A LIFTING OF GRAPHS TO 3-UNIFORM HYPERGRAPHS, ITS GENERALIZATION, AND FURTHER INVESTIGATION OF HYPERGRAPH RAMSEY NUMBERS

A thesis presented to the faculty of the Graduate School of Western Carolina University in partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics

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A LIFTING OF GRAPHS TO 3-UNIFORM HYPERGRAPHS, ITS GENERALIZATION, AND FURTHER INVESTIGATION OF HYPERGRAPH RAMSEY NUMBERS

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Ramsey theory has posed many interesting questions for graph theorists that have yet to be solved. Many different methods have been used to find Ramsey numbers, though very few are actually known. Because of this, more mathematical tools are needed to prove exact values of Ramsey numbers and their generalizations. Budden, Hiller, Lambert, and Sanford have created a lifting from graphs to 3-uniform hypergraphs that has shown promise. They believe that many results may come of this lifting, and have discovered some themselves. This thesis will build upon their work by considering other important properties of their lifting and analogous liftings for higher-uniform hypergraphs. We also consider ways in which one may extend many known results in Ramsey Theory for graphs to the r-uniform hypergraph setting.

1 INTRODUCTION

1.1 Purpose and Significance of the Study

Ramsey theory for graphs began when Frank Ramsey posed the question "How many people must be gathered to guarantee that there are three mutual acquaintances or three mutual strangers?" [12] The answer to this question turns out to be six. It was determined by graph theorists by trying to find the maximum number of vertices such that every red/blue coloring of the edges of K_n (a complete graph on n vertices) will either have a red K_s-subgraph or a blue K_t-subgraph. Note that a graph G is a finite, non-empty set V together with a set E, of distinct two-element subsets of distinct elements of V. Each element of V is called a vertex, while each element of E is called an edge. We may denote V and E by $V(G)$ and $E(G)$ respectively. An edge of a graph G is said to join two vertices a and b and the vertices a and b are said to be adjacent. A complete graph is a graph G where every set of two vertices are adjacent. We will denote the minimum number of vertices needed to guarantee that every red/blue coloring of the edges of K_n (the complete graph on n vertices) contains either a red K_s-subgraph or a blue K_t-subgraph by the Ramsey number $R(K_s, K_t)$. So in the case of the party problem, $R(K_3, K_3) = 6$. The proof for this theorem is as follows.

Consider a red-blue coloring of a K_6. In this K_6 there is a vertex u that is adjacent to the other five vertices in the graph.

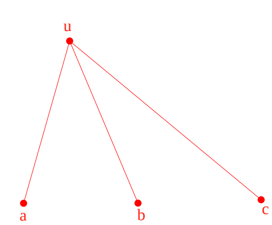

Figure 1: Red edges emanating from u

From this vertex u, we are guaranteed by the Pigeonhole Principle that at least three out of the five vertices must have edges of one color. Without loss of generality assume that there are three red edges emanating from u, say $\{ua, ub, uc\}$. If any one of the edges $\{ab, bc, ac\}$ are colored red then we are guaranteed to have a red K_3. Otherwise these edges are all colored blue, and we are guaranteed a blue K_3. Therefore $R(K_3, K_3) \leq 6$.

To show that $R(K_3, K_3) > 5$, we will show that there exist a K_5 that has no red or blue K_3. Consider coloring the outer edges of the K_5 with only red edges and all interconnecting edges blue.

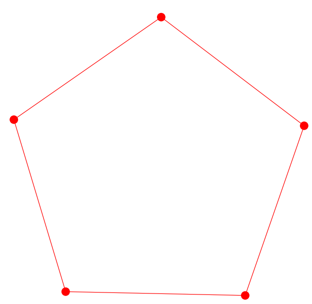

Figure 2: Red Outer Edges of K_5

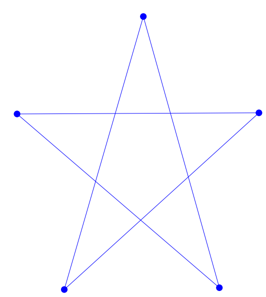

Figure 3: Blue Interconnecting Edges of K_5

If we combine these two, we will have a complete K_5 with no red or blue K_3 as a subgraph. Thus $R(K_3, K_3) > 5$. Since $R(K_3, K_3) \leq 6$ and $R(K_3, K_3) > 5$ then $R(K_3, K_3) = 6$. [4]

The study of Ramsey Numbers becomes extremely complicated since arguments used in the proof for $R(K_3, K_3) = 6$ become too complex once s and t become large enough. Note that we may use $R(s,t)$ to denote a Ramsey Number instead of $R(K_s, K_t)$. The following are the only known Ramsey Numbers [11]:

- $R(1,t) = 1$
- $R(2,t) = t$
- $R(3,3) = 6$ (Ramsey, 1930)
- $R(3,4) = 9$, $R(3,5) = 14$, $R(4,4) = 18$ (Greenwood and Gleason 1955)

- $R(3,7) = 23$ (Kalbfleisch 1966)
- $R(3,6) = 18$ (Graver and Yackel 1968)
- $R(3,9) = 36$ (Grinstead and Roberts 1982)
- $R(3,8) = 28$ (McKay and Min 1992)
- $R(4,5) = 25$ (McKay and Radziszowski 1995)

After these simpler cases, the opportunity to find an exact Ramsey number becomes extremely difficult due to all possible combinations of vertices to form an edge in each graph; there are $\binom{n}{2}$ edges in a complete graph, and $2^{\binom{n}{2}}$ ways to two color them, without considering isomorphic graphs. What mathematicians have been doing is creating bounds for Ramsey Numbers, to eventually reach a single number. The following are known upper and lower bounds for some Ramsey Numbers [11]

$$\begin{array}{cc} 36 \leq R(4,6) \leq 41 & 49 \leq R(4,7) \leq 61 \\ 43 \leq R(5,5) \leq 49 & 58 \leq R(5,6) \leq 87 \\ 102 \leq R(6,6) \leq 165 & 113 \leq R(6,7) \leq 298 \\ 565 \leq R(9,9) \leq 6588 & \end{array}$$

Once both s and t start to become numbers greater than 5, there is even more ambiguity to what the actual Ramsey Number really is. In fact, closing the bounds on Ramsey Numbers listed above are extremely difficult. Erdős, one of the most published researchers in this field, is famously quoted in [9] with an explanation on the difficulty of determining these numbers:

"Suppose that evil aliens land on the earth and say that they are going to come back in five years and blow it up, unless humankind can tell them the value of $R(5,5)$ when they come back. Then all the mathematicians and computer scientists of the world should get together, and using all the computers in the world, we would probably be able to compute $R(5,5)$ and save the earth. But what if the aliens had instead said that they would blow up the earth unless we could calculate $R(6,6)$ in five years? In that case, the best strategy that humankind could follow would be to divert everyone's energy and resources into weapons research for the next five years."

– P. Erdős (paraphrased)

This quote greatly describes the difficulty of finding specific Ramsey numbers. Mathematicians so far have been able to find Ramsey numbers for specific types of graphs such as trees, stars, paths, and a few others, usually given a condition on the number of vertices. With these Ramsey numbers, it is possible to take graphs, defined with edges that are a combination of two vertices, and generalize them to graphs where an edge is a combination of r vertices instead.

1.2 r-Uniform Hypergraphs

Remember that a graph G is a finite, non-empty set V together with a set E, of distinct two-element subsets of distinct elements of V. We will take this definition of a 2-uniform graph and generalize it for an r-uniform hypergraph. So an r-uniform hypergraph G is a finite, non-empty set V together with a set E, of distinct r-element subsets of distinct elements of V. Each element of V is called a vertex, while each element of E is called a hyperedge. We may denote V and E by $V(G)$ and $V(G)$ respectively. A hyperedge of a graph G is said to join r vertices $v_1, v_2, ..., v_r$ and the vertices $v_1, v_2, ..., v_r$ are said to be adjacent.

For r-uniform hypergraphs, depicting hyperedges is not as straight forward as connecting two vertices with a line. For any hypergraph we will depict a hyperedge by circling the group of vertices within that certain hyperedge. For example, consider a 3-uniform hypergraph with the vertex set $V(G) = \{a,b,c,d,e\}$ and hyperedge set $E(G) = \{abc, ade\}$ depicted in Figure 4.

By how we depict these hypergraphs, adding in more edges means that it will become harder to depict. In this hypergraph, we only depict two out of the possible $\binom{5}{3} = 10$ hyperedges, so depicting more hypergraphs can only become more clustered. To illustrate this, assume the same hypergraph as before, but with the hyperedge set $E(G) =$

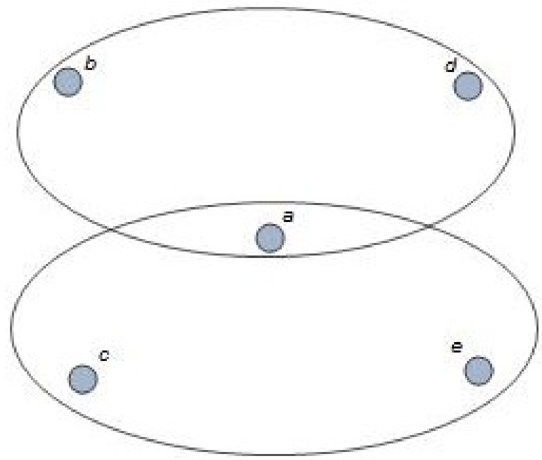

Figure 4: 3-Uniform Hypergraph, $V(G) = \{a,b,c,d,e\}$, $E(G) = \{abc, ade\}$

$\{abc, ade, ace, abd\}$, as seen in Figure 5. As we can see, when we include four of the

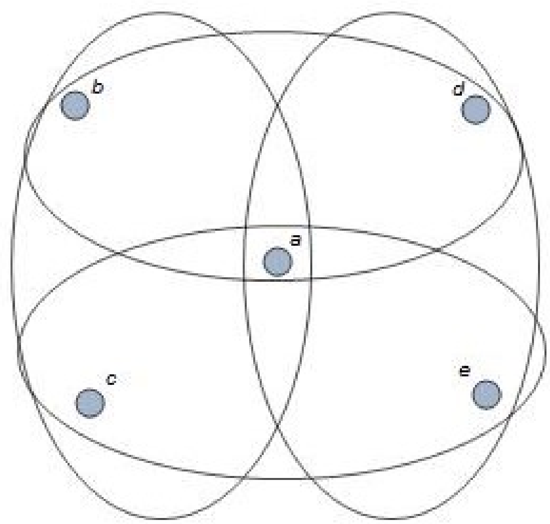

Figure 5: 3-Uniform Hypergraph, $V(G) = \{a,b,c,d,e\}$, $E(G) = \{abc, ade, ace, abd\}$

possible ten hyperedges, the hypergraph starts to become clustered. If we were to include nine of the ten possible hyperedges, it becomes hard to discern the different hyperedges and what vertices they contain. It also becomes more difficult to draw since by including the hyperedge *bed* in the hypergraph above, we are not able to draw a regular oval over the vertices; it would be an L-shaped oval.

This problem becomes amplified when dealing with r-uniform hypergraphs since each hyperedge will contain r vertices. Any complete r-uniform hypergraph becomes so clustered that it will be close to impossible to discern the different hyperedges. If we try coloring the different hyperedges as well, any visualization or argument based on drawing or picturing hypergraphs becomes extremely difficult to do. This leads to complications with proving Ramsey numbers since we cannot necessarily depict the process, as was seen with proving $R(3,3) = 6$.

In r-uniformity, the hypergraph Ramsey numbers are defined similarly to their counter parts in regular graphs, though they are denoted $R(s,t;r)$ where s and t denote the order of the complete hypergraphs in an r-uniform setting. Though a few Ramsey numbers are known for graphs, there is only one known Ramsey number for hypergraphs: $R(4,4;3) = 13$. [11] There do exist bounds for other two color, hypergraph Ramsey numbers, mostly only lower bounds, which are listed below from [11].

- $33 \leq R(4,5;3)$
- $58 \leq R(4,6;3)$
- $82 \leq R(5,5;3)$
- $34 \leq R(5,5;4)$

Though this list is meager, it has been improved upon by Graham, Rothschild, and Spencer [8] who have found a relation between Ramsey Numbers for graphs and Ramsey Numbers for r-uniform hypergraphs. This theorem, called the Stepping-Up Lemma, will be discussed in the next section.

Lastly, for r-uniform hypergraphs $H_1, H_2, \ldots, H_k$, define the t-color Ramsey number

$$R(H_1, H_2, \ldots H_t; r)$$

to be the least $n \in \mathbb{N}$ such that every arbitrary coloring of the hyperedges of $K_n^{(r)}$ (a complete r-uniform hypergraph on n vertices) using t colors results in a subhypergraph isomorphic to H_i for some color $i \in \{1,2,\ldots,t\}$. If $H_1 = H_2 = \cdots = H_t$, then we may write $R_t(H_1;r)$ for the corresponding Ramsey number. It is also standard to write $R(k_1,k_2,\ldots,k_t;r)$ whenever $H_i = K_{k_i}^{(r)}$ for all $i \in \{1,2,\ldots,t\}$. When $r = 2$, it is standard to reduce the notation to $R(G_1,G_2,\ldots,G_t)$ for graphs $G_1,G_2,\ldots,G_t$, or to $R(k_1,k_2,\ldots,k_t)$ when $G_i = K_{k_i}$ for all $i \in \{1,2,\ldots,t\}$.

Another type of graph that we can generalize is a path. For $1 \leq t < r$, a t-tight r-uniform path, denoted $P_{t,n}^{(r)}$, is a connected r-uniform hypergraph that is formed by each consecutive hyperedge including t vertices from the previous hyperedge. $P_{t,n}^{(r)}$ has a vertex set $V(P_{t,n}^{(r)}) = \{v_1,v_2,...,v_n\}$ and a hyperedge set $E(P_{t,n}^{(r)}) = \{e_1,e_2,...,e_p\}$ such that for $1 \leq i \leq p$,

$$e_i = \{v_{(r-t)(i-1)+1}, v_{(r-t)(i-1)+2}, \ldots, v_{(r-t)(i-1)+r}\}$$

and $p = \dfrac{n-t}{r-t}$. Through this definition, we only allow for leaves to exist at the ends of the t-tight r-uniform path, similarly to what can be found in graphs. Note that we lose a part of the strictness of paths found in graphs since we can have repeated vertices in multiple hyperedges of $P_{t,n}^{(r)}$. This is a consequence of the generalization, though if we only allow for each consecutive hyperedge to share a single vertex ($t = 1$) then we will have a stricter generalization of a path that follows closer to the original definition.

An extension of a 1-tight r-uniform path is an r-uniform tree, denoted $T_m^{(r)}$. This is a connected r-uniform hypergraph on m vertices that can be formed hyperedge-by-hyperedge, with each new hyperedge including exactly one vertex from the previous hypergraph. Of course, $K_n^{(r)}$ is unique (up to isomorphism), but there can be many r-uniform trees on a given number of vertices. It is also easily observed that the number of hyperedges in $K_n^{(r)}$ and $T_m^{(r)}$ are $\frac{n!}{r!(n-r)!}$ and $\frac{m-1}{r-1}$, respectively.

An r-uniform hypergraph $H = (V,E)$ will be called bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 with every hyperedge including vertices from both V_1 and V_2. The complete bipartite r-uniform hypergraph $K_{m,n}^{(r)}$ has vertex sets V_1 and V_2 with cardinalities m and n, respectively, and includes all r-uniform hyperedges that include vertices from both V_1 and V_2. In particular, we call the hypergraph $K_{1,n}^{(r)}$ a hyperstar, and note that it contains $\frac{n!}{(r-1)!(n-r+1)!}$ hyperedges. We can further generalize the idea of an r-uniform hyperstar, denoted $S_{t,n}^{(r)}$, such that the set of vertices V can be partitioned into two sets V_1 and V_2 such that $|V_1| = t$ and each hyperedge contains all t vertices from V_1 and $r-t$ vertices from V_2. When $r = 2$, we may write K_n, T_m, and $K_{m,n}$ in place of $K_n^{(2)}$, $T_m^{(2)}$, and $K_{m,n}^{(2)}$.

1.3 The Lifting of Graphs to 3-Uniform Hypergraphs

To find Ramsey numbers, mathematical tools are needed to help decrease the upper bounds or increase the lower bounds. Budden, Hiller, Lambert, and Sanford [2] have defined a map from graphs of order at least three, that are composed of edges with two vertices ($\mathcal{G}_2$) to hypergraphs of order at least three, that are composed of hyperedges with three vertices ($\mathcal{G}_3$). We denote this by $\varphi : \mathcal{G}_2 \to \mathcal{G}_3$, the lifting of a graph to a 3-uniform hypergraph with vertices $V(\varphi(\Gamma)) = V(\Gamma)$ and hyperedges

$$E(\varphi(\Gamma)) := \{abc|\text{ exactly one or all of } ab, bc, ac \in E(\Gamma)\}.$$

From this lifting, there are a few major properties and theorems that were found in [2]. The first is as follows.

Theorem 1.1. *If* $\Gamma \in \mathcal{G}_2$*, then* $\overline{\varphi(\Gamma)} = \varphi(\overline{\Gamma})$.

This theorem shows a very interesting and important relation for this lifting. This importance can be applied to Ramsey numbers because it preserves complements and thus, shows some promise that it will convert some lower bounds for Ramsey numbers from $\mathcal{G}_2$ to lower bounds for Ramsey numbers in $\mathcal{G}_3$. Note, for the following theorem we define the notation $\Gamma[S]$ to be the subgraph of Γ induced by the set of vertices S. The following is Theorem 4 from [3].

Theorem 1.2. *Let $\Gamma \in \mathcal{G}_2$, $S \subset V(\Gamma)$ a subset containing at least three elements, and $K := \Gamma[S]$. Then $\varphi(\Gamma)$ is complete if and only if K is complete or K is the union of exactly two disjoint complete subgraphs.*

This has led to an interesting result regarding complete bipartite graphs and their lifting, specifically that they lift to an empty hypergraph.

An important theorem that is discussed in [2] is the Stepping-Up Lemma. A version of the Stepping-Up Lemma from [8] is the following.

Theorem 1.3. *If $r > 3$ and N is a lower bound for $R(n,n;r)$ then 2^N is a lower bound for $R(2n+r-4, 2n+r-4; r+1)$.*

This lemma is extraordinary in that once a lower bound is found for one specific case, it will lead to new lower bounds for higher dimensional cases. There does exist a problem with this theorem in that it does not step up from $\mathcal{G}_2$ to $\mathcal{G}_3$. In fact, Colon, Fox and Sudakov [7] provided a method for stepping-up from $\mathcal{G}_2$ to $\mathcal{G}_3$, but it jumps from a two coloring to a four coloring. Thus the lifting introduced in [2] fills in the hole in the Stepping-Up Lemma. In [2], they are able to further improve upon the Stepping-Up Lemma from $\mathcal{G}_2$ to $\mathcal{G}_3$, by using the knowledge that a 3-uniform hypergraph of the form $K_n^{(3)} - e$ does not exist in the range of φ, where e is a single hyperedge.

2 Lifting to Hypergraphs

2.1 Extensions of the 3-Uniform Lifting

Since Budden, Hiller, Lambert, and Sanford were able to improve some Ramsey numbers by identifying which 3-uniform hypergraphs do not exist in the image of φ, this leads to further questions on what other 3-uniform hypergraphs do not exist in the image of φ? We can go one step further than what was addressed in [2] and show that a complete 3-uniform hypergraph that is missing a subhypergraph of a lesser order will not appear in the image of φ.

Theorem 2.1. *Let $n > m \geq 3$ and assume that H is a subhypergraph of $K_n^{(3)}$ of order m having at lease one hyperedge. If $\Gamma \in \mathcal{G}_2$, then the lifting $\varphi(\Gamma)$ does not contain any induced subhypergraph isomorphic to $K_n^{(3)} - H$.*

Proof. Suppose false, then $\varphi(\Gamma)$ contains a subhypergraph isomorphic to $K_n^{(3)} - H$. Denote by abc some hyperedge in H and let x be some vertex in $K_n^{(3)}$ that is not contained in H (the existence of such a hyperedge and vertex are assumed). Then the subhypergraph of $\varphi(\Gamma)$ induced by $\{x, a, b, c\}$ is isomorphic to $K_4^{(3)} - e$, which cannot happen by Theorem 7 of [2]. $\square$

Theorem 2.1 greatly restricts the hypergraphs that appear in the range of φ. It also enables us to improve upon some of the Ramsey number results obtained in [2]. In their proofs, Budden, Hiller, Lambert, and Sandford used a class of graphs known as Turàn graphs, which possess certain optimal parameters, escpecially when dealing with Ramsey Numbers. Suppose $n \geq 3$ and $q \geq 2$ are integers. By the Division Algorithm, there exist integers $m \geq 0$ and $0 \leq r < q$ such that $n = mq + r$. The Turàn graph $T_q(n)$ is the complete q-partite graph whose vertices are partitioned into balanced sets. Such graphs contain K_q-subgraphs but lack K_{q+1}-subgraphs. In fact, out of all graphs of order n, they possess the maximal number of edges possible without containing a K_{q+1}-subgraph. This was demonstrated by Turàn [14] in 1941 when he proved that every other graph of order n and size equal to that of $T_q(n)$ contains a K_{q+1}-subgraph. When considering the lifting of Turàn graphs, we obtain the following theorem, which are extensions of some Ramsey number results from [2].

Theorem 2.2. *Let $n \geq 4$, $q \geq 2$, and $n = mq + r$, where $0 \leq r < q$. Then we have the following:*

(1) If $n = qm$, then $R(K^{(3)}_{q+1} - H_1, K^{(3)}_{2m+1} - H_2; 3) > n$,

(2) If $n = qm + 1$, then $R(K^{(3)}_{q+1} - H_1, K^{(3)}_{2m+2} - H_2; 3) > n$,

(3) If $n = qm + r$, with $r > 2$, then $R(K^{(3)}_{q+1} - H_1, K^{(3)}_{2m+3} - H_2; 3) > n$,

where H_1 and H_2 are subhypergraphs of the respective complete subhypergraphs having smaller orders and containing at least one hyperedge.

Proof. Regardless of the value of r, note that $T_q(n)$ contains a K_q-subgraph, but not a K_{q+1}-subgraph. Also, at most one vertex of a complete subgraph can come from any one connected set of vertices. So, $\varphi(T_q(n))$ contains a $K^{(3)}_q$-subhypergraph, but not a $K^{(3)}_{q+1}$-subhypergraph. Note that $T_q(n)$ consists of disconnected complete subgraphs of orders m and $m+1$. By Theorem 1.2, we obtain the following cases. If $n = qm$, then all of the sets of vertices have cardinality m and $\varphi(T_q(n))$ contains a $K^{(3)}_{2m}$-subhypergraph, but not a $K^{(3)}_{2m+1}$-subhypergraph. If $n = qm + 1$, then exactly one vertex set has cardinality $m+1$ and $\varphi(T_q(n))$ contains a $K^{(3)}_{2m+1}$-subhypergraph, but not a $K^{(3)}_{2m+2}$-subhypergraph. For the remaining cases in which $n = qm + r$ with $2 \geq r > q$, at least two vertex sets have cardinality $m+1$, and we find that $\varphi(T_q(n))$ contains a $K^{(3)}_{2m+2}$-subhypergraph, but not a $K^{(3)}_{2m+3}$-subhypergraph. These results along with the implication of Theorem 2.1 proves the theorem. □

We will move on to determining the connection between graphs and 3-uniform hypergraph Ramsey numbers for complete 3-uniform hypergraphs missing a subhypergraph of a lesser order.

Theorem 2.3. *Let $s, t \in \mathbb{N}$ with $s \geq 3$ and $t \geq 3$, then*

$$R(K^{(3)}_{2s-1} - H_1, K^{(3)}_{2t-1} - H_2; 3) > R(K_s, K_t)$$

where H_1 and H_2 are subhypergraphs of the respective complete subhypergraphs having smaller orders and containing at least one hyperedge.

Proof. Assume that $m = R(K_s, K_t)$, then there exists a graph G of order $m-1$ that does not contain a K_s-subgraph, and whose complement does not contain a K_t-subgraph. From Theorem 1.2, it follows that $\varphi(G)$ does not contain a $K^{(3)}_{2s-1}$ subhypergraph, and whose complement does not contain a $K^{(3)}_{2t-1}$-subhypergraph. For any subhypergraphs H_1 and H_2 that contain at least one hyperedge and have a lesser order than their respective complete graphs K_{2s-1} and K_{2t-1}, Theorem 2.1 then implies that $\varphi(G)$ does not contain a $(K^{(3)}_{2s-1} - H_1)$-subhypergraph, and its complement does not contain a $(K^{(3)}_{2t-1} - H_2)$-subhypergraph. Thus,

$$R(K^{(3)}_{2s-1} - H_1, K^{(3)}_{2t-1} - H_2; 3) > m - 1 = R(K_s, K_t)$$

□

From this, we can imply that the Stepping-Up Lemma from [8] may have some connection to Theorem 2.3. This is because Theorem 2.3 implies

$$R(K^{(3)}_{2s-1} - H_1, K^{(3)}_{2t-1} - H_2; 3) \geq R(K^{(3)}_{2s-1}, K^{(3)}_{2t-1}) > R(K_s, K_t)$$

which shows somewhat of a similar result to the Stepping-Up Lemma. In fact, in [2] they were able to improve upon the Stepping-Up Lemma in some fashion with the result that there exist no 3-uniform hypergraph that is missing an edge in the image of φ. Again, we will further this result using the fact that $K^{(3)}_n - H$ does not exist in the image of φ for any subhypergraph H of order less than n containing at least one hyperedge.

Theorem 2.4. *If $q \geq 3$, then*

$$R(K^{(3)}_5, K^{(3)}_{q+1} - H_1, K^{(3)}_{2s-1} - H_2, K^{(3)}_{2t-1} - H_3; 3) > q(R(K_s, K_t) - 1),$$

where H_1, H_2, and H_3 are subhypergraphs of the respective complete subhypergraphs having smaller orders and containing at least one hyperedge.

Proof. Suppose that $m = R(K_s, K_t)$, $q \geq 3$, and let $n = q(m-1)$. Denote the partitioned vertex sets in $T_q(n)$ by $V_1, V_2, \ldots, V_k$. We have already noted that $\varphi(T_q(n))$ contains a $K_q^{(3)}$-subhypergraph, but not a $K_{q+1}^{(3)}$-subhypergraph. From Theorem 2.2, it follows that it does not contain a $(K_{q+1}^{(3)} - H_1)$-subhypergraph, for any subhypergraph H_1 that has an order less than $q+1$ and has at least one hyperedge. Color the hyperedges in $\varphi(T_q(n))$ yellow. Note that $T_q(n)$ consists of q disconnected K_{m-1}-subgraphs. Since $R(s,t) = m$, there exists a red/blue coloring of the edges of K_{m-1} that does not contain a red K_s-subgraph or a blue K_t-subgraph. When lifting just a single K_{m-1} colored in this way, the lifted hypergraph contains at most a red $K_{2s-2}^{(3)}$-subhypergraph or a blue $K_{2t-2}^{(3)}$-subhypergraph by Theorem 1.2. In fact, by Theorem 2.4, the lifted hypergraph does not contain red $(K_{2s-1}^{(3)} - H_2)$-subhypergraph or a blue $(K_{2t-1}^{(3)} - H_3)$-subhypergraph, where H_2 and H_3 have order less than their respective complete hypergraphs and contain at least one hyperedge. We apply this coloring to the hyperedges in $\varphi(T_q(n))$ that arise from the individual liftings of the disjoint vertex sets. The remaining hyperedges in $\varphi(T_q(n))$ are precisely those that include one vertex from V_i and the other two vertices from V_j, where $i \neq j$. Color these hyperedges green. A complete subhypergraph formed using only these hyperedges includes at most two vertices from any V_i and vertices from no more than two of the partitioned vertex sets. Hence, the green hyperedges may contain a $K_4^{(3)}$-subhypergraph, but not a $K_5^{(3)}$-subhypergraph. From this coloring, we find that $R(K_5^{(3)}, K_{q+1}^{(3)} - H_1, K_{2s-1}^{(3)} - H_2, K_{2t-1}^{(3)} - H_3; 3) > n = q(m-1)$, from which the theorem follows. $\square$

2.2 Extension of Hypergraphs Not in the Image of φ

With Theorem 2.1, we are able to identify a vast amount of hypergraphs that do not exist in the image of our lifting. Though this covers a vast majority of cases, there still exist some other types of hypergraphs that are not in the image of φ. Recall the definition of a t-tight r-uniform path from Section 1.2. In the 3-uniform setting, such a path can only be 1-tight (called a loose path) or 2-tight. With this definition, we are able to identify anohter type of subypergraph that does not exist in the image of φ. This type of subhypergraph is not included in Theorem 2.1, since the t-tight paths we consider may have the same order as the hypergraph they are contained in.

Theorem 2.5. *Let $n \geq 4$, $n \geq k \geq 3$, and $\Gamma \in \mathcal{G}_2$. The lifting $\varphi(\Gamma)$ cannot contain an induced subhypergraph isomorphic to $K_n^{(3)} - P_{t,k}^{(3)}$, where $P_{t,k}^{(3)}$ is a t-tight path with k vertices, in a 3-uniform hypergraph.*

Proof. If $k = 3$ then $(K_n^{(3)} - P_{t,3}^{(3)}) = \left(K_n^{(3)} - e\right)$ and since $(K_n^{(3)} - e)$ is not in the image of φ then $(K_n^{(3)} - P_{t,3}^{(3)})$ is not in the image either. For some arbitrary $k > 3$, consider the complement of $(K_n^{(3)} - P_{t,k}^{(3)})$. Note that the compliment of this is just a loose, 3-uniform path containing at least two hyperedges. Take a vertex x from an end hyperedge of the path. Also, for the other end hyperedge, we will have a set of vertices $\{a,b,c\}$ that constitute this hyperedge. Notice that the induced subhypergraph $\Gamma[\{x,a,b,c\}]$ in the complement has only one edge. Therefore in $(K_n^{(3)} - P_k^{(3)})$, the induced subhypergraph $\Gamma[\{x,a,b,c\}] \cong (K_4^{(3)} - e)$, which cannot exist in the image of φ, thus $(K_n^{(3)} - P_{t,k}^{(3)})$ does not exist in the image of φ. $\square$

Though a majority of the hypergraphs $(K_n^{(3)} - P_{t,k}^{(3)})$ will be of an order less than $K_n^{(3)}$ (thus it will be contained in Theorem 2.1), there still exist cases where a $P_{t,k}^{(3)}$ may have the same order as the complete $K_n^{(3)}$. For example consider $P_{1,7}^{(3)}$ in Figure 6.

As depicted above, the order of $P_{1,7}^{(3)}$ is 7, which is the same order of a complete hypergraph on the same vertices. Therefore, there still exist some cases of t-tight paths in a 3-uniform setting that are not covered by Theorem 2.1. Thus, we can improve upon the Ramsey numbers stated above, with these special cases. Note that the proofs are almost

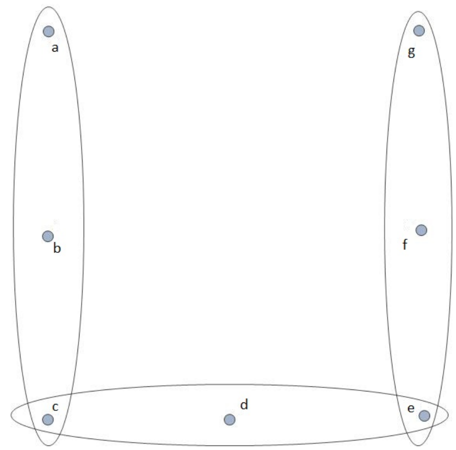

Figure 6: 3-Uniform Loose Path of Order 7

identical to the ones given above, except that we will use Theorem 2.5 instead of Theorem 2.1.

Theorem 2.6. *Let $n \geq 4$, $q \geq 2$, and $n = mq + r$, where $0 \leq r < q$. Then we have the following:*

(1) If $n = qm$, then $R(K^{(3)}_{q+1} - P^{(3)}_{t,k}, K^{(3)}_{2m+1} - P^{(3)}_{t,k}; 3) > n$;

(2) If $n = qm + 1$, then $R(K^{(3)}_{q+1} - P^{(3)}_{t,k}, K^{3}_{2m+2} - P^{(3)}_{t,k}; 3) > n$;

(3) If $n = qm + r$, with $r > 2$, then $R(K^{(3)}_{q+1} - P^{(3)}_{t,k}, K^{(3)}_{2m+3} - P^{(3)}_{t,k}; 3) > n$.

Theorem 2.7. *If $q \geq 3$, then*

$$R(K^{(3)}_5, K^{(3)}_{q+1} - P^{(3)}_{t,k}, K^{(3)}_{2s-1} - P^{(3)}_{t,k}, K^{(3)}_{2t-1} - P^{(3)}_{t,k}; 3) > q(R(K_s, K_t) - 1).$$

2.3 Lifting to r-Uniform Hypergraphs

The theorems that have been discussed before, and in [2], have all been for only 3-uniform hypergraphs. Since we have been able to find a relation between $\mathcal{G}_2$ and $\mathcal{G}_3$ through this lifting φ, there should be a way to generalize this lifting to hypergraphs where hyperedges are made up of r vertices, $\mathcal{G}_r$. A r-uniform hypergraph $\Gamma = (V, E)$ consists of a finite set V of vertices and a set E of distinct unordered r-tuples of different vertices (called hyperedges). Define $\varphi_2^n : \mathcal{G}_2 \mapsto \mathcal{G}_r$ to send a graph Γ to graph $\varphi_2^r(\Gamma)$ with the same vertex set $V(\Gamma) = V(\varphi_2^r(\Gamma))$ and $x_1x_2x_3...x_r$ is a r-Uniform Hyperedge in $\varphi_2^r(\Gamma)$ if and only if $\Gamma[\{x_1, x_2, x_3, ..., x_r\}]$ is a disjoint union of at most $r-1$ complete subgraphs, including the possibility that it is complete itself. Throughout the rest of this thesis, we assume graphs in $\mathcal{G}_2$ contain at least r vertices when considering φ_2^r. Note that by how this generalized lifting is defined, if $r = 3$, all of the previous theorems in this paper and in [2] still hold. Thus it is important to notice that if we have a complete graph in $\mathcal{G}_2$, then it will lift to a complete graph in $\mathcal{G}_r$ through φ_2^r. It is important to determine what other types of graph will lift to complete graphs in $\mathcal{G}_r$ through this generalized lifting.

Lemma 2.8. *Let $\Gamma \in \mathcal{G}_2$, $S \subseteq V(\Gamma)$ a subset containing at least r elements, and $H := \Gamma[S]$. If H is the disjoint union of at most $r-1$ complete subgraphs, then $\varphi^{(r)}(H)$ is complete.*

Proof. Assume that H is the disjoint union of complete subgraph $C_1, C_2, \ldots, C_\ell$, where $1 \leq \ell \leq r-1$. Let $x_1, x_2, \ldots, x_r$ be any subset of r distinct vertices in S. If $x_i \in C_i$ and $x_j \in C_j$ with $i \neq j$, then $x_ix_j \notin E(H)$ since C_i and C_j are disconnected. Also, for any two vertices $y, z \in C_i$, $yz \in E(H)$ since C_i is complete. Thus, any collection of r distinct vertices in S is a disjoint union of at most $r-1$ complete subgraphs in H, and $\varphi^{(r)}(H)$ is complete. $\square$

This lemma and the following theorem classify what types of subgraphs can be lifted to complete subhyper graphs.

Theorem 2.9. *Let $\Gamma \in \mathcal{G}_2$, $S \subset V(\Gamma)$ a subset containing at least r elements, and $H := \Gamma[S]$. Then $\varphi^{(r)}(H)$ is complete if and only if H is the disjoint union of at most $r-1$ complete subgraphs.*

Proof. Lemma 2.8 provides one direction for the biconditional statement in the theorem. It remains to be shown that if $\varphi^{(r)}(H)$ is complete, then H is the disjoint union of at most

$r-1$ complete subgraphs. Since $\varphi^{(r)}(H)$ is assumed to be complete, it follows that the induced subgraph for every subset of exactly r vertices in H is the disjoint union of at most $r-1$ complete subgraphs. Draw H one vertex at a time, beginning with $k=r$ vertices and including all edges incident with each new vertex and the vertices contained in the previous graph. Let H_k be the graph after k vertices have been drawn. We proceed by induction on k. In the initial case $k=r$, the r vertices lift to a hyperedge, so the preimage is a disjoint union of at most $r-1$ complete subgraphs by definition. Now suppose that for $k \geq r$, H_k is the disjoint union of at most $r-1$ complete subgraphs and consider H_{k+1}, where $V(H_{k+1}) = V(H_k) \cup \{x\}$. Assume that H_k is composed of ℓ disjoint complete subgraphs $C_1, C_2, \ldots, C_\ell$, where $1 \leq \ell \leq r-1$. The first case we consider is when x is disjoint from H_k. Then H_{k+1} is the disjoint union of $\ell+1$ complete subgraphs. If $\ell = r-1$, then picking a vertex x_i from each C_i, we find that $xx_1x_2\cdots x_{r-1}$ is not a hyperedge in $\varphi(\Gamma)$, contradicting the assumption that it is complete. Hence, $\ell < r-1$ and we find that H_{k+1} is the disjoint union of at most $r-1$ complete subgraphs. In the remaining cases, x is incident with some vertex in H_k, so H_{k+1} has the same number of components as H_k (or possibly fewer). It remains to be shown that if x is incident with a vertex in C_i, then it must be incident with every vertex in C_i and that x cannot be incident with vertices from more than one copy of C_i. If x is incident with $x_i \in C_i$ and $x_j \in C_j$ for $i \neq j$, then any subset of r vertices from H_{k+1} that contains x, x_i, and x_j cannot be the disjoint union of complete subgraphs since x_i is not adjacent to x_j. Finally, suppose that x is adjacent with $x_i \in C_i$ for only one value of i. If $C_i = \{x_i\}$, then $\{x\} \cup C_i$ forms a K_2 and H_{k+1} is still the disjoint union of ℓ complete subgraphs. Otherwise, C_i must contain at least r vertices. Suppose that x is adjacent with some $y \in C_i$. Then for any other distinct vertices $x_1, x_2, \ldots, x_{r-2}$ in C_i, the subgraph of H_{k+1} induced by $\{x, y, x_1, \ldots, x_{r-2}\}$ must be complete since it is connected. Thus, x must be adjacent to all vertices in C_i. Thus, we have shown that H_{k+1} must be the disjoint union of at most $r-1$ complete subgraphs and the same must be true for H. $\square$

From this theorem, we can imply some important aspects from graphs that can be mapped to their r-uniform hypergraph through the lifting φ_2^r. Using the previous theorem, we are able to construct r-uniform hypergraphs that have certain clique sizes, as well as address the clique size in an r-uniform hypergraph and identify what could have lifted to it.

Corollary 2.10. *If* $\omega(\Gamma) = k$ *then* $\omega(\varphi_2^r(\Gamma)) \leq (r-1)k$.

Proof. Let $\omega(\Gamma) = k$. For $\varphi_2^r(\Gamma)$ to be complete, we know that Γ can have at most $r-1$ disjoint complete subgraphs. Since $\omega(\Gamma) = k$, then each complete subgraph of Γ will have a clique size of at most k. Thus since there will be at most $(r-1)k$ vertices in Γ, then there can be at most $(r-1)k$ vertices in $\varphi_2^r(\Gamma)$, thus $\omega(\varphi_2^r(\Gamma)) \leq (r-1)k$. □

Corollary 2.11. *If* $\omega(\varphi_2^r(\Gamma)) = k$ *then* $\omega(\Gamma) \leq k$.

Proof. Let $\omega(\varphi_2^r(\Gamma)) = k$. Since the max clique size in $\varphi_2^r(\Gamma)$ is k, then the largest complete subhypergraph of $\varphi_2^r(\Gamma)$ will have clique size k. Since this is true, then Γ can have a maximum clique size of k by Theorem 2.9. Thus $\omega(\Gamma) \leq k$. □

This is important since Ramsey numbers, in any uniformity, are a different way of determining the number of vertices need to have a certain (hyper)subgraph of certain clique size in the red coloring of the (hyper)graph or the blue coloring of the (hyper)graph. The existence of these bounds are helpful, but do not encourage to much hope in helping find reasonable bounds on hypergraph Ramsey numbers since the bounds can become pretty large. This, compounded with the loss of the complement property found in the lifting $\varphi(\Gamma)_2^3$, makes finding hypergraph Ramsey numbers from these results even more difficult. Further investigation might result in bounds created for hypergraph Ramsey numbers, though they would most likely be for obscure Ramsey numbers, and the bounds might not be that helpful.

3 Hypergraph Ramsey Numbers

3.1 2-Color Ramsey Theorems

For any r-uniform hypergraph H whose hyperedges all consist of two or more vertices, one can define the weak chromatic number $\chi_w(H)$ to be the minimal number of colors needed to color the vertices of H so that no hyperedge is monochromatic. The strong chromatic number $\chi_s(H)$ is the minimal number of colors needed to color the vertices of H so that all adjacent vertices (contained within a common hyperedge) have different colors. It is easily observed that for any r-uniform hypergraph H,

$$\chi_w(H) \leq \chi_s(H),$$

and whenever $r = 2$, $\chi_w = \chi_s = \chi$, where χ is the chromatic number for graphs. Finally, we denote by $\lceil x \rceil$ and $\lfloor x \rfloor$ the ceiling and floor functions for $x \in \mathbb{R}$, respectively. The reader should note that the material found in Section 3, is also discussed in [1].

In 1972, Chvátal and Harary [6] proved a general Ramsey inequality for graphs:

$$R(G_1, G_2) \geq (c(G_1) - 1)(\chi(G_2) - 1) + 1, \tag{1}$$

where $c(G_1)$ is the order of the largest connected component of G_1 and $\chi(G_2)$ is the chromatic number of G_2. Using this result, Chvátal [5] was then able to prove the explicit Ramsey number

$$R(T_m, K_n) = (m-1)(n-1) + 1, \tag{2}$$

where T is any tree on m vertices. In this section, we focus on extending these two results to r-uniform hypergraphs. First, we generalize (1) to r-uniform hypergraphs.

Theorem 3.1. *Let H_1 and H_2 be r-uniform hypergraphs with $r \geq 2$. Then*

$$R(H_1, H_2; r) \geq (c(H_1) - 1)(\chi_w(H_2) - 1) + 1,$$

where $c(H_1)$ is the order of the largest connected component of H_1 and $\chi_w(H_2)$ is the weak chromatic number of H_2.

Proof. Let $k = (c(H_1) - 1)(\chi_w(H_2) - 1)$ and consider $K_m^{(r)}$ composed of $\chi_w(H_2) - 1$ copies of $K_{c(H_1)-1}^{(r)}$. Color the hyperedges within each copy of $K_{c(H_1)-1}$ red and the remaining hyperedges blue. No red copy of H_1 can exist since the largest red connected component has order $c(H_1) - 1$. Also, no blue copy of H_2 can exist since one can obtain a weak coloring of any blue hypergraph by assigning a single color to the vertices in each $K_{c(H_1)-1}$. Hence, $R(H_1, H_2; r) \geq k + 1$. □

When considering an analogue for (2) using r-uniform hypergraphs, we will find that it is no longer possible to obtain an exact value for $R(H_1, H_2; r)$ when $r > 2$. The exact value in the $r = 2$ case was due to the fact that the weak and strong chromatic numbers agree in this setting. When considering complete r-uniform hypergraphs, we have the following.

Lemma 3.2. *If $n \geq r \geq 2$, it follows that $\chi_w(K_n^{(r)}) = \lceil \frac{n}{r-1} \rceil$ and $\chi_s(K_n^{(r)}) = n$.*

Proof. The chromatic number evaluations for complete r-uniform hypergraphs follow from the fact that every weak coloring of $K_n^{(r)}$ contains at most $r - 1$ vertices of a given color. For a strong coloring, no two distinct vertices can have the same color since there exists some hyperedge that includes both vertices. □

Theorem 3.3. *If $n \geq r \geq 2$ and $T_m^{(r)}$ is any r-uniform tree on m vertices, then*

$$(m-1)\left(\left\lceil \frac{n}{r-1} \right\rceil - 1\right) + 1 \leq R(T_m^{(r)}, K_n^{(r)}; r) \leq (m-1)(n-1) + 1.$$

Proof. Letting $H_1 = T_m^{(r)}$ and $H_2 = K_n^{(r)}$ in Theorem 3.1 and using the weak chromatic number result from Lemma 3.2, we obtain the first inequality

$$(m-1)\left(\left\lceil \frac{n}{r-1} \right\rceil - 1\right) + 1 \leq R(T_m^{(r)}, K_n^{(r)}; r).$$

To prove the second inequality, consider a 2-coloring of the edges on $K_k^{(r)}$, where $k = (m-1)(n-1)+1$. First, we handle the base cases in which $m = r$ or $n = r$. If $m = r$, then $T_m^{(r)}$ consists of a single hyperedge and it is easily seen that

$$R(T_m^{(r)}, K_n^{(r)}; r) = n \leq (r-1)(n-1)+1.$$

If $n = r$, then $K_n^{(r)}$ consists of a single hyperedge and we have

$$R(T_m^{(r)}, K_n^{(r)}; r) = m \leq (m-1)(r-1)+1.$$

Now we proceed by using strong induction on $m+n$. Assume that

$$R(T_{m'}^{(r)}, K_{n'}^{(r)}; r) \leq (m'-1)(n'-1)+1$$

for all $m'+n' < m+n$ and any r-uniform tree $T_{m'}^{(r)}$ on m' vertices. Now, for a fixed r-uniform tree $T_m^{(r)}$ on m vertices, form the r-uniform tree T' by removing a single "leaf." That is, for some hyperedge containing only a single vertex of degree greater than 1, remove the hyperedge and the $r-1$ vertices of degree 1, resulting in T having order $m-(r-1)$. Call the one remaining vertex from the removed leaf x. By the inductive hypothesis, we have that the red/blue coloring of the edges of $K_k^{(r)}$ contains either a red T' or a blue $K_n^{(r)}$. In the latter case, we are done, so assume the former case. Now, consider the red/blue coloring of the edges of $K_{k-(m-(r-1))}^{(r)}$ formed by removing the $m-(r-1)$ vertices in the red T'-

subgraph from the original $K_k^{(r)}$. It is easily confirmed that

$$k-(m-(r-1)) \geq (m-1)(n-2)+1$$

from which we obtain a red/blue coloring of the edges of $K_{(m-1)(n-2)+1}^{(r)}$. Applying the inductive hypothesis again, we find that this hypergraph contains either a red $T_m^{(r)}$ or a blue $K_{n-1}^{(r)}$. In the former case, we are done, so assume the latter case. Thus, the original red/blue coloring of the edges of $K_k^{(r)}$ contains a red T' and a blue $K_{n-1}^{(r)}$ that are disjoint. Consider the possible colors that can be assigned to the hyperedges that contain x and $r-1$ vertices from the $K_{n-1}^{(r)}$ subgraph. If any of them are red, then there exists a red $T_m^{(r)}$. Otherwise, all of them are blue and there exists a blue $K_n^{(r)}$. Hence,

$$R(T_m^{(r)}, K_n^{(r)}; r) \leq (m-1)(n-1)+1,$$

completing the proof of the theorem. □

In 1974, Burr [3] proved that when $m-1$ divides $n-1$,

$$R(T_m, K_{1,n}) = m+n-1, \tag{3}$$

where T_m is any tree on m vertices. We extend this result to r-uniform hypergraphs in the following two theorems, and corollary.

Theorem 3.4. *If $r \geq 2$, $k \geq 1$, and $T_m^{(r)}$ is any r-uniform tree on $m \geq r$ vertices, then*

$$R(T_m^{(r)}, K_{1,k(m-1)+r-1}^{(r)}; r) \geq (k+1)(m-1)+1.$$

Proof. Form a 2-coloring of the hyperedges in $K_{(k+1)(m-1)}^{(r)}$ by taking $k+1$ copies of $K_{m-1}^{(r)}$.

Let all of the hyperedges in each copy of $K_{m-1}^{(r)}$ be colored red and all interconnecting hyperedges be colored blue. No red $T_m^{(r)}$ has been formed since $T_m^{(r)}$ has m vertices and the largest connected component in the hypergraph spanned by the red hyperedges has order $m-1$. When considering the largest value of t for which there exists a blue $K_{1,t}^{(r)}$, note that if x is the vertex that is alone in its bipartite vertex set, then at most $r-2$ other vertices in the same copy of $K_{1,t}^{(r)}$ can be included in the other vertex set. Thus, our coloring includes a blue $K_{1,k(m-1)+r-2}^{(r)}$, but not a blue $K_{1,k(m-1)+r-1}^{(r)}$, resulting in the lower bound stated in the theorem. □

Note that in the special case in which $n-1$ is divisible by $m-1$, we can let $k = \frac{n-1}{m-1}$ to obtain the lower bound

$$R(T_m^{(r)}, K_{1,n+r-2}; r) \geq n+m-1. \tag{4}$$

This result agrees with the lower bound necessary to prove (3) when $r = 2$. Now we turn our attention to finding an upper bound.

Theorem 3.5. *If $t+1 \geq r \geq 2$, and $T_m^{(r)}$ is any r-uniform tree on m vertices, then*

$$R(T_m^{(r)}, K_{1,t}^{(r)}; r) \leq m+t-(r-1).$$

Proof. Let $m = r+\ell(r-1)$ (that is, $\ell+1$ is the number of hyperedges in $T_m^{(r)}$). We proceed by induction on $\ell \geq 0$. In the case $\ell = 0$, it is easily seen that

$$R(T_r^{(r)}, K_{1,t}^{(r)}; r) = t+1 = r+t-(r-1).$$

Now assume that the inequality is true for the $\ell-1$ case:

$$R(T^{(r)}_{m-(r-1)}, K^{(r)}_{1,t}; r) \leq m+t-2(r-1),$$

for all r-uniform trees on $m-(r-1)$ vertices. For a given r-uniform tree $T^{(r)}_m$, let T' be the tree formed by removing a single leaf (a hyperedge and the $r-1$ verities of degree 1 contained in that hyperedge) and let x be the vertex in T' that was incident with the removed leaf. Consider a red/blue coloring of the hyperedges in $K^{(r)}_{m+t-(r-1)}$. By the inductive hypothesis, this coloring contains either a red T' or a blue $K^{(r)}_{1,t}$. Assume the former case and note that besides the vertices in T', the graph $K^{(r)}_{m+t-(r-1)}$ contains

$$m+t-(r-1)-(m-(r-1)) = t$$

other vertices. Now consider the hyperedges that include x along with all $r-1$ subsets of vertices from the t not included in T'. If any one of these hyperedges is red, we obtain a red copy of $T^{(r)}_m$. Otherwise, they are all blue, and we have a blue $K^{(r)}_{1,t}$. $\square$

If we assume that $n-1$ is divisible by $m-1$ and let $t = n+r-2$, then combining (4) with Theorem 3.5, we obtain the following corollary.

Corollary 3.6. *If $n+1 \geq r \geq 2$, $T^{(r)}_m$ is any tree on m vertices, and $m-1$ divides $n-(r-1)$, we have that*

$$R(T^{(r)}_m; K^{(r)}_{1,n}; r) = m+n-(r-1).$$

Proof. We proceed by induction on $m \geq r$. First consider the case $m = r$. If a 2-coloring of the hyperedges in a complete r-uniform hypergraph lacks any red edges, then they must all be blue and $n+r-1$ vertices are required to have a blue $K^{(r)}_{1,n+r-2}$. Thus,

$$R(T^{(r)}_r, K^{(r)}_{1,n+r-2}; r) = n+r-1 \leq n+2r-3.$$

Now suppose that

$$R(T_j^{(r)}, K_{1,n+r-2}^{(r)}; r) \leq j + n + r - 3$$

for all $j < m$ and consider a 2-coloring of the hyperedges of $K_{m+n+r-3}^{(r)}$. Let T' be a hypergraph formed by removing a single leaf (a hyperedge and the $r-1$ vertices of degree 1 in that hyperedge) from $T_m^{(r)}$. Suppose that x is the remaining vertex from the removed leaf. Since T' has $m-(r-1) < m$ vertices, the inductive hypothesis implies that there exists a red T' or a blue $K_{1,n+r-2}^{(r)}$. In the latter case, we are done, so assume the former. Note that there are $n+2r-4 \geq n+r-2$ vertices in $K_{m+n+r-3}^{(r)}$ that are not in the red T'. Consider all of the hyperedges that include these vertices and the vertex x. If any one of them are red, we obtain a red $T_m^{(r)}$. Otherwise they are all blue and there exists a blue $K_{1,n+r-2}^{(r)}$. □

Since we are able to generalize some results from Burr [3], this indicates that there might exist other Ramsey numbers that include stars and paths that can be generalized. A path, denoted P_m, is a set of vertices $V(P_m) = \{v_1, v_2, ..., v_m\}$ such that its edges are $E(P_m) = \{v_1v_2, v_2v_3, ..., v_{m-1}, v_m\}$. A complete bipartite graph, denoted $K_{s,t}$, has a set of vertices that can be partitioned into two subsets U and W such that each edge joins a vertex of U and a vertex of W. A star is a complete bipartite graph where either $s = 1$ or $t = 1$. [4] Parsons has defined bounds for these specific Ramsey numbers for any path of length m and $star$ with order $n+1$, $R(P_m, K_{1,n})$. In his paper [10], he was also able to determine a specific Ramsey number, depending on the relationship of the orders of the Path and Star. When in an r-Uniform setting the definitions of a star and path become a bit more ambiguous.

It is important to note for both of these definitions given in Section 1.2, we must be careful about picking the number of vertices that we can use. For a t-tight r-Uniform path, it is best to determine, how many edges you will want, and the tightness of the path, and from there you will be able to determine the number of vertices needed. If you start with declaring the number of vertices in the set, it may be impossible for each vertex to be included in a hyperedge, by how they are defined.

With those two definitions in mind, we will generalize the upper bound found in [10].

Theorem 3.7. *For some* $1 \leq t < r$,

$$R(P_{t,n}^{(r)}, S_{t,m}^{(r)}; r) \leq m + n - 1$$

.

Proof. By induction on k, where k is the number of hyperedges in $P_{t,n}^{(r)}$ and let $k = r$. Then $R(P_{t,r}^{(r)}, S_{t,m}^{(r)}; r) = m < m + r - 1$. Next suppose that for $n = r + (k-1)(r-t)$, we have $R(P_{t,n}^{(r)}, S_{t,m}^{(r)}; r) \leq m + n - 1$ and consider $P_{t,r+k(r-t)}^{(r)}$. Consider a red/blue coloring of the hyperedges in $K_{m+r+k(r-t)-1}^{(r)}$. Remove a hyperedge that is at an end of $P_{t,r+k(r-t)}^{(r)}$. Without loss of generality, assume that we have removed the hyperedge $\{v_1, v_2, ..., v_r\}$. Note that the vertices $v_1, v_2, ..., v_{r-t}$ are not contained in $P_{t,r+k(r-t)}^{(r)}$, but $v_{r-t+1}, v_{r-t+2}, ..., v_r$ are still contained in $P_{t,r+k(r-t)}^{(r)}$. By the induction hypothesis, we have that $K_{m+r+k(r-t)-1}^{(r)}$ contains a red $P_{t,r+(k-1)(r-t)}^{(r)}$ or a blue $S_{t,m}^{(r)}$. Assume the former, then there exist $m + r + k(r - t) - 1 - (r + (k-1)(r-t)) = m + (r-t) - 1$ vertices that are not contained in the red $P_{t,r+(k-1)(r-t)}^{(r)}$. Next consider the vertices $\{v_{r-t+1}, v_{r-t+2}, ..., v_r\}$. Note that there are t vertices, and consider subsets of $r-t$ vertices such that all $r-t$ vertices are not contained in $P_{t,r+(k-1)(r-t)}^{(r)}$. If a hyperedge containing these vertices is colored red, then we will have a red $P_{t,r+(k)(r-t)}^{(r)}$. If not, then all such hyperedge will be colored blue and we will have a blue $S_{t,m}^{(r)}$. Therefore $R(P_{t,n}^{(r)}, S_{t,m}^{(r)}; r) \leq m + r + k(r-t) - 1 = m + n - 1$. □

3.2 Multicolor Hypergraph Ramsey Numbers

In 2002, Robertson (Theorem 2.1, [13]) proved that if $n \geq 3$ and $k_i \geq 3$ for $i = 1, 2, \ldots, n$, then

$$R(k_1, k_2, \ldots, k_n; 2) > (k_1 - 1)(R(k_2, k_3, \ldots, k_n; 2) - 1).$$

Here, $R(k_1,k_2,\ldots,k_n;r)$ denotes the least natural number m such that every coloring of the r-uniform edges of the complete graph K_m on m vertices using r colors results in a complete K_{k_i} for some i. When $k_1 = k_2 = \cdots = k_n$ (the "diagonal" case), we write $R_n(k_1;r)$ in place of $R(k_1,k_2,\ldots,k_n;r)$. Robertson's result followed from a "Turán-type" coloring and implied four improved lower bounds for diagonal multicolor Ramsey numbers: $R_5(4;2) \geq 1372$, $R_5(5;2) \geq 7329$, $R_4(6;2) \geq 5346$, and $R_4(7;2) \geq 19261$. All of these bounds have since been improved (see [11] for a current list of best bounds), but very little is known about their r-uniform analogues.

In this short note, we generalize Robertson's theorem in two different directions. First, there is no need to assume that the monochromatic subgraphs are complete subgraphs, so our generalization will allow for arbitrary subgraphs of order 3 or more, with the exception of the first color. The significant generalization that we make is to extend Robertson's constructive method of proof to the r-uniform case.

Theorem 3.8. *Let $q \geq 2$, $n \geq 3$, and suppose that $R(H_2,H_3,\ldots,H_n;r) \geq n$. Then*

$$R(K^{(r)}_{(n-1)q+1},H_2,H_3,\ldots,H_n;r) > q(R(H_2,H_3,\ldots,H_n;r)-1).$$

Proof. Suppose that $R(H_2,H_3,\ldots,H_n;r) = m$, where it is assumed that $m \geq n$, and fix a coloring ϕ of maximal $(n-1)$-Ramsey coloring of $K^{(r)}_{m-1}$ (lacking subhypergraphs isomorphic to H_2, H_3, …, H_n in colors 2, 3, …, n, respectively). Consider the r-uniform hypergraph Γ of order $t = q(m-1)$, where $q \geq 2$, consisting of q disjoint copies of $K^{(r)}_{m-1}$, each colored according to ϕ. Denote the vertex sets in the partition by V_1, V_2, …, V_q. The hyperedges $(x_1,x_2,\ldots,x_r)$ in the complement of Γ are those in which not all of the x_i come from the same V_j. Thus, every hyperedge in $\overline{\Gamma}$ contains at most $r-1$ vertices coming from a single V_j. Coloring all of these hyperedges with the same color produces a clique of order at most

$(r-1)q$. Thus, we find that

$$R(K^{(r)}_{(r-1)q+1}, H_2, H_3, \ldots, H_n; r) > q(m-1) = q(R(H_2, H_3, \ldots, H_n; r) - 1),$$

completing the proof of the theorem. $\square$

As an example of the utility of Theorem 1 the following lower bounds follow immediately from the explicit lower bounds given in Section 7.1 of Radziszowski's dynamic survey [11]:

$$\begin{aligned}
R(K_4^{(3)}, K_4^{(3)}; 3) = 13 \quad &\Longrightarrow \quad R(K_{2q+1}^{(3)}, K_4^{(3)}, K_4^{(3)}; 3) > 12q,\\
R(K_4^{(3)}, K_5^{(3)}; 3) \geq 33 \quad &\Longrightarrow \quad R(K_{2q+1}^{(3)}, K_4^{(3)}, K_5^{(3)}; 3) > 32q,\\
R(K_5^{(3)}, K_5^{(3)}; 3) \geq 82 \quad &\Longrightarrow \quad R(K_{2q+1}^{(3)}, K_5^{(3)}, K_5^{(3)}; 3) > 81q,\\
R(K_5^{(4)}, K_5^{(4)}; 4) \geq 34 \quad &\Longrightarrow \quad R(K_{2q+1}^{(4)}, K_5^{(4)}, K_5^{(4)}; 4) > 33q,\\
R(K_4^{(3)} - e, K_4^{(3)} - e; 3) = 7 \quad &\Longrightarrow \quad R(K_{2q+1}^{(3)}, K_4^{(3)} - e, K_4^{(3)} - e; 3) > 6q,\\
R(K_4^{(3)} - e, K_5^{(3)}; 3) \geq 14 \quad &\Longrightarrow \quad R(K_{2q+1}^{(3)}, K_4^{(3)}, K_5^{(3)}; 3) > 13q,\\
R(K_4^{(3)}, K_4^{(3)}, K_4^{(3)}; 3) \geq 56 \quad &\Longrightarrow \quad R(K_{2q+1}^{(3)}, K_4^{(3)}, K_4^{(3)}, K_4^{(3)}; 3) > 55q.
\end{aligned}$$

For example, letting $q = 2$ in the second inequality gives us the following immediate corollary.

Corollary 3.9. $R(K_5^{(3)}, K_5^{(3)}, K_5^{(3)}; 3) \geq 163$.

The next corollary is proved by induction on the number of colors. To simplify the statement, we define the notation $R^r(K_m^{(n)}, H_1, H_2; n)$ to denote the $r-$color $n-$uniform hypergraph Ramsey number for $r-2$ copies of $K_m^{(n)}$ along with nonempty hypergraphs

H_1 and H_2. Note that we are using superscripts to denote these semi-diagonal Ramsey numbers, in contrast to using subscripts for their diagonal counterparts.

Corollary 3.10. *If $r \geq 3$ and $q \geq 2$, then*

$$R^n(K^{(r)}_{(r-1)q+1}, H_1, H_2; r) > q^{n-2}(R(H_1, H_2; r) - 1).$$

Proof. The proof of Corollary 2 follows from a simple inductive argument on n. For the $n = 3$ case, Theorem 3.8 implies

$$R^3(K^{(r)}_t, H_1, H_2; r) > q(R(H_1, H_2; r) - 1),$$

where $t = (r-1)q + 1$. Assume now that

$$R^k(K^{(r)}_t, H_1, H_2; r) \geq q^{k-2}(R(H_1, H_2; r) - 1) + 1,$$

for $3 \leq k \leq n-1$. Applying Theorem 3.8 again, we have

$$R^{k+1}(K^{(r)}_t, H_1, H_2; r) \geq q(R^k(K^{(r)}_t, H_1, H_2; r) - 1) + 1 \geq q(q^{k-2}(R(H_1, H_2; r) - 1)) + 1,$$

implying the statement of the corollary. □

Of course, when $H_1 = H_2 = K^{(r)}_{(r-1)q+1}$, we obtain the following *diagonal* case:

$$R_n(K^{(r)}_{(r-1)q+1}; r) > q^{n-2}(R_2(K^{(r)}_{(r-1)q+1}; r) - 1).$$

In 2004, Xiaodong, Zheng, Exoo, and Radziszowski (Theorem 2, [15]) proved the following multicolor Ramsey number inequality for graphs:

$$R(k_1, k_2, \ldots, k_t) \geq (R(k_1, k_2 \ldots, k_i) - 1)(R(k_{i+1}, k_{i+2}, \ldots, k_t) - 1) + 1, \tag{5}$$

for $k_j \geq 2$, $1 \leq j \leq t$, and $2 \leq u \leq t-2$. Their proof was constructive and described a method for coloring the edges in K_{mn} with t colors, avoiding the necessary monochromatic subgraphs, where

$$m = R(k_1, k_2 \ldots, k_i) - 1 \quad \text{and} \quad n = R(k_{i+1}, k_{i+2}, \ldots, k_t) - 1.$$

Although their approach does not easily generalize to hypergraphs, the following two theorems make use of the constructive method used in [15] to provide multicolor Ramsey number inequalities. Theorem 3.11 also makes use of the approach used in Theorem 3.1.

Theorem 3.11. *Let $r \geq 2$, $t \geq 3$, and H be an r-uniform hypergraph. Then*

$$R(H, K_{k_2}^{(r)}, \ldots, K_{k_t}^{(r)}; r) \geq (\chi_w(H) - 1)(R(K_{k_2}^{(r)}, \ldots, K_{k_t}^{(r)}; r) - 1) + 1.$$

Proof. Let $n = R(K_{k_2}^{(r)}, K_{k_3}^{(r)}, \ldots, K_{k_t}^{(r)}; r) - 1$ and consider a coloring of the hyperedges of $K_{(\chi_w(H)-1)n}^{(r)}$ formed by considering $\chi_w(H) - 1$ copies of $K_n^{(r)}$. Within each copy of $K_n^{(r)}$, the hyperedges are colored such that no no copy of $K_{k_i}^{(r)}$ appears for any color $2 \leq i \leq t$. Color all of the hyperedges that interconnect the different copies of $K_n^{(r)}$ with color 1. Note that no copy of H appears in color 1 since one can obtain a weak coloring of the vertices of any color 1 hypergraph by coloring the vertices according to which copy of $K_n^{(r)}$ they lie within. Thus, we find that

$$R(H, K_{k_2}^{(r)}, \ldots, K_{k_t}^{(r)}; r) > (\chi_w(H) - 1)n,$$

from which the result follows. $\square$

The next theorem is a true generalization of Xiaodong, Zheng, Exoo, and Radziszowski's Theorem as it reduces to (5) when $r = 2$ since

$$R(2, k_2, \ldots, k_t; r) = R(k_2 \ldots, k_t; r).$$

Theorem 3.12. *Let $r \geq 2$ and $t-2 \geq i \geq 3$. Then*

$$R((r-1)^2+1, k_2, \ldots, k_t; r) \geq (R(k_2, \ldots, k_i; r) - 1)(R(k_{i+1}, \ldots, k_t; r) - 1) + 1.$$

Proof. Let

$$m = R(k_2, \ldots, k_i; r) - 1 \quad \text{and} \quad n = R(k_{i+1}, \ldots, k_t; r) - 1$$

and form a t-coloring of the hyperedges is $K_{mn}^{(r)}$ by considering m copies of $K_n^{(r)}$. Color the hyperedges within each copy of $K_n^{(r)}$ so that no copy of $K_{k_j}^{(r)}$ exists in color j for any $i+1 \leq j \leq t$. The remaining hyperedges are those those interconnect the different copies of $K_n^{(r)}$. Give color 1 to the hyperedges that have at least two vertices within a common copy of $K_n^{(r)}$. So, all hyperedges in color 1 include at most $r-1$ vertices from any given copy of $K_n^{(r)}$ and can include vertices from at most $r-1$ different copies of $K_n^{(r)}$. Thus, the maximum clique in color 1 has order $(r-1)^2$. Finally, the remaining hyperedges are those whose vertices are all in different copies of $K_n^{(r)}$. If we identify the vertices in $K_m^{(r)}$ with the distinct copies of $K_n^{(r)}$, we can form a coloring of the remaining hyperedges with colors 2 through i that avoids a copy of $K_{k_j}^{(r)}$ in color j for all $2 \leq j \leq i$. Thus, our t-coloring of the hyperedges of $K_{mn}^{(r)}$ has avoided all of the necessary monochromatic subhypergraphs. □

Lastly, we are able to determine another conjectured extension of Theorem 2 of [15].

Conjecture 3.13. *If $r \geq 2$ and $t > i+1 > 2$, then*

$$R(k_1, k_2, \ldots, k_t; r) \geq \left\lfloor \frac{R(k_1, k_2, \ldots, k_i; r) - 1}{r-1} \right\rfloor (R(k_{i+1}, k_{i+2}, \ldots, k_t; r) - 1) + 1.$$

To provide some support for our conjecture, consider the following construction. Let

$$m = R(k_1, k_2, \ldots, k_i; r) - 1, \qquad n = R(k_{i+1}, k_{i+2}, \ldots, k_t; r) - 1,$$

$a = \left\lfloor \frac{m}{r-1} \right\rfloor$, and form a t-coloring of the hyperedges of $K_{an}^{(r)}$ using a copies of $K_n^{(r)}$. Within each copy of $K_n^{(r)}$, color the hyperedges so that no copy of $K_{k_j}^{(r)}$ exists in color j for any $i+1 \leq j \leq t$. The remaining hyperedges each have at most $r-1$ vertices within a single copy of $K_n^{(r)}$, forming a clique of order at most

$$\left\lfloor \frac{m}{r-1} \right\rfloor (r-1) \leq m.$$

It is clear that for any choice of $r-1$ vertices from each copy of $K_n^{(r)}$, the resulting $K_{\lfloor \frac{m}{r-1} \rfloor (r-1)}^{(r)}$ has an i-coloring of the hyperedges that lack a copy of $K_{k_j}^{(r)}$ in color j for all $1 \leq j \leq i$. Of course, it is not clear whether or not this can be done in a well-defined manner for all choices of $r-1$ vertices from each $K_n^{(r)}$.

4 Further Topics of Interest

This paper has generalized many Ramsey numbers to hypergrah Ramsey number of $\mathcal{G}_3$ and others to any uniformity $\mathcal{G}_r$. Further investigation has shown that some of the results from Section 2, may be able to be generalized for any uniformity. If this is the case, then it is possible to not only improve on bounds for these numbers at any uniformity, but also have exact numbers. Also further investigation of Parsons [10] might show promise for finding a Path-Star Ramsey number for any uniformity, depending on how m and n are selected.

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