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Coarse topology is the study of interesting topological properties of discrete spaces. In this dissertation, we will discuss a coarse analog of dimension and several generalizations. We begin by extending the class of metric spaces for which these properties are known. The next few chapters are devoted to generalizing these properties to all coarse spaces and exploring the relationships between these generalizations. Finally, we give a brief discussion of computational topology, highlighting how to generate the Rips and Čech simplicial complexes from a set of data. We end with some code written to generate these complexes, and present some thoughts on how to use this to compute certain coarse properties.

PERMANENCE RESULTS FOR DIMENSION-THEORETIC COARSE NOTIONS

by

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CHAPTER I

INTRODUCTION

1.1 The Philosophy of the Coarse Approach to Metric Spaces

The goal of coarse geometry is to bring to bear the power of topological ideas to discrete spaces. By their nature, discrete spaces have no interesting topology. As a motivating example, we would like the coarse geometry of $\mathbb{Z}$ to be a direct analog of the topology of $\mathbb{R}$. One way to effect this is to examine both $\mathbb{Z}$ and $\mathbb{R}$ from a metaphorical distance. As the distance increases, the two spaces appear increasingly alike.

To be more precise, let $(X, d_X), (Y, d_Y)$ be metric spaces. Then we say that a function $f : X \to Y$ is coarse if, for positive numbers R and S, there exist numbers R' and S' so that $d_Y(f(x), f(x')) < S'$ whenever $d_X(x, x') < S$ and $d_Y(f(x), f(x')) > R'$ whenever $d_X(x, x') > R$. Two spaces are coarsely equivalent if there is a coarse function f from X to Y and a coarse function g from Y to X where $\sup\{d_Y(f \circ g(y), y) \mid y \in Y\} < \infty$ and $\sup\{d_X(g \circ f(x), x) \mid x \in X\} < \infty$. Thus the inclusion map $f : \mathbb{Z} \to \mathbb{R}$ and the greatest integer map $g : \mathbb{R} \to \mathbb{Z}$ show that $\mathbb{Z}$ and $\mathbb{R}$ are coarsely equivalent. We define these notions in more detail in Section 2 of Chapter II.

A fundamental topological property is dimension. We begin this dissertation by considering a notion of dimension that is a coarse invariant, i.e. is invariant under this notion of coarse equivalence. This so-called asymptotic dimension is the coarse analog to topological covering dimension. We will also be considering other coarse invariants related to asymptotic dimension, such as property A, asymptotic property C and finite decomposition complexity in Chapters III, IV and V.

The original motivation for this asymptotic approach comes from the geometry of finitely generated groups. Asymptotic dimension itself was introduced by Gromov in [Gro93] as an invariant of finitely generated groups, that is, not dependent on the presentation of the group. Smith showed in [Smi06] that countable groups carry a unique left-invariant proper metric coarse structure, discussed further in Section 3 of Chapter II. Thus, this large-scale setting is the natural one for considering metric properties of such objects.

1.2 Our Main Focus

Asymptotic dimension rose to prominence after Yu proved the Novikov higher signature conjecture (see [FRR95]) for finitely generated groups with finite asymptotic dimension, in [Yu98]. For his result, having finite asymptotic dimension (FAD) was a sufficient but not necessary condition, as there are finitely generated groups with infinite asymptotic dimension that satisfy the conjecture. This motivated the introduction of similar properties such as Yu's property A in [Yu00], Dranishnikov's asymptotic property C in [Dra00], and Guentner, Tessera and Yu's finite decomposition complexity in [GTY12]. In [Roe03], Roe introduces coarse structures, which unify a number of notions of topological control. In the metric setting, these coarse structures reduce to the coarse approach defined above.

We have three main goals in this dissertation. The first is to present a number of permanence results for finite asymptotic dimension, asymptotic property C, finite decomposition complexity and property A; i.e. to determine to what extent these properties are preserved by unions, direct products, free products and other such constructions. Our second goal is to generalize asymptotic property C and finite decomposition complexity to all coarse structures and explore the relationships between them. Our third goal is to implement algorithms in Sage for building certain constructions from computational topology.

1.3 Our Results

In Chapter II, we recall some basic definitions needed for the rest of the paper.

In Chapter III, we recall the definitions of two large-scale metric invariants: asymptotic dimension (asdim) and property A (PA). We present numerous previously known permanence results for these invariants that lead up to our main result of that section:

Theorem III.27. *Let Γ be a countable graph that contains no complete subgraph with more than k vertices. Let $\mathfrak{G}$ be a collection of finitely generated groups with asymptotic dimension bounded above by some positive number n indexed by the vertices of Γ. Then the asymptotic dimension of the graph product $\Gamma\mathfrak{G}$, defined in Section 1 of Chapter II, is at most nk.*

This extends Antolín and Dreesen's result in [AD13], where Γ is presumed to be finite, but requires a completely new set of tools. On the other hand, their techniques can be directly applied to prove the following theorem:

Theorem III.25. *Let Γ be a finite graph. Let $\mathfrak{G}$ be a collection of finitely generated groups with property A. Then $\Gamma\mathfrak{G}$, defined in Section 1 of Chapter II, has property A.*

In Chapter IV, we recall the definition of asymptotic property C (aPC) from [Dra00] and introduce a generalization to the coarse category as follows. The precise definitions of a coarse space and entourages can be found in Section 4 of Chapter II.

Definition IV.6. *A coarse space* $(X, \mathcal{E})$ *has* coarse property C *(cPC) if for any sequence* $L_1 \subset L_2 \subset L_3 \subset \cdots$ *of entourages there is a finite sequence* $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ *so that*

(1) $\mathcal{U} = \bigcup_{i=1}^{n} \mathcal{U}_i$ *covers* X*;*

(2) *each* $\mathcal{U}_i$ *is uniformly bounded; and*

(3) *each* $\mathcal{U}_i$ *is* L_i*-disjoint.*

We discuss the relationship between cPC and the notions of coarse finite asymptotic dimension (cFAD) and coarse property A (cPA) introduced in [Roe03]. In addition, a number of previously known permanence properties are presented in this chapter, and we add to that collection a union theorem (Theorem IV.3). Although Theorem IV.3 contains several technical assumptions, it immediately implies the following simple finite union theorem.

Corollary IV.4. *Let* (Z, d) *be a metric space with* $Z = X \cup Y$*. If* (X, d) *and* (Y, d) *have asymptotic property C, then so does* Z*.*

In Chapter V, we recall the definitions of finite decomposition complexity (FDC) and straight finite decomposition complexity (sFDC) from [GTY12] and introduce generalizations of both to the coarse category. A number of previously known permanence results are presented here as well. We also discuss the relationships between the different varieties of finite decomposition complexity.

We will prove an analog to Guentner, Tessera and Yu's fibering theorem from [GTY12]. In particular, we will show the following:

Theorem V.7. *Let X and Y be metric spaces and let $f : X \to Y$ be a coarse map. Assume that Y has sFDC and that for every bounded family $\mathcal{V}$ in Y, the inverse image $f^{-1}(\mathcal{V})$ has sFDC. Then, X has sFDC.*

A number of permanence results follow readily from this theorem. This includes graph products, using the same machinery as in [AD13] and in Theorem III.25.

Table 1. Permanence results in the metric setting

Operation	FAD	APC	sFDC	PA
Direct Product	[BD08]		[GTY13]	[Yu00]
Unions	[BD11]	Prop. IV.3	[GTY13]	[Bel03]
Free Product	[Dra08]		[GTY13]	[Bel03]
Amalgamated Product	[Dra08]		[GTY13]	[Bel03]
Finite Graph Product	[AD13]		Cor. V.10	Thm. III.25
Infinite Graph Product	Thm. III.27			

Table 2. Permanence results in the coarse setting

Operation	cFAD	cProp C	cFDC
Direct Product	[Gra05]		Thm V.25
Unions	Thm IV.23	Thm IV.21	Thm V.27

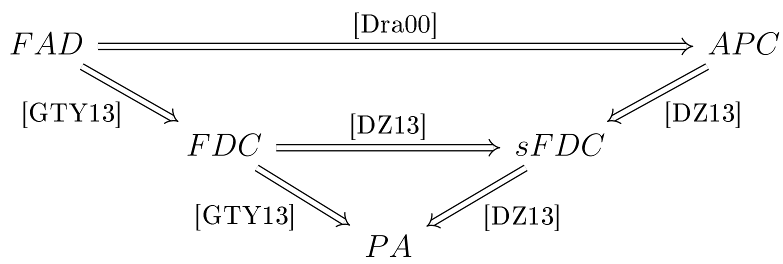

Figure 1. Relationships in the metric setting

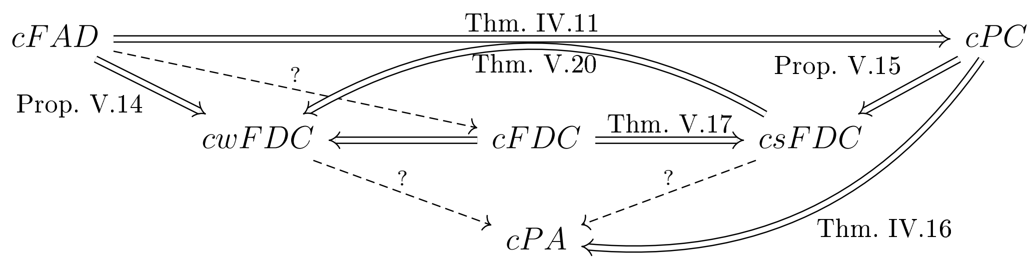

Figure 2. Relationships in the coarse setting

In Chapter VI, we discuss two constructions that allow us to build a simplicial complex whose vertices are points in a metric space: the Čech complex and the Rips complex. We present algorithms for constructing both of these complexes. A Sage implementation of these algorithms appears in Appendix A. Finally, we consider how one might use these algorithms to compute another large-scale invariant, simply called Gromov's invariant in [BD08]. This was originally defined by Gromov in [Gro93] and is a measure of control of the sets in a cover.

CHAPTER II

PRELIMINARIES

2.1 The Word Metric on a Finitely Generated Group

Let $(G, \cdot)$ be a group and let S be a non-empty subset of G. The set of *S-words* in G is the set $\{s_1 \cdot s_2 \cdot \cdots \cdot s_k \mid s_i \in S,\ k \in \mathbb{N}\}$ consisting of all formal finite products of elements in S. We will call k the length of the word $s_1 \cdot s_2 \cdot \cdots \cdot s_k$. As usual, we will suppress the product notation to concatenation in what follows. Let $g \in G$ be a group element. We say that the S-word $s_1 s_2 \cdots s_k$ is a *presentation of g* if g and $s_1 s_2 \cdots s_k$ are equal as elements of G. It is possible that many different S-words could present the same element g. By convention, every set of S-words contains a presentation of the identity element, denoted by e, as the empty word. We call S a *generating set* for G if every $g \in G$ has a presentation as an S-word. In this case we will also say that S *generates* G.

We say that S is a *symmetric* generating set if whenever $s \in S$, then the group element s^{-1} is also in S.

A group is called *finitely generated* if it has a finite generating set. Observe that any finite group is finitely generated. It is easy to show that the group of rational numbers under addition is not finitely generated.

Fix a finitely generated group G with finite symmetric generating set S. There is a natural notion of distance that can be associated to the pair (G, S). For every element $g \in G$, let $\|g\|_S$ denote the length of the shortest S word presenting g. The *left-invariant word metric* on G corresponding to S is defined by

$$d_S(g, h) = \|g^{-1}h\|_S.$$

The metric is called left-invariant because, for every $g \in G$, the map $x \mapsto gx$ is an isometry from G to G, i.e. it preserves this metric: $d_S(x, y) = \|x^{-1}y\|_S = \|x^{-1}g^{-1}gy\|_S = d_S(gx, gy)$

The *Cayley graph* of a pair (G, S) is a graph Γ_G so that

(1) the vertex set of Γ_G is G;

(2) for any element $g \in G$ and generator $s \in S$, there is an edge between g and gs.

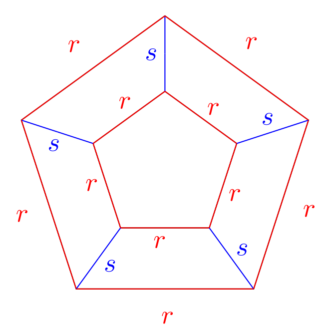

Figure 3. Cayley graph of $D_{10} = \langle r, s \mid r^5 = s^2 = e \rangle$

If we assign each edge of Γ_G length 1, then the distance between any two elements of G will be the same in the word metric and the edge length metric, which is defined as the length of a shortest path between the two vertices. Any path between g and h in the Cayley graph is given by a sequence of vertices as follows, $g, gs_1, gs_1s_2, \ldots, gs_1s_2 \cdots s_k = h$. Therefore, $s_1s_2 \cdots s_k$ is a presentation of $g^{-1}h$. So we have that the shortest path between g and h corresponds to the shortest presentation of $g^{-1}h$.

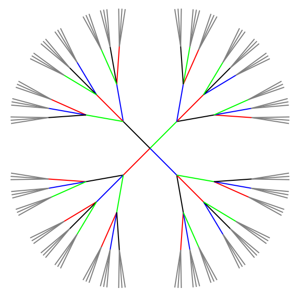

Figure 4. Cayley graph of the free group on two letters.

We will be dealing with four main operations on groups: the direct product, the amalgamated product, the free product and the graph product. We use the notation of [LS01] to define the amalgamated product of groups and let $A = \langle S_A \mid R_A \rangle$ and $B = \langle S_B \mid R_B \rangle$ where S_A and S_B are generating sets and R_A and R_B are sets of relations, and let C be a group with injective homomorphisms $\phi_A : C \to A$ and $\phi_B : C \to B$. The *free product of* A *and* B *amalgamated over* C is denoted $A *_C B$ and is defined to be the group generated by the disjoint union of S_A and S_B with a set of relations that is the disjoint union of R_A and R_B, with the additional relations that $\phi_A(c) = \phi_B(c)$ for all $c \in C$.

The *free product of* A *and* B, denoted $A * B$, is defined as the amalgamated product $A *_C B$ where $C = \{e\}$.

Finally, we define the graph product of groups. Let Γ be an undirected graph without loops or multiple edges. Let $V(\Gamma)$ and $E(\Gamma)$ be the set of vertices and edges of Γ, respectively. Suppose that $\mathfrak{G} = \{G_v \mid v \in V(\Gamma)\}$ is a collection of groups indexed by the elements of $V(\Gamma)$. The *graph product* $\Gamma\mathfrak{G}$ of the collection $\mathfrak{G}$ over the graph Γ is defined to be the free product of the G_v with the additional relations that whenever $\{v, v'\}$ is an edge in Γ, then $gg' = g'g$ for all $g \in G_v$ and $g' \in G_{v'}$. Thus, if $E(\Gamma) = \emptyset$, $\Gamma\mathfrak{G}$ is the free product of the vertex groups. If Γ is the complete graph on n vertices, $\Gamma\mathfrak{G}$ is the direct product of the vertex groups. Graph products were introduced by Green in [Gre90] and were the focus of her dissertation.

Let $g \in \Gamma\mathfrak{G}$. We say that $g = g_1 \cdots g_\ell$ is an expression of g in syllables if each g_i is a non-trivial element of a single vertex group, and no two consecutive g_i and g_{i+1} belong to the same vertex group.

2.2 Coarse Equivalence

Because the definition of the word metric relies on a generating set, a single group can be endowed with many different metric structures. We would like to define an equivalence relation on metric spaces so that two metric structures placed on the same group are equivalent. The notion of coarse equivalence introduced in Chapter I provides such a relation, which we will make precise as follows.

Definition II.1 ([Roe03])**.** Let (X, d_X) and (Y, d_Y) be metric spaces. We say that $f : X \to Y$ is a *coarse embedding* if there exist positive valued, non-decreasing maps ρ_1 and ρ_2 that go to infinity so that for every two points x_1, x_2 in X, $\rho_1(d_X(x_1, x_2)) \leq d_Y(f(x_1), f(x_2)) \leq \rho_2(d_X(x_1, x_2))$.

Definition II.2 ([Roe03])**.** Let (X, d_X) and (Y, d_Y) be metric spaces. If $f : X \to Y$ is a coarse embedding, and there exists a $R > 0$ such that for all $y \in Y$ there exists an $x \in X$ with $d(f(x), y) < R$, then f is called a *coarse equivalence* and X and Y are said to be *coarsely equivalent.*

Using the notation from Chapter I, we have that given R and S, one can take $\rho_1(R) = R'$ and $\rho_2(S) = S'$. It is easy to show that coarse equivalence defines an equivalence relation on metric spaces.

Example II.3. Let G be a finitely generated group. The metric spaces (G, S) and (G, T) are coarsely equivalent, where S and T are two finite generating sets for G. We let f be the identity map and see that the definition is satisfied by $R = 1$, $\rho_1(x) = \frac{1}{\lambda}x$ and $\rho_2(x) = \lambda x$ with $\lambda = \max\{\lambda_1, \lambda_2\}$, where $\lambda_1 = \max\{||s||_T \mid s \in S\}$ and $\lambda_2 = \max\{||t||_S \mid t \in S\}$.

Remark. In geometric group theory, one encounters a similar notion of equivalence: A *quasi-isometric embedding* is a map $f : X \to Y$ for which there exists $\lambda \geq 1, \varepsilon \geq 0$ such that $\frac{1}{\lambda}d_X(x_1, x_2) - \varepsilon \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + \varepsilon$. This is more restrictive that we require however.

Example. Any metric space of finite diameter is quasi-isometric (and thus coarsely equivalent) to a point. We let f be any function that identifies the single point to some point in X and see that the definition is satisfied by $\rho_1(x) = \rho_2(x) = x$ and $R = \mathrm{diam}(X)$.

Example. In the Euclidean metric, $\mathbb{Z}^n$ and $\mathbb{R}^n$ are quasi-isometric (and thus coarsely equivalent). We let $f : \mathbb{Z}^n \to \mathbb{R}^n$ be the inclusion. Then we see that the definition is satisfied by $\rho_1(x) = \rho_2(x) = x$ and $R = 1$.

Example. The map $f : \mathbb{Z} \to \mathbb{Z}, f(n) = n^2$, where $\mathbb{Z}$ is equipped with the standard metric, is not a coarse equivalence, since $d(n, n+1) = 1$ and $d(f(n), f(n+1)) = 2n+1$. As $2n+1$ will grow to infinity with n, there can be no ρ_i that will satisfy the definition.

Example II.4. Let X be any unbounded space and let Y be any bounded space with $\operatorname{diam} X = k$. Then X and Y are not coarsely equivalent. Let $f : X \to Y$. Then we note that $d_Y(y, y') \leq k$ for all y, y'. Thus, for any ρ_1 such that $\rho_1(d_X(x_1, x_2)) \leq d_Y(f(x_1), f(x_2))$, $\rho_1(d_X(x_1, x_2)) \leq k$ and thus ρ_1 does not go to infinity.

2.3 Countable Groups and Coarse Invariants

When G is not finitely generated, it is no longer the case that the identity map from the group to itself with different generating sets must be a coarse equivalence. For example, we can consider $S = \{\frac{1}{n} \,|\, n \in \mathbb{Z}\}$ and $T = \mathbb{Q}$ both as generating sets of $\mathbb{Q}$. Then $(\mathbb{Q}, d_T)$ is bounded, while $(\mathbb{Q}, d_S)$ is unbounded. Therefore, as we saw above in Example II.4, $(\mathbb{Q}, d_T)$ and $(\mathbb{Q}, d_S)$ cannot be coarsely equivalent.

To work around this, we modify the definition of the word metric in the following way. We define a *weight function* on a generating set $S = S^{-1}$ for a group to be a function $w : S \to [0, \infty)$ for which

(1) if $w(s) = 0$ then $s = e$;

(2) $w(s) = w(s^{-1})$; and

(3) for each $N \in \mathbb{N}$, $w^{-1}([0, N])$ is finite.

One then defines a norm by $\|g\| = \inf\{\sum w(s_i) \mid g = s_1 s_2 \cdots s_n\}$, where the norm of the identity is defined to be 0 (i.e. it is presented by the empty product). Then as before, we define the metric by $d(g, h) = \|g^{-1}h\|$. This metric is *proper*; i.e. every closed ball is compact. It is also left-multiplication invariant. We note that if $w \equiv 1$, then S must be finite and this definition reduces to the word metric as previously defined.

In [Smi06], J. Smith showed that on any countable group G any left-invariant proper metrics arises from a weight function in this way. Moreover, such a metric is unique up to coarse equivalence. Therefore, any property that is invariant under coarse equivalence can be seen as a property of G, that is independent of the choice of metric.

Later in this dissertation, we introduce the notions of finite asymptotic dimension from [Gro93], property A from [Yu00], asymptotic property C from [Dra00] and finite decomposition complexity from [GTY12]. These properties are invariant under coarse equivalence and therefore can be seen as properties of groups.

2.4 The Coarse Category

Let (X, d) be a metric space. For any $r > 0$, we define $E_r = \{(x, y) \in X \times X \mid d(x, y) \leq r\}$. The sets E_r are symmetric and contain the diagonal $\Delta_X = \{(x, x)\}$. Finite unions of two of these sets result in another set of this form, as we have that $E_r \cup E_s = E_t$ where $t = \max\{r, s\}$. If we define the collection $\mathcal{E} = \bigcup_{r \geq 0} \mathcal{P}(E_r)$, then we also have that $\mathcal{E}$ is closed under the composition $E_r \circ E_s := \{(x, z) \mid \exists_{y \in X} \text{ with } (x, y) \in E_r \text{ and } (y, z) \in E_s\}$.

Without relying on a metric, we can define a collection of subsets of $X \times X$ with similar properties.

Definition II.5 ([Gra05, Roe03])**.** A *coarse structure* on a space X is a collection $\mathcal{E}$ of subsets of $X \times X$ called *entourages* or *controlled sets* such that:

(1) a subset of an entourage is an entourage;

(2) a finite union of entourages is an entourage;

(3) the diagonal $\Delta_X := \{(x, x) \mid x \in X\}$ is an entourage;

(4) the inverse $E^{-1} := \{(y, x) \mid (x, y) \in E\}$ of an entourage E is an entourage; and

(5) the composition of two entourages E_1 and E_2 as defined above is an entourage.

We call the pair $(X, \mathcal{E})$ a *coarse space*.

In this context, spaces are said to be *connected* if every point of $X \times X$ is contained in some entourage. For $E \in \mathcal{E}$ and $A \subset X$, we define $E[A] := \{x \in X \mid (x, a) \in E \text{ for some } a \in A\}$ and denote $E[\{x\}]$ as $E[x]$. Then, a set is said to be *bounded* if it is of the form $E[x]$ for some $x \in X$ and $E \in \mathcal{E}$. If X is a topological space, we call a subset $E \subset X \times X$ *proper* if $E[K]$ and $E^{-1}[K]$ are relatively compact whenever K is relatively compact.

To give an idea of what coarse structures can look like, we present the following list of examples from [Gra05].

Example. Let X be any set and let $\mathcal{E} = \mathcal{P}(X \times X)$. Then $(X, \mathcal{E})$ is a coarse space and $\mathcal{E}$ is called the *maximal coarse structure* on X.

Example. Let (X, d) be a metric space, and let $\mathcal{E}$ be the collection of subsets E of $X \times X$ such that $\sup\{d(x, y) \mid (x, y) \in E\} < \infty$. Then $(X, \mathcal{E})$ is a coarse space and $\mathcal{E}$ is called the *bounded coarse structure* on X associated with d.

Example. Let X be any set and let $\mathcal{E}$ be the collection of all subsets of $X \times X$ that contain only finitely many points not in Δ_X. Then $(X, \mathcal{E})$ is a coarse space and $\mathcal{E}$ is called the *discrete coarse structure* on X.

Example. Let X be a topological space and let $\mathcal{E}$ be the collection of all proper $E \subset X \times X$. Then $(X, \mathcal{E})$ is a coarse space and $\mathcal{E}$ is called the *indiscrete coarse structure* on X. If X is compact, this is the same as the maximal coarse structure.

Example. Let $(X, \mathcal{E})$ be a coarse space and $Y \subset X$. Then we define the coarse structure inherited from X to be $\mathcal{E}_Y = \{E \cap (Y \times Y) \mid E \in \mathcal{E}\}$. Then $(Y, \mathcal{E}_Y)$ is a coarse space.

Example. Let X and Y be coarse spaces. Then we can get the *product coarse structure* on $X \times Y$ by saying that a subset of $(X \times Y) \times (X \times Y)$ is controlled if and only if both its projection to $X \times X$ and to $Y \times Y$ is controlled.

In the next definition, we establish terminology for function between coarse spaces.

Definition II.6 ([Gra05])**.** Let X and Y be coarse spaces and $f : X \to Y$ be a function.

(1) We call f *coarsely proper* if the inverse image of every bounded set is bounded.

(2) We call f *coarsely uniform* if the image of each entourage of X under the map $f \times f$ is an entourage of Y.

(3) We call f a *coarse map* if it is coarsely proper and coarsely uniform.

(4) We call f a *coarse embedding* if it is coarsely uniform and the inverse image of an entourage of Y under $f \times f$ is an entourage of X. We note that a coarse embedding is a coarse map.

(5) Let S be a set. Then the maps $f : S \to X$ and $g : S \to X$ are called *close* if the set $\{(f(s), g(s)) \mid s \in S\}$ is an entourage of X.

(6) We call f a *coarse equivalence* if f is a coarse map, and if there exists a coarse map $g : Y \to X$ such that $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y.

CHAPTER III

ASYMPTOTIC DIMENSION AND PROPERTY A

3.1 Introduction

We begin by recalling some well-known definitions from coarse geometry. For the following definitions, let (X, d_X) and (Y, d_Y) be metric spaces.

Definition III.1. A function $f : X \to Y$ is called *uniformly expansive* if there is a non-decreasing $\rho_2 : [0, \infty) \to [0, \infty)$ such that

$$d_Y(f(x), f(x')) \leq \rho_2(d_X(x, x')).$$

Definition III.2. The function $f : X \to Y$ is called *effectively proper* if there is some proper, non-decreasing $\rho_1 : [0, \infty) \to [0, \infty)$ such that

$$\rho_1(d_X(x, x')) \leq d_Y(f(x), f(x')).$$

In these terms, the definition of coarse embedding given in Chapter II can be reformulated by saying that $f : X \to Y$ is a coarse embedding if f is both uniformly expansive and effectively proper.

Let $R > 0$ be a (large) real number. A collection $\mathcal{U}$ of subsets of the metric space X is said to be *R-discrete* if there is a uniform bound on the diameter of the sets in $\mathcal{U}$ and if, whenever $U \neq U'$ are sets in $\mathcal{U}$, then $d(U, U') > R$, where $d(U, U') = \inf\{d(x, x') \mid x \in U, x' \in U'\}$. We will often refer to such families as being *uniformly bounded* and *R-disjoint*. Gromov [Gro93] describes this situation by saying that $\bigcup_{U \in \mathcal{U}} U$ is 0-*dimensional on R-scale*.

Definition III.3 ([Gro93])**.** We say the *asymptotic dimension* of the metric space X does not exceed n, and write $\operatorname{asdim} X \leq n$, if for each (large) $R > 0$, X can be written as a union of $n+1$ sets with dimension 0 at scale R.

In [Yu00], G. Yu defined property A for discrete metric spaces as a generalization of amenability of groups.

Definition III.4 ([Yu00])**.** A discrete metric space X has *property A* if for any $r > 0$ and any $\varepsilon > 0$, there is a collection of finite subsets $\{A_x\}_{x \in X}$, where $A_x \subset X \times \mathbb{N}$, so that

(1) $(x, 1) \in A_x$ for each $x \in X$;

(2) for every pair x and y in X with $d(x, y) < r$, $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon$; and

(3) there is some R so that if $(y, n) \in A_x$, then $d(x, y) \leq R$.

There are a number of equivalent characterizations of both asymptotic dimension and property A. In order to state them, we give some preliminary definitions. We will primarily be concerned with spaces X such that X is a discrete metric space with bounded geometry, that is, every ball of finite radius has finite cardinality. We begin with a set of definitions that concern themselves with $\mathcal{U}$, a cover of X, that is, a collection of subsets of X such that $\bigcup_{U \in \mathcal{U}} U = X$. We do not require the subsets be open.

Definition III.5. Let X be a discrete metric space with bounded geometry and $\mathcal{U}$ a cover of X.

(1) For $d \geq 0$, we say the *d-multiplicity* of $\mathcal{U}$ is $\sup_{x\in X}\{\mathrm{card}\{U \in \mathcal{U} \mid U \cap B_d(x) \neq \emptyset\}\}$. The 0-multiplicity is also called the *multiplicity*.

(2) A *Lebesgue number* of $\mathcal{U}$ is a number $\delta > 0$ such that every subset of X having diameter less than δ is contained in some member of $\mathcal{U}$.

(3) A cover $\mathcal{U}$ is said to be *uniformly bounded* if there exists a $D > 0$ such that $\mathrm{diam}(U) \leq D$ for all $U \in \mathcal{U}$.

Let K be a simplicial complex. We say that K is a *uniform simplicial complex* when it is given the metric inherited from an affine embedding into $\ell^2(\mathbb{N})$ obtained by sending each vertex v to a distinct basis element..

Definition III.6. Let K be a uniform simplicial complex and let X and Y be metric spaces.

- A map $\phi : K \to \ell^2$ is *uniformly cobounded* if $\mathrm{diam}(\phi^{-1}(\sigma))$ is uniformly bounded for all simplexes σ.
- A map $\phi : X \to Y$ is *ε-Lipschitz* if $d_Y(\phi(x_1), \phi(x_2)) \leq \varepsilon d_X(x_1, x_2)$.

We will use the notation $d < \infty$ to indicate that d is a large, positive number.

Theorem III.7 ([BD11])**.** *Let X be a discrete metric space with bounded geometry. The following conditions are equivalent:*

(1) $\mathrm{asdim}\, X \leq n$*;*

(2) *for every $d < \infty$ there exists a uniformly bounded cover $\mathcal{V}$ of X with d-multiplicity $\leq n+1$;*

(3) *for every* $\lambda < \infty$ *there is a uniformly bounded cover* $\mathcal{W}$ *of* X *with Lebesgue number* $> \lambda$ *and multiplicity* $\leq n+1$*; and*

(4) *for every* $\varepsilon > 0$ *there is a uniformly cobounded,* ε*-Lipschitz map* $\phi : X \to K$ *to a uniform simplicial complex of dimension* n*.*

Theorem III.8 ([HR00])**.** *Let* X *be a discrete metric space with bounded geometry. Then* X *has property A if and only if for each* $n \in \mathbb{N}, x \in X$ *there exists a functions* $a_x^n : X \to [0,1]$ *satisfying:*

(1) $\Sigma_{z \in X} a_x^n(z) = 1$*;*

(2) *for every* $n > 0$ *there is an* $R = R(n) > 0$ *such that* $\mathrm{supp}(a_x^n) \subset B_R(x)$ *for all* $x \in X$*; and*

(3) *for every* $K > 0$*,*

$$\lim_{n\to\infty} \sup_{d(z,w)<K} ||a_z^n - a_w^n||_1 = 0.$$

Higson and Roe define this condition for all metric spaces. If it happens that the metric space is discrete, with bounded geometry, then their definition is equivalent to the one given by Yu.

Theorem III.9 ([HR00])**.** *Let* X *be a discrete metric space with bounded geometry. If* X *has finite asymptotic dimension, then* X *has property A.*

In addition to permanence results for certain topological constructions such as direct products and unions, we wish to prove some permanence results for more group-theoretic constructions. These group theoretic constructions still grow from a topological root however, and have topological applications, as we see with the case of the amalgamated product, the first of the group theoretic constructions we will consider. For example, if we let $\Gamma = \pi_1(X)$, then if we have $X = U \cup V$ where U and V are open path-connected subspaces of X and $U \cap V$ is path-connected and non-empty, we have that, by Seifert Van Kampen, $\Gamma = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$. This is of use to us thanks to the following important theorem:

Theorem III.10 (Švarc-Milnor Lemma, [dlH00]). *Let X be a proper geodesic space and let Γ act properly on X (that is, for all compact K, $|\{\gamma \mid \gamma.K \cap K \neq \emptyset\}| < \infty$) such that $\Gamma \backslash X$ is compact. Then, for any $x_0 \in X$, the map $\Gamma \to X : \gamma \mapsto \gamma.x_0$ is a quasi-isometry.*

As an example, take a compact proper geodesic space X, $\pi_1(X)$ acts properly on the universal cover of X, $\mathrm{E}X$, by deck transformations, and since $\pi_1(X) \backslash \mathrm{E}X = X$, we have that $\pi_1(X)$ is quasi-isometric to $\mathrm{E}X$.

We will also consider how both asymptotic dimension and property A are preserved by group actions, group extension and finally, graph products of groups.

3.2 Background Results

We will begin this section by establishing that finite asymptotic dimension and property A are in fact coarse invariants, and then present a number of permanence results.

Theorem III.11 ([BD11]). *If $f : X \to Y$ is a coarse equivalence and if $\operatorname{asdim} X = n < \infty$, then $\operatorname{asdim} Y = n$.*

Theorem III.12 ([Wil06]). *If $f : X \to Y$ is a coarse equivalence and if Y has property A, then so does X.*

The following variation of the definition of asymptotic dimension will be useful in a number of permanence results we wish to prove. A family of metric spaces $\{X_\alpha\}$ satisfies the inequality $\operatorname{asdim} X_\alpha \leq n$ *uniformly* in α if for $d < \infty$ one can find an R and R-bounded d-disjoint families $\mathcal{U}_\alpha^0, \ldots, \mathcal{U}_\alpha^n$ of subsets of X_α such that the union $\bigcup_i \mathcal{U}_\alpha^i$ is a cover of X_α.

We wish to show that these properties are preserved by finite unions and certain infinite unions. To do so, we require a preliminary definition and proposition. Both are from [BD11]. Later we will use similar techniques in the coarse setting, so we will reproduce the proofs here.

Let $\mathcal{V}$ and $\mathcal{U}$ be families of subsets of a metric space X. Given $V \in \mathcal{V}$ and $d > 0$, we denote by $N_d(V, \mathcal{U})$ the union of V and all sets $U \in \mathcal{U}$ where $d(U, V) = \inf\{d(x, y) \mid x \in U, y \in V\} \leq d$. The *d-saturated union* of $\mathcal{U}$ and $\mathcal{V}$ is denoted $\mathcal{U} \bigcup_d \mathcal{V} = \{N_d(V, \mathcal{U}) \mid V \in \mathcal{V}\} \cup \{U \in \mathcal{U} \mid d(U, V) > d \ \forall\ V \in \mathcal{V}\}$.

Proposition III.13 ([BD11])**.** *Assume $\mathcal{U}$ is d-disjoint and R-bounded with $R \geq d$. Assume that $\mathcal{V}$ is $5R$-disjoint and D-bounded. Then $\mathcal{V} \cup_d \mathcal{U}$ is d-disjoint and $D + 2(d + R)$-bounded.*

Proof. First we note that there are two types of elements in $\mathcal{V} \cup_d \mathcal{U}$, coming from the two different collections in the definition. Pairs of elements of type U (that is, that are also elements of $\mathcal{U}$) are clearly d-disjoint. Also, an element of type U and an element of type $N_d(V, \mathcal{U})$ are also clearly d-disjoint. Now, consider elements $N_d(V, \mathcal{U})$ and $N_d(V', \mathcal{U})$, with $V \neq V'$. They are contained within the $(d+R)$-neighborhoods of V and V' respectively. Since V and V' are $5R$-disjoint, and $R \geq d$, the neighborhoods will be d-disjoint.

Finally, we have that $\operatorname{diam} N_d(V, \mathcal{U}) \leq \operatorname{diam} V + 2(d + R) \leq D + 2(d + R)$. □

Theorem III.14 ([BD11]). *Let $X = \bigcup_\alpha X_\alpha$ where $\operatorname{asdim} X_\alpha \leq n$ uniformly in α. Suppose that for any r there exists $Y_r \subset X$ with $\operatorname{asdim} Y_r \leq n$ and such that the family $\{X_\alpha \setminus Y_r\}$ is r-disjoint in the sense that if $\alpha \neq \alpha'$, then $d(X_\alpha \setminus Y_r, X_{\alpha'} \setminus Y_r) \geq r$. Then $\operatorname{asdim} X \leq n$.*

Proof. Let d be given. Consider R-bounded families $\mathcal{U}_\alpha^0, \ldots, \mathcal{U}_\alpha^n$ from the definition of the uniform inequality $\operatorname{asdim} X_\alpha \leq n$. We may take $R > d$ as necessary. Let $r = 5R$ and consider Y_r given by our assumptions and find r-disjoint, D-bounded families $\mathcal{V}^0, \ldots, \mathcal{V}^n$ from the definition of $\operatorname{asdim} Y_r \leq n$. Let $\overline{\mathcal{U}}_\alpha^i$ be the restriction of $\mathcal{U}_\alpha^i$ to $X_\alpha \setminus Y_r$. Let $\overline{\mathcal{U}}^i = \cup_\alpha \overline{\mathcal{U}}_\alpha^i$. We note that the family $\overline{\mathcal{U}}^i$ will be d-disjoint and R-bounded. For each i, we define $\mathcal{W}^i = \mathcal{V}^i \cup_d \overline{\mathcal{U}}^i$. By the above proposition, the family $\mathcal{W}^i$ is d-disjoint and uniformly bounded. As the original $\mathcal{U}_\alpha^i$ covered X, we have that the saturated union with also cover X and therefore, $\operatorname{asdim} X \leq n$. □

Corollary III.15 ([BD11]). *Let (Z, d) be a metric space with $Z = X \cup Y$. If (X, d) and (Y, d) have finite asymptotic dimension, then so does Z. Specifically, $\operatorname{asdim} Z \leq \max\{\operatorname{asdim} X, \operatorname{asdim} Y\}$.*

Proof. To see this, we consider the family of spaces $\{A, B\}$ and let $Y_r = B$ and apply the previous theorem. □

Theorem III.16 ([Bel03]). *Let $X = \bigcup_\alpha X_\alpha$ where the X_α have property A uniformly, that is $R(n)$ is independent of α. Suppose that for any r there exists $Y_r \subset X$ with property A such that the family $\{X_\alpha \setminus Y_r\}$ is r-disjoint. Then X has property A.*

Corollary III.17 ([Bel03]). *Let (Z, d) be a metric space with $Z = X \cup Y$. If (X, d) and (Y, d) have property A, then so does Z.*

Proof. Similarly, we consider the family of spaces $\{A, B\}$ and let $Y_r = B$ and apply the previous theorem. □

Theorem III.18 (Hurewicz Theorem, [BD06]). *Let X be a geodesic metric space and let $f : X \to Y$ be an ε-Lipschitz map such that for every $R > 0, \operatorname{asdim} f^{-1}(B_R(x)) \leq n$ uniformly in x. Then $\operatorname{asdim} X \leq \operatorname{asdim} Y + n$.*

Theorem III.19 ([BD08]). *Let X and Y be two discrete metric spaces with bounded geometry and finite asymptotic dimension. Then $X \times Y$ has finite asymptotic dimension. Specifically, $\operatorname{asdim} X \times Y \leq \operatorname{asdim} X + \operatorname{asdim} Y$*

Theorem III.20 ([Yu00]). *Let X and Y be two spaces with property A and let $X \times Y$ be given the ℓ^2 product metric. Then $X \times Y$ has property A.*

Theorem III.21 ([Dra08]). *Let A and B be two groups with finite asymptotic dimension. Then $A *_C B$ has finite asymptotic dimension. Specifically,*

$$\operatorname{asdim} A *_C B \leq \max\{\operatorname{asdim} A, \operatorname{asdim} B, \operatorname{asdim} C + 1\}.$$

This bound is sharp, as we can see in the following example from [BD04]. Let $A = B = C = \mathbb{Z}$ (and therefore have $\operatorname{asdim} A = \operatorname{asdim} B = \operatorname{asdim} C = 1$) and let both inclusions, $C \to A$ and $C \to B$ be given by multiplication by 2. Then $A *_C B$ is isomorphic to the fundamental group of the Klein bottle. By the Švarc-Milnor Lemma, this means that $A *_C B$ is quasi-isometric to the universal cover of the Klein bottle, which is $\mathbb{R}^2$. Therefore, $\operatorname{asdim} A *_C B = 2$.

Theorem III.22 ([Dyk04, Tu01, Bel03]). *Let A and B be two groups with property A. Then $A *_C B$ has property A.*

3.3 Graph Products

In this section we extend the result of Antolín and Dreesen in [AD13] concerning asymptotic dimension of graph products of groups in two directions. First, we show that one can replace finite asymptotic dimension everywhere with property A and arrive at the corresponding conclusion. Second, we extend the asymptotic dimension result to include certain infinite graphs.

We begin by recalling their result.

Theorem III.23 ([AD13, Theorem 6.3])**.** *Let Γ be a finite graph and let $\mathfrak{G}$ be a family of finitely generated groups indexed by vertices of $V(\Gamma)$. Let $G = \Gamma\mathfrak{G}$. Let $\mathcal{C}$ be the collection of subsets of $V(\Gamma)$ spanning a complete graph. Then*

$$\operatorname{asdim} G \leq \max_{C\in\mathcal{C}} \sum_{v\in C} \max(1, \operatorname{asdim} G_v).$$

For our present purposes, we need a slightly weaker result that we state as a corollary. For a graph Γ, we recall that the *clique number* $\omega(\Gamma)$ is the maximum number of vertices in a clique in Γ; i.e., the size of the largest set of vertices for which each pair is connected by an edge in Γ.

Corollary III.24. *Let Γ be a finite graph with $\omega(\Gamma) \leq k$ and let $\mathfrak{G}$ be a collection of finitely generated groups indexed by $v \in V(\Gamma)$ such that $0 < \operatorname{asdim} G_v \leq n$ for all $v \in V(\Gamma)$. Then, $\operatorname{asdim} \Gamma\mathfrak{G} \leq nk$.*

Proof. We have that $\max(1, \operatorname{asdim} G_v) = \operatorname{asdim} G_v$ for each v. Also, there is at least one $C \in \mathcal{C}$ with $\omega(\Gamma)$ elements. Thus,

$$\operatorname{asdim} G \leq \max_{C\in\mathcal{C}} \sum_{v\in C} \max(1, \operatorname{asdim} G_v) \leq \omega(\Gamma) \max_{v\in V(\Gamma)} \{\operatorname{asdim} G_v\} \leq kn. \qquad \square$$

All that is necessary for the preceding proof to work is that at least one of the G_v should be infinite, forcing $n > 0$. If all G_v are finite, then $\operatorname{asdim} G \leq k$ instead of the estimate given above, which would be $0 = nk$.

The techniques of proof in [AD13, Theorem 6.3] immediately imply the following.

Theorem III.25. *Let Γ be a finite graph. If all the G_v have property A, $\Gamma\mathfrak{G}$ has property A.*

Proof. We proceed by induction on $|V(\Gamma)|$. We note that if $|V(\Gamma)| = 1$, then $\Gamma\mathfrak{G} = G_v$ which is assumed to have property A.

Now we suppose that $|V(\Gamma)| = n > 1$ and also that the theorem holds for graphs with fewer than n vertices.

Then let $v \in V(\Gamma)$ be any vertex, and put $A = \{v\} \cup \operatorname{lk}(v), B = \Gamma - \{v\}, C = \operatorname{lk}(v)$. Then, by [Gre90] we have that $\Gamma\mathfrak{G} = G_A *_{G_C} G_B$.

Now, we have two cases, either $A = \Gamma$ or $A \subset \Gamma$. In the first case, $A = \Gamma$. In that case, we must have that $\Gamma\mathfrak{G} = G_v \times G_C$. Now G_v has property A by assumption. Since $|V(C)| < |V(\Gamma)|$ the induction hypothesis implies that G_C has property A. Since property A is preserved by direct products as we saw above, $\Gamma\mathfrak{G}$ has property A.

In the second case, where $A \neq \Gamma$, we have then that $|V(A)| < |V(\Gamma)|$. By definition, we have that $|V(B)| < |V(\Gamma)|$. And so, by our induction hypothesis, G_A and G_B both have property A. Since by Theorem III.22, amalgamated free products preserve property A, we conclude that $\Gamma\mathfrak{G}$ has property A. □

We can go further when we consider finite asymptotic dimension and allow Γ to be a countable graph, rather than a finite one.

Let Γ be a countable graph. Define a weight function $\bar{w} : V(\Gamma) \to \mathbb{N}$ by taking any one-to-one correspondence between $V(\Gamma)$ and $\mathbb{N}$. For each vertex $v \in V(\Gamma)$, let G_v be a finitely generated group, with generating set S_v. We insist that the set S_v be closed under inverses and not contain the identity element. Define a weight function from the disjoint union of S_v as follows: $w : \bigsqcup S_v \to \mathbb{N}$ by $w(s) = \bar{w}(v)$, where $s \in S_v$ is a generator. Clearly, w is a weight function.

Next, suppose that $r > 0$ is given. Define a graph Γ_r by setting the vertex set of Γ_r equal to $\bar{w}^{-1}([0, r])$. The edge set of Γ_r contains precisely those edges in Γ for which both vertices are also in Γ_r. Let $g \in \Gamma\mathfrak{G}$. We will say that a reduced word $g_1 \cdots g_k$ is a presentation of g in Γ_r-*standard form* if

(1) $g = g_1 \cdots g_k$ with each g_i a reduced syllable and

(2) Whenever $g = h_1 \cdots h_k$ is a reduced word in reduced syllables presenting g we have

$$\max\{i \mid g_i \notin \Gamma_r\mathfrak{G}\} \geq \max\{i \mid h_i \notin \Gamma_r\mathfrak{G}\}.$$

This second condition amounts to saying that each Γ_r syllable is commuted as far to the right of the word as possible. Call an element x of $\Gamma\mathfrak{G}$ *permissible* if the standard form of x does not end with a non-trivial element of $\Gamma_r\mathfrak{G}$. In other words, x is permissible if no reduced word that presents the element x can be made to end with any non-trivial Γ_r syllable. In this way, we will consider the identity to be permissible.

Lemma III.26. *Let $\Gamma\mathfrak{G}$ be a graph product of finitely generated groups $\mathfrak{G} = \{G_v\}$ with the metric described above. Let $r > 0$ be given and take Γ_r as above. Then, each* *element of* $\Gamma\mathfrak{G}$ *can be written in the form* xb*, where* x *is permissible and* $b \in \Gamma_r\mathfrak{G}$*. Moreover, if* $x \neq x'$ *are permissible, then* $d(xb, x'b') > r$*.*

Proof. First, we check that each element has such a form. To this end, let $g \in \Gamma\mathfrak{G}$ be given and write $g = g_1 \cdots g_t$ as an expression in syllables. We proceed by induction on the number of syllables t. If $t = 1$, then either g_1 is in $\Gamma_r\mathfrak{G}$ or not. In the first case, it can be written as xg_1, where $x = e$. In the latter case, $x = g_1$ is permissible.

Suppose now that every word of syllable length at most $t-1$ can be written in the form xb with x permissible and $b \in \Gamma_r\mathfrak{G}$. Then, consider $g = g_1 \cdots g_t$. Since $g_1 \cdots g_{t-1}$ has syllable length shorter than t it can be written in the form xb. Therefore, express x and b in syllables so that we have $g = x_1 \cdots x_p b_{p+1} \cdots b_{t-1} g_t$. If g_t itself is in $\Gamma_r\mathfrak{G}$, then this word is already in permissible form.

Suppose therefore, that $g_t \notin \Gamma_r\mathfrak{G}$. If it commutes with b_{t-1}, then we can write $b_{t-1}g_t = g_t b_{t-1}$ and therefore we have $g = x_1 \cdots b_{t-1} g_t = x_1 \cdots g_t b_{t-1}$. Now, since its length is less than t, the element $x_1 \cdots g_t$ can be written as some $x'b'$ in permissible form. But, then $g = x'b'b_{t-1}$ is a permissible presentation of g.

Finally, we consider the case in which g_t does not commute with b_{t-1}. If any rearrangement of this word allows g_t to commute past a syllable, then we apply the argument of the preceding paragraph to obtain a word in permissible form. Otherwise, $x = g$ is already permissible.

Now, we show the disjointness condition holds. Suppose that x and x' are distinct, but permissible. Then, write $x^{-1}x' = z$ for some $z \in \Gamma\mathfrak{G}$. Observe that $z \notin \Gamma_r\mathfrak{G}$, as, if it were, then xz would be a presentation of x' that ends with a non-trivial element of $\Gamma_r\mathfrak{G}$, which is not allowed. Thus, z must contain some element that is not in $\Gamma_r\mathfrak{G}$. Hence it contains a generator s from a group with weight $> r$. Thus, $d(xb, x'b') = \|b^{-1}zb'\| \geq \|s\| > r$. $\square$

Theorem III.27. *Let Γ be a countable graph with clique number $\omega(\Gamma) \leq k$. Suppose that $\{G_v\}_{v\in V(\Gamma)}$ is a collection of finitely generated groups with $0 < \operatorname{asdim} G_v \leq n$ for all $v \in V(\Gamma)$. Then, in a left-invariant proper metric,* $\operatorname{asdim} \Gamma\mathfrak{G} \leq nk$.

Proof. For a given $r > 0$ we will construct a cover by $nk + 1$ uniformly bounded, r-disjoint families of subsets of $\Gamma\mathfrak{G}$. Since $\Gamma\mathfrak{G}$ is a countable group that is not finitely generated, we endow it with a metric arising from a weight function $\bar{w} : V(\Gamma) \to \mathbb{N}$ as described above.

Define a subgraph Γ_r of Γ by setting $V(\Gamma_r) = \bar{w}^{-1}([0, r])$ and by defining an edge between two vertices of Γ_r if and only if there is an edge between these vertices in Γ. By Corollary III.24, we know that $\operatorname{asdim} \Gamma_r\mathfrak{G} \leq nk$. Thus, there is a cover by $nk+1$ r-disjoint families of uniformly bounded sets, say $\mathcal{U}^0, \mathcal{U}^1, \ldots, \mathcal{U}^{nk}$. Let $P \subset \Gamma\mathfrak{G}$ denote the set of all Γ_r-permissible elements.

For each i define the collection $\{xU \mid x \in P, U \in \mathcal{U}^i\}$. We claim that for each i, the collection is r-disjoint and uniformly bounded. Moreover, we claim that the union of these collections covers $\Gamma\mathfrak{G}$.

Since the metric on $\Gamma\mathfrak{G}$ is left-invariant, we know that $d(xu, xu') = d(u, u')$, for all xu and xu' in xU. Since $\operatorname{diam}(U)$ is uniformly bounded, we have that $\operatorname{diam}(xU)$ is also uniformly bounded.

Next, suppose that xU and $x'U'$ are distinct sets, where $U, U' \in \mathcal{U}^i$. If $x = x'$, then we have $d(xU, x'U') = d(xU, xU') = d(U, U')$, and since these sets must be different (yet still in the same family $\mathcal{U}^i$), they are at least r-disjoint. If $x \neq x'$, then by the previous lemma $d(xu, x'u') > r$ and so these two sets are r-disjoint.

Finally, we show that the collection of all such families covers $\Gamma\mathfrak{G}$. To this end, let $g \in \Gamma\mathfrak{G}$ be given. Then, by the lemma $g = xb$, where $x \in P$ and $b \in \Gamma_r\mathfrak{G}$. Thus, there is some i and some $U \in \mathcal{U}^i$ so that $b \in U$. Thus, $g \in xU$, as required. □

We note by the following examples that both bounds k and n from the above Theorem are required.

Example. Let Γ be the Cayley graph of $\langle a, a^{-1} | \rangle$. Let $G_v = \mathbb{Z}^{|v|}$. Then there is no n such that $0 < \operatorname{asdim} G_v \leq n$ for all $v \in V(\Gamma)$. Also, we have that $\operatorname{asdim} \Gamma\mathfrak{G}$ is not finite, as there is a quasi-isometrically embedded copy of $\mathbb{Z}^n$ in $\Gamma\mathfrak{G}$ for all n.

Example. Let Γ_0 be the Cayley graph of $\langle a, a^{-1} | \rangle$. Let Γ be the graph that replaces each vertex v with a complete graph on $|v|$ vertices. Then there is no k such that $\omega(\Gamma) \leq k$. Also, we have that $\operatorname{asdim} \Gamma\mathfrak{G}$ is not finite, as there is a quasi-isometrically embedded copy of $\mathbb{Z}^n$ in $\Gamma\mathfrak{G}$ for all n.

3.4 Open Questions

Another related invariant of groups, discussed by Gromov in [Gro93], is the asymptotic behavior of the dimension function, which is defined as follows.

Definition III.28. Let Γ be the Cayley graph of a group G. Let $\delta > 1$. Let $k = k(\delta)$ be the minimal number of colors so that we can color vertices of Γ in k colors and there are no arbitrary long monochromatic δ-paths without repeated vertices. Then $k(\delta) - 1$ is called the dimension growth function of Γ. We note that $\operatorname{asdim} G = \max_k \{k(\delta) - 1\}$.

Question III.29. How does the dimension function of $\Gamma\mathfrak{G}$ grow?

Now that it has been shown that finite graph products preserve property A, it seems plausible that the result could be extended to certain infinite graph products, as we did for finite asymptotic dimension.

Question III.30. Let Γ be a countably infinite graph with $\omega(\Gamma) < \infty$ and suppose that all $G_v \in \mathfrak{G}$ have property A. Then in a proper, left-invariant metric, does $\Gamma\mathfrak{G}$ have property A?

As we mentioned in the first chapter, one of the key reasons finite asymptotic dimension and property A are so important is that they imply that the group is coarsely embeddable into Hilbert space, which itself implies that the group satisfies the Novikov higher signature conjecture. In [AD13], they show that finite graph products of groups that are coarsely embeddable into Hilbert space are themselves coarsely embeddable into Hilbert space, which leads to the following question.

Question III.31. Let Γ be a countably infinite graph and suppose that all $G_v \in \mathfrak{G}$ are uniformly coarsely embeddable in an ℓ^p space. Then in a proper, left-invariant metric, is $\Gamma\mathfrak{G}$ coarsely embeddable in an ℓ^p space?

CHAPTER IV

PROPERTY C

4.1 Permanence Properties of Asymptotic Property C

Dranishnikov defined the notion of *asymptotic property C* for metric spaces in his work on asymptotic topology in an effort to extend the class of properties of metric spaces that imply coarse embeddability into Hilbert space.

Definition IV.1 ([Dra00]). A metric space X has *asymptotic property C* if for any number sequence $R_1 \leq R_2 \leq R_3 \leq \cdots$ there is a finite sequence of uniformly bounded families of open sets $\{\mathcal{U}_i\}_{i=1}^k$ such that the union $\bigcup_{i=1}^k \mathcal{U}_i$ is a covering of X and every family $\mathcal{U}_i$ is R_i-disjoint.

It is clear that a metric space with finite asymptotic dimension will have asymptotic property C. Dranishnikov showed that a discrete metric space with bounded geometry and asymptotic property C also has property A [Dra00, Theorem 7.11].

Asymptotic property C is another large-scale invariant and is also preserved by a number of other constructions. For this section, we will show it is preserved by certain infinite unions and free products. Notably, it is not preserved by direct products.

We begin by proving that it is a large-scale invariant. The proof is similar to the corresponding result for asymptotic dimension.

Theorem IV.2 ([Dra00]). *If $f : X \to Y$ is a coarse equivalence and if X has asymptotic property C, then Y has asymptotic property C.*

Proof. As X has asymptotic property C, for any given number sequence $R_1 \leq R_2 \leq R_3 \leq \cdots$ we can find a a finite sequence of uniformly bounded families of sets $\{\mathcal{U}_i\}_{i=1}^k$ such that $\bigcup_{i=1}^k \mathcal{U}_i$ covers X and every family $\mathcal{U}_i$ is R_i-disjoint. Let the uniform bound be D. Now, as $N_R(f(x)) = Y$ we have that $N_R(f(\mathcal{U}^i))$ collectively cover Y.

So, since $f(\mathcal{U}^i)$ is $\rho_1(R_i)$-disjoint and $\rho_2(D)$-bounded, we have that $N_R(f(\mathcal{U}^i))$ is $(\rho_1(R_i) - 2R)$-disjoint and $2R + \rho_2(D)$-bounded. As $\rho_i \to \infty$ and R is fixed, we can choose a number sequence R_i in order to satisfy the requirements that Y have asymptotic property C. □

Now, we consider the case where X can be expressed as a union of a collection of spaces with uniform property C as defined below with the additional property that for each $r > 0$ there is a "core" space with asymptotic property C whose removal leaves the families r-disjoint. We will be following the scheme used in [BD01]; the same scheme we used to prove a similar result for asymptotic dimension.

We will say that the family X_α satisfies asymptotic property C *uniformly in* α if for every sequence $R_1 < R_2 < \cdots$ there exist $B_1 < B_2 < \cdots$ so that for each α there exist families $\mathcal{U}_\alpha^i$ of R_i-disjoint, B_i-bounded families ($i = 1, \ldots, n$) so that $\cup_{i=1}^n \mathcal{U}_\alpha^i$ covers X_α.

Theorem IV.3. *Suppose that* $X = \bigcup_\alpha X_\alpha$ *is a countable union of spaces that have uniform asymptotic property C. Suppose further that for each* $r > 0$ *there is a* $Y_r \subset X$ *so that* Y_r *has asymptotic property C and such that the family* $\{X_\alpha - Y_r\}_\alpha$ *is* r*-disjoint. Then,* X *has asymptotic property C.*

Proof. Let $d_1 < d_2 < \cdots$ be a sequence of positive numbers. For each α, choose families $\mathcal{U}_i^\alpha$ of d_i-disjoint, R_i-bounded sets, $i = 1, 2, \ldots, n$. Since R_i are upper bounds on diameters, we may take them to be increasing and insist that $R_i \geq d_i$. Put $r = 5R_n$. Take Y_r as in the statement of the theorem.

Let $\mathcal{V}^1, \mathcal{V}^2, \ldots, \mathcal{V}^k$ be $5R_i$-disjoint, B_i-bounded families of sets whose union covers Y_r.

Let $\overline{\mathcal{U}^i_\alpha}$ denote the restriction of $\mathcal{U}^i_\alpha$ to $X_\alpha - Y_r$. Next, put $\overline{\mathcal{U}^i} = \cup_\alpha \mathcal{U}^i_\alpha$. Note that $\overline{\mathcal{U}^i}$ is R_i-bounded and d_i disjoint. Finally, set $\mathcal{W}^i = \mathcal{V}^i \cup_{d_i} \overline{\mathcal{U}^i}$, for $i = 1, 2, \ldots, \max\{k, n\}$. Here, we take $\mathcal{V}^i = \emptyset$ or $\mathcal{U}^i = \emptyset$ if $i > k$ or $i > n$, respectively. Thus, in these cases, we have $\mathcal{W}^i = \overline{\mathcal{U}^i}$ or $\mathcal{W}^i = \mathcal{V}^i$, respectively. By Theorem III.13, $\mathcal{W}^i$ is d_i-disjoint and uniformly bounded. It is clear that this collection covers X. □

Corollary IV.4. *Let (Z, d) be a metric space with $Z = X \cup Y$. If (X, d) and (Y, d) have asymptotic property C, then so does Z.*

Proof. To see this, we consider the family of spaces $\{A, B\}$ and let $Y_r = B$ and apply the previous theorem. □

We note that we have no permanence result for the direct product of two spaces with asymptotic property C. This is still unknown, and common thought has it that it is likely not true, as topological property C is not preserved by direct product. In fact, Pol and Pol, in [PP09], have an example of a space X with topological property C where $X \times X$ does not have topological property C.

We do have the following weaker result from [She11].

Theorem IV.5 ([She11])**.** *Let X be a metric space such that X has asymptotic property C and let Y be a metric space such that* $\operatorname{asdim} Y = n < \infty$*. Then $X \times Y$ has asymptotic property C.*

Proof. Let $R_1 \leq R_2 \leq \ldots$ be a number sequence. We consider the subsequence $R_{n+1} \leq R_{2(n+1)} \leq \ldots$. As X has asymptotic property C, there must exist a finite sequence of uniformly bounded families of open sets $\{\mathcal{U}_i\}_{i=1}^k$ such that the union $\bigcup_{i=1}^k \mathcal{U}_i$ is a covering of X and every family $\mathcal{U}_i$ is $R_{i(n+1)}$-disjoint.

Let $R = R_{k(n+1)}$. As $\operatorname{asdim} Y = n$, we can find a collection $\{\mathcal{V}^0, \ldots \mathcal{V}^n\}$ of R-disjoint, D-bounded sets such that the collection covers Y.

Now, we define $\mathcal{W}_{j(n+1)+i-1} = \mathcal{U}_{j+1} \times \mathcal{V}^i$ for $j = 0, 1, \ldots, k-1$ and $i = 0, 1, \ldots, n$. We note that $\bigcup_{i=1}^{k(n+1)} \mathcal{W}_i$ covers $X \times Y$ as $\bigcup_{i=1}^k \mathcal{U}_i$ covers X and $\bigcup_{i=1}^{n+1} \mathcal{V}^i$ covers Y. We also note that each $\mathcal{W}_i$ is uniformly bounded, as each $\mathcal{U}_i$ and $\mathcal{V}^i$ is uniformly bounded.

To show that each $\mathcal{W}_i$ is R_i-disjoint, we let $U_1 \times V_1, U_2 \times V_2$ be distinct elements of $\mathcal{W}_i$. We have that, as $\mathcal{U}_i$ is $R_{i(n+1)}$-disjoint and $\mathcal{V}^j$ is R-disjoint, $\mathcal{U}_i \times \mathcal{V}^j$ is $\min\{R_{i(n+1)}, R\}$-disjoint. By our choice of R, $\min\{R_{i(n+1)}, R\} \geq R_{i(n+1)} \geq R_i$. So, we have that $d(U_1 \times V_1, U_2 \times V_2) \geq R_i$ and therefore $\mathcal{W}_i$ is R_i-disjoint. And so $X \times Y$ has asymptotic property C. $\square$

4.2 Coarse Property C

If we translate the notions from the category of metric spaces to coarse spaces in the sense of Roe [Roe03], we obtain the following definition, which we call *coarse property C*. See also [Gra06].

Definition IV.6. A coarse space $(X, \mathcal{E})$ has *coarse property C* if for any sequence $L_1 \subset L_2 \subset L_3 \subset \cdots$ of entourages there is a finite sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ so that

(1) $\mathcal{U} = \bigcup_{i=1}^n \mathcal{U}_i$ covers X;

(2) each $\mathcal{U}_i$ is uniformly bounded; and

(3) each $\mathcal{U}_i$ is L_i-disjoint.

We can also define a coarse analog to asymptotic dimension.

Definition IV.7 ([Roe03, Gra05])**.** A coarse space $(X, \mathcal{E})$ satisfies the inequality $\operatorname{asdim} X \leq n$ if for any entourage L there exists a finite sequence $\mathcal{U}^1, \mathcal{U}^2, \ldots, \mathcal{U}^n$ so that

(1) $\mathcal{U} = \bigcup_{i=1}^{n} \mathcal{U}^i$ covers X;

(2) each $\mathcal{U}^i$ is uniformly bounded; and

(3) each $\mathcal{U}^i$ is L-disjoint.

In this setting, we similarly have that coarse property C is a coarse invariant, along with permanence results along the same lines as in the metric setting. Grave and Roe also show this for coarse asymptotic dimension [Roe03, Gra06].

Proposition IV.8. *Coarse property C is a coarse invariant.*

Proof. Let $f : X \to Y$ be a coarsely uniform embedding and suppose that $(Y, \mathcal{F})$ has coarse property C. Let $L_1 \subset L_2 \subset L_3 \subset \cdots$ be a sequence of entourages in $\mathcal{E}$. Then we have that $(f \times f)(L_i) = K_i$ is a sequence of entourages in $\mathcal{F}$ such that $K_1 \subset K_2 \subset K_3 \subset \cdots$.

Therefore, since $(Y, \mathcal{F})$ has coarse property C, there is a finite sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ as above. Let $\mathcal{V}_i = \{f^{-1}(A) | A \in \mathcal{U}_i\}$. Since $\mathcal{U} = \bigcup_{i=1}^{n} \mathcal{U}_i$ covers Y, we have that $\mathcal{V} = \bigcup_{i=1}^{n} \mathcal{V}_i$ covers X.

Now, denoting $\bigcup_{V \in \mathcal{V}} V \times V$ by $\Delta_{\mathcal{V}}$, we have that $\Delta_{\mathcal{V}} = \bigcup_{U \in \mathcal{U}} f^{-1}(U) \times f^{-1}(U) = (f \times f)^{-1}(\bigcup_{U \in \mathcal{U}} U \times U) = (f \times f)^{-1}(\Delta_{\mathcal{U}})$. Since $(Y, \mathcal{F})$ has coarse property C, $\mathcal{U}$ is uniformly bounded and so $\Delta_{\mathcal{U}}$ is an entourage. Since f is coarsely proper, we have that $(f \times f)^{-1}(\Delta_{\mathcal{U}}) = \Delta_{\mathcal{V}}$ is an entourage and therefore $\mathcal{V}$ is uniformly bounded.

It remains to show that V_i is L_i disjoint. Let $A, B \in V_i$, with $A \neq B$. Then $A = f^{-1}(A')$ for some $A' \in U_i$ and $B = f^{-1}(B')$ for some $B' \in U_i$, with $A' \neq B'$. So $A \times B \cap L_i \subset (f \times f)^{-1}(A' \times B' \cap K_i) = (f \times f)^{-1}(\emptyset) = \emptyset$ since U_i is K_i disjoint.

Therefore the sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ satisfies our requirements, and $(X, \mathcal{E})$ has coarse property C □

Coarse property C also passes nicely to subsets.

Proposition IV.9. *If $Y \subset X$ where X has coarse property C and Y has the coarse structure inherited from X, then Y has coarse property C.*

Proof. Let $L_1 \subset L_2 \subset L_3 \subset \cdots$ be a sequence of entourages in Y. Then $L_1 \subset L_2 \subset L_3 \subset \cdots$ is a sequence of entourages in X, and since X has property C, we have a finite sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ so that $\mathcal{U} = \bigcup_{i=1}^{n} \mathcal{U}_i$ covers X, each $\mathcal{U}_i$ is uniformly bounded and each $\mathcal{U}_i$ is L_i-disjoint.

We consider the finite sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ where $\mathcal{V}_i = \{U \cap Y | U \in \mathcal{U}_i\}$. Then each $\mathcal{V}_i$ is still L_i-disjoint and since $\Delta_{\mathcal{V}} = \Delta_{\mathcal{U}} \cap (Y \times Y)$ we have that $\mathcal{V}$ is uniformly bounded. Since $\mathcal{U}$ covers X, we also have that $\mathcal{V}$ covers Y. Therefore, Y has property C. □

Our coarse definition reduces to the asymptotic case when the metric space is given the bounded coarse structure, that is, the structure $\mathcal{E}$ where $E \in \mathcal{E}$ if and only if $\sup\{d(x, x') \mid (x, x') \in E\}$ is finite.

Proposition IV.10. *Let (X, d) be a metric space. Let $\mathcal{E}$ denote the bounded coarse structure. Then (X, d) has asymptotic property C if and only if $(X, \mathcal{E})$ has coarse property C.*

Proof. Suppose first that (X, d) has asymptotic property C. Let $L_1 \subset L_2 \subset \cdots$ be a sequence of controlled sets. For each i, put $R_i = \sup\{d(x, x') \mid (x, x') \in L_i\}$. Then each R_i is finite, by the definition of the bounded coarse structure and moreover $R_1 \leq R_2 \leq \cdots$.

Since (X, d) has asymptotic property C, there are families $\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_k$ that cover X, that consist of uniformly bounded sets, and that are R_i-disjoint $(i = 1, 2, \ldots, k)$.

We need to show that the $\mathcal{U}_i$ are coarsely uniformly bounded and L_i-disjoint.

The collection $\mathcal{U}_i$ is coarsely uniformly bounded if and only if $\Delta_{\mathcal{U}_i} = \bigcup_\alpha U^i_\alpha \times U^i_\alpha$ is in $\mathcal{E}$. But, $\Delta_{\mathcal{U}_i} \in \mathcal{E}$ if and only if

$$\sup\{d(x, y) \mid (x, y) \in \Delta_{\mathcal{U}_i}\} < \infty,$$

which is implied by our assumption that the family has uniformly bounded diameter, i.e.

$$\sup_\alpha \{\mathrm{diam}(U^i_\alpha)\} < \infty.$$

Next, to show that the $\mathcal{U}_i$ are L_i-disjoint, we must show that $\left(U^i_\alpha \times U^i_\beta\right) \cap L_i = \emptyset$ whenever $U^i_\alpha \neq U^i_\beta$. Suppose that $a \in U^i_\alpha$ and $b \in U^i_\beta$ and $(a, b) \in L_i$. Then, we have $d(a, b) \leq R_i$, which contradicts the fact that the family $\mathcal{U}_i$ is R_i-disjoint.

Suppose now that $(X, \mathcal{E})$ has coarse property C and let $R_1 \leq R_2 \leq \cdots$ be given. Define a sequence of controlled sets L_i as follows:

$$L_i = \{(x, y) \in X \times X \mid d(x, y) \leq R_i\}.$$

Using this sequence, we find a cover of X by uniformly bounded $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_k$, where each $\mathcal{U}_i$ is L_i-disjoint.

As above, we see that the set $\mathcal{U}_i$ is coarsely L_i-disjoint if and only if it is metrically R_i-disjoint and coarsely uniformly bounded if and only if it is metrically uniformly bounded.

□

We can also prove some relationships between coarse property C and coarse asymptotic dimension, as we had in the metric case.

Theorem IV.11. *Let $(X, \mathcal{E})$ be a coarse space such that* $\operatorname{asdim} X \leq n$. *Then X has coarse property C.*

Proof. Given a sequence of entourages $L_0 \subset L_1 \subset L_2 \subset L_3 \subset \cdots$, we let $L = L_n$. Then we can find a finite sequence $\mathcal{U}^0, \mathcal{U}^1, \mathcal{U}^2, \ldots, \mathcal{U}^n$ that satisfies the definition of $\operatorname{asdim} X \leq n$. This sequence satisfies the requirements of the definition of coarse property C as well, since $L_i \subset L_n = L$ for all $i \leq n$. □

In the metric case, we showed that asymptotic property C implies property A. A coarse definition of the property A is not so easily constructed. However, we can consider a series of maps similar to those in the definition of property A such that coarse property C implies thhe existence of such a series of maps. We require a few preliminaries first.

Proposition IV.12. *Let $(X, \mathcal{E})$ be a coarse space and let $E \in \mathcal{E}$ such that $E = E^{-1}$. Define $D : X \times X \to \mathbb{R}^+ \cup \{+\infty\}$ by $D(x, y) = \min\{k \geq 0 \mid (x, y) \in E^{k+1}\}$. Then we have that D is symmetric, $D(\Delta) = 0$ and $D(x, y) \leq D(x, z) + D(z, y) + 1$.*

Proof. As $E = E^{-1}$, we have that $E^k = (E^k)^{-1}$ and so if $(x, y) \in E^{k+1}$ then $(y, x) \in E^{k+1}$ and thus $D(x, y) = D(y, x)$. As $E \in \mathcal{E}$, we have that $\Delta \subset E$ and therefore $D(\Delta) = 0$.

For $D(x, y) \leq D(x, z) + D(z, y) + 1$, we note that if $D(x, z) = k$, then $(x, z) \in E^{k+1}$ and if $D(z, y) = l$, then $(z, y) \in E^{l+1}$. Therefore, $(x, y) = (x, z) \circ (z, y) \in E^{k+1} \circ E^{l+1} = E^{k+l+2}$ and thus $D(x, y) \leq (k + l + 2) - 1 = D(x, z) + D(z, y) + 1$. $\square$

We will use this function to define a function based on a nice cover of X.

Proposition IV.13. *Let $(X, \mathcal{E})$ be a coarse space and let $E \in \mathcal{E}$ such that $\Delta = E^0 \subsetneq E^1 \subsetneq E^n \subsetneq \cdots$. Fix $n \in \mathbb{N}, n > 1$ and suppose X has a cover $\mathcal{U}_1, \ldots, \mathcal{U}_k$ by E^{n^i}-disconnected sets. Put*

$$\phi^i_j(x) = \max\{0, \frac{n^i}{4} - D(x, U^i_j)\}.$$

Then

(1) *for each i and x, there exists at most one $j = j_x(i)$ such that $\phi^x_j(i)(x) \neq 0$*

(2) *$|\phi^i_j(z) - \phi^i_j(w)| \leq D(z, w) + 1$ for all i, j, z, w.*

Proof. For (1), we suppose $j \neq j'$ and $\phi^i_j(x) \neq 0 \neq \phi^i_{j'}(x)$. Then, $\frac{n^i}{4} > D(x, U^i_j)$ and $\frac{n^i}{4} > D(x, U^i_{j'})$. If z^i_j and $z^i_{j'}$ in U^i_j and $U^i_{j'}$, respectively, realize $D(x, U^i_j) + 1$ and $D(x, U^i_{j'}) + 1$, then we see that $(x, z^i_j) \in E^{D(x, U^i_j)+1}$ and $(x, z^i_{j'}) \in E^{D(x, U^i_{j'})+1}$ together imply that $(z^i_j, z^i_{j'}) \in E^{2\left(\max\{D(x, U^i_j), D(x, U^i_{j'})\}+1\right)}$ and yet $\max\{D(x, U^i_j), D(x, U^i_{j'})\} <$

$\frac{n^i}{2}$, so that $\max\{D(x, U^i_j), D(x, U^i_{j'})\} + 1 < n^i$. Thus, there are u and v in $\mathcal{U}^i_j$ and $\mathcal{U}^i_{j'}$, respectively, so that $(u, v) = (u, x) \circ (x, v) \in E^{D(x,U^i_j)} \circ E^{D(x,U^i_{j'})} \subseteq E^{n^i}$. This is a contradiction since $\mathcal{U}^i$ is n^i-disconnected; i.e., $U^i_j \times U^i_{j'} \cap E^{n^i} = \emptyset$.

For (2), we consider by cases. First, we consider the case $j = j_z(i) = j_w(i)$. There are three possibilities. If z and w are both in U^i_j, then $|\phi^i_j(z) - \phi^i_j(w)| = 0 \leq D(z, w) + 1$. If $z \in U^i_j$ and $w \notin U^i_j$, then $|\phi^i_j(z) - \phi^i_j(w)| = |D(w, U^i_j)| \leq D(w, z) < D(w, z) + 1$, since $z \in U^i_j$. Finally, when both z and w are not in U^i_j, then $|\phi^i_j(z) - \phi^i_j(w)| = |D(z, U) - D(w, U)|$, which, by an elementary argument applying the "triangle inequality" from Proposition IV.12, does not exceed $D(z, w) + 1$, as required.

If $j_z(i) \neq j_w(i)$, then the fact that $U^i_{j_z(i)} \times U^i_{j_w(i)} \cap E^{n^i} = \emptyset$ implies that $D(z, w) \geq n^i$ and so $\phi^i_{j_z(i)}(z) < n^i \leq D(z, w)$. Similarly, $\phi^i_{j_w(i)}(w) \leq D(z, w)$.

□

Next, we will construct a function into $\ell^1(X)$ that will provide the underlying basis of our analog to property A.

Proposition IV.14. *Let $(X, \mathcal{E})$ be a coarse space and let $E \in \mathcal{E}$ such that $\Delta = E^0 \subsetneq E^1 \subsetneq E^n \subsetneq \cdots$. Fix $n \in \mathbb{N}, n > 1$ and suppose X has a cover $\mathcal{U}_1, \ldots, \mathcal{U}_k$ by E^{n^i}-disconnected sets. For each pair (i, j) take $x^i_j \in U^i_j$. Define $b^n : X \to \ell^1(X)$ by*

$$b^n_x(y) = \Sigma^k_{i=1} n^{k-i+1} \phi^i_{j_x(i)}(x) \delta_{x^i_{j_x(i)}}(y).$$

Then

(1) $0 < \|b^n_x\|_1 < \infty$

(2) *for each n, $\{(x, y) \mid y \in \mathrm{supp}(b^n_x)\}$ is controlled;*

Proof. For (1), we consider $\|b_x^n\|_1 = \sum_{y\in X}\sum_{i=1}^k n^{k-i+1}\phi^i_{j_x(i)}(x)\delta_{x^i_{j_x(i)}}(y)$. Each term $\sum_{i=1}^k n^{k-i+1}\phi^i_{j_x(i)}(x)\delta_{x^i_{j_x(i)}}(y) = 0$, unless $y \in \{x^1_{j_x(1)}, x^2_{j_x(2)}, \dots, x^k_{j_x(k)}\}$, so $\|b_x^n\|_1 \leq k\sum_{i=1}^k n^{k-i+1}\phi^i_{j_x(i)}(x) \leq k^2 n^k \dfrac{n^k}{4} < \infty$.

Similarly, we claim that $\|b_x^n\|_1 \geq \frac{n^{k+1}}{4}$ for any $x \in X$. Indeed,

$$\begin{aligned}\|b_x^n\|_1 &= \left\|\sum_{i,j} n^{k-i+1}\phi^i_j(x)\delta_{x^i_j}\right\|_1 \\ &= \sum_{i=1}^k n^{k-i+1}|\phi^i_{j_x(i)}(x)|.\end{aligned}$$

But, there is some i_0 for which $x \in U^{i_0}_{j_x(i_0)}$ and so

$$\sum_{i=1}^k n^{k-i+1}|\phi^i_{j_x(i)}(x)| \geq n^{k-i_0+1}\cdot\frac{n^{i_0}}{4} = \frac{n^{k+1}}{4}.$$

For (2), we again have that if $\sum_{i=1}^k n^{k-i+1}\phi^i_{j_x(i)}(x)\delta_{x^i_{j_x(i)}}(y) \neq 0$, then y belongs to $\{x^1_{j_x(1)}, x^2_{j_x(2)}, \dots, x^k_{j_x(k)}\}$. Fix some t and consider $x^t_{j_x(t)}$; then $n^{k-t+1}\phi^t_{j_x(t)}\delta_{x^t_{j_x(t)}}(x^t_{j_x(t)}) \neq 0$ is equivalent to $\phi^t_{j_x(t)} > 0$. Thus, there is some $m < \frac{n^i}{4}$ and some $z \in U^t_{j_x(t)}$ so that $(x,z) \in E^m$. Now $(x, x^t_{j_x(t)}) = (x,z)\circ(z, x^t_{j_x(t)}) \in E^m \circ (U^t_j \times U^t_j)$, which is controlled. Thus, the set of all (x,y) for which $a^n_x(y) \neq 0$ is contained in a finite union of controlled sets (for a fixed x) and so (2) holds. □

Proposition IV.15. *Let $(X,\mathcal{E})$ be a coarse space and let $E \in \mathcal{E}$ such that $\Delta = E^0 \subsetneq E^1 \subsetneq E^n \subsetneq \cdots$. Fix $n \in \mathbb{N}, n > 1$ and suppose X has a cover $\mathcal{U}_1, \dots, \mathcal{U}_k$ by*

E^{n^i}-disconnected sets. For each pair (i,j) take $x^i_j \in U^i_j$. Define $b^n : X \to \ell^1(X)$ as above. Then

$$\|b^n_z - b^n_w\|_1 \leq \frac{3n(n^k-1)}{n-1}\left(D(z,w)+1\right).$$

Proof. We begin by estimating $\|b^n_z - b^n_w\|_1$ using ϕ^i_j as follows:

$$\begin{aligned}
\|b^n_w - b^n_z\|_1 &= \left\| \sum_{i=1}^{k} n^{k-i+1} \phi^i_{j_w(i)} \delta_{x^i_{j_x(w)}} - \sum_{i=1}^{k} n^{k-i+1} \phi^i_{j_z(i)} \delta_{x^i_{j_x(z)}} \right\|_1 \\
&\leq \left\| \sum_{\substack{i=1 \\ j=j_z(i)=j_w(i)}}^{k} n^{k-i+1} \left(\phi^i_j(z) - \phi^i_j(w)\right) \delta_{x^i_j} \right\|_1 \\
&+ \left\| \sum_{\substack{i=1 \\ j=j_z(i)\neq j_w(i)}}^{k} n^{k-i+1} \phi^i_j(z) \delta_{x^i_j} \right\|_1 + \left\| \sum_{\substack{i=1 \\ j=j_w(i)\neq j_z(i)}}^{k} n^{k-i+1} \phi^i_j(w) \delta_{x^i_j} \right\|_1 \\
&\leq \sum_{\substack{i=1 \\ j=j_z(i)=j_w(i)}}^{k} n^{k-i+1} \left|\phi^i_j(z) - \phi^i_j(w)\right| \\
&+ \sum_{i=1}^{k} n^{k-i+1} |\phi^i_j(w)| + \sum_{i=1}^{k} n^{k-i+1} |\phi^i_j(z)|
\end{aligned}$$

Now, by Proposition IV.13 we have that $|\phi^i_j(z) - \phi^i_j(w)| \leq D(z,w)+1$. Thus, we conclude that

$$\begin{aligned}
\|b^n_z - b^n_w\|_1 &\leq \sum_{i=1}^{k} n^{k-i+1} \left(D(z,w)+1\right) + 2\sum_{i=1}^{k} n^{k-i+1} \left(D(z,w)\right) \\
&\leq 3n\frac{n^k-1}{n-1}\left(D(z,w)+1\right).
\end{aligned}$$

□

Finally, we will show that coarse property C implies the existence of a sequence of functions reminiscent of asymptotic property A.

Theorem IV.16. *Let $(X, \mathcal{E})$ be a coarse space with coarse property C. Let $E \in \mathcal{E}$ such that $\Delta = E^0 \subsetneq E^1 \subsetneq E^n \subsetneq \cdots$. Then, there is a sequence a^n of maps $a^n : X \to \ell^1(X)$ such that*

(1) $\|a^n_x\|_1 = 1$ *for each* $x \in X$ *and* $n \in \mathbb{N}$*;*

(2) *for each* n*,* $\{(x, y) \mid y \in \operatorname{supp}(a^n_x)\}$ *is controlled;*

(3) *for each* $K > 0$*,*

$$\lim_{n\to\infty} \sup_{(x,y)\in E^K} \{\|a^n_x - a^n_y\|\} = 0.$$

Here, we write a^n_x *for the function* $a^n(x) \in \ell^1(X)$*.*

Proof. We may assume that $E = E^{-1}$. If not, replace E with $E \cup E^{-1}$. Fix n and form the (increasing) sequence $E, E^n, E^{n^2}, \ldots$ and observe that each element of the sequence is controlled. Using this sequence, we can find a finite family $\mathcal{U}^1, \mathcal{U}^2, \ldots, \mathcal{U}^k$ covering X so that each $\mathcal{U}^i$ is E^{n^i}-disconnected, as X has coarse property C.

Define b^n_x as above. Then put $a^n_x = \dfrac{b^n_x}{\|b^n_x\|}$. This is well defined, by Proposition IV.14, as $0 < \|b^n_x\| < \infty$ and clearly, $\|a^n_x\|_1 = 1$ for each $x \in X$ and $n \in \mathbb{N}$.

For (2), if $y \in \operatorname{supp}(a^n_x)$, then $y \in \operatorname{supp}(b^n_x)$ and so by Proposition IV.14, $\{(x, y) \mid y \in \operatorname{supp}(a^n_x)\}$ is controlled.

For (3), we have, by Proposition IV.14 and Proposition IV.15 that

$$\begin{aligned}
\|a_z^n - a_w^n\|_1 &= \frac{1}{\|b_z^n\|_1}\|b_z^n - \|b_z^n\|_1 a_w^n\|_1 \\
&\leq \frac{1}{\|b_z^n\|_1}\left\| \frac{\|b_z^n\|_1}{\|b_w^n\|_1} b_w^n - b_w^n \right\|_1 + \|b_w^n - b_z^n\|_1 \\
&= \frac{1}{\|b_z^n\|_1}\|b_w^n\|_1 \frac{|\|b_z^n\|_1 - \|b_w^n\|_1|}{\|b_w^n\|_1} + \|b_w^n - b_z^n\|_1 \\
&\leq \frac{1}{\|b_z^n\|_1} 2\|b_w^n - b_z^n\|_1 \\
&\leq \frac{6n(n^k-1)}{\frac{n^{k+1}}{2}(n-1)}(D(z,w)+1) \leq \frac{12(D(z,w)+1)}{n-1}
\end{aligned}$$

which goes to zero as $n \to \infty$ for all z, w with $D(z,w) \leq K$.

□

As before, we wish to show that coarse property C is preserved by unions. For finite unions, we require a coarse analog of saturated unions, such as we had in the metric case.

Definition IV.17 ([Gra05]). Let $\mathcal{U}$ and $\mathcal{V}$ be families of subsets of X. Let $V \in \mathcal{V}$ and L be an entourage. We define

$$N_L(V,\mathcal{U}) := V \cup \bigsqcup_{\substack{U \in \mathcal{U}, \\ L \cap U \times V \neq \emptyset}} U.$$

The *L-saturated union of $\mathcal{V}$ in $\mathcal{U}$* is denoted $\mathcal{V} \cup_L \mathcal{U}$ and is given by $\mathcal{V} \cup_L \mathcal{U} := \{N_L(V,\mathcal{U}) | V \in \mathcal{V}\} \cup \{U \in \mathcal{U} | L_i \cap U \times V = \emptyset \, \forall \, V \in \mathcal{V}\}$.

The following two results closely follow Grave in [Gra05], leading to an analog of Theorem 3.29 in that paper. We use $L\Delta_{\mathcal{U}} L \Delta_{\mathcal{U}} L$ as in [Gra05], replacing the $5R$ we used earlier in the metric case, as that provides us with the appropriate contradiction in Proposition IV.18, providing the analogous extension of L necessary to ensure the L-saturated union remains L-disjoint.

Proposition IV.18. *If $\mathcal{U}$ is uniformly bounded and L-disjoint, for some symmetric entourage L and $\mathcal{V}$ is uniformly bounded and $L\Delta_{\mathcal{U}}L\Delta_{\mathcal{U}}L$-disjoint then $\mathcal{V} \cup_L \mathcal{U}$ is L-disjoint and uniformly bounded.*

Proof. We begin by observing that $N_L(V,\mathcal{U}) \subseteq \Delta_{\mathcal{U}}L[V]$ and so $\mathcal{V} \cup_L \mathcal{U}$ is uniformly bounded.

To show that $\mathcal{V} \cup_L \mathcal{U}$ is L-disjoint, let $A, B \in \mathcal{V} \cup_L \mathcal{U}$, with $A \neq B$. We will proceed by cases.

Case 1: $A, B \in \{U \in \mathcal{U} | L \cap (U \times V) = \emptyset \forall V \in \mathcal{V}\}$. In this case, we have that $L \cap A \times B = \emptyset$ since $\mathcal{U}$ is L-disjoint.

Case 2: $A \in \{N_L(V,\mathcal{U}) | V \in \mathcal{V}\}, B \in \{U \in \mathcal{U} | L \cap (U \times V) = \emptyset \forall V \in \mathcal{V}\}$. Then $L \cap (A \times B) = \emptyset$ since $L \cap (V \times B) = \emptyset$ and $L \cap (U \times B) = \emptyset, \forall U$ such that $L \cap (U \times V) \neq \emptyset$.

Case 3: $A, B \in \{N_L(V,\mathcal{U}) | V \in \mathcal{V}\}$. Let $A = N_L(V_A,\mathcal{U}), B = N_L(V_B,\mathcal{U})$. We note that we then have that $V_A \neq V_B$. Then $(V_A \times V_B) \cap L = \emptyset$ since $\mathcal{V}$ is $L\Delta_{\mathcal{U}}L\Delta_{\mathcal{U}}L$-disjoint and $L \subset L\Delta_{\mathcal{U}}L\Delta_{\mathcal{U}}L$. Also, we have that $(V_A \times B \setminus V_B) \cap L = \emptyset$ by the construction of $N_L(V_B,\mathcal{U})$, and similarly $(A \setminus V_A \times B) \cap L = \emptyset$. Finally, $(A \setminus V_A \times B \setminus V_B) \cap L = \emptyset$ by the L-disjointedness of $\mathcal{U}_i$ since we claim that if U was part of $N_L(V_A,\mathcal{U})$, then it could not be part of $N_L(V_B,\mathcal{U})$.

To prove our claim, it remains to show that if $(V \times U) \cap L \neq \emptyset$ then $(V' \times U) \cap L = \emptyset$ whenever $V' \neq V$. If we suppose that $(V' \times U) \cap L \neq \emptyset$, then since L is symmetric, we must have that $(U \times V') \cap L \neq \emptyset$. Let $(u, v') \in (U \times V') \cap L$ and $(v, u) \in$

$(V \times U) \cap L$. Then $(v \times u)(u \times u')(u' \times v) = (v \times v') \in (V \times V') \cap L\Delta_{\mathcal{U}}L\Delta_{\mathcal{U}}L$ which is a contradiction. □

Theorem IV.19. *Let $X = X_1 \cup X_2$ be a coarse space. If X_1 and X_2 have coarse property C then X does.*

Proof. We follow closely the techniques of [Gra05] in Theorem 3.29 where he proves that finite coarse asymptotic dimension is preserved by finite unions. Let $L_1 \subset L_2 \subset L_3 \subset \cdots$ be a sequence of symmetric entourages containing Δ_X.

Since X_1 has coarse property C, there exists a finite sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ such that $\mathcal{U} = \bigcup_{i=1}^{n} \mathcal{U}_i$ covers X_1, each $\mathcal{U}_i$ is uniformly bounded and each $\mathcal{U}_i$ is L_i-disjoint. Since X_2 has coarse property C, there exists a finite sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ such that $\mathcal{V} = \bigcup_{i=1}^{n} \mathcal{V}_i$ covers X_2, each $\mathcal{V}_i$ is uniformly bounded and each $\mathcal{V}_i$ is $L_i\Delta_{\mathcal{U}}L_i\Delta_{\mathcal{U}}L_i$-disjoint.

Set $\mathcal{W}_i = \mathcal{V}_i \cup_{L_i} \mathcal{U}_i$. We observe that since $\mathcal{U}$ and $\mathcal{V}$ cover X_1 and X_2 respectively, $\mathcal{W} = \bigcup_{i=1}^{n} \mathcal{W}_i$ covers X, since $V \subset N_{L_i}(V, \mathcal{U}_i)$ and if $U \in \mathcal{U}_i$ is such that $L \cap U \times V \neq \emptyset$ then $U \subset N_{L_i}(V, \mathcal{U}_i)$ and if $U \in \mathcal{U}_i$ is such that $L \cap U \times V = \emptyset$ then $U \in \mathcal{W}_i$.

By Proposition IV.18, we have that $\mathcal{W}_i$ will be uniformly bounded and L_i-disjoint, so we have that X has coarse property C. □

In order to show that coarse property C is preserved by some infinite unions, we also require a definition of a uniform coarse property C.

Definition IV.20. A family of coarse spaces $(X_\alpha, \mathcal{E}_\alpha)$ has *uniform coarse property C* if for any sequence $L_1 \subset L_2 \subset L_3 \subset \cdots$ of entourages there is a sequence of entourages $K_1 \subset K_2 \subset K_3 \subset \cdots$ and a $N \in \mathbb{N}$ such that for each α there exists a finite sequence $\mathcal{U}_\alpha^1, \mathcal{U}_\alpha^2, \ldots, \mathcal{U}_\alpha^n$ with $n \leq N$ so that

(1) $\mathcal{U}_\alpha = \bigcup_{i=1}^n \mathcal{U}_\alpha^i$ covers X_α;

(2) for each $\alpha, \Delta_{\mathcal{U}_\alpha^i} \subset K_i$, that is $\mathcal{U}_\alpha^i$ is K_i-bounded; and

(3) each $\mathcal{U}_\alpha^i$ is L_i-disjoint.

We note that the families $\mathcal{U}_\alpha^i$ will also be uniformly bounded, since a subset of an entourage is also an entourage. An example of this, in a bounded coarse structure derived from a metric on the space, this corresponds to a uniform bound on the diameter of the covers.

Theorem IV.21. *Suppose that $X = \bigcup_\alpha X_\alpha$, where the family X_α has uniform coarse property C and for each entourage $L \in \mathcal{E}$ there is a subset $Y_L \subseteq X$ with coarse property C such that $\{X_\alpha \setminus Y_L\}$ forms an L-disjoint collection. Then, X has coarse property C.*

Proof. Let $L_1 \subseteq L_2 \subseteq \cdots$ be a sequence of entourages. For each α, choose families $\mathcal{U}_\alpha^i$ of L_i-disjoint, K_i-bounded sets, where $\mathcal{U}_\alpha = \cup_{i=1}^n \mathcal{U}_\alpha^i$ is a cover of X_α. Let $\mathcal{U} = \cup_\alpha \mathcal{U}_\alpha$ and put $K = L_n \Delta_{\mathcal{U}} L_n \Delta_{\mathcal{U}} L_n$. Take Y_K as in the statement of the theorem.

Since Y_K has coarse property C, let $\mathcal{V}^1, \mathcal{V}^2, ..., \mathcal{V}^k$ be $L_i \Delta_{\mathcal{U}} L_i \Delta_{\mathcal{U}} L_i$-disjoint, uniformly bounded families of sets whose union covers Y_K.

Let $\overline{\mathcal{U}_\alpha^i}$ denote the restriction of $\mathcal{U}_\alpha^i$ to $X_\alpha \setminus Y_K$ and put $\overline{\mathcal{U}^i} = \cup_\alpha \overline{\mathcal{U}_\alpha^i}$. Since $\overline{\mathcal{U}_\alpha^i}$ are each L_i-disjoint and $X_\alpha \setminus Y_K$ are K-disjoint and thus L_i-disjoint $\forall i$, we have that $\overline{\mathcal{U}^i}$ is L_i-disjoint. We note that $\Delta_{\overline{\mathcal{U}^i}} \subset K_i$, since each $\overline{\mathcal{U}_\alpha^i} \subset K_i$ and therefore $\overline{\mathcal{U}^i}$ is uniformly bounded.

Now, set $\mathcal{W}^i = \mathcal{V}^i \cup_{L_i} \overline{\mathcal{U}^i}$ for $i = 1, 2 \cdots, \max\{k, n\}$. By Proposition IV.18, $\mathcal{W}^i$ is L_i-disjoint and uniformly bounded. Clearly, $\mathcal{W} = \cup \mathcal{W}^i$ covers X and so X has coarse property C. $\square$

We can similarly show that finite coarse asymptotic dimension is preserved by some infinite unions.

Definition IV.22. A family of coarse space $(X_\alpha, \mathcal{E}_\alpha)$ satisfies the inequality $\operatorname{asdim} X_\alpha \leq n$ uniformly if for any entourage L there is an entourage K such that for each α there exists a finite sequence $\mathcal{U}_\alpha^1, \mathcal{U}_\alpha^2, \ldots, \mathcal{U}_\alpha^n$ so that

(1) $\mathcal{U}_\alpha = \bigcup_{i=1}^n \mathcal{U}_\alpha^i$ covers X_α;

(2) for each $\alpha, \Delta_{\mathcal{U}_\alpha} \subset K$, that is $\mathcal{U}_\alpha^i$ is K-bounded; and

(3) each $\mathcal{U}_\alpha^i$ is L-disjoint.

Theorem IV.23. *Suppose that* $X = \bigcup_\alpha X_\alpha$*, where* $\operatorname{asdim} X_\alpha \leq n$ *uniformly and for each entourage* $L \in \mathcal{E}$ *there is a subset* $Y_L \subseteq X$ *with* $\operatorname{asdim} Y_L \leq n$ *such that* $\{X_\alpha \backslash Y_L\}$ *forms an* L*-disjoint collection. Then,* $\operatorname{asdim} X \leq n$*.*

Proof. We will follow the techniques in [BD01, Theorem 1]. Let L be an entourage. For each α, choose families $\mathcal{U}_\alpha^1, \cdots, \mathcal{U}_\alpha^n$ of L-disjoint, K-bounded sets, where $\mathcal{U}_\alpha = \cup_{i=1}^n \mathcal{U}_\alpha^i$ is a cover of X_α. Let $\mathcal{U} = \cup_\alpha \mathcal{U}_\alpha$ and put $M = L\Delta_{\mathcal{U}} L \Delta_{\mathcal{U}} L$. Take Y_M as in the statement of the theorem.

Since $\operatorname{asdim} Y_M \leq n$, let $\mathcal{V}^0, \mathcal{V}^1, ..., \mathcal{V}^n$ be M-disjoint, uniformly bounded families of sets whose union covers Y_K.

Let $\overline{\mathcal{U}_\alpha^i}$ denote the restriction of $\mathcal{U}_\alpha^i$ to $X_\alpha \setminus Y_K$ and put $\overline{\mathcal{U}^i} = \cup_\alpha \overline{\mathcal{U}_\alpha^i}$. Since $\overline{\mathcal{U}_\alpha^i}$ are each L-disjoint and $X_\alpha \setminus Y_M$ are M-disjoint and thus L-disjoint, we have that $\overline{\mathcal{U}^i}$ is L-disjoint. We note that $\Delta_{\overline{\mathcal{U}^i}} \subset K$, since each $\Delta_{\overline{\mathcal{U}_\alpha^i}} \subset K$ and therefore $\overline{\mathcal{U}^i}$ is uniformly bounded.

Now, set $\mathcal{W}^i = \mathcal{V}^i \cup_L \overline{\mathcal{U}^i}$ for $i = 0, 1 \cdots, n$. By Proposition IV.18, $\mathcal{W}^i$ is L-disjoint and uniformly bounded. Clearly, $\mathcal{W} = \cup \mathcal{W}^i$ covers X and so $\operatorname{asdim} X \leq n$. □

4.3 Open Questions

Question IV.24. Is asymptotic property C preserved by free products?

Our current plan to pursue the previous question relies on some of the tools from the proof of the permanence of finite asymptotic dimension in regards to free products. It uses the action of the group on a tree of cosets to create the cover necessary.

Question IV.25. Is asymptotic property C preserved by amalgamated products?

Question IV.26. Is asymptotic property C preserved by direct products?

If the answers to the previous two questions are both yes, then it would immediately follow that the following question also has a positive answer.

Question IV.27. Let Γ be a finite graph. If all the G_v have asymptotic property C, does $\mathfrak{G}\Gamma$ have asymptotic property C?

If the answer to that question is yes, one could additionally ask the following.

Question IV.28. Let Γ be a countably infinite graph with bounded clique number. Suppose that all G_v have asymptotic property C. Then, in a proper, left-invariant metric, does G have asymptotic property C?

CHAPTER V

DECOMPOSITION COMPLEXITY

5.1 Metric Notions of Decomposition Complexity

Guentner, Tessera and Yu [GTY13, GTY12] defined another coarse invariant of groups that is applicable when the asymptotic dimension is infinite: finite decomposition complexity. Following this, Dransihnikov and Zarichnyi defined a related notion in [DZ13]: straight finite decomposition complexity. Let $\mathcal{X}$ and $\mathcal{Y}$ be familes of metric spaces. For a positive R, we say that $\mathcal{X}$ is R-decomposable over $\mathcal{Y}$ and write $\mathcal{X} \xrightarrow{R} \mathcal{Y}$ if for any $X \in \mathcal{X}$ one can write

$$X = Y^0 \cup Y^1 \text{ where } Y^i = \bigsqcup_{R\text{-disjoint}} Y^{ij}, \text{ for } i = 0, 1,$$

where the sets $Y^{ij} \in \mathcal{Y}$ and the notation means $d(Y^{ij}, Y^{ij'}) > R$ if $j \neq j'$.

We begin by describing the metric decomposition game for X. In this game two players take turns. First, Player 1 asserts a number R_1. Player 2 responds by finding a metric family $\mathcal{Y}^1$ and a R_1-decomposition of $\{X\}$ over $\mathcal{Y}^1$. Then, Player 1 selects a number R_2 and Player 2 again finds a family $\mathcal{Y}^2$ and an R_2-decomposition of $\mathcal{Y}^1$ over $\mathcal{Y}^2$. Player 2 wins if the game ends in finitely many steps with a family that consists of uniformly bounded subsets.

Definition V.1 ([GTY13])**.** The metric space X is said to have *finite composition complexity* or FDC, if there is a winning strategy for Player 2 in the metric decomposition game for X.

Definition V.2 ([DZ13]). The metric space X has *straight finite decomposition complexity* sFDC if for every sequence $R_1 \leq R_2 \leq \cdots$ there exists an n and metric families $\mathcal{Y}^i$ $(i = 1, 2, \ldots, n)$ so that $X \xrightarrow{R_1} \mathcal{Y}^1$, $\mathcal{Y}^{i-1} \xrightarrow{R_i} \mathcal{Y}^i$ for $i = 2, 3, \ldots, n$, and such that $\mathcal{Y}^n$ is uniformly bounded.

It follows clearly from these definitions that finite decomposition complexity implies straight finite decomposition complexity.

Theorem V.3 ([GTY13, Theorem 4.1]). *Let X be a metric space. Then, if X has finite asymptotic dimension, X has finite decomposition complexity.*

Theorem V.4 ([DZ13]). *Let X be a metric space. Then, if X has asymptotic property C, X has straight finite decomposition complexity.*

Proof. Let $R_1 \leq R_2 \leq \cdots$ be a number sequence. Then, as X has asymptotic property C, there exists uniformly bounded families of open sets $\{\mathcal{U}_i\}_{i=1}^k$ such that the union $\bigcup_{i=1}^k \mathcal{U}_i$ is a covering of X and every family $\mathcal{U}_i$ is R_i-disjoint. Let $\mathcal{Y}^1 = \mathcal{U}_1 \cup \{X \setminus (\bigcup \mathcal{U}_1)\}$. Then $X = Y^0 \cup Y^1$ where $Y^0 = \bigsqcup \mathcal{U}_1$ and $Y^1 = \{X \setminus (\bigcup \mathcal{U}_1)\}$.

For any set $Y \in \mathcal{Y}^1$, if $Y \in \mathcal{U}_1$ then Y is bounded and we can decompose it. If $Y = \{X \setminus (\bigcup \mathcal{U}_1)\}$, then we decompose it by considering the intersection of $\mathcal{U}_2$ and Y.

In the k-th step, the decomposition will be uniformly bounded, as each $\mathcal{U}_i$ is. □

Since finite asymptotic dimension implies asymptotic property C, we have the following easy corollary.

Corollary V.5. *Let X be a metric space. Then, if X has finite asymptotic dimension, X has straight finite decomposition complexity.*

The goal of this section is to apply the techniques of Guentner, Tessera and Yu [GTY13, GTY12] to the notion of straight finite decomposition complexity defined by Dranishnikov and Zarichnyi [DZ13]. It is shown in [DZ13] that sFDC is a coarse invariant, is preserved by finite unions, and is preserved by some infinite unions (analogous to our theorem above about property C). We extend these results to show that sFDC is preserved by fiberings and conclude that it is preserved by amalgamated products and graph products.

We begin by recalling some of the results from [DZ13].

Theorem V.6. *[DZ13, Theorem 3.1] If $f : X \to Y$ is a coarse equivalence between the metric spaces X and Y and if Y has sFDC, then so does X.*

We include a proof for the reader's convenience and also because we will use the same technique to prove our fibering theorem.

Proof. Let $f : X \to Y$ be uniformly expansive and effectively proper. Suppose that $\rho : [0, \infty) \to [0, \infty)$ is an increasing function for which $d(f(x), f(x')) \leq \rho(d(x, x'))$ for all x and x' in X.

Let $R_1 < R_2 < \cdots$ be given and set $S_i = \rho(R_i)$ for each i. By way of notation, put $\{Y\} = \mathcal{V}^0$. Then, since Y has sFDC, there is some $m \in \mathbb{N}$ and metric familes $\mathcal{V}^1, \mathcal{V}^2, \ldots, \mathcal{V}^m$ so that $\mathcal{V}^0 \xrightarrow{S_1} \mathcal{V}^1 \xrightarrow{S_2} \mathcal{V}^2 \xrightarrow{S_3} \cdots \xrightarrow{S_m} \mathcal{V}^m$ with $\mathcal{V}^m$ bounded. According to [GTY13, Lemma 3.1.1], if $\mathcal{V}^{i-1} \xrightarrow{S_i} \mathcal{V}^i$ then $f^{-1}(\mathcal{V}^{i-1}) \xrightarrow{R_i} f^{-1}(\mathcal{V}^i)$.

More explicitly, write $Y = V_0^1 \cup V_1^1$, where

$$V_i^1 = \bigsqcup_{S_1\text{-disjoint}} V_{ij}^1,$$

and $V_{ij}^1 \in \mathcal{V}^1$. Then $X = f^{-1}(Y) = f^{-1}(V_0^1) \cup f^{-1}(V_1^1)$, with

$$f^{-1}(V_i^1) = \bigsqcup_{R_1\text{-disjoint}} f^{-1}(V_{ij}^1).$$

Then, for each $V \in \mathcal{V}^1$, write $V = V_0^2 \cup V_1^2$ where

$$V_i^2 = \bigsqcup_{S_2\text{-disjoint}} V_{ij}^2,$$

and $V_{ij}^2 \in \mathcal{V}^2$. Then, as above, obtain an R_2-decomposition of $f^{-1}(\mathcal{V}^1)$ over $f^{-1}(\mathcal{V}^2)$. We continue in this way until we eventually find an R_m-decomposition of $f^{-1}(\mathcal{V}^{m-1})$ over $f^{-1}(\mathcal{V}^m)$. Since f is effectively proper and $\mathcal{V}^m$ is bounded, we apply [GTY13, Lemma 3.1.2] to conclude that $f^{-1}(\mathcal{V}^m)$ is bounded, as required. □

Next, we obtain a version of [GTY13, Theorem 3.1.4] for straight finite decomposition complexity.

Theorem V.7. *Let X and Y be metric spaces and let $f : X \to Y$ be a uniformly expansive map. Assume that Y has sFDC and that for every bounded family $\mathcal{V}$ in Y, the inverse image $f^{-1}(\mathcal{V})$ has sFDC. Then, X has sFDC.*

Proof. Let $R_1 < R_2 < \cdots$ be given. Since Y has straight finite decomposition complexity, and since f is uniformly expansive, we take $S_i = \rho(R_i)$ as in the previous theorem to find families $\mathcal{V}^1, \mathcal{V}^2, \ldots, \mathcal{V}^m$ so that $\mathcal{V}^{i-1} \xrightarrow{S_i} \mathcal{V}^i$ and for which $\mathcal{V}^m$ is bounded. Then, as before, we pull these families back to X to obtain $f^{-1}(\mathcal{V}^{i-1}) \xrightarrow{R_i} f^{-1}(\mathcal{V}^i)$. Since we assume that $f^{-1}(\mathcal{V}^m)$ has straight finite decomposition complexity, we take the sequence $R_{m+1}, R_{m+2}, \ldots$, and find n and families $\mathcal{U}^{m+1}, \mathcal{U}^{m+2}, \ldots, \mathcal{U}^{m+n}$ so that $\mathcal{U}^{m+j-1} \xrightarrow{R_{m+j}} \mathcal{U}^{m+j}$ with $\mathcal{U}^{m+n}$ bounded. Then, with $\mathcal{U}^i = f^{-1}(\mathcal{V}^i)$ for $i = 1, 2, \ldots, m$, we have $\mathcal{U}^{i-1} \xrightarrow{R_i} \mathcal{U}^i$ for all $i = 1, 2, \ldots, m+n$, as required. □

Proposition V.8. *Let G be a countable group expressed as a union of subgroups $G = \cup G_i$ where each G_i has straight finite decomposition complexity. Then, G has straight finite decomposition complexity.*

Proof. We equip G with a proper, left-invariant metric. Let $R_1 < R_2 < \cdots$ be given. Since the metric is proper, there is some G_i that contains $B_{R_1}(e)$. Then, the decomposition of G into cosets of G_i is R_1-disjoint and each coset is isometric to G_i, which is assumed to have sFDC. □

The fibering theorem and the fact that the map $g \mapsto g.x$ for a group acting by isometries on a metric space is uniformly expansive [GTY13, Lemma 3.2.2] immediately imply:

Proposition V.9. *Let G be a countable group acting on a metric space X with straight finite decomposition complexity. If there is a $x_0 \in X$ so that for every $R > 0$ the R-coarse stabilizer of x_0 has straight finite decomposition complexity, then G has straight finite decomposition complexity.* □

Corollary V.10. *The following results easily follow from this theorem.*

(1) *sFDC is closed under group extensions.*

(2) *sFDC is closed under free products with amalgamation and HNN extensions.*

(3) *sFDC is closed under finite graph products.*

(4) *FDC is closed under finite graph products.*

Proof. (1) Suppose that $1 \to K \to G \xrightarrow{\phi} H \to 1$ is an exact sequence of countable groups with H and K both having straight finite decomposition complexity. Let G act on H by the rule $g.h = \phi(g)h$. The R-coarse stabilizer is coarsely equivalent to K, so it has sFDC. Thus, by the theorem, G has sFDC.

(2) This follows from the Bass-Serre theory of graphs of groups. More precisely, if G is an amalgamated product (or HNN extension), then there is a tree T and an action of G on that T by isometries with vertex stabilizers isomorphic to the factors of the amalgam. The coarse stabilizers of the action will therefore have sFDC and so G itself will.

(3) This follows from parts (1) and (2) using the technique of Corollary III.8 or [AD13].

(4) This is immediate from the results of [GTY13] using the technique of Corollary III.8 or [AD13].

□

5.2 Coarse Notions of Decomposition Complexity

Following the scheme of Chapter 3, we can translate these notions to coarse spaces in the sense of Roe [Roe03]. Doing so, we obtain the following definition for coarse version of finite decomposition complexity. As above, we can modify this definition to get us a coarse version of straight finite decomposition complexity. Also, we can define a weak version of finite decomposition complexity that is necessary for this setting.

Let $(X, \mathcal{E})$ be a coarse space. Let $L \in \mathcal{E}$ be a controlled set. An *L-decomposition* of X over the coarse family $\mathcal{Y}$ is a decomposition

$$X = X_0 \cup X_1 \qquad X_i = \bigsqcup_L X_{ij}$$

where each $X_{ij} \in \mathcal{Y}$ and the union is L-disjoint in the sense that $X_{ij} \neq X_{ij'}$ implies $X_{ij} \times X_{ij'} \cap L = \emptyset$. We call the coarse family $\mathcal{Y}$ *bounded* if $\cup_{Y \in \mathcal{Y}} Y \times Y$ is controlled.

We say that the family $\mathcal{X}$ admits an L-decomposition over $\mathcal{Y}$ if every $X \in \mathcal{X}$ admits an L-decomposition over $\mathcal{Y}$.

The decomposition game for the coarse space X works as follows. Player 1 asserts a controlled set L_1. Player 2 responds with a family $\mathcal{Y}_1$ and an L_1-decomposition of X over $\mathcal{Y}_1$. Then, Player 1 asserts another controlled set L_2 and Player 2 responds with an L_2-decomposition of $\mathcal{Y}_1$ over a family $\mathcal{Y}_2$. The game ends and Player 2 wins if at some finite stage, the family over which the decomposition can be taken to be bounded.

Definition V.11. The coarse space X is said to satisfy the *coarse finite decomposition complexity* if Player 2 has a winning strategy in the decomposition game.

Definition V.12. The coarse space X is said to satisfy the *coarse straight finite decomposition complexity* if for any sequence of controlled set $L_1 \subset L_2 \subset \cdots$ there exists some finite sequence $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$ so that $\mathcal{Y}_{i-1} \xrightarrow{L_i} \mathcal{Y}_i$ with $\mathcal{Y}_n$ bounded.

In the metric setting, the proof that finite asymptotic dimension implies finite decomposition complexity relies on embedding the space with finite asymptotic dimension into a product of trees. The construction of these universal spaces relies on a sequence of anti-Cech approximations, which are not guaranteed to exist in the coarse setting. This motivates two lines of questioning. The first is whether we can prove the implication as in the metric case, if we assume the existence of anti-Cech approximations. The second is whether there is a way to weaken the definition of finite decomposition complexity so that the implication holds without needing anti-Cech approximations. We have followed this second line of questioning for the following definition.

Given an entourage $L \in \mathcal{E}$, we say that X admits a *weak (L, d)-decomposition over $\mathcal{Y}$* if

$$X = X_0 \cup X_1 \cup \cdots \cup X_d$$

so that, for each $i = 0, \ldots, d$,

$$X_i = \bigsqcup_L X_{ij}$$

where each $X_{ij} \in \mathcal{Y}$ and the union is L-disjoint in the sense that $X_{ij} \neq X_{ij'}$ implies $X_{ij} \times X_{ij'} \cap L = \emptyset$.

Definition V.13. We will then say that the space X has *weak coarse finite decomposition complexity* if the second player has a winning strategy in the weak coarse decomposition game.

We have some implications among these properties and the ones we have mentioned in previous chapters.

Proposition V.14. *Let $(X, \mathcal{E})$ be a coarse space such that* $\operatorname{asdim}(X, \mathcal{E}) \leq n$. *Then $(X, \mathcal{E})$ has coarse weak finite decomposition complexity.*

Proof. Given an entourage $L \in \mathcal{E}$, we have that there exists a L-disjoint uniformly bounded cover $\mathcal{U}_1, \cdots, \mathcal{U}_n$ of X, by the definition of finite asymptotic dimension. Then we have that $X = \cup X_i$ where $X_i = \bigsqcup_L U_{ij}$, with $\mathcal{U}_i = \{U_{ij}\}$. Therefore we have that X admits a weak (L, n)-decomposition over $\mathcal{U}$ and therefore the second player can win on the first phase of the coarse weak finite decomposition game. $\square$

The following proof follows the same scheme as in [DZ13] for the corresponding result in the metric case.

Proposition V.15. *Let $(X, \mathcal{E})$ be a coarse space with coarse property C. Then $(X, \mathcal{E})$ has coarse straight finite decomposition complexity.*

Proof. Given a sequence of entourages $L_1 \subset L_2 \subset \cdots$, we have that there exists families $\mathcal{U}_1, \cdots, \mathcal{U}_n$ such that $\mathcal{U} = \cup_{i=1}^{n} \mathcal{U}_i$ covers X and each $\mathcal{U}_i$ is uniformly bounded and L_i-disjoint.

We define $\mathcal{Y}_i = \{X \setminus \cup_{j=1}^{i} \mathcal{U}_j\} \bigcup (\cup_{j=1}^{i} \mathcal{U}_j)$. For the first stage, we can decompose X the following way: $X = X_0 \cup X_1$ with $X_0 = X \setminus \cup \mathcal{U}_i$ and $X_1 = \bigsqcup_{L_1} \mathcal{U}_1$.

For the following stages, we can decompose $Y \in \mathcal{Y}_{i-1}$ the following way: if $Y \neq X \setminus \cup_{j=1}^{i-1} \mathcal{U}_j$, then $Y \in \mathcal{Y}_i$ and so $Y = Y_0 \cup Y_1$ with $Y_0 = \emptyset$ and $Y_1 = Y$. If $Y = X \setminus \cup_{j=1}^{i-1} \mathcal{U}_j$, then $Y = Y_0 \cup Y_1$ with $Y_0 = X \setminus \cup_{j=1}^{i} \mathcal{U}_j$ and $Y_1 = \bigsqcup_{L_i} \mathcal{U}_i$.

With such construction, we have that $\mathcal{Y}_{i-1} \xrightarrow{L_i} \mathcal{Y}_i$ and, since $\mathcal{U}$ covers X, we have that $\mathcal{Y}_n = \mathcal{U}$ and is therefore bounded. $\square$

Corollary V.16. *Let $(X, \mathcal{E})$ be a coarse space such that* $\operatorname{asdim}(X, \mathcal{E}) \leq n$. *Then $(X, \mathcal{E})$ has coarse straight finite decomposition complexity.*

Proof. Since coarse finite asymptotic dimension implies coarse property C, by the previous theorem, we have that coarse finite asymptotic dimension implies coarse straight finite decomposition complexity. $\square$

Theorem V.17. *Let $(X, \mathcal{E})$ be a coarse space such that X has coarse finite decomposition complexity. Then $(X, \mathcal{E})$ has coarse straight finite decomposition complexity.*

Proof. Given a sequence of entourages $L_1 \subset L_2 \subset \cdots$, we can play the decomposition game where Player 2 always gives the next entourage in the sequence on their turn. Since X has coarse finite decomposition complexity, Player 1 has a winning strategy, which will provide a finite sequence $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$ so that $\mathcal{Y}_{i-1} \xrightarrow{L_i} \mathcal{Y}_i$ with $\mathcal{Y}_n$ bounded. □

Coarse straight finite decomposition itself implies coarse weak finite decomposition complexity. To show this, we will use the following lemma.

Lemma V.18. *Let $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ be families of coarse spaces. Then if $\mathcal{X}$ admits an (L_1, d_1)-decomposition over $\mathcal{Y}$ and $\mathcal{Y}$ admits an (L_2, d_2)-decomposition of $\mathcal{Z}$, then $\mathcal{X}$ admits a weak $(L_1 \cap L_2, d_1 * d_2)$-decomposition over $\mathcal{Z}$.*

Proof. Given any space $X \in \mathcal{X}$, we have that $X = X_0 \cup \ldots \cup X_{d_1}$ where $X_i = \bigsqcup_{L_1} Y_{ij}$, each $Y_{ij} \in \mathcal{Y}$ and if $j \neq j'$ then we have that $(Y_{ij} \times Y_{ij'}) \cap L_1 = \emptyset$. Since $Y_{ij} \in \mathcal{Y}$, we have that $Y_{ij} = Y_0^{ij} \cup \ldots \cup Y_{d_2}^{ij}$ where $Y_k^{ij} = \bigsqcup_{L_2} Z_{kl}^{ij}$, each $Z_{kl}^{ij} \in \mathcal{Z}$ and if $l \neq l'$ then we have that $(Z_{kl}^{ij} \times Z_{kl'}^{ij}) \cap L_2 = \emptyset$.

Therefore, we can write $X = \cup_{i=0}^{d_1} \cup_{k=1}^{d_2} \bigcup_{j,l} Z_{kl}^{ij}$. We will proceed by cases to show that the collection $\{Z_{kl}^{ij}\}_{j,l}$ is $L_1 \cap L_2$-disjoint. We take two distinct elements of that collection Z_{kl}^{ij} and $Z_{kl'}^{ij'}$.

Case 1: $j \neq j'$ In this case, we have that $Z_{kl}^{ij} \subset Y^{ij}$ and $Z_{kl'}^{ij'} \subset Y^{ij'}$ and so therefore $(Z_{kl}^{ij} \times Z_{kl'}^{ij'}) \cap L_1 = \emptyset$. Since $(L_1 \cap L_2) \subset L_1$, we therefore have that $(Z_{kl}^{ij} \times Z_{kl'}^{ij}) \cap (L_1 \cap L_2) = \emptyset$.

Case 2: $j = j'$ and therefore, since the two elements are distinct, we much have that $l \neq l'$ In this case, we have that $(Z_{kl}^{ij} \times Z_{kl'}^{ij}) \cap L_2 = \emptyset$. Since $(L_1 \cap L_2) \subset L_2$, we therefore have that $(Z_{kl}^{ij} \times Z_{kl'}^{ij}) \cap (L_1 \cap L_2) = \emptyset$. □

As a corollary, we can simplify the definition of coarse weak finite decomposition complexity.

Corollary V.19. *Let $\mathcal{X}$ be a family of coarse spaces. If $\mathcal{X}$ has coarse weak finite decomposition complexity, the weak decomposition game can be won by Player 2 on the first turn.*

Proof. Suppose that the game for $\mathcal{X}$ could be won in a finite number of rounds, say k. Then Player 2 will play the game as follows: when Player 1 asserts a controlled set L_i, Player 2 will find a decomposition using the controlled set $L_1 \cup \ldots \cup L_i$. This will satisfy the game, as $L_i \subset L_1 \cup \ldots \cup L_i$. Then in the last round, Player 2 will have found an $(L_1 \cup \ldots \cup L_k, d_k)$-decomposition of $\mathcal{Y}_{k-1}$ over $\mathcal{Y}_k$, with $\mathcal{Y}_k$ bounded. By the above, Player 2 then can find an $(L_1, d_1 * \ldots * d_k)$-decomposition of $\mathcal{X}$ over $\mathcal{Y}_k$ as $L_1 = \bigcap_i \cup_{j=1}^{i} L_j$. Therefore, Player 2 could have won on turn one. □

Theorem V.20. *Let $(X, \mathcal{E})$ be a coarse space such that X has coarse straight finite decomposition complexity. Then $(X, \mathcal{E})$ has coarse weak finite decomposition complexity.*

Proof. Given $L \in \mathcal{E}$, take $L_1 \subset L_2 \subset \cdots$ be a sequence of controlled sets with $L_i = L$ for all i. Then there exists some finite sequence $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n$ so that $\mathcal{Y}_{i-1} \xrightarrow{L} \mathcal{Y}_i$ with $\mathcal{Y}_n$ bounded. Then by the above, X admits an $(L, 2^n)$-decomposition over $\mathcal{Y}_n$ and $\mathcal{Y}_n$ is bounded. Therefore, X has coarse weak decomposition complexity. □

As in the metric case, we can show that these properties are preserved by coarse embeddings and subspaces.

Theorem V.21. *Let $f : X \to Y$ be a coarsely uniform embedding. Then,*

(1) *X has coarse finite decomposition complexity if and only if Y has coarse finite decomposition complexity ;*

(2) *X has coarse straight finite decomposition complexity if and only if Y has coarse straight finite decomposition complexity;*

(3) *X has coarse weak finite decomposition complexity if and only if Y has coarse weak finite decomposition complexity .*

Proof. Let $f : X \to Y$ be a coarsely uniform embedding and suppose that Y has coarse weak finite decomposition complexity. To construct a winning strategy for the decomposition game for X, we will play a parallel game for Y as follows: For the first stage, Player 2 is given a controlled set L_1 and we take as our initial controlled set in the parallel game to be $K_1 = (f \times f)(L_1)$. Then, as Y has coarse finite decomposition complexity, we can find a family $\mathcal{Y}_1$ and a K_1-decomposition of Y over $\mathcal{Y}_1$. We claim that X has an L_1-decomposition over the family $\mathcal{X}_1 = \{(f \times f)^{-1}(B) | B \in \mathcal{Y}_1\}$.

Now, since

$$Y = Y_0^1 \cup \cdots \cup Y_{d_1}^1 \qquad Y_i^1 = \bigsqcup_{K_1} Y_{ij}$$

where each $Y_{ij} \in \mathcal{Y}_1$ and the union is K_1-disjoint in the sense that $Y_{ij} \neq Y_{ij'}$ implies $Y_{ij} \times Y_{ij'} \cap K_1 = \emptyset$, we have that

$$X = X_0^1 \cup \cdots \cup X_{d_1}^1 \qquad X_i^1 = \bigsqcup_{L_1} X_{ij}$$

where $X_i^1 = (f \times f)^{-1}(Y_i^1)$ and $X_{ij} = (f \times f)^{-1}(Y_{ij})$. So therefore each $X_{ij} \in \mathcal{X}_1$ and the union is L_1-disjoint, as $X_{ij} \neq X_{ij'}$ implies $Y_{ij} \neq Y_{ij'}$ and $X_{ij} \times X_{ij'} \cap L_1 \subset (f \times f)^{-1}(Y_{ij} \times Y_{ij'} \cap K_1) = (f \times f)^{-1}(\emptyset) = \emptyset$. This proves our claim.

On the i-th stage, Player 2 has a family $\mathcal{X}_{i-1}$ over X and is given a controlled set L_i. We note that $\mathcal{Y}_{i-1} = \{(f \times f)(A) | A \in \mathcal{X}_{i-1}\}$ by the previous construction and define $K_i = (f \times f)(L_i)$. As before, we can find a family $\mathcal{Y}_i$ and a K_i-decomposition of $\mathcal{Y}_{i-1}$ over $\mathcal{Y}_i$, and again, it is true that $\mathcal{X}_{i-1}$ has an L_i-decomposition over the family $\mathcal{X}_i = \{(f \times f)^{-1}(B) | B \in \mathcal{Y}_i\}$.

Since Y has coarse weak finite decomposition complexity, after a finite number of stages, $\mathcal{Y}_i$ will be bounded, and therefore so will $\mathcal{X}_i$ since f is a coarse embedding. Therefore, Player 2 has a winning strategy for the decomposition game over X and so X has coarse weak finite decomposition complexity.

We note that since d_i for X in any given stage is the same as d_i for Y, this also proves that coarse finite decomposition complexity is a coarse invariant, since in that case $d_i = 2$ for all i.

If instead we have that Y has coarse straight finite decomposition complexity, then we begin with $L_1 \subset L_2 \subset \cdots$ a sequence of controlled sets of X. Then we have that $K_i = (f \times f)(L_i)$ is a sequence of controlled sets such that $K_1 \subset K_2 \subset \cdots$ as f is coarsely uniform. Therefore, since Y has coarse straight finite decomposition complexity, there exists $\mathcal{Y}_1, \mathcal{Y}_2 \cdots, \mathcal{Y}_n$ such that $\mathcal{Y}_{i-1} \xrightarrow{K_i} \mathcal{Y}_i$ with $\mathcal{Y}_n$ bounded.

Let $\mathcal{X}_i = \{(f \times f)^{-1}(B) | B \in \mathcal{Y}_i\}$. As above, we then have that $\mathcal{X}_{i-1} \xrightarrow{L_i} \mathcal{X}_i$ and also that X_n is bounded. Therefore, X has coarse straight finite decomposition complexity. $\square$

Proposition V.22. *If $Y \subset X$ and Y has the coarse structure inherited from X, then if*

(1) *X has coarse finite decomposition complexity then Y has coarse finite decomposition complexity;*

(2) *X has coarse straight finite decomposition complexity then Y has coarse straight finite decomposition complexity;*

(3) *X has coarse weak finite decomposition complexity then Y has coarse weak finite decomposition complexity.*

Proof. Given an entourage L of Y, it is also an entourage of X and therefore we have a coarse family $\mathcal{X}$ and an L-decomposition of X over $\mathcal{X}$ with $X = X_0 \cup X_1$, $X_i = \bigsqcup_L X_{ij}, X_{ij} \in \mathcal{X}$. Then we define $\mathcal{Y} = \{U \cap Y | U \in \mathcal{X}\}$ and have that $Y = Y_0 \cup Y_1$ with $Y_i = X_i \cap Y = \bigsqcup_L X_{ij} \cap Y = \bigsqcup_L Y_{ij}, Y_{ij} \in \mathcal{Y}$.

Therefore, any winning strategy for Player 2 in any version of the coarse finite decomposition game for X gives us a winning strategy for Player 2 in that version of the coarse finite decomposition game for Y and so the theorem holds. $\square$

Provided we have a nice fibering, these properties are preserved by any coarsely uniform map.

Theorem V.23. *Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces where Y has coarse finite decomposition complexity. Let $f : X \to Y$ where f is coarsely uniform such that, for every bounded family $\mathcal{V}$ in Y, $f^{-1}(\mathcal{V})$ has coarse finite decomposition complexity. Then X has coarse finite decomposition complexity. If Y and $f^{-1}(\mathcal{V})$ have coarse weak finite decomposition complexity, then X has coarse weak finite decomposition* *complexity. If Y and $f^{-1}(\mathcal{V})$ have coarse straight finite decomposition complexity, then X has coarse straight finite decomposition complexity.*

Proof. If we suppose that Y has coarse (weak) finite decomposition complexity, then we can construct a winning strategy for the decomposition game for X by playing a parallel game for as above for the first n stages, until Player 2 wins in Y.

At that point, we have $\mathcal{Y}_n$ is bounded, and thus $f^{-1}(\mathcal{Y}_n) = \mathcal{X}_n$ has coarse (weak) finite decomposition complexity and we can therefore find a winning strategy to finish the decomposition game for X.

For Y with coarse straight finite decomposition complexity, we let $L_1 \subset L_2 \subset \cdots$ be a sequence of entourages in X. Then $K_i = (f \times f)(L_i)$ gives us an increasing sequence of entourages in Y, so there exists $\mathcal{Y}_1, \cdots, \mathcal{Y}_n$ such that $\mathcal{Y}_{i-1} \xrightarrow{K_i} \mathcal{Y}_i$ with $\mathcal{Y}_n$ bounded.

Then, as before, we pull these back to X to obtain $\mathcal{X}_1, \ldots, \mathcal{X}_n$, $\mathcal{X}_{i-1} \xrightarrow{L_i} \mathcal{X}_i$. By assumption, $\mathcal{X}_n$ has coarse straight finite decomposition complexity, so we can take the sequence $L_{n+1} \subset L_{n+2} \subset \cdots$ and have that there is an m and families $\mathcal{X}_{n+1}, \mathcal{X}_{n+2}, \cdots, \mathcal{X}_{n+m}$ so that $\mathcal{X}_{n+j-1} \xrightarrow{L_{n+j}} \mathcal{X}_{n+j}$ with $\mathcal{X}_{n+m}$ bounded.

Together we have $\mathcal{X}_{i-1} \xrightarrow{L_i} \mathcal{X}_i$ for $i = 1, 2, \cdots, n+m$ and therefore X has coarse straight finite decomposition complexity. □

To show that coarse straight finite decomposition is preserved by products, we require a basic result on products with bounded sets.

Proposition V.24. *Let E be a bounded subset of some coarse space X, and give $E \times Y$ the product coarse structure and E the subspace coarse structure. Then $E \times Y$ is coarsely equivalent to Y.*

Proof. Let $f : E \times Y \to Y : (e, y) \mapsto y$. We will show this is a coarsely uniform embedding.

Let K be an entourage in Y. Then $(f \times f)^{-1}(K) = (E \times K) \times (E \times K)$ which is an entourage in $E \times Y$ since E is an entourage in E, as it is bounded, and K is an entourage in Y.

Let L be an entourage in $E \times Y$. Then $L = E \times Y \cap M \times K$ where M is an entourage in X and K is an entourage in Y. So, $(f \times f)(L) = K$ and is therefore an entourage in Y.

Therefore, f is a coarsely uniform embedding and therefore $E \times Y$ is coarsely equivalent to Y □

Now we are in a position to prove that these properties are preserved by direct products.

Theorem V.25. *Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be coarse spaces where X and Y have coarse straight finite decomposition complexity. Then $X \times Y$ has coarse straight finite decomposition complexity.*

Proof. We note that the projection map $f : X \times Y \to X$ is coarsely uniform and if $\mathcal{V}$ is a bounded family in X, then the family $f^{-1}(\mathcal{V}) = \{V \times Y | V \in \mathcal{V}\}$.

By Proposition V.24, $f^{-1}(V) = V \times Y$ is equivalent to Y since $V \in \mathcal{V}$ is bounded, as $\mathcal{V}$ is a bounded family. Therefore, $f^{-1}(\mathcal{V})$ has coarse straight finite decomposition complexity, since each element is equivalent to Y and thus has coarse straight finite decomposition complexity. So, by Theorem V.23, $X \times Y$ has coarse straight finite decomposition complexity. □

As in the previous chapters, the finite decomposition complexity is preserved by unions. Although the finite union theorem is a corollary of our infinite union theorem, we state it separately and give an alternate proof that cannot be extended to the infinite case.

Theorem V.26. *Let $X = X_1 \cup X_2$ be a coarse space. If X_1 and X_2 have coarse finite decomposition complexity then X does. If X_1 and X_2 have coarse straight finite decomposition complexity then X does. If X_1 and X_2 have coarse weak finite decomposition complexity then X does.*

Proof. Let X be a coarse space, $X = X_1 \cup X_2$. Suppose that both X_1 and X_2 have coarse (straight) finite decomposition complexity. Then, given an entourage L_1 we write

$$X = X_1 \cup X_2$$

as an L_1-decomposition of X over the family $\mathcal{Y}_1 = \{X_1, X_2\}$. Then, apply the coarse (straight) finite decomposition complexity property to the family $\mathcal{Y}_1$ to find that X has this property. □

Now, we consider the case where X can be expressed as a union of a collection of spaces with the property that for each $r > 0$ there is a "core" space such that removing this core from the families leaves the families L-disjoint. We will be following the same scheme we used to prove the corresponding results for asymptotic dimension and property C, adapting it for use in the coarse case. In this situation, however, we do not require a separate uniform version of the property.

Theorem V.27. *Let $X = \cup \mathcal{X}$. If for each entourage L, there exists $Y_L \subseteq X$ such that $\{X_\alpha \setminus Y_L\} = \mathcal{X}_L$ forms an L-disjoint collection, then*

(1) *if $\mathcal{X}$ has coarse straight finite decomposition complexity and Y_L has coarse finite decomposition complexity $\forall L$, then X has coarse finite decomposition complexity;*

(2) *if $\mathcal{X}$ has coarse straight finite decomposition complexity and Y_L has coarse straight finite decomposition complexity $\forall L$, then X has coarse straight finite decomposition complexity; and*

(3) *if $\mathcal{X}$ has coarse weak finite decomposition complexity and Y_L has coarse weak finite decomposition complexity $\forall L$, then X has coarse weak finite decomposition complexity.*

Proof. For the first part, we will follow the techniques in [DZ13], Theorem 3.6. Given $L_1 \subseteq L_2 \subseteq \cdots$ a sequence of entourages, then we consider the family $\mathcal{Y}_1 = \{Y_{L_1}\} \cup \mathcal{X}_{L_1}$ and write $X = X_0 \cup X_1$, where $X_0 = Y_{L_1}$ and $X_1 = \cup \mathcal{X}_{L_1}$. Since X_0 is a single element of the family, it is L_1-disjoint and we have that $\mathcal{X}_{L_1}$ is L_1-disjoint by assumption, so therefore we have an L_1-decomposition of X over $\mathcal{Y}_1$.

Now, $\mathcal{X}$ has coarse straight finite decomposition complexity and thus $\mathcal{X}_{L_1}$ has coarse straight finite decomposition complexity. Therefore, since $\mathcal{Y}_{L_1}$ also has coarse straight finite decomposition complexity, we have a natural number n and families $\mathcal{Y}_i, i = 2, 3, \cdots, n$ such that $\mathcal{Y}_{i-1} \xrightarrow{L_i} \mathcal{Y}_i$ for $i = 2, \cdots, n$ and $\mathcal{Y}_n$ is a bounded family. Therefore, X has coarse straight finite decomposition complexity.

For the second part, we are given an entourage L. We consider the family $\mathcal{Y} = \{Y_L\} \cup \mathcal{X}_L$ and write $X = X_0 \cup X_1$, where $X_0 = Y_L$ and $X_1 = \cup \mathcal{X}_L$. Since X_0 is a single element of the family, it is L-disjoint and we have that $\mathcal{X}_L$ is L-disjoint by assumption, so therefore we have an L-decomposition of X over $\mathcal{Y}$.

Now, $\mathcal{X}$ has coarse finite decomposition complexity and thus $\mathcal{X}_L$ has coarse finite decomposition complexity. Therefore, since $\mathcal{Y}_L$ also has coarse finite decomposition complexity, Player 2 has a winning strategy for each element of $\mathcal{Y}$ and therefore, X has coarse finite decomposition complexity. $\square$

If one tries to play the decomposition game with property C, then one obtains a coarse version of Dranishnikov and Zarichnyi's game-theoretic property C, [DZ13].

Definition V.28. The coarse space $(X, \mathcal{E})$ has *game-theoretic coarse C* if there is a winning strategy for Player 2 in the following game. Player 1 selects an entourage L_1 and Player 2 finds a uniformly bounded family $\mathcal{U}_1$ of sets that are L_1-disjoint. Then, Player 1 gives an entourage L_2 and Player 2 responds with an L_2-disjoint, uniformly bounded family $\mathcal{U}_2$. The game ends and Player 2 wins if there is some k for which $\mathcal{U} = \cup_{i=1}^{k} \mathcal{U}_k$ covers X.

As in the case with Dranishnikov and Zarichnyi's metric version, the attempt to define game-theoretic property C gives rise to precisely the same class of spaces with coarse asymptotic dimension 0.

Proposition V.29. *A coarse space* $(X, \mathcal{E})$ *has game theoretic coarse C if and only if* $\operatorname{asdim}(X, \mathcal{E}) = 0$.

Proof. If $\operatorname{asdim}(X, \mathcal{E}) = 0$, then it is clear that X the game ends in one step regardless of what L is asserted by Player 1.

On the other hand, suppose that X has game theoretic coarse C and there is some entourage L for which X has no uniformly bounded L-disjoint cover. Then, Player 1 selects any entourage L_1 that properly contains L. Player 2 responds with a uniformly bounded family $\mathcal{U}_1$ that is L_1-disjoint. Player 1 then responds with the entourage $L_2 = L_1 \Delta_{\mathcal{U}_1} L_1 \Delta_{\mathcal{U}_1} L_1$. Player 2 responds with $\mathcal{U}_2$ and Player 1 gives $L_3 = L_2 \Delta_{\mathcal{U}_2} L_2 \Delta_{\mathcal{U}_2} L_2$. This continues until at some point Player 2 returns $\mathcal{U}_k$ so that the family $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k$ covers X.

Set $\mathcal{V}_{k-1} = \mathcal{U}_k \cup_{L_{k-1}} \mathcal{U}_{k-1}$. Then, this family is uniformly bounded and L_{k-1} disjoint. Also, it covers $\mathcal{U}_k \cup \mathcal{U}_{k-1}$. Next, put $\mathcal{V}_{k-2} = \mathcal{V}_{k-1} \cup_{L_{k-2}} \mathcal{U}_{k-2}$ and so on. Finally, one obtains a single family $\mathcal{V}_1$ that is uniformly bounded, L_1-disjoint, and covers X. This contradicts the choice of L. □

5.3 Open Questions

As we mentioned in chapter 3, an analog to property A for coarse spaces is difficult to define and manipulate. However, we can attempt to look for a sequence of maps reminiscent of those for property A.

Question V.30. Does coarse straight finite decomposition complexity imply the existence of a sequence a^n of maps $a^n : X \to \ell^1(X)$ such that

(1) $\|a^n_x\|_1 = 1$ for each $x \in X$ and $n \in \mathbb{N}$;

(2) for each n, $\{(x, y) \mid y \in \operatorname{supp}(a^n_x)\}$ is controlled;

(3) for each $K > 0$,

$$\lim_{n\to\infty} \sup_{(x,y)\in E^K} \{\|a^n_x - a^n_y\|\} = 0?$$

As we mentioned above, there is a second line of questioning regarding the relationship between finite coarse asymptotic dimension and coarse finite decomposition complexity.

Question V.31. If a coarse space X has finite coarse asymptotic dimension and an anti-Cech approximation, does it have coarse finite decomposition complexity?

Alternately, we could use the tools of the proof in the metric case.

Question V.32. If X is a coarse space with finite asymptotic dimension, does it embed into a product of trees?

CHAPTER VI

COMPUTATIONS

6.1 Introduction

In [Car09], Gunnar Carlsson presents the idea that data has a shape and that the shape of the data matters. This shape is given by a distance function. In some situations, our data might be known to be a sample from a manifold, and we might use the inherited distance function to analyze the data. In that case, one hopes to recover the topological properties of that underlying space from the sample. In other situations, we might define a metric on the data that reflects how similar any two data points are. In this case, we might discover the topological properties on the newly created metric space to make statements about the overall similarity of the set of data.

In both these situations, we would like to place a topological structure on top of our data set that reflects the sample space. There are a few ideas that one holds in mind when attempting to construct these spaces. One is that our structures should not depend on the given coordinates of the data, but only on the distance between points. We also want to make sure that we can continue to represent the space on a computer. There are a number of constructions that satisfy these requirements. For the purposes of this dissertation, we focus on two, the Rips complex and the Čech complex. The definitions below follow the notation found in [EH10].

A k*-simplex* σ is the convex hull of $k+1$ affinely independent points in Euclidean space, denoted $S = \{v_0, \dots v_k\}$. We call those points *vertices*. Any subset $T \subset S$

defines a *face* of σ. A *simplicial complex* K is a finite set of simplices such that every face of a simplex in K is in K and also the intersection of any two simplices in K is in K. A *flag complex* is a simplical complex K such that if σ is a subset of the vertices of K, and each pair of vertices in σ is itself a simplex of K, then σ is a simplex of K.

Definition VI.1 (Vietoris, Rips)**.** Given a finite metric space S and a positive real number r, we define the *Rips complex* of S at scale r by:

$$\mathrm{Rips}_r(S) = \{\sigma \subset S \mid \operatorname{diam} \sigma \leq r\}.$$

Definition VI.2 (Čech)**.** If we have a finite set S of points of $\mathbb{R}^n$ and a positive real number r, we can define the *Čech complex* of S at scale r by:

$$\check{\mathrm{C}}\mathrm{ech}_r(S) = \{\sigma \subset S \mid \bigcap_{x \in \sigma} B_r(x) \neq \emptyset\}.$$

We note that if $r_0 \leq r$, we have that $\check{\mathrm{C}}\mathrm{ech}_{r_0}(S) \subseteq \check{\mathrm{C}}\mathrm{ech}_r(S)$.

These two constructions are closely related. The Čech complex does rely on embedding the finite set on points into some Euclidean space, while the Rips complex merely requires that there be a metric on the space of points. The Rips complex is also a flag complex, while the Čech complex need not be. However, by [EH10], we do have that $\mathrm{Rips}_r(S) \subseteq \check{\mathrm{C}}\mathrm{ech}_{\sqrt{2}r}(S)$ and also that $\check{\mathrm{C}}\mathrm{ech}_r(S) \subseteq \mathrm{Rips}_r(S)$.

Another way of stating the definition of the Čech complex is that it is the nerve of the collection of sets $\mathcal{U} = \{B_r(x) \mid x \in S\}$.

Theorem VI.3 (Nerve Theorem, [Bor48])**.** *Given X a metric space and $\mathcal{U}$ a cover of X by closed, convex sets, then the nerve of $\mathcal{U}$ and X are homotopy equivalent.*

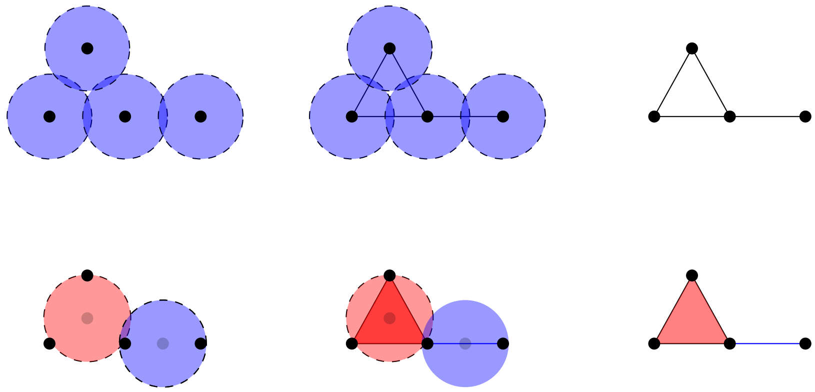

Figure 5. A comparison of the Čech and Rips complex. On the top is the Čech complex on 4 points with r=.56 and below is the Rips complex on the same 4 points

Therefore, provided we can find a suitable r, we can approximate X by a sampling S and have some relationship between the homotopy type of X and of $\check{\mathrm{C}}\mathrm{ech}_r(S)$.

6.2 Algorithm

The central idea of the algorithm for constructing the Čech complex relies on a slightly different formulation of what a simplex in $\check{\mathrm{C}}\mathrm{ech}_r(S)$ looks like. We note that $\sigma \in \check{\mathrm{C}}\mathrm{ech}_r(S)$ if and only if $\bigcap_{x\in\sigma} B_r(x) \neq \emptyset$ which is the case if and only if the centers of each of these balls live inside a single ball of radius r. We define the *miniball* of a set S to be the unique smallest closed ball containing S. We then note that $\sigma \in \check{\mathrm{C}}\mathrm{ech}_r(S)$ if and only if the miniball of σ has radius $\leq r$.

So therefore our basic algorithm is as follows:

Algorithm 1 Calculate Čech(r) of S

for all $\sigma \subseteq S$ **do**

  if the radius of MiniBall$(\sigma, \emptyset) \leq r$ **then**

    Put $\sigma \in$ Čech(r)

  end if

end for

The bulk of the calculations are hidden in the MiniBall$(\sigma, \emptyset)$ subroutine. An algorithm based on [EH10] allows us to find both the center and the radius of the miniball, though we mostly concern ourselves with the radius. The two inputs are to allow us to proceed recursively. We split our point set as $T \cup N$ where T is the set of points that are allowed to be interior points and N is the set of points that must be on the boundary.

Algorithm 2 Welzl's miniball algorithm

```
if T = ∅ then
  if N = ∅ then
    return ball with radius 0, centered at the origin
  end if
  if N ≠ ∅ then
    return B = Ball(N)
  end if
end if
if T ≠ ∅ then
  let P ∈ T
  B = Miniball(T \ {P}, N)
  if P ∉ B then
    return B = Miniball(T \ {P}, N ∪ {P})
  end if
  if P ∈ B then
    return B
  end if
end if
```

We can be sure the recursion terminates, since the terminating condition is that $T = \emptyset$ and each non-terminating step results in removing an element from T. However, again, the bulk of the calculations are hidden away, this time in the Ball(N) subroutine. The Ball(N) subroutine finds the center and radius of the ball that contains the points in the set N on its boundary. The calculations there are not hidden, and the matrices and computation involved are derived from [CHM06].

This algorithm makes much use of the SimplicialComplex package included in Sage. That package is well suited for this project, as it asks for a set of vertices, and allows one to define a complex by setting subsets of the vertices as faces. So, in order to find the Čech complex of a set of data, once we find the Miniball of a subset of the given data and compare its radius to r, then to include that subset as a simplex, we simply define it to be a face of our final SimplicialComplex.

The computations within the Miniball algorithm were easily handled with Sage's implementation of the matrix arithmetic, along with some use of the set and list data types.

The programming of the Rips complex algorithm benefited greatly from the underlying commands in Sage. The Rips complex can be constructed entirely based on pairwise distances, as whether a subset is a face of the complex depends on the diameter of the subset, which is computed by taking a maximum of the pairwise distances between elements. Thus, once we have calculated the 1-skeleton of the complex as a graph object in Sage by adding an edge if two data points are within r of one another, the pre-existing method "clique_ complex()" generates the Rips complex for the data set.

6.3 Persistent Homology

Once we have built these complexes on top of our data, we can recover a large amount of information from them. One piece of information that we often focus on are the Betti numbers, where the n-th Betti number is the rank of the n-th homology group. For low n - that is, 0, 1, and 2 - these give us the number of connected components, the number of holes and the number of voids, three very important features of the space.

However, converting these data sets into a complex requires a choice of our parameter r. If we set r to be too small, we are likely to generate a 0-dimensional complex. If we set r large enough, we will generate a complex consisting of a single, high-dimensional simplex. Even between these two extremes we are unlikely to find a single optimal r that captures every feature of the underlying set exactly, with no artifacts.

A solution to this problem is to construct a profile of the space, which tracks the features of the complexes over a sequence of radii. One such is called the barcode of the space, which [Ghr08] calls an analog to Betti numbers. This keeps track of the components, holes and voids that give rise to $\beta_0, \beta_1, \beta_2$ as r increases along the horizontal axis. Once we build the barcode, we then have that the longer bars should correspond to real features, and the shorter bars are more likely to represent artifacts.

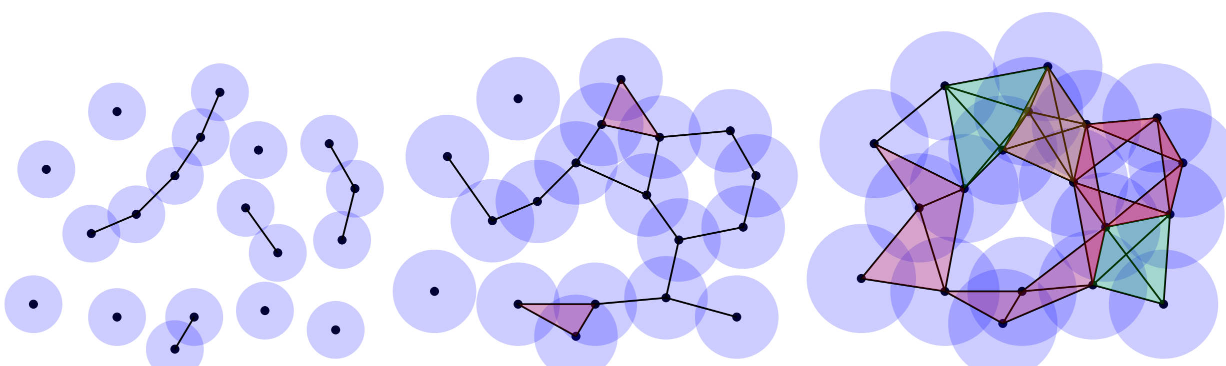

Figure 6. The Čech complex. Given at three different stages based on an example from [Ghr08].

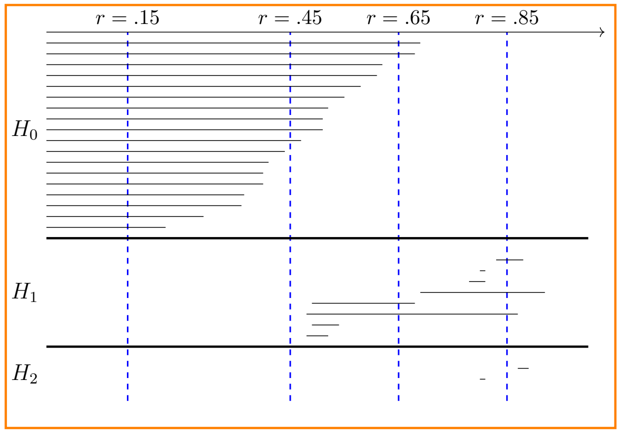

Figure 7. The set of barcodes for Figure 6

6.4 Gromov Invariant

We recall from above that a space X has asdim $X \leq n$ if it can be covered by $n+1$ sets with dimension 0 on R-scale. If we denote the uniform bound on these sets by D, then we can find a new quasi-isometric invariant defined in [Gro93] by considering the behaviour of the function $D_p(R)$ as $R \to \infty$, where $D_p(R)$ is the minimal D so that X can be covered by $p+1$ D-bounded R-disconnected sets.

Example ([Gro93])**.** If $X = \mathbb{R}^n$ then $D_p(R) = \text{const}_{n,p} R$.

Example ([Gro93])**.** If X is an infinite tree, then $D_p(R) = p^{-1}R$.

Let X be some simply connected space, and take a sample S of p points of X. We can use my code to find a series of Čech or Rips complexes for an increasing sequence of R-values. Let r_0 be the smallest r such that $\text{Čech}_r(S)$ (or, equivalently, $\text{Rips}_r(S)$) is simply connected. Testing could be done to see if there is any relationship between $D_p(R)$ and r_0.

For sufficiently complicated spaces X, we might need to consider S a sample of a closed and bounded subset $Y \subset X$. In this case, $d = \text{diam}(Y)$ will most likely need to figure into the relationship between $D_p(R)$ and r_0.

6.5 Open Questions and Extensions

Question VI.4. Can we use a quick computation of the Čech complex to approximate Gromov's invariant for a sampled space?

Extension VI.5. The algorithm as given uses the standard Euclidean metric, but some work could be done to extend it to any metric. This could be useful either in the case that our data can be embedded into a known space with a different metric, or in the case that our data comes equipped with its own concept of "close-ness".

Extension VI.6. This version of the algorithm bogs down when working in $\mathbb{R}^d$ with $d > 20$. Optimization of the Ball subroutine in particular could extend the useful range of dimensions.

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APPENDIX A

CODE

Miniball computation

```
"""
FUNCTIONS:
 Miniball(T,N)
 Ball(N)
 _containedIn(B,P)
 _rotationContained(N, pair)
"""

def Miniball(T, N, n, **kwds):
    r"""
    This function finds the smallest ball with the point set T
       in its interior and N on its boundary.

    INPUT:

    - ``T`` - a set of points in R^n
    - ``N`` - a set of points in R^n
    - ``n`` - the dimension of the Euclidean space being worked
        in
    - ``dimension check`` - boolean (optional, default True)
```

```
OUTPUT: pair [r, c] with r being the radius of the miniball
    with the point set T in its interior and N on its
   boundary and c being its center

``T`` and ``N`` should be lists or tuples or sets (anything
    which may be converted to a set) whose elements are
   tuples (or lists, etc) of real numbers

``n`` should be an integer

If ``dimension_check`` is True, check that each tuple (or
   list, etc) in N and T have the same length and that
   length is ``n``.

EXAMPLES:

::

    sage: Miniball([(2,1,3,-1),(-2,3,-1,0),(1,3,-1,-2)
       ,(0,2,3,-3),(2,-1,3,5),(0,0,0,0),(3,1,2,0)],[],4)
    [4.61737819233406, (0.957177989548109,
       0.966830617891177, 1.65244389794036,
       1.18576698432216)]
    sage: Miniball([(2,1,3,-1),(-2,3,-1,0),(1,3,-1,-2)
       ,(0,2,3,-3),(2,-1,3,5)],[(0,0,0,0)],4)
    [6.10292806326925, (1.70816326530612, 4.47142857142857,
        2.99795918367347, 2.31224489795918)]
```

```
    sage: Miniball([(0,0,0,0,1), (0,0,0,0,-1),(0,0,0,0,0)
       ],[],5)
    [1.00000000000000, (0.000000000000000,
       0.000000000000000, 0.000000000000000,
       0.000000000000000, 0.000000000000000)]

NOTES:

Based on the algorithm given in "Computational Topology: An
    Introduction" by Edelsbrunner and Harer, otherwise
   known as Welzl's miniball algorithm.  Could use
   improvement in the case of high dimension points,
   possibly using the algorithm in "Fast Smallest-Enclosing
   -Ball Computation in High Dimensions" by Fischer,
   Gartner and Kutz.

AUTHORS:

- Dani Moran (2011-05-24)

"""

# process kwds
if 'dimension_check' in kwds:
    dimension_check = kwds['dimension_check']
else:
```

```
        dimension_check = True
    # done with kwds

    if dimension_check:
        for i in xrange(len(T)):
            if len(T[i]) != n:
                raise ValueError, "The point T[%s]does not have
                    the appropriate dimension." %i
        for j in xrange(len(N)):
            if len(N[j]) != n:
                raise ValueError, "The point N[%s]does not have
                    the appropriate dimension." %j

    sT = Set(T)
    lT = list(sT)
    sN = Set(N)
    lN = list(sN)

    if sT.cardinality() == 0:
        if sN.cardinality() == 0:
            c = tuple([0 for i in xrange(n)])
            return [0, c]
        else:
            B = Ball(N, dimension_check = False)
    else:
        P = lT.pop()
```

```
        nT = sT.difference(Set([P]))
        B = Miniball(nT, N, n, dimension_check = False)
        if _containedIn(B, P) == False:
            nN = sN.union(Set([P]))
            B = Miniball(nT, nN, n, dimension_check=False)

    return B

def _containedIn(B,P):
    r"""
    This function determines whether or not a given point is
       contained in a given closed ball

    INPUT:

    - ``B`` - a ball defined by its radius r and its center c
    - ``P`` -  a point in R^n

    OUTPUT: boolean value

    EXAMPLES:

    ::

    NOTES:
```

```
    AUTHORS:

    - Dani Moran (2011-05-24)

    """
    r = B[0]
    c = B[1]
    n = len(B[1])

    d = 0
    for i in xrange(n):
        d += (abs(c[i] - P[i]))^2
    d = sqrt(d)

    if d <= r: return True
    else: return False

def Ball(N, **kwds):
    r"""
    This function finds the smallest ball with the point set N
       on its boundary.

    INPUT:

    - ``N`` - a set of points in R^n
    - ``dimension check`` - boolean (optional, default True)
```

```
OUTPUT: pair [r, c] with r being the radius of the miniball
    with the point set N on its boundary and c being its
   center

``N`` should be a list or tuple or set (anything which may
   be converted to a set) whose elements are tuples of real
    numbers

If ``dimension_check`` is True, check that each tuple (or
   list, etc) in N and T have the same length and that
   length is ``n``.

EXAMPLES:

::
    sage: Ball([(2,1,3,-1),(-2,3,-1,0),(1,3,-1,-2)
       ,(0,2,3,-3),(2,-1,3,5)])
    [108.806020054039, (-19.5000000000000,
       -94.5000000000000, -35.0000000000000,
       -29.5000000000000)]
    sage: Ball([(0,0,0,0,1), (0,0,0,0,-1)])
    [1.00000000000000, (0.000000000000000,
       0.000000000000000, 0.000000000000000,
       0.000000000000000, 0.000000000000000)]
    sage: Ball([(0,0,0,0,1), (0,0,0,0,-1),(0,0,0,0,0)])
```

```
    There is no ball with N on its boundary.

NOTES:

Matrices based on the computations in "On the Smallest
   Enclosing Balls" by Cheng, Hu and Martin

AUTHORS:

- Dani Moran (2011-05-24)

"""
# process kwds
if 'dimension_check' in kwds:
    dimension_check = kwds['dimension_check']
else:
    dimension_check = True
# done with kwds

if dimension_check:
    dim = len(N[0])
    for j in xrange(len(N)):
        if len(N[j]) != dim:
            raise ValueError, "The point N[%s]does not have
                the appropriate dimension." %j
```

```
sN = Set(N)
lN = list(sN)
n = len(lN[0])
k = len(lN) - 1

if k == 0:
    return [0, lN[0]]

Q = matrix(RR, k, n)
M = matrix(RR, k, k)
for i in xrange(k):
    for j in xrange(n):
        Q[i,j] = lN[i+1][j] - lN[i][j]
        if j < k: M[i,j] = lN[i+1][j] - lN[i][j]

if k == n: A = Q
else:
    mu = det(M)
    MU = matrix(RR, k, n-k)
    for i in xrange(k):
        MU_I = copy(M)
        for j in xrange(n-k):
            for l in xrange(k):
                MU_I[l,i] = lN[l+1][k+j]-lN[l][k+j]
            MU[i,j] = det(MU_I)
```

```
    H = matrix(RR, n-k, n)
    for i in xrange(n-k):
        for j in xrange(n):
            if j < k: H[i,j] = MU[j][i]
            elif j == k+i: H[i,j] = -mu

    A = Q.stack(H)

if A.is_invertible() == False:
    FirstKs = Subsets(n,k)
    for sub in FirstKs:
        a, r, c = _ColSwap(lN, sub)
        if a == True:
            return [r, c]
    print "There is no ball with N on its boundary."
    return None

B = matrix(RR, k,1)
for i in xrange(k):
    for j in xrange(n):
        B[i,0] += lN[i+1][j]^2 - lN[i][j]^2
    B[i,0] = (1/2)*B[i,0]

if k != n:
    h = matrix(RR, n-k,1)
    for i in xrange(n-k):
```

```
            for j in xrange(n):
                h[i,0] += H[i][j]*(lN[1][j] + lN[0][j])
            h[i,0] = (1/2)*h[i,0]

        B = B.stack(h)

    d = A.inverse()*B
    d = list(d)

    c=[]
    for i in xrange(len(d)):
        c.append(d[i][0])

    c=tuple(c)

    r = 0
    for i in xrange(n):
        r += (abs(c[i] - lN[0][i]))^2
    r = sqrt(r)

    return [r,c]

def _ColSwap(N, firstK):
    n = len(N[0])
    k = len(N) - 1
    last = Subsets(n,n)[0].difference(firstK)
```

```
perm = list(firstK) + list(last)

swappedN = []
tempElt = [0 for i in xrange(n)]

for i in xrange(k+1):
    for j in xrange(n):
        tempElt[j] = N[i][perm[j] - 1]
    swappedN.append(tuple(tempElt))
    tempElt = [0 for l in xrange(n)]

Q = matrix(RR, k,n)
M = matrix(RR, k, k)
for i in xrange(k):
    for j in xrange(n):
        Q[i,j] = swappedN[i+1][j] - swappedN[i][j]
        if j < k: M[i,j] = swappedN[i+1][j] - swappedN[i][j
            ]

mu = det(M)
MU = matrix(RR, k, n-k)
for i in xrange(k):
    MU_I = copy(M)
    for j in xrange(n-k):
        for l in xrange(k):
            MU_I[l,i] = swappedN[l+1][k+j]-swappedN[l][k+j]
```

```
            MU[i,j] = det(MU_I)

    H = matrix(RR, n-k, n)
    for i in xrange(n-k):
        for j in xrange(n):
            if j < k: H[i,j] = MU[j][i]
            elif j == k+i: H[i,j] = -mu

    A = Q.stack(H)

    if A.is_invertible() == False:
        return False, 0, 0



    B = matrix(RR, k,1)
    for i in xrange(k):
        for j in xrange(n):
            B[i,0] += swappedN[i+1][j]^2 - swappedN[i][j]^2
        B[i,0] = (1/2)*B[i,0]



    h = matrix(RR, n-k,1)
    for i in xrange(n-k):
        for j in xrange(n):
            h[i,0] += H[i][j]*(swappedN[1][j] + swappedN[0][j])
        h[i,0] = (1/2)*h[i,0]
```

```
    B = B.stack(h)

    d = A.inverse()*B

    d = list(d)

    c=[0 for i in xrange(n)]

    for i in xrange(len(d)):
        c[perm[i]-1] = d[i][0]

    c=tuple(c)

    r = 0
    for i in xrange(n):
        r += (abs(c[i] - N[0][i]))^2
    r = sqrt(r)

    return True, r, c
```

Čech complex computation

```
"""
FUNCTIONS:
  CechComplex(S,r)
```

```
"""

def CechComplex(S, r, **kwds):
    r"""
    This function finds the Cech complex of a point set T with
       radius r.

    INPUT:

    - ``S`` - a set of points in R^n
    - ``r`` - the desired radius
    - ``dimension check`` - boolean (optional, default True)

    OUTPUT: simplicial complex of S with radius r isomorphic to
        the Cech complex of S and r

    ``S`` should be a list or tuple or set (anything which may
       be converted to a set) whose elements are tuples (or
       lists, etc) of real numbers

    ``r`` should be a real number

    If ``dimension_check`` is True, check that each tuple (or
       list, etc) in N and T have the same length and that
       length is ``n``.
```

```
EXAMPLES:

::

    sage: CechComplex([(2,1,3,-1),(-2,3,-1,0),(1,3,-1,-2)
       ,(0,2,3,-3),(2,-1,3,5),(0,0,0,0),(3,1,2,0)],4)
    Simplicial complex with 7 vertices and 2 facets
    sage: S=CechComplex([(2,1,3,-1),(-2,3,-1,0),(1,3,-1,-2)
       ,(0,2,3,-3),(2,-1,3,5),(0,0,0,0),(3,1,2,0)],4)
    sage: S.facets()
    {((-2, 3, -1, 0), (0, 0, 0, 0), (0, 2, 3, -3), (1, 3,
       -1, -2), (2, 1, 3, -1), (3, 1, 2, 0)), ((0, 0, 0, 0)
       , (2, -1, 3, 5), (2, 1, 3, -1), (3, 1, 2, 0))}
    sage: CechComplex([(0,0,1),(0,0,-1)],.5)
    Simplicial complex with vertex set ((0, 0, -1), (0, 0,
       1)) and facets {((0, 0, 1),), ((0, 0, -1),)}
    sage: CechComplex([(0,0,1),(0,0,-1)],1.5)
    Simplicial complex with vertex set ((0, 0, -1), (0, 0,
       1)) and facets {((0, 0, -1), (0, 0, 1))}

NOTES:

AUTHORS:

- Dani Moran (2011-05-24)

"""
```

```
if S == []:
    return SimplicialComplex(S)

n = len(S[0])

# process kwds
if 'dimension_check' in kwds:
    dimension_check = kwds['dimension_check']
else:
    dimension_check = True
# done with kwds

if dimension_check:
    for j in xrange(len(S)):
        if len(S[j]) != n:
            raise ValueError, "The point S[%s]does not have
                the appropriate dimension." %j

for i in xrange(len(S)):
    if i == 0:
        faces = list(Subsets(S,1))
    elif i == 1:
        for sub in Subsets(S,2):
            if _distance(sub[0],sub[1]) <= 2*r:
                faces.append(sub)
    else:
```

```
            for sub in Subsets(S,i+1):
                B = Miniball(sub,[],n)
                if B[0] <= r:
                    faces.append(sub)

    return SimplicialComplex(S, faces)

def _distance(P,Q):
    r = 0
    n = len(P)
    if n != len(Q):
        print "Unmatched dimensions for distance."
    for i in xrange(n):
        r += (abs(P[i] - Q[i]))^2
    r = sqrt(r)

    return r
```

Rips complex computation

```
"""
FUNCTIONS:
  RipsComplex(S,r)

TODO: Non-euclidean metrics?
"""
```

```
def RipsComplex(S, r, **kwds):
    r"""
    This function finds the Vietoris Rips complex of a point
       set T with radius r.

    INPUT:

    - ``S`` - a set of points in R^n
    - ``r`` - the desired radius
    - ``dimension check`` - boolean (optional, default True)

    OUTPUT: simplicial complex of S with radius r isomorphic to
        the Rips complex of S and r

    ``S`` should be a list or tuple or set (anything which may
       be converted to a set) whose elements are tuples (or
       lists, etc) of real numbers

    ``r`` should be a real number

    If ``dimension_check`` is True, check that each tuple (or
       list, etc) in N and T have the same length and that
       length is ``n``.

    EXAMPLES:
```

```
::

NOTES:

AUTHORS:
- Dani Moran (2014-04-14)
"""
if S == []:
    return SimplicialComplex(S)

n = len(S[0])

# process kwds
if 'dimension_check' in kwds:
    dimension_check = kwds['dimension_check']
else:
    dimension_check = True
# done with kwds

if dimension_check:
    for j in xrange(len(S)):
        if len(S[j]) != n:
            raise ValueError, "The point S[%s]does not have
                the appropriate dimension." %j

g = Graph()
```

```
    k = len(S)

    for i in xrange(k):
        for j in range(i+1,k):
            if _distance(S[i],S[j]) < r:
                g.add_edge(S[i],S[j])

    return g.clique_complex()

def _distance(P,Q):
    r = 0
    n = len(P)
    if n != len(Q):
        print "Unmatched dimensions for distance."
    for i in xrange(n):
        r += (abs(P[i] - Q[i]))^2
    r = sqrt(r)

    return r
```