previously used color to achieve $D(G) = \mu$. If every vertex, $v_i \in V(\mu)$ has at least one neighbor not in $V(\mu)$ then the rest of the graph can be assigned the color $\mu + 1$ to achieve $D(G) > \mu$. □

Theorem 3.7. *For a graph G with subgraph H there is no strict relation between $D(G)$ and $D(H)$.*

Proof. First we will show that $D(G)$ can be larger than $D(H)$. Consider the complete graph of order n, K_n. Let m be a positive integer such that $m < n$. Then there is a subgraph K_m of the graph K_n. As shown previously, $D(K_n) = n$. Hence, $D(K_n) > D(K_m)$.

Second we will show that $D(H)$ can be larger than $D(G)$. Consider the graph G shown in Figure 3.2. By Theorem 3.2 we can see that $D(G) \leq 4$. But because of the isolate vertex, there must be a set that dominates all other sets in the upper domatic partition of G. This can not be achieved in this graph. However, an upper domatic partition of size 3 can be easily achieved in several ways. Now consider the subgraph, H, of G shown in Figure 3.2

The removal of the isolate in the subgraph H allows for an upper domatic partition of size 4 to be achieved for H. Hence, $D(H) > D(G)$. □

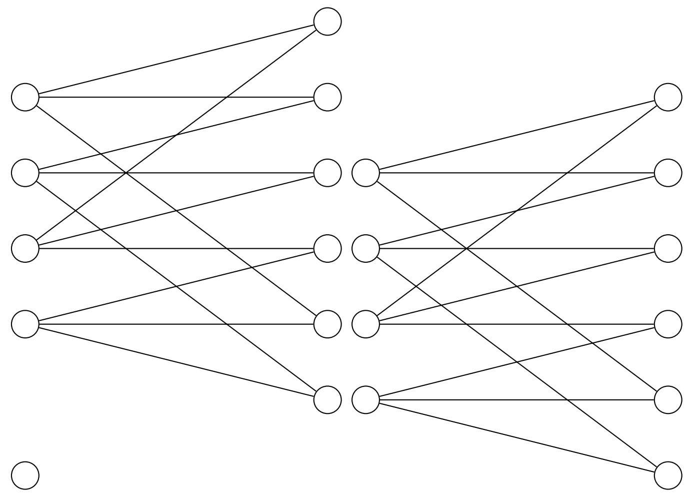

Figure 3.2: Left: the graph G. Right: the subgraph H

3.2.2 Joins

The graph $G + H$, called a *join graph*, is formed from graphs G and H with an edge added between every vertex in G and every vertex in H. An example is shown in Figure 3.3.

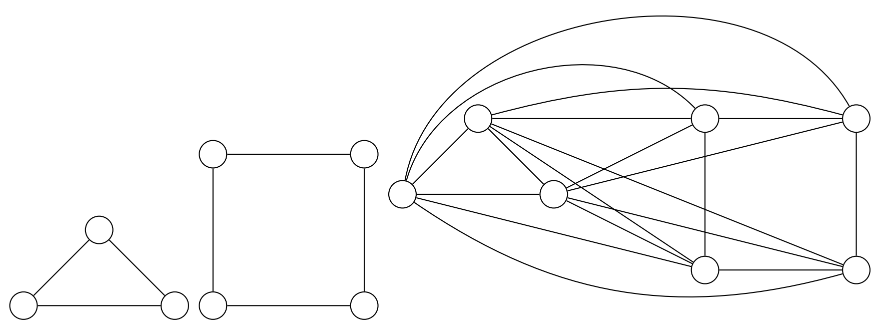

Figure 3.3: From left to right: the graph G, the graph H, the join graph $G + H$

Let G be an undirected graph. Let H be a subgraph of G such that $D(H)$ is the highest upper domatic number for any subgraph of G.

Theorem 3.8. $D(G + K_1) \geq D(H) + 1$.

Proof. Let π be a D-partition of H. Then we can color all remaining vertices in $G + K_1$ a new color. This new color dominates all other colors in π because every vertex in H has at least one neighbor in $V(G + K_1) - V(H)$. Hence, $D(G + K_1) \geq D(H) + 1$. $\square$

Let G and H be two undirected graphs.

Theorem 3.9. $D(G) + D(H) \leq D(G + H)$.

Proof. Consider the graph, G, shown in Figure 3.4.

Figure 3.4: A graph G

Clearly, $D(G) = 1$.

Now consider the join of G with itself, as shown in Figure 3.5. Note that we will simply refer to the second copy of G as H.

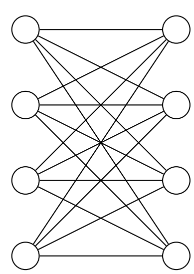

Figure 3.5: The join of G with itself, i.e. $G + H$

We can assign one color to the top pair of vertices, a second color to the next pair of vertices, and so on. This results in a $D - partition$ using 4 colors. Thus, $D(G + H) = 4$. But $D(G) + D(H) = 2$. Therefore, we can see that $D(G) + D(H) \leq D(G + H)$. $\square$

3.3 Corona

In this section we we will discuss the corona $G \circ K_1$. The corona is constructed by using a graph G and attaching a leaf vertex to every vertex in G. We will refer to these leaf vertices as being *on the corona* and the vertices of G as being *inside the corona*. When talking about a specific vertex on the corona, we will call the adjacent vertex that is inside the corona as the *supporting vertex*. An example corona can be seen in Figure 3.6.

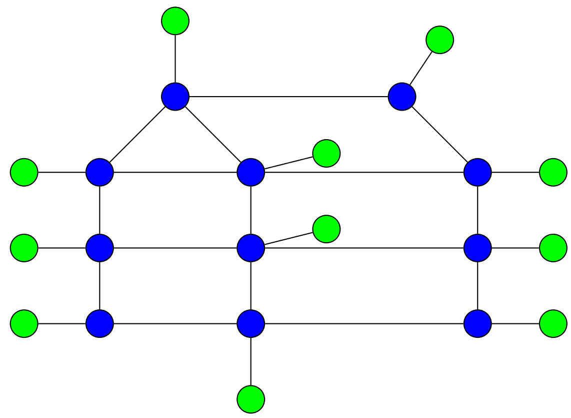

Figure 3.6: A corona with blue vertices *inside* the corona and green vertices *on* the corona

Lemma 3.1. *If $D(G \circ K_1) \geq 4$ then there are no more than two distinct colors assigned to the vertices on the corona.*

Proof. Assume $D(G \circ K_1) \geq 4$. Suppose for the sake of contradiction that there are three or more distinct colors assigned to the vertices on the corona. Let X, Y, and Z be three of the colors on the corona. If a vertex on the corona is assigned the color X, then we know that X must dominate all colors with the possible exception of whatever color is assigned to the supporting vertex. This is true for any color on the corona. So suppose the supporting vertex is assigned the color Y. This leaves 2 cases, either Y dominates X or Y does not dominate X.

If Y dominates X, then X must dominate Z and therefore Z must dominate Y. This the corona colors form a cycle domination digraph.

Else if Y does not dominate X then X must dominate Y and thus Y must dominate Z and Z must dominate X. Again, the corona colors form a cycle domination digraph.

Now suppose some fourth color, A, is assigned to a vertex inside the corona. The corona vertex adjacent to this vertex cannot be assigned any of the corona colors because it would sever the relationship between that color and the color that dominates it. This is a contradiction. Hence, if $D(G \circ K_1) \geq 4$ then there are no more than two distinct colors assigned to the vertices on the corona. □

Let G be an undirected graph. Let H be a subgraph of G such that $D(H)$ is the highest upper domatic number for any subgraph of G.

Theorem 3.10. *$D(G \circ K_1) = D(H) + 1$ for $D(G \circ K_1) \geq 4$.*

Proof. We will first show that $D(G \circ K_1) \geq D(H)+1$ and then that $D(G \circ K_1) \leq D(H)+1$
Case 1: $D(G \circ K_1) \geq D(H) + 1$.

Let π be a D-partition of H. Then we can color all remaining vertices in $G \circ K_1$ a new color. This new color dominates all other colors in π because every vertex in H has at least one neighbor in $V(G \circ K_1) - V(H)$. Hence, $D(G \circ K_1) \geq D(H) + 1$.
Case 2: $D(G \circ K_1) \leq D(H) + 1$.

Note that $D(G \circ K_1) \leq D(H) + 1$ is equivalent to $D(H) \geq D(G \circ K_1) - 1$. Also note that we want to find a subgraph H of G such that there is an upper domatic partition of H that uses at least $D(G \circ K_1) - 1$ colors. Let π be a D-partition of $G \circ K_1$.

Suppose there is one color and one color only on the vertices of the corona in $G \circ K_1$, call it X. Remove all vertices that are in set X. This leaves a subgraph, H such that there is one less color in π. Therefore $\pi - \{X\}$ is still an upper-domatic partition for H. Hence, $D(H) \geq D(G \circ K_1) - 1$.

Suppose there are two colors on the vertices of the corona in $G \circ K_1$, call them X and Y. Without loss of generality let X dominate Y since one must dominate the other. Remove all vertices in the set X and all corona vertices to create a subgraph, H. Clearly, none of the colors aside from X and Y are affected. Since Y was dominated by X then any corona vertex that was color Y was adjacent to one and only one vertex which had to be color X. Therefore all remaining colors had to be dominated by Y vertices not on the corona and Y still dominates all colors. Hence, $D(H) \geq D(G \circ K_1) - 1$. □

3.4 Nordhaus-Gaddum

Here we will look at Nordhaus-Gaddum results for the upper domatic number. Nordhaus-Gaddum results relate properties of a graph and the complement of a graph in inequalities.

The *complement* of a graph G, denoted $\overline{G}$, has the same vertices as G and all possible edges not found in G.

Note that S_n is the star graph with n leaf vertices.

Theorem 3.11. *$D(S_n) + D(\overline{S_n}) = 2 + n$.*

Proof. As was previously shown, $D(S_n) = 2$. Therefore we must show that $D(\overline{S_n}) = n$. Note that the graph $\overline{S_n}$ is a single isolate vertex and a clique of order n. Thus, $D(\overline{S_n}) \geq n$. The only other vertex in the graph is an isolate which must be a member of a dominating set. Since every color used is in a clique, all colors are dominating sets. Thus, the isolate must be assigned one of the colors already used and can not be a new color. Therefore, $D(\overline{S_n}) \leq n$. Hence, $D(S_n) + D(\overline{S_n}) = 2 + n$. $\square$

Theorem 3.12. *$D(P_n) + D(\overline{P_n}) \leq \left\lfloor \frac{3(n+1)}{4} \right\rfloor + 3$ for $n \geq 4$.*

Proof. As was previously shown, $D(P_n) = 3$ if $n \geq 4$. Therefore we must show that $D(\overline{P_n}) \leq q \left\lfloor \frac{3(n+1)}{4} \right\rfloor$. The graph P_n has an independent set of $\left\lceil \frac{n}{2} \right\rceil$ vertices starting with one leaf vertex and adding every second vertex after that. This same set of vertices is a clique in $\overline{P_n}$. We know that every vertex in a clique can belong to it's own set so $D(\overline{P_n}) \geq \left\lceil \frac{n}{2} \right\rceil$. Of the remaining vertices, every new color must have at least two vertices in order to have some domination relation with each color in the clique. Therefore $D(\overline{P_n}) \geq \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{4} \right\rfloor$. $\square$

Theorem 3.13. *$D(C_n) + D(\overline{C_n}) \leq \left\lfloor \frac{3n}{4} \right\rfloor + 3$ for $n > 4$.*

Proof. As was previously shown, $D(C_n) = 3$ for $n \geq 3$. Therefore we must show that $D(\overline{C_n}) \leq \left\lfloor \frac{3n}{4} \right\rfloor$. The graph P_n has an independent set of $\left\lfloor \frac{n}{2} \right\rfloor$ vertices starting with any vertex and adding every second vertex after that. This same set of vertices is a clique in $D(\overline{C_n})$. Thus $D(\overline{C_n}) \geq \left\lfloor \frac{n}{2} \right\rfloor$. Of the remaining vertices, every new color must have at least two vertices in order to have some domination relation with each color in the clique. Therefore $D(\overline{C_n}) \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor$. $\square$

Chapter 4: Inequalities for the Upper Domatic Number

In this chapter we will evaluate the upper domatic number's relationships with various other previously studied graph parameters. We will relate the upper domatic number to the domatic number, the transitive number, achromatic number, and the pseudoachromatic number. As previously defined, the domatic number of a graph G is the maximum integer k such that $V(G)$ can be partitioned into k dominating sets, denoted $d(G)$.

Theorem 4.1. *$d(G) \leq D(G)$.*

Proof. First note that every d-partition is also a D-partition but every D-partition is *not* a d-partition. Therefore we know that at least $d(G) = D(G)$. Now consider the graph C_4. We know that $D(C_4) = 3$ by corollary 3.2.2. But $d(C_4) = 2$ since there are only two dominating sets in C_4. Hence $d(G) \leq D(G)$. □

We can also remember the transitive number of a graph G is the largest positive integer k such that G can be partitioned into k sets where for every pair of sets, $V_i, V_j \in \pi$ if $i < j$ then $V_i \to V_j$.

Theorem 4.2. *$Tr(G) \leq D(G)$.*

Proof. First note that every transitive partition is also a D-partition but every D-partition is *not* a transitive-partition. Therefore we know that at least $Tr(G) = D(G)$. Now consider the graph G shown in Figure 4.1. Then consider the partition of G and the resulting domination digraph shown in Figure 4.2.

Clearly, $D(G) = 4$. Now we must determine $Tr(G)$. First note that a transitive partition of size k requires two adjacent vertices each with degree at least $k - 1$. This is required so that the sink set in a transitive partition and the set dominated by every other set aside from

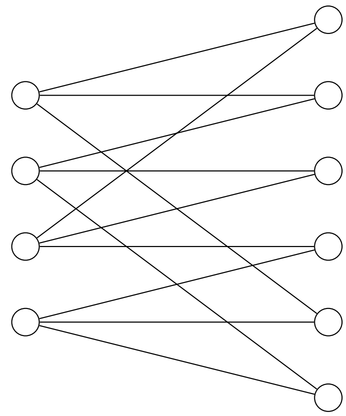

Figure 4.1: The graph G

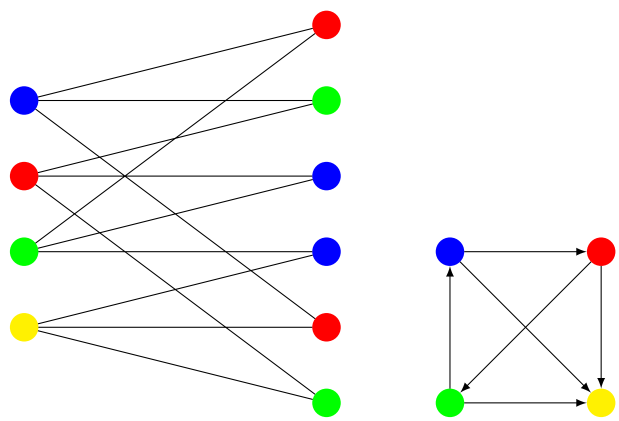

Figure 4.2: G and resulting domination digraph

the sink set can each have a neighbor from every other set. Otherwise, the set dominated by every set but the sink set would not be able to dominate the sink set and by dominated by all other sets. But G only has two adjacent vertices of at least degree 2. Thus, $Tr(G) = 3$. Hence, $Tr(G) \leq D(G)$. □

Recall that the achromatic number of a graph is the largest positive integer k such that G has a complete k-coloring and is denoted $\alpha(G)$.

Theorem 4.3. *There is no relation between the upper domatic number of a graph and the achromatic number of a graph.*

Proof. To show that there is no relation between $D(G)$ and $\alpha(G)$ we will show that $D(G)$ can be either arbitrarily larger or smaller than $\alpha(G)$. First we will show that $D(G)$ can be arbitrarily larger than $\alpha(G)$. Consider the NK_n class of graphs described in the previous chapter. We know by theorem 3.1 that $D(NK_n) = \left\lfloor \frac{3n}{4} \right\rfloor$. So we must show that $D(NK_n)$ can be arbitrarily larger than $\alpha(NK_n)$. Let S be a set of half of the vertices in $V(NK_n)$ such that S is a clique of order $\frac{n}{2}$ and let S' be the remaining vertices. It is clear that S' is also a clique of order $\frac{n}{2}$. It is also clear that for any vertex $v \in S$ there exists a vertex $u \in S'$ such that $N[v] = V(G) - \{u\}$. Now, assign each vertex in S a unique color. If a new color is assigned to one of the vertices $v \in S'$ then there will be some vertex $u \in S$ such that $vu \notin E(NK_n)$. So there must be at least two vertices assigned the new color in S'. But they will be neighbors so the coloring is not proper. Hence, there can be no new colors in S'. Each pair of vertices, one in S and one in S' that are not adjacent must be assigned the same color. Thus, $\alpha(NK_n) = \frac{n}{2}$. Therefore $\alpha(NK_n) < D(NK_n)$.

Now we must show that $D(G)$ can be arbitrarily smaller than $\alpha(G)$. Consider a graph, G, that is comprised of s disjoint stars, each of which has n leaves. We know that $D(S_n) = 2$ but that is only for a single star. Thankfully, $D(G) = 2$ as well since having two leaves with two different colors both adjacent to the same center vertex means that the center vertex must be one of those two colors in order to have some domination relationship between the two colors. This means that any additional disjoint stars in a graph can not increase the upper domatic number since no new colors can be introduced. Hence,$D(G) = 2$. On the other hand, the achromatic number can take advantage of the additional disjoint stars. Let $s = n + 1$, in other words every disjoint has as many leaves as there are disjoint stars in the graph plus one. Then a star can have a color in the center and a different color on every leaf. So if every star has a different color in the center and the remaining colors on the leaves, then there is at least one edge between every pair of colors. Thus, $\alpha(G) = s$. Therefore, $\alpha(G)$ can scale with the number of disjoint stars and be arbitrarily larger than $D(G)$. Hence, there is no relation between $D(G)$ and $\alpha(G)$. □

Also recall that the pseudoachromatic number of a graph, denoted $\psi(G)$ is the same as the achromatic number but that the color classes do not need to be independent.

Theorem 4.4. *For any graph, G, $D(G) \leq \psi(G)$.*

Proof. Clearly any D-partition of a graph is also a pseudoachromatic coloring of the graph. Therefore we know that at least $D(G) = \psi(G)$. Consider the graph C_8 with the coloring as shown:

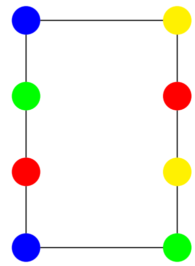

Figure 4.3: The C_8 graph with partition $\pi = \{B, R, G, Y\}$

Here we can see that $\psi(C_8) = 4$ since there is at least one edge between every pair of distinct color classes. But we know that $D(C_8) = 3$ from corollary 3.2.2. Hence, $D(G) \leq \psi(G)$. $\square$

From these results, two inequality chains become clear:

$$d(G) \leq D(G) \leq \psi(G)$$

and:

$$Tr(G) \leq D(G) \leq \psi(G)$$

Therefore we must determine the relationship between $Tr(G)$ and $d(G)$.

Theorem 4.5. *$d(G) \leq Tr(G)$.*

Proof. Clearly, by the definition of the domatic and transitive numbers, we can see that every d-partition is also a transitive partition but not vice versa. Therefore we know that at least $d(G) = Tr(G)$. Consider the house graph, G, shown in Figure 4.4 with two different partitions, π and π'. If we examine the graph G we see that there are two dominating sets possible, the red and blue sets in π are both dominating sets. Then, if we note that π is a transitive partition and that π' is a d-partition, we can clearly see that $Tr(G) = 3$ and $d(G) = 2$. Hence, $d(G) \leq Tr(G)$. $\square$

Now we can construct a single inequality chain to relate the domatic, transitive, upper domatic, and pseudoachromatic numbers:

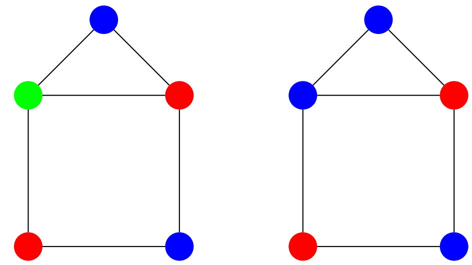

Figure 4.4: G with partition π (left) and partition π' (right)

Corollary 4.5.1. *$d(G) \leq Tr(G) \leq D(G) \leq \psi(G)$.*

Proof. This result follows directly from Theorem 4.5 combined with the previous inequality chains. $\square$

Chapter 5: Open Problems

- Is there a graph G with $|V(G)| < 10$ where $D(G) < \alpha(G)$?
- What is the smallest $\delta(G)$ such that $D(G) < \delta(G)$?
- Is there a high level algorithm for finding an upper domatic partition of a graph?
- How much larger can $D(H)$ be than $D(G)$ where H is a subgraph of G?
- Do there exist graphs with no D-partition that contains a sink set?
- What is the time complexity of deciding if a given graph has $D(G) \geq 4$? $D(G) \geq 5$?
- What classes of graphs can be described as "upper domatically full," if any?

Bibliography

[1] Gerard J. Chang. The domatic number problem. *Discrete Mathematics*, 125(1):115–122, 1994.

[2] Claude A. Christen and Stanley M. Selkow. Some perfect coloring properties of graphs. *Journal of Combinatorial Theory Series B*, 27(1):49–59, 1979.

[3] E. J. Cockayne and S. T. Hedetniemi. Towards a theory of domination in graphs. *Networks*, 7:247–263, 1977.

[4] Peter Dankelmann and Neil Calkin. The domatic number of regular graphs. *Ars Comb.*, 73, 2004.

[5] Paul Erdos, Stephen T. Hedetniemi, Renu C. Laskar, and Geert C.E. Prins. On the equality of the partial grundy and upper ochromatic numbers of graphs. *Discrete Mathematics*, 272(1):53–64, 2003.

[6] Nicolas Gastineau, Hamamache Kheddocui, and Oliver Togni. On the family of r-regular graphs with grundy number r+1. *Discrete Mathematics*, 328:5–15, 2014.

[7] Wayne Goddard, Sandra M. Hedetniemi, Stephen T. Hedetniemi, and Alice A. McRae. The algorithmic complexity of domination digraphs. *Journal of Combinatorial Mathematic and Combinatorial Computing*, 80:367–384, 2012.

[8] Frank Harary and Stephen Hedetniemi. The achromatic number of a graph. *Journal of Combinatorial Theory*, 8(2):154–161, 1970.

[9] Frank Haray and Teresa W. Haynes. Combinatorics nordhaus-gaddum inequalities for domination in graphs. *Discrete Mathematics*, 155(1):99–105, 1996.

[10] Sandra M. Hedetniemi, Stephen T. Hedetniemi, Alice A. McRae, Dee Parks, and Jan Arne Telle. Iterated colorings of graphs. *Discrete Mathematics*, 278(1-3):81–108, 2004.

[11] Stephen T. Hedetniemi, Alice A. McRae, and Nicholas K. Phillips. The transitivity of a graph. *Author's Notes*.

[12] Tung-Lin Lu, Pei-Hsin Ho, and Gerard J. Change. The domatic number problem in interval graphs. *Society for Industrial and Applied Mathematics*, 3(4):531–536, 1990.

[13] E. A. Nordhaus and J. W. Gaddum. On complementary graphs. *American Mathematical Monthly*, 63:175–177, 1956.

[14] Vitaly I. Voloshin. Graph coloring: History, results, and open problems. *Alabama Journal of Mathematics*, Spring/Fall, 2009.

[15] V. Yegnanarayanan and B. Logeshwary. On vertex coloring of graphs. *International Journal of Mathematical Analysis*, 9(17):857–868, 2015.

[16] Bohdan Zelinka. On k-domatic numbers of a graph. *Czech. Math. J.*, 33:309–313, 1983.

Appendix: Alphabetized Vocabulary

Here we provide an alphabetized list of most of the terms used in this thesis for easy reference.

Definition 1. The *achromatic number* of a graph G is the largest positive integer k such that G has a complete k-coloring, denoted $\alpha(G)$.

Definition 2. Two distinct vertices v and u are *adjacent* if $vu \in E(G)$ and we say v and u are neighbors.

Definition 3. The *chromatic number* of a graph G is the smallest positive integer k for which G has a proper k-coloring and is written as $\chi(G)$.

Definition 4. A *clique* is subgraph of a graph that is complete, i.e. an induced K_n subgraph of a graph G.

Definition 5. The *closed neighborhood* of a vertex v is the set of adjacent vertices plus v, denoted $N[v]$.

Definition 6. The *complement* of a graph G, denoted $\overline{G}$, has the same vertices as G and all possible edges not found in G.

Definition 7. A *complete graph* K_n is a graph of order n where there is an edge between every pair of vertices.

Definition 8. A *complete k-coloring* is a proper k-coloring such that for every pair of distinct colors, there exists two adjacent vertices assigned these two colors, i.e. there exists at least one edge between every pair of color classes.

Definition 9. A *cycle graph* C_n is a graph of order and size n and every vertex is of degree 2. In other words, a cycle graph is an unbroken chain of adjacent vertices that start and end at the same vertex.

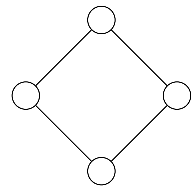

Figure 1: The C_4 graph

Definition 10. The *degree* of a vertex v is the number of adjacent vertices, denoted $deg(v)$.

Definition 11. A *digraph*, or directed graph, is a graph where edges have direction, called arcs, and every arc has a start and end vertex.

Definition 12. The *domatic number* of a graph G is the maximum integer k such that $V(G)$ can be partitioned into k dominating sets, denoted $d(G)$.

Definition 13. A *d-partition* of a graph G is a partition π that achieves $d(G)$.

Definition 14. A *D-partition* of a graph G is a partition π that achieves $D(G)$.

Definition 15. A *dominating set* of vertices is a set where every vertex in a graph G that is not in the dominating set, is adjacent to at least one vertex from the dominating set.

Definition 16. A *domination digraph*, denoted $D(\pi)$, is a digraph representation of a partition π of a graph G where the vertices represent the k sets of π, and there is an arc from V_i to V_j if $V_i \rightarrow V_j$.

Definition 17. A *graph* G is a set $V(G)$ of vertices and a set $E(G)$ of edges.

Definition 18. An *independent set* of vertices is a set in which no two vertices are adjacent to one another.

Definition 19. A *k-coloring* is a coloring of the vertices of a graph G using k colors.

Definition 20. A *k-regular graph* is a graph G where $\Delta(G) = \delta(G)$, i.e. all vertices have the same degree.

Definition 21. A *leaf* is a vertex of degree 1.

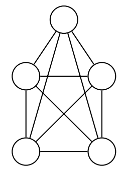

Figure 2: The K_5 graph which is also an example of a 4-regular graph

Definition 22. The *open neighborhood* of a vertex v is the set of adjacent vertices, denoted $N(v)$.

Definition 23. The *order* of a graph G is the number of vertices, i.e., $|V(G)|$.

Definition 24. A *path graph* P_n is a graph of order n with exactly two leaves and $n-2$ vertices of degree 2.

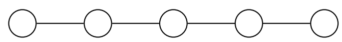

Figure 3: The P_5 graph

Definition 25. A *proper k-coloring* is a k-coloring where no two adjacent vertices are the same color.

Definition 26. The *pseudoachromatic number* of a graph G is the largest positive integer k for which G has a k-coloring where there exists at least one edge between every distinct pair of color classes but the coloring does not have to be proper, denoted $\psi(G)$.

Definition 27. A *star* S_n is a graph containing only a single vertex with n leaves.

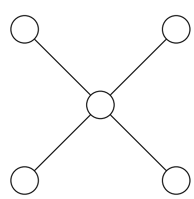

Figure 4: The S_4 graph

Definition 28. The *size* of a graph G is the number of edges, i.e., $|E(G)|$.

Definition 29. A *subgraph*, typically denoted as H, is a set of vertices from a graph G with all edges from G that are between any pair of vertices in H.

Definition 30. The *transitive number* of a graph G is the largest positive integer k such that G can be partitioned into k sets where for every pair of sets, $V_i, V_j \in \pi$ if $i < j$ then $V_i \to V_j$.

Definition 31. A *transitive partition* is a partition that achieves $Tr(G)$.

Definition 32. The *upper domatic number* of a graph G is the maximum integer k such that $V(G)$ can be partitioned into k sets where for every $V_i, V_j \in \pi$, where π is a partition of $V(G)$, either $V_i \to V_j$ or $V_j \to V_i$ or both.

Vita

Nicholas Phillips was born in Denver, Colorado, before moving to North Carolina a few months later. He graduated from Mooresville High School in 2012 and entered university at Appalachian State University that fall. In may 2016, he earned two Bachelor's of Science degrees, in Computer Science and Computational Mathematics. Nicholas remained at Appalachian to begin pursuing a Master's of Science degree. He received this degree in May 2017.