STUDENTS' CONCEPTUAL KNOWLEDGE OF LIMITS IN CALCULUS:
A TWO-PART CONSTRUCTIVIST CASE STUDY

by

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ABSTRACT

MARGARET SMOLINKA ADAMS. Students' conceptual knowledge of limits in calculus: a two-part constructivist case study. (Under the direction of DR. VICTOR CIFARELLI)

This case study investigated students' conceptual knowledge of limits in calculus by implementing semi-structured interviews. The constructivist learning principles of Piaget and Inhelder as well as theories of understanding by Skemp guided the study. In Phase I, a pilot study was conducted with 15 students from a Calculus III class. By using 41 traditional textbook type tasks and non-traditional tasks, various ways students think about functions, limits at a point, limits at infinity and limits that do not exist were explored. Tasks included continuous and non-continuous functions; piecewise, rational and trigonometric functions, including those with oscillatory end behaviors. In Phase I, two students with different conceptions of limits and ability levels were selected for the initial analysis. The findings gave rise to a more in-depth follow-up investigation referred to as Phase II , which explored what students know about limits with respect to the definition of limit and infinity as well as knowledge of domains. Four student cases were selected out of nine based on similar emerging themes of understandings. The results are interpreted in terms of the constructivist framework and suggest that students assimilate information about limits into a variety of unique conceptual knowledge structures that ultimately develop into operational schemas. Given the nature of the content knowledge, these schemas are either appropriate or altered. Implications related to improving instructional practices, differentiating instruction for diverse learners and teaching limits in meaningful ways across the curriculum, are discussed.

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CHAPTER 1: INTRODUCTION

"What knowledge is of most worth?" Herbert Spencer

This long-term, two-part qualitative case study involving in-depth interviews was designed to explore how undergraduate mathematics and engineering majors think about limits in calculus. The project evolved from a personal interest in why students struggle with the topic of limits. Understanding limits appears to be contingent upon knowledge of functions. This study reveals that students do not always incorporate into their schemas what their calculus instructors had intended. Instead, students develop alternate conceptual knowledge structures of functions and limits that develop into different operational schemas.

Calculus has played a large role in the history of mathematics but has always been a difficult subject to teach and learn (Confrey, 1980; Davis & Vinner, 1986; Edwards, 1979). The concept of limit is the most important topic in calculus, being prerequisite for definitions of both differentiation, integration, and the convergence of sequences and series (Boyer, 1968). Yet, the concept of limit is difficult to understand, even for very good students (Francis, 1992). Allendoerfer (1963) stressed how important the limit is:

> "Many people assume that calculus is chiefly concerned with differentiation and integration, but this is a superficial view. The essential idea of calculus is that of the limit, and without a clear exposition of limits, any calculus course is a failure" (p. 484).

The limit concept is stated as the Conceptual Underpinnings of Calculus in the NCTM Curriculum and Evaluation Standards (1989) and may form, with the concept of function, the foundation of students' understanding and success in calculus. These standards suggest that the limit concept is based on infinite processes and introduces students to another mode of mathematical thinking. Therefore, it is not difficult to convince the reader how important it is to understand limits.

Definition and Types of Limits

A limit is a number, the number that the function values are "tending to or heading toward" as x gets closer or "nearer" to a particular value x=a, or as x gets larger either in the positive or negative sense. Limit at a point refers to the behavior of the function values for $x's$ nearer and nearer to $x = a$. Limit at infinity refers to the behavior of the function values for larger and larger values of x in the positive or negative direction. Being the heart of calculus, limits permeate the calculus course sequence including derivatives, integrals and series. In fact, derivatives are limits, integrals are limits and infinite series are limits.

This study considers both types of limits in calculus, limits at a point and limits at infinity. The first is limit at a point. As x approaches a number, a, the limiting behavior of the function values refers to how the function values' behavior for $x's$ near $x = a$. The point $x = a$ does not have to be in the domain of the function in order for a function to have a limit at that point, a fact which seems to not be emphasized in the classroom. The case of limit at a point can be further dissected into left- and right-hand limits. In these cases, x is restricted to be near $x = a$ on the left of $x = a$ or to the right, respectively. However, the function values may or may not be getting closer to some number L. If they do, the limit exists and is equal to L, if not, we say the limit does not exist. On a graph that has a hole whose first coordinate is $x = a$, the limit will exist and be equal to the second coordinate of the point. The same will be true if the graph goes through that hole. The limit does not exist typically because of a jump discontinuity or the function values could get larger in either direction and tend toward infinity. The function values "blow up".

The second is limit at infinity, which refers to the behavior of function values for large x increasing without bound in either the positive or negative directions. A limit does not exist when function values increase in the positive or negative directions without bound because infinity is not a number. Limiting behavior of function values increasing without bound is referred to as an “infinite limit”. Limits at infinity and infinite limits are not synonyms by any means. In fact, “limits at infinity” pertain to what the first coordinate or x-value is approaching, whereas “infinite limits” pertain to the limiting behavior of the second coordinate or function values. Also, limits might not exist for oscillating functions.

Limit Notation

The notation for limit, $\lim_{x \to \infty} f(x)$, is confusing; however, it is important in order for students to make connections between function values and limits (Monoghan,1991). As a result, difficulty occurs when starting the problem solving process since it is unlike any notation seen in prior mathematical experiences (Tall, 1981; Davis and Vinner,1986). Students have vague interpretations of “lim” if the referents do not represent the mathematical meaning well enough, or if the connection between the referent and written notation is not appropriate (Hiebert & Carpenter, 1992). An understanding of the notation is built gradually in order to make connections necessary to understand its meaning and so there must be an understanding of this notion of "nearness", what happens to the function values on the y-axis when x-values get close or near a given point on the x-axis. Also for large x, as x is tending toward infinity, the focus is on what number the function values are getting near.

While standard written symbols play an important role in student learning of mathematics, students often experience difficulties in constructing meanings of symbols (Skemp, 1987). In the context of mathematics, Skemp (1971) contended that communication is a central function to the use of symbols. Similarly, Layzer (1989) noted "a peculiar synergy between mathematics and ordinary language" (p. 311). When Skemp used the term "symbols" in this context, he referred both to mathematical notation and to language. Moreover, he also stressed the importance of "recording knowledge" with oral and written communication and stated,

> "Whereas the spoken communication usually, though not always, takes place in circumstances which allow questions and explanations to be given, written or printed symbols have to convey all the required meaning, without a second chance on either side. So the communicator has to take more trouble to try to ensure this" (p. 73).

Ensuring that students understand the notation by being able to communicate what it means collaboratively among others might result in less confusion and more positive problem solving outcomes.

Background on the Topic of Limits

The topic of limit is very difficult for most students but is one of the foundational ideas in the study of calculus, particularly as the basis for understanding the derivative. According to the National Council of Teachers of Mathematics (2000), students struggle with learning limits in calculus and exit calculus courses with little more than algorithmic or procedural knowledge of how to compute limits, not understanding the meaning behind limiting behavior of function values. This is a problem not just at UNC Charlotte but throughout the country and world (Tall, 1981). Unfortunately, students do not complete calculus courses with a good understanding of what a limit is and what limiting behavior means (NCTM, 2000).

When students are asked to define a limit, most offer vague definitions (Tall, 1981). More recently from personal experience, students also claim they do not know, forgot, or that a limit is “something” that one computes in calculus. This is partly because textbooks do not explain what a limit is in colloquial language, unlike a definition of function which seems rather unanimously defined in any calculus book on the market (Stewart, 2005; Strang, 1991). It is rare in a calculus textbook to see a limit defined as a “number”. Therefore it is up to the teacher to supplement the content in the textbook. For instance, when E.A. Eagle (personal communication, July 2009) teaches limits, she states the limit is “the height of the y-axis--what y is doing”. Students are expected to develop their understanding of limit on their own and are subject to variability of instructors’ pedagogical skill and content knowledge. As a result, they are subject to assimilating information into their conceptual structures other than what is intended.

Beginning calculus students usually develop an intuitive understanding of limits of a function at a point but have misconceptions about limits at infinity and infinite limits (Tall, 1981; Juter, 2006). Although students can typically compute limit problems, they often have difficulty understanding, explaining and articulating the meaning of their solutions (Skemp, 1987). In order to be successful in calculus-based courses, students should be able to compute limit problems algorithmically or instrumentally, but more important, understand the meaning of limiting behavior.

Limits appear in introductory calculus classes in high school or college. Students in Calculus II study limits of sequences and series, such as the Taylor, Telescoping and Fourier series. Therefore, over the course of two semesters, students supposedly acquire knowledge about limits. By the time students enroll in Calculus III, they have encountered and solved numerous problems involving derivatives and integrals, both of which involve limits. Calculus III students were selected as subjects in this study because of their prior knowledge of limits and other mathematical concepts, and are able to describe their understandings about limits through problem solving tasks.

In learning mathematics, one constructs concepts by direct instruction, collaborative group learning, use of definitions, and examples. Therefore, learning mathematics, including functions and limits, is contingent upon good teaching. Since 1888, there have been repeated calls for school curricula to place greater emphasis on functions (College Entrance Examination Board, 1959; Klein, 1883; Hendrick, 1922; Hamley, 1934; NCTM 2000). In spite of this, students continue to emerge from high school and freshman college courses with a weak understanding of this important concept (Carlson, 1998). Students who think about functions only in terms of symbolic manipulations and procedural techniques are unable to comprehend a more general mapping of a set of input values to a set of output values. They also lack the conceptual structures for modeling function relationships in which the function value changes continuously in tandem with continuous changes in the input variable (Carlson, 1998). This impoverished understanding of such a central concept in secondary and undergraduate mathematics can result in losing interest in mathematics. The primarily procedural orientation of using functions to solve specific problems is absent of meaning and coherence for students, and causes frustration (Carlson, 1998). Instructional shifts promoting richer conceptions and powerful reasoning abilities may stimulate students' curiosity and interest in mathematics, and can positively affect the decision to continue studying mathematics (Oehrtman, et al., 2008).

The National Council of Teachers of Mathematics (2000) emphasizes the importance of learning mathematics with understanding. Mathematics educators have called for a shift from an instructional approach that emphasized algorithmic manipulations, to one that emphasized conceptual understanding (Skemp, 1987). Researchers and educators alike have emphasized the importance of building on prior knowledge, developing connections among key ideas in mathematics, engaging students in discussions of mathematical thinking and reasoning, and offering challenging problems to students to use what they know in novel situations (Carpenter & Lehrer, 1999, 2002; Grunow, 1992; Hiebert & Carpenter, 1992); Moyer, 2001).

Calculus is the first course in which students are confronted with the concept of limit which is the first instance whereby they must extend their knowledge beyond the algebraic properties venture into the topological properties of the real numbers. Therefore, primary calculations go beyond simple arithmetic and algebraic operations. In calculus, an important transition occurs for students when they move from the algebra-based courses they have encountered to a course such as calculus that contains more sophisticated concepts. Research suggests that the predominant direct instructional approach to calculus is not achieving the foundational understandings and problem solving behaviors that are needed for students' continued mathematical development and course taking (Carlson, 1998). If algebraic and procedural methods were more connected to conceptual learning, students would be better equipped to apply their algebraic techniques appropriately in solving novel problems and tasks (Oehrtman, et al., 2008).

Teachers often circumvent the problem of teaching limits meaningfully by using approaches that may draw from their own misconceptions of limit (Bukova-Guzel, 2007). Moreover, teachers may reduce the rigorous formal definition and compromise conceptual understanding by focusing solely on the computational algorithms, leading to unsatisfactory outcomes for calculus students (Carlson, 1997; Tall, 1992a; Juter, 2005).

Role of the Domain

Although domains are a part of functions, personal teaching experiences reveal that students tend not consider the domains when working with functions (Oehrtman et al., 2008; Hohensee, 2006). In fact, they dislike being asked any questions about domains. Although in textbooks it is established practice to for domains to be specified along with the graph of a function (Stewart, 2005), the domain is not often emphasized during classroom instruction. Instead, due to time restrictions and a syllabus to cover, there is more focus on how to compute various functions, such as piecewise, rational, logarithmic or trigonometric functions. Therefore, when students are solving limit problems there is reason to suspect that they do not consider the domains of these functions, either. This might offer clues as to why students struggle with limits of rational and piecewise functions and possess various types of conceptual knowledge about limits. Information about the domain of continuous functions might also render important underlying clues regarding any difficulties that emerge in the problem solving process.

Although there is sufficient research done on more advanced aspects of limits, no studies were found that specifically examined how students think about the domain when looking at graphs of limits doing the computations.

Figure 1.1 provides a diagram of the understanding between functions, domains and limits. Although it is essential to understand this connection, many students of mathematics typically do not consider domains when problem solving. The visual representation shows that all functions have domains. Limits are, in fact, limits of functions. All limits of functions have domains and a function cannot be mentioned without acknowledging the domain. An alternative figure appears in Appendix Q.

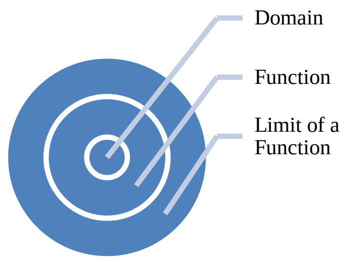


Figure 1.1: Visual Representation of Domain in Relation to a Function and Limit.

When learning about limits at a point for a continuous function or for the non-continuous type in which there is a hole in the line instead of a function value, one must consider the left hand limits and right hand limits. When these have the same value regardless of a hole or a dot, the limit exists. At other times when there are restricted intervals in a domain, there are one-sided limits in which case function values get close to and sometimes equal the function value where the domain begins or ends. In this case, f(x) left and/or right continuous, one describes the behavior of function values from the left and/or from the right, i.e., as x approaches “a”, denoted $x \to a$, then the function values at “x” approach the function value of “a”, denoted $f(x) \to f(a)$. Given this is the case of a one-sided limit, there is no need for the superscript (+ or -) on “a” since by definition, x must be in the domain of the function thereby automatically making it a left- or right-hand limit. A graph of the arccosine is an example because as $x \to a$, “a” need not be in the domain for the limit to exist, though the results will show that students have difficulty with this idea. As a result, a part of this study investigates what students think about the role of the domain while describing their knowledge about limits.

Importance of Domains using Real-World Example

Hooke's Law presents a real life application which demonstrates the necessity of always considering the domain of a function (Cutnell & Johnson, 2004, p. 286). The domain of the function is determined by the physical system. Numbers from the domain are the numbers that can be plugged into the function, maintaining a valid model of the system. For a spring, the formula for the function that models the system is $F(x) = kx$, whereby $x$the displacement of the spring is (how far it is stretched out) and $F(x)$ is the magnitude of the restoring force that the spring exerts. As a mathematical formula, this function makes sense for any x that is plugged in; however, it is not a valid model for all displacements.

Clearly, if the spring is only 12 inches long and an attempt was made to stretch the spring out to one mile, the spring would break long before it would reach that displacement. Even if the spring was only stretched out to 10 inches, it might exceed the elastic limit of the spring and therefore, the formula for the function would have to change since Hooke's Law would no longer describe the spring. This imposes a finite interval for the domain of the function, something like $[0,8]$ for this example. What this illustrates is that since a function's job is to model the behavior of a physical system, one cannot separate the formula for the function from the domain of the function. Domains that have both infinite and finite intervals are considered in this study.

Piecewise Functions: Real-World Applications

The absolute value function $|x| = \begin{cases} x, & \text{if x} \geq 0 \\ -x, & \text{if x<0} \end{cases}$ is as example of a piecewise function used in many applications, not just in mathematics but in areas such as navigation and transport, architecture, engineering, science and sports. Piecewise functions occur very often in business applications. For example, the cost of goods per item can differ depending on how many items are purchased given by the formula:

$$c(x) = \begin{cases} 10, & \text{if } 0 < x < 100 \\ 10 - 0.001x, & \text{if } 100 \leq x < 1000 \\ 10 - 0.001x - 0.000001x^2, & \text{if } 1000 \leq \text{x} \leq 2000 \end{cases}$$

Here is another real-world example of a piecewise function. An electric motor draws a lot of current on start up if the full voltage is applied. The voltage is ramped up to the running voltage and then ramped down to stop, as illustrated in the Figure 1.2. The up and down lines can be replace by nonlinear functions.

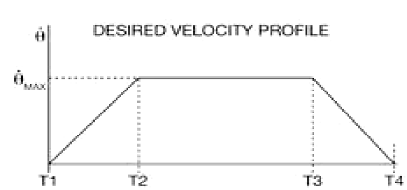


Figure 1.2: Velocity piecewise function.

Limits of Piecewise Functions

Piecewise functions are common models of physical systems studied in algebra and calculus, but not very well understood. A real-world application of a naturally occurring piecewise function would be the gravitation force, E, on a point mass both inside and outside the earth. Newton's Law of Gravitation says that the force on a mass, E, is inversely proportional to the square of the distance, r, between two masses. When inside the earth the mass of the earth that lies above our point of interest plays no role in the force on the point mass. Below the point, there is a sphere containing the mass that acts on one's point of interest. The total mass below the point is proportional the volume of that sphere, while the volume of the sphere is proportional to r^3 , where r is the distance of the point from the center of the earth. By Newton's Law, the force, E, is proportional to the product and hence varies linearly with the distance in r. So inside the earth, using the proportionality symbol, $\propto$, we have the force inside, $E_{in} \propto r$. Outside of the earth, the force is inversely proportional to the square of the distance, r^2. So outside of earth we can write the force, $E_o \propto 1/r^2$. At the surface of the earth we have $E_s \propto 1/R^2$, whereby, R is the radius of the earth hence the distance from the center, i.e. $r = R$. So there are two formulas for the force that depend on r thereby requiring a piecewise function to describe this behavior. The graph of this function would resemble that of Figure 1.3.

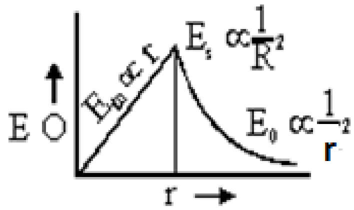


Figure 1.3: Graph of a Piecewise Function.

The limiting behavior of gravity can also be seen from this function. The limit at infinity would be equal to zero. That is to say, "gravity dies off with distance". Note that no function value for large positive x is zero but the limit is equal to zero. Exploring how students think about limits of piecewise functions are is one major focus of this study.

Limits of Rational Functions

Limits of rational functions are also problematic for students to grasp since more skill mastery is required, yet students often enter calculus with inadequate knowledge of rational functions (NCTM, 2000). This is unfortunate, since knowledge of rational functions has many useful real world applications (Hornsby, 1984). As a general example, many engineering systems are described by differential equations, and solutions to these equations are the functions that describe the system (Rasmussen, 2000). Laplace Transforms, a method used to solve differential equations, gives the transform of the solutions as rational functions that describe solutions for usual engineering systems; however, the actual solutions to the differential equations are not rational functions. The limiting behavior of these rational functions describes the limiting behavior of the solutions and hence, the behavior of the underlying physical system (Rasmussen, 2000).

As a specific example, Ohm's Law, $I = \frac{E}{R}$ involves the rather simple rational function used in this study, $f(x) = \frac{1}{x}$. In this example, I is the current, E is a fixed voltage for a particular circuit and R is the resistance giving, $I(R) = \frac{E}{R}$, whereby the current is a function of resistance. So the formula appears like, $I = f(x) = \frac{E}{x}$, where x represents the resistance. If E=1, then the formula is $f(x) = \frac{1}{x}$, and so the value f(x) represents the current in the circuit. This relation is important, as the current is inversely proportional to the resistance with the proportionality constant being the voltage.

The limit at a point would have to be considered coming from the right as R has to be positive. If E is positive, the limit would be plus infinity and so the limit does not exist because the current gets larger and larger without bound. A current produces heat in a resistor, so the heat and hence the temperature get larger without bound. The material in the resistor cannot sustain high temperatures and hence catches fire, which is the source of a lot of home fires. It is called a short circuit (0 or very low resistance).

Sometimes this simple rational function might involve time dependence, such that "t" would replace "x", generating $\lim_{t \to 0^+} \frac{1}{t} = \infty$ meaning as t approaches 0 from the right in which case this is a right-handed limit, the limit would not exist. Heading in the other direction, $\lim_{x \to \infty} \frac{1}{t} = 0$, if $\frac{1}{t}$ is the position of the system at time t, then theoretically the system's position approaches 0 but the function value is never 0. Once the position is sufficiently close to 0, though, the difference can't be measured and so for technical purposes, the function value is considered to be 0. Therefore, learning about the limiting behavior of rational functions at a point and at infinity is very important in many applications areas, which is why exploring what students know about limits of rational functions is another focus of this study.

Learning Theories of Mathematics

Any theory of learning mathematics must consider the structure of the subject (Gallagher & Reid, 1981; Skemp, 1987). It is not possible to learn about differentiation and integration before algebra skills are mastered and before the limit concept is understood. If the notion of limiting behavior is presented earlier in one's mathematical education prior to calculus, there might be a greater chance that appropriate knowledge structures could be developed (Gallagher & Reid, 1981, p.5). Sufficient time must be invested as well as experience for meaning to be integrated within the knowledge structure, as "learning is an internal process of construction" (Gallagher & Reid, 1981, p.2). Perhaps then with enough time and sufficient introduction to limiting behavior in the elementary years, the introduction of limits in calculus might be an easier transition.

In order to teach students mathematical concepts, "we need to know what our students are thinking, how they produce the chain of little marks we see on their papers, and what they do (or want to do) with the material we present to them" (Noddings, 1990, p. 15). In other words, a key aspect of effective teaching is to know what and how students are thinking (Dunham & Osborne, 1991) which is why a case study method involving interviews is appropriate for this investigation.

Rationale of the Two-Part Study

This constructivist case study provides insight into how students think about limits and describes what conceptual knowledge structures emerge with problem solving. There are two parts or phases involved. The first part, Phase I, considers two initial pilot cases of how students think about functions, limits at a point limits at infinity and limits that do not exist. The second part, Phase II, evolves from results in Phase I and focuses more specifically on the students' definition of limit, definition of infinity and the role of the domain. The theoretical framework utilizes the learning principles of Piaget and Inhelder (Gallagher & Reid, 1981), and theories of understanding by Skemp (1987).

Yin (1994, p. 74) states pilot studies can help researchers refine data collection plans with respect to both the content of the data and the procedures to be followed. Pilots can help develop relevant questions, possibly providing some conceptual clarifications for the research design. In this particular case, pre-pilot planning helped inform the Phase I pilot study in terms of deciding which traditional and non-traditional tasks would be used. Next, the Phase I pilot study collected initial data to inform the subsequent investigation in Phase II, which probed deeper into various tasks to discover how thoughts and beliefs about limits are constructed and exemplified through problem solving. In order to understand the knowledge a learner possesses, the researcher needs to focus on the learner's actions while completing mathematical tasks (Cobb & Steffe, 1983). In order to determine prior knowledge students have, how they think about limits and how they understand mathematical concepts in general, interviews are an ideal methodology (Yin, 1994: Aspinwall, 1994). In this study, different ways students conceptualize knowledge about limits are explored.

Origin of the Study

The project evolved from personal interest in why students struggle with learning limits in calculus. My journey began as an undergraduate psychology where my classmates and I found limits out of reach, beyond our comprehension because of the "limit' notation and the ideas of "nearness and approaching". Math phobic to begin with, we sat in class confused, often guessed answers to homework problems, rarely ever understanding what we were doing or what any of it meant. Given the mystery about limits, this topic was too intriguing to let go and leave behind once the course ended.

While remaining determined over many decades to find out why limits are difficult to understand, an opportunity finally emerged years later while pursuing a degree in mathematics and certification for secondary mathematics teacher licensure. The delight and honor of having taken calculus again was met by the same mystery of limits. As a novice, it is frustrating to see a discontinuity on a graph, and wonder how a limit could exist if there is a hole in the graph since the hole appears empty.

There are at least three particular sources of confusion with limits that are quite common. The first deals with limits at a point of piecewise functions, when there is a discontinuity (hole) above or below a function value (point). Another problem deals with asymptotic behavior of functions or limits at infinity. The third deals with infinite limits and the belief that limits exist if they equal infinity. Figures 1.3 and 1.4 provided examples of these cases and the study builds upon these and other basic types of functions that present sources of difficulty not just to novices, but to even the more experienced calculus students.

One popular question that typically appears on a first calculus test is the graph of a function followed by instructions to compute the limit at a point and determine if the limit exists. The fact that the function is piecewise is not disclosed. Figure 1.5 presents an example of the graph of a quadratic function with a discontinuity (hole) and point (function value) located above the discontinuity. Students refer to the graph then compute the limit as x approaches 2, $\lim_{x \to 2} f(x)$, then must explain if the limit exists. Although there is only one correct answer, that the limit is equal to 4 because the left hand limit equals the right hand limit, this graph is subject to many different interpretations.

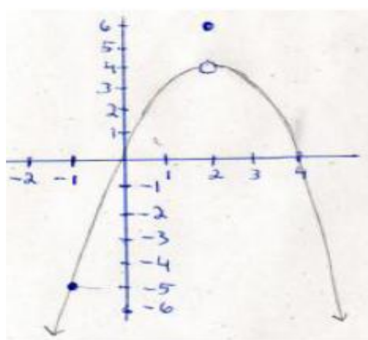


Figure 1.5: Common test question for limit at a point, compute $\lim_{x \to 2} f(x)$.

After studying the graph for quite some time, a typical student compares the hole and point, wondering what the limit is and if it exists. Although it does not seem necessary to identify what kind of function for which this is the graph, it would have been useful to know the function is piecewise because one could identify two distinct parts on the graph. The quadratic is one part, and the function value above the discontinuity (hole) is another. By looking at the graph, the hole is a confusing element; it does not seem possible to some students that a limit could exist where a hole is. Some students erroneously think that when there is a discontinuity (hole) with a function value above it, there is no limit anywhere; others think the limit is the function value and worse, the limit is infinity because the left and right sides of the quadratic never make it to the hole. Using definitions, identifying domains or relying on prior algebra skills was not practiced by the novice student, as limit problems seemed unique, having nothing to do with any math previously learned.

Another typically confusing problem is presented in Figure 1.6 which contains the graph of a rational function. On a test, the student would be asked to compute the limit as x approaches infinity, $\lim_{x \to \pm\infty} \frac{1}{x}$, and explain if the limits exist.

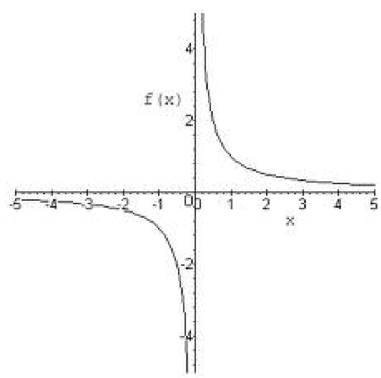


Figure 1.6: Common test question for limit at infinity, compute $\lim_{x \to \pm\infty} \frac{1}{x}$.

In this case, the student tends to think about the graph visually, not mathematically. As a result, the common response is that the limit equals infinity because the line on the graph keeps going and never actually touches the x-axis. It is natural to follow points on the graph and conclude that the limit is infinity because the points keep going. The fact of the matter is that the limit exists and equals 0 because the function values keep decreasing, getting closer and closer to 0.

The concept of infinity is another source of trouble in calculus when describing the behavior of graphs. In the case of the graph in Figure 1.6, it is common for a student to erroneously state that as x approaches infinity, the limit equals infinity and exists. First, their conclusion would be wrong, because the limit is really 0. Next, infinity is not a number and so by definition of limit, a limit can only exist if it is a number. Therefore, stating that a limit exists when it equals infinity is an erroneous conclusion. Yet, it happens all the time in calculus classes, even from instructors which I discovered from personal experiences as well as from students interviewed. These three main types of misunderstood conceptions of the graphs just presented are addressed in this study.

The purpose of this study is to explore what students know about limits in terms of the definition of limit, infinity and their perceived role of the domain. Semi-structured interviews facilitate discussion and feedback, and fosters an environment to reveal prior knowledge while constructing problem solving actions in response to given tasks. Semi-structured interviews were conducted with the pilot and semi-structured studies in order explore students' thinking about limits and mathematical concepts more completely. A number of questions emerged during the analysis of the Phase 1 study. The interviews provided the opportunity to unpack some of the ideas and capture a students' actions and experiences. Traditional and non-traditional tasks were designed and piloted on volunteer calculus students prior to the implementing the pilot study.

Significance of the Study

Results of this study will hopefully result in the development of more effective instructional practices related to teaching limits. By exploring how students think about limits with respect to the definitions of limit and infinity and role of the domain, various conceptual knowledge structures can be identified which can ultimately foster meaningful instructional interventions. Students often pursue non-technical majors due to unsuccessful endeavors in mathematics courses. If more students can be successful with learning limits and other introductory topics in calculus, the retention rates in calculus courses might increase, thereby attracting more students into technical, math-related majors. Therefore, the long term goal of this study is to improve curriculum and instructional practices, so that more students would be successful in calculus.

Research Questions

The research questions in the Phase I pilot study were: How do students think about limits, and what do students know about limits? Based on the results and analysis of Phase I, a more focused research question was developed for Phase II: Incorporating the intuitive definition of limit, prior knowledge of domains and definition of infinity, in what ways do students solve problems and reveal what they know in response to tasks involving limits at a point and limits at infinity?

Outline of Chapters

The chapters in this document are presented in narrative form as follows. Chapter One was an introductory overview of the study. Chapter Two presents theoretical framework that guided the study. Chapter Three presents the literature review. Chapter Four details the methodology. Chapter Five contains the pilot study results. Chapter Six reports the results and discussion of the secondary investigation. Chapter Seven contains the conclusions and teaching implications.

CHAPTER 2: THEORETICAL FRAMEWORK

Introduction

The theoretical framework for this study is constructivism, a general theoretical position in philosophy and psychology that characterizes perceptual experiences as being constructed from prior knowledge (Reber, 1985). The process of constructing knowledge is contingent upon a person's subjective interpretation of the experience instead of reality in terms of what events actually occurred. Since knowledge is a subjective construct rather than a collection of empirical data, it is impossible to know to what particular aspects of knowledge reflect an ontological reality.

In order to conduct the study and answer the main research question, Piaget and Inhelder's Principles of Learning and Skemp's Model of Instrumental and Relational Understanding are combined. Therefore, the framework is a hybrid of Piaget, Inhelder and Skemp, given similarities and compatibilities between their terminology and philosophies. In this paper, I refer to the word "schema" as conceptual knowledge structure, being consistent with Skemp (1987, p. 187) who refers to "schema" as a "conceptual structure stored in memory".

Piaget is considered the pioneer of the constructivist approach to cognition in this century (Von Glasersfeld, 1995), having offered powerful insights into the human mind and its development along with our understanding of how children view the world (Confrey & Smith, 1994). Piagetian theory emphasizes individual knowledge construction is stimulated by internal cognitive conflict as learners strive to resolve mental disequilibrium (Applefield, Huber, & Moallem, 2001). A gap or internal conflict may appear between a learner's existing knowledge and formal instruction. As a result, this may lead to difficulties in learning (Baroody & Ginsburg, 1990).

Constructivism a Philosophy of Knowledge, Learning and Teaching

Constructivism is also referred to as a philosophy of knowledge, learning and teaching (Jaworski, 1994; Simon, 1995; Steffe & D'Ambrosia, 1995). As a philosophy of knowledge, no objective reality can be attained by individuals or society since both have only a limited access to reality through the human senses. However, what individuals or society describe as "truth" may be constructed through information gained through the senses. This particular view informs my study by acknowledging that students bring prior experiences as well as prior knowledge to the interviews.

As a philosophy of learning, constructivism may be categorized into cognitive and social. In the cognitive domain, individuals construct knowledge and cognition is an adaptive mechanism for understanding and interacting with the environment (Ernest, 1996; Richards & Von Glasersfeld, 1980). In this regard, the cognitive domain incorporates the above mentioned constructivism as a philosophy of knowledge, acquired through the senses because interacting with the environment depends on sensory input. Though reality cannot be known, it is believed that individuals construct knowledge through interaction and reflection with the experiential world. Sometimes, this version of constructivism is referred to as radical or psychological constructivism. This particular view informs my study acknowledging students bring prior experiences as well as prior knowledge to the interviews.

As a philosophy of teaching, constructivism holds that teachers themselves actively construct knowledge as they prepare for work in the classroom (Steffe & D'Ambrosia, 1995; Simon, 1995). In this model, teachers create situations (problems, tasks, discussions, etc) based on their beliefs of what students know, what they believe students are capable of learning and upon student actions and reactions within these situations. Added to these beliefs are opportunities for reflection upon lesson planning and results, in particular, those reflections as a result of student responses that generate a dialogue and discussion of confirmations, contradictions, as well as new meanings and understandings.

Constructivist teaching begins when students are engaged in ongoing construction of mathematical knowledge (Ferrini-Mundy & Graham, 1991). Inquiry-based learning is considered to be at the heart of a constructivist classroom (Wood, Cobb, & Yackel, 1995), as students work collaboratively to explore and discuss mathematical tasks at hand in order to facilitate learning. Many mathematics terms have precise meanings that are highly abstract. Learning mathematics vocabulary is a challenge for many students given its abstract, unequivocal, collective and compact nature (Monroe, 1997). Mathematics language is unique and tends to be used only in mathematical contexts, with very few occurrences outside of school (Thompson & Rubenstein, 2000). With the absence of cooperative group learning and social interaction in class whereby students can collaborate with peers and construct knowledge, there are fewer opportunities to use and reinforce newly learned concepts. As a result, either appropriate conceptions or altered conceptions can occur. In many cases, the type of schema developed goes unnoticed. According to the Department of Public Instruction, conceptual understanding in math is a measurable outcome but limited. Constructing knowledge and developing appropriate schemas informs this study as the results have implications regarding curriculum and instruction on the topic of limits.

Piaget and Inhelder's Developmental-Constructivist Epistemology

Piaget and Inhelder shared a developmental-constructivist epistemological view on learning. Epistemology is a branch of philosophy which deals with the relationship between knowledge (what we know) and different forms of reality, which includes "apparent, real, possible, and even impossible" (Gallagher & Reid, 1981, p.14). This theory differs from other approaches to learning given it does not postulate that growth in knowledge is only the "result of experience" (Cross, 1974, p.7), but rather, emphasizes the active role of the person (Gallagher & Reid, p. 10).

Philosophers or researchers interested in epistemological studies would ask questions such as "how do we know about objects in the external world, how valid is our understanding, and by what methods can we study our interactions with these objects" (Gallagher & Reid, p. 14). According to Piaget, epistemology must be studied developmentally and then one must ask, by what means does a person transition from less knowledge to greater knowledge.

Piaget and Inhelder's Six Principles of Learning

Piaget and Inhelder discuss six principles of learning (Gallagher & Reid, p. 11) which serve as a foundation for this study (Table 2.1).

Table 2.1: Piaget's Six Principles of Learning.

Principle 1: Learning is an internal process of construction.
Principle 2: Learning is a function of development.
Principle 3: Learning as a Higher-Level of Reorganization.
Principle 4: Growth in knowledge follows feedback from an individual's questions, contradictions, and consequent mental reorganization.
Principle 5: Questions, contradictions and the consequent reorganization of thought are initiated by social interaction.
Principle 6: Awareness is a process of reconstruction rather than sudden insight, and actions precede understandings.

Principle 1: Learning is an internal process of construction.

Certain phenomena cannot be learned by observation in the environment, but rather by an internal mental process. If a child is given a set of objects to count in different ways, he or she subsequently reflects on the fact that in spite of ordering the objects differently, the sum remains constant. The commutative law cannot be understood unless the child thinks about what he or she has done. Therefore, the information learned from his or her actions is used to construct the mental idea that the sum is independent of order. Since this involves going from general manipulations with objects to specific theories, this is a deductive process. This is why Piaget and Inhelder reported that "knowledge deduced from activity is called logico-mathematical knowledge" (Gallagher & Reid, p.3). Relative abstraction is the key learning process that informs this development. It occurs when a person reflects on the results of his or her actions, and learns by constructing rules from activities involved when interacting with objects (Gallagher & Reid, p. 3).

Principle 2: Learning is a function of development.

Learning cannot occur from observation or experience alone and so children cannot be expected to improve their understanding or gain new skills by an accumulation of experiences alone, due to developmental levels of maturation. “Experience is considered indispensable to learning but does not ensure it” (Gallagher & Reid, 1981). This principle is the place where the terms “assimilation” and “accommodation” originate. According to Piaget, in order to learn, there must be an ability to respond to a new experience or concept, by assimilating it into one’s repertoire. One cannot assimilate something, though, if he or she is not receptive to the stimulus or does not have the competence to understand due to ability or due to the new concept being contradictory or esoteric. However, once a connection is made and one understands what the stimulus means, then one is said to accommodate the information into his or her knowledge base and apply this knowledge to new situations. Afterwards, one can later judge if a situation appears correct or incorrect.

According to Piaget, action is central to cognitive development. Development is the result of assimilation and accommodation. When the action occurs without causing any change in the child, assimilation occurs. When the child adjusts to the environment in some ways, accommodation is involved. Both adaptive processes occur simultaneously despite they are very different. Initially, they are adaptive processes of behavior, but become processes of thinking.

Cognitive development involves assimilation and accommodation as a method to adapt to the environment. McLaghlin (1992) reports that “accommodation is an important idea that has been taken into second language learning under the label “restructuring” used to refer to the reorganization of mental representations of a language.”

Principle 3: Learning as a Higher-Level of Reorganization.

A person reorganizes knowledge in their minds, “seeing internally” (Gallagher & Reid, 1981, p. 6). The process of seeing inside, then self-correcting or self-regulating is referred to as “equilibration” because it is associated with advanced states of equilibrium in the cognitive system. Equilibration is the central concept of genetic epistemology, and reveals a reciprocal relationship between a person and the environment. This means that children don’t just record events in the environment, but also act on them. As a result, children learn not just by observation but by reorganizing information on a higher mental level, coordinating their activities with constructed rules and principles (Gallagher & Reid, p. 7).

Principle 4: Growth in knowledge follows feedback from an individual’s questions, contradictions and consequent mental reorganization.

Piaget believed feedback was very important to learning. If a student reports that 1+8=18, then this constitutes a procedural error. Feedback would be the teacher can use objects to demonstrate how the sum of 1+8 objects equals 9 objects. Piaget was criticized for his studies on conservation. According to Donaldson (1978), the child tries to make sense of the world by asking questions and from an early age, has purposes and intentions; however, asking the same question twice forces children to give an answer that is against their better judgment and the ability to make sense of something is limited by their experience.

Principle 5: Questions, contradictions and the consequent reorganization of thought are initiated by social interaction.

An example might be when a student asks the teacher or peers questions about a math problem, and everyone within a given collaborate group gets a different answer.

Principle 6: Awareness is a process of reconstruction rather than sudden insight, and actions precede understandings.

Being aware that a math problem yields a wrong answer originates being able to go back and trace the steps to identify the particular step with the error. In order to understand the error, the student must discover his or her flaw with their algorithm.

These various ideas about learning are important to the study. The first three principles provide the framework to explain the prior knowledge students bring to the study in terms of their schemes. The last three principles relate to the interviewing process in terms of how students construct actions, ask questions and receive feedback during problem solving. These also help explain how students either form or avoid misconceptions, by asking questions, receiving feedback and organize information into either appropriate or inappropriate schemes.

Concept of Structure

Piaget was primarily interested in how knowledge developed or was constructed genetically in humans and so the concept of structure was central to his theory. Cognitive structures are patterns of physical or mental actions that underlie specific acts of intelligence, and they correspond to four stages of child development. Structure is a concept that Piaget acquired from mathematics, "A structure is a system with sets of laws that apply to the system as a whole; not only to its elements" (Gallagher & Reid, p. 31).

Perturbations and Disequilibrium

Piaget believed that learners experience perturbations and therefore, constantly try to reach a state of equilibrium. They test the adequacy of their ideas through assimilation and accommodation, the latter of which occurs when there is an actual change of behavior. Meanwhile, as the learner experiences conflict or perturbations between pre-existing conceptions and newer ones, disequilibrium occurs thereby making the student uncomfortable (Appleton, 1993). Once a new conception is accommodated, equilibrium occurs. Constructivists believe that through accommodation, actual learning takes place. This occurs when students change their existing ideas in response to new information.

Studies by Piaget and others led to the constructivist philosophy. Constructivism purports that learners bring their personal experiences into the classroom and these experiences impact the students' views of how the world works. Students come to learning situations with a variety of prior knowledge, feelings, attitudes, preferences and skills. This knowledge exists within the student and is developed as individuals interact with peers, teachers and the environment. Learners construct understanding or meaning by making sense of their experiences and fitting their ideas into reality.

Children construct thoughts, ideas, questions, and explanations about natural phenomena to make sense of their everyday experiences. Their explanations form an intricate framework that often differs from scientific views, and these differing views are referred to as misconceptions, alternative conceptions or alternative frameworks. Studies indicate that alternative conceptions common to elementary students are also found among high school, college students and even adults (Kyle & Shymansky, 1989). These misconceptions often interfere with learning because students resist change unless they are dissatisfied with their current explanations and can find sensible alternatives with supporting evidence. The more dissatisfied students are, the more likely they will search for and accept new explanations. By the same token, the more satisfied they are, they less likely they will question their notions or views. Dissatisfied students ultimately reorganize ideas, requiring them to discard old views and construct new ones (Posner, Strike, & Hewson et al., 1982). The learner must first recognize that his or her current knowledge is insufficient to explain an experience.

Action Schemes and Schemas

Piaget believed children are active and motivated learners. “Schemes” are plans to execute some behavior a recurrent action pattern that leads to practical knowledge about one’s environment. Through action schemes, they construct schemas which are clusters of knowledge structures in memory, used to understand and respond to the world. For example, the behavior of planning to add the prices of three items to get a sum before paying on a debit card is an action scheme. Knowing what particular retail stores are expensive constitute knowledge structures which comprise a schema. A scheme is a plan of action, where as a schema is a specific body of knowledge stored in memory.

By interacting with the environment and using certain recurrent action patterns, one comes to understand and appreciate much of the world in terms of what can be understood. In adult life, schemes for understanding the world become more complicated and less dependent on overt actions as with younger children, and so Piaget calls these “interiorized schemes”. This means they are carried out mentally without physical action. For instance, mental arithmetic replaces counting on the fingers. So the scheme of an older learner is internalized.

Piaget’s genetic epistemological efforts were devoted to understanding how the child’s intelligence is transformed into the intelligence of the adult, becoming increasingly complex, abstract and subtle with maturity (Inhelder & Piaget, 1958). Intelligence was not considered some fixed trait, but a process by which a person constructs an understanding of reality. Constructions of reality mature through encounters between children and their environments, in which case children experience a discrepancy between what they already understand and what new things their environment presents. The resolution of this discrepancy or disequilibrium transforms children's previous view of how objects and events are related into a new and more mature understanding.

Mental growth is basically an increased ability to adapt to new situations and this growth takes place because of two key processes that Piaget calls assimilation and accommodation which is part of his second principle of learning. Assimilation is the process of incorporating a new stimulus into one's existing cognitive view of the world. Accommodation is the process of changing one's cognitive view and behavior when new information dictates such a change. Both processes work in a complementary fashion trying to establish a state of equilibrium when new stimuli are introduced. Through the process of accommodation, schemes gradually become more refined and therefore, intelligence grows.

Learning is not so much a process of acquiring new knowledge, but reconstructing our existing schemas, which affect how organisms make sense of the world and learn new things. Assimilation involves fitting new information into existing schemas, often by distorting, transforming and imposing meaning on the information. On the other hand, accommodation involves modifying, transforming, and reconstructing existing schemas.

When assimilation and accommodation do not work together simultaneously, there are perturbations in the mind. As a result, disequilibrium occurs as schemas do not match up with reality of the world. Equilibration occurs when one restructures their schemas. Piaget believed equilibration is what leads to conceptual change, makes a person smart, and leads to the development of more complex levels of thought such as critical and abstract thinking.

Sometimes, people hold onto various conceptions that do not align with reality and are typically referred to as “misconceptions”. Reasons could include the following: a newer idea does not make sense; language barriers; or lack of motivation. Sometimes one might believe something is true and be resistant to being wrong or might hold onto prior beliefs. Reasons could include habits; particular experiences; one’s personality, values or identity; as well as the possibility that one may discount evidence, opinions, or ideas that are different. Attempting to change one’s beliefs may cause perturbations because the prior belief makes sense. Prior beliefs, too, may be socially embedded. Piaget’s principles of learning can help to explain how learning occurs with knowledge structures and schemas, through the processes of assimilation, accommodation and equilibration.

Skemp’s Theories of Understanding

Skemp (1987) is the second constructivist theorist who informs my study, having developed a model of instrumental and relational understanding. Skemp (1987, p. 153) defines “understanding’ as "ways of knowing" (Figure 2.1). Instrumental understanding is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works, or in other words, "rules without reasons". It can be applied to very specific situations, and can be acquired through “habit learning” (p. 32) and therefore, is “not really understanding” at all.

Figure 2.1: Skemp’s Model of Instrumental and Relational Understanding.

Instrumental mathematics is easier to understand as it involves algorithmic procedures such as “factor the numerator and denominator to find common factors then divide out any factors there are in common” (Skemp 1987, p. 153). After solving the problem, the learner can receive feedback. Skemp’s notion coincides with Piaget, who considered feedback to be very important when constructing knowledge, as noted in his fourth principle of learning. Skemp also contended that when less knowledge is involved, one can answer questions quicker with instrumental understanding. An example is solving a differential equation easily by using differential forms such as a LaPlace Transform rather than using a more complicated algorithm and requires a lot more mathematical skill as well as procedural and conceptual knowledge (Rasmussen, 2000).

In order for instrumental understanding to occur, “the student must be able to identify the problem type and then associate it with a solution procedure” (Skemp, 1987). Unfortunately, there are many problem types that a particular mathematical concept can be used to solve; however, memorizing all of them could be arduous and inefficient. Still, it’s common to memorize formulas, procedures and problem types. Skemp notes the deficiency of this method is that the connection between procedures and problem types is likely to deteriorate, leaving the student with no way of matching the problem with the concept. As a result, instrumental understanding lacks the two qualities relational understanding: adaptability and integration.

Relational understanding is "knowing both what to do and why" (p. 153) and is considered "the ability to deduce specific rules or procedures from more general mathematical relationships, "the process of achieving stability". A student who attempts to understand relationally will try to link a new concept with other concepts he or she has developed and then reflect on the similarities and differences between the new concept and those previously understood. Thus a student who understands relationally has resources to draw from when he or she gets stuck in a problem. This understanding, though, is not always accurate because schemas are never complete. "As schemas enlarge", the awareness of possibilities expands and so "this process often becomes self-continuing and self-rewarding" (Skemp, 1987, p. 163). Keeping in mind that schemas are never complete, this informs the study because it appears normal and acceptable to elicit incorrect or inaccurate answers given the nature of the cognitive growth process.

Skemp (1987) noted that often both students and teachers operate out of an instrumental view of understanding, and that there are occasions when relational learners use instrumental approaches for the sake of efficiency. In a discussion of Skemp's framework, Reason (2003) reported that the ability to explain "why" as well as a willingness to ask "why" is characteristics of relational learners. In contrast, the inability to explain "why", a reliance on memorization, and a dependency on teacher-produced examples, served as indicators of instrumental learners.

Later, Skemp included logical or formal understanding into his model of instrumental and relational understanding which appears in Table 2.2. Logical understanding is "is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning" (Skemp, 1987, p. 166).

Table 2.2: Skemp's Revised Model of Instrumental and Relational Understanding.

Instrumental Understanding	Relational Understanding	Logical Understanding
Rules without reasons; ideas unconnected to schema.	Knowing what to do and why. Deducing specific rules or procedures from more general mathematical relationships. Schemas subject to change or expand.	Accurate syllogistic reasoning based on empirical knowledge alone.

Logical understanding allows a student to communicate mathematically and be understood by others. In this type of understanding, students are able to "connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning" (p. 166). Although a student can solve a problem correctly and can understand it, this is no guarantee that he or she could prove formally that the sequence of actions is based on a series of logical inferences used in mathematical proofs. This could be due to inexperience as learning to think logically is a behavior that can be acquired through practice and reinforcement. This model seems to be fit the current study given the nature of students' conceptual knowledge of definitions and the notation for limit.

Relational Understanding, Conceptual Structure and Schema

According to Skemp, "to understand something means to assimilate it into an appropriate schema" (1987, p. 29). By schema, Skemp means "conceptual structures stored in memory (p. 187). Schemas contain a group of connected structures or concepts, each of which has been formed by abstracting invariant properties from sensory motor input or from other concepts. The concepts are then connected by relations or transformations. According to Piaget, information from the environment gets assimilated into mental structures which form schemas in memory. These structures are ultimately used to construct additional knowledge and used to develop conceptual understanding.

Many authors have discussed understanding in the realm of mathematics (Skemp, 1971; Hiebert and Carpenter, 1992; Hiebert, Carpenter & Fennema, et al., 1997; Grunow, 1992). In particular, the work of Hiebert and Carpenter (1992) pertains to this study as it provides additional insight into the philosophies of Piaget and Skemp. They described changes in the networks as "reorganizations" that occur when representations are rearranged; new connections are formed, and old connections may be modified or abandoned" (p. 69).

> "A mathematical idea or procedure or fact is understood if it is a part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of the connections" (Hiebert and Carpenter (1992), p. 67). Therefore, "understanding is not an all or none phenomenon" (p. 69; also Carpenter & Lehrer, 2001, p. 22).

The quote is important to the results and implications of the current study because it emphasizes that understanding is not "all or none" but a process of building up conceptual structures or schemas which are never complete (Skemp, 1987, p. 163).

Schema Theory

Schema theory, which is connected to Ausubel (1968) and Bartlett (1932), emphasizes the importance of hierarchical cognitive structures centered around "anchoring ideas" as essential to meaningful learning. According to Driscoll (2000), Ausubel's notion of meaningful learning refers to the "process of relating meaningful information to what the learner already knows in a nonarbitrary and substantive way" (p. 117). Based on this definition, three conditions are essential: the learner must approach the task with a meaningful learning mindset (as opposed to a memorization mindset); the content must be relevant and potentially meaningful to the learner, and prior knowledge must be connected to new material.

Like Hiebert and Carpenter's (1992) description of new information included in existing networks, or Piaget's description of assimilation and accommodation, schema theorists believe that new ideas are incorporated into existing networks by subsumption (subordinate ideas), superordinate learning (more abstract than other ideas in the network), or combinatorial learning (new relationship between coordinate ideas). This is known as assimilation theory (Ausubel, 1968). In each case, the network is altered in some way. Processes of accretion (comparable to assimilation), tuning (comparable to minor accommodation), and restructuring (comparing to major accommodation), affect this network alteration (Driscoll, 2000).

Relational Understanding, Misconceptions and Inappropriate Schemas

An "inappropriate schema" refers to not making a correct connection to another schema whereas an "appropriate schema" refers to making a connect correction, thereby establishing a state of equilibrium (Skemp, 1987, p. 29). Given the nature of the learning process, people have both types. A subjective feeling of understanding is acquired can occur with either appropriate or inappropriate schemas by assimilation (p. 29).

An example of an inappropriate schema would be "the Greeks understood thunderstorms by assimilating thunder noise to the schema of Zeus getting angry and throwing things". An appropriate schema would "involve the idea of an electric spark". Later on, Ben Franklin assimilated concepts of thunderstorms to electrical discharges. Subsequently, knowledge of the ionization process in the atmosphere was assimilated into a more extensive schema. Therefore, several centuries before relational understanding of thunderstorms occurred. The basic initial schema had become enlarged with more information linked to the “original points of assimilation” (p.29).

According to Skemp, there are two ways to decide if a schema is appropriate or not. First, it must be appropriate to the subject matter and second, it must be “appropriate to the task at hand, to the goal to be achieved” (p. 167). In math, if someone asks “What is a differential?” The person being asked must know if the subject matter is math or motor cars before understanding what it means (p. 167). The same subject matter may be involved in other kinds of tasks, which may imply important differences within what appears to be superficial as the same subject matter.

Inappropriate schemas evolve when the mental structures contain misconceptions which Skemp uses to explain why students may think that they understand a concept when they do not (Skemp, 1987, p. 29). Suppose a student thinks that the notation $f(x)$ means $f \cdot x$. The student may believe he or she understands the notation, so this is assimilated into an inappropriate schema for multiplication. However, this assimilation could be detrimental to the student’s understanding of the concept of function.

All is not lost, though, because a student can “reconstruct” their schema if he or she encounters situations for which his or her existing schemas are not adequate. Skemp studied what happens when two conflicting concepts are aroused in the student’s mind by the same basic data, contending that students will attempt their own interpretation in an attempt to achieve stability or equilibrium. Piaget would refer to this as a perturbation or a state of disequilibrium.

Skemp notes that changing one's thinking is not an easy or a comfortable process because of the strength of existing schema. "If situations are then encountered for which they are not adequate, this stability of the schemas becomes an obstacle to adaptability" (Skemp 1987, p. 27). Subsequently, a change in the schematic structure is necessary so adaptation can occur. Piaget would have referred to this as "accommodation" since a change of thinking behavior is required. Skemp contends "reconstruction is required before the new situation can be understood" and in some cases, "it could fail and then the new experience can't be successfully interpreted". If this happens, then "adaptive behavior breaks down and the individual cannot cope" (Skemp, 1987, p. 27). This means that one's beliefs do not change to align with the truth of reality.

Relational understanding involves constructing schemas from stimuli that may be new encountered concepts. The goal is connecting the stimuli into an appropriate relational schema. In addition, one can deduce specific methods for particular problems, or specific rules for classes of tasks. The ability to do either of these is considered to be relational understanding. Meanwhile, another goal is to improve the schemas a person already has, by reflecting on them to make them better organized and more cohesive, so more effective for the goals just mentioned (p. 169).

Better internal organization of a schema can improve relational understanding, and there is no stage at which this process is complete (p. 29). Skemp thought that the only obstacle to understanding was one's belief that something is already understood fully. The importance of the schema as a tool for learning means that inappropriate schemas will make the assimilation of later ideas much more difficult, and perhaps, impossible (p.33).

Students continue to develop new schemas as they construct knowledge, by assimilating information from the environment. However, they may develop either appropriate or inappropriate schemas. Sometimes students seeking a relational understanding as part of a long term schema may form their own unique schema that might be correct or incorrect. This may be unsuitable for future accommodation because it could contain sources of conflict. In other cases, an inappropriate schema develops due to conflict arising from a genuine mathematical distinction. Therefore, a student can have still relational understanding with schemas that contain misconceptions but their ability to construct solutions to problems and explain their actions can be disastrous.

Schemas and Teaching Implications

An appropriate schema means one which takes into account the long-term learning task and not just the immediate one. A teacher must look beyond the present task, and whenever possible, communicate new ideas so appropriate long-term schemas can be formed. Teachers must look for schematic learning, not just memorizing or manipulating symbols, is taking place. They must know which stages require only straightforward assimilation and when reconstruction is needed, since at the latter stages the pace must be slower and progress more carefully checked for conceptual understanding. So they must plan on a long-term basis the kinds of schemas which will be most adaptable to both present and future needs (Skemp (1987, p.34).

Exposing students' thinking processes reveals important information about their difficulties, referred to as systematic errors, bugs, gaps or learning difficulties (Lehn, 1983; Baroody & Ginsburg, 1990). Ginsburg (1977) and Baroody & Ginsburg (1990) suggested that most mathematical errors are not simple or random mistakes. On the contrary, errors in students' conceptualizations "are all strong indicators of the existence of cognitive obstacles" (Herscovics, 1989, p. 80) and "provide a window to the learner's thinking" (Baroody & Ginsburg, 1990, p. 56). Baroody and Ginsburg (1990) describe a gap that prevents assimilation between formal instruction and a student's existing knowledge. Herscovics (1989) describes an internal conflict or "cognitive obstacle" that might appear between the existing learner's constructed knowledge and the new conceptual schemata in the accommodation of new knowledge.

Cognitive obstacles are difficulties which learners come up against in their thinking because of the complexity of the idea or topic. They are also known as epistemological obstacles because they involve difficulties in understanding an idea Zaslavsky (1997). When new information is taken in, it becomes interpreted, distorted, and elaborated upon. Then meaning is imposed on it such that information is never coded in a pure, unaltered form. When information is recalled, it is often distorted, elaborated and transformed because there is no pure, unaltered memory recall. In fact, memories of things can be constructed that never happened, and yet believed they are true.

Similarities of Piaget and Skemp

Piaget and Skemp share similar views on how knowledge is constructed and about how new information is assimilated into existing schemas (McLaughlin, 1992). Both refer to schemas building blocks of knowledge with which new, more elaborate schemas are formed. While Piaget makes reference to these learning principles applying to adult learners, Skemp makes this more explicit. As for misconceptions, Piaget's view is that perturbations occur, a process which cause a state of disequilibrium; Skemp refers to this process as "stability as an obstacle to adaptability". Piaget's term "accommodation" is similar to Skemp's term "restructuring", involving a change in behavior or actions. Accommodation as a growth process seems to coincide with what Skemp defines as relational understanding.

Piaget and Inhelder's Six Principles of Learning can essentially be condensed into two main categories: Adaptation and Organization. Skemp refers to adaptation occurring when a good schema and new knowledge are assimilated, whereas Piaget refers to adaptation as involving assimilation and accommodation. As for organization, Skemp refers to this as a process of "restructuring" which is synonymous with Piaget's process of "accommodation"; however, Piaget refers to organization as "the nature of adaptive mental structures" in which case the "mind organizes mental structures into schemas".

Piaget primarily studied children whereas Skemp studied adult learners. According to Piaget, intellectual development occurs in stages, in each of which the child thinks and behaves in a quite different fashion than earlier. This can also be conceptualized from Skemp's viewpoint with adult learners attempting to study a new complex task for the first time. Piaget maintains that the child grows intellectually not like a leaf, which simply gets larger every day, but like a caterpillar that is eventually transformed into a butterfly. This can be true for adult learners as well, because adult learners can either be in the stage of concrete or formal operations, depending on their ability as some adults do not make it to the stage of formal operations.

Within each level of development, a learner can understand and profit from only such experiences and pieces of information as a match to what they already know, or that are just a little in advance of their existing information and skills. Similarly, learners seek stimuli that are complex enough to be intriguing, but not so complex as to overwhelm and be baffling. Learners have an innate curiosity, but only to the point where curiosity is rewarding rather than frustrating and sometimes knowledge is not received to the learner as intended by the teacher. As a result, altered conceptions can become reality. According to Skemp, altered conceptions (p. 29) can help explain why different students from the same classroom can exhibit a variety of unique written and articulated actions during problem solving. Piaget would contend the information is assimilated—incorporating a new stimulus into the existing cognitive view to meet new situations; however, he does not stipulate whether ideas must meet new situations correctly. Skemp, though, would contend that if the idea does not meet a new situation correction, then an altered conception occurs. In both cases, by believing what one thinks they know to be true, inappropriate schemas or misconceptions in learning can occur.

In the case of learning limits, if the student internalizes at least an intuitive definition that the limit can “get near, approach and at times equal”, this could lead to a correct cognitive schematic representation. On the other hand, if a student has a vague notion of a limit as only “approaching” then this can result in an altered conception or incorrect schematic representation. In both cases, whether or not the conceptions were correct, relational understanding would occur because these constructed understanding are based on the student’s beliefs about reality, whether correct or incorrect.

Moreover, if the student memorizes a definition correctly but applies it to a problem incorrectly, this could suggest that the definition was assimilated into mental structures and into an altered schema. The student knows the definition but cannot do a problem, struggles to start the problem and gets it wrong. This would be a case in which the student is in a state of disequilibrium (Piaget) or stability “(Skemp). In this case, when the student is stuck just trying to begin solving a problem, they neither have instrumental nor relational understanding.

There is always tension, Piaget concluded, between assimilation, which in essence represents the use of old ideas to meet new situations, and accommodation which in essence is a change of old ideas to meet new situations. It is this resolution of this tension or disequilibrium that results in intellectual growth or relational understanding (Skemp), and growth takes places in a series of stages. Therefore, I hypothesize that these stages can also occur with adult learners from the time a novel idea is introduced and assimilated to the time it becomes accommodated.

Summary

Constructivism is the guiding theoretical framework of this study, emphasizing a focus on the learner’s internal mathematical actions. Having evolved from work by Piaget on genetic epistemology, constructivism is still a major influence in math education research. The focus is on how the learner builds or constructs new knowledge upon previously learned ideas and makes sense of the questions and challenges that emerge from actively engaging in math tasks. Piaget and Inhelder’s principles inform the study by providing a framework for learning whereas Skemp’s model provides a framework for understanding, both of which are used to explain different conceptual structures students possess about limits in calculus.

CHAPTER 3: LITERATURE REVIEW

The review of literature is partitioned into several sections. First, there is a brief historical development of the limit concept. Next the research is presented on limits and on functions, including piecewise and rational functions. Afterwards, research on domains and the importance of domains is presented, as well as various misconceptions and obstacles to learning limits. When appropriate, there is reference to how previous research informs the current study.

Historical Development of the Limit Concept

Over the years, mathematics educators found the concept of limit to be a difficult notion for the student to understand and for the teacher to teach. The historical development of the concept of limit provides insight into why students have difficulty comprehending the idea of limit (Boyer, 1968).

Many of Zeno's paradoxes relate to limits or to what are called completed infinities and based on this and the awareness of incommensurables, arguments stemming from infinity of infinitesimals were not acceptable (Boyer, 1968). Zeno's paradox depicted the impossibility of intuitively conceiving the limit of the sum of an infinite series (Boyer, 1969). While the paradoxes relate to issues of limits and the nature of real numbers, they can enhance students' introduction to infinite series. Unable to explain the infinite process, the Greeks tried to find ways of avoiding infinitesimals and restricted themselves to geometric methods. Archimedes, a Greek mathematician, surmounted this difficulty with his "method of exhaustions" which relied on indirect argument and purely finite constructions (Davis & Hersh, 1981). The “method of infinitesimals’ is another method for dealing with geometric problems. It’s seen in work by Kepler and Cavalieri, who used the methods and results of Archimedes, using intuitive infinitesimal reasoning to a variety of problems, such as the cone which is essentially a pyramid whose base is a regular polygon of infinitely many sides (Boyer, 1968).

Issac Newton and Gottfried Liebniz are the founding fathers given credit for the invention of calculus and application to outstanding problems in mathematics due to the fact that they both worked on the topic of limits with an intuitive approach during the same time period (Boyer, 1968). Newton is considered to be the first to come up with basic ideas such as the product and chain rules, Taylor series and higher derivatives, applying calculus to general physics, while Leibniz developed much of the notation used in calculus today and was the first to publish Newton’s results. Leibnitz put Newton’s ideas into practice and developed a clear set of rules for working with infinitesimal quantities. Leibnitz also devoted a lot of time to formalizing concepts, determining appropriate symbols.

Liebniz based his work on infinitesimals, contending it was useful to consider quantities infinitely small such that when their ratio is sought, it will not be considered zero. Leibniz made these infinitesimal quantities the basic concepts in his differentials (Graves, 1910). The “method of limits’ is attributed to Newton, who presented a geometric interpretation of the ratios dy/dt and dx/dt as the slope of both the secant line through two points of a curve and of the tangent line that it approaches as the two points coincide. “Newton’s attempted definition of the concept of limit: Quantities, and the ratios of quantities, which in any finite time converge continually to equally, and before the end of that time approach nearer to each than by any given difference, become ultimately equal" (Edwards, 1979, p.225). It is from this idea of limit that differential calculus begins, and from which it later is more formally developed. To summarize, Newton dealt with a finite quantity that is the ratio of two infinitely small quantities, the ratio of velocities whereas Leibniz dealt with the sum of an infinite number of infinitely small quantities that has a finite value (Schrader, 1962).

Formal proofs came with later mathematicians, primarily Augustin Louis Cauchy, who formalized the theory of limits with a rigorous definition, and used it as the basis for rigorous definitions of continuity and convergence, the derivative and the integral. Cauchy's focus was to reconcile the need for rigor with the simplicity that results from the direct consideration of infinitely small quantities, and defined infinitesimals to be a function that tends to zero (Tall, 1981).

Karl Weierstrauss restated Cauchy's original definition of the limit in strict arithmetical terms, using only absolute values and inequalities, giving the epsilon-delta definition currently used today in some analysis courses. Weierstrass is credited for having made limits static and independent of time, and for having provided the formal epsilon-delta definition which states: $\forall \varepsilon > 0\ \exists \delta > 0 \ni 0 < |x-a| < \delta \Rightarrow |f(x)-L| < \varepsilon$. To interpret the notation, the definition states that a number L is the limit of a function f(x) at $x = x_0$ if, given an arbitrary small number epsilon ε, another number delta δ can be found such that for all values of x differing from x_0 by less than δ but greater than 0, then the values of f(x) will differ from L by less than ε.

During the last century, the epsilon-delta definition has led to fruitful advances in analysis and to a clarification of the meaning of certain concepts for the professional mathematician. It is occasionally still taught and used in some analysis courses, but for the beginning student in calculus, it proved to be too complex and intractable (Davis & Vinner, 1986). Therefore, the formal definition is seldom taught in undergraduate calculus courses. Instead, instructors use an intuitive version that emphasizes "nearness" and "approaching", but still contains elements of complexity and vagueness (Tall, 1981; Juter, 2005).

Over a period of time, many definitions of limit have been cited which could be directly related to some of the confusion students have today. First are presented definitions from the original founders of calculus. According to Newton, the definition of limit was "Quantities, and the ratios of quantities, which in any finite time converge continually to equally, and before the end of that time approach nearer to each than by any given difference, become ultimately equal" (Edwards, 1979, p.225). Leibnitz defined limit as "by infinitely small, we understanding something….indefinitely small, so that each conducts itself as a sort of class, and not merely as the last thing of a class" (Dunham, 2008, p. 24). Euler defined limit as "any quantity can be diminished until it all but vanishes and then goes to nothing, but a vanishing quantity, and so it is really equal to zero" (Dunham, 2008, p. 53). Cauchy defined limit as "when the values successively attributed to a particular variable approach indefinitely a fixed value so as to differ from it by as little as one wishes, this latter value is called the limit as the others" (Dunham, 2008, p 77). Finally, Weierstrauss defined limit as "$\lim_{x \to a} = L$ if and only if, for every $\varepsilon > 0$ there exists a $\delta > 0$ so that, if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$" (Dunham, 2008, p. 130).

With all due respect to these predecessor's definitions, the abstractness and ambiguity also seems to exist among various calculus textbooks used in high school and colleges (Doyle, 1996; Vinner, 1991). The informal definition is defined as follows in Thomas' Calculus by Finney, Weir & Gordano (2001, p. 89): "Let f(x) be defined on an open interval about $x_{0,}$ except the possibility at x_0 itself. If f(x) gets arbitrarily close to L, for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0 and we write $\lim_{x \to x_0} f(x) = L$. This definition is informal because phrase like 'arbitrarily close' and 'sufficiently close' are imprecise; their meaning depends on its context". A few pages later, the formal definition of limit occurs, "To show that the limit….equals the number L, we need to show that the gap between f(x) and L can be made "as small as we choose' if x is kept 'close enough' to x_0 (p. 91). In another textbook, the first chapter begins with Section 1.2 on "calculus without limits" then once limits appears in Section 2.6, the author states "if x is close to a, then f(x) is close to L….When Plato can find a delta for every epsilon, Socrates concedes that the limit is L" (Strang, 1991, p. 82). Still, it is not known what exactly "L" refers to but another hint appears with the statement, "the correct limit L comes by substituting x=a into the function" (Strang, 1991, p. 83). According to Stewart (2005, p. 99), a definition in a red box is given: "We write $\lim_{x \to a} f(x) = L$ and say the limit of f(x), as x approaches a, equals L if we can make the values of f(x) arbitrarily close to L…by taking x to be sufficiently close to a but not equal to a". In each of these examples, none of the authors tell the reader what "L" is, leaving it up to interpretation and thereby reinforcing the mystery. Hence, this appears to set the stage for the modern day struggles that students have with learning limits. Understanding the definition of limit is one aspect that informs this study.

Research on Limits

Williams (2001) describes how much of the work on understanding limits can be classified into three major areas. The first involves the work of Tall and Vinner (1981) and Vinner (1983), who distinguishes between concept image which is the "total cognitive structure associated with the concept, including all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152) and concept definition which is the "formal definition as understood and accepted in the mathematical community". Tall & Vinner (1981) also found that students think limits involve motion and that 54 out of 70 students used dynamic approaches to solve limit problems. These students think the function value is never equal to the limit because a function approaches, but never reaches, its limiting value. As a result, the researchers found that students hold preconceived notions of the terms and definitions used when learning limits and assume students hold underlying mental models of cognitive notions of the limit concept that exist prior to formal instruction in the classroom. Confrey (1980) also contends that students view whether or not a limit is attainable is related to their previous knowledge, including their interpretations of division by zero, asymptotes, geometric series and lines as sets of points. Implicit in these studies is the belief that prior knowledge of limits influences construction of the concept.

The second area focuses on obstacles to learning the limit concept, which is subcategorized into psychological, didactical, epistemological obstacles, as well as obstacles due to weaknesses in mathematics' content essential to the construction of the limit concept. This area focuses on understanding why the coordination of concept image and concept definition can be so difficult.

As the third area, Williams (2001) identifies is the work from a Piagetian tradition, exemplified by work of Dubinsky (1991, 1994) and Cottrill et al., 1996) on reflective abstraction. Growing from these three main areas of work is a fairly consistent set of standard conceptions, or mental models, that students seem to have about limits. Some of the relevant studies pertaining to these three areas will be subsequently discussed in this literature review.

The concept of limit is among the more important topics in mathematics when calculus is introduced but stands to be the most difficult even for very good students, who never become aware of the importance of limits when constructing their knowledge about derivatives and integrals (Orton, 1983; Francis, 1992). Allendoefer (1963) claims people don't identify the limit as being central to calculus and so without a clear exposition of limits, a calculus course is a failure. The research presented confirms the difficulties that students encounter, sheds light on why understanding limiting behavior is important throughout the calculus course sequence, and offers implications for improving instructional practices.

Jackson (1916) suggests that a student need only have a clear understanding of the concepts of variables and functions to understand limits, believing that once the idea of functional dependence is established, learning limits is a simple undertaking. Jackson contends that graphical methods should be investigated when teaching limits, which has been done over the years. These results are useful to the study with respect to knowing that functional dependence and algebra skills are essential to computing limits. As a result of Jackson's contentions, graphs of functions are presented to students in this study.

Dosemagen (2004) examined the mathematical understanding of AP Calculus Students using Skemp's model. Methods were explored, to improve classroom instruction and to help students develop deep understanding of mathematical concepts rather than simply manipulate formulas or expressions. The study was based on the notion that students can accurately complete mathematical tasks but without knowing why they did what they did. They learned to connect a type of problem with a type of process, without understanding the connection between them. This is an example of what Skemp refers to as instrumental understanding. Dosemagen’s study involved action research using individual interviews to explore perceptions of students own conceptual understanding. Interventions were then developed to help students acquire a more relational understanding of the material, but the results did not turn out as expected. This study is important because it incorporates Skemp’s model and replicates former findings about various misconceptions with conceptual understanding (Davis & Vinner, 1986; Tall, 1992b; Juter, 2006). Although the study did not specifically report how students think about limits, it described teaching interventions that can potentially move students from instrumental to relational understanding.

Several researchers have explored students’ cognitive notions of limits held prior to the introduction of formal mathematics instruction (Tall & Vinner, 1981; Confrey, 1980; Juter, 2006). Tall and Vinner (1981) use the terms “concept image and concept definition” and proposed this cognitive model to explain how students learn mathematical concepts and why they have difficulty distinguishing between limits and continuity. A concept image is a “total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (p. 152). A person's concept image includes all personal conscious and unconscious mental pictures and attributes of the given concept. The word "limit" might bring about images of something that cannot be crossed; therefore, a function may never go beyond the limit at a certain point. These studies inform the current one, revealing the importance of prior knowledge and insight as to why students confuse limits with continuity.

Tall and Vinner (1981) conducted interviews with college students to decide how they develop the concept of limit. A goal of their study was to determine how concept images of limits and continuity lead to the concept definition, and what potential conflicts may occur in this development. In this dynamic view of limit, the value of the function is moving towards but is never equal to the limit, which is a potential conflict with the concept definition. Students expressed the belief that a false statement was true based on the idea that a function approaches, but never reaches, its limiting value. These results are important to my study to learn more about why discontinuities (lines with holes) mean to some students that the limit does not exist as well as considering how the limit concept might be introduced in the classroom.

Beliefs about mathematics are related to learning the limit concept. Williams (1991) studied college students' understanding of limits and factors that affect changes in understanding. One of the factors in acquiring the concept of limit was found to be students' beliefs about mathematics. Richgels (1992) also studied beliefs about mathematics in high school calculus students. The students constructed a definition of limit, and then applied that definition to solve elementary limit problems and prove theorems. Both studies show that certain beliefs about mathematics can be obstacles and offers insight into the notion of believing that in the case of the discontinuity with the hole, the limit does not exist because nothing is there. Their studies suggest that belief systems can interfere with learning limits which also informs the current study.

Students taking calculus courses often experience difficulty with learning limits at a point and limits at infinity (Cornu, 1991; Davis & Vinner, 1986; Tall, 1981; Tall, 1992a; Vinner, 1991). Juter (2005) did extensive research on what students know and understand about limits and functions and found most students' foundations were not sufficiently strong for them to understand the concept of limit well enough to be able to form coherent concept images, and there was confusion about different features of the limit concept. These studies revealed consistent patterns of misconceptions about limits at a point and limits at infinity that constitute obstacles to conceptual understanding. (Tall & Vinner, 1981; Tall, 1991; Juter, 2005; Juter, 2006).

Juter (2005) studied how students learn limits of functions during their first semester of mathematics, and examined how they reason and show inconsistencies in their representations. Results show that students consider mathematics to be a series of facts and processes to remember, and that mathematics is about solving problems and coming up with new ideas. Most students can synthesize ideas they learned but must work very hard to understand mathematics. Limits of functions are considered the most important topic in the analysis course, but also the most difficult to understand since students must work hard to integrate new knowledge to existing concept images. This validates Tall and Vinner's (1981) claims about the importance of prior knowledge. Juter (2005) found justification for working through limits thoroughly and for using various problems at different levels to help students understand the concept's features. This study was important because it shows how students can make strong and meaningful mental connections to other concepts.

Juter (2006) later found that students have trouble when given limit tasks that were slightly different than what they were accustomed to seeing. Students were presented with limits of functions and were asked to examine the functions for attainability. Some students claimed that the function could not attain the value even though it obviously could. This informs my study knowing that some students may either confuse limits with continuity, and may not think that a limit can exist a point on a continuous function, because it can only “approach”.

Another important finding from this study is that an insufficient mathematical base to work from can result in constraints on the individual, in that he or she is not sure what operations are allowed and how to carry them out, thereby explain the reason for wrong or non-existent answers. These are the deficiencies in algebra skills that were also reported earlier by Cornu (1991).

Infinity is another source of confusion (Tall, 1980). Some students reason about “local limits” for the functions $\frac{1}{x}$ *and* $\frac{2}{x}$ separately or decompose the function in other ways and “consider the limits locally”. Even though students can usually decompose the function, they appear to lack “parts in the mental web that represents this fraction” in terms a concept image (Tall & Vinner, 1981) as they do not have access to the essential information about the properties of the limit process and functions. According to Hiebert & Lefevre (1986), the reason is because the development of conceptual knowledge has not occurred. In Skemp’s model, the student would only have instrumental, not relational understanding.

In a study on student's understanding of integration, Orton (1983) contended that "students had little conception of the power of limiting processes in mathematics" (p. 7), and stated that few students realized that the limit of a sequence of successive approximations to the area under a curve would give the correct area. Bratina (1983) developed a meaning of the understanding of the limit of a sequence based upon students' behavior and concluded students do not have a good formal understanding of limits, though they had low levels of understanding. These studies confirm other studies that have shown the lack of conceptual knowledge students have about limits.

Graham & Ferrini-Mundi (1989) contend that students are successful at evaluating limits of the form $\lim_{x \to a} f(x)$ where the function is continuous at "a". When given a simple limit problem in this format along with the graph, most students solved the problem by plugging a number into a formula and evaluating the limit, but showed very little geometric understanding. Their results confirm previous findings, that students confuse limits with continuity (Tall, 1981; Davis & Vinner, 1986). This informs my study by offering insight into problem solving of continuous and non-continuous functions.

Mamona-Downs (1990) compared Greek and English students with their interpretations of the limit concept. The English students used more intuitive, infinitesimal reasoning based on the Leibniz-Cauchy Model, while Greek students had the facility to use the Weierstrauss Model involving formal definitions and standard procedures. This research informs the current investigation and implications of the current study, as it shows that there is more than one way to teach and learn about limits. Emphasizing the intuitive version of limits involving infinitesimals has different outcomes than more modern day standard procedures involving the formal definition.

Smith (1959) explored factors that affect proficiency in conceptualizing the limit and found that experience is a major factor. Piaget and Inhelder (1967), in their continuity experiment, examined the child's ability to conceptualize the infinite subdivision of certain geometric figures, including the line segment, square, circle and triangle, to arrive at a point. The children were also asked to insert as many points as possible between the two points, with the aim of finding out whether they think they will eventually form a line. Though their experiment embodies a notion of convergence, Piaget & Inhelder neither designed nor evaluated their research in terms of the child's understanding of a line segment. They concluded, though, that not until they reach the formal operational stage of development around the age of 11 or 12 do children see that subdivision is an infinite process. Therefore, Piaget and Inhelder provide mathematics educators with some introductory knowledge of the child's understanding of the limit concept.

Taback (1975), guided by the work of Piaget who reported that students' understanding of mathematics is dependent on their cognitive development which is incremental in nature, studied the concept of limit in children ages 8, 10 and 12 years old. He provided tasks where they had to recognize certain concepts that led to an overall understanding of limit. The concepts he considered are: functional rule of correspondence, neighborhood, limit point and convergence. He found that only about 50% of 12 year olds could conceptualize convergence, the infinite process and the limit point. Hence, Piaget's claim of developmental stages of understanding was confirmed by the work of Taback. Research from the Piagetian perspective informs this study using genetic epistemology to explain how constructing knowledge is contingent upon principles of learning and stages of cognitive development.

Limits of Rational Functions

Students experience difficulty with determining limits of rational functions. For example, when exploring the limiting behavior of the rational function $f(x)=\frac{1}{x}$ there are two main results. First, for very large x (as x approaches infinity), the function values tend to 0 but instead, many students think the function goes to infinity. Second, when x is very small and approaches zero, the function values increase without bound. Understanding how to work with rational functions requires certain skills and mastery with algebra, number sense, reciprocal relationships and mathematical reasoning, as rational functions are often being a student's first encounter with the concept of discontinuities. In order to understand why a rational function is discontinuous, a student needs to utilize knowledge of division by zero (Hornsby & Cole, 1986). Discontinuities appear again with limits of functions and so the study by Hornsby & Cole is useful as it appears to be the only one in the literature on rational functions and pertinent to the current investigation.

Juter (2006) reported that students did not consider what x tends to, and suggested that rational functions require reasoning rather than an algorithm. Algebra is the reason for a number of mistakes with rational functions, and many of the errors are serious. These errors reveal a lack of knowledge in basic calculation rules that should not exist at the university level yet mistakes with algebra and other numerical computations are quite common. Students have to be aware of the problems before they feel a need to alter anything they are trying and if they do not discover their errors, then no progress is made (Juter, 2006). This researcher suggested that students need to experience a larger variety of problems to better understand the rules and properties of mathematics. Students should be able to see the basic foundations of the mathematics they work with and recognize characteristic features. Juter contends such abilities can be developed if teachers provide an environment allowing students to collaborate and discuss mathematical issues. This study told me that students need good algebra skills when thinking about limits and that without them, they will have difficulty.

Functions

Functions are foundational throughout the secondary mathematics curriculum (NCTM, 2000), which provide an excellent opportunity for students to study various relationships that are central to the study of mathematics and related areas of the sciences. Yet, even though the function concept is considered to be one of the most important notions in mathematics, it is difficult for students to understand and has been the subject of extensive research in mathematics education (Vinner, 1983; Dreyfus & Eisenberg, 1983; Markovits, Eylon & Bruckheimer, 1986; Vinner & Dreyfus, 1989; Adams, 1997; Carlson, 1997; Janvier, 1998; Even, 1998; Hitt, 1998; Clement, 2001). In fact, research indicates that college and high school students do not have a well developed understanding of the function concept (Vinner & Dreyfus, 1989; Mousoulides & Gagatsis, 2004).

A strong understanding of functions is essential for students hoping to understand calculus, which is a critical course for the development of future scientists, engineers and mathematicians (Oehrtman et al., 2008). Even understanding functions in terms of input and output can be a major challenge for many students. As an example, 43% of A-students at the completing college algebra attempted to find f(x+a) by adding "a" onto the end of the expression for "f" rather than substituting x+a into the function (Carlson, 1998). When probed to explain their thinking, they typically provided some memorized rule or procedure to support their answers, and were not thinking of x+a as a value of the function's argument at which the function is being evaluated. Another misconception is thinking that constant functions, e.g. y=5, are not functions because they do not vary. According to Rasmussen (2000), not viewing y=5 as an example of a function can become problematic for students, such as when considering equilibrium solution functions for differential equations such as $\frac{dy}{dt} = 2y(y-5)$. Understanding more advanced mathematics contingent upon understanding functions.

Earlier in this report, I mentioned the work of Jackson (1916) who suggested investigating how to teach limits with a graphical approach. As time went on, this idea was explored. For instance, Markovits et al. (1986) contends that students have more facility with understanding functions when represented graphically rather than algebraically. Dreyfus and Eisenberg, 1981 report that high ability students prefer using the graphical representation, while low-ability students prefer a formula or "tabular" representation of the function. Their claim might or might not be true, depending on the diversity of teaching strategies employed and various grade levels. These days, there are many accommodations for diverse learners and many new teaching strategies are in place, as students bring to the classroom a multitude of learning and thinking styles (Wood & Blanton, 2009, pp. 144-169). Nevertheless, these results have important implications with presenting limits and inform the current study given that high ability students in advanced calculus are recruited and since graphs of functions are presented at the beginning of most tasks. The research cited here gives justification for capitalizing on the graphical approach.

Xiaobao (2006) explored why students have difficulty learning functions. The causes of errors were explored by comparing high achieving and low achieving students' understanding of concepts at the object (structural) or process (operational) levels. High achieving students were found to prefer using object thinking to solve problems even if the problems could be solved through both algebra and arithmetic approaches. The relationship between students' misconceptions and object process thinking explained why some misconceptions were hard to change. The misconception of equal sign was misunderstanding that either side of the equation is a process rather than an object. This study informs the current one by revealing some misconceptions students have with functions.

Rational Functions

Rational functions are important and are of interest to this study because student problem solving actions reveal significant knowledge about domains. Few researchers have done work specifically on rational functions (Krakowski, 2000), though it constitutes a significant element of the curriculum in undergraduate mathematics (Hornsby & Cole, 1986). In fact, the topic of rational functions was not part of the American high school mathematics curriculum until the early 1980's (Hornsby & Cole, 1986). Such lack of experience in working with rational functions was reflected in the research results that focused on learners' concept image of function. Even (1993) found one common misconception among prospective secondary math teachers is that all functions are continuous. These studies inform the current one by revealing the difficulties students have with understanding rational functions, as well as some questionable teaching skills with respect to appropriate content knowledge.

Algebra concepts of domains and zeros are important because they are significant prerequisites for learning elementary calculus, at which time students are expected to apply their knowledge about the domain and zero(s) of rational functions in various situations. Typical problems would include locating a removable discontinuity, determining the limit, and/or finding critical points when working with rational functions.

Piecewise Functions

According to Breidenbach et al. (1992), many students believe that all functions should be definable by a single algebraic formula, a focus which hinders flexible thinking about function situations and can lead to erroneous conclusions such as thinking that all functions must behave "nicely" in some sense. For instance, many students argue that a piecewise defined function like $f(x)=\begin{cases}0, & \text{if } x \le 0\\ e^{\frac{-1}{x^2}}, & \text{if } x>0\end{cases}$ is actually two separate functions or that a function such as the Dirichlet Function $g(x)=\begin{cases}1, & \text{if x is rational}\\ 0, & \text{if x is irrational}\end{cases}$ is not even a function at all because it "behaves badly".

Many students struggle with graphing piecewise functions in spite of having the necessary prerequisite skills, such as the ability to graph individual functions from equations and apply the vertical line test for functions (Chazan & Yerushlamy, 2003; Markovits et al., 1986). Graphing and reasoning about piecewise functions is important because analysis of discontinuous functions as well as the topic of one and two-sided limits in calculus cannot be appreciated without knowing about piecewise functions. Also, a strong understanding of piecewise functions is important because it leads to a deeper understanding of domain that more regular functions cannot offer (Oehrtman, Carlson & Thompson, 2008). According to Bell and Janvier (1981), students focus more on individual points than on functions or graphs in a more global way, such as by considering the domain. Markovits et al. (1986) found that translations of piecewise and most functions were easier for going from equations to graph than from graph to equation. This informs my study as the former are referred to as "traditional tasks" and the latter, "non-traditional tasks". Moreover, Markovits et al. found that students showed a lack of attention to the domain and range restrictions when asked to decide of a discrete function was in fact a function and whether the points should be connected. Hohensee (2006) found that students were overwhelmed when piecewise functions contained multiple domains relied more on recognizing functions than they did on definitions.

Research on Understanding Domains

Hohensee (2006) conducted a two-phase study on high school student difficulties with understanding domains which involved graphing and piecewise functions. The goal was to gain a better understanding of how students think about a domain is related to its range and to explore how students think about functions with multiple domains. Results revealed that students relied on the recognition of functions more than they did on a definition of function, and students did not consider the domain when they thought about or graphed functions. This informs the current study as students were asked to write the domains for all functions.

Markovits et al. (1986) found that translations of most functions were easier for students going from equations to graphs than from graphs to equations, and also found that students show a lack of attention to the domain and range restrictions when asked to decide if a discrete function was in fact a function, and whether the points of a discrete function should be connected. Hohensee (2006) found this lack of attention to be compounded in graphing piecewise functions. Moreover, Hohensee contends that the way domain is normally taught, where regular graphs can be graphed without consulting the domain and often the domain being determined as a post-graphing exercise, leads students to view domain as a superficial descriptor. As a result, Hohensee altered the way he taught functions in Algebra 2, exposing students to piecewise functions sooner and more frequently so that piecewise functions would be more familiar and that the domain would be perceived as a vital component of a function. In another study by Bell and Janvier (1981), students focused more on individual points than on functions or graphs in a more global way, such as by considering the domains. These studies inform the current one given students seem to overlook the importance of domains and also justifies the use of the traditional and non-traditional problem-solving tasks used in this study.

Importance of Domains

In Chapter 1, an example which illustrates the importance of the domain using Hooke's Law was provided. There are additional mathematical implications regarding knowledge of the domain (Taylor, 2005). Brook Taylor (1715) developed a useful tool, the Taylor series, in part to help solve differential equations (Diefenderfer & Nelsen, 2010) and showed that from the properties of finite differences, one could write as f(x+h) in terms of f(x) and its quotients of differences of various orders. By letting the differences get smaller, they ultimately passed the limit. The importance of this property of derivatives was soon recognized by Colin Maclaurin, Leonhard Euler and Joseph-Louis Lagrange. In their hands, the Taylor series became a powerful tool for studying functions and in approximating the solutions of equations (Diefenderfer & Nelsen, 2010, p.100). The Taylor series is one example in mathematics where limiting behavior occurs as one continues adding terms to get better approximations.

Hooke's Law pertains to displacements within the elastic limit of the spring, a law which changes once the elastic limit is exceeded. The term "elasticity" refers to when an elastic body is distorted the restoring force is proportional to the displacement. The elastic limit determines the right-hand endpoint of the domain. Within the specified interval of the finite domain, e.g. [0,8], the spring function behaves one particular way, linear. Outside of this interval where the spring could snap, the domain changes due to the elastic limit of the spring having been exceeded and so the force is no longer a linear function of displacement (Cutnell & Johnson, 2004, p. 268). In this situation, more terms in a Taylor series are added to obtain better approximations of the system outside of the domain.

A general relationship in physics is that the mechanical force exerted by a spring is the negative of the derivative of the potential energy as a function of the spring's position. As with most functions, the potential energy can be expanded in a Taylor series (Taylor, 2005, p. 162), whereby $P(x) = P(0) + P'(0)x + \frac{1}{2}P''(0)x^2 + \frac{1}{3!}P'''(0)x^3 + \ ...$ Since the force of the spring is the derivative and the derivative of a constant is 0, the constant term is $P(0) = 0$. One measures the potential from the equilibrium position which is the minimum for the potential, hence $P'(0) = 0$ which is the case with any minimum. With these values of the coefficients in the first two terms, the potential's Taylor Series is as follows: $P(x) = \frac{1}{2}P''(0)x^2 + \frac{1}{3!}P'''(0)x^3 + \ ...$. The derivative would be the force and becomes, $F(x) = -P'(x) = -P''(0)x - \frac{1}{2}P'''(0)x^2 - \ ...)$. The constant k in Hooke's law is $P''(0)$. The general theory of Taylor series is that kx is a good approximation for small x. The domain would be the largest set of x's for which the approximation is good enough for the application at hand. In order to get a better approximation, then the next term in the Taylor series is taken. So now the force law is approximately: $F(x) = P'(x) = -kx - \frac{1}{2}P'''(0)x^2$ in which case the domain would be larger than with the linear law. In order to get an even better approximation, the cubic term would be taken next, and this pattern would continue.

These approximations from the Taylor series could be used to construct a piecewise function to describe the spring's behavior. On the first part of domain, one would use the linear term only up to the point that the approximation is not good enough for the problem. Then on the next piece of the domain, the quadratic formula above would be used. If needed, one would then have to include another piece of the domain with the cubic term added, and so on. Knowing this might be useful, for instance, if the computation was to be done in real time on a small embedded processor. In this case, the computational power is very limited so if the linear term gives an appropriate approximation, one need not compute the quadratic or higher degree term; however, the higher degree terms would be needed at the extreme of the domain for the required accuracy. The Taylor series represents a form of limiting behavior and this example shows why knowledge about the domains of functions is important in real-world applications, as well as shows how mathematics and physics can interact.

Misconceptions

Students possess a variety of misconceptions and beliefs that range from continuity issues to conflicts stemming from the representations of functions (DeMarois, 1997; Doyle, 1986; Dreyfus & Eisenberg, 1982; Vinner, 1983). A student's misconceptions about the function concept may occur for several reasons. Students may not understand the formal definition of a function. Lack of understanding a definition may lead to conflicts between students' images and their concept definition (DeMarois, 1996; Vinner, 1991; Vinner & Dreyfus, 1989). Early in their high school mathematics courses, students are typically introduced to a formal definition of function: "a correspondence between two non-empty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the co-domain)" (Vinner & Dreyfus, 1989, p. 357). This definition is not unique. A survey of mathematics textbooks revealed many different definitions for the function concept, as well as for the limit concept. In both algebra and calculus courses, the definitions are noted, and the instructor moves on in the instruction to help the students create and build concept images. Moreover, it was found that students do not refer to the formal definition of a concept when presented with an unfamiliar function (Doyle, 1996; Slavit, 1997; Vinner, 1983; Vinner, 1991). As a result, they tend to rely on their concept images for a function when answering questions and solving problems. This informs the current study by giving rationale to investigate how students use definitions when solving various limit problems.

Obstacles to Learning Functions and Limits

Several researchers have looked at obstacles to learning functions and limits (Sierpinska, 1987; Cornu, 1991; Davis & Vinner, 1986). Learning difficulties have been shown to be associated with cognitive obstacles (Brousseau, 1983). Various obstacles can result from students' own psychological or social development; from inadequate or misleading though usually well-intended instruction; or from the nature of the concepts themselves (Cornu, 1991). These later obstacles have been called epistemological obstacles by Bachelard (1938) and discussed in the context of pedagogical theory by Brousseau (1983) who describes them as well-established pieces of knowledge that are useful in one arena of activity but not in another and hence, stands in the way of proper functioning in the second arena.

According to Williams (2001), epistemological obstacles are inherent in the limit concept itself, which can be examined from a historical perspective as well as by studying the seemingly unavoidable challenges of constructing an accurate conception of the limit concept. Williams (2001) also contends that external psychological and didactical obstacles interfere with constructing the limit concept as well as the influence of the student's own mathematical weaknesses in areas essential to understanding the limit concept.

Cornu (1991) identified three types of obstacles to learning limits: "genetic and psychological obstacles" which occur naturally as a result of a person's growth; "didactical obstacles" which result from a student's interaction with the educational environment; and "epistemological obstacles" which occur due to the nature of mathematical concepts themselves. Cornu (1991) also identified mathematical weakness due to algebra deficits as another obstacle. Davis and Vinner (1986) also contend that students' prior informal experiences with concepts that are formally introduced in a calculus class, such as limits, may conflict with construction of new knowledge. Ervynck (1981) found some students possess a dynamic notion of functions which leads to dynamic notions with the limit concept. Davis and Vinner (1986) contend that some students think a sequence may possess a last term, so the limit of a sequence is the last term of the sequence. Incomplete or incorrect ideas about infinity may lead to the idea that limits may never be reached (Sierpinska, 1987; Williams, 2001). These studies are important to the current study as they reveal many reasons why students have difficulty learning limits.

Sierpinska (1987) analyzed the epistemological obstacles related to limits as evidenced in discussions held with a group of 17-year-old humanities students in Poland. Epistemological attitudes including attitudes toward infinity were identified in students regarding mathematical knowledge and eight specific models of limit were discussed. Using these models as a framework, she described the actions and statements of two students as they struggled to present and defend their ideas in the classroom. It was found that students' attitudes towards mathematical knowledge are a major factor inhibiting the altering of misconceptions about limits, and that students who viewed mathematics as "a piece of objective, impersonal knowledge which we should know and not have options on" (p. 136) are largely unaffected by counterexamples or arguments of a mathematical nature.

Not all obstacles to learning functions are simple errors and some are hard to accommodate because of the fixedness of earlier thinking (Sierpinska, 1987). For example: having multiple names for the same things, like "solutions, roots, and zero's" in polynomials. Cognitive conflict is demonstrated in students who believe that 0.333…. is equal to the fraction $\frac{1}{3}$ but that 0.999… is less than one or is approximately equal to 1. With counting infinite sets, students may not realize that there are as many even integers as there are integers in total. Other erroneous ideas that can occur include the following; adding makes bigger; subtracting makes smaller; multiplying makes bigger and dividing makes smaller; or thinking that the larger the 'arms' the bigger the angle.

Erroneous ideas such as universal linearity, e.g. $(a + b)^2 = a^2 + b^2$ or $sin(A + B) = sin(A) + sin(B)$ and non-distributivity, e.g. 2(3 + 5) = 2 x 3 + 5, are also cognitive obstacles to learning.

Davis (1984, 1990) suggests that if care is taken in how things are spoken about in lessons, some cognitive obstacles about functions can be avoided. An example might be: trying not to say things that are later going to turn out to be false (such as “addition makes bigger”). Classic obstacles were reported to be controllable with a suitably chosen task. Davis (1984) called these "torpedo tasks" because they are designed to confront learners with a contradiction which needs to be resolved.

Unfortunately some learners apparently develop the impression early on that mathematics is not self consistent (perhaps because they have confused or been confused by similar situations that are treated very differently). Zaslavsky (1997) contended that once an error has been internalized, it is likely to emerge again when the learner is under pressure (having to think deeply about what they are doing), and so they make a slip in the midst of some other activity. It is not always easy to distinguish between a deep-seated misunderstanding and a slip due to attention being focused elsewhere.

Sierpinska’s (1987) work on student’s informal notions of limits and infinity; Cornu’s (1991) description of epistemological obstacles related to limits; Davis and Vinner’s (1986) discussion of “seemingly unavoidable misconception stages” in learning of limit; William’s (1991) research on informal models and the tenacity with which students hold them; Lauten, Graham and Ferrini-Mundy’s (1994) work on interactions with technology; and finally, Szydlik’s (2000) work on beliefs about mathematics and their affect on understanding limits inform this current study by providing the history of what has been studied, including the interview methodology.

Conclusions

Several studies have been done on understanding limits and functions. Though a few were done on rational and quadratic functions, no studies were available on students' understandings of piecewise functions, a very important function used in calculus. Many researchers have identified misconceptions that students have about limits, but do not explore some of the findings in any particular depth to find out why these misunderstandings occur. No previous research appears to have been done on what students reported to know about limits, nor did any prior studies focus on how knowing the definition of limit and infinity, as well as knowledge of the definition, affect problem solving. Using a constructivist lens, the current study probes deeper into identifying precisely how college students think about limits of functions and how their understandings emerge through their articulated and written actions.

CHAPTER 4: METHODOLOGY

Introduction

The purpose of the study was to explore how students think about limits in calculus. Students' problem solving activities were investigated using interviews while working on tasks that involved functions and limits. The study was divided into two phases. Phase I was a pilot study where emergent themes of students' understanding of limits were identified. Phase II was a deeper follow-up investigation in which various conceptual knowledge structures and schemas were described and explained.

Research Questions

Phase I

In Phase I, there were two initial exploratory research questions. In what ways do students think about limits, and how do student's understandings about limits manifest through various tasks?

Phase II

Given the results found in the analysis of Phase I, a new, more focused research question was developed. Incorporating the intuitive definition of limit, prior knowledge of domains and definition of infinity, in what ways do students solve problems and reveal what they know in response to tasks involving limits at a point and limits at infinity?

Participants

In the initial Phase I analysis, two college students enrolled in Calculus III courses were selected from a sample size of 15 stipulated in the original dissertation proposal. The two students were selected based on their mathematical proficiency and ability to articulate their thoughts. Data from self-reported actions of problem solvers are valid provided that the solvers are able to express their thoughts with limited intervention from interviewers (Ericsson & Simon, 1993). In the Phase II follow-up, deeper probing into understanding limits was needed so nine additional students were recruited from a Calculus III class. All students were enrolled at the University of North Carolina at Charlotte.

Rationale

Studying the problem solving actions of college students can be useful in explaining a developmental array of internal and overt mathematical actions (Carlson & Bloom, 2005; Cifarelli & Cai, 2005). For instance, Cifarelli & Cai (2005) examined the problem solving actions of college students and identified a pattern of recursive reasoning while problem solving, finding that the student's reflections on their actions provided on-going feedback to the development of their understandings.

The rationale for selecting students in upper level calculus courses is that they typically have had more exposure and experience with limit problems than just those in introductory. Such topics would have included integration, iteration, series and sequences.

Design of Study

A qualitative exploratory multiple case study method was designed as a two-phase study, enabling the researcher to gather initial pilot data (Phase I) and then probe deeper into the pilot's significant findings with a more in-depth investigation (Phase II). Semi-structured interviews were implemented to collect data that focused on what students know about limits. Yin (1994, p. 84) stated that interviews are one of the most important sources of cased study information.

Instruments and Tasks

Piloting Phase

Prior to Phase I, two years were spent developing and piloting potential traditional and non-traditional tasks to be used in Phases I and II. Traditional tasks typically involved doing computations and sketching graphs, whereas non-traditional ones often involved starting with graphs then deriving their functions but not in all cases. They had been piloted on volunteer calculus students to determine feasibility, so that tasks too difficult or abstract could be eliminated. For instance, algebraic functions involving square roots (traditional task) or rational functions involving only variables (non-traditional task) were too difficult (Figure 4.1). Given their level of difficulty, they were deemed unnecessary to get to the root of the basic underlying problems students have with limits, which was the ultimate focus of the study. A comprehensive list of all of the initial tasks developed along with their corresponding solutions appear in Appendix C. Due to the nature of the developmental phase, not all tasks were defined as traditional versus non-traditional due to potential overlapping of both.

Traditional Task	Non-Traditional Task
Compute the limit, sketch a graph and explain the behavior of the function values as x approaches plus or minus infinity. $\lim_{x\to\pm\infty}\frac{\sqrt{4x^2+2}}{3x+1}$ Is this function rational? Why or why not?	Compute the limits and identify all values of "n" for the following functions, such that $\lim_{x\to n} f(x)=\infty$ or $\lim_{x\to n} f(x)=-\infty$. $\lim_{x\to n}\frac{(2x+a)(x-b)}{(cx+d)(3x-k)}$. Next, compute $\lim_{x\to\infty}\frac{(2x+a)(x-b)}{(cx+d)(3x-k)}$ and describe the behavior of the function values for large x.

Figure 4.1: Tasks ruled out due to difficulty in nature.

Phase I

Following the initial task development and piloting period, 41 traditional and non-traditional mathematics tasks involving limits of functions were selected. These were created by the researcher with collaboration of a mathematician, and specifically designed to enable students to demonstrate their knowledge of limits in a variety of problem-solving situations. A combination of computational problems and graphing tasks were carefully selected for limits at a point, limits at infinity, and for limits that do not exist. Given the exploratory nature of the study, the tasks addressed several areas of difficulty for students. These included limits of rational, quadratic, trigonometric and polynomial functions, including formulas and graphs of functions. All of the tasks used in Phase I appear in Appendix E. They are separated into four tables corresponding to the content covered in the interview. Appendix E-1 contains the tasks for functions; Appendix E-2 for limits at a point; Appendix E-3 for limits at infinity; and Appendix E-4 for limits that do not exist. The four tables show the tasks on the left along with a summary of results on the right for CL and NS.

In order to enable the researcher to document the solvers' actions accurately and efficiently, students were presented with tasks drawn and organized in a large spiral notebook then drew each task onto the large 18” x 24” drawing sheets provided. Colored markers and a calculator were used to complete the computations and to construct various graphs. They were asked to do the problem solving aloud.

Traditional tasks included those similar to textbook exercise problems, including computing limits and sketching graphs of functions. Non-traditional tasks were those that asked students to consider problems in contexts that are different from those of traditional textbook problems. Figure 4.2 contains an example of each type.

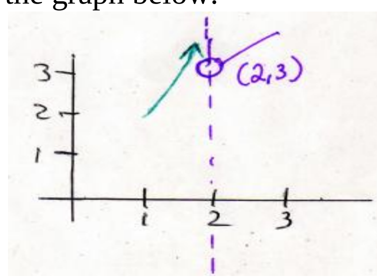
Traditional Task	Non-Traditional Task
Particle Problem: Given $s(t)=1-e^{-t}$ a. Compute the limit: $\lim\limits_{x\to\infty} 1-e^{-t}$ b. How does the particle behave for positive values of t? c. Graph the function.	a. Construct a possible function from the graph below.  Note: piecewise (one piece linear, other rational) b. Compute the limits as x approaches 2 from the left and from the right. $\lim\limits_{x\to 2^-} f(x) = \_\_\_$ $\lim\limits_{x\to 2^+} f(x) = \_\_\_$ c. Explain if these limits exist.

Figure 4.2: Sample of traditional and non-traditional tasks.

A traditional task is a textbook problem that would involve computing a limit and then sketch its graph; however, a nontraditional task works in an opposite way, possibly providing students with a graph of a function first followed by instructions such as to name what kind of function the graph contains and to derive the function. While traditional problems give rise to action-oriented elicitations of their knowledge and understanding of limits, non-traditional tasks can elicit student actions that provide an instantaneous psychological snapshot of how the student views the limit concept and the variety of ways they can apply their knowledge of limits to solve novel problems. For both types of tasks, students need to explain what limiting behavior means for limits at a point and limits at infinity, with both the function and the graph (relational understanding) rather than just generate a computation of the limit (instrumental understanding). The purpose of using these tasks was to provide a rich differentiated opportunity for students to demonstrate their understanding of limits and observe the various ways they solve problems.

Phase II

In Phase II of the study, 10 pre-drawn tasks were presented to students on large 18" x 24" drawing sheets of paper. These tasks appear in Appendix F along with a synopsis of protocol questions. All of the functions and limit formulas computed were pre-written to save time and expedite the interview process. Students wrote responses using various colorful markers.

A few previous tasks were used from Phase I, but mostly new tasks were designed. Several of these began with graphs followed by a consistent repetitive series of questions, sequenced from a function to its associated limiting behaviors. In essence, the questions started with a particular function such as $f(x) = e^{-x}\cos x$, then using that function, expanding it into questions about limits at a point (compute $\lim_{x \to 0} e^{-x}\cos x$), limits at infinity (compute $\lim_{x \to \infty} e^{-x}\cos x$) and limits that do not exist (compute and describe the end behavior for $\lim_{x\to-\infty} e^{-x}\cos x$). Explicit examples of the format of questions can be found in the students' work in Appendix F.

All interviews were recorded with a Samsung camcorder and stored on HDSC cards for subsequent transcription and analysis. Each drawing sheet had students' initials written at the top for later identification. Sheets were later photographed in order and images saved as PDF files, in order to be later imported into text documents.

Data Collection

Phase I

Protocol analysis is a rigorous data collection methodology for eliciting verbal reports of thought sequences as a valid data source on thinking (Ericcson & Simon, 1993). In Phase I, subjects were interviewed individually on three separate occasions. Each interview lasted approximately two hours each. The interviews were spaced about two weeks apart so the researcher could review and interpret the video recordings and written work. All interviews were videotaped with a camcorder on a tripod and data was stored on HDSC cards.

Students copied down the traditional and non-traditional tasks onto large 18 x 24 inch sheets of paper. A camcorder recorded the responses of students solving a variety of function and limit problems while thinking aloud. Students used colored markers to draw graphs or show their work for computational tasks. They worked individually while solving the problems and were given as much time as they needed to complete each task and explain their thinking.

The first interview consisted of problem-solving tasks involving functions and limits at a point. The second interview focused on mostly limits at infinity and some involving infinite limits. The third interview focused on more advanced limit problems, including other limits that do not exist such as those with oscillatory behavior as well as when the left hand side of the function does not equal the right hand side (piecewise functions). Tasks began with graphs or computations, requiring students to think aloud while writing task solutions with colorful markers on the provided paper sheets. Some tasks involved rational functions and focused on the asymptotic behaviors in which case students needed to explain the limiting behavior of the function values.

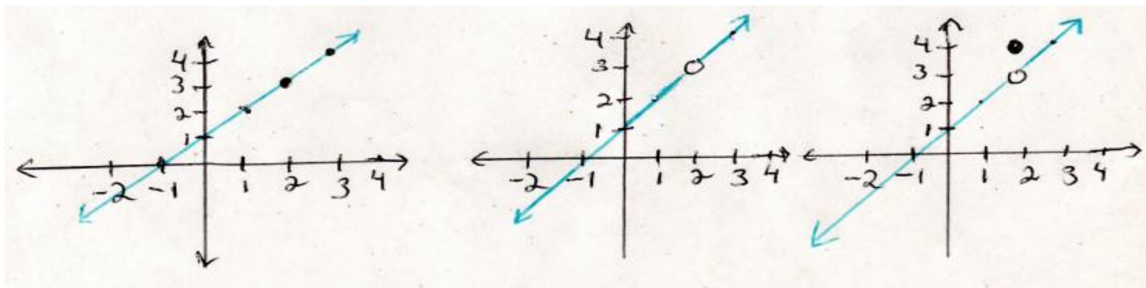
After carefully analyzing the initial data from Phase I, the researcher found specific artifacts worthy of further more in-depth exploration, and so the format for questioning had been refined to semi-structured interviews rather than collecting completely open-ended responses.

Phase II

Following the results of the initial analysis in Phase I, a more focused research question emerged for the follow-up data collection in Phase II. Semi-structured interviews were conducted to extract ways students applied their definitions of limit and infinity, and how they considered the role of domains. In order to expedite data analysis such that transcript evidence could be easily located across subjects for particular questions and responses, the same questions were asked of all students in consecutive order. Therefore, the episode numbers in the transcripts for any particular researcher's question or students' responses for that matter are the same. For example, in "turn 11" the question for all students appears as "Can limits and function values be the same?" Following in "turn 12", the student's response to that question was transcribed.

In the Phase II follow-up analysis, each person participated in a 90-minute interview whereby they solved a variety of function and limit problems while thinking aloud. The students were presented with pre-constructed tasks on large sheets on which to construct to construct their responses. Students were given as much time as they needed to complete each task and explain their thinking. The nine cases were then categorized into four themes or patterns of similarities among students. As a result, there are four cases presented that are representative of all nine subjects. They were categorized based on their conceptions of limits and infinity, and their perceived roles of the domain.

The analysis of the case studies in Phase II begins with what was referred to as Task 4 in the study (Figure 4.3). This task contains consecutive graphs that start with a continuous function and proceed into those with discontinuities, and is considered to contain a comprehensive description of their work pertaining to the phenomena under investigation. During this task, students examined three graphs in which the limit was exactly the same though the graphs were different. The graphs progressed from continuous, to discontinuous at x=2 to having a jump discontinuity at x=2 with a function value at x=6. In “turn 97” the question asked by the researcher was: “Study these three graphs…State the domains and ranges for each one of these and compute the limits. Then explain the behavior of the function values”.



Graph A	Graph B	Graph C
Domain: $(-\infty,\infty)$	Domain: $(-\infty,2)\cup(2,\infty)$	Domain: $(-\infty,\infty)$

Figure 4.3: Task with continuous and non-continuous graphs.

Data Analysis of Phase I and Phase II

The following analyses apply to both Phase I and Phase II. First, video and written protocols were examined to isolate problem solving episodes that reveal aspects of the student's understanding, such as whether or not they understand the notion of nearness, continuity, discontinuity, definitions, infinity and the role of the domain. Second, students' protocols were examined to identify critical processes such as various overt actions observed that occur along a continuum of instrumental and relational understanding. Actions included pauses where they encountered cognitive obstacles, frustration, as well as facility with drawing graphs or constructing the limits of functions given graphs. Also, ways of knowing were identified, such as whether students could identify a piecewise function as being piecewise. Given graphs of functions, students had to explain if a limit exists or not. They also had to explain the meaning of limiting behavior. After analyzing the cases, emergent themes, knowledge structures and schemas were described and explained.

Rationale of Data Collection for Phases I and II

The interviewing protocol followed the constructivist principles of Steffe and Cobb (1983) that restrict questions to those asking for clarification and further explanations of particular responses made by the students. These types of questions have been documented to have minor influence on the subject's natural flow of thought processes (Ericcson & Simon, 1993) and thus ensure that the verbal reports on thinking constitute valid data. Students wrote their responses on large sheets with pre-drawn tasks and functions and needed to verbalize their thinking while problem solving. Both verbal and written responses were examined and compared (Pugalee, 2004).

Using a constructivist framework similar to that used by Cobb and Steffe (1983) for analyzing case studies of students, the videotaped interviews were transcribed for each student in this study. The video protocols were important to the analysis because they represent a significant resource to isolate and examine verbal and nonverbal processes that may play important roles in the ways students explain what they are thinking an how they solve the limit problems. Moreover, research on problem solving shows that students apply their conceptual knowledge in problem solving experiences within chunks or clusters of activity (Schoenfeld, 1992) that are both situational and episodic in structure (Hall, Kibbler, Wenger & Truxaw, 1989) and that their conceptual knowledge "unfolds" in the course of on-going activity (Pirie and Kieren, 1991). For instance, Schoenfeld (1992), found that students developed localized goals and purposes within these episodes of activity and that a student's solution to a task may be constructed upon several of these episodes. Hall et al. (1989) explored these episodes as situational expressions that unfold as the student became engaged within the situation and began to develop goals and purposes. Kieren and Pirie (1991) discovered that students develop their understanding in problem situations by unfolding their actions in the course of formulating problems, and then they reconstructed their actions at an increased level of understanding during which time they carried out the solutions. This study was based on the methods used in these prior studies in order to capture pre-conceived ideas about limits as well as their overall verbal reports on thinking.

The use of recorded camcorder videos proves to be more effective than written protocols alone when analyzing such diverse examples of overt actions of student thinking during the tasks. Also, since an interview is a social interaction in which the interviewer and student participate in a dialogue, later viewing of the recordings lets the researcher get the opportunity to look back and analyze the dialogue from a typical observer's perspective and allows for on-going interpretation, reflection and revision of the subject's activity in the course of the analysis (Roth, 2006). Once a phenomenon has been identified that could cause for a revision, the recording can be re-analyzed in light of the new results, allowing for continued interaction between the theory and the data (Cobb and Steffe, 1983; Roth, 2006).

CHAPTER 5: RESULTS OF PHASE I

Introduction

The two original research questions of this study were: (a) In what ways do students think about limits and (b) how do their understandings about limits manifest in their task solutions? In order to explore how students think about limits and how their understandings were revealed in their task solutions, a 96 hour pilot study was conducted. There were 41 original traditional and non-traditional tasks designed by this researcher that appear in Appendix E, consisting of functions, limits at a point, limits at infinity and limits that do not exist. Some tasks were previously piloted on other students to ensure selecting feasible and appropriate tasks. The following process was followed.

1. Three interviews were conducted on each student, lasting two hours each.
2. Students received graphs and computations on a spiral notebook and then transcribed them on a white board and on paper sheets.
3. Students articulated their thoughts aloud and wrote on a white board.
4. The rationale for collecting initial data was to identify common themes of misconceptions that emerged between and within subjects, as well as to probe deeper into knowledge associated with particular tasks. This included, but not limited to, knowledge of the definitions of limit and infinity, and knowledge of the domain.

5. In addition to interview transcripts, photographs of their written solutions were analyzed.

According to Cobb (1988), analyses that employ clinical interviews are most effective when the interviews document examples of task-involved activity. The nature of the research questions considered the validity of their mathematical correctness as further probing of students' understanding evolved. Given the nature of this evolving model of understanding, new hypotheses and research questions were formulated in response to this pilot data and were subsequently explored with an additional nine students.

Two students in Calculus III with contrasting levels of mathematical proficiency were selected out of a larger group of 15 for the initial pilot cases. Student CL has minimal algebra proficiency whereas student NS has a more comprehensive background in mathematics, but both students could articulate their thoughts and actions very well. Nicholls (1984) and Cobb (1988) discussed different levels of engagement during clinical interviews. Subjects were said to be "task involved" if their verbal and non-verbal actions indicate that they are fully engaged and motivated. A detailed narrative description of these two initial cases with the transcript evidence and results appear in Appendix D. A synopsis of how they thought about functions and limits along with a summary of tasks is provided in Appendix E.

Functions

The first interview involved responding to eight different tasks on functions that appear in Appendix D. Some tasks were traditional but others were more nontraditional in nature, including piecewise and rational functions. CL and NS drew correct examples of functions, and used the vertical line test to demonstrate when the figure did not represent the graph of a function, such as a circle. However, neither student correctly interpreted the meaning of 1-1, a term reserved for inverse functions. They said in order for the relation to be a function, it had to be 1-1 which was not correct. I realized what they meant, though, which was that there can only be one y-value for any x-value so in this case they used the term 1-1 to describe this relationship. Neither student could distinguish between a function and function value. They were given both traditional textbook like tasks on functions, as well as some nontraditional types in which case they were given graphs and had to derive the function by hand or with the assistance of a graphing calculator. Such nontraditional tasks seemed difficult for CL and so she would often work backwards, taking each of the multiple choices and plugging them into the calculator to see if they matched the graph that was provided. Table 5.1 presents a summary of the most important findings that emerged.

Table 5.1: Results for Functions.

Actions by CL	Actions by NS
• A function is f(x)= x^2 . • A function must be 1-1 (1 y for each x). • Function values refer to a set of coordinates, x and y. • Functions and function values are synonymous terms.	• A function is an equation that maps x into a single y. • A function must be 1-1 (1y for each x) • Function values are what functions equal at specific points in the form (x,y). • Functions and function values are synonymous terms.

Limits at a Point

The second part of the first interview involved 12 tasks involving limits at a point. Some tasks were traditional but others were more nontraditional in nature and included piecewise, linear, cubic, rational and one involving a domain with a finite interval. Both students seemed to have vague ideas about what a limit was and neither student could pinpoint a limit as being a number. NS recognized a piecewise function whereas CL did not. CL thought the isolated point above a quadratic piece with a discontinuity did not belong to the graph of the function. With rational functions, both students compared the left and right hand limits, which are not done with infinite limits. CL revealed that given a discontinuity with a hole, “the limit exists because there is a hole to fall into but then the limit did not exist once one falls into the hole because they could keep going down indefinitely”. On the other hand, NS referred to the left hand limit being equal to the right hand limit for the limit to exist. When CL saw a continuous function, she said the contended the limit does not exist because “one could walk right over it, like a line on the highway, with no hole to fall into. CL associated limits with being holes and brick walls, which were vertical asymptotes, whereas NS referred to the function’s behavior near the point. Interestingly, CL thought limits are exclusively about x, not y whereas NS acknowledged that limits were about the behavior of the y-coordinate. A summary of the most important findings that emerged appear in Table 5.2.

Table 5.2: Results for Limits at a Point.

Actions by CL	Actions by NS
• Limits are “physical barriers, restraints” • Limits are vertical asymptotes (brick walls) and holes one can fall into. • Limits are about x, not y. • Left and right hand limits are not compared for limits at a point. • Limits exist where there are holes in discontinuous functions. • Limits do not exist for continuous functions where there are solid dots because there is no hole to fall into. One can walk right over it.	• A limit is a value a function approaches as x approaches a limit value. • Limits are about y, not x. • Left and right hand limits are compared for limits at a point, including rational functions. • Limits exist when there are holes in discontinuous functions because the left hand limit equals the right hand limit. • Limits and function values are different given a piecewise function with a discontinuity and a function value.

• The limit exists for $\lim_{x \to 0} = \frac{1}{x}$ because as x approaches 0 from the left, there is a brick wall it can't go past.	• The limit does not exist for $\lim_{x \to 0} = \frac{1}{x}$ because the left hand limit $-\infty$ $\neq$ the right hand limit ∞.

Limits at Infinity

The second interview involved 11 tasks involving limits at infinity. These comprised of linear, quadratic, cubic, exponential, trigonometric, piecewise and rational functions. Some tasks were traditional but others were more nontraditional in nature. The limit notation was confusing for CL, who perceived the arrow beneath "lim" to be directional in nature, pointing to the right only and she also revealed that whatever appears below the "lim" notation is what the limit is going to be. So if $\lim_{x \to \infty} \frac{1}{x}$, CL thinks the limit will be ∞ because of $x \to \infty$ appearing beneath "lim". NS had no problem with the notation. NS erroneously used mathematical operations with infinity though infinity is not a number. A summary of the most important findings appear in Table 5.3.

Table 5.3: Results for Limits at Infinity.

Actions by CL	Actions by NS
• Infinity is a large number. • Limit notation implies direction from left to right. Ex: $\lim_{x \to 3}$ means approach 3 from the left. • The limit will be infinity because that's what is below the lim notation so no math is involved: $\lim_{x \to \infty} \frac{1}{x} = \infty$ • The limit will be whatever appears beneath the "lim" notation.	• Infinity is an indefinable large number. • Limit notation not confusing.

Limits that Do Not Exist

The third interviewed focused on limits that do not exist. These included infinite limits in which case the function values increase without bound as well as other types including trigonometric functions (oscillatory and periodic), rational and piecewise functions with jump discontinuities. Some tasks were traditional but others were more nontraditional in nature. Both students had misconceptions about infinity in terms of thinking infinity is a very large number, and erroneously thought that a limit exists if it equals infinity, most likely because of the infinity symbol representing a large number. Oscillatory functions provided interesting information because CL erroneously contended that there were two limits for the sine function, y=-1 and y=1, because they were the "horizontal brick walls or lines one could not go past". On the other hand, NS correctly articulated that the limit does not exist for the sine function as x approaches infinity due to its oscillatory behavior. These are quite different interpretations. A summary of the most important findings that emerged appear in Table 5.4.

Table 5.4: Results for Limits that Do Not Exist.

Actions by CL	Actions by NS
• A limit exists if it equals infinity. The $\lim_{x\to\infty}\frac{1}{x}=\infty$ because it never stops or touches the x-axis. • There are 2 limits for sin x, 1 and -1 because you can't go past those 2 walls. $\lim_{x\to\infty}\sin x = 1$ and -1. • Given $\lim_{x\to 0}\frac{1}{x^4}$ the limit does not exist because function not defined at 0.	• A limit exists if it equals infinity.. • The $\lim_{x\to\infty}\frac{1}{x}=0$ because values get smaller. • $\frac{2x^3}{x^2}=\frac{2\infty}{\infty}=\infty$ so limit exists and equals infinity • $\lim_{x\to\infty}\sin x = d.n.e.$ due to oscillations • Given $\lim_{x\to 0}\frac{1}{x^4}$ the left hand limit equals the right hand limit, so the limit exists and equals $=\infty$, i.e., $+\infty=+\infty$

• Given $\lim_{x \to 0} \frac{1}{x}$ the limit is the y-axis, a brick wall where the points go up to infinity; limit exists and $= \infty$	• Given $\lim_{x \to 0} \frac{1}{x}$ correctly stated limit d.n.e. for whole problem, but for wrong reason. Incorrectly compared left and right sides near 0, $-\infty \neq +\infty$

Common Themes of Knowledge

Many of the results compiled from this study replicates what has been found in prior studies that appear in the Literature Review. For instance, Tall & Vinner (1981) and Juter (2006) document the vague ideas students have about the definition of limit and meaning of infinity, and that they use colloquial language to explain limits. When students see a hole representing a discontinuity, they claim the limit does not exist because "nothing is there" in which case they confuse limits with continuity. Table 5.5 lists the misconceptions that occurred in this study, some of which has been previously cited in the literature.

Table 5.5: Areas of Misinterpretations with Functions and Limits.

• Properties of the real number system • Functions and domains • Graphs and the coordinate system • Algebra with polynomials • Piecewise functions • The symbol for infinity • The definition of limit	• Limits versus continuity • The notation for limit • Understanding logic • Reading comprehension • Use of colloquial language to replace mathematical terms • Instructor competence

Summary

In the pilot study, students constructed solutions to both traditional and non-traditional type tasks for functions, limits at a point, limits at infinity and limits that do not exist. While CL and NS share some similarities in terms of their responses, there were some contrasting differences. Even though NS had more mathematical experience and skill sets, she still demonstrated some misconceptions that were worthy of investigating further.

They both seemed to have a fairly good understanding of what a function was, even though they used the wrong terminology at times and could not distinguish between a function and a function value. NS computed limits at a point correctly and constructed correct graphs, but her interpretation was incorrect for infinite limits with rational functions, in which case she would incorrectly contend that a limit exists if it equals infinity. CL also had the same incorrect interpretation about the behavior of the function values as x approached 0. In addition, CL incorrectly contended that the limit did not exist as x approached positive infinity, rather than 0, because the line just kept going and did not stop. The limit notation was problematic for CL, as she believed the arrow beneath "lim" implied direction from the left. Oscillatory functions were of interest since CL described the sine function as having two limits as x approached infinity: y=-1 and y=1 whereas NS correctly contended that the limit did not exist due to oscillatory behavior.

Using Skemp's model, CL fit the description of instrumental understanding and NS as having relational understanding. Upon further consideration, they both appear to show relational understanding. They both elicited relational understandings based on their actions in the interview sessions. They were both quite able to articulate their thoughts and ideas in a very consistent, coherent fashion. Piaget and Skemp's notion of developing internal mental structures fits here, because not all mental structures are developed correctly. Once these structures get assimilated into one's schema, the structures are fixated there unless there is a reason to change their thinking, in which case they recognize being in a state of disequilibrium (Piaget) or state of instability (Skemp). If this happens, they would learn the correct way of thinking about the problem and would accommodate that into their schemata. Meanwhile, if they possess incorrect mental structures, then those get assimilated into their thinking, which are revealed through their actions. Both students had quite differentiated schemas evidenced through their constructed actions, which leads me to conclude they had relational understanding that included both appropriate and altered conceptions.

Many of these misconceptions occurred in the other 13 students as well, though not reported here in the pilot, and so it is best to say that CL and NS could possibly represent the more general population of Calculus III students at UNCC that possess altered conceptions of limits, explainable with Piaget and Skemp's constructivist theories. It seems that both students in the pilot revealed through their actions inappropriately developed mental structures, which evolved into altered conceptions. A rationale is now provided for additional data collection to probe deeper into some of the mental structures themselves, such as the definition of limit, definition of infinity and role of the domain, and explore students' schematic conceptions, either appropriate or altered.

Rationale for New Data Collection

The pilot study served as the beginning of this evolving model of understanding. Having found several common patterns of correct and incorrect responses within subjects, I decided to probe deeper by studying more carefully how students apply their definition of limit to their tasks since all actions constructed come from the definition.

Moreover, given the unanimous response that limits exist if they equal infinity, I wanted to further explore the conception of infinity and meaning of the infinity symbol. To further understand why students think the limit does not exist where there is a hole or why limits do not exist for the rational function, $\lim_{x\to\infty}\frac{1}{x}$, I thought it would be insightful to study how they think about the domains of functions. Specifically, I wanted to find out if they think the point *a* has to be in the domain of a function as $x \to a$. Given the various emerging patterns of responses and themes of understandings that occurred with CL and NS, some additional questions and hypotheses emerged.

1. How consistently definitions of limit and infinity are applied in task solutions;
2. How the role of the domain is perceived with respect to the existence of limits;
3. What skill sets and conceptual understandings are needed to know about functions to understand limits, particularly piecewise and rational functions;
4. Why students perceive infinity to be a very large number and think a limit exists if it is equal to infinity.

Hypothesis 1

Students may or may not use their definitions of limit and infinity in their task solutions, meaning, they may define or conceptualize these one way but might apply them differently.

Hypothesis 2

Students may apply their knowledge about the domains of functions through tasks involving graphs. Since the domain is a part of the function and functions are a part of limits, students might or might not look at the graphs and explain as $x \to a$ whether or not “a” had to be in the domain in order for limits to exist. Moreover, by using graphs, understandings of one-sided limits that occur with domains of finite intervals could be explored.

Hypothesis 3

Understanding limits might be contingent upon being able to identify piecewise functions. When there is a single point sitting above or below a discontinuous function with a hole at the same input value, students do not think that a single point constitutes a piece of the graph and they tend to not think that that particular point is even on the graph of the function. More abstractly with an even deeper probe, students might not know that with a discontinuity, the point is still in the domain, but just not on the graph of the function.

Hypothesis 4

Given rational functions, students are uncertain what procedures to use with limits at a point and limits at infinity and how to construct the corresponding graphs. I speculated they were not sure when to factor the numerator and denominator versus when to factor out the highest power of x separately from the numerator and denominator. By probing deeper using specific tasks that required such factoring, their algebra proficiency could be explored, in addition to their understanding of when a hole would occur versus a vertical asymptote in a graph. Their ability to translate information from the computation to sketching the graph could be assessed.

Hypothesis 5

Students have ambiguous notions about infinity and some even consider it to be a large number. A deeper investigation into their understanding of infinity through graphs involving infinite limits facilitated exploring why students claim that limits exist if they are equal to infinity.

Hypothesis 6

Students have vague notions of when limits do not exist in other cases. This includes infinite limits in which case the function values increase or decrease without bound, as well as with jump discontinuities, periodicities and oscillatory behaviors. For instance, with the cosine function, it was typical for students to say that as $x \to \infty$, that the "limit was equal to infinity" because the function "kept on going and did not stop." However, the correct interpretation would be that the limit does not exist due to oscillator behavior, or the function values not settling down to any one particular point. So this was a focus of the deeper investigation. As a result of developing new hypotheses, I combined all of the single artifacts into one research question that would effectively provide the details regarding the definition of limit, infinity and role of the domain. The research question appears at the end of this chapter following the rationale for additional data collection and a description of how the methodology of the pilot informs Phase II.

How Methodology of Pilot Informs Phase II

Overall the pilot was very effective and informative. As a result of the pilot, I made some changes with the new data collection and refined the research question so that I could do a more semi-structured investigation of particular tasks. In the pilot, I used a constructivist Piagetian framework to explore their actions, by asking questions and providing feedback as they constructed solutions. Using Skemp's framework, I classified CL as having instrumental understanding and NS as having relational understanding. In the Phase II, I keep both Piaget's and Skemp's constructivist framework, but I now classify all students as having relational understanding, who are able to construct their actions based on schematic prior knowledge.

In the pilot, there were 41 tasks given over the course of 3 interviews, and each interview took over 2 hours each. Since many students tended to construct similar actions, I recruited fewer students for the semi-structured investigation. In addition, I designed fewer but more concise tasks, which I explain below.

In the pilot study, I began by having students write tasks on a white board, and subsequently on large paper sheets. This was very time consuming and so in the new study, I had all of the tasks and formulas pre-drawn to expedite data collection and maximize the time spent observing and recording their actions. Instead of using 43 separate tasks, students worked with 10 pre-drawn sheets that contained between two and five tasks. Unlike in the pilot where students had separate interviews for functions, limits at a point, limits at infinity and limits that do not exist, I redesigned the tasks so that each task would cover each of the above as part of a single problem. The reason for doing this was to see where difficulties emerged as they constructed their actions.

The pilot study was exploratory in nature, to acquire an overall understanding of the types of tasks that present difficulties. I discovered that in all cases linear, piecewise and rational functions were particularly problematic, as were the limiting behaviors of function values both near a point and as x approached infinity. Given so many definitions of limits and infinity, I probed deeper into finding out why. In addition, given the nature of the responses such that the “limits are holes”, “limits do not exist inside the hole because one can fall into the hole and keep going without stopping and “if there is a solid dot on a line the limit doesn’t exist because you can walk right over it”, I began to think there might be some missing information to explain why such unique responses were constructed. To further explore why students think a limit does not exist where there is a hole or why they think limits exist if it equals infinity, this suggested to me that an semi-structured look at linear, piecewise and rational functions could give specific insight into how students might perceive the role of the domain. The reason is if students think as $x \to a$, "*a*" must be in the domain for a limit to exist, this could explain why they think if x=a is not in the domain, there is a hole in the graph. Therefore, their knowledge of the definition of limit, infinity, and domains were formed into a single research question. Their solutions are analyzed using a constructivist lens of Piaget and Skemp, who share similar ideas involving conceptual structure.

Development of New Tasks

Since this study is of an evolving nature, more focused research questions and tasks were developed pertaining to the interpretation of graphs, computing limits, stating domains and describing limiting behavior. Graphs of functions or functions alone were presented so questions could be asked across the board, starting with functions to limits that do not exist (Appendices E & I). This approach gave insight into an evolving understanding from functions to limits of those functions and was completely different from the pilot study, in which 43 discrete tasks were drawn by hand and solved (Appendix E).

In Phase II, pre-composed tasks were shown whereby students referred to a graph, looked down a column, computed various limits, and wrote down answers to some open ended questions. Graphs of continuous, piecewise and rational functions which appear in

Appendix F were used as a tool to pursue a follow-up investigation warranted by data from the initial two cases.

Development of New Research Question

In the pilot study, the interest was in exploring ways students think about limits and how their understandings are revealed in their actions. This was accomplished by exploring how they think about limits in categories, including tasks exclusive to functions, limits at a point, limits at infinity and limits that do not exist. Given that there was still additional exploration to be done at a deeper level with some of the tasks, new tasks were designed and nine new students were recruited in order to probe deeper into aspects of understandings that were not done in Phase I, the pilot study.

The new research question is: Incorporating the intuitive definition of limit, prior knowledge of domains and definition of infinity, in what ways do students solve problems and reveal what they know in response to tasks involving limits at a point and limits at infinity?

The following chapter describes the inquiry into this new question and explores more deeply the answer to this question.

CHAPTER 6: RESULTS AND DISCUSSION OF PHASE II

Introduction

This section presents four cases selected from which data was collected. Several problem-solving tasks are presented along with a description and explanation of each of the four cases. At times, an additional exploration occurred in which instance only one or two particular student cases are described and explained further. A discussion of the findings occurs following the descriptions and explanations for each problem-solving task. It should be noted that in this discussion, several original tasks from the study are combined and compared. They are labeled accordingly and correspond to the original tasks of the study that appear in Appendix F as well as to those in Appendix J which contains pictures of the hand-drawn tasks along with the students' written responses. The comprehensive results of this study that include an analysis of students' work on each task in the study appear in Appendix G.

Research Question:

Incorporating the intuitive definition of limit, prior knowledge of domains and definition of infinity, in what ways do students solve problems and reveal what they know in response to tasks involving limits at a point and limits at infinity?

Importance of Definitions and Understanding Domains

Mathematics is based on definitions which can help students solve problems as they construct solutions and develop conceptual understanding. In the interviews, students were asked to provide definitions for domain, range, function, limit and infinity as a basis for subsequently constructing their actions and solutions to tasks. Students gave described and defined what a limit is; in one case a formal definition as provided.

They also defined and described what infinity means to help explain different interpretations about infinite limits, e.g., as $\lim_{x \to \infty} f(x) = \infty$ and whether or these limits exist. Infinity has vague meaning for students and it is not until they study limits that they define infinity for the first time. In both Phase I and Phase II of this study, many students did not articulate that the limits that equal infinity do not exist because the function values increase without bound. Students who report that infinity is a very large number tend to interpret limiting behavior differently than students who identify the infinity symbol to represent "increasing without bound".

Domains were investigated for continuous and non-continuous functions. In the tasks throughout this study, students identified and wrote down the domains for all of the tasks presented as well as explained as $x \to a$ whether or not the particular value of delta ("a") has to be in the domain in order for the limit to exist at a point.

Some examples of the types of piecewise, rational and linear functions that were used in the study are presented in Figure 6.1, while all of the tasks used along with the protocol appear in Appendix F. Descriptions of the tasks and detailed results with evidence can be found in Appendix G. Summaries of all nine students recruited and interviewed for Phase II appear in Appendix H. Appendix I contains the transcript evidence of the four cases selected out of the nine students for the study. Appendix J presents images of the four students' work on all of the tasks presented in the study.

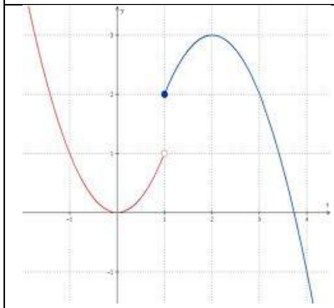
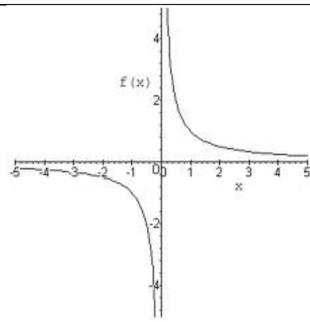
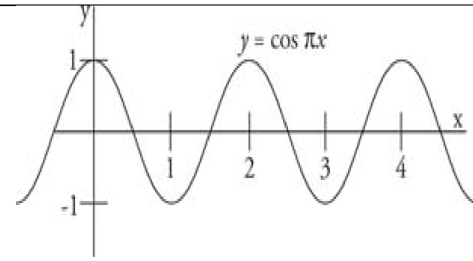
Example of Piecewise Function	Example of Rational Function	Example of Continuous Function
		

Figure 6.1: Examples of tasks.

Description of Four Student Cases

Responses and thought patterns of nine students are described in Appendix H; however, based on common themes of responses, four student cases were selected to represent the most common knowledge structures. These four cases are summarized in Table 6.1. The selections were initially based on the definition of limit and infinity that they provided, their understandings of domain, and the nature of their actions. There is one exemplary case representing what Piaget and Skemp would describe as an appropriate schema and three unique non-exemplary cases representing what Piaget and Skemp would refer to as altered schemas. The term “unremarkable” is what I selected to describe appropriate conceptual knowledge structures whereas “remarkable” represents knowledge structures of alternate forms. Table 6.1 illustrates the four ways that students have perceived, interpreted and now recall prior knowledge constructed in response to formal classroom instruction on limits in their calculus classes. Although calculus instructors expect students to learn and understand limits based on what they’ve taught, the data shows that there is more than one way the instruction is perceived, further constructed and conceptualized in students’ minds.

Table 6.1: Description of the Four Cases.

Name	Conceptual Structure Knowledge	Application of Conceptual Structural Knowledge
Brendon	• Defined limit as the y-value approached and gave formal definition of limit $\forall\varepsilon > 0\ \exists\delta > 0 \ni 0 <\lvert x-a\rvert < \delta \Rightarrow\lvert f(x)-L\rvert < \varepsilon$ • As $x \to a$, "a" does not have to be in the domain for the limit to exist. • Given graphs with two parts, quadratic and function value, are piecewise. • A limit does not exist if it equals infinity. • Infinite limits (end behaviors) are not compared to decide if limit exists. Only one side is needed.	Exemplary case. Solved and interpreted tasks unremarkably, based on knowledge of functions, definition of limit and infinity and understanding of the domain.
Jean	• Defined limit as a number or amount approaching 0 (or number) but never equals it. • Limits approach but cannot equal a function value. • As $x \to a$, "a" cannot be in the domain for a limit to exist. • Limits exist only at holes, not points, if the left and right sides are equal. • Graphs with two parts, quadratic and function value, are piecewise. • Infinity is too large to count. • Limits do not exist if they equal infinity. • Infinite limits (end behaviors) are not compared to decide if limit exists. Only one side is needed.	Non-exemplary case 1. Solved and interpreted tasks remarkably, based on knowledge of functions, conception of limit, infinity and understanding of domain.
Amanda	• Defined limit as a number or infinity. • As $x \to a$, "a" must be in the domain for the limit to exist. • Limits do not exist where there are holes. • Given graphs with 2 parts, function value and quadratic, are not piecewise; function value is not part of the function. • Infinity is a place with no end. • Limits exist if they equal infinity. • Infinite limits (end behaviors) must be compared. If left side does not equal right side, $\infty \neq -\infty$, then the limit does not exist.	Non-exemplary case 2. Solved and interpreted tasks remarkably, based on knowledge of functions, conception of limit, infinity and understanding of domain.

Linsey	• Defined limit as the left side equals the right side and does not exist at holes. • As $x \to a$, “a” must be in the domain for the limit to exist. • Given graphs with 2 parts, function value and quadratic, are not piecewise; function value is not part of the function. • Infinity is a place with no end. • A limit exists if it equals infinity. • Infinite limits (end behaviors) must be compared. If left side does not equal right side, $\infty \neq -\infty$, then limit does not exist.	Non-exemplary case 3. Solved and interpreted tasks remarkably, based on knowledge of functions, conception of limit, infinity and understanding of domain.

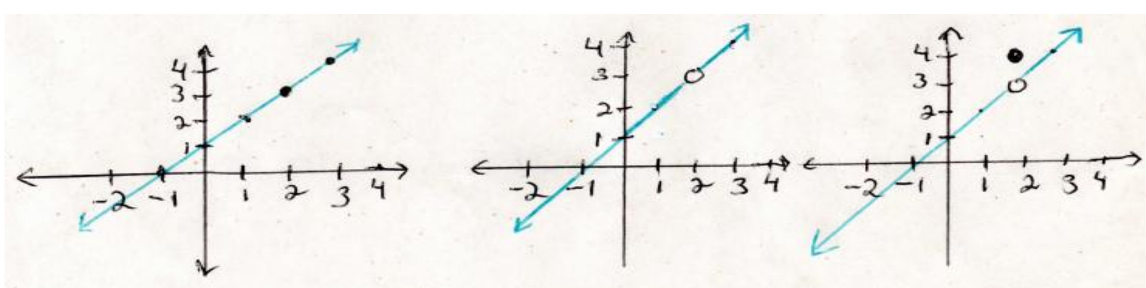
The results for the cases appear in sections. The first section deals with continuous versus non-continuous functions for both limits at a point and limits at infinity. The next section stems from the first and reports more specifically on domain investigation tasks which focus on domains with finite intervals. Following that are results for students’ knowledge of end behaviors with a final specific section devoted to the question, “do limits exist if they equal infinity?” All or some of the four cases are described and immediately analyzed in each of these sections, depending on the nature of their responses.

Continuous versus Non-continuous Task Investigations

Figure 6.2 represents Task 4 used in the study, which contains examples of graphs of continuous to non-continuous functions. All three graphs have the same limit, 3. The labels (Graphs A, B, and C) correspond exactly to those in the actual task. Graph A is continuous over a domain $(-\infty, \infty)$. Graph B is considered piecewise or rational, but x=2 is not in its domain. Graph C is piecewise and has the same domain as Graph A $(-\infty, \infty)$. In fact, x=2 is in the domain but the function value is 4. Hence, the point (2,4) is on the graph of the function. Students refer to the graphs while articulating their solutions.

One reason for having selected this particular task to present in the analysis was is it captures several important aspects of limits at a point and limits at infinity. In Phase I of the study, students showed difficulty recognizing piecewise functions and when shown a non-continuous function with a jump discontinuity, students reported that the function value was not on the graph of the function. In fact, functions involving discontinuities were not regarded as piecewise.

Therefore in Phase II, students were asked to compare all three graphs. First they had to state the domain for each one then compute the limits as x approached 2 and as x approached plus and minus infinity. They were asked to explain for each graph if x=2 had to be in the domain for the limit to exist. Finally, they were asked to explain the behavior of the function values both near the point x=2 and as x approached plus and minus infinity. The particular task was designed to determine if and how students perceive the change in domain that occurs from graph A $(-\infty, \infty)$ to Graph B $(-\infty, 2) \cup (2, \infty)$, then to Graph C $(-\infty, \infty)$.



Graph A
Domain: $(-\infty, \infty)$

Graph B
Domain: $(-\infty, 2) \cup (2, \infty)$

Graph C
Domain: $(-\infty, \infty)$

Figure 6.2: Task with continuous and non-continuous graphs.

Description and Analyses of Student Responses

Four student cases are presented along with descriptions and analyses of how they responded to various tasks. The first task, Task 4 from the study, appears in Figure 6.3, exploring how students solve problems for limits at a point then for limits at infinity. Subsequently, two of the cases, Amanda and Linsey, are explored, described and analyzed further given the nature of their responses. The actual transcript evidence is provided in Appendix I and images of their written drawings appear in Appendix J.

Brendon

Brendon defines limit as “the y-value approached”. He also wrote down and explained the formal definition, $\forall\varepsilon > 0\ \exists\delta > 0 \ni 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ then reported that “as $x \to a$ that “a” does not have to be in the domain of the function because the limit is about nearness and one cannot get any closer to near than being equal”. He states the limit equals 2 and exists in all three cases at x=2 because “hypothetically as $x \to a$ the point “a” does not have to be in the domain”. In this case “as $x \to 2$, “2” does not have to be in the domain”. When looking at any of the graphs as x approaches plus or minus infinity, Brendon indicates that the “limit does not exist because infinity is undefined” and that “a limit has to be a number to exist, and infinity is not a number”.

In Brendon’s case, he knew both the formal epsilon-delta definition of limit as well as the more intuitive version, in which case he understood that limits involve both notions of approaching and equals. He clearly knew that the point (2,3) does not have to be in the domain for the limit to exist because the limit is about nearness. So having assimilated the definitions of limit into an appropriate schema, he exhibited no perturbations and was not in a state of disequilibrium. As part of his beliefs, infinity is not a number. He applied his understandings to his problem solving tasks. As a result of his problem solving actions, he seems to have developed appropriate knowledge structures and schemas.

Jean

Jean defines limit as “approaches but cannot equal function values” so for the linear function in Graph A, she says “the point cannot be in the domain for the limit to exist because limits are only about nearness, not equals”. Jean said a limit is only about “approaching, not equals”, and so she applies this reasoning to the graphs. She also contends that the point x=2 cannot be in the domain for the limit to exist. In Graph B, though, she says the limit exists where the hole is at (2,3) because it “doesn’t equal a function value,” which is consistent with her claim that x=2 cannot be in the domain for the limit to exist. In Graph C, Jean notes that the limit exists where the hole is for the same reason in Graph B, that the x=2 cannot be in the domain. As x approaches infinity from either direction, Jean indicated that the limit does not exist, her reason being “infinity is too large to count”.

Jean presents a very interesting case because given a linear function in Graph A, she correctly claims that the limit is equal to 3 but when asked whether or not the limit exists, she reported that it didn’t. If she had not been asked the latter question, it would have appeared on the surface that she got the answer correct and understood this simple graph, but this is not the case. She states “in the first graph, limit does not exist but value is 3.” Figure 6.3 provides evidence whereby she writes that the limit does not exist as x approaches 2. In spite of computing a number for the limit, in this case 3, Jean reports that this number does not exist. This is not a logical outcome because a limit is a number and therefore, by definition, it must exist.

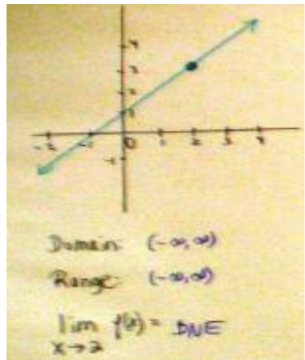


Figure 6.3: Jean's work on Graph A.

Based on my understanding of Piaget and Skemp, Jean assimilated the definition of limit into a knowledge structure and hence, developed an altered schema. She assimilated the word "approaching" at face value. I inferred this from her definition of limit, "A certain number or amount is approaching 0 but never equals 0……It approaches but never equals it." Furthermore, she later stated "Straight lines or curves with dots, no limit but there is a value." Using Piaget's principles, this suggests an absent perturbation. Due to her belief system that her knowledge is the truth, there is no need to change her behavior in order to would accommodate "equals" into her schema. These particular mental structures are organized into an altered schema that she used to construct solutions to the interview tasks.

Her prior knowledge of defining limit interfered with constructing the correct answer, which would have been that for a continuous function, as x approaches 2, the limit exists because f(x) approaches f(2). Yet when she looked at Graph B which had a discontinuity, she correctly decided that the x=2 was not in the domain and that the limit existed but this was because she used the prior knowledge of "approaching" from her definition of limit to construct her response. Overall, I inferred that Jean knows the definition of limit, but she has an altered concept which excludes the notion of "equals". As a result, she is in disequilibrium and incorporates these erroneous ideas into a schema that corresponds to her beliefs about reality.

As x approaches plus or minus infinity, Jean correctly interpreted the graph indicating the limit does not exist because infinity is not defined. Since she constructed the meaning of infinity correctly prior to the interview, I inferred that she assimilated the notion of "undefined" into her schema, thereby accommodating it into an appropriate schema about the existence of limits. The fact that infinity is not a number is part of her beliefs, and she revealed her knowledge and her understanding about limiting behavior of function values through problem solving.

Amanda

Amanda contends a limit is "either a number or infinity'. In Graph A, she indicated the limit exists and equals 3 at x=2 but in Graph B, she reported the limit did not exist because of the hole in the graph. She also claims that as $x \to 2$, the limit did not exist because x=2 was not in the domain. For Graph C, she states the limit did not exist where the hole is, but claims the limit exists at the point (2,4). As x approached plus infinity, she reported the "limit equals infinity and exists, and as x approaches minus infinity, the limit exists and is minus infinity". She stated infinity as "numbers go on forever so there is no particular number for the answer. It's not a reachable number, it is just a symbol that represents 'forever'. This suggests that the infinity symbol serves as a place holder for a number and so Amanda's solutions were affected by her interpretation of the infinity symbol.

Earlier in the results, Amanda elicited a vague, incomplete understanding of what a limit is claiming that "a limit can be a number or can be infinity". She also did not recognize a piecewise function as such in Graph's B and C and contended that the "limit does not exist because of the hole….which is because x=2 is not in the domain". In her conceptual structure, as $x \to a$ this means that *a* must be in the domain for the limit to exist. The reason she thinks a limit exists if it equals infinity goes back to her definition of infinity, "It means the numbers go on forever so there is no particular number for the answer". Based on her erroneous thinking that infinity is a "number" and on her definition of limit, "a limit can be a number or infinity", this seems to explain why she believes that a limit exists if it equals infinity. Based on my understanding of Piaget and Skemp, Amanda assimilated information into knowledge structures that formed an altered schema.

Overall, Amanda formed an initial conceptual structure of a linear function. Therefore, she perceives any graph that appears linear to actually be linear. This is why she did not recognize Graphs B and C to be piecewise. The reason could be that she is in a state of disequilibration (Piaget) and has not accommodated the change of domain into an appropriate schema and so "restructuring" has not occurred (Skemp).

Linsey

In Linsey's case, she defined limit as "when the left hand side equals the right hand side" and claims the limit does not exist where holes at are at (2,3) because "as x approaches 2, the point x=2 must be in the domain for the limit to exist". If there is a point on a continuous function as in Graph A, she contends the limit exists because the point is in the domain. In Graphs B and C, she says if there is a hole then the limit does not exist because “the point x=2 is not the domain”. When looking at any of the graphs as x approaches plus infinity or minus infinity, Linsey contends that the limit exists and equals infinity. Earlier, she reported that infinity was a very large number.

In Linsey’s case, she knows the definition of limit and clearly identifies the limit existing at the point in Graph A. She also correctly identifies x=2 as being in the domain. When she switches focus to Graph B, although she knows that x=2 is not in the domain, she erroneously cites this as the reason for the hole and why the limit does not exist. In this case, Linsey did not assimilate standard mathematical content into her knowledge structures, which is why her definition of limit got organized into an altered schema. In Graph C, she correctly recognizes that the domain is the same as in Graph A, but once again, she focuses on the hole in the graph and contends that limit does not exist where the hole is at (2,3). Interestingly, unlike Amanda, she contends that the limit exists where function value appears at (2,4). One explanation for this is that Linsey formed an alternate conception that limits do not exist when there are holes. A reason might be her belief that something cannot exist if it is not there in which case she thinks the limit exists 4 is the limit because the point x=2 is in the domain. This corresponds to the belief that something can only exist if it is there. Overall, Linsey knows the definition of limit such that it can both approach and equal a function value, but she erroneously contends the limit does not exist when there is a hole and that the reason is because x=2 is not in the domain. Knowledge about limits was assimilated into knowledge structures that organized into an altered schema.

Earlier in the results I discussed the students' conceptions of end behaviors. I will now discuss the interpretation of these results. Since Linsey did not elicit the correct definition of infinity, I inferred that she assimilated the erroneous notion "infinity is a large number" into her schema. The fact that infinity is a large number is part of her beliefs and is one of her mental structures, and so she applied this knowledge to her understanding about limiting behavior of function values. As a result, this perturbation prevents her from accommodating the truth about infinity into her schema.

Further Exploration of Graph C with Jump Discontinuity

Since Amanda and Linsey had unique perceptions of Graph C, further probing was done to find out more about their understandings about this graph. In the initial pilot study, piecewise functions were problematic and given the same patterns of understandings occurred, this warranted further investigation.

Amanda

Amanda claims the graph has two parts but reported the wrong domain and formula. She claimed the limit would not exist because of the hole but also refers to the point (2,4) and claims the limit is the function value, 4. "The domain is gonna be $(-\infty,2)\cup(2,\infty)$. The limit as x approaches 2 is 4, where the solid dot is above the hole. There is no limit at the hole." Furthermore she explains "hole means that a point is missing, it didn't include 2, so 2 is not in the domain. But in the first graph A, the domain was across the board all the x values". This was interesting because while she contends the limit exists in Graph C at (2,4), this contradicts her claim that "x=2 is not in the domain". The reason is that if there is an x value of 2, then surely x=2 is in the domain. It is conflicting that on one hand she contends x=2 is not in the domain in general but yet says x=2 is in the domain when she refers to the point (2,4).

Probing deeper into this, she was found to construct her actions from the linear piece of the graph, and considers the point (2,4) as a disjoint separate entity. Therefore, even though she acknowledges two parts to the graph, at no time does she identify the graph as the piecewise function with the formula $f(x) = \begin{cases} (x+0)+1, & \text{if x} \neq 2 \\ 4, & \text{if x=2} \end{cases}$. Being able to construct the formula from the graph, which is a nontraditional type of problem, is one way of determining whether or not a student knows such a function is piecewise. She was not able to do this. Instead, she only constructed the formula for the linear piece which was incorrect, stating "The formula is f(x)=(x-1)+1."

Earlier when Amanda named the kinds of functions she saw in Graphs A, B and C, she contends that all three are linear in spite of the discontinuity at x=2 in Graphs B and C stating, "the function is still linear even with holes or no holes". Moreover, she contends the domains of Graphs B and C are the same though they are different.

Even though there was a tentative explanation that fit both Piaget and Skemp's frameworks discussed in the next section, there was a curiosity to find out why Amanda thought all three graphs were linear and, rather than acknowledging that two of them were discontinuous. So once again, I probed deeper into elements of Graph C itself by focusing on the meaning of the point (2,4) and asked her as well as the other students in the study, if the point (2,4) is on the graph of the function. This question is typically very difficult for students to answer and they typically answer it incorrectly because they do not know what it means for a point to be on the graph of a function and because they do not identify the function as being piecewise. In any event, the additional probing into

Amanda's thinking reveals that she does not think the point (2,4) is on the graph of the function and I offer reasons for this.

I asked if she thought the point (2,4) was on the graph, and she stated, "No, because it's not within or on the line." That suggested to me that she did not think the function was piecewise. I did not know whether or not she knew in formal terms the definition of piecewise. In any event, I asked why she contended the limit was equal to 4 yet why she considered the function to be linear. She answered this by saying "There is no limit at the hole but there is one where the dot is because the dot is not a hole. She reported that "x=2 has to be in the domain for the limit to exist". If this were a true statement, it could explain why she put focus on the function value and identified the limit to be equal to 4. She also contends "since the graph was linear, the point (2,4) did not actually belong with the graph".

I inferred from the results that Amanda constructs her actions based on visual representations of linearity because she perceives all three graphs to be linear in appearance and therefore, calls them linear. Since she reports an incorrect domain for Graph C but sees two parts of the graph, this suggests that she did not construct knowledge linking the domain with the function. Identifying both pieces but without correct knowledge of the domain, she is unable to construct the formula of the function as one that is piecewise. The mathematical content in her knowledge structures, therefore, was organized into an altered schema.

Linsey

Linsey identified the domain but said the function was linear, as she only gave a formula for the linear piece. I inferred she did not know what to with the point (2,4), being the other piece of the function. Linsey explains "the domain is $(-\infty, \infty)$ and range is $(-\infty, \infty)$. As x approaches 2, the limit does not exist because of the hole. The formula is f(x)=(x-1)+1 I think. I confuse what goes first, the shift up or down." Upon asking her whether or not x=2 has to be in the domain for the limit to exist, she claims "x=2 must be in the domain for the limit to exist because if there is a hole, that means nothing is there—nothing in the domain and no limit".

Unlike Amanda, Linsey contends the point (2,4) is on the graph but "not on the function or graph of the function" and she did not recognize the graph as piecewise (Figure A). "It's just some random point. It's got nothing to do with the linear function since it's above the function. If you slid it down or whatever, then it would be part of the function". This reasoning was consistent with CL in the pilot study and so I inferred that Linsey lacks a particular knowledge structure for piecewise functions in her schema.

Analysis of Amanda and Linsey for Graph C

Neither Linsey nor Amanda considered Graph C to represent a piecewise function. They were not specifically asked what a piecewise function was, though, as it was assumed that being in an advanced calculus class they would know and asking this question would have influenced their responses. Based on their focusing selectively on separate pieces of the graph, I inferred that they did understand the essence of piecewise functions but this did not include a graph in which one piece was only a function value. Interestingly, each student only focused on one piece of the graph, the linear piece.

First, students defined the domain. Linsey stated the domain is $(-\infty,\infty)$ whereas Amanda claimed the domain is $(-\infty,2)\cup(2,\infty)$. In Linsey's case, she correctly identified the domain and acknowledged there were two parts on the graph, but she did not associate the point (2,4) as not being a part of the linear function, stating "it's on the graph but just not on the function". So .I then asked Linsey if she could write a formula for the graph. She was not able to construct the two pieces, only the first piece but wrote an incorrect formula, f(x)=(x-1)+1. She did not use a calculator to check her work; otherwise she would have seen the graph would intersect at x=2 and y=2. Not able to demonstrate the necessary algebra skills as well as the ability to consider constructing two pieces to the function, this suggested she did not know the function was piecewise.

Amanda was not able to produce the correct formula, either, In fact, she wrote the wrong domain $(-\infty,2)\cup(2,\infty)$; wrote the same incorrect formula as Linsey which is f(x)=(x-1)+1; did not use the provided graphing calculator to check her formula's accuracy; and did not incorporate the domain's piecewise intervals she indicated earlier. Like Linsey, she only identified one part of the function, disregarding the other part of the function which is the point (2,4). From these results, I inferred that Amanda needs different conceptual structures for algebra, as she was unable to derive the correct formula for the linear piece and does not consider the domain when constructing the formula for the function.

In order for Amanda to have recognized the graph as piecewise, she would have had to know that the point (2,4) is actually another piece of the graph. Being unable to construct the function from the graph containing both pieces, this gives some insight into why neither student thought Graph C represented a piecewise function with jump discontinuity.

Summary of Results from Graph C

In summary, two students had quite different types of perceptions about Graph C. Neither student identified the function as piecewise, but instead considered the function to only be linear. Each student reported a different domain for the graph but both agreed that the point x=2 must be in the domain for the limit to exist. Therefore, they contended the limit did not exist where the hole was because x=2 was not in the domain.

Neither student was sure about what it meant for the point (2,4) to be on the graph of the function. Amanda viewed the function value at (2,4) as being the limit, contending x=2 must be in the domain for the limit to exist. The only possible place this could happen would be the site of the function value, yet she contended x=2 was not in the domain. This reasoning constitutes an oxymoron, so I inferred that Amanda did not incorporate the point (2,4) into her schema of domains. If the limit exists at 4 because x=2 must be in the domain, then this domain is different from the one that she wrote down for the graph. Meanwhile, Linsey thought the point (2,4) was just a random point on the graph that had no meaning, other than being there since x=2 is in the domain.

Finally, as x approaches plus infinity, both students contended the limit exists and equals infinity, and similarly as x approaches negative infinity, the limit exist, equaling negative infinity. The results of these two students suggest that mathematical content got assimilated inappropriately into knowledge structures that got organized into altered schemas.

Domain Investigation Tasks

One-sided limits were used in the study to explore conceptual knowledge about domains involving both open and finite intervals. In addition to studying the graphs presented and computing limits, students explained the limiting behavior of function values for both limits at a point and for limits at infinity.

The two tasks in Figure 6.4 were selected from Task 2 in Appendix F. Presented first is what I am labeling Graph D to keep a consecutive order with the previous tasks. Following that is Graph E—both of these are officially from Task 2, but are not labeled as Graph D and E in the actual Task 2. Graph D represents the graph of the arccosine over a finite interval [-1,1], whereas Graph E contains the arccosine on an open interval (-1,1).

Differences in conceptual knowledge structures for both graphs are described for each of the four cases. Each case starts off with comparisons of Graphs D and E for limits at a point. Following this, the end behaviors of Graphs D and E are compared for limits at infinity.

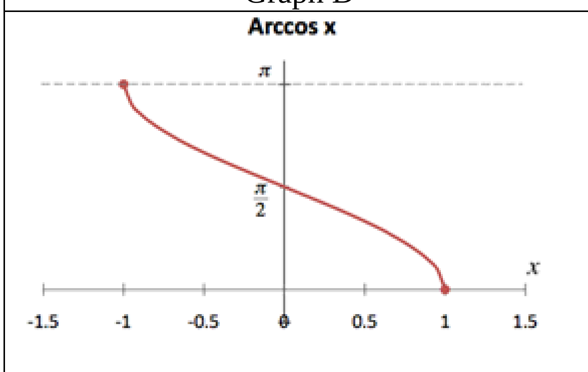
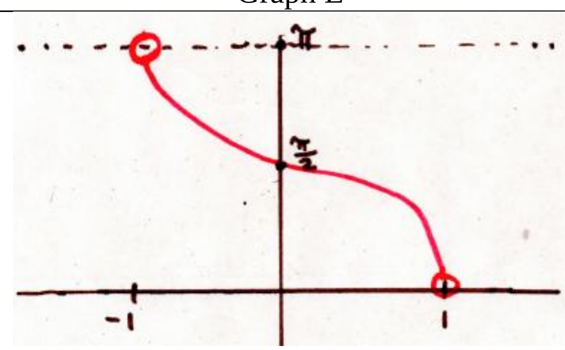
<table>
<tr><th>Graph D</th><th>Graph E</th></tr>
<tr><td>
</td><td>
</td></tr>
</table>

Figure 6.4: Graphs of arccosine with finite and open intervals.

Brendon

When presented with Graphs D and E, Brendon was asked to determine if the limit existed at the endpoints. When presented with the arccosine in Graph D, Brendon determined that the right hand limit was equal to 0, and that the left hand limit was equal to pi/2. He also reported that both of these limits exist at each of the endpoints. His responses were similar for Graph E, determining that the limits were exactly the same as those in Graph D and that the right hand limit exists at 0 and the left hand limit exists at pi. He was not confused by the open hole on each of the endpoints and solved the tasks unremarkably. Here is his complete response:

> “For arccosine, the domain is [-1,1] and range is [0,pi]. As x goes to 0, the limit is pi/2 and exists because both left and right sides equal the same. If you have an open dot you would not have a limit there, if I remember right. I am not sure though. As x goes to -1, it’s a one sided limit so the limit is pi and exists. As x goes to 1, it’s a one sided limit approaching from the left so the limit exists and is 0. When it’s continuous like this you don’t need the little superscripts to show limit as x approaches from the left only or from the right only, because we are not evaluating continuity here, only the limit. I’m thinking about continuity at the same time, but the limit would still be pi/2 on the left and so it exists. ”

Next, Brendon was asked to consider the limiting behavior of the function values as x approached positive and negative infinity for Graphs D and E. He reported that as x approached infinity or negative infinity, the limits did not exist. His response to Graph D was, “As x goes to infinity the limit doesn’t exist because if it’s out here past the domain then it’s not going to have an actual y-value”. Then for Graph E he stated,

> “For arccosine with holes at endpoints, the domain is now (-1,1) and range is (0,pi). The domain does not include the endpoints. So the limit as x approaches 1 is 0, and as x approaches -1 the limit is pi. The limits exist even though there are open dots because limits are about what happens near the x value, not at x. So even though x is not in the domain on either endpoint, the limit still exists.”

Brendon’s knowledge structures seem to contain the necessary components to construct interpretations of the limiting behavior for one-sided limits over closed and open finite interval domains. I inferred from his work that his knowledge structure contains information about the limit existing at a point when there is a closed interval with Graph D in which case there is a point, or an open interval with Graph E in which case there is a hole. I also inferred that his knowledge structures contain appropriate mathematical components for limits at infinity, with respect to “no y-values” going past either of the finite-interval domains in both graphs. Given the assimilation of mathematically standard content knowledge, Brendon developed appropriate schemas.

Jean

When presented with Graphs D and E, Jean was asked to determine if the one-sided limits existed at the endpoints. Jean reported:

> “the domain for arccosine is fixed between [-1,1]. Range is [0,pi]. The limit is about what x is approaching, not equal to, in the domain. As x approaches 0, the limit is pi/2 but it does not exist. As x goes to positive 1, the limit is converging to 0 and as x goes to -1 the limit is converging to pi but these limits do not exist.”

Recalling the definition of limit she gave earlier, “limits are about what x is approaching and not equals”, this corresponded to her response to the open endpoints on Graph E. When she saw the open endpoints, Jean reported that the limit existed at both endpoints because the function value was absent. In her words,

> “The domain is (-1,1). The limits exist even though there are open dots because limits are about what happens near the x value, not at x per se. So even though x is not in the domain on either endpoint, the limit still exists.”

As a result, she reported the left hand limit as -1 and the right hand limit as 1, claiming that they both of these one-sided limits at a point exist.

Next, when Jean was asked to consider the limiting behavior of the function values for arccosine as x approached positive and negative infinity for Graphs D and E, she reported the limits did not exist for both cases. In response to Graph D, she stated:

> "For arccosine with closed dots, I would say since there is no fixed value outside of that as you approach infinity, so the limit either does not exist. As x approaches negative infinity, the limit does not exist because it's not in the domain so this one doesn't have meaning. I say does not exist for both because it has a certain value of y, so it is fixed in this interval and so it is not approaching anything. The limits don't exist because there is no x in the domain. So the question makes no sense."

Similarly, Jean contended the same for Graph E with the open endpoint when she claims "I say the limit does not exist for both at plus or minus infinity, because it is fixed in the interval and so it can't approach anything".

Jean consistently incorporates her definition of limit in her work, claiming that the limit can exist only where there is a hole but not a function value for both the limits at a point and limits at infinity. Given her beliefs about the definition of limit, she states that "as x approaches positive or negative infinity, the limit does not exist" for graph D, but "the limit exists" for graph E. Meanwhile, she makes the appropriate determination that it "makes no sense" to ask even about end behaviors approaching infinity over either of the finite intervals, because there are no x-values beyond the domain. From her responses, I inferred that her conceptual structures contain an established definition of limit and appropriate knowledge about domains for one-sided limits.

Amanda

When asked about limits at a point and presented the arccosine containing closed endpoints in Graph D, she reported, "the domain is [-1,1]. The range is [0,pi]. As x goes to minus 1, the limit is pi. As x goes to 1, the limit is 0". When presented with the arccosine with open endpoints in Graph E she stated, "the domain is (-1,1) and the limit does not exist as x approaches 1 and -1 because it is not in the domain. The hole messes it up."

Next, Amanda was asked to consider the limiting behavior of the function values as x approached positive and negative infinity for Graphs D and E. For Graph D, she was asked, "For arccosine of x, what's the limit as x approaches plus or minus infinity? Can a limit exist to the right of 1 or to the left of -1? Explain." Her response was, "as x approaches positive infinity, the limit is 1. As x goes to minus infinity, the limit is -1". Next, we referred to Graph E and she stated, "The domain is (-1,1) and the limit does not exist as x approaches 1 and -1 because it is not in the domain. As x approaches infinity and minus infinity, the limit won't exist either because of the hole, it's not in the domain."

From these results, I inferred that Amanda has a unique conceptual structure for one-sided limits at infinity, since she contends the limits would be -1 and 1 as x approaches positive or negative infinity if there were closed endpoints (Graph D) and the limit does not exist with open endpoints (Graph E) because of the hole. At no time did she acknowledge that the limits at infinity actually did not exist and so for neither graph did Amanda mention what was happening with x-values past the finite intervals. However, Graph E reveals an important knowledge structure, which is that limits do not exist when there are holes. Amanda appears to have different knowledge structures and hence, an altered schema for limits at a point for functions over closed and open interval domains.

Linsey

When presented with Graphs D and E, Linsey was asked to determine if the limit existed at the endpoints. She stated for Graph D, “The domain is [-1,1] and range is (0,pi). Limit as x goes to 0 is pi/2. Limit as x goes to -1 is pi, and limit as x goes to 1 is 0. Then for Graph E she stated, “the domain is (-1,1) and the limits as x goes to -1 and 1 does not exist because of the holes”. It seems that Linsey uses her definition of limit while solving these tasks, sticking to the belief that limits do not exist where there are holes, and this includes on domains with finite intervals.

Next, Linsey was asked to consider the limiting behavior of the function values as x approached positive and negative infinity for Graphs D and E. First, she was asked about Graph D, “For arccosine of x, what’s the limit as x approaches plus or minus infinity? Can a limit exist to the right of 1 or to the left of -1? Explain”. Her response was, “No the limit does not exist outside of 1 and -1 because there aren’t any x-values out there to put into the function to find the limit. Next, for graph E she said, “the domain is (-1,1) and the limit as x goes to minus infinity doesn’t exist because of the hole and neither does the limit as x goes to positive infinity. There is no limit because of the hole I think.”

From these results, I inferred that Linsey has a unique knowledge structure about one-sided limits with closed endpoints on a finite interval, which is there were no x-values past that domain to consider. Another common knowledge structure she has is that limits do not exist if there are holes. In this case, her reason for the limit not existing as x approaches infinity on a domain with the open endpoints is because of the hole. For open endpoints, she does not consider the x-values beyond the domain. Therefore, Linsey appears to have different knowledge structures and hence, an altered schema for limits at a point for functions that have finite interval domains.

Do Limits Exist that Equal Infinity?

In both Phase I and Phase II, there were rather consistent responses from students when presented with the case of infinite limits, a lot of which has already been described and explained. However, there are additional aspects of students' responses worth mentioning. Earlier with Task 4 in Figure 6.2, results were reported for limits at a point for the discontinuities as well as limits at infinity resulting in the case of infinite limits for each of the four cases. Recall, infinite limits refer to function values that increase without bound in either the positive or negative directions. At that time, it was shown that Brendon and Jean reported that limits do not exist if they equal infinity, whereas Amanda and Linsey reported that limits do exist.

First, though, here is some background information to anyone without mathematical expertise. After a limit is computed and the behavior of the function values increase without bound, it is correct to state that the approaches infinity and therefore, does not exist. The reason is that infinity is not a number, and by definition, a limit must be a number in order to exist. At no time would $| f(x) - \infty | < \varepsilon$. The reason is that infinity cannot be subtracted from a real number and be smaller than the number defined as epsilon (on the y-axis). Therefore, given the importance of this implication, additional questioning was performed to find out how their definitions of limit and infinity are used when contending that a limit exists if it equals infinity.

While Graph F, the rational function, serves as a perfect example for an infinite limit that occurs for a limit at a point, a few other particular graphs of infinite limits are discussed for individual cases to illustrate the different ways of knowing. Students were also asked directly if a limit exists if it equals infinity and it will also be shown how students actually compare end behaviors of infinite limits, a procedure that is supposed to be done for limits at a point.

Brendon

Brendon presents the exemplary case so when asked if a limit is equal to infinity, does that limit exist"? He stated, "when a limit goes to infinity, I just say the limit does not exist but on graphs I'll write "$=\infty$" but it means d.n.e.…as x goes to infinity and minus infinity the limit is 0…as x approaches a, the limit does not exist because the left hand limit does not equal the right hand limit". When shown graphs A, B, and C that were presented earlier in Task 4, he said "the limit as x approaches plus infinity is positive infinity and the limit as x approaches minus infinity is negative infinity. Those limits do not exist since infinity cannot be reached". He elaborated further about teachers sharing different opinions about this and notes the following:

> "It depends on who the teacher is. The high school teacher said it equals infinity the value exists even though infinity is technically not a real number. The teacher said it depends on if you consider infinity existing or not. But the college professors said the limits don't exist if they go to infinity, and I tend to believe them".

Jean

Jean simply stated "no" when asked "if a limit is equal to infinity, then does that limit exist"? Going back to her definition of infinity, she said "it's too large to count and the infinity symbol represents something that is too large to count". It is apparent that her conceptual structure does not include a "large number" but contains a more vague description, "something too large". In Graphs A, B and C presented earlier, she said "As x goes to plus infinity the limit is infinity and for x goes to minus infinity, the limit is minus infinity so they don't exist". I inferred from Jean's statements that her knowledge structure combines infinity being "too large" with the limit cannot exist since infinity is not a number, by definition.

Amanda

When asked "if a limit is equal to infinity, does the limit exist"? She responded, "usually, yes. My instructor JT said 'no' in the first chapter but later said 'yes' it exists later on in the course. He taught us that if the limit goes to infinity, then it doesn't exist but it does." This seemed to be an oxymoron but an important point giving insight into instructional practices. Next, I asked what "infinity" is, she responded, "it's not a reachable number, it is just a symbol. It represents 'forever'…. It means the numbers go on forever so there is no particular number for the answer".

Returning to Graphs A, B and C that appear in Task 4 (Figure 6.2), Amanda stated "as x goes to infinity, the limit exists and is infinity and as it goes to negative infinity the limit exists is gonna be negative infinity". During Task 3 (Appendix F), there were also some problems with infinite limits. So again I asked her "explain the behavior of the function values as x approaches plus or minus infinity, do these limits exist"? She replied,

> "It's negative something and to the right it's positive something. Whether or not it exists depends what chapter we were in. In algebra, it's no. In calculus, the answer is yes that it exist. But they don't teach you anything so I guess it's yes, the limit exists depending on teacher error".

What is interesting is that this is at least the second time the issue of instructional practices arises, that teachers have different opinions about the limit existing if it is equal to infinity. As a result, I inferred that both instructors and students have knowledge structures containing beliefs that limits exists if they equal infinity. Amanda's definition of infinity, going back, was “forever, not a reachable number”. This response seems to infer that infinity is a number, but just not a reachable one. Therefore, I inferred that Amanda, and possibly instructors who do this, have conceptual structures that include infinity as some type of large number. Since a limit has to be a number in order to exist, it would make sense in this conceptual structure for a limit to exist if it is some type of very large, unreachable number.

Linsey

When asked, “If a limit is equal to infinity, then does that limit exist?” Linsey responded, “yes”. As a result, I asked what “infinity” is, she responded: “Infinity is a place that has no end to it. So if a limit equals infinity then infinity is where the limit would exist. There is no assigned number. It just keeps going”. From these responses, I inferred that Linsey has a knowledge structure and altered schema in which the infinity symbol represents a place that can be reached.

Earlier in this discussion, Graph A presented which is part of Task 4 in the study (Figure 6.2). Linsey states, “As x goes to plus infinity the limit exists and is plus infinity and as x goes to minus infinity the limit exists and is minus infinity”. A similar finding occurred with the piecewise function presented in Figure 6.5 which refers to Task 3 (Appendix F). It will be noted that she makes similar claims and wrote “limit exists” next to the infinity symbol on the graph. She was asked to “Explain the behavior of the function values as x approaches plus or minus infinity and explain if these limits exist or not”. Her response was, “I think as x goes to plus infinity the limit exists and is negative infinity. As x goes to negative infinity the limit exists because it’s negative infinity”.

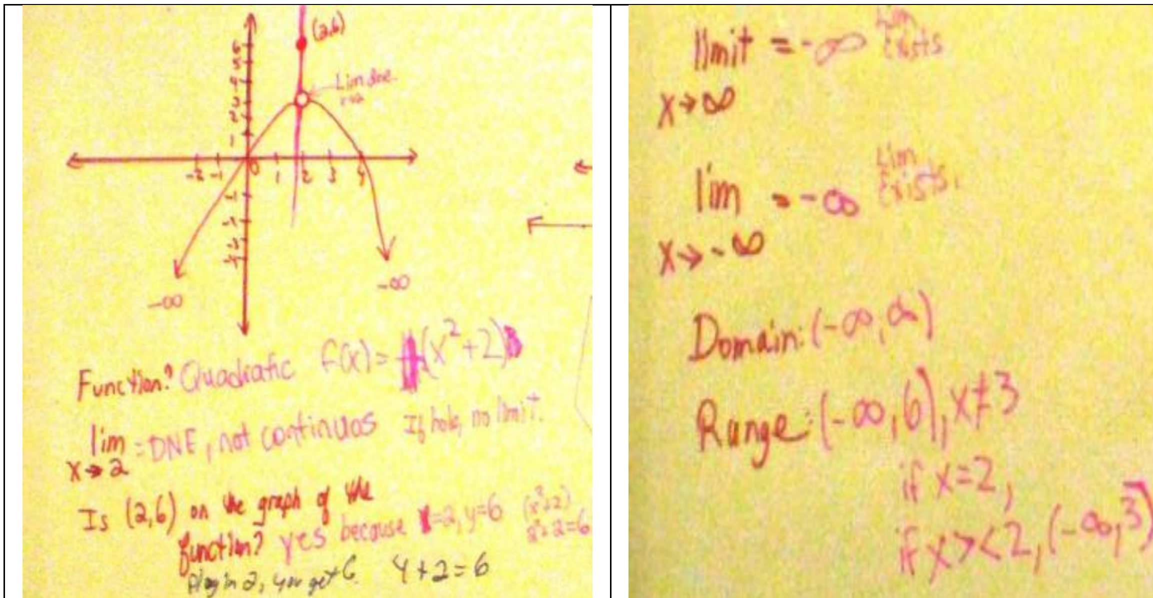


Figure 6.5: Linsey's work on Task 3--the limit exists if it equals infinity.

Given the nature of the responses, I inferred that Linsey's conceptual structures contain knowledge of limits existing when they are equal to infinity. In her definition of infinity, she contends "infinity is a place with no end to it". From this, I inferred that since infinity is a place, then the place must exist. So whenever the infinity symbol is written, she thinks the limit exists.

Earlier, in her definition of limit, she indicated **"**A limit is when you see if the left side is equal to the right side. I don't know how to explain it other than that. There is no limit if there is a hole". From this definition, I inferred that Linsey's conceptual structures do not contain the notion that a limit has to be a number to exist. Moreover, I also asked her, **"**Do you say that the limit equals infinity", or do you say "the function values are approaching infinity"? Explain. Her response was as follows:

> "I say the limit equals infinity. It takes too long with function values. I would get confused saying all that and since you are finding a limit not a function value, I would just say the limit equals infinity".

The statement, “you are finding a limit not a function value”, this is a significant finding because it appears her conceptual structure is one in which limits are distinct from function values. This would explain why she is not associating the behavior of the function values with the infinity symbol, and instead, decides the limit exists just by the fact that the infinity symbol represents a place as she mentioned earlier.

Comparing Left Hand Limit with Right Hand Limit for Limit at a Point

Linsey

Referring now to the actual rational function used in Task 9, which is similar to the rational function Graph A in Figure 6.6, there is another important point worth noting which occurred in the pilot study as well as in the second phase. NS stated in the pilot (Appendix D) when given $\lim_{x \to 0} \frac{1}{x}$, the left hand limit, minus infinity, is not equal to the right hand limit, plus infinity, and since they are not equal the limit does not exist. Other students also do this, comparing the left hand infinite limit with the right hand infinite limit when deciding if the limit at a point exists, rather than just consider one side to make the determination. In Figure 6.6, there is an example of this generated by Linsey.

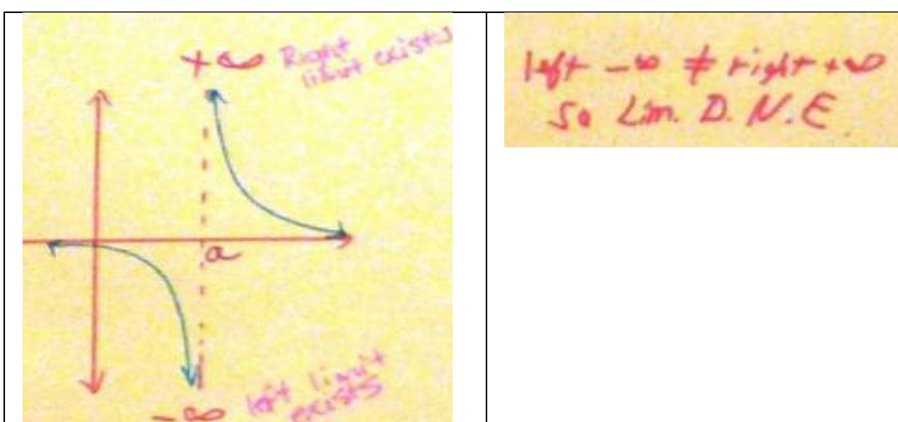


Figure 6.6: Linsey’s work on Task 9.

In Task 9 where the rational function was shown, students were asked to describe the end behaviors for limit at a point as x approached a, $\lim_{x \to a} \frac{1}{x-2}$. As it can be seen, Linsey wrote on the top of the graph that the right limit exists as it equals infinity, and the left hand limit exists as it equals negative infinity. Then she proceeded to compare both end behaviors, determining that the limit does not exist because negative infinity on the left side did not equal positive infinity on the right side. She stated,

> "the left limit equals minus infinity an it exists, but the right limit equals plus infinity and that exists so you have to compare the left with the right and since minus infinity does not equal plus infinity the whole limit as x approaches a does not exist".

The results of Linsey's work suggest that her knowledge structure extends the procedure of comparing the right hand limit with the left hand limit for limit at a point to end behaviors involving infinity. This might be because her knowledge structures include both a limit and infinity as numbers to be compared. In her definition of limit, she mentions "when the left and right come together" so it appears that her knowledge structure includes this particular definition of limit, given she compares both sides. Consequently, her knowledge structures organized into an altered schema.

Amanda

Amanda also compares infinite limits for limits at a point. When presented with this same task, she was asked "as $x \to a$ do you have to compare the left hand limit with the right hand limit to decide if the limit exists"? Her response was "yes because they go in 2 different directions so in each direction the limit exists since it equals infinity, but comparing them together it doesn't exist because one is plus infinity and the other is minus infinity.

Here it can be seen that in addition to Linsey, Amanda demonstrates the same type of conceptual knowledge structure. In both cases, the conceptual structure incorporates a limit as being a number or infinity as well as the procedure of comparing the left hand limit with the right hand limit.

Another task Amanda completed is also presented as there is visual evidence of her work which appears in Figure 6.7 (Task 10). First, Amanda identifies the graph as exponential rather than rational then claims as x approaches “a” that the limit exists if it equals infinity. Amanda looks at the asymptotic behavior by the y-axis and acknowledges that both sides approach positive infinity, but then compares the left and right infinite limits to decide if the limit exists for the whole function. She states the following:

> “Domain: x not equal to 2. It’s an exponential function. As x approaches 2, the limit equals infinity on both sides and so it exists. The limit exists as x approaches infinity and is equal to infinity and as it approaches negative infinity, the limit exists and is also negative infinity. You plug in 2 and get 0, so you do 1.999 and 2.0001 to see what it’s doing. Do 1.9 – 2 and square it. Do 1.99 – 2. Then do 1.999-2. You are going to positive infinity. Under 2, you do 2.1, 2.01, so take 2.1-1 and then 2.01-2 and you also get plus infinity. The limit exists and equals positive infinity. Since both sides are infinity, the left side is the same as the right side, the limit exists”.

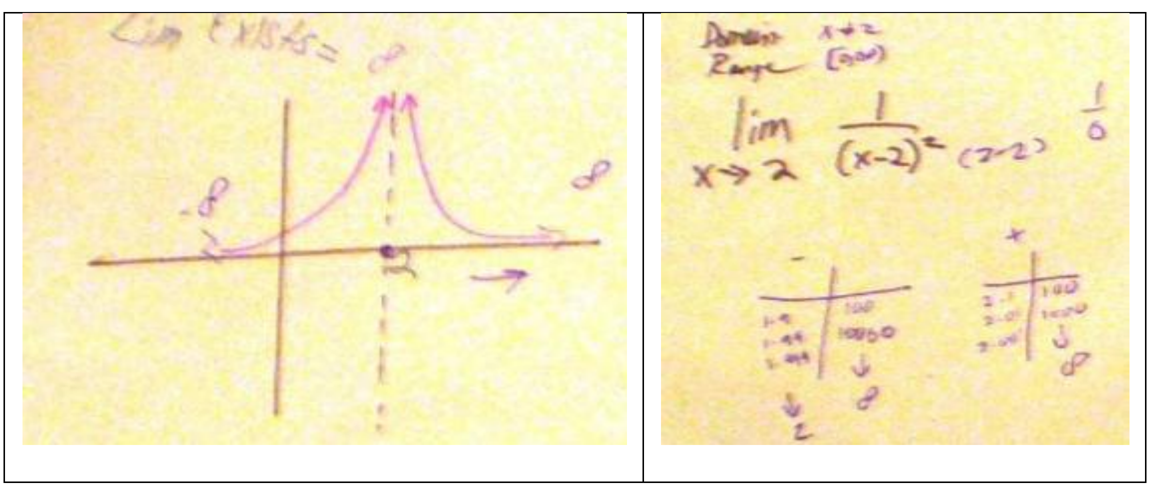


Figure 6.7: Amanda’s work on Task 10.

As a result of these statements from Task 10, I inferred that Amanda's conceptual structure incorporates the notion that limits exist if they are equal to infinity, and that the left hand limit (infinity) must be compared with the right hand limit (infinity) to decide if the limit exists for the whole graph. The type of conceptual structure is the same for both cases. The first is if both sides tend toward positive infinity; the second is as seen with the rational function, when one side tends toward positive infinity and the other toward negative infinity. Even though she claims the infinity symbol does not represent a number since "you can add 1 to infinity and still get infinity" and claims infinity is "not a reachable number", it seems she still associates the symbol with representing a place, being labeled an unreachable number. Making this association might explain why she claims the limit exists if it equals infinity. In general, her conceptual structures seem to involve a focus on the symbol instead of on the behavior of the function values. Similar findings occur for her interpretations of infinity with the other two graphs in Task 10 which appear in Appendix F for the tasks and Appendix J-4 for images of her work. The results suggest her conceptual knowledge structured organized into an altered schema.

Asymptotic and End Behavior Investigation

The conceptual knowledge structures of end behaviors were also investigated. In addition to infinite limits, there are other types of end behaviors to be discussed that appear in Figures 6.8 and 6.9. The rational function in Figure 6.8 is presented first. A similar type with a horizontal shift right two units was presented in Task 9. Here it is labeled Graph F to keep a consecutive order listed in these results with the previous tasks. Next is the graph of the cosine function used in Task 1, referred to here as Graph G which also appears in Figure 6.8. Last, the damped cosine function used in Task 2 is labeled Graph H and appears by itself in Figure 6.9.

The graph of the rational function presented first contains both a limit at a point which results in an infinite limit, as well as a limit at infinity that equals 0. The cosine function's limits do not exist in either direction as x approaches positive or negative infinity due to oscillatory behavior. The damped cosine function is unique in that as x approaches negative infinity, the limit does not exist but as x approaches positive infinity, the limit is zero. Therefore, these three graphs selected from different tasks in the study are presented to illustrate various perceptions and conceptual knowledge structures students have about end behaviors. In addition to computing limits, students explained the limiting behavior of function values for both limits at a point and for limits at infinity, including whether or not a limit exists if it equals infinity for these tasks.

Conceptual knowledge of end behaviors is presented by comparing their perceptions of three different types of graphs: the rational, cosine and damped cosine functions, respectively. The tasks appear in Appendix F and their work on these tasks appears in Appendix J.

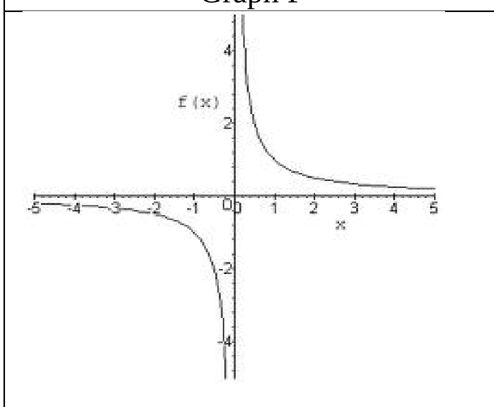
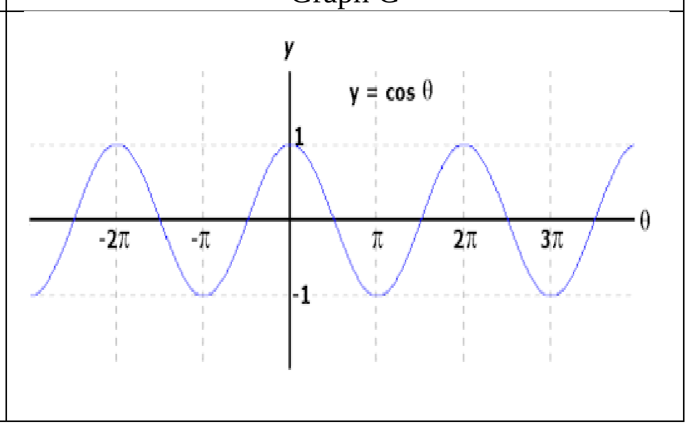
Graph F	Graph G
	

Figure 6.8: Graph F, rational function $f(x) = \frac{1}{x}$; Graph G is the cosine function.

Graph H

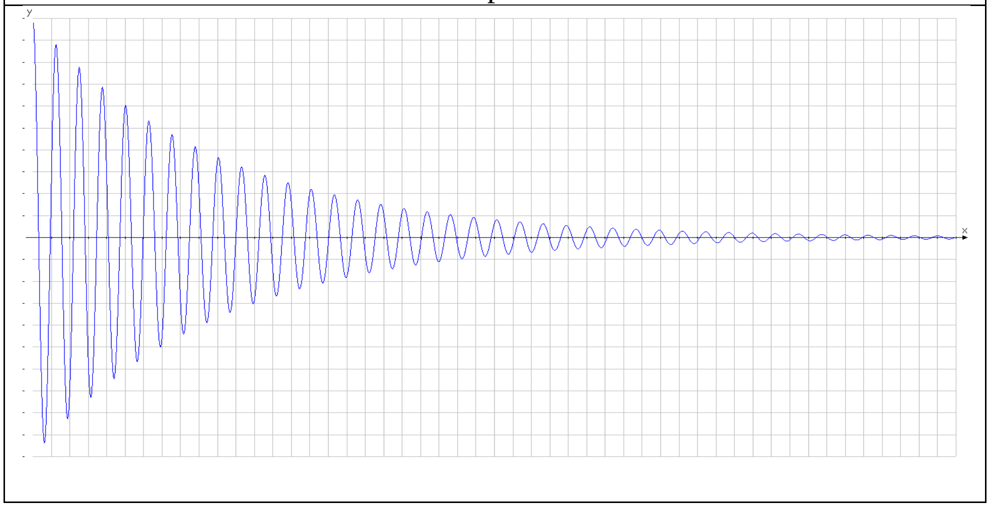


Figure 6.9: Graph of damped cosine $\lim_{x \to \infty} e^{(-x)} cos(x)$ and $\lim_{x \to -\infty} e^{(-x)} cos(x)$.

Brendon

Brendon focused on the behavior of the function values and concluded that for the rational function in Graph F, the limit is 0. "As x goes to infinity and minus infinity the limit is 0… The function values go to 0 and that's why the limit is 0. The values get smaller and smaller as x increases". From his responses, I inferred that his conceptual structures contain a mathematical perspective involving the behavior of function values, rather than a visual interpretation of the graph. Moreover, his definition of limit is part of his conceptual schema because he defines limit intuitively as "the y-value that is approached". This coincides with his remarks about the "function values that go to 0". From this I inferred that he has two knowledge structures for understanding end behaviors, one for the definition of limit and the other for the behavior of function values.

When shown the cosine function in Graph G, he stated, "The limit as x goes to infinity has no set value it settles down at so since it keeps oscillating the limit does not exist….You can like move your finger along the x-axis and watch the y-values they range between 1 and -1 and don't stop at any point so that is why the limit does not exist". From his responses, I inferred that his conceptual structures contain a mathematical perspective involving the definition of limit previously stated "the y-value that's approached" as well as the behavior of function values not settling down to one point, rather than a visual interpretation of the graph.

When shown the damped cosine function in Graph H, he contends that as x approaches positive infinity the limit is equal to 0. "Domain is all x and the range also looks like it's all y or minus infinity to infinity. The limit as x goes to infinity converges to 0 because the values keep oscillating and get smaller and smaller." He also explained that as x approaches negative infinity, "the limit doesn't exist because it's like the cosine on the left side of the graph it keeps oscillating but gets larger and larger. This reminds me of a problem with springs and dampers in systems dynamics—values of chasse and shock systems…" In both cases, Brendon's conceptual structures contain the definition of limit along with knowledge of how and why the limit does not exist due to the function values not settling down to any given point rather than a visual interpretation of the graph.

Jean

Jean gave similar responses to Brendon, stating for Graph F with the rational function, **"**as x goes to infinity the limit is 0. Same for minus infinity, the limit is 0". Then for graph G the cosine, the **"**limit as x goes to infinity or minus infinity keeps bouncing up and down so the limit does not exist". In her definition of limit, she stated that a limit is "a certain number or amount that is approaching (0 or number) but never equals (0 or that number). It approaches but never equals it". From these responses, I inferred that Jean's knowledge structures contain the definition of limit which she uses when deciding that the limit is 0. Since in her case the limit is only about approaching but not "equals", this might explain the nature of her perception of the limit being 0. If the graph had touched the x-axis, then she possibly would have said that the limit did not exist because it was equal to 0. So her knowledge structure is established, containing a very unique definition of limit that involves "approaching but not equals".

Lastly for graph H, the damped cosine function, she stated,

> "The limit as x goes to infinity diverges to 0 because the values keep bouncing up and down on left and get smaller and smaller on right. It is becoming less and less but is approaching 0.As x goes to minus infinity then the limit doesn't exist because it's like the cosine on the left side of the graph it keeps bouncing up and down but gets larger and larger".

From her responses, I inferred that Jean also has knowledge structures for the function values, because she knows the truth of the matter that the function values on the left do not settle down one particular point whereas on the right, the function values "diverge".

On this note, I think she uses the word "diverge" in an interesting way which includes her definition of limit "approaching by not equal". In mathematics, a series either converges to a number, or it diverges to infinity. So what Jean says the function values "diverge to 0" is that they approach but never equal zero, and this is also consistent with her definition of limit.

Linsey

Linsey's perception was somewhat different. When studying the graph F of the rational function, she stated "as x goes to infinity the limit is 0 because the lines get closer to 0. Same for minus infinity, the limit is 0". It is not the answer that is the concern but rather the fact that she mentions "the lines get closer to 0". This suggests her conceptual knowledge structure processes visual cues, as she explains the graph from a visual perspective rather than from a mathematical one which would focus on the behavior of function values. The conceptual structure containing her definition of limit does not apply to end behavior in this case as the definition "when the left side is equal to the right side" since her definition of limit is exclusive to limit at a point. Therefore, for end behaviors, it seems she does not have a knowledge structure in place incorporating the definition of limit. One reason might be due to this alternative visual knowledge structure she uses. Therefore, it seems her knowledge structure relies on visual images.

When studying Graph G, cosine, Linsey indicated that

> "as x goes to plus infinity the limit doesn't exist because it could be anything. It's a whole bunch of points and it keeps going forever. So it's hard to determine where it is going when it goes to infinity. If I plugged in 3 pi it's that number -1 but for 4 pi it is up there at 1 so the left doesn't equal the right because it keeps going".

Here, the reason given "the left doesn't equal the right because it keeps going" shows evidence of a different conceptual structure, possibly an oxymoron since these two events cannot occur simultaneously. In particular, "the left doesn't equal the right" applies to limit at a point, whereas "keeps going" implies the end behavior. There seems to be some confusion and blending of both ideas when explaining the end behaviors.

When studying Graph H, the damped cosine function, she claims "the domain is $(-\infty,\infty)$. Range is also $(-\infty,\infty)$. As x goes to plus infinity the limit is 0. As x goes to negative infinity the limit does not exist because it keeps oscillating". Again, I inferred from her explanation that her knowledge structure consists of visual interpretations.

Amanda

Amanda had a unique take on these graphs. When presented with the rational function in Graph F, instead of reporting that the limit was zero she indicated that “as x goes to infinity, the limit is infinity and so it exists and the limit exists and is minus infinity the other way”. Her responses are consistent with those from Task 4 involving the infinite limit as her definition of limit is “either a number or infinity” and contending that limits exist if they equal infinity.

When presented with Graph G, the cosine function, she stated, “As x approaches infinity, the limit exists because it equals infinity. As x goes to negative infinity the limit exists which is negative infinity because it keeps going and doesn’t stop”. Although this response is different from the others, it is an important one and was also seen in other interview cases from both the pilot and second phase of the study. This is why Amanda was selected to represent other students who fit into this particular category. In any event, I inferred that Amanda has knowledge structure that relies on visual images rather than mathematical behavior of function values. Earlier, Amanda described a limit as “a number or infinity”. Here, with the rational function, it appears to her that the line of the function never touches the x-axis; therefore, the two points do not come together. In this regard, her conceptual structure contains her own unique definition of limit, which explains the nature of her responses. Since she defines limit as being “infinity” this might explain why she reports that a limit equals infinity. These knowledge structures organized into an altered schema.

Next, considering what C.L. reported in Phase I about the sine function having 2 limits, I asked Amanda if for cosine there can there be two limits, at -1 and 1". Her response was, "yes because it bounces off -1, then bounces off 1, then bounces again off -1 so it keeps bouncing off these 2 numbers so I think these would both be limits". This is important because it confirms and validates results from elicited by C.L. in Phase I, from which I inferred that Amanda has another conceptual structure of limits for the cosine function containing a graphical images of the function's range [-1,1].

When presented with Graph H, the damped cosine function, Amanda stated, "as x approaches infinity the limit is gonna be closer to 0 so the limit is 0. As x goes to minus infinity, the limit is gonna be negative infinity". I inferred she inspects the graph visually rather than mathematically because on the right, the graph visually appears to oscillate but levels off at 0. On the left as x approaches negative infinity, she drew the same conclusion as with the cosine function in Graph G, claiming that the limit equals infinity. It seems her knowledge structures consist of visual interpretations as well as elements involving a limit being a "number or infinity". Therefore, the reason she says the limit equals negative infinity is because she is not studying the behavior of the function values, but rather, is focusing on the visual aspects only of the graph. Alternatively, she might also be focusing on the x-axis going to infinity in either direction rather than on the limiting behavior of the function values.

Asymptotic Behaviors of Limits at a Point

One last result involves algebra proficiency with understanding limits. Earlier it was shown that for the limit at a point for the rational function, Brendon and Jean reported that the limits do not exist as x approaches zero because the function values increase without bound, whereas Linsey and Amanda reported that the limits did not exist because the left hand limit, minus infinity, did not equal the right hand limit, positive infinity.

What is missing was their knowledge about the domain and understanding about the function itself when there is a 0 in the denominator and so a further investigation using a very specific task probed deeper into how they think about rational functions when there is a 0 in the denominator. Figure 6.10 contains a graph of cosine  which will be referred to as Graph I.

This graph appeared in Task 2 and students were asked to describe the limiting behavior of the function values as x approached 0 as well as when x approached infinity. They entered the function into a calculator and adjust the windows for detailed information about the graph. In the Phase I pilot study, CL reported that there is no limit when zero is in the denominator because the function is undefined. Her counterpart, NS, contended that the limit had nothing to do with zero being in the denominator. Therefore, this particular task offers some insight into how students think about limits of functions when there is a zero in the denominator as well as the asymptotic behavior for its limit at infinity.

Graph I

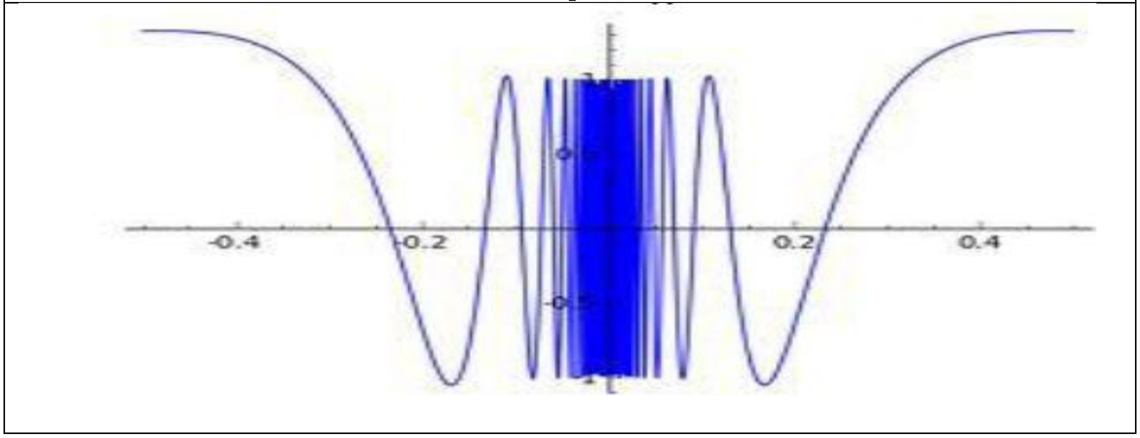


Figure 6.10: Graph of $\lim_{x \to 0} \cos\frac{1}{x}$ and $\lim_{x \to \infty} \cos\frac{1}{x}$.

Brendon

Brendon thought the domain is "all x-values and the range is [-1,1]" but studied the graph visually rather than consider the denominator. Regardless, it did not affect the outcome of his interpretation. He did not report that the function was undefined when 0 was in the denominator. In fact, he stated, "as x approaches 0, with zero in the denominator, the limit does not exist because it keeps fluctuating and oscillating, getting infinitely smaller, not settling down to any particular number. The function may not be defined at zero but the limit is about what is going on near zero, so nobody cares what's in the denominator". As for the limiting behavior of the function values as x approaches infinity, he claimed "as x goes to minus infinity and plus infinity the limit is 1". From the results, I inferred that Brendon's knowledge structures consist of mathematical notions such that he is able to describe the limiting behavior of the function values near 0. Given the nature of his knowledge structures for the limit at a point and limit at infinity, he has an appropriate schema.

Jean

Jean had a similar interpretation about the domain of this function to Brendon, contending that “ the domain is all x-values and the range is [-1,1]” . Then interestingly, she compared this function to the cosine function and stated “cosine (0) is 1 so the answer cannot be 0”. After further thought without use of the calculator, she studied the function values near zero and stated, “As x approaches 0, the limit does not exist because values keep bouncing. Limit is about what is near not at zero. As x goes to minus infinity and plus infinity the limit is 1”.

Jean’s conceptual knowledge structure focuses on information about the limiting behavior of the function values near 0. She also has a knowledge structure that contains information about asymptotic behavior of the function values, with a limit being equal to one as x approaches positive or negative infinity. Given the nature of her knowledge structures for the limit at a point and limit at infinity, she has an appropriate schema.

Amanda

Amanda also studied the graph visually at first and claimed the “domain and range is [-1,1]”. Then she reported difficulty working with the function. “As x approaches 0, the limit is gonna be hard because I don’t understand 1/x. I guess the limit does not exist because of 0 in the denominator”. Here is evidence of a student focusing on what happens with the denominator rather than on what happens near 0 for the limit. She also claims the domain is all x values, yet mentions the zero in the denominator. So the domain stated is not consistent with her interpretation of the function.

Next, she considered what happens as x approaches infinity and showed evidence of various knowledge structures because she stated, “as x goes to infinity, the limit is infinity. As x goes to minus infinity, the limit is gonna be infinity. Well, what is it going to, 2 here on the x-axis? Wait. As x goes to infinity, the limit is 1 and the other way it's -1 because I am just looking at where it stops on the x-axis and can't go any further.

Amanda has an altered schema that contains various knowledge structures. One of the knowledge structures stems from visual cues present on the graph and another knowledge structure is exclusive to algebra of rational functions. There does not seem to be a knowledge structure that focuses on limiting behavior of function values of a rational function, and this is consistent with the evidence shown earlier for the rational function $f(x) = \frac{1}{x}$ at which time she claims that as x approaches infinity, the limit is infinity.

As a result, Amanda shows evidence of an altered schema consistent with what she reported earlier about the point having to be in the domain for a limit to exist. She applies this notion to her problem solving. Her knowledge structure contains the notion that zero is in not in the domain of this particular function; therefore, the limit doesn't exist. There are other knowledge structures involving infinity that also fit in this altered schema.

Linsey

Linsey also reported the domain being $(-\infty, \infty)$ and range [-1,1]. She then stated "as x goes to 0 the limit doesn't exist, because you can't pinpoint any point on there. It goes crazy by 0. If you plug in 0 it's undefined but close to 0 there are all different numbers. She also declared that "as x goes to minus infinity and to infinity the limit is one". From her responses, I inferred that her knowledge structures allow her to focuses on the limit rather than on the function being defined. Given the nature of her knowledge structures for the limit at a point and limit at infinity, she has an appropriate schema.

In summary, this problem revealed that most students have an appropriate schema for studying this particular task, there are some who possess an altered schema by focusing on the function when there is a zero in the denominator. One interesting observation, though, is that all students reported the domain to be all x-values. This could be that they were distracted by the visual appearance of the graph and therefore did not acknowledge that zero was excluded from the domain.

Discussion

Brendon appeared to possess the necessary knowledge structures for the definition of limit and showed evidence of possessing an appropriate schema. The other three students knew parts of the definition of limit. The information contained in their knowledge structures was applied consistently when problem solving in which case altered schemas were involved. For instance, when Jean said a "limit can only approach but not equal", she demonstrated this consistently and so at no time was a limit ever equal to a function value. Neither Amanda nor Linsey thought a limit could exist where there was a hole representing a discontinuity. This was the case for functions over infinite intervals or those with finite interval domains. Both students showed evidence of altered schemas.

The inquiry into their knowledge of infinity and infinite limits also generated interesting results. Recall by definition, a limit must be a number and infinity is not a number. Therefore, when a limit "equals infinity" the limit does not exist because infinity is not a number. When the x values get larger in the positive (or negative) directions, this is represented with the notation $x \to \infty$. The end behaviors of the graphs in Figure 6.2

(Task 4) show that as x is getting larger, the function values are also increasing without bound and so the limit do not exist.

Infinity was problematic for Amanda and Linsey who considered it to be large number or a place. When infinite limits occurred, these students reported that limits exist if they equal infinity. Moreover, when given a rational function for limit at a point and reported on the asymptotic behavior near the point, they compared the left hand limit with the right hand limit and conclude that the limit does not exist, because $-\infty \neq \infty$.

Knowledge of end behaviors also generated significant information. Brendon, Jean and Linsey knew that the cosine function’s limit did not exist, but for different reasons. Linsey referred to the definition of limit at a point claiming that “it never reaches a particular point” whereas Brendon and Jean indicated that the limit did not exist due to the oscillatory nature of the function values not settling down. Amanda thought the limit existed and was equal to infinity in which case it seems she had interpreted the graph visually rather than studied the mathematical behavior of the function values. This also occurred for the rational function as x approached infinity, in which case she said that as x approached infinity, the limit was equal to infinity. It is reasonable to suspect that for end behaviors, Amanda and other students may be visually focusing on the x-axis rather than pursuing a mathematical analysis of the function values.

The limiting behavior of function values for Graph I with cosine $\frac{1}{x}$ revealed that students have an altered schema for the domain. Yet, most have an appropriate schema containing knowledge structures that consider the behavior of function values near the point x=0. Amanda has an altered schema for this task which consists of knowledge structures that focus on the function being undefined when there is a zero in the denominator. A more appropriate schema would be that the limit does not depend on what is in the domain because the point does not have to be in the domain for a limit to exist.

Summary and Conclusion

Four student cases were represented on behalf of nine interviewed to reveal various conceptual knowledge structures that students possess about limits. There was one exemplary case and three non-exemplary given the unique natures of their responses. Although not all tasks used in the interviews were cited here as evidence, a few representative tasks were selected to illustrate what students reportedly know about limits as they captured many of the outcomes generated by some of the other tasks used in the interviews.

Knowledge in mathematics is constructed, but not always in a logical sequence meeting the expected outcomes. According to Piaget and Skemp, the conceptual knowledge structures acquired do not develop into appropriate schemas, and as a result, such structures are assimilated into altered schemas. This describes the behavior of the three non-exemplary cases in the study. I attribute this, too, to individual differences in learning and thinking. Not all students will learn or think alike; hence, can develop different schemas containing information about what they know.

One reason altered schemas might occur is that a student is simply comfortable with his or her pre-existing knowledge structures and as a result, does not experience a perturbation. Hence, in the absence of a perturbation, there is no motivation to change their internal actions or thinking so they do not accommodate the knowledge established by the mathematical community. Unfortunately, the conceptual structures contained in these schemas remain and erroneous solutions continue. On the other hand, if students are genuinely in a state of disequilibrium and disturbed by the perturbation, they will actively try to change their thinking behavior. In this case, they will accommodate the knowledge established by the mathematical community into new schemas, thereby leading to more successful learning outcomes.

The knowledge structures that students develop as they learn new material and solve problems, through collaborative inquiry or through independent practice, are essential for continued conceptual growth and development. To maximize opportunities for success, content knowledge of the instructor and the ability to articulate conceptual ideas are essential. It is important for instructors to identify potential misconceptions a priori and design meaningful lesson plans that will help students develop knowledge structures that ultimately develop into appropriate schemas.

CHAPTER 7: CONCLUSIONS AND TEACHING IMPLICATIONS

This final chapter discusses the conclusions based on the results of this study. After stating the problem and research questions, the results are further discussed in light of the constructivist framework. The discussion includes how the case-study methodology used provides an in-depth exploration and description into what students know about limits. The results infer that students assimilate different information into their knowledge structures and form either appropriate or altered schemas with respect to the definition of limit, conception of infinity and knowledge about domains. Lastly, limitations of the study as well as several implications for teaching are presented.

Statement of the Problem

This study explored the conceptual knowledge structures and operational schemas that students have about limits, using a hybrid constructivist framework involving the learning principles of Piaget and Inhelder, and theories of understanding by Skemp. Two questions that framed the study were (a) what do students know about limits and (b) incorporating the intuitive definition of limit, prior knowledge of domains and definition of infinity, in what conceptual knowledge structures do students have involving limits at a point and limits at infinity? The goal of the study was to describe and explain problem solving according to the learning theories of Piaget and Skemp. Ultimately, as a teacher and researcher, my goal is to surface and unpack ideas that would inform instructional practices.

Review of the Methodology

This is a two-part qualitative case study that utilized video-recorded relied chiefly on semi-structured student interviews according to the research methodology of Yin (1994). The first part of the study, Phase I, was a pilot designed to collect initial data from mathematics and engineering majors enrolled in a Calculus III class to explore how students think about limits at a point, limits at infinity and limits that do not exist. CL and NS were the two initial cases selected from a total of 15 interviewed. These two students had contrasting conceptual knowledge structures and so they were selected as the initial cases.

The results of Phase I suggested that deeper probing into certain tasks would reveal additional evidence of the types of conceptual structures students have about limits. As a result, a more focused research question was developed: "incorporating the intuitive definition of limit, prior knowledge of domains and definition of infinity, in what ways do students solve problems and reveal what they know in response to tasks involving limits at a point and limits at infinity? This commenced the second part of the study, Phase II at which time nine students from a Calculus III were recruited and semi-structured interviews helped to further investigate some of the phenomena uncovered in Phase I. Given common patterns of responses and similar themes, four students were selected out of the nine as they were representative of the others in the sample. Following completion of the study, students were debriefed and informed of any misconceptions they had about limits.

The results of this study confirmed findings of other researchers. Allendoefer (1963), Orton (1983) and Francis (1992) reported that although limits are central to the study of calculus, the topic is difficult even for very good students. Algebra deficiencies constitute one major problem (Carlson, 1997; Tall, 1981; Davis and Vinner, 1986; Juter, 2005; Juter, 2006; and Williams, 1991). Jackson (1916) suggested that students need a clear understanding of the concepts of variables and functions which seems to be the case, as the current study showed how knowledge about functions and their domains affects problem solving. Jackson suggested that more graphical methods should be taught when teaching limits, and although this has been done during the last century, students can still develop altered schemas. Exploring how students interpret graphs and the limiting behavior of functions through semi-structured interviews was a very rich, revealing method of determining what knowledge structures and operational schemas students possess (Bloom, 2005; Cifarelli & Cai, 2005; Yin, 1994).

How prior knowledge influences the construction of the limit concept was studied by Davis and Vinner (1986) as well as by Williams (2001). It was found that students' pre-conceived notions of limits and prior informal experiences with limits can interfere with formal instruction of this topic in calculus. Moreover, Juter (2005) found inconsistencies in their representations. Such notions and experiences appear to constitute conceptual knowledge structures. Interfering with formal instruction appears to be what Piaget describes as "disequilibrium" as they try to make sense of reality and accommodate these knowledge structures into either appropriate or altered schemas. Several pre-conceived notions were evident in the interviews when they articulated their thoughts out loud and solved problems. The results of this study are also consistent with what Sierpinska (1987) and Williams (2001) reported, that preconceived notions may lead to the ideas that limits can never be reached. This was the case with Jean, Amanda and Linsey for limits at a point.

Tall and Vinner (1981) explored concept image and concept definition to explain why students had difficulty distinguishing between limits and continuity. The current study expanded upon this by looking at how student think about the definition of limit and consider the role of the domain with continuous and non-continuous functions. The concept image Tall and Vinner (1981) discuss were personal conscious and unconscious mental pictures and attributes of the given concept. In this study, these would be the conceptual knowledge structures that form into schemas. They mention that during concept formation, the concept may not be coherent at all times, which is what Piaget would refer to as a state of “disequilibrium”. The word limit may provoke an image of something that cannot be crossed; therefore, a function may never go beyond the limit at a certain point. CL in the pilot study possessed this kind of reasoning, as did Jean in the follow-up, which suggests that information assimilated on limits into their knowledge structures ultimately exited the state of disequilibrium and got accommodated into altered schemas.

Debriefing Student Participants

Upon debriefing, many students in the study admitted having no idea that Graph C from Task 4 (Appendix F) was piecewise. When shown how to construct the formula, many seemed surprised and claimed they were learning something new for the first time. Moreover, explaining why a limit would or would not exist was quite a revelation for many of them. When I explained that writing $=\infty$ on a graph means that the function values are increasing without bound and therefore the limit does not exist, they were surprised to hear this. Their reactions reinforced my findings about various conceptual structures and underlying belief that this study was an important undertaking, providing insight into improving instruction with limits in the introductory level calculus courses and thereafter.

In fact, a few students mentioned to me that some instructors say in class that "limits exist if they equal infinity". Brendon heard this in high school in AP Calculus, but then when took Calculus III, he heard the correct version from his professor, that "the limit does not exist if it equals infinity". Amanda revealed that "it depends which course one is taking". She started with business calculus then later took calculus for science and engineers. She informed me that "in business calculus", the class was told "a limit does not exist if it equals infinity but then in calculus for science and engineers, the class was told that it does exist".

Discussion of Results for Phases I and II

In this section, a discussion of the results for both parts of the study is presented. Overall, the results reveal various ways in which students think about limits. A brief review of the theoretical framework precedes the discussion so that the results can be interpreted in the context of the framework.

According to Piaget and Inhelder (Gallagher & Reid, 1981), human beings possess mental structures that assimilate external events and convert them to fit their mental structures. These mental structures accommodate themselves to new, unusual, and constantly changing aspects of the environment and get organized into schemas. Skemp (1987) shares the same views and contends that there are useful as well as unsuitable or inappropriate schemas, the latter of which makes assimilation of later ideas more difficult, if impossible. Schema can be a hindrance when new experiences do not fit an existing schema which results in what is learned is soon forgotten. Unsuitable schemas are a hindrance for future learning. In Skemp's view, to understand something means to assimilate it into an appropriate schema (Skemp, 1987, p. 29). Understanding is not all or none, but is subjective in nature and so one can get a feeling of understanding by also assimilating information into an inappropriate schema. Piaget and Skemp both contend that better internal organization of a schema may improve understanding, especially with more knowledge getting assimilated into a more extensive schema.

Given the synonyms, ineffective and inappropriate schemas, I refer to these with the term "altered schema" to avoid any negative connotations. There are instances in which altered schemas simply represent an alternative set of knowledge structures, or view of reality that may be different but not necessarily wrong. Students appear to have their own mathematical beliefs and truths based on their cognitive styles and personal philosophies. For instance, Platonists view mathematical objects as real objects that can be explored, like exploring a new country, whereas formalists would easily adapt to the formal definitions. In the case of CL from the pilot study, she has her own unique perceptions of reality and sets of beliefs about limits that I classify as an altered schema.

Students construct knowledge into conceptual knowledge structures and create schemas. The mind is organized by schemas. Having appropriate schemas consistent with knowledge in the mathematical community facilitate constructing solutions to tasks. The conceptual knowledge structures within the schemas include understanding the definition of limit; the meaning of the infinity symbol; and knowledge of domains. Having developed appropriate knowledge structures leads to success in the mathematics classroom. Students with altered or inappropriate schemas lack information consistent with the field of mathematics. The result might be they construct inaccurate solutions and encounter obstacles with continued learning of that topic. In either case, given an appropriate or altered schema, relational understanding can exist because solutions are constructed and explained based on prior knowledge and beliefs. Hence, what a student thinks he or she knows is based on their perception of reality. The conclusions of this study suggest important teaching implications.

The students in this study provided answers to the research question by solving problems and receiving feedback at times, which Piaget thought was important to learning (Gallagher & Reid, 1981). Feedback is more important, though, in the teaching environment for obvious reasons. All students in the study demonstrated relational understanding given they were able to articulate their thoughts and were able to construct solutions and explanations by means of their actions. Nobody provided answers without explanations, or rules without reasons, which is why I am convinced none of the students exclusively demonstrated instrumental understanding.

In the pilot study, CL was a prime example of how a student can develop an altered schema and construct solutions based on the knowledge structures that exist within this altered schema. CL thought a limit was a barrier or constraints. Limits were considered to be vertical asymptotes and she perceived them as being brick walls. She also considered limits to be holes that one could fall into. So when she saw a hole on a graph, she would correctly say the limit exists, but for the wrong reason, i.e., because there was a hole. Interestingly when she saw a solid dot on a line instead of a hole, she would say the limit does not exist because one could "walk right over it without falling into it". CL consistently demonstrated this type of understanding and constructed all of her actions accordingly.

CL represents a student who has an altered schema given her definition of limit. She solved problems based on the truths that encompass her sense of reality. She very confident when articulating her thoughts and was quite consistent with explaining her reasoning across tasks. In Skemp's terms, CL was in a state of "stability" and without any reason to think about changing her thinking, no change in behavior would occur to reorganize or restructure her mental representations and assimilate them into a correct schema. According to Piaget, accommodation only occurs if there if a person is in disequilibration due to a perturbation and attempts to change their thinking behavior. CL was not in disequilibrium. In fact, she was satisfied with her thinking in spite of altered conceptions and so there was no evidence of perturbations. This explains why CL could articulate her solutions consistently across tasks without considering that her beliefs contain altered conceptions.

In Phase II of the study, students either had exemplary or non-exemplary conceptual knowledge structures based on the nature of their responses. In the non-exemplary case, the students had knowledge structures that had been assimilated into altered schemas. There were numerous permutations of knowledge structures, for example, with the definition of limit and interpretation of infinity. All of the students demonstrated relational understanding based on their ability to explain what they know along with the reasons why. Their relational understanding was based on their beliefs about limits and prior knowledge acquired.

In Brendon's case, he used a combination of an intuitive and formal definition of limit to construct his responses and consistently indicated that a limit was a number. In addition, he held that a limit was either a number or did not exist. He explained that what he learned about limits was part of the classroom instruction he received and even indicated that in high school, the calculus teacher told students that a limit exists if it equals infinity, whereas in college, the professor told the class that limits do not exist in that case. Therefore, two professionals in the field have different interpretations which are passed onto students.

Jean provides evidence of an interesting altered schema, but obviously based on the definition of limit provided by Stewart (2005) that was discussed in Chapter 3 where this textbook definition states, "we write $\lim_{x \to a} f(x) = L$ and say the limit of f(x), as x approaches a, equals L if we can make the values of f(x) arbitrarily close to L…by taking x to be sufficiently close to a but not equal to a". Having learned the definition of limit from this source would explain why she consistently reports that a "limit can only approach but not equal a function value" on a continuous graph. In Amanda and Linsey's cases, they took parts of an intuitive definition of limit to describe what a limit is and they both reported that a limit could either be a number or equal infinity. Yet, they did not specifically state that a limit, first and foremost, is a number. This was also true for the students in the pilot study, CL and NS. Collectively most students gave vague descriptions of limit that models the founders of calculus as well as modern textbook language.

Referencing Piaget and Inhelder's learning theory (Gallagher & Reid, 1981), if students assimilate correct information, such as definition of limit, notion of infinity, knowledge of domains, understanding of the limit notation, algebra proficiency, experience with piecewise and rational functions, and knowledge of domains into their mental structures, they will organize these structures appropriate schemas to use as they construct their problem solving actions. If there is either a lack of information about piecewise functions or incorrect information assimilated into the mental structures, then they develop altered schemas, which are ultimately exhibited through their problem solving actions.

According to Skemp (1987), either appropriate or altered schemas constitute relational understanding when students are very consistent with articulating their understandings through their actions. If Piaget's six step paradigm (Gallagher & Reid, 1981) for the process by which all human learning occurs is used to guide teaching methodology, teachers would be able to help students learn in the most natural, authentic, and effective ways. To transform some of Piaget's principles of learning into practice, the teacher should decide what to teach first and the order in which to present the rest of the lesson. Then the instructor should devise tasks that will give students collaborative inquiry-based opportunity to practice activities and thinking operations that allow them to construct knowledge.

The Definition of Limit

As described in Chapter 3, historically there have been many variations on how limits are defined, often with language-related abstractness and ambiguity. Textbook definitions model this pattern with similar ambiguities, not only with the language, but by not directly specifying that a limit is a number (Stewart, 2005; Strang, 1991 and Finney, Weir & Giordano, 2001). Most authors do not state that L is a number because it is assumed that introductory level calculus involves the field of real numbers—not the surreal. Since an explicit definition is missing, it is expected to be understood from the context in which it appears, in which case |f(x)-L| only makes sense for numbers. The definition remains subject to multiple interpretations as evidenced with the students' work.

Therefore, it is no wonder that inexperienced teachers may encounter difficulties teaching this topic, given they may not have learned it correctly or fully understand it themselves (Bukova-Guzel, 2007). Meanwhile, their interpretations of the definition of limit are passed onto students every semester, resulting in misconceptions and a variety of conceptual structures that represent content knowledge not appropriately recognized in the field of mathematics. Therefore, the results seem to suggest that students assimilate different information into their knowledge structures and form either appropriate or altered schemas with respect to the definition of limit.

The Interpretation of Infinity

Similar to the ambiguities associated with the definition of limit, students have different interpretations of infinity. While some think infinity represents something undefined, others think it represents a very large number as well as a place far away or something too large to measure. This could be avoided by understanding the definition of limit, and having internalized the fact that a limit is a number, L, so that this fact is retrieved from memory automatically while problem solving. By definition, |f(x)-L| only makes sense for numbers. Therefore, in this expression, L cannot be replaced with the infinity symbol as it is senseless to write $| f(x) - \infty |$. If computing a limit at a point, there is no number one could obtain that is within epsilon of L. By definition, $| f(x) - L | < \varepsilon$

and if $L = \infty$, then $|f(x) - \infty| < \varepsilon$ is definitely not a true statement for any number, ε, let alone for small ε.

In advanced mathematics, infinity has representations outside of the real number system and extends into the surreal numbers. In calculus, though, only the field of real numbers is involved. Therefore, infinity is not a number and the infinity symbol is only used to represent the behavior of the function values increasing without bound.

Careful attention must be given when teaching concepts involved with the use of the infinity symbol. If students understood that infinity is not a number as well as referred to the definition of limit, then students and instructors alike might be less likely to erroneously state that a limit exists if it equals infinity. If instructors have altered conceptions of their own with the mathematical content, then students can develop altered schemas which could result in difficulty constructing solutions to new mathematical tasks then encountered (Bukova-Guzel, 2007). The results seem to suggest that students assimilate different information into their knowledge structures and form either appropriate or altered schemas with respect to the conception of infinity.

Importance of Understanding the Domain

Understanding domains of functions in mathematics is very important, but seems to be overlooked or ignored by students when they solve problems involving functions as they do not consider the domain. Yet, without a domain, there is no function. All physical systems determine the domain of the function used to model the system. In Chapter 1, a real world application was described involving Hooke’s Law, which involves displacements within the elastic limit of the spring, a law which changes once the elastic limit is exceeded. In this case, there is a domain with a finite interval which is stipulated and makes sense for a particular physical system. Beyond the elastic limit, a Taylor series best describes the behavior of the function values. One result of the study showed that when looking at graphs, students reported for limits at a point that the function value had to be in the domain for a limit to exist, meaning as $x \rightarrow a$, "a" must be in the domain. This, of course, is not true. Yet, this was the case for Amanda and Linsey, whereas for Jean, she reported that the "a" cannot be in the domain based on the definition of limit which only means "approaching and not equals". Brendon reported that limits can sometimes be function values, as in the case of continuous functions, but that the point "a" does not have to be in the domain for a limit to exist, as in the case of the non-continuous functions. Therefore, students assimilate different information into their knowledge structures and form either appropriate or altered schemas with respect to the domain.

Importance of Piecewise Functions and Rational Functions

Piecewise functions were a major source of difficulty and are very important for math instructors to take into consideration when lesson planning in order to eliminate the development of altered schemas. Students did not identify the graph of the function as being piecewise and could not construct a formula for the function. They typically pretended the solid dot situated above the discontinuity was not there and just wrote a formula for the larger piece. Piecewise functions were selected for this study was because at the beginning of most calculus courses, the graph of a quadratic with a hole and solid dot above or below the hole are shown to students and they are asked to determine if the limit exists as x approaches some number (the first coordinate of the hole). They study the graph and seem to get confused when they see the hole and the dot. By focusing on the hole, they erroneously contend the limit does not exist because of the hole. To many students, the hole "means nothing is there". This led me to explore how they perceived the role of the domain of a function in order to ascertain if they thought that *a* had to be in the domain as $x \to a$.

Considering the hole on a straight line, a philosophical question that emerges is "how something can exist if there is nothing there"? Evidence presented earlier suggests students think a point has to be in the domain for a limit to exist. In this case if their knowledge structures had assimilated the definition of limit and an understanding of how to apply it, the students would have been able to successfully construct their solutions by using the definition. By drawing on the fact that a limit is a number, one begins by looking at points on both sides close to *a*, let x approach a ($x \to a$) and then study the behavior of the function values. If they understood the definition of limit, they would know that the point *a* need not be in the domain for the limit to exist and therefore, even if there is a hole, the limit exists but is just not equal to the function value. They also seem to think that the hole is "missing" so nothing is there. The fact is, the hole is missing from the graph of the function but it is still a point in the plane.

Limits of rational functions are also important because students showed evidence of having focused on zero being in the denominator rather than having explained the limiting behavior of the function values near a point. Given a limit problem for a simple rational function in the form, $\lim_{x \to 0} \frac{1}{x}$, CL thought vertical asymptote at x=0 was a brick wall, which constituted the limit while AK reported a limit cannot exist because of zero being in the denominator. CL focused on the vertical asymptote and Amanda focused on the denominator, but neither focused on the limiting behavior of the function values near zero. With rational functions, there are also issues with algebra. Factoring is problematic for some students and so they are not able to complete the task correctly due to these limitations. Meanwhile, rational functions have real-world applications, such as in business with cost and production; in science with Ohm's Law, as well as in the medical field where dosage calculations are calculated and the time at which a drug exits a body is computed.

Limitations

A limitation of this study was no opportunity to have studied these students longitudinally to find out more about their backgrounds in mathematics. It would have been useful to have followed them through elementary and high school to see how they were taught and to what extent the instruction was constructivist in nature as well as to gain insight about the conceptions and beliefs they hold about mathematics. Knowing specifically what calculus textbooks they used when they learned limits as well as what textbooks their instructors had used would have provided insightful information regarding the various definitions of limits and infinity that students report.

A quantitative component to this study would have been useful, by applying the tasks from the interview in multiple choice or true/false form and administering them to several large classes of calculus III students. Using a grounded theory approach to identify common themes, reliability and validity of the data could have been assessed. A quantitative component could have provided the frequency of which certain conceptions and beliefs appear so that such problems could be identified and prevented in the classroom by careful lesson planning.

Teaching Implications

Constructivism, a theory of knowledge based on the idea that people construct their own realities by interacting with the world, offers a framework for ultimately designing lessons that incorporate suggestions from the research for improving mathematics education. Constructivists engage in the development of mathematical knowledge. One challenge in teaching is to create experiences that engage the learner and support his or her explanation, evaluation, communication, and application of the mathematical models needed to make sense of these experiences. Unlike discovery learning in which the generalization sought after is presumed to be predictable prior to the investigation, the constructivist is engaged in a process of inventing models for explaining students' actions and words.

There are some important implications for teachers designing and implementing instruction. The first is the critical role of students' prior knowledge in a new learning situation. If an appropriate internal network does not exist whereby prior knowledge is absent, then new material has nothing with which to connect (Skemp, 1971). In this case, it is unlikely the learner will understand it, since understanding is defined by a learner's ability to "assimilate it [new material] into an appropriate schema" (Skemp, 1971, p. 46). Also, learners may need help activating their prior knowledge or accessing their schema. Instruction should include consideration of explicit connections between new material and related mathematical ideas. Many researchers noted, though, that students benefit more from experiences that help them to discover or construct these connections rather than from connections simply identified by the teacher.

Results of the study showed that students develop various conceptual knowledge structures that form either appropriate or inappropriate schemas. In spite of good intentions of instructors who anticipate appropriate learning to occur, this is not always the case given individual differences in perceptions and understandings (Bukova-Guzel, 2007). The question remains, then, why are students developing altered schemas and how can this be rectified? Some possible answers appear below along with some teaching strategies to improve instructional practices associated with learning limits.

1. One indirect explanation seems related to mathematical literacy, not just of the students but of the calculus instructors as well. This explanation might be based on the instruction in mathematics education that the instructors received during their own formal study of mathematics in college. Since understanding definitions are important to understanding limits, included in Appendix P is a Literacy Toolkit designed by this researcher that contains specific definitions students and teachers should know with respect to this topic.

It seems as though research in reading education and mathematics education has caused scholars within each field to have a comprehensive view of contributing factors, underlying constructs and instructional implications related to content development and cognitive strategy use in their respective fields. However, there remains a disconnection in the dialogue among researchers from these disciplines (Hyde, 2006). Due to the disconnection across reading and mathematics education literature (Hyde, 2006), university faculty in these fields may have been unfamiliar with the similarities of relevant cognitive strategy instruction within each discipline. As a result, the connections may not be readily included in their instructional planning and instructional delivery methods. Texts integrating reading and mathematics are limited in publication (Hyde, 2006) and the underlying research bases across these domains remains disjointed to date. Therefore, it is unlikely that the textbooks and other reading materials to which the clinicians were exposed during their formal education and professional development experiences illustrated similarities among the use of cognitive strategies in reading and mathematics contexts. Such literatures synthesizing the similar, self-regulatory cognitive bases that are currently being advocated by the national Council of the Teachers of Mathematics (2011), International Reading Association (2011) and recent research in each content area would be useful. In addition to offering information, these resources could provide an opportunity to align university-based teacher educators from these content areas who have the earliest opportunities to inform future teachers' content knowledge, perceptions and practices.

2. The language that teachers use to teach limits could be rectified (Monoghan, 1991). When a student is asked to "find a limit" there is a lot of confusion over what they are finding because what they should be doing is describing the behavior of function values. They do not understand this. A better way to teach limits might be to start off by replacing the word "limit" with the phrase "limiting behavior of function values", since this is a more accurate and meaningful description. Therefore, when tests are given, one suggestion would be to ask students to perhaps compute the limit, and then describe the function's limiting behavior in a few sentences. All answers go back to definitions in math and so if students are taught to write out definitions of function, function values, domain, range, limit, etc., they will assimilate these vocabulary words into their literacy toolboxes. By doing so, they will acquire the facility to construct correct knowledge

absent of misconceptions, and be able to incorporate this knowledge into appropriate schemas of relational understanding.

3. Students need to understand domains to understand functions and ultimately limits because the domain is a part of the function. In the classroom, domains should always be mentioned when teaching functions. Often, when functions are taught by inexperienced or novice teachers, there is little or no mention of domains. The reasons could be due to time constraints or content knowledge restrictions (Bukova-Guzel, 2007).

4. More instructional emphasis should be placed on understanding piecewise and rational functions in algebra, as well as graphing (Hornsby & Cole, 1986; Adams, 1997; Dunham & Osborne, 1991).The difficulties students have with piecewise functions can be attributed to not having knowledge about the domains of such functions among their mental structures within their schemas.

Piecewise functions are very common in engineering courses, modeling various physical systems, and Brendon informally mentioned to me that students in engineering "get stuck on these all the time" and that "piecewise functions show up all the time in engineering". Perhaps it is the case of the solid dot above a discontinuity that leads a student to think that a solid dot cannot possibly be part of a function. If this is the case, that means they do not know what it means for a point to be on the graph of the function, which is that the second coordinate, y, is a function of the first coordinate, x. The problem is they do not recognize a point as a piece of a function. More emphasis also needs to be placed on computing rational functions and interpreting the asymptotic behavior as limiting behavior of function values is very apparent from the time rational functions are introduced in the Algebra 2 curriculum (NCTM, 2000).

Teaching how to construct functions from graphs would help students work backwards to derive the original function (Dunham & Osborne, 1991). Collaborative group activities where students can take turns using the calculator to create graphs and have their partners construct the functions is one way to initiate this in class. This type of formative assessment would provide teachers with feedback with which they can construct more precise lesson plans or re-construct follow-up re-teaching exercises.

5. Instructional activities that address the meaning of infinity early on in the elementary and middle school curriculum should be implemented so that students develop a rich conceptual understanding of infinity (Juter, 2005;Tall & Vinner, 1991; Tall, 1991; Williams, 1991). Understanding infinity is difficult for students because of the vague notions they acquire, assimilate and accommodate into schemas. Because it is not until the topic of limits that they learn how infinity is defined, many students do not learn that infinity is not a number. If taught correctly, students will learn that infinity is a symbol used with limits to represent numbers increasing in a positive sense without bound.

Teaching about limiting processes and infinity can begin in elementary school. For instance, as part of unit lesson plan, students can be presented with a sequence of regular polygons, starting with a triangle, so that each new polygon has one more side that the previous polygon in the sequence. In this case the idea of a never-ending process and of limit may be discussed (Orton, 1987).

Middle school and high school students would benefit from some formal instruction about infinity, rather than be left on their own to assume that which is not true, i.e., that infinity is a large unreachable number. In order for students to understand that infinity is not a number, they must return to the definition of limit and recognize that a limit is a number by this definition. In order for a limit to exist, the limit has to be a number and so if the limit "equals" or tends toward infinity, then the limit does not exist because infinity is not a number. Proper instruction would encourage accommodation of knowledge into an appropriate correct schema, so that relational understanding can occur.

6. Real-world applications involving limiting behavior needs to be emphasized in the classroom also beginning in elementary school. An example is when a teaching place value, fractions or decimal approximations one could argue that 1.9999….=2 but that $1.9999 \neq 2$. The underlying idea is to see how close one can actually get to 2.

In middle school, a teacher could incorporate into unit lesson plans how functions are models of physical systems which exhibit limiting behavior. The input or independent variable gets manipulated and the output or dependent variable is measured. For example, turning the volume knob on a radio to different positions affects the output volume, and so limiting behavior occurs as the knob is turned up to the last physical notch until it cannot be turned up any further. When this occurs, the maximum volume for that particular physical system is reached.

Moreover, a teacher could explain that sometimes physical systems need time to equilibrate. For instance, when a stove burner is turned on it does not come on at the temperature to which the thermostat is set. This is limiting behavior. The set temperature is the limit. In another example, limiting behavior occurs when a band tunes instruments in unison. Some instrumentalists are either sharp or flat, and so it takes a series of fine tunings to get closer and closer to the correct pitch so that all players are in tune.

In mathematics, irrational numbers such as e and π are also examples in which limiting behavior occurs. In this case the limit exists because adding the next decimal place in the decimal expansion of these numbers gets closer to the actual value. The decimal values keep increasing but ultimately converge to e and π .Outdoors on the road, if the speed limit posted is 55 mph then the driver can be slightly below or slightly above this limit and adjusts the gas pedal so the speedometer gets close to 55mph, though it may be hard to be precisely achieve 55mph.

Most people possess knowledge of these everyday real-world occurrences of limiting behavior, and at some level, most of us suspect that mathematics is at the heart of these ideas. This is why I recommend capitalizing upon teaching limiting behavior early on, beginning in the elementary mathematics curriculum so that these conceptual ideas become assimilated into students' conceptual structures and subsequently develop appropriate schemas.

7. Graphs need to be featured when teachers begin to teach limits and limiting behavior is to start off with graphs (Dunham & Osborne, 1991). The computations should be secondary and the harder computations are unnecessary at the beginning. Once the definitions and basic ideas behind limits are visualized, then this knowledge can be assimilated into one's schema and restructured or accommodated to new problem situations, so that appropriate solutions can be constructed.

Pictures are efficient ways to communicate the idea of what is happening mathematically and for many students depending on their philosophical viewpoint, that may be exactly what mathematics is about. Some of our students are not visual learners, but for many, accompanying algebra with a picture makes the situation unambiguous. If one if a formalist, there would be a formal system for using pictures, as they are a device to help visual learners develop formal proofs. If one is a Platonist, pictures may play an important role in how people as physical beings can have contact with the mind, with space and with time-independent realms of mathematical objects.

A sample lesson plan that I designed on piecewise functions appears in Appendix L followed by a sample assessment in Appendix M. Also, there is a sample lesson plan I designed on rational functions in Appendix N followed by a sample assessment in Appendix O. In Appendix P, I have included my mathematical literacy toolkit for limits in calculus that contains various teaching strategies for reading, writing, oral and technological literacy. One section is devoted just to definitions. These teaching tools include visual graphic organizers and vocabulary learning strategies that would be usable with diverse learning populations, including students who are English Language Learners and students with learning or behavioral-emotional disabilities.

8. Considering cognitive behavioral and information-processing theories (Atkinson & Shiffrin, 1968) in addition to Piaget's genetic epistemology, all of which are neurophysiologically-based, I think that all students are capable of constructing knowledge; however, recalling what they constructed as well as the growth of new knowledge is contingent upon practice and reinforcement. Ongoing sensory input involving interaction with the material, communicating and collaborating with classmates as well as independent practice, results in storage and retrieval of information in long-term memory.

Future Research

A replication study incorporating Calculus I students could be done explore their understandings while their conceptual structures are being developed. Unlike the students in Calculus III who have been exposed to limits throughout topics of differentiation, integration and series, Calculus I students study limits for the first time.

A future study could gather data from a larger sample size by implementing a questionnaire that contains many of the same tasks used in the interviews (see Appendix K-1 and K-2). By keeping the open-ended definitions of limit and infinity but changing question types from open ended to true/false and by using quantitative methods, a lot of data could be obtained in minutes and possibly confirm many of the findings. This questionnaire could also be used as a formative assessment to determine what aspects of limits need to be re-taught. The goal would be that students ultimately assimilate feasible mathematical problem-solving conceptions into conceptual structures that develop into appropriate schemas, thereby preventing altered conceptions from occurring.

In another study, it would be important to find out how certain conceptions can be prevented by implementing various instructional strategies for mathematical literacy, including those for reading, writing, vocabulary building, reading comprehension and technological literacy. By having a comparison group that receives traditional instruction, I would measure observable outcomes with grades as well as assessments of procedural knowledge and conceptual understanding.

Summary and Conclusion

This two-part study was originally designed to explore why students struggle with learning limits in calculus, but was ultimately modified to describe and explain what students think and know about limits. Prior to Phase I, two years were devoted to interviewing calculus students and piloting potential tasks to use in the actual study. Once the tasks were decided upon based on their feasibility of not being too difficult or abstract, the pilot study was developed. Although 15 students were interviewed over the course of 96 hours, two students with different perceptions and abilities were selected as initial cases to explore how students think about limits at a point, limits at infinity and limits that do not exist. Phase II evolved out of these results, with the goal of probing deeper into some aspects of problem solving that was not done in Phase I. In Phase II, nine students were interviewed, but four were selected as cases to represent the actions of the sample, given common emerging themes.

Using a constructivist lens incorporating the learning principles of Piaget and Inhelder, and theories of understanding of Skemp, the results suggest that students assimilate information differently into knowledge structures and form either appropriate or altered schemas with respect to the definition of limit, conception of infinity and knowledge about domains. Solving problems are contingent upon knowing definitions and understanding the role of domains. If the definitions of limit and infinity are not recognized as done so in the mathematical community, and if the knowledge of domain is not complete, then students might assimilate inappropriate information into their conceptual knowledge structures and develop altered schemas that directly conflict with learning expectations and assessment outcomes in the classroom. It is hoped that in the classroom, only information that is considered to be mathematically precise will be assimilated into conceptual structures and that future learning is successful, such that other appropriate schemas are developed.

Limits have been a difficult concept to understand since its conception, and given the abstractness and vagueness in definitions by the founding fathers of calculus as well as in modern textbooks, this might explain why students most often cannot readily define a limit as being a number and have trouble articulating exactly what a limit is. The problem is compounded by the content knowledge of instructors, who also might possess some incomplete or inappropriate ideas. It is imperative that teachers know their content areas well enough so that they can identify potential misconceptions and develop lesson plans and activities that can minimize or eliminate such occurrences (Bukova-Guzel, 2007).

Implementing teaching strategies that incorporate mathematical literacy is one possible way to rectify the situation. Reading and writing in mathematics, oral discourse, vocabulary building exercises, the use of non-text technological resources, concept maps, collaborative group-work and inquiry project-based learning activities are among some of the teaching strategies that could improve conceptual understanding and assessment outcomes. In addition to fostering an environment in which knowledge is constructed, I also think that on-going practice will reinforce these ideas both cognitively and neurophysiologically, fostering continued growth of axons and dendrites in the brain, since the brain is the site at which cognitive development occurs. Implementing a variety of teaching strategies in mathematics would also accommodate for diversity in the classroom including differences in culture, socio-economic status, learning style and cognitive style. As a result, these would facilitate meeting state requirements of differentiating instruction in the classroom but could also be implemented in college classrooms to ensure that pre-service teachers and all students receive outstanding learning opportunities.

In summary, this study focused on what students know about limits and reveals that students assimilate mathematical information about functions and limits into knowledge structures in many different ways. As a result, students possess altered schemas which may either help or hinder problem solving, depending on the specific nature of the knowledge structures. Knowledge structures are analogous to the neurons, axons and dendrites that develop in the brain during learning, which coincides with Piaget's developmental epistemology.

Content knowledge delivered in the classroom must be done with careful attention to detail and instructors must be able to identify potential misconceptions students might have and prepare lesson plans accordingly. Students such as CL, NS, and the others seem to possess different cognitive, learning and thinking styles. Therefore, it is important to acknowledge the needs and challenges of diverse learners by improving the curriculum and instruction of the limit concept so that all students in the classroom have an opportunity to be successful with learning and acquiring an accurate conceptual understanding these important mathematical ideas.

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APPENDIX A: PARTICIPANT CONSENT FORM

UNC CHARLOTTE
College of Liberal Arts & Sciences

Department of Mathematics and Statistics

9201 University City Blvd, Charlotte, NC 28223-0001
Telephone: 704.687.2580 fax: 704.687.6415

Consent to Participate in Study on Understanding Limits

I, ______________________________________, hereby give my consent of participation to Margaret Adams, the principal investigator of the study who is under the direction of faculty dissertation advisor, Dr. Victor Cifarelli, in the Department of Mathematics and Statistics. This is a non-invasive study on limits in calculus being done to fulfill the requirements for the degree, Doctorate of Philosophy.

In this study, I would be required to attend 3 interview sessions in the Math Dept. conference room lasting approximately but not exceeding 90 minutes each. The 3 interview sessions will be spread about two weeks apart. During the sessions, I will be given colored markers to write limit problems on a white board. I will explain my thinking aloud in response to the questions about the tasks presented on white index cards. I acknowledge that some problems may be difficult and that it is OK to pause when necessary. A reasonable break will be given if requested. I will make every honest effort though to explain aloud what I am thinking about as I work the problems.

I acknowledge that no hints will be given about whether my responses are right or wrong, but I will be prompted as necessary in order to help facilitate my responses.

These sessions will involve video recordings from a camcorder, which will later be downloaded to a DVD. My face will not be shown, and my identity will not be revealed verbally or on tape. Only my back, hand movements while writing and verbal responses to the questions will be recorded, as the goal is to capture close up shots of what is being written on the white board while thinking aloud. The purpose of the recordings is that later on Margaret will play them each from beginning to end, taking notes, analyzing the data and classifying the nature of the responses, in order to identify what the struggles and strengths that exist among the students. The video recordings may be shared with Margaret's dissertation advisor and dissertation committee members but the video recordings will not be shared with other staff, students or faculty members, including those I currently have for my math course. The DVDs will be locked in my advisor's office, inside a locked plastic box until the completion of the dissertation, at which time the DVDs will be destroyed. My identity will remain anonymous and information obtained will remain confidential for the duration of the study.

Other than being asked some questions and going over some problems with colored markers on a white board, there are no invasive procedures. The benefit to me would be to potentially understand more about limits and it could ultimately mean a change in instructional practices so that other students who take calculus can be more successful and possibly consider the pursuit of more technical majors otherwise not considered because of struggles with the math courses.

During the experiment, I have the right to withdraw at any time without any notice as my participation is completely voluntarily. No explanation would have to be provided about my reasons for withdrawing. As a courtesy, I agree to contact Margaret in case I cannot attend a session for any reason. I will be debriefed at the end of the study.

Since I agree to voluntarily participate in this study, both parties will sign below and I will be given a copy for my records. This gives me a way to contact the investigator, as well, in case I have any questions or need to reschedule any scheduled meetings.

Student Participant:

Print Name ____________________________
Date__________________________________

Signature: _____________________________

Principal Investigator

Print Name ____________________________
Date__________________________________
Signature: _____________________________

Contact Information:

Margaret Adams, Doctoral Candidate
meadams@uncc.edu 704-299-2475

Dr. Victor Cifarelli, Professor, Department of Mathematics and Statistics
(Dissertation advisor)
vvcifare@uncc.edu 704-687-4579

Compliance Office for the Institutional Review Board Approval of this Study
Cat Runden | Office of Research Compliance
UNC Charlotte | Cameron 321F
9201 University City Blvd. | Charlotte, NC 28223
Phone: 704-687-3309 | Fax: 704-687-2292
crunden@uncc.edu | http://research.uncc.edu/comp/human.cfm

APPENDIX B: STUDENT INFORMATION QUESTIONNAIRE

Nickname____________________________	List Previous College Math Courses Taken or any AP High School Math Courses:
Freshman, Sophomore, Junior, Senior Graduate Student	Do you have trouble with "limits" in calculus? yes_____ no_____
Current Math Course:	Do you find calculus courses to be difficult at times? yes_____ no______
Do you do required homework on a regular basis or fall behind? Explain; _________ ___________________________________	Have you ever repeated a math course? yes______ no______ If so, which one(s)______________________________
Do you prefer on-line homework or written homework in math? _________________ Why__________________________________ ______________________________________	Do you struggle with math courses in general? yes______ no______
Do you have math or test anxiety, and go blank during tests? yes_____ no______	Have you ever been "put down" or insulted in a math course by an instructor? yes______ no______
Was the topic of limits in general easy to understand? yes______ no______ Explain:_____________________________	If you look at a graph, can you tell if a limit exists or not? yes____ no____ sometimes__________

Write down a definition to each of the following to the best of your every day knowledge: You should write a statement and can also sketch a graph if you wish.

1. What is a function?

2. What is a limit?

APPENDIX C: TASKS ORIGINALLY DEVELOPED AND PILOTED

Potential Tasks for Inteview 1: Functions and Limits at a Point

Interview 1

Describe the difference between

A) $f(x) = \dfrac{2x^2+5}{x-1}$ and

B) $\lim_{x \to 1} \dfrac{2x^2+5}{x-1}$

Are you finding x or y?

(A) Function

(B) Limit of a function for values of x near 1.

Finding y or f(x)

What do these expressions mean? Are you finding x or y?

$\lim_{x \to 3} \dfrac{x+1}{x-3}$

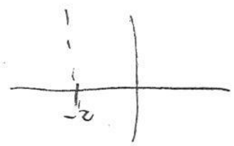
$\lim_{x \to 3} \dfrac{x^2-2x-3}{x-3}$

Behavior of f(x) near 3.

Finding y or f(x)

Means finding behavior of f(x) near 3.

Can you come up with a graph of a function that has limiting behavior near x = −2?



Given,

$\lim_{x \to \pm\infty} \dfrac{2x^2+x-1}{2-x}$

Can you explain what you are looking for?

Are you seeking x or y?

Can you graph this?

Looking for behavior of function values as x goes to ±∞

Seeking y or f(x).

Given V.A at $x = -\frac{3}{2}, x = \frac{3}{2}$, and H.A at $y = \frac{1}{2}$, which below could be the equation of the graph?

(a) $f(x) = \frac{x^2}{x^2 - \frac{9}{4}}$ (c) $f(x) = \frac{1}{2x^2 - \frac{9}{4}}$

(b) $f(x) = \frac{x^2}{2x^2 - \frac{9}{2}}$ (d) $f(x) = \frac{x^2}{2(x-\frac{3}{2})^2}$

Ans: (b)

$\frac{2x^2}{2} - \frac{\frac{9}{2}}{2}$

$x^2 - \left(\frac{9}{2} \cdot \frac{1}{2}\right)$

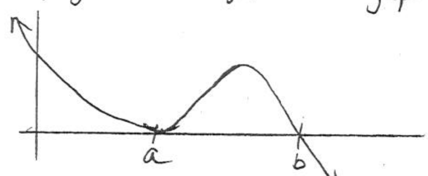
$x^2 - \frac{9}{4}$

$x^2 = \frac{9}{4}$

H.A $= \frac{1}{2}$ $\frac{x^2}{2x^2}$

Same powers of x

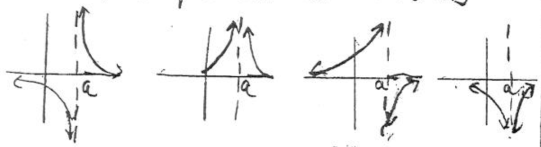
Which poly. function might have the graph shown?



1) $y = x(x-a)^2(x-b)$
2) $y = (x-a)^3(b-x)$
3) $y = (x-a)^2(x-b)^3$
4) $y = (x-a)^2(b-x)$
5) $y = (x-a)^3(b-x)$

Ans: 4

Match the Graphs with the Functions

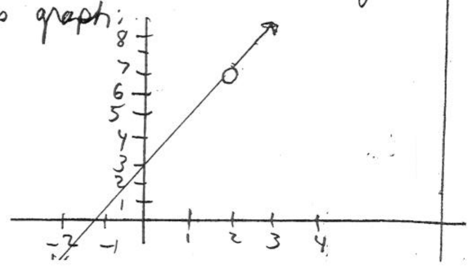


a) $f(x) = \frac{-1}{x-a}$ (c) $f(x) = \frac{-1}{(x-a)^2}$

b) $f(x) = \frac{1}{x-a}$ (d) $f(x) = \frac{1}{(x-a)^2}$

ANS: 1: B 2: D
3: A 4: C

Construct a Rational Function for this graph:



Ans:

$g(x) = \frac{2x^2 - x - 6}{x - 2} = \frac{(x-2)(2x+3)}{x-2}$

$= 2x + 3$ if $x \neq 2$

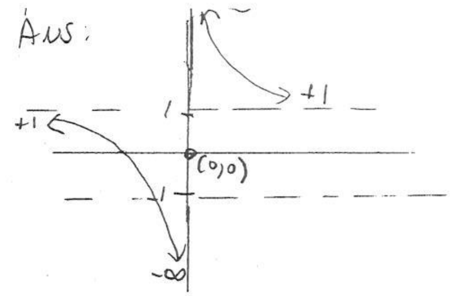
$x - 2$ common factor gives hole when $x = 2$ due to restriction $x \neq 2$.

(61)

Sketch $f(x) = \frac{1}{x} + 1$

Explain what x and f(x) are doing.

Ans:



Construct a rational function f(x) such that f(x) satisfies these 4 properties.

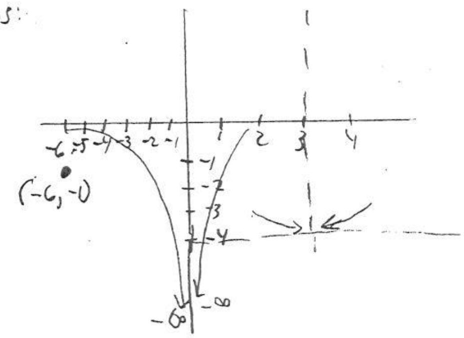
ⓐ As $x \to 3$, $f(x) \to -4$

ⓑ As $x \to 0^+$, $f(x) \to -\infty$

ⓒ As $x \to 0^-$, $f(x) \to -\infty$

ⓓ $f(-6) = -1$

Ans:



Suppose a rational function has

End behavior: $x \to +\infty$ as $f(x) \to 0$

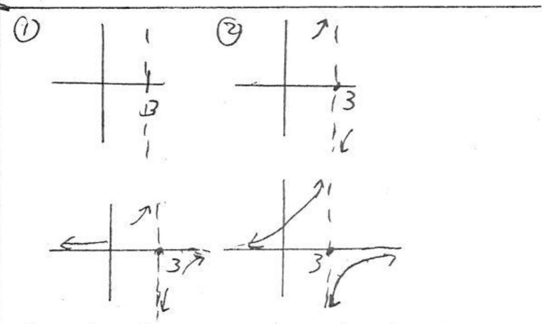
$x \to -\infty$ as $f(x) \to 0$

Single discontinuity at $x = 3$.

$f(x) \to \infty$ as $x \to 3^-$

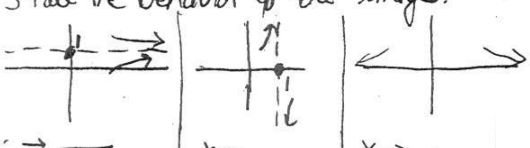
$f(x) \to -\infty$ as $x \to 3^+$

Sketch the graph. What elementary function does it resemble?



$f(x) = \frac{1}{x}$ flipped over x-axis, translated 3 units right. $f(x) = \frac{-1}{(x-3)}$

State the behavior of the images:

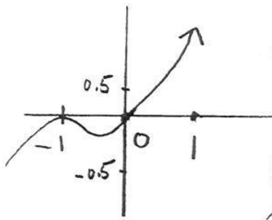


$x \to$ ___	$x \to$ ___	$x \to$ ___
$f(x) \to$ ___	$f(x) \to$ ___	$f(x) \to$ ___

Ans:

$x \to \infty$	$x \to 1$	$x \to \pm\infty$
$f(x) \to 1$	$f(x) \to \pm\infty$	$f(x) \to 0$

Which poly. below might have the graph shown?



ⓐ $y = x^2(x+1)^3$

ⓑ $y = -x^3(x+1)^2$

ⓒ $y = x^2(x+1)^2$

ⓓ $y = -x(x+1)^3$

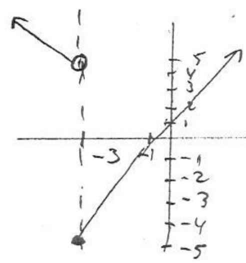
ⓔ $y = x(x+1)^2$

ANS. E

Construct a limit of a function with the following conditions:

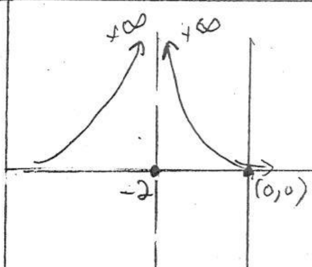
1) V.A at $x = -3$

2) $f(x) = \begin{cases} 5 - x & \text{for } x < -3 \\ 2x+1 & \text{for } x \geq -3 \end{cases}$



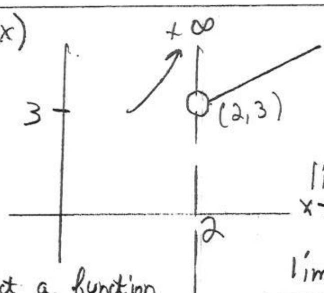
Sketch $\frac{1}{(x+2)^2}$

Explain what x and f(x) are doing. Describe any asymptotic behavior.



Domain: $x \neq -2$ $y > 0$

$y = f(x)$



$\lim_{x \to 2^-} f(x) =$ ____

$\lim_{x \to 2^+} f(x) =$ ____

Construct a function for this graph.

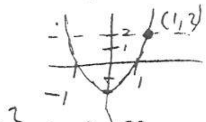
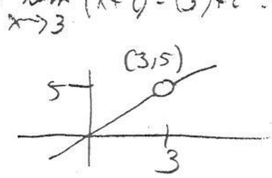
Ans: $\lim_{x \to 2^-} f(x) = +\infty$

$\lim_{x \to 2^+} f(x) = 3$

$f(x) = \begin{cases} \frac{1}{2-x} & x < 2 \\ x+1 & x > 2 \end{cases}$

(D.3)

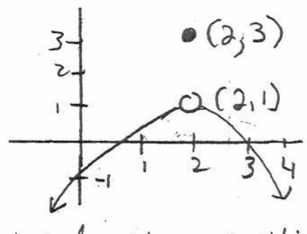
Potential Tasks for Interview 2: Limits at Infinity

Limit concept: What is a limit and how might it be used?	→ Ans: Describes a function's behav as the I.V. approaches a given val
$\lim_{x \to a} f(x) = L$ What does this mean? What do X and a mean here? What is L?	→ Ans: If values of $f(x)$ can be made close to L, by taking values of X close to a (but not $\neq a$) we have $\lim_{x \to a} f(x) = L$. $f(x) \to L$ as $x \to a$.
Provide a definition of continuity of a function at a point where x=a.	ANS. A function is continuous at x=a if ① $f(a)$ exists ② $\lim_{x \to a} f(x)$ exists ③ $\lim_{x \to a} f(x) = f(a)$.
Compute the limit if it exists. $f(x) = 3x^2 + x - 2$ at x=1	Ans: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x^2 + x - 2) = 2$ $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3x^2 + x - 2) = 2$ $f(1) = 3(1)^2 + 1 - 2 = 2$  So $\lim_{x \to 1} f(x) = \lim_{x \to 1} (3x^2 + x - 2) = 2$
Compute $\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3}$ Do you get a V.A or hole and what does it mean? As $x \to 3^-$, $f(x) =$ _____ As $x \to 3^+$, $f(x) =$ _____	Ans: den=0, x=3; num=0, x=3 $\lim_{x \to 3} \frac{(x-3)(x+2)}{(x-3)} = \lim_{x \to 3} (x+2) = (3)+2 = 5$ $\frac{x^2 - x - 6}{x - 3}$ 

(64)

) Does the limit exist?

) Is (2,3) on the graph of the function? why / why not?



Ⓒ What function could this be?

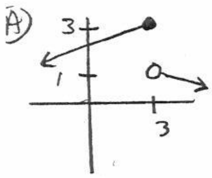
Ⓐ yes LLH = RHL
Lim exists but not equal to the val of the function.

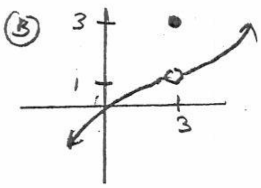
Ⓑ yes. It's a Function value

Ⓒ $f(x) = -(x-2)^2 + 3$

x	f(x)
0	-1
1	2
2	3
3	2
4	-1

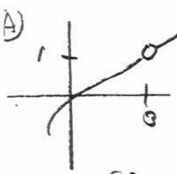
Do the limits exist?

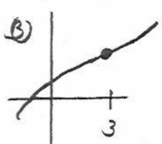
Ⓐ


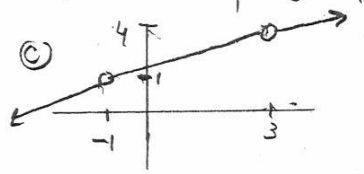
Ⓑ


Ⓐ No Ⓑ yes

Do the limits exist?

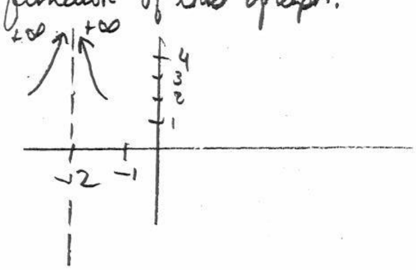
Ⓐ


Ⓑ


Ⓒ


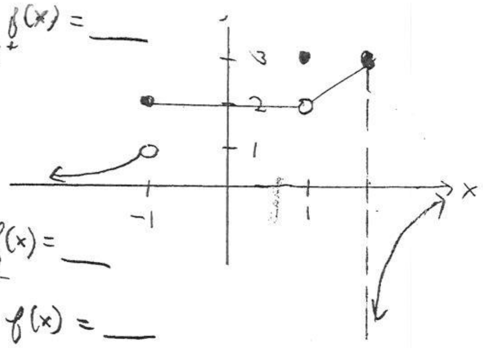
Ⓐ yes Ⓑ yes Ⓒ yes

What could be a possible limit of the function of this graph?



Ans: $\lim_{x \to -2} \frac{1}{(x+2)^2}$

$\lim_{x \to -1^+} f(x) =$ ___



B) $\lim_{x \to -1^-} f(x) =$ ___

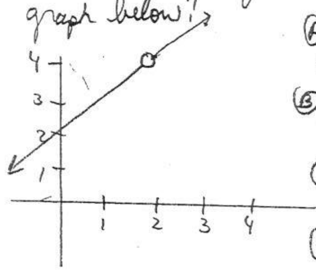
C) $\lim_{x \to 1} f(x) =$ ___

(A) 2

(B) 1

(C) 2

What could be the function that matches the graph below?



(A) $\lim_{x \to 4} \frac{x+2}{x^2-2}$

(B) $\lim_{x \to 2} \frac{x^2-4}{x-2}$

(C) $\lim_{x \to 2} x+2$

(D) $\lim_{x \to 2} \frac{x-2}{x^2-4}$

Does the limit exist?

Ans: $\lim_{x \to 2} \frac{x^2-4}{x-2}$

yes, it exists

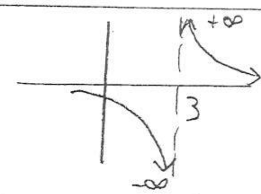
Consider the function $f(x) = \frac{1}{x-3}$

Find $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$.

Graph this function.

Does the limit exist? Why / why not?

Explain behaviors of x and $f(x)$.



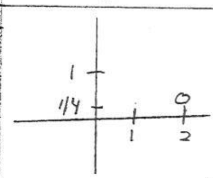
$\lim_{x \to 3^-} \frac{1}{x-3} = -\infty \quad \lim_{x \to 3^+} \frac{1}{x-3} = +\infty$

As x gets closer to 3 from left, $f(x) \to -$

As x gets closer to 3 from right, $f(x) \to +$

Compute the limit and describe the behavior near $x = 2$. Construct a graph.

$\lim_{x \to 2} \frac{x-2}{x^2-4}$



Don't get a polynomial, Point is that the denom is no longer zero at the limit point.

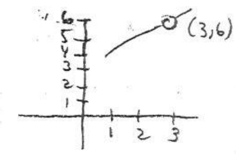
$\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x+2)(x-2)} = \lim_{x \to 2} \frac{1}{x+2} =$

Limit is $\frac{1}{4}$, and get hole at $(2, \frac{1}{4})$

Compute this limit and describe the behavior near x = 3. Construct a graph.

$$\lim_{x \to 3} \frac{x^2-9}{x-3}$$

$\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x \to 3} (x+3) =$



Get a polynomial. Limit exists, functi[on] is not continuous. Common factor: hole.

Describe the behavior of this function and write it as a limit of a function for values of x near 1.

$$f(x) = \frac{x-1}{\sqrt{x}-1}$$

$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1}$ Indeterminate. Corresponding values of f(x) get closer to 2, to 6 decim[al] places

$\frac{x-1}{\sqrt{x}-1} = \frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{(x-1)(\sqrt{x}+1)}{x-1} = \sqrt{x}+1 \quad (x \neq$

$\lim_{x \to 1} \sqrt{x}+1 = \boxed{2}$

Compute the limit and describe what happens near x = 3.

$$\lim_{x \to 3^+} e^{\frac{6}{3-x}}$$

For $x \to 3^-$, $3-x$ is neg.

$\lim_{x \to 3^+} e^{\frac{6}{3-x}} = 0 \qquad e^{\text{large neg \#'s}} = \text{Small \#'s}$

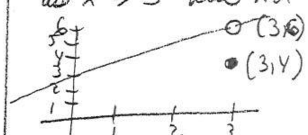
Closer x gets to 3 on the right, the larger e is on the top.

$e^{\frac{6}{3-4} = -6}, \quad e^{\frac{6}{3-3.5} = \frac{6}{-.5} = -2} \quad e^{\frac{6}{3-3.00001}}$

Let $r(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{for } x \neq 3 \\ 4 & \text{for } x = 3 \end{cases}$

[a]) [G]raph and discuss continuity at x = 3.
Is (3,4) on the graph of this function?
Does the limit exist?

r(x) is not contin. at x=3 b/c limit as $x \to 3$ does not equal r(3)

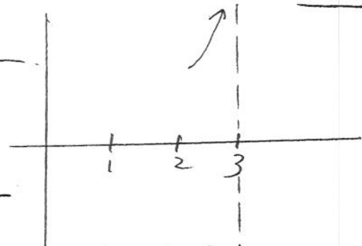


Open circle (3,6)
Closed circle (3,4)

(B) yes, (3,4) is on the graph of the function
(C) yes limit exists but not ≠ to value of the function

$\lim_{x \to 3^-} f(x) =$ ___

$\lim_{x \to 3^+} f(x) =$ ___



Find a limit of a function to match the graph.

Non-trad. Informal sense

$\lim_{x \to 3^-} f(x) = +\infty$

$\lim_{x \to 3^+} f(x) = -\infty$

$f(x) = \frac{1 + x - 4x^2}{x-3}$

Evaluate: $\lim_{x \to 5} \frac{x-5}{\sqrt{x-5}}$

$\lim_{x \to 5} \frac{x-5}{\sqrt{x-5}} = \lim_{x \to 5} \frac{(x-5)^1}{(x-5)^{\frac{1}{2}}} = \lim_{x \to 5} (x-5)$

$= \lim_{x \to 5} (x-5)^{\frac{1}{2}} = \lim_{x \to 5} \sqrt{x-5}$. Substitute x=

$\lim_{x \to 5} \sqrt{x-5} = \sqrt{5-5} = \sqrt{0} = \boxed{0}$

Particle Problem on Position

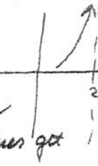
$S(t) = \frac{1}{2-t}$

How does the particle behave near 2 sec. but less than 2 sec.

Write as a limit. Graph, explain the particle's behavior

$\lim_{t \to 2^-} \frac{1}{2-t} = +\infty$ Particle gets further away from origin

At t=0, there's no way to get past the origin. No good for t to the right. When t=0, particle is at $\frac{1}{2}$. t progresses → 2 but stays left, and goes to ∞. Function values get large in the positive direction.



Particle Problem on Velocity

$v(t) = \frac{1}{(2-t)^2}$

How does the particle behave near t=2.

Write as a limit, graph, explain the particle's behavior.

$\lim_{t \to 2^-} \frac{1}{(2-t)^2} = +\infty$

Particle gets farther from origin, moves faster in that direction. When t→2, particle goes to ∞ and speeds up.

Here they associate limiting behavior with particle motion

(P8)

Potential Tasks for Interview 3: Infinite Limits and Limits that Do Not Exist

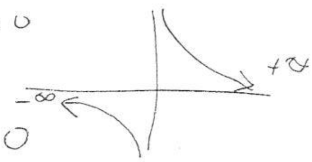
Compute and Graph:

$\lim_{x \to -\infty} \frac{1}{x} =$

$\lim_{x \to +\infty} \frac{1}{x} =$

$\lim_{x \to -\infty} \frac{1}{x} = 0$

$\lim_{x \to +\infty} \frac{1}{x} = 0$

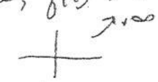


Describe the end behaviors of each:

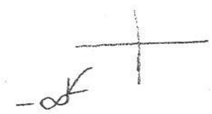
(A) $\lim_{x \to +\infty} \frac{x^2 - x - 6}{x - 3}$

(B) $\lim_{x \to -\infty} \frac{x^2 - x - 6}{x - 3}$

(A) $\boxed{+\infty}$ As $x \to \infty$, $f(x) \to \infty$



(B) $\boxed{-\infty}$ As $x \to -\infty$ $f(x) \to -\infty$



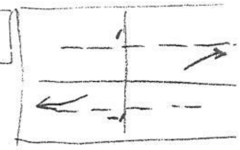
Evaluate the following limits and sketch both on the same graph.

A) $\lim_{x \to +\infty} \frac{\sqrt{x^2+1}}{x+1}$

3) $\lim_{x \to -\infty} \frac{\sqrt{x^2+1}}{x+1}$

(A) $\lim_{x \to +\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x \to +\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x(1+\frac{1}{x})}$ Factor out x^2 in num and x in den

$= \lim_{x \to +\infty} \frac{x\sqrt{1+\frac{1}{x^2}}}{x(1+\frac{1}{2})} = \boxed{1}$



(B) $\lim_{x \to -\infty} \frac{|x|\sqrt{1+\frac{1}{x^2}}}{x(1+\frac{1}{x})} =$

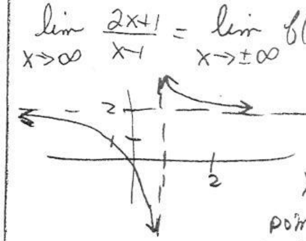
$\lim_{x \to -\infty} \frac{-x\sqrt{1+\frac{1}{x^2}}}{x(1+\frac{1}{x})} = \lim_{x \to -\infty} \frac{-\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}} = \boxed{-1}$

As $x \to +\infty$, $f(x) \to 1$, As $x \to -\infty$, $f(x) \to -1$

Write this function as a limit, as $x \to \pm\infty$, graph it and explain your results

$f(x) = \frac{2x+1}{x-1}$ Find: HA's, V.A's, Hole's, Limits

$\lim_{x \to \infty} \frac{2x+1}{x-1} = \lim_{x \to \pm\infty} f(x) = \frac{2+0}{1-0} = 2$



when x gets closer to $\pm\infty$, x gets large, so the points on the graph get closer to the horizontal line $y = 2$.

(P9)

(A) Horizontal Asymptotes represent limits at ______

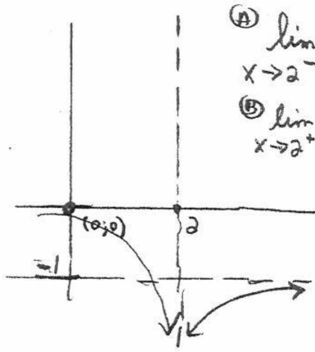
(B) Vertical Asymptotes represent limits at ______

(C) Write an example of a "limit at infinity." Is it $\lim_{x \to 2} f(x) = \infty$ or $\lim_{x \to \infty} f(x) = 2$

(A) ∞

(B) A point,

(C) $\lim_{x \to 2} f(x) = \infty$



(A) $\lim_{x \to 2^-} f(x) =$ ____

(B) $\lim_{x \to 2^+} f(x) =$ ____

(C) $\lim_{x \to 2} f(x) =$ ____

(D) $\lim_{x \to +\infty} f(x) =$ ____

(A) $-\infty$

(B) $-\infty$

(C) DNE

(D) -1

N.T.

(A) Identify all values of n such that $\lim_{x \to n} f(x) = \infty$ or $\lim_{x \to n} = -\infty$ for

$$f(x) = \frac{(2x+a)(x-b)}{(cx+d)(3x-k)}$$

(B) What is "n" in this problem?

(A) An infinite limit indicates the presence of a V.A. To find values of x that ma[kes] the den = 0, set both factors in den = 0, th[en] Solve, if x-value makes num and den = 0 there's usually a hole, not V.A. so limit ex[ists]

$cx + d = 0$, $cx = -d$, $x = -\frac{d}{c}$

$3x - k = 0$, $3x = k$, $x = \frac{k}{3}$

Set den = 0

Then set Num = 0

$2x + a = 0$, $2x = -a$, $x = -\frac{a}{2}$

$x - b = 0$, $x = b$.

(B) $n = -\frac{d}{c}$ $n = \frac{k}{3}$ from denom

With the same problem, write the limit of the function now using these values for n.

$\lim_{x \to -\frac{d}{c}} f(x) = -\infty$ or ∞

$\lim_{x \to \frac{k}{3}} f(x) = -\infty$ or ∞

As long as $-\frac{d}{c}$ or $\frac{k}{3}$ equals $-\frac{a}{2}$ or b.

(P(10))

Particle Problem:

$$S(t) = 1 - e^{-t}$$

A) Compute this as a limit as $t \to +\infty$.

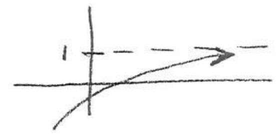
B) How does the particle behave for positive values of t?

C) Graph the function

(A) $\lim_{t \to +\infty} (1 - e^{-t}) = 1$

(B) Behavior gets close to 1

Function values (position of particle) get close to 1.



Explain the difference between finding a limit at a point, limit at infinity and what an infinite limit means.

Examples

$\lim_{x \to a} = L$

$\lim_{x \to \infty} = L$

$\lim_{x \to a} = \pm\infty$

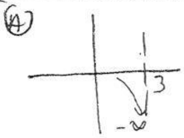
$\lim_{x \to \pm\infty} = \pm\infty$

Explain how these differ, using graphs

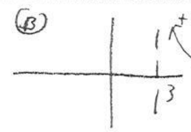
(A) $\lim_{x \to 3^-} f(x) = -\infty$ and (B) $\lim_{x \to 3^+} f(x) = +\infty$

(C) $\lim_{x \to \infty} f(x) = +\infty$ and (D) $\lim_{x \to -\infty} f(x) = +\infty$

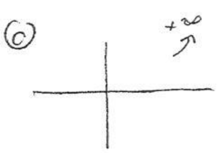
(A)



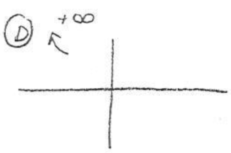
(B)



(C)



(D)

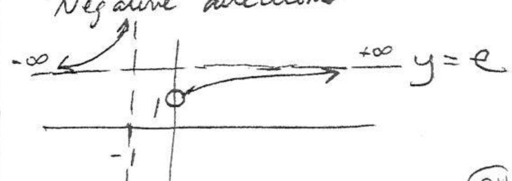


Compare and graph the following.

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$$

$y = e$ is a H.A. for the function in both positive and Negative directions



(p11)

Revisiting a previous problem, evaluate $\lim_{x\to\infty} f(x)$ for

$$f(x) = \frac{(2x+a)(x-b)}{(cx+d)(3x-k)}$$

Write this as a limit

$$\lim_{x\to\infty} \frac{(2x+a)(x-b)}{(cx+d)(3x-k)} = \lim_{x\to\infty} \frac{2x^2-2bx+ax-ab}{3cx^2-ckx+3dx}$$

Deg Num = deg. denom so lim. at ∞ equals fuotient of leading coef's, a, b, c, d, k a constants

$$\lim_{x\to\infty} \frac{2x^2-bx+ax-ab}{3cx^2-ckx+3dx-dk} = \frac{2}{3c}$$

Compute this limit, graph then explain the end behavior, $x \to \pm\infty$. Is this function rational? Why/why not?

$$f(x) = \frac{\sqrt{4x^2+2}}{3x+1}$$

Algebraic function, not rational,

$$\sqrt{4x^2+2} = \sqrt{x^2\left(4+\frac{2}{x^2}\right)} = |x|\sqrt{4+\frac{2}{x^2}}$$

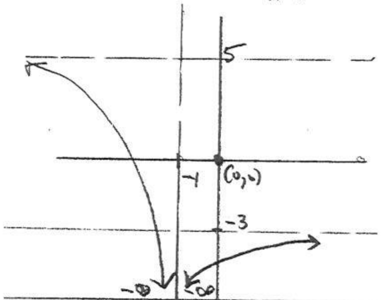
$$\lim_{x\to+\infty} \frac{|x|\sqrt{4+\frac{2}{x^2}}}{3+\frac{1}{x}} = \frac{\sqrt{4+0}}{3+0} = \boxed{\frac{2}{3}}$$

When $x \to +\infty$, then $x>0$ implies $|x| = x$.

$$\lim_{x\to-\infty} \frac{-x\sqrt{4+\frac{2}{x^2}}}{3+\frac{1}{x}} = \frac{-\sqrt{4+0}}{3+0} = -\frac{2}{3}$$

when $x \to -\infty$, then $x<0$ so $|x| = -x$

Evaluate: $\lim_{x\to\infty} g(x)$, $\lim_{x\to-\infty} g(x)$ and $\lim_{x\to-1} g(x)$.



$\lim_{x\to\infty} g(x) = -3$ $g(x) \to$ closer to $y=-3$

$\lim_{x\to-\infty} g(x) = 5$ As x becomes more negative the function approaches H.A. $y=5$

$\lim_{x\to-1} g(x) = \infty$ As $x \to -1$ from both left + right, the fnct. values decrease w/o bound

$$\lim_{x\to-1^-} g(x) = \lim_{x\to-1^+} = -\infty$$

Limits are the same but do not represent a finite limit at $x=-1$ so $\lim_{x\to-1} g(x)$ d.n.e.

Find V.A. and H.A. for the graph of:

$$f(x) = \frac{\sqrt{x^2+1}}{x-2}$$

Write as a limit. Explain what kind of problem(s) we have here and sketch the graph.

Have limit at a point (V.A) and limit at infinity (H.

$$\lim_{x\to2^-} \frac{\sqrt{x^2+1}}{x-2} = \frac{\sqrt{5}}{0^-} = -\infty$$

$$\lim_{x\to2^+} \frac{\sqrt{x^2+1}}{x-2} = \frac{\sqrt{5}}{0^+} = \infty$$

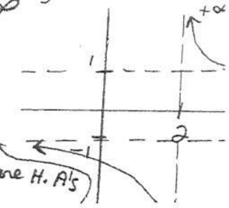
} So $x=2$ is V.A.

Now there's a limit at ∞:

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x-2} = 1$$

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+1}}{x-2} = -1$$

$y=1$, $y=-1$ are H.A's



Interview 1

Describe the difference between

A) $f(x) = \frac{2x^2+5}{x-1}$ and

B) $\lim_{x \to 1} \frac{2x^2+5}{x-1}$

Are you finding x or y?

Ⓐ Function

Ⓑ Limit of a function for values of x near 1.

Finding y or $f(x)$

What do these expressions mean? Are you finding x or y?

$\lim_{x \to 3} \frac{x+1}{x-3}$

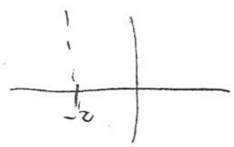
$\lim_{x \to 3} \frac{x^2-2x-3}{x-3}$

Behavior of $f(x)$ near 3.

Finding y or $f(x)$

Means finding behavior of $f(x)$ near 3.

Can you come up with a graph of a function that has limiting behavior near $x=-2$?



Given,

$\lim_{x \to \pm\infty} \frac{2x^2+x-1}{2-x}$

Can you explain what you are looking for?

Are you seeking x or y?

Can you graph this?

Looking for behavior of function values as x goes to $\pm\infty$

Seeking y or $f(x)$.

Given V.A at $x = -\frac{3}{2}$, $x = \frac{3}{2}$, and H.A at $y = \frac{1}{2}$, which below could be the equation of the graph?

ⓐ $f(x) = \frac{x^2}{x^2 - \frac{9}{4}}$ ⓒ $f(x) = \frac{1}{2x^2 - \frac{9}{4}}$

ⓑ $f(x) = \frac{x^2}{2x^2 - \frac{9}{2}}$ ⓓ $f(x) = \frac{x^2}{2(x - \frac{3}{2})^2}$

ANS: (b)

$\frac{2x^2}{2} - \frac{\frac{9}{2}}{2}$

$x^2 - \left(\frac{9}{2} \cdot \frac{1}{2}\right)$

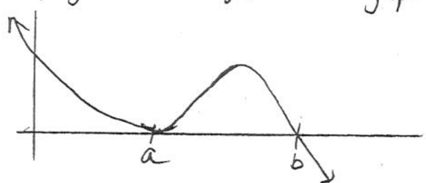
$x^2 - \frac{9}{4}$

$x^2 = \frac{9}{4}$

H.A $= \frac{1}{2}$ $\quad \frac{x^2}{2x^2}$

Same powers of x

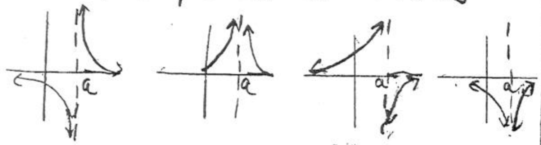
Which poly. function might have the graph shown?



1) $y = x(x-a)^2(x-b)$
2) $y = (x-a)^3(b-x)$
3) $y = (x-a)^2(x-b)^3$
4) $y = (x-a)^2(b-x)$
5) $y = (x-a)^3(b-x)$

Ans: 4

Match the Graphs with the Functions

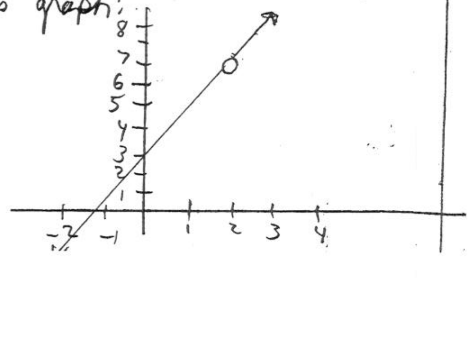


a) $f(x) = \frac{-1}{x-a}$ ⓒ $f(x) = \frac{-1}{(x-a)^2}$

b) $f(x) = \frac{1}{x-a}$ ⓓ $f(x) = \frac{1}{(x-a)^2}$

ANS: 1: B 2: D
3: A 4: C

Construct a Rational Function for this graph:



Ans:
$g(x) = \frac{2x^2 - x - 6}{x - 2} = \frac{(x-2)(2x+3)}{x-2}$

$= 2x + 3$ if $x \neq 2$

$x - 2$ common factor gives hole when $x = 2$ due to restriction $x \neq 2$.

(61)

APPENDIX D: PHASE I RESULTS

Analysis of Carrie and Nicole

Analysis of Interview on Functions: Carrie

A narrative summary of how Carrie solved function tasks is presented. Transcript evidence from her problem solving activities will be discussed to highlight her underlying ideas about functions. This summary serves as a foundation for discussing her ideas about limits. The rationale for selecting Carrie for this initial analysis is that her knowledge of functions was minimal. Significant revelations and misunderstandings about limits also emerged. Carrie referred to the term "restraint" to describe certain regions on graphs that fail the vertical line test. With limits, these restraints referred to imaginary vertical lines that function values can't go past. Her perceptions are interesting but not unique which this researcher hypothesized might be observed in other students using the same tasks.

Defined function. Write or draw examples of functions. What is a function value? What's the difference between a function and a function value? How are functions and function values related?

Figure D.1.1: Task 1 Problem Statement.

C: The f(x) is the big thing that comes to mind. I think f of x or g of x. Function values are the dots on the graph of x and y. A function is something like f(x) is x-squared so a function value is a dot on that curve. *(Turn 4, 1:00 SDV_0001*)

R: Can you sketch the graph of what a function might look like as an example or perhaps draw an example of something that's not a function? *(Turn 5, 1:25, SDV_0001)*

C: See like this, it's a function (drew what looks like x-cubed). I forgot what this kind is called, but it's a function. It doesn't have restraints. But if you change it so something's up and down you get this. I want to say you have something like this and then you put restraints on it in some way. There is a restraint if it comes down there (*Carrie drew a vertical line through the graph with blue horizontal line through it*) and you look at some area of it. So it's not a function right there where it goes up and down. The rest of it would still be a function though without the restraint. (*Turn 6, 1:30, SDV_0001)*

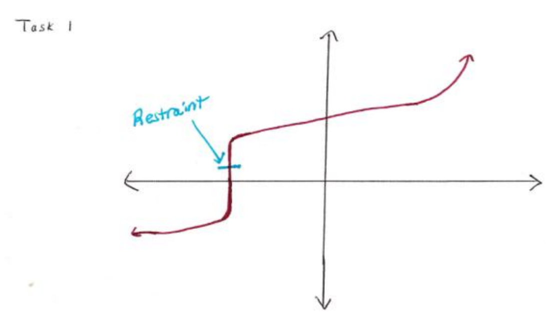


Figure D.1.2: Carrie's work on Task 1.

R: Can you name some types of functions? *(Turn 7, 2:16, SDV_0001*)

C: Absolute value, x-squared and linear ones and that stinkin' e to the x thing. *(Turns 8 & 13, 2:20, SDV_0001*

Analysis:

Inference. Carrie did not give a definition of function but described what a function was and recalled some examples such as absolute value and quadratic functions. As she modified an example of a cubic function to include a vertical line through the middle, she introduced the term "restraints" to describe certain regions of the graph that fail the vertical line test. She split the function into two separate parts. Only the middle part with the restraint was not a function, but the regions to the left and right were still considered to be a function. Carrie considered restraints as regions of a graph when the relation was not a function and she uses the word "constraint" interchangeably. She used the vertical line test on regions with restraints to identify where part of a graph was not defined, or not the graph of a function.

Explores the circle, and the top and bottom separately, to decide which are graphs of functions.

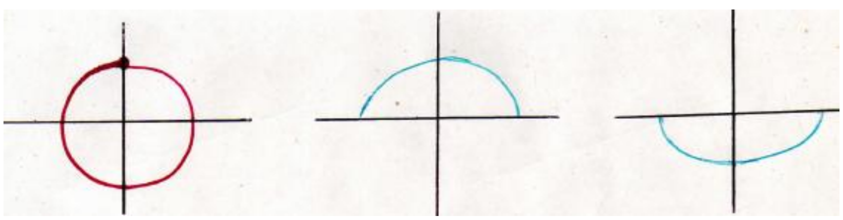


Figure D.1.3: Task 2 Problem Statement

R: What about the upper half? Would you say this is the graph of a function?

C: Yes.

R: How about the whole circle, is this the graph of a function?

C: The upper part, yes. Bottom part, no. It's got to do with this area down here (under x axis) because of restraints put on the function it's just the positive not necessarily the negative ones. Just the bottom part is the graph of a function because it doesn't have any constraints. *(Turn 16, 5:00, SDV_0001)*

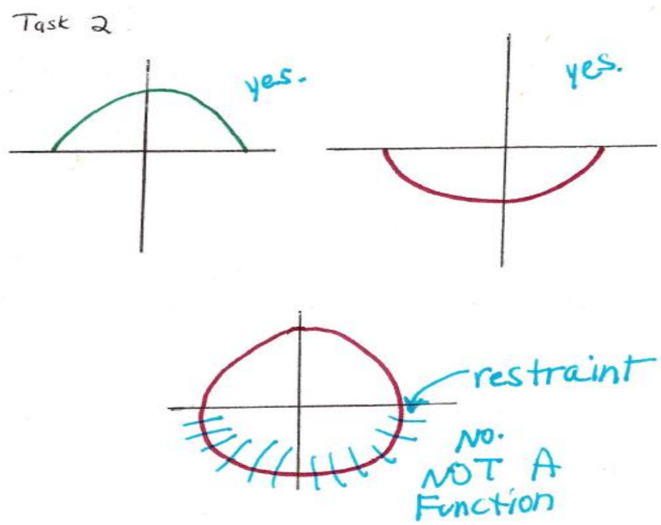


Figure D.1.4: Carrie’s work on Task 2.

Analysis:

Inference: Carrie consistently used the vertical line test to determine functional relationships. She referred to the term "restraint" to identify portions of the graph that disqualify the relationship from being the graph of a function. Her knowledge of functions and understanding of functional relationships were limited beyond the vertical line test. Carrie continually refers to "restraints" or “constraints” to identify regions of the graph where she believes the curve is not the graph of a function because there was more than one value of y for each x. Carrie marked up the bottom part of circle with straight lines to illustrate how the bottom half constitutes a restraint which means the relationship was not a function.

Hypothesis: Other students might share similar notions and so new data would focus on questioning about understanding domains of functions.

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Explores the graph of a discontinuous function with an isolated point on the same graph.

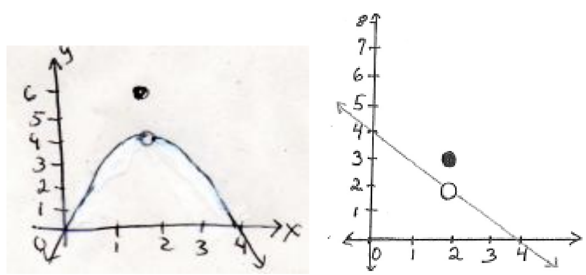


Figure D.1.5: Task 3 Problem Statement

R: Is this point on the graph of the function? Also can you name or describe this type of function? (*Turn 17, 6:20 SDV_0001,)*

C: No it's not on the graph. Because this function is -- I get functions and limits confused, the function is going down --it can go any direction it's gonna go, but (*she draws* *vertical line through* the point and open circle) I get confused. The point is on the graph of the function, it's just not on the function. The point is not on the graph because it doesn't intersect with the function. The point is not on the function. I see it more as a limit than a function. Oh, the type of function. It's one of those x-squared kinds. *(Turn 18, 6:45, SDV_0001)*

R: Does the point or solid dot above it have anything to do with the function?

C: No. It has nothing to do with the function. The point is just in the wrong place. If you slide it down the dotted line, then it is on the function and so then I'd say the point is on the graph of the function which is why it looks more like a limit than a function problem. *(Turn 94, 49:00 SDV_0001,)*

R: So you see a function here on this graph. Can you tell me how many different pieces do you see and show me where they are?

C: I see two pieces. The curve to the left of the hole going up is the first piece, and the curve going to the right of the hole going up is the second piece. *(Turn 96, 50:15)*

R: Does the dot have anything to do with the function?

C: No. The dot is above the x-squared function but it's not part of the function. The only function on this graph is x-squared. The dot is just some random thing put there to be confusing like with limit graphs but it doesn't affect the x-squared function in any way. The dot is by itself up there doing nothing. *(Turn 98, 50:55)*

R: Is there any way this graph could be a piecewise function? Explain why or why not.

C: No because piecewise functions have straight lines going straight across to a solid dot or to a hole. There are 2 separate pieces to the graph. And this graph down here only has one piece, not two pieces. It only has a x-squared function with a dot in the wrong place. It looks like it just got put in the wrong place and doesn't go with the graph, so the only function that's on there is x-squared. If you had a straight line going across to the left of the dot and a straight line going to the right of the hole that's the only time it can be piecewise. *(Turn 100, 51:40, SDV_0001)*

Analysis:

Inference: When shown the graph and asked if the isolated point is on the graph of the function, Carrie thought that the point was not on the graph because it didn't intersect with the function. Carrie saw a quadratic function, not piecewise. Since the isolated point above the curve was not on the function, it was not on the graph of the function. Carrie thought that the point had to be a point of continuity to be on the graph of the function. Moreover, she saw no relationship of the isolated point with the quadratic function, and claimed it was just a "random point." She did not know what kind of function this was and by focusing on the curve alone, she thought it was quadratic. She did not see two pieces each having a different domain, but she saw two pieces, one on each side of the hole of the quadratic function. She was not able to identify the function as piecewise. Therefore, her knowledge of piecewise functions is minimal.

Hypothesis: Students may not understand what it meant for a point to be on the graph of a function because they don't know that the second coordinate is the function value of the first coordinate.. This type of piecewise function may be unfamiliar to other students because they view it as quadratic instead of piecewise. Once again, further data might focus on their understanding of domains.

The problem should have been identified as: $f(x) = \begin{cases} 6 & \text{if x = 2} \\ -(x-2)^2 + 4, & \text{if x} \neq 2 \end{cases}$

Follow up questions about the graph of a discontinuous function with an isolated point. A point was added to the graph of a piecewise function so that it was now not the graph of a function.

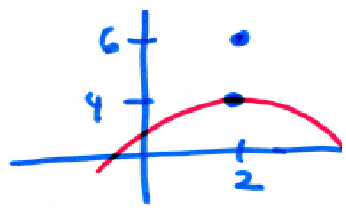


Figure D.1.6: Task 4 Problem Statement

R: Is the point (-2,4) on the graph of this function on the left? (*Turn 13, 10:30, SDV_0001, 5/5/2011)*

C: It's on the graph of the function. It's just not on the function. *(SDV_0001, Turn 20, 9:58, 11/2/2010 and SDV_0001,Turn 14, 11:00, 5/5/2011)*

R: Explain how many pieces of the graph you see here?

C: There are 2 pieces down here on the curve, to the left of the hole going up and to the right of the hole going up.

R: What about the point, could it be part of the graph of the function?

C: No, because it's not on the function. It's just a random dot above the function.

R: Is there any way this function could be piecewise? Explain how this might or might not be considered to be a piecewise function. *Turn 19, 12:20, 5/5/2011*

C: It has two pieces to the left and right of the hole, but it's not a piecewise function because it doesn't look like one. Piecewise functions have to have an open hole with a line going like to the left, and then a solid dot above it with a line going out to the right. So because of the x-squared part and the fact that the point above it doesn't have any straight lines attached, it cannot be piecewise. *SDV_0001,Turn 20, 14:20, 5/5/2011)*

R: With the figure on the right, is this the graph of a function? *(Turn 21, 14:45, 5/5/2011*

C: No, it's not the graph of a function because you can draw a vertical asymptote through it. There's no hole to worry about so everything so you can just look straight up and down where x is and see 2 different y's which isn't legal. (*Turn 22, 14:55, 5/5/2011)*

Analysis:

Inference: When Carrie saw just a point in the plane, she said it was on the graph but not on the function. When she saw the discontinuous quadratic shaped curve below, she also said that it was also on the graph. When both were shown, she stated that both pieces were on the same graph. She did not identify the whole function, though, as being a piecewise function. Instead, she only acknowledged the quadratic piece. With the graph on the right, where the hole was replaced by a solid dot, she correctly recognized this as not being the graph of a function because for one value of x there were two different values of y. This confirms her interpretation of the circle not being the graph of a function as well.

The initial problem should have been identified as: $f(x) = \begin{cases} 4 & \text{if x = -2} \\ (x+2)^2+1, & \text{if x} \neq -2 \end{cases}$

Hypothesis: Other students may lack knowledge about piecewise functions and about their domains so additional data could address underlying notions about domains.

The results appear to suggest the need for more exploration with similar piecewise functions since problems with discontinuities are common with limits. One subsequent question might be to ask the student to draw different piecewise functions.

Given VA at x= - $\frac{3}{2}$, x= $\frac{3}{2}$ *and* HA at y= $\frac{1}{2}$ identify the correct function below.

$a.\ f(x) = \dfrac{x^2}{x^2 - \frac{9}{4}}$ b. $f(x) = \dfrac{x^2}{2x^2 - \frac{9}{2}}$ c. $f(x) = \dfrac{1}{2x^2 - \frac{9}{4}}$ d. $f(x) = \dfrac{x^2}{2(x-\frac{3}{2})^2}$

Figure D.1.7: Task 5 Problem Statement

R: This one is a more challenging problem with vertical asymptotes at x=-3/2 and x=3/2 with a horizontal asymptote at y=1/2. Which one of the following functions could be the graph or contain these features? Here are the 4 answer choices. Explain how you would find out. Would you use a calculator, draw a graph?

C: I actually would work it backwards looking at the different answers. I would go that way, from the equations to the information because looking at this I'm thinking this is nothing like I've ever seen lately or before so answer choices are more familiar to me so I'd try to piece that apart. "B and C" are unfamiliar to me so I would try to factor. I'd say no to 'B' because the 2x squared is throwing me off. I'd say no to 'D' because the 2 in the denominator throws me off and won't give me a negative. I feel like I'm missing a number. I have odd reasoning when it comes to this stuff. Honestly when I look at this, I don't know where to start in terms of working it. It's been a while. I would try to reduce it. To set it up is a big thing for me, but once it is set up I can easily get through it usually. *(Turn 26, 12:55 SDV_0001)*

At this point, I asked if she could do any factoring and she said she could by looking at the solutions. Then she tried to factor out x squared and changed her mind from using factoring to reducing instead. (*Turn 34, 16:03 SDV_0001*)

R: How would you reduce it? *(Turn 36, 16:5 SDV_00012)*

C: That's my "split in half theory" and it doesn't work either. I might have to use a calculator because I can't do math in my head. If you put in (x-3/2) it doesn't work. You get a zero at the bottom so it doesn't work. *(Turn 37, 17:05)*

Carrie tried to plug in the two x values into the denominator and the y value given in the problem and thinks that if the answer choices are equal to zero, then they would not be correct. After she used the calculator to graph each one of the answer choices, she saw vertical asymptotes and also the horizontal asymptote but they were not what she thought they should look like. She then finally selected answer choice B because she would end up with a zero in the denominator. This approach was not in line with the spirit of the problem to use the properties given to identify the function. After she constructs what she thinks the graph should look like, she claimed there was a part in the middle of the calculator's graph that confused her because it did not appear as she had thought it would. I then asked her to explain what this unaccounted middle region was about.

C: The restraints. That's where the function would not exist I would say. The function won't exist be between -3/2 and 3/2. *(Turn 47, 24:25 SDV_0001.)*

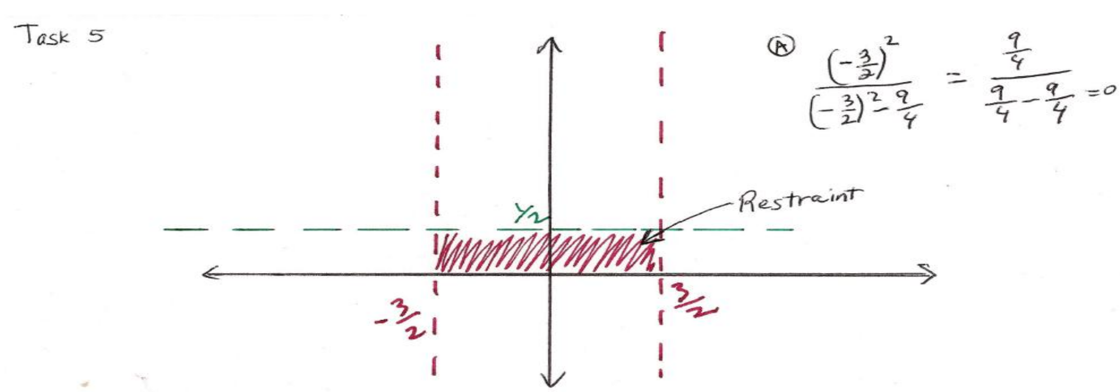


Figure D.1.8: Carrie's work on Task 5.

Analysis:

Inference: This problem involved challenges beyond Carrie's skill level and so there were a lot of pauses and long silences while she worked on this problem. Carrie drew the asymptotes as given in the problem to try constructing the graph. Her drawing, though, does not match the correct answer choice, B. She did not use the rules of finding a horizontal asymptote by comparing the degree of the numerator to the degree of the denominator. If the powers are equal, then the ration of the leading coefficients will be the horizontal asymptote. If she had known this rule, she could have identified the 1/2 among the answer choices and ruled out at least 2 possibilities.

She thought that the function did not exist between the two given vertical asymptotes and the horizontal asymptote. She worked backwards using the process of elimination and started with the answer choices to come up with the graph she visualized. She focused on answers that yielded zero in the denominator.

She used the calculator to see which answer choice generated the asymptotes given in the problem but did not enter them correctly using parentheses in the right places so her graphs were not correct. As Carrie got increasingly more frustrated and started guessing how to reconcile her incorrect graphs with the properties given, she stated there were vertical asymptotes as restraints and so the function would not exist between -3/2 and 3/2. The horizontal asymptote also served as a restraint in which the function could not rise above or go past on the y-axis.

She computed the value of the denominator at the vertical asymptotes and said the function didn't exist because she got zero in the denominators but could not graph this correctly to illustrate her claim. Carrie referred to squaring the fraction in the denominator as a "split in half theory" because she was splitting up the two terms. Carrie clearly expressed frustration with this task. She could not explain why the function did not exist in the region between -3/2 and 3/2 other than say that region was the restraint. She appeared confused over what it means in terms of the graph to have zeros in the denominator.

Hypothesis: Algebra proficiency and knowledge of rational functions may be a problem for other students.

R: Which polynomial function might represent the graph shown?

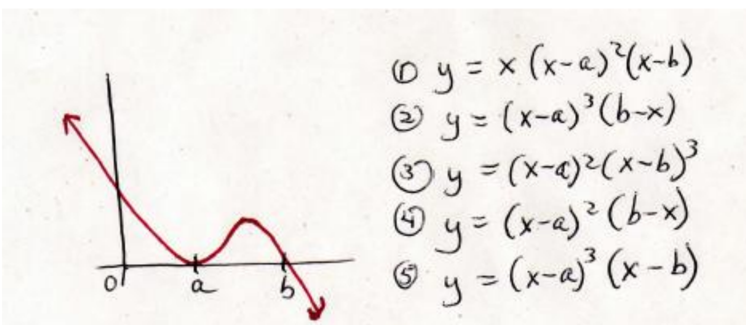


Figure D.1.9: Task 6 Problem Statement

C: What? I have no clue. I'm supposed to look at this graph and figure out the function? This is a backwards question. Usually it's the other way around, like here's the function, go graph it. Not do something like this though. I see it crosses the x-axis twice, but there's not enough here on the graph to pick out answers. I don't think any of these answer choices would work. I would just plug all of them in the calculator and see if I get that same graph. That's the only way. *(SDV_0001, Turn 227, 21:50, 5/5/2011)*

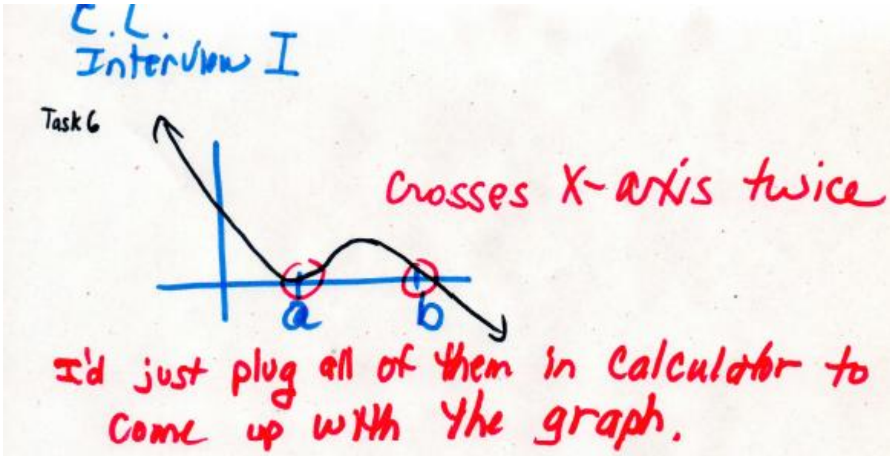


Figure D.1.10: Carrie's work on Task 6.

Analysis:

Inference: Carrie worked backwards, plugging in given answer choices to reproduce the graph rather than study the features of the graph to come up with the correct equation. She did not know the graph crosses once and touches once. Carrie can only solve problems in a more traditional format, going from the function to the graph and this is because she lacks algebraic skills necessary to work backwards, such as recognizing the slope being negative, and a cubic function having 2 turning points. She might not know enough about finding zero's or solutions especially given in more abstract format with letters, which is why she did not recognize there were two solutions (x=a) and (x=b). She also could not manipulate the answer choices, such as factor out a negative from (b-x) leading her to get the negative slope. She also confuses crossing with touching. Touches once and crosses once. It is speculated that if no answer choices were given, it would have been impossible for her to come up with a correct form of the function due to weak algebra skills and low algebra proficiency.

Hypothesis: Students have difficulty when asked to derive the original function from a graph and this is associated with a lack of algebra proficiency.

Match the graphs with the corresponding functions.

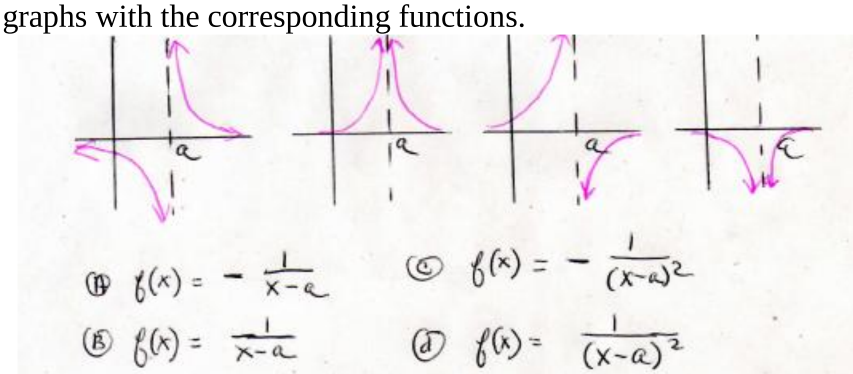


Figure D.1.11: Task 7 Problem Statement

R: First, can you tell me what kinds of functions these are in the answer choices?

C: You mean give them a name? Well, I call them fractions with letters in them to confuse me. *(Turn 229, 23:20, SDV_0001 5/5/2011)*

R: How can you find out which of the functions goes with the graphs?

C: This is another strange problem. I can't start from graphs. I have to start from the function things and then plug it into the calculator to find the graph, so that's what I am going to do again. The only one I know is negative is the last graph on the right. Maybe the first and 3rd but I always get those confused so they both could be, but I will know after I put the in the calculator. *(Turn 231, 28:00, SDV_0001 5/5/2011)*

R: Can you at least tell me which ones you think will be positive and which will be negative?

C: The only one I know is negative is the last graph on the right. Maybe the first and 3rd but I always get those confused so they both could be, but I will know after I put the in the calculator. *(SDV_0001, Turn 231, 28:00, 5/5/2011)*

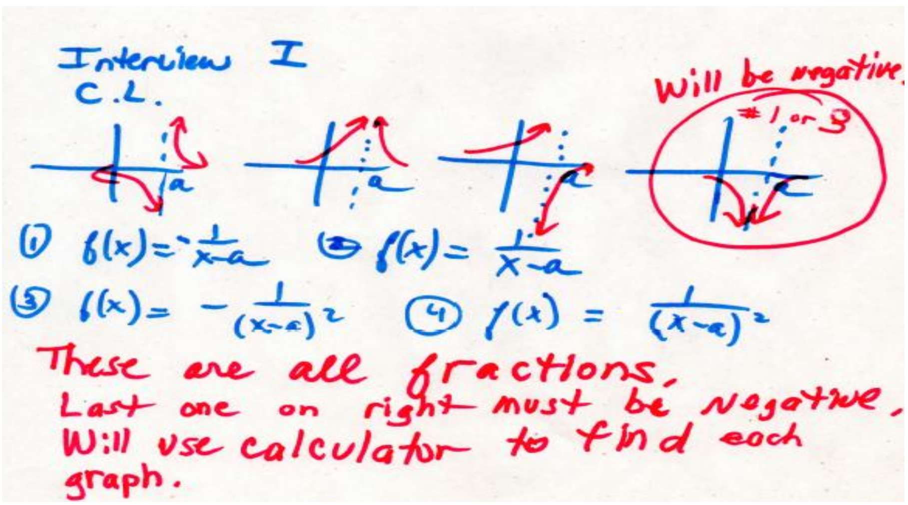


Figure D.1.12: Carrie's work on Task 7

Analysis:

Inference: Carrie did not correctly identify these as rational functions and instead called them fractions. She considers them fractions instead of quotients because of the fact they have a numerator and denominator.

She could not ascertain which graphs corresponded to the given functions below so she used trial and error with the calculator to guess each one. She didn't explain the difference between the rational functions raised to the first or second powers, and how they might behave. Since she did not recognize the shifted function f(x)= $\frac{1}{x}$, this suggested she doesn't recognize the underlying elementary function translated right "a" units. She guessed the last one was negative based on visual inspection of both arrows pointing downward, but did not explain the function values were decreasing without bound. The reason could be she doesn't understand the properties of this basic reciprocal function.

Carrie could not explain the second graph with both arrows pointing up and the fourth graph with both arrows pointing down because lacks an understanding of how function values behave when squared. If she squared the denominator, positive function values would have resulted and then she might have matched choice 4 with the second graph. Then by placing a negative in front of it, the function values would have been negative hence answer choice 3 would have matched the fourth graph. Lacking number sense, knowledge of reciprocal functions and understanding of the coordinate system, she had difficulty with this problem and her only recourse was to work backwards by plugging each function into the calculator to get the graphs.

Hypothesis: Students might have trouble with this task because they need number sense and basic skills about rational functions.

Construct a possible rational function for this graph with a hole at (2,7).

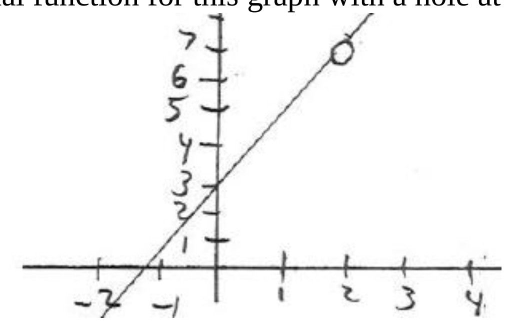


Figure D.1.13: Task 8 Problem Statement.

C: I honestly have no idea. I know it is linear and it has a positive slope so I would just guess and say f(x) = x+3. *(Turn 233, 30:45, SDV_0001 5/5/2011)*

R: How did you come up with that?

C: I look at the y intercept and the line crosses at 3 and it is a straight line so it has a slope of 1. *(Turn 235, 31:20, SDV_0001 5/5/2011)*

R: Does the hole mean anything about the function?

C: It means the limit exists. I know we didn't do limits yet but when I see a hole with a line, I think limits. And since there is a hole a person can fall into, then down in the hole the limit doesn't exist because they can keep falling down further and further into the hole and it doesn't stop. They can go to infinity. *(Turn 237, 34:00, SDV_0001 5/5/2011)*

R: If this problem was just about functions and not limits, would the hole mean anything about the function?

C: Honestly, no. Because with regular functions like straight lines, linear functions, they don't have holes. Only limit problems have holes. *(Turn 239, 34:00, SDV_0001 5/5/2011)*

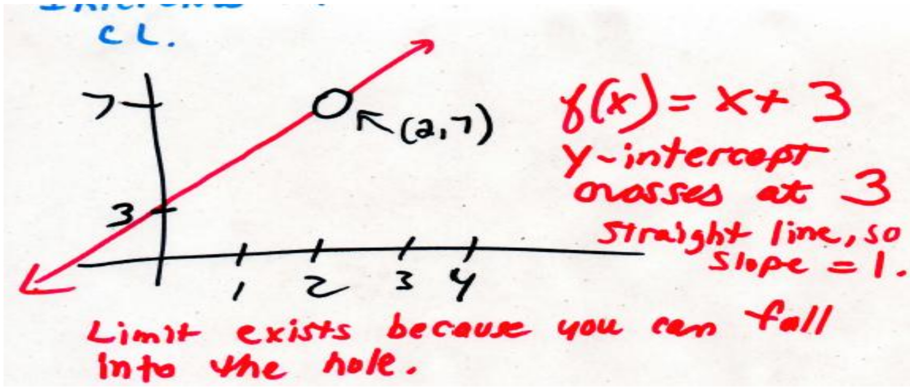


Figure D.1.14: Carrie's work on Task 8.

Analysis:

Inference: Carrie was not able to construct a rational function from this graph and did not know why there was a hole. She did not recognize that there is no function value for x=2. This is of course symptomatic of a zero in the denominator. Basic math and algebra deficiencies were probably the source of not correctly find the slope. She came up with a potential linear function y=x+3 then worked backwards to state the slope was equal to 1 due to the fact that the graph was a straight line.

Hypothesis: Students struggle with this type of problem because they lack knowledge about domains. They may also lack basic math and algebra proficiency, and not know properties of rational functions.

Summary

Carrie's understanding of functions from solving the tasks is summarized in Table 1. Eight tasks were presented on functions. Carrie could not define a function but described what one could look like on a graph, and gave some examples of functions. . She used the vertical line test to decide if a function existed or not. She also drew a curve on a graph with a red marker which represented a function but noted that an area where she could draw a vertical straight line constituted a "restraint" in which case the function didn't exist at that particular value of x. Carrie consistently used the term "restraint" to describe regions on the graphs that do not represent functions, for instance, a vertical asymptote as part of a curve,

Carrie knew that a circle was not the graph of a function, and that a function can't have two values of y for the same x, yet she did not use the same rationale for the first task she drew that had infinitely many y-values for a single x. This could be a result of having familiarity with the circle. Given a discontinuous function with a hole, Carrie had trouble deciding if a point above the hole was on the graph of the function because she did not know the definition of the graph of a function. She also could not come up with a formula for the functions. In this case she worked backwards and put the answer choices into the calculator instead to come up with the graphs. Carrie exhibited significant limitations with number sense, mastery and proficiency of algebra, knowledge of elementary functions, piecewise functions, and knowledge of how function values behave for rational functions. This was often accompanied by evidence such as long pauses, changing her mind quite often, as well as "I don't know" remarks. Overall, her understanding of functions is very limited.

Analysis of Interview on Limits at a Point: Carrie

A narrative summary is presented of how Carrie solved tasks involving limits at a point. Transcript evidence from her problem solving activities will be discussed to highlight her underlying ideas about limits at a point. This summary serves as a foundation for discussing her ideas about limits at infinity and limits that do not exist.

Limits at a point involved tasks in which a student looked at graphs of functions and described whether or not limits existed. Other tasks involved computing limits, constructing their graphs and describing the limiting behavior of the function values near a point. Significant misunderstandings emerged from these tasks. Carrie exhibited difficulty with the notation for limit; perceived limits to be only about how the first coordinate x was behaving, rather than the second coordinate; visualized limits on graphs as "barriers or restraints" to describe certain regions of graphs including vertical asymptotes and holes; drew erroneous conclusions that limits existed when they did not, or correctly stated limits existed but for the wrong reasons; elicited algebra deficiencies; used everyday language to replace mathematical terminology; and was not able to construct graphs without use of the calculator.

Using Skemp's theoretical model, Carrie would be categorized as having minimal instrumental understanding. Although she could provide creative believable solutions and explanations, her responses were generally incorrect. Many of her perceptions are unique but not isolated, as I hypothesized similar types of understandings to be present in other students using the same tasks.

First Carrie was asked to give a definition of limit or describe what a limit is, using some examples. When a definition could not be provided then an intuitive description was accepted. Carrie was asked to give examples of limits that did and did not exist. She was also asked if limits referred to either the first or second coordinate on a graph, or both. Then she was asked how functions and limits were related.

What is a limit? Provide a definition or an intuitive explanation. Explain whatever comes to mind when you think of finding a limit. Write or draw examples of problems involving limits, both that exist and do not exist. Do limits refer to either the first or second coordinate on the graph, or do they refer to both? How are functions and limits related?
Figure D.1.15: Task 1 Problem Statement

R: Explain what a limit is. You can give a definition or just describe whatever comes to mind. *(Turn 26, 59:34, SDV_0001.mp4)*

C: I'm thinking of constraints or barriers the function can't go past. A wall, like a vertical asymptote. It's also a hole you can fall into like an open man hole or hole on a golf course. I hear limits and I figure out where it is those moving lines and function values can't go to. I look for the wall or hole. Where's the stopping point for the graph, that's the limit. *(Turn 27, 59:51, SDV_0001.mp4)*

R: When you find limits, are you looking for the first coordinate "x" or the second coordinate "y"? Do limits involve what y is doing? *Turn 32, 1:02:13, SDV_0001.mp4.*

C: Limits are about what x is doing, not y. I don't think of y when I see limits. Unless it was written find the limit as y approaches 3 or something, then I'd look at the y but I only look at the x for these. I don't even look at the y. *(Turn 33, 1:02:15, SDV_0001.mp4)*

C: When we were doing limits in other classes, we had not yet gotten to functions. I didn't even know what the functions were *(Turn 145, 00:10 SDV_0006.mp4)*

R: They taught limits before functions? *(Turn 72 (146), 00:12, SDV_0006.mp4)*

C: Yeah. So when you started talking functions and then with limits, I thought, huh? To me, functions and limits never went together because we didn't use functions with the graph at that level. We didn't get to functions and graphing until brief calculus and limits were all the way back in the beginning. They didn't teach them together. *(Turn 147, 0:10, SDV_0006.mp4)*

R: Are functions and limits related somehow?

C: Functions are different from limits. With a function, f(x)=y. With a limit, I am just thinking about x. I honestly have no clue what the relationship is between a function and a limit. *(Turn 151, 01:23, SDV_0006.mp4)*

R: What comes to your mind when you see the "lim" notation?

C: I just know it means limit. Nothing comes to my head. The instant I see $\lim_{x \to 3}$ I think of barriers or walls at x=3 and which way am I going. It tells you that the limit is 3 because the x points there. Limits are about what x is doing because it's written in the notation x$\rightarrow$ *whatever* . *(Turn 25, 57:40, SDV_0001.mp4)*

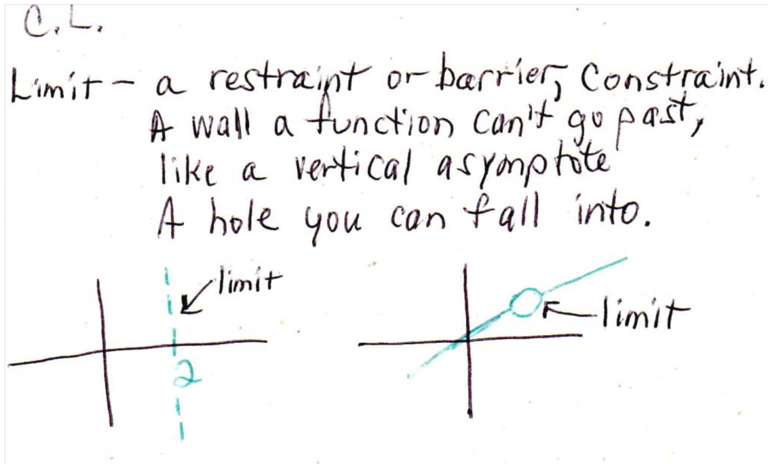


Figure D.1.16: Carrie's description of limit.

Analysis:

Inference: Carrie considers limits to be barriers, restraints, constraints, brick walls or holes one can fall into. She claimed limits were taught before functions. Moreover, Carrie could not describe a relationship between functions and limits and thought that limits refer to x-values. She associates limits only with the x-coordinate because of what appears beneath the "lim" notation. This could be why she claims that limits have nothing to do with y-values.

Hypothesis: Students don't know if they are finding x or y when finding a limit and so more data would reveal what students consider and look for when finding limits.

Compute the limit if it exists as x approaches 3, then sketch a graph of this function.

$$\lim_{x \to 3}(2x+1) \quad \text{or} \quad \lim_{x \to 3}(5x+2)$$

Figure D.1.17: Task 2 Problem Statement.

C: Ok, for the first one, plug in 3 get a 7. You get a 7 on the x-axis and 0 on the x-axis and there is your limit (in between). You see, f(x) is 3 and then there is a vertical line at 0 and a vertical line at 7. There's another vertical asymptote line at 3 because as x is approaching 3 the limit is at 7. So the limit is at 3 and 7 and at 0 and 7. So as x approaches 3, the limit is gonna be here at 7, another vertical asymptote. The limit doesn't exist because it is only approaching the 3 from this side from the left. It is only going to 3 from left and not beyond so it stops at 3. Then between 3 and 7 this area is also a limit because it's a restraint of what is constrained. *(Turn 86. 1:35, SDV_0039.mp4)*

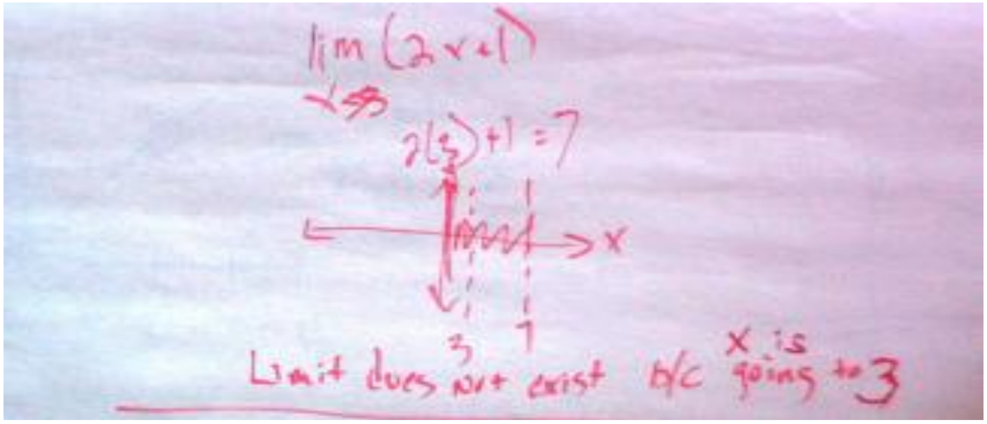


Figure D.1.18: Carrie's work on Task 2.

In order to try to figure out her logic and reasoning, we did another similar problem only changing the x to approach 2 instead of 3. The purpose was to further explore her interpretation of a limit being a barrier or restraint, headed toward 2 from the left, as she claimed. The arrow is directional, so as $x \to 2$ means going from left to right only, up to the vertical asymptote she drew at 2. She is firm that whatever appears under the "lim" notation is a limit, because that's what the notation suggests. She knew to plug in the x values into the function and she got a correct answer of 5. Even though she said that 5 is a limit, she drew it on the x-axis as another vertical asymptote. Finally, she said there is a 3rd limit which is the region between 2 and 5 because that area is a restraint.

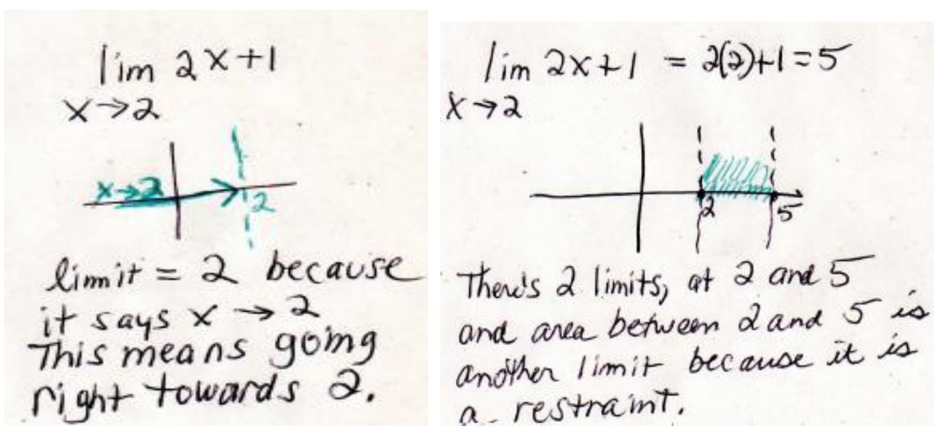


Figure D.1.19: Carrie's evidence of a similar linear function having 3 limits.

Analysis:

Inference: Carrie thinks that limits are x-values so computed limits are drawn on the x-axis in the form of vertical asymptotes and saw 3 limits: at x=3, x=7 and the region between 3 and 7. Yet, she said these limits did not exist and provided erroneous reasons for the limit not existing. She does not know a limit is a number and refers to the y-coordinate. She thinks the limit does not exist at 3 is because she perceives the graph approaching 3 from the left given the "lim" notation. She seems to see $x \to 3$ as having order involved, from numbers less than 3 approaching 3 on the number line. Her sense of nearness is directional and one-sided.

Hypothesis: Students may also think limits are about finding x-values, are directional, may not be thinking about nearness, and not know when a limit exists or does not exist.

Compute the limit if it exists. Explain the behavior of the function values near x=2.

$$\lim_{x \to 2} \frac{x^2 + 4}{x - 2}$$

Figure D.1.20: Task 3 Problem Statement

C: It factors into x+2 and x-2 in numerator, so x-2 cancels with x-2 in denominator. I substitute 2 (into x+2) in so I get a limit at zero. The calculator says quit. It's doing something funky like this, curve in 3rd and 4th quadrant, and up in the air in 1st quadrant. The limit is at 2 on the calculator and it says so in the problem as x goes to 2, but in my head it is 0. So I did something wrong. I still say it is 0. You can't have zero in the fraction. If I plug in x=0 anywhere, it doesn't work so that's why the limit is zero. *(Turn 26, 00:17, SDV_0044.mp4)*

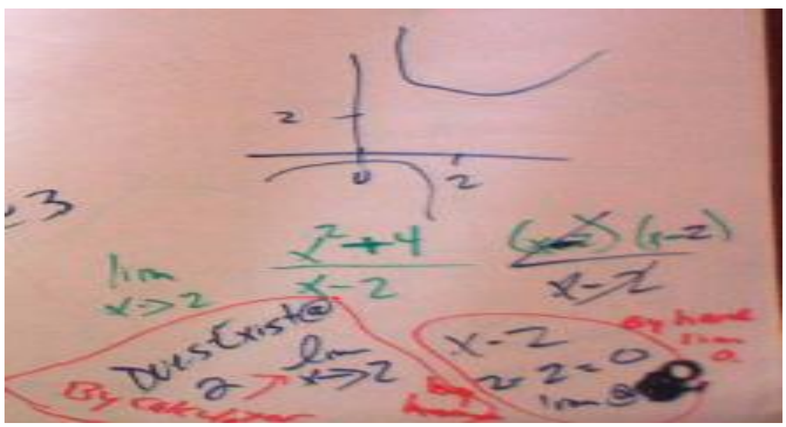


Figure D.1.20: Carrie's work on Task 3.

Analysis:

Inference: Carrie tried to factor the sum of squares, which is not possible so her result was incorrect. The limit does not equal zero in this problem, and her graph does not correspond to what she factored. In fact, the limit does not even exist on either side because the function values increase without bound in the positive and negative directions. She claimed by plugging in 0 does not work anywhere so the limit is 0. This did not make sense since there was no reason to plug in 0 after she got 0 as a result. She did not use any numbers near 2 to explore how the function values would behave to the left and right of 2. Meanwhile, she said there is a limit at 2 because the "lim" notation provided that information. She sketched the graph exclusively by entering this function into the calculator. She lacks knowledge of mathematical terminology as she considers a rational function to be a fraction. Because she thinks limits are about what x is doing, she considers 2 to be the limit. As a result of getting the wrong limit, she could not reconcile the 0 with the graph, but insisted that there was still a limit at 2 since it happened to come with the problem. Although she could get a graph on the calculator, she could not reconcile the limit of 0 with the graph, since the graph showed behavior going on near 2.

Hypothesis: Other students may lack algebra proficiency by trying to factor a sum of squares and may also think limits are about what lies beneath the “lim” notation.

Compute the limit if it exists. Explain the behavior of the function values near x= -3. $\lim_{x\to -3} \frac{x^2-9}{x+3}$

Figure D.1.21: Task 4 Problem Statement

Carrie factored the numerator, then set (x-3)=0. Given the "lim" notation, she said the limit exists at 3 in the form of a vertical asymptote or brick wall. She perceived this problem as being like a rational function, 1/x, and then from here many misconceptions occurred.

C: This is where x=3, my point is at 3 and the vertical asymptote is the brick wall. The limit is more like this, a rational function 1/x. *(Turn 5, 51:30, SDV_0001)*

R: What happens with the function values as x approaches 3?

C: They start decreasing. It's not gonna exist, not that's not right. Coming this way (right), they don't exist. x=3, come in this way (left) my limit is right here but it doesn't necessarily go to infinity. It's gonna start going down it won't go past this point. It comes right from left, then at x=3 it drops down and doesn't get past this point (x=3). I'd say the y values constantly drop. So f(x)=y so function decreases. You can write negative numbers. For this (right side), you say it continuously increases from the right. It stops at 3 and can't go past 3. The x values stop at 3 but y values still increase. Function still increases. *(Turn 5, 52:00, SDV_0001)*

R: So the limit computed tells you that you have a vertical asymptote. Why is it a vertical asymptote and not a hole? How do you know when you get a hole? *(Turn 20, 55:35, SDV_0001)*

C: Well it makes sense but it would be, because when I think of a hole, I think more of this (less than or equal to symbol). *(Turn 21, 55:43, SDV_0001)*

R: What would your problem have to look like to have a hole in a graph? *(Turn 22, 56:00, SDV_0001)*

C: You need a limit saying it can't equal 3, and x cannot equal negative 3. Because I would set this denominator equal to zero and that's going to be when this x equals negative 3. The bottom is zero and therefore it's not a rational number. It's undefined. That's why it tells me right there x can't be negative 3 which is why I have the barrier or constraint at x=3. Limits are about what x equals not what y equals. If was what equals, would draw it this way, a horizontal line at y=3. (*Turn 23, 56:30, SDV_0001)*

The 3 is he constraint. That's the limit. *(Turn 29, 1:01:15, SDV_0001)*

R: When you compute a limit, does it say what y is doing? When you find limits, are you looking for the first coordinate x or the second coordinate y? *(Turn 32, 1:02:13)*

C: Limits are about what x is doing, not y. I don't think of y for limits when I play with limits and stuff. I'll be honest. I don't see it. Unless it was written, find the limit as y approaches 3 or something, then I'd look at the y but I only look at the x for these. *(Turn 33, 1:02:15, SDV_0001)* Instead of x approaches 3, I'd say as y approaches 3. Then I would take into account the y. This is how I would have drawn it. I would have put a horizontal line at y=3. *(Turn 35, 1:03:21, SDV_0001)*

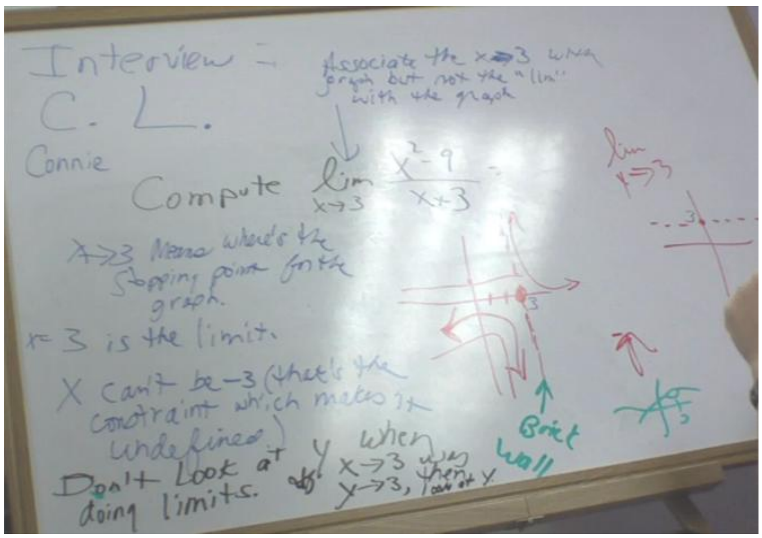


Figure D.1.22: Carrie's work on Task 4.

Carrie thinks that the limit is what appears under the "lim" notation. She perceives a vertical asymptote x=3 as the limit or restraint. She said if below the "lim" notation it had $y \to 3$ then she would have drawn a horizontal line at y=3 as the restraint.

As a follow up task, I asked Carrie to take a look at the problem $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ and she revealed the same consistent misconceptions, except that she drew her graph more in line with the previous graph involving the linear function and ended up with 3 limits, the first one at x=2, x= -2, and the last one at x=4 all of which were drawn as vertical asymptotes. None of her drawings revealed any relationship between limits and second coordinates as function values. She thinks that since 2 in the denominator would make the function undefined, she said that 2 is the limit, the reason being a limit is a constraint. When she sees a limit problem, the question in her mind is if there is a restraint. As a result, she draws vertical asymptotes for what appears beneath the "lim" notation because these are the given restraints. The thinks that since x cannot be equal to 2, then 2 is the limit. Originally she could not decide how to translate the information to the graph so she first drew a hole at (2,-2) and horizontal asymptote at y=-2 but later in the interview, changed her mind and gave different interpretations.

R: So you divided out the common factor (x-2) and you are left with x+2. And so then you said that in the numerator x cannot be 2. How would you graph this?

C: This is my 2, that's my wall. My limit is at x=-2 so draw a horizontal asymptote at x=-2. x cannot equal 2 so right there that's a limit. Since x can't be 2, then the 2 is a limit. So you're asking me what is the limit when x equals 2? I say the limit is -2. *(Turn 43,1:08:32, SDV_0001)*

R: What does minus 2 have to do with any of this? *(Turn 50, 1:09:56, SDV_0001)*

C: The x=2 is gonna be my dotted line thing. This is gonna be my solid line (HA at y=-2) The vertical asymptote at x=2, and this point right here at x=2 or (2,-2) is going to be pen. It's an open circle. I might have it backwards though, so let me think. *(Turn 51, 1:10:45, SDV_0001)*

C: Then at x=-2, there is a VA so I'd say there is no limit. At x = -2 I'd say OK maybe this is what x cannot equal, whether it's s constraint or a limit or something. When I graph this, that's what I think of because that's how I am seeing it, that's what I am left with. So when I see this solid line right here at x=-2, I don't see a constraint because it's going up and down this way vertically. It's not running into the wall x=2 vertically where dotted line is. There is no constraint at x equals = -2 because it keeps going up and down.

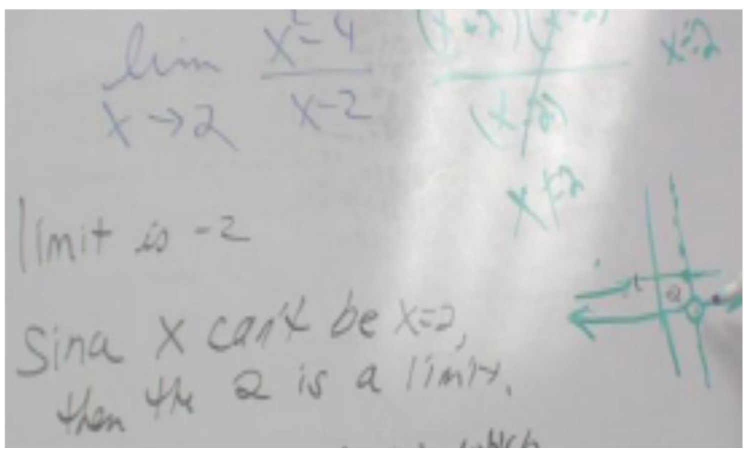


Figure D.1.23. Carrie’s work on Task 4.

Analysis:

Inference: In this initial attempt at the problem, she decided that x= -2 was not a limit but later on, she claims it is. She said it has no restraint because it keeps going up and down at x= -2. Drawing the circle appeared to be a guess to address that the function was not defined at 2. She had difficulty translating the information from the problem onto the graph. She thinks x= -2 has no restraints so it's not a limit. Meanwhile, she considers a vertical asymptote at x=-2 similar to a hole in the graph, as there is no restraint when going up or down the vertical asymptote similar to how when you fall into the hole there is no restraint on how far down you keep going. Carrie does not know under what conditions holes or vertical asymptotes appear in graphs, due to weak algebra proficiency and knowledge of rational functions. Her understanding of limit is restricted to being a restraint, constraint, wall or barrier.

Hypothesis. Students lack algebra proficiency with rational functions and knowledge about domains.

R: When you see "lim" as x approaches 2, does that tell you that the limit is 2?

C: Not always. It's asking me when x is 2, is there a limit or restraint. At x=2 or x approaches 2, is there a limit. The restraint is at 2, so it's a vertical asymptote. *(Turn 57, 1:14:00, SDV_0001)*

R: You computed a limit of 4. What's the relationship of the 4 with the graph?

C: I know the (x-2) will cancel then I plug a 2 in there and say 2+2=4 and plot a 4 somewhere, and heavens know what I do from there. My instinct is to write a 4 where x is. *(Turn 62, 1:16:40, SDV_0001)*

R: Why is there a vertical asymptote at -2 on the graph? *(Turn 65, 1:17:46,SDV_0001)*

C: You have a restraint at -2 which is a limit because if you plug it in the denominator and numerator, you get -4 and the -4's cancel on top and bottom. *(Turn 66, 1:18:19, SDV_0001)*

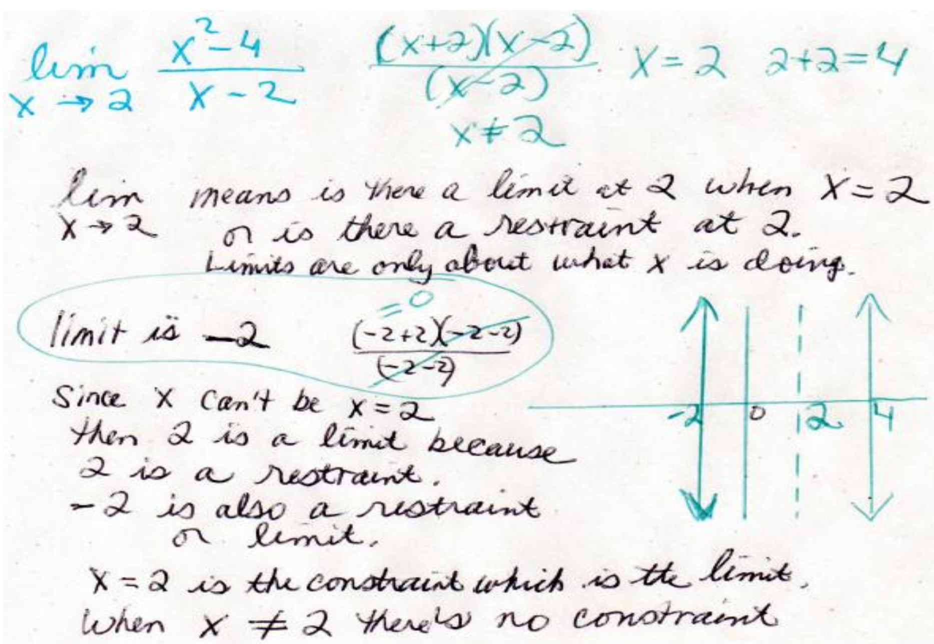


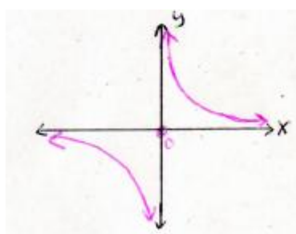
Figure D.1.24: Carrie's work on Task 4.

Analysis:

Inference: Carrie thinks that the first limit x=2 comes with the problem from the notation itself with $\lim_{x \to 2}$. This line is dotted by itself to represent this limit exclusively. Next she drew vertical asymptotes at -2 and at 4, but drew solid lines instead of dotted ones to acknowledge that these are about what is computed. At x=4, there is a solid line for the computed limit, and then a solid vertical line at x=-2 to represent a limit she would get if she plugged 2 into the denominator, making the function itself undefined. She specifically states that when $x \neq 2$ there are no constraints. Her understanding is based on physical characteristics of barriers, constraints, restraints and walls, and refers to these as limits. She draws all of her limits as vertical asymptotes because she considers limits to be only about x. It is interesting how she uses a dotted line for the restraint at x=2 but solid lines for the other ones. This is because she denotes what is beneath the "lim" notation separately. Given the solid lines for x=4, the computed limit, and the solid line at x= -2, the plugged in value, it seems Carrie does not know the difference between input x-values and output y=values.

Hypothesis: Students may not know that limits are numbers that involve the behavior of function values near a point and may confuse x and y values.

Compute the limit. Explain how the function values behave near 0. Does the limit exist as x approaches 0 from the left, from the right, and from both sides together?



$$\lim_{x \to 0} \frac{1}{x}$$

Figure D.1.25: Task 5 Problem Statement

C: When it goes to zero, it can't touch it. I don't know why. It would cross over and it wouldn't be a limit or a function anymore. The graph would start crossing over like a natural log does. It hits that brick wall. I see it as stopping at 0. I don't have anything stopping it. The limit doesn't exist. There wouldn't be a limit. That's because you can't have 1/0 if you plug in. Where the line y=0 there is nothing stopping it so y=0 is a brick wall and there's no limit there. From the left and right, It's not touching. The lines go down and up but never touch zero. *(Turn 12, 3:02, SDV_0044.mp4)*

R: In either direction up or down, does this curve you drew ever reach a number?

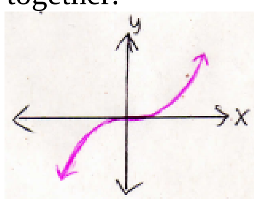
C: Down here it gets smaller, up there they keep on going. It doesn't ever cross over the line x=0 on both sides. As it's coming right, it's like 0.12... This is the y value. Wait, I think it's the x. Then the y gets bigger. *(Turn 18, 3:20 SDV_0044.mp4)*

Analysis:

Inference: Carrie seems to understand that the limit does not exist because the function values increase in the positive direction without bound as x approaches 0 from the right and as x approaches 0 from the left, increase in the negative direction without bound. A zero in the denominator tells her that the function is undefined, so that constitutes her restraint or brick wall. She claims the limit does not exist because nothing hits the brick wall. Carrie referred to y=0 instead of x=0 because she gets confused and mentally switches x with y.

Hypothesis: Students may not know under what conditions limits exist and do not exist, and may confuse x with y on the graph. They may also follow points along the graph instead of look at the function values.

First, can you tell what kind of function this is, then explain how the function values behave near 0. Explain if the limit exists as x approaches 0 from the left, from the right, and from both sides together.



$f(x) = ?$ $\lim_{x \to 0} f(x) = ?$

Figure D.1.26: Task 6 Problem Statement

C: It looks cubed. I just ask as x approaches infinity, is there another limit because infinity is the first limit. You'd have to plug a number into there, any one that's positive, you can plug this into here so it can be any number like 2 cubed, so it would be 8 and then 3 cubed is 9. The limit notation is a tool to figure out a number. *(Turn 20, 6:10, SDV_0008)*

C: I think as x goes to 0, there are 2 limits. I would say -3 and 3 are the restraints, and the region between where the flat line is. *(Turn 20, 6:10, SDV_0008)*

R: How does this function behave as x gets larger? *(Turn 37, 10:05, SDV_0008)*

C: Increasing, going to infinity up and down. *(Turn 42, 11:00, SDV_0008)*

R: What are function values anyway? Can you explain more about them going to infinity? *(Turn 43, 11:08, SDV_0008)*

C: Function values are the whole point. Values are both x and y. So the points on the graph are going to infinity. That's what I mean by function values are going to infinity. *(Turn 44, 11:15,SDV_0008)*

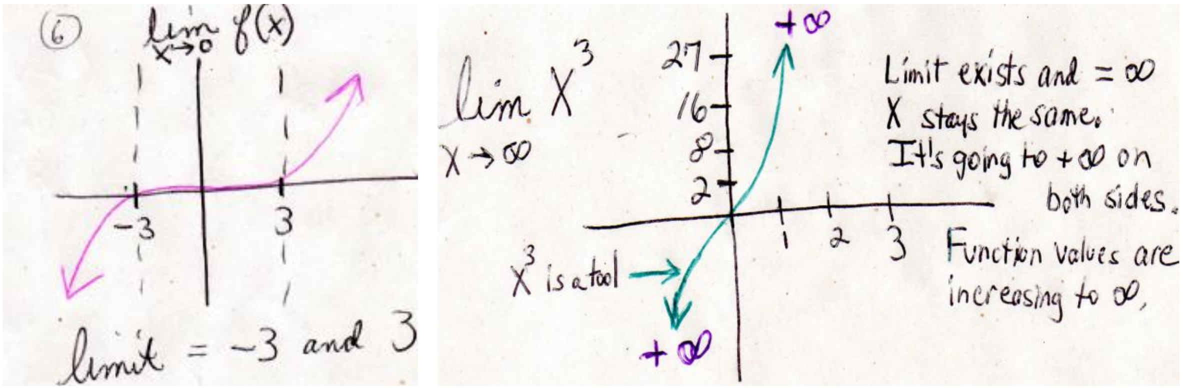


Figure D.1.27. Carrie's work on Task 6 the $\lim_{x\to 0} x^3$ *and* $\lim_{x\to\infty} x^3$

Analysis:

Inference: Carrie continues to compute limits as though they were constraints. She thinks of limits as a tool to get a number (turn 20). She correctly identified this function as cubic, but labeled incorrect function values on the y-axis and said 3-cubed is 9, instead of 27. She decided the limits were -3 and 3 by visual inspection only, yet re-drew her sketch with a curve now looking like stops at x=1 saying x stayed the same. Moreover, she does not know that function values in the 3rd quadrant were getting larger (or decreasing) in the negative direction thereby approaching negative infinity. Instead of minus infinity, she put plus infinity on the graph. She also said the limit exists and equals infinity, when in fact, the limit does not exist because the function values increase without bound. Carrie follows points on the graph going to infinity, instead of function values (Turn 44). Carrie confuses x and y at times and thinks function values are the values of x and y, or points. This is why she watches points on the graph going to infinity.

Hypothesis: Students don't know that if limits equal infinity, then they do not exist because the function values keep increasing without bound and therefore, do not get close to any particular number.

Look at the following graphs and explain if the limits exist as x approaches 3 from the left and the right.

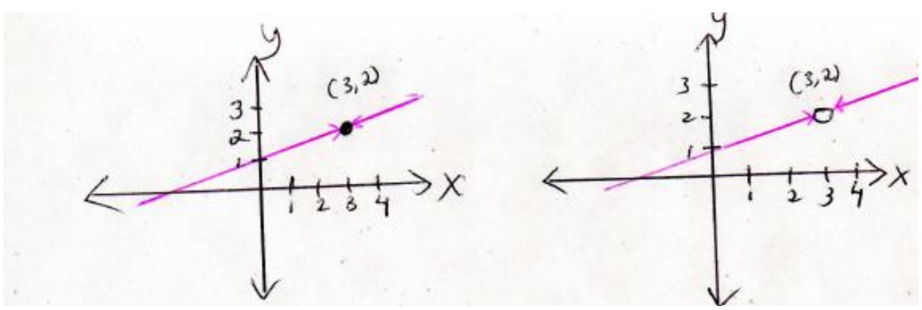


Figure D.1.28: Task 7 Problem Statement

C: For the one on the right with the hole, the limit exists because there is a hole in the graph. The hole is where the limit is but the limit in the hole does not exist because once you fall in you keep on going down. But if it was solid like on the

> left, there wouldn't be a limit. Solid dot means limit does not exist b/c the line goes right through it. This point is a part of the line. Above with the hole, the hole is removed. *(Turn 49, 25:30, SDV_0001.mp4)*

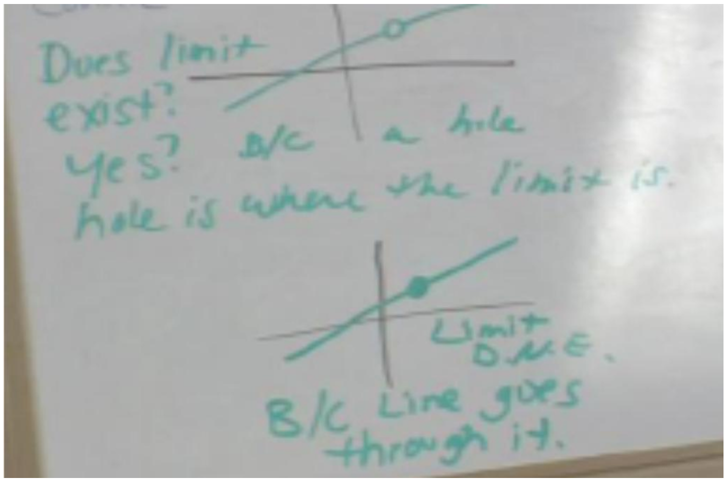


Figure D.1.29: Carrie's work on Task 7.

Analysis:

Inference: When a hole appeared in the straight line, she said the limit exists but for the wrong reason. She did not compare the left hand limit with the right hand limit to explain that the limit exists because both sides are approaching 3. Carrie thinks holes and vertical asymptotes are limits, so if there's a hole, the limit exists. Next, she focuses on the hole itself and claims that the limit does not exit inside the hole because one could fall down and keep going. A solid meant that the limit does not exist because one can walk across it. Since there's no restraint or hole to fall into, there is no limit. Just like with vertical asymptotes, Carrie thinks holes are limits because holes constitute barriers or restraints that prevent getting to the other side.

Carrie considers limits from a visual perspective as barriers only, not a mathematical one, possibly due to insufficient understanding of functions and the behavior of function values near a point. This is also because she thinks limits have nothing to do with functions. Later, she shifts attention to the hole and says that the limit does not exist inside the hole because you can fall down and keep going down without stopping. When she sees the hole, she appears to know the line is discontinuous but thinks this is due to the limit being in the way because a limit is a barrier. When she sees the solid dot, she knows the line is continuous but claims that no barrier or limit exists there. Her primary focus with graphs is finding restraints, and holes are the restraints that she calls limits.

Hypothesis: Students might confuse limits with continuity and don't think a limit can exist when there is a hole within a straight line.

What kind of function is this? Does the limit exists as x approaches 2 from the left and from the right. Why or why not?

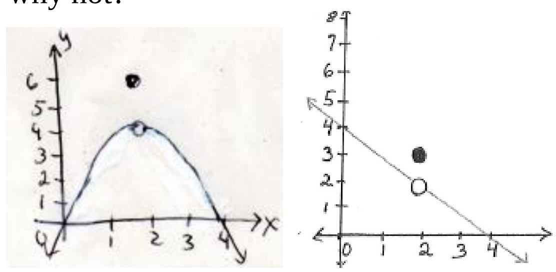


Figure D.1.30: Task 8 Problem Statement

C: It's x-squared. The point has nothing to do with the function. The point is just in the wrong place. No. The dot is above the x-squared function but it's not part of the function. The only function on this graph is x-squared. The dot is just some random thing put there to be confusing like with limit graphs but it doesn't affect the x-squared function in any way. The dot is by itself up there doing nothing. *(Turn 98, 50:55, SDV_0001.mp4)*

R: Is there any way this graph could be a piecewise function? Explain why or why not. Does the limit exists for this function? *(Turn 99, 51:15, SDV_0001.mp4)*

C: No because piecewise functions have straight lines going straight across to a solid dot or to a hole. There are 2 separate pieces to the graph. And this graph down here only has one piece, not two pieces. It only has a x-squared function with a dot in the wrong place. It looks like it just got put in the wrong place and doesn't go with the graph, so the only function that's on there is x-squared. If you had a straight line going across to the left of the dot and a straight line going to the right of the hole that's the only time it can be piecewise. The limit exists because there is a hole, but it really doesn't exist at 2 because the dot could fall into the hole and keep going down. *(Turn 100, 51:40)*

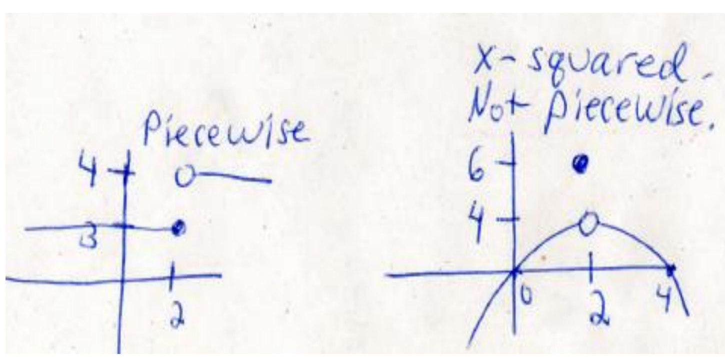


Figure D.1.31: Carrie's drawing of Task 8

Analysis:

Inference: Carrie did not know this was a piecewise function. She said the limit exists at the hole because one can fall into it but then once inside the hole, the limit doesn't exist because one keeps falling without stopping. For the piecewise component, she does not know that the limit exists but is not equal to the value of the function.

Hypothesis: Students confuse limits with continuity as well as lack understanding about piecewise functions and their domains.

Explain what type of function this is and then explain if the limit exists as x approaches 2.

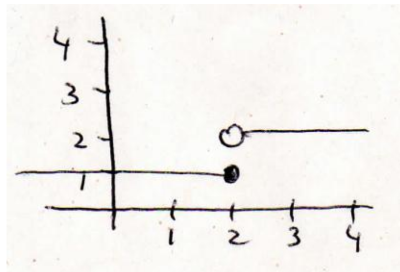


Figure D.1.32: Task 9 Problem Statement

C: It's piecewise .A limit exists from 1 to 2 and limit would exist from 2 to infinity because the line is going to the right. Then (2,1) is included in the limit to infinity. It's saying from this point 2 to negative infinity left, that point is included with the limit. From the point 2 to infinity, it excludes the point so it starts right there at 2 so your limit starts right there. So there are 2 limits or constraints. Limit doesn't exist for whole graph Area in red at 2 between y=3 and y=1, limit doesn't exist but there is a line there. There is a limit for the bottom one with the hole. I would say no, the limit doesn't exist, simply because if you're graphing a point that's your limit (VA between hole and dot). The limit for the whole thing doesn't exist because if you are graphing a point, that's your limit. This is included (dot and line going left), that's included (circle and line going right), but the area in between the dot and open hole is not. So your limit would be between (2,1) and (2,2). *(Turn 51, 28:00, SDV_0001)*

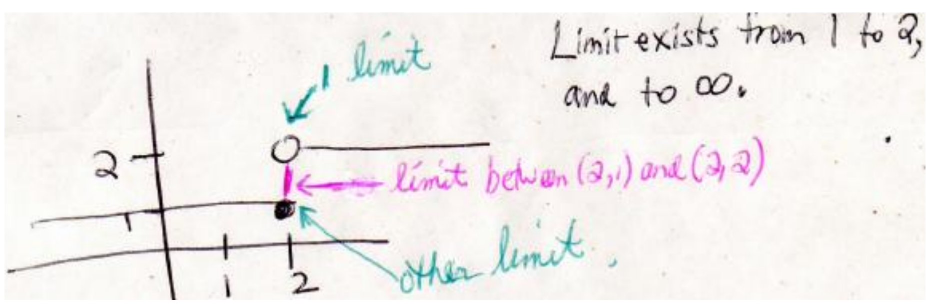


Figure D.1.33. Carrie's work on Task 9

Analysis:

Inference: Carrie recognizes this as piecewise, but did not articulate that the limit does not exist due to jump discontinuity, in which the left hand limit does not equal the right hand limit. Holes and solid dots are limits, according to Carrie, and the there is a vertical line connecting them, constituting a third limit. Carrie does not use the simple tool for limits at a point which involves comparing the right hand limit to the left hand limit.

Hypothesis: This function is commonly recognized by students as piecewise but they may lack an understanding about the domain of this function.

Given a restricted domain, explain if the one-sided limit exists as x approaches 2 from the left. $D:\{x \mid 0 \le x < 2\}$ $\lim_{x \to 2^-} f(x)$

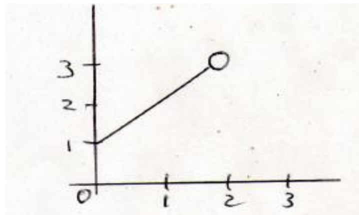


Figure D.1.34: Task 10 Problem Statement

C: This one looks easy but there must be a catch to it, maybe in the hole. So I'd say the limit exists between 1 and 3 and then another limit exists where the hole is because you can fall into it. *(Turn 241, 35:14, SDV_0001.mp4)*

R: What if it had a solid dot where the hole is. *(Turn 242, 35:35, SDV_0001.mp4)*

C: Then the limit would not exist because there is nothing to fall into and the line could keep shooting up if it wanted. *(Turn 243, 35:40, SDV_0001.mp4)*

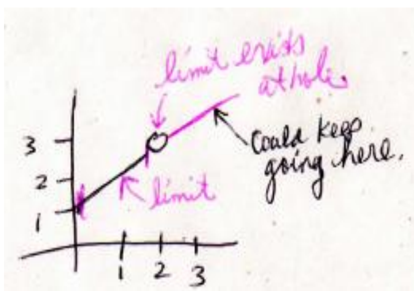


Figure D.1.35: Carrie's work on Task 10.

Analysis:

Inference: Carrie looks at the y-axis and sees a limit between 1 and 3. She thinks the limit exists up until the hole, then the limit does not exist because one can fall into the hole and keep going down. Meanwhile she perceives an imaginary line on the right side of the hole. If a solid dot replaced the hole, the limit does not exist because there is no restraint or nothing to fall into.

Hypothesis: Students may not know about the domain of this one-sided limit.

What is the limit as x approaches 3? Study the following answer choices and explain your reasoning.

$$\lim_{x \to 3}(5) = ? \quad \text{a. } 3 \quad \text{b. } 5 \quad \text{c. } 0 \quad \text{d. } \infty$$

Figure D.1.36: Task 11 Problem Statement

C: I'd say there is no limit. There would be nothing because there is nothing to do. There's no x, just a number from what I can tell. This problem looks weird like it's some trick question because there is nothing to compute so there is no way to find a limit. (Turn 145, 7:50, SDV_0010.mp4)

C: This is not a normal limit problem because there's no letters. Well on the graph, there is a vertical asymptote at x=3 with a dotted line but then there is another vertical asymptote at 5 that is a restraint it can't go past. Then here before 3 on the left, that is what x is approaching and can't go past, so that's why 3 is a limit or a restraint. Then the area between 3 and 5 is a limit because it's constrained between those 2 numbers, so there the limit is 0 because it's on the x-axis where y is 0. *(Turn 244, 36:10, SDV_0001.mp4)*

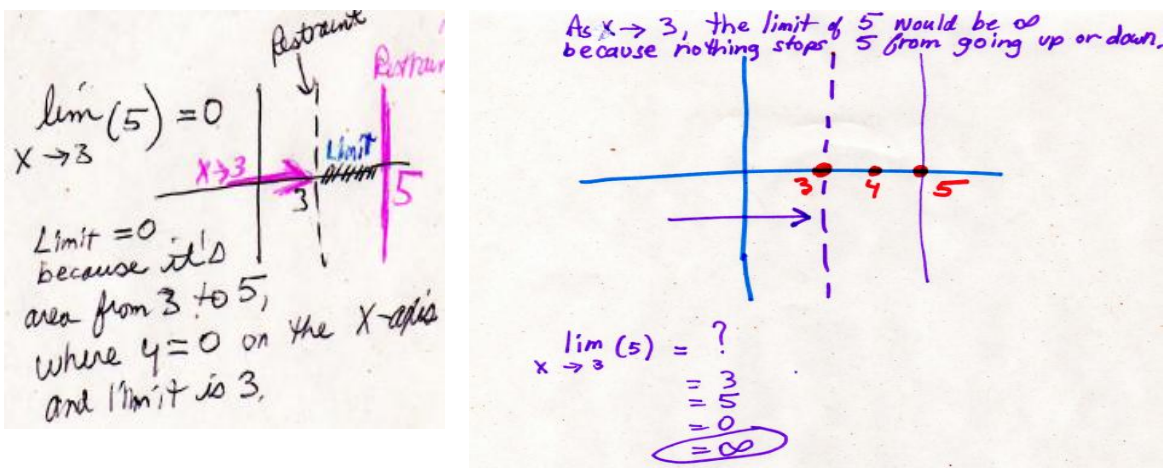


Figure D.1.37: Carrie's work on Task 11

C: So x approaches 3, what's the limit of 5? Wait, there is no limit, it could be infinity b/c 5 could go on forever. As x approaches 3, the limit of 5 would be infinity because there is nothing stopping it from going up or down. *(Turn149, 12:44, SDV_0010.mp4)*

Analysis:

Inference: Carrie does not know this is a constant function. At first, she incorrectly drew the graph with limits in the form of vertical asymptotes at x=3 and x=5. She claimed the area between 3 and 5 was also a limit. Also, the $x \to 3$ notation is perceived as directional, approaching 3 from the left. Later she changes her mind to say the limit equals infinity. She consistently drew the dotted line to represent what she thinks is a vertical asymptote for what appears beneath the "lim" notation and then drew the 5 on the x-axis because she thinks limits are all about x-values. The reason she perceives a restraint at x=3 is because she perceives the $x \to 3$ as directional. Therefore, if the notation were written $3 \leftarrow x$ under the "lim" notation, she would probably say this means x approaching 3 from the right. She later changed her mind to say the limit of 5 was equal to infinity because she looks at the vertical line at x=5 and thinks nothing can stop the line from going up and down.

Hypothesis: Constant functions are unfamiliar to students and so they confuse x with y when trying to find the limit. Students may perceive the arrow to be directional, as x is approaching 3 from the left only.

What type of function is this? Construct its graph. Explain if the limit exists as x $\to 3$.

$$f(x) = \begin{cases} 5x+2 & \text{if x} \neq 3 \\ 20 & \text{if x=3} \end{cases} \qquad \lim_{x\to 3} f(x) = ?$$

Figure D.1.38: Task 12 Problem Statement

C: I see 2 different graphs. There is a vertical asymptote at 3 going through a hole because it can't be 3 there. The dot at (3,20) doesn't do anything. I'd say, to be honest, the limit exists at 3 because a hole is a limit you can fall into. The limit does not exist at 20 because there is no line taking you there and even if it did, the solid dot's not a barrier or restraint. *(Turn 245, 36:55, SDV_0001.mp4)*

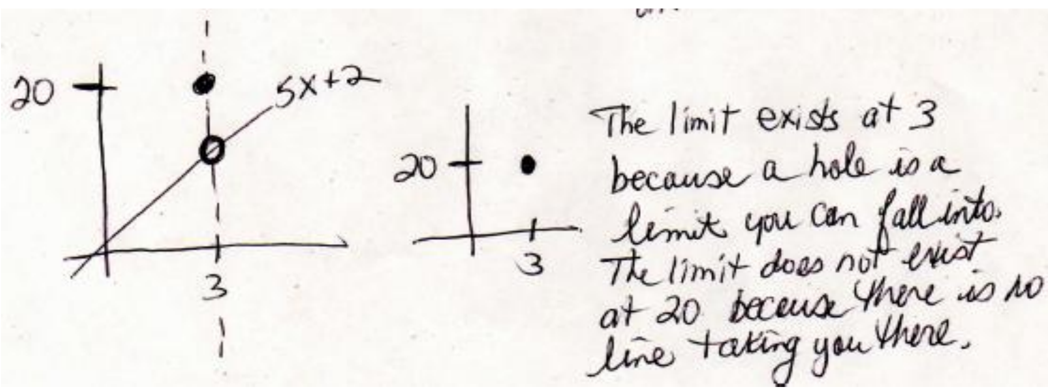


Figure D.1.39: Carrie's work on Task 12

Analysis:

Inference: Carrie did not recognize that this function is piecewise, though she saw 2 parts. She drew a vertical asymptote at x=3 because that came with the problem, under the "lim" notation. She drew a separate graph to plot (3,20) before putting them on the same graph. Carrie insists that if there is a hole, then the limit exists because it's something to fall into. She does not think solid dots can be a limit because you can walk right over them and not fall down. Because there is no line through the point (3,20), she claims the limit doesn't exist.

Hypothesis: Students lack familiarity with piecewise functions and their domains and may also not be familiar with discrete functions—isolated points on graphs with one value of y for each x-value.

Summary

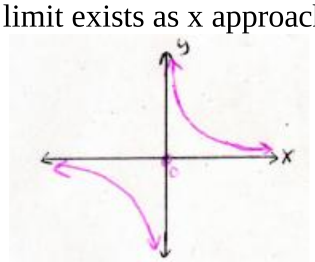
Carrie possesses deficient algebra skills and demonstrates major misunderstanding about limits at a point, mostly because she thinks limits are about the x's, instead of y's. Therefore she is not able to describe the behavior of function values near a point with any of the linear, piecewise or rational functions given. She uses ever day language to describe limits, such as brick walls, holes, barriers, restraints and constraints. Holes are considered limits or restraints because they're something you can fall into. She thinks the "lim" notation explicitly dictates what her first limit will be and denotes that with a dotted vertical line. Anything she computes after that ends up on the x-axis and forms a vertical asymptote with a solid vertical line, and is considered to be a restraint or constraint. The x→a appears directional, as x approaches a from the left.

Carrie has two different opinions about the limits at times. In one instance she claims a vertical asymptote is a restraint, but also says later is not a restraint because you can go up and down the line in a vertical position. She does this with holes as well. When she sees a hole on a line, she will say the limit exists because you can fall into the hole. However, she perceives a second step regarding what happens at the hole. She says the limit does not exist inside the hole because once you fall down you keep going. In contrast, solid dots on a line mean that the limit does not exist because there is nothing to fall into, so one can walk right across it. Most of the time there are consistent responses but no logic or reason behind them.

Limits at Infinity: Carrie

Eleven tasks involving limits at infinity were investigated. Some tasks involved displaying graphs and asking about the limit as x approached infinity; others started with computations followed by constructing the graphs. A cubic and quadratic function were each presented, but the majority of tasks were rational functions involving different types of asymptotic behavior which required knowing rules associated with finding horizontal, oblique, parabolic and vertical asymptotes. Ways in which students think about the behavior of function values as x increased in the positive and negative directions was explored, as well as reasons why students think a limit exists if it approaches infinity.

Compute the limit as x gets larger in the positive and negative directions. Explain whether or not the limit exists as x approaches positive and negative infinity.



$$\lim_{x\to\infty}\frac{1}{x} \qquad \lim_{x\to -\infty}\frac{1}{x}$$

Figure D.1.40: Task 1 Problem Statement.

C: The limit exists because it would continue on all the way through infinity. If you plug any number into here (x), it would be all right. You could do 2, 3, 4, 5. *(Turn 4, 1:25, SDV_0008.mp4)*

R: What if you look at the opposite side, the limit as x approaches negative infinity of 1/x. *(Turn 8, 2:16, SDV_0008.mp4)*

C: It goes to minus infinity. (*Turn 9, 2:20, SDV.0008.mp4)*

R: How do you decide this? Do you just look at the graph, do any math?

C: No, you don't need any math for these. All I do is see what it says under "lim"and that says the limit. It's where the line is going, so the limit is infinity in the first one, and minus infinity for the second one and the limits exist because they go to infinity. *(Turn 11, 3:00 SDV.0008.mp4)*

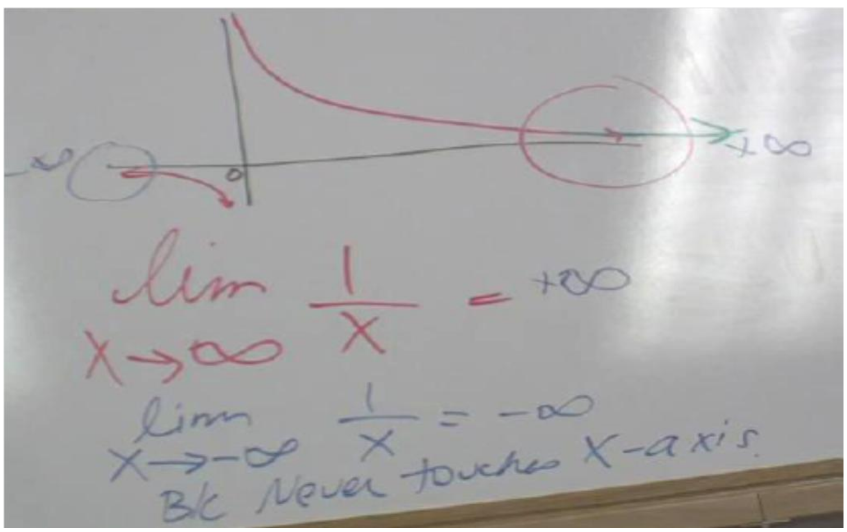


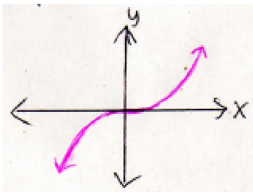
Figure D.1.41: Carrie's work on Task 1.

Analysis:

Inference: Carrie considers values for x, such as 2, 3, 4, 5, etc. but does not plug in those values of x and study the second coordinate to see that the function values get smaller and smaller, tending toward zero as x goes to infinity. Instead, she looks at points on the graph itself that appear to be going out toward infinity. Meanwhile, she looks at $\lim_{x\to\infty}$ notation and decides the limit is whatever is written under the "lim" notation.

Hypothesis: Students may think function values are actual points (x,y) and follow the points on the graph instead of studying the behavior of the function values.

Recall this type of function. Describe the limiting behavior as x gets larger in the positive and negative directions. Explain whether or not the limit exists as x approaches positive and negative infinity.



$\lim_{x\to\infty} f(x)$ $\lim_{x\to -\infty} f(x)$

Figure D.1.42: Task 2 Problem Statement.

C: It looks like x-cubed and you could set it equal to zero but that's not gonna work. *(Turn 16, 3:30, SDV_0008.mp4)*

C: I just ask as x approaches infinity, that's the first limit because it's given. It tells you under "lim." is there another limit because infinity is the first limit. It's given. *(Turn 20, 6:10, SDV_0008.mp4)*

R: Can you put the function in the graphing calculator and compute function values to help explain the behavior of the function values as x goes to positive and negative infinity? *(Turn 25, 7:20, SDV_0008.mp4)*

C: I can see where it is as I am approaching x, this is going to infinity right here. As I come right towards x (3rd quadrant), this part right here (3rd quad) goes down to infinity. So the limit would probably be 3. Not 3 here but going to infinity. *(Turn 28, 8:06, SDV_0008.mp4)* So the points on the graph are going to infinity. *(Turn 44, 11:15)*

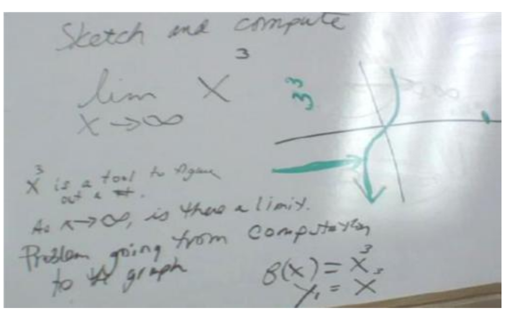


Figure D.1.43: Carrie's work on Task 2

Analysis:

Inference: Carrie identifies the function as x-cubed and entered it into the calculator without difficulty but she admitted having a hard time transitioning from the function to the graph seen on the calculator. She claimed the first limit was infinity because it came with the problem, but she sees this as directional, which is why she drew the green arrow to show that x is going to the right. However, there was a restraint that occurred at x=3 which she identified as being another limit. She claimed the limit was at 3 but drew the green arrow in the area that looks like x= -3. Carrie has several misconceptions with the limit aspect but also with graphing and the x-y axis. She says the limit is x=3 but is draws a green arrow to x=-3 because she sees numbers as symmetric about the y-axis. The statement that a limit exists at 3 is incorrect, but this is because she sees the cubic function as a barrier. No matter what she calls the limit, x=3 or x=-3, the fact is that she perceives this x-value as a barrier which is why it's a limit. Carrie thinks the "lim" notation reveals the limit for the entire problem which is infinity, and this is because x $\rightarrow\infty$ appears under the "lim" notation and dictates the direction that x is going, denoted by her green arrow. She seems to think that in figure 2a the graph is on the x-axis from x= -3 to x=3. So these points become her barriers.

Hypothesis: Students follow points on the graph instead of look at function values and might think the x→a under the “lim” notation is directional from the left only.

Describe the limiting behavior as x approaches plus or minus infinity and explain whether or not the limit exists.

$$\lim_{x\to\infty}\frac{3x}{2x^2+1} \qquad \lim_{x\to-\infty}\frac{3x}{2x^2+1}$$

Figure D.1.44: Task 3 Problem Statement

C: The first thing I'd want to do is take the bottom part and set it equal to zero and solve it. It's like some funky thing with an "i" so you can't graph it. It doesn't work. I get frustrated with that. *(Turn 54, 14:04, SDV_0008.mp4)*

C: Here using the calculator, Here using the calculator, the limit crosses twice so that makes it not even a function. It's not the horizontal line test, it's the vertical. I'm crossing over, there's a limit going to infinity here, it's not gonna cross the x- axis. It gets smaller and smaller but never touches or crosses the x-axis. *(Turn 62, 17:23, SDV_0008.mp4)* The limit exists I would say, and it would be infinity because it keeps going. *(Turn 70, 18:15, SDV_0008.mp4)*

R: Is the limit you found from the graph or from your computation? *(Turn 71, 18:40, SDV_0008.mp4)*

C: The graph. I took this, limit as x goes to infinity (not the function part circled) and said as it's going this way (right), where is it gonna go? The (3x/2x squared +1) did not even matter. *(Turn 72, 19:00, SDV_0008.mp4)*

R: So the function itself didn't really matter? (Turn 73, 19:07, SDV_0008.mp4)

C: You don't have to do any math. Just look at the 'lim' and it tells you what the limit is. The answer is already right there, "limit as x goes to infinity." So going the other way, the limit is minus infinity because the 'lim" tells you that. *(Turn 74, 19:20, SDV_0008.mp4)*

R: Are there any horizontal asymptotes on this graph*? (Turn 75, 19:30)*

C: There's one between -.5 and -1. Because you would have it here, I'm so lost. So there's the restraint. The limit is between -.5 and -.1, the area in between them 2 numbers. *(Turn 78, 20:30, SDV_0008.mp4)*

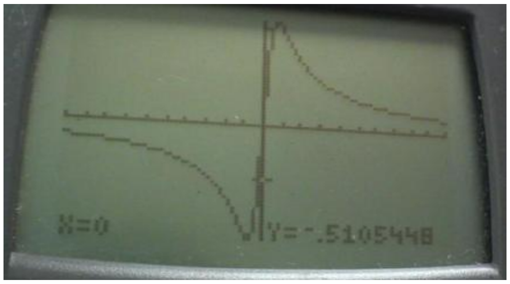


Figure D.1.45: Carrie's misconceived horizontal asymptote between -0.51 and -0.1

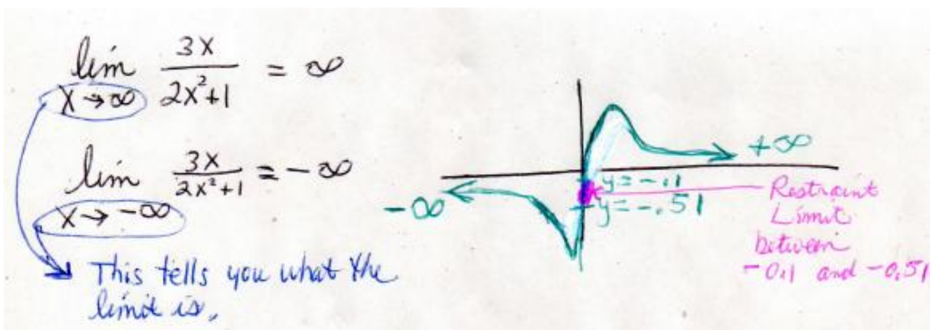


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Figure D.1.46: Carrie's work with Task 3.

Analysis:

Inference: Carrie does not have skills working with rational functions, as she did not mention any rules for asymptotes or use any mathematics to compute function values. Had she known that the degree of the numerator for the leading term in the numerator was less than the degree of the denominator's leading term, she would have determined that that the limit is 0. She said the limit crosses twice so it wasn't a function. Also, she did not see how this problem resembles 1/x with the denominator blowing up faster than the numerator. If she had seen this, she could have confirmed the limit was 0. Instead of doing mathematics, she used the graphing calculator. From the graph, she decided that the limit was equal to infinity and justified this by explaining the "lim" notation tells what the limit will be. She said no math is required to do this problem because the answer is under the "lim" notation. Later, I directly asked her about if there was a horizontal asymptote, hoping she would say y=0, but instead, she looked at the calculator which said y=-.51 and said there is a horizontal asymptote between y= -.51 and y=-.1, the latter of which appeared to be a random estimate.

Hypothesis: Students need algebra proficiency and know rules of rational functions to be successful with this problem. Some students might confuse x and y and follow points on the graph rather than study the behavior of the function values.

Describe the limiting behavior as x approaches plus or minus infinity and explain whether or not the limit exists. $\lim_{x\to\infty}(1+\frac{1}{x})^x \qquad \lim_{x\to -\infty}(1+\frac{1}{x})^x$

Figure D.1.47: Task 4 Problem Statement

C: Well, if you put 0 in the denominator then there's no limit, then the 1/x would cancel out and you have 1 raised to the 0 power which is 1**.** So you would have 3 limits: plus infinity, minus infinity and 1. If you put in other numbers like x=5, then you get 2.448. If I put in -5, I get .32768. *(Turn 178, 25:50)*

R: How do the function values behave for large x, either positive or negative? And, can you describe the limiting behavior for large x? *(Turn 179, 26:45)*

C: They keep getting bigger so I would say the limit exists because there are infinitely many limits, or infinitely many vertical lines up and down. You get infinitely many when x is not equal to 0. When x equals 0, though, you get 3 limits. *(Turn 180, 27:00)*

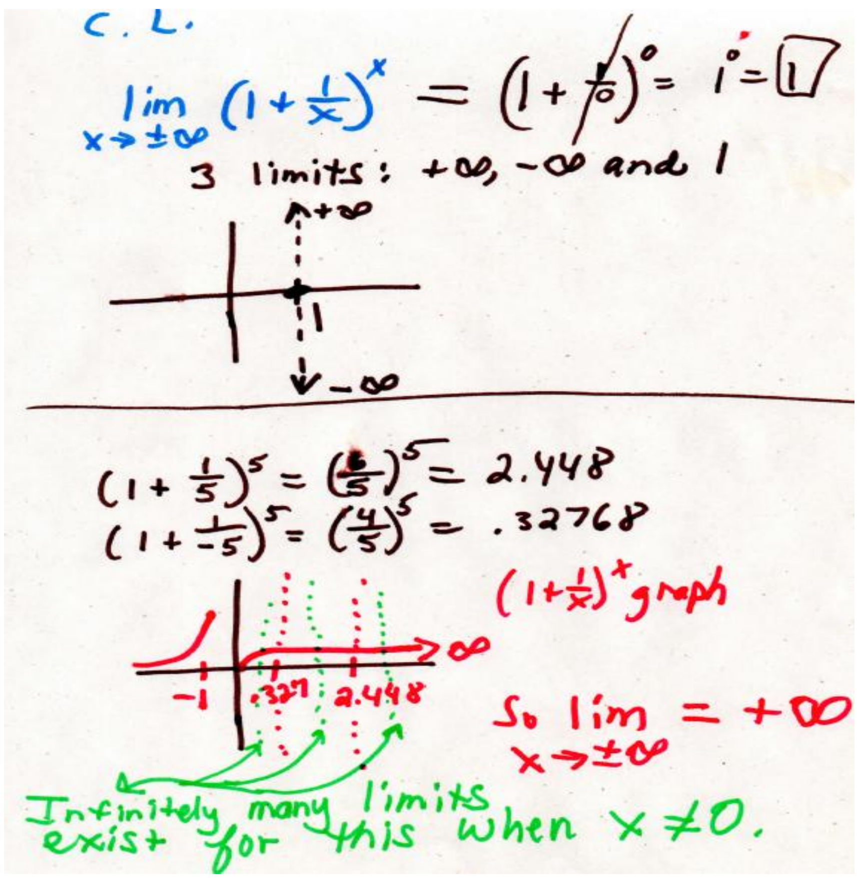


Figure D.1.48: Carrie's work with Task 4

Analysis:

Inference: Carrie thinks that if x =0, the two main limits are plus infinity and minus infinity, as they come from the "lim" notation, but also computed an incorrect limit equal to 1 and plotted this result as a vertical line on the x-axis. When x $\neq 0$, there were infinitely many limits. Later computations and the graph seemed to be correct when she used the calculator. However, she drew dotted vertical lines on the x-axis to represent infinitely many limits. She did not study the behavior of the function values as x got larger in the positive and negative directions, so did not discover the limit was equal to approximately 2.718 (actually e). She thinks limits have nothing to do with function values or the y-axis.

Hypothesis: Students may lack technology proficiency with using parentheses and tables in the graphing calculator and may confuse x with y throughout the problem.

Particle Problem: Given s(t) represents the position of a particle at time "t", sketch a graph and describe the limiting behavior as t approaches positive infinity for s(t)=$1-e^{-t}$.

$$\lim_{x \to \infty}(1-e^{-t})$$

Figure D.1.49: Task 5 Problem Statement

C: I suck at word problems. I hate things with negatives and special letters and totally hate word problems. But I'll put it in the calculator with an "x" instead of "t". (pauses). It goes up to the right then has a horizontal asymptote there so the limit exists and is infinity because it keeps going out to the right. So it never reaches any particular altitude because it keeps going. This is right, I'm sure,because it also says what the limit is under the "lim" sign so you don't need any math. *(Turn 151, 13:30)*

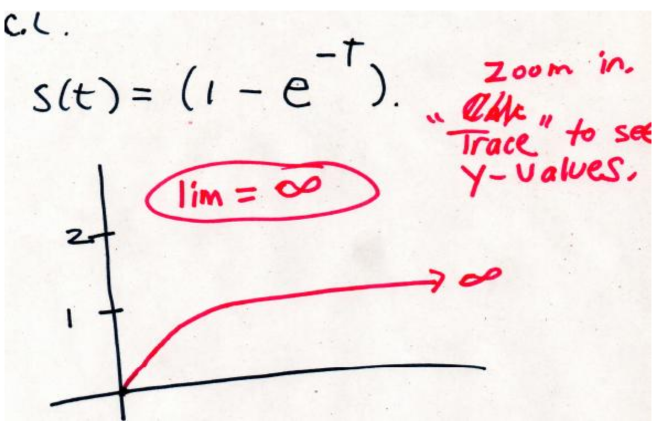


Figure D.1.50: Carrie’s work on Task 5

Analysis:

Inference: Carrie used the calculator to graph the function then decided the limit was equal to infinity instead of 1. She did not consider making the negative exponent a positive one by rearranging that term into $\frac{1}{e^t}$, at which time she could have determined that as t got large, that term would go to zero so the limit would be 1. She followed the points along the x-axis instead of looking at the behavior of the function values on the y-axis. She thinks limits are about x, so she does look at the behavior of the function values, which would tend toward 1 in this problem for large values of t.

Hypothesis: Deficits with basic math and algebra cause students to use the calculator. Carrie relies heavily on the calculator due to deficits with algebra and basic math. Students might follow points along the line on the graph and think the line is going to infinity.

Describe the limiting behavior of the function below as x gets larger in the positive and negative directions. Sketch a graph of the result.

$$\lim_{x\to\infty}(5x^2+2) \qquad \lim_{x\to -\infty}(5x^2+2)$$

Figure D.1.51: Task 6 Problem Statement

C: Limit is 2 because you start at 2 so you don't even have a function anywhere below 2. Everything is up here. *(Turn 28, 00:45, SDV_0045.mp4)*

C: As x is going to infinity, x is going this way (right), as x is going from the positive side, y is increasing and so is x. Then as x is going to infinity negative, y is increasing as the x values are decreasing. *(Turn 30, 2:09, SDV_0045.mp4)*

C: Yes. It's at 2. (LDE means limit does exist) at x=2 because that's where everything stops at. If you set it equal do you get an answer, no. You get no real solution. *(Turn 32, 4:00, SDV_0045.mp4)*

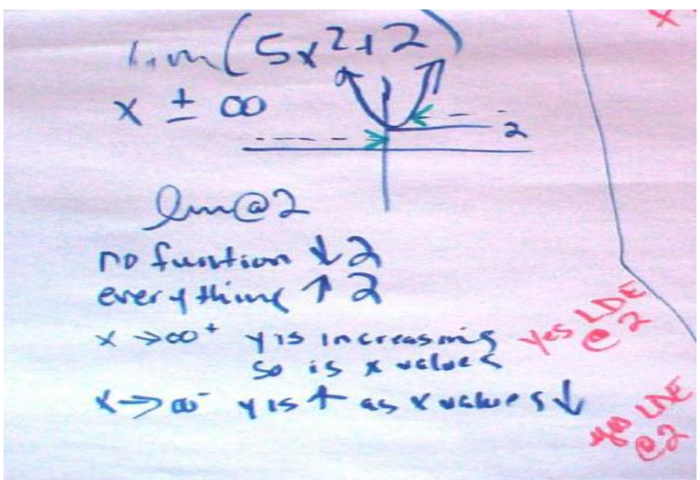


Figure D.1.52: Carrie’s work on Task 6.

Analysis:

Inference: Her typical answer changed in this problem, such that now the limit was equal to 2, on the y-axis instead of on the x-axis. She claims that as x approaches positive infinity, both x and y are increasing, and as x approaches negative infinity, she says that y is increasing as the x-values are decreasing. This is true, except the limit is not equal to 2. Her barrier might equal 2, but the limit does not exist. She had nowhere to draw 2 on the x-axis. This one appeared to be more of a horizontal restraint because nothing could go below y=2, and this is why she said the limit existed and was equal to 2 instead of plus infinity. Although she correctly talks about the function values increasing, she did not elaborate more and translate that into meaning that the function values approach infinity and therefore the limit does not exist.

Hypothesis: The y-intercept may be considered to be a limit from a visual perspective.

Describe the limiting behavior of the functions. Find any horizontal asymptotes, vertical asymptotes, holes and limits. Graph the function and explain the limiting behavior as x approaches infinity and as x approaches 1. $\lim_{x\to\infty}\frac{2x+1}{x-1}$ $\lim_{x\to 1}\frac{2x+1}{x-1}$

Figure D.1.53: Task 7 Problem Statement

C: You can't factor, I don't think. But there is a vertical asymptote at x=1 because that gives you a zero in the denominator. I think the first one the limit equals infinity because the "lim" says so and the other one as x approaches 1 has a restraint at 1, so I know there the limit will also be 1 because under the "lim" it says the limit is 1. So in the calculator it looks like the 1/x thing. *(Turn 247, 37:55)*

R: What happens as x is approaching 1. Does the limit exist? *(Turn 248, 38:25)*

C: At 1, the limit exists because it equals infinity. You can slide up and down the vertical line at x=1 because it's a brick wall. The limit there is also 1 because it's a brick wall that's a restraint you can't go past. The wall is the limit. *(Turn 249, 38:48)*

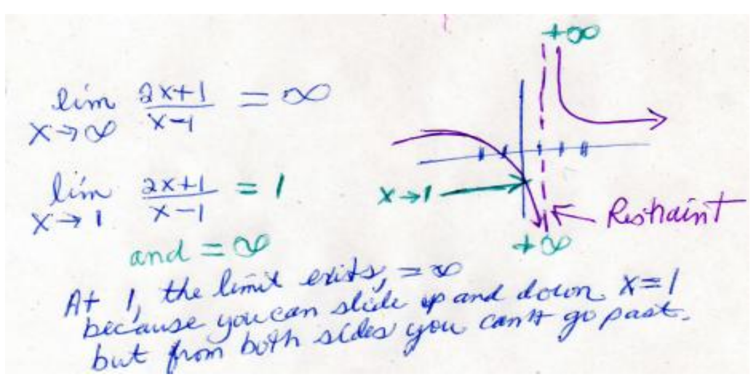


Figure D.1.54: Carrie's work on Task 7.

Describe the limiting behaviors. Sketch the graph and explain how the function behaves for large x in the positive and negative directions. Do the limits exist?

$$\lim_{x\to\infty}\frac{3x^3-x^2-3}{2x+3} \qquad \lim_{x\to-\infty}\frac{3x^3-x^2-3}{2x+3}$$

Figure D.1.55: Task 8 Problem Statement

C: Looking at this (dotted line) between the 2, there is a limit between the two curves, a limit at y=-1. Looking at the graph, that is what I would say. But your limit goes straight across not up and down, so it throws me off. Nope, limit can't go straight across, it has to be y so it can't be straight across. So coming from infinity, the limit is still gonna be -1 roughly. So the limit is going down on the left, and going up on the right, with a straight line between the 2 graphs. *(Turn 139, 5:00, SDV_0010.mp4)*

R: So the limit is equal to minus 1? *(Turn 140, 6:00, SDV_0010.mp4)*

C: Right. Based on graphing, not on math. *(Turn 141, 6:15, SDV_0010.mp4)*

R: What's the limit as x approaches minus infinity. *(Turn 142, 6:30))*

C: That's when I would draw arrow pointing left, so it would still be -1 based on the graph because that's where the curve crosses the y-axis. So -1 is the restraint the limit can't go past. *(Turn 143, 6:50, SDV_0010.mp4)*

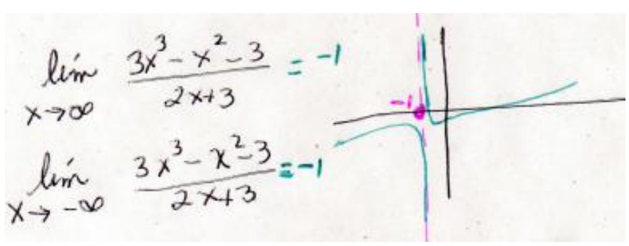


Figure D.1.56: Carrie's work on Task 8

Analysis:

Inference: This task was particularly difficult for Carrie and so she was not able to describe the parabolic asymptotic behavior of this function. She did not enter any large positive or large negative x-values to compute function values, and she was not able to rely on the algebraic rules for finding asymptotes of rational functions. She entered the wrong function into the calculator because her graph was not accurate and she did not check what she entered. She found a limit by looking at the graph. Despite this limit being incorrect, she was consistent with plotting it on the x-axis and drew the vertical line. Then instead of looking at the graph to see how the function values behave as x gets larger in the positive and negative directions, she focused on what was happening around x=-1 only.

Hypothesis: Algebra proficiency with rational functions and knowing rules for finding asymptotes facilitates explaining the asymptotic behavior.

Describe the limiting behaviors. Sketch the graph and explain how the function behaves for large x in the positive and negative directions. Do the limits exist?

$$\lim_{x\to\pm\infty} \frac{3x^2+2}{9x^2-2x+5} \quad \text{or} \quad \lim_{x\to\pm\infty} \frac{9x^2+2}{3x^2-2x+5}$$

Figure D.1.57: Task 9 Problem Statement

C: Now I put 3 into the limit where the x's are. So denominator 3 times 3 squared is 27, numerator 9*9=81+2, that gives 83/26. If you divide this you get 2.30. Yeah. I'd say, can I take the 3, put it into here (the function in the denominator) and not come up with zero. So since we got 83/26, it's telling me that the limit does exist, and this is what it is 83/26 = 2.30 and that's the number.If you get a zero in the denominator then the limit doesn't exist. *(Turn 113, 36:15)*

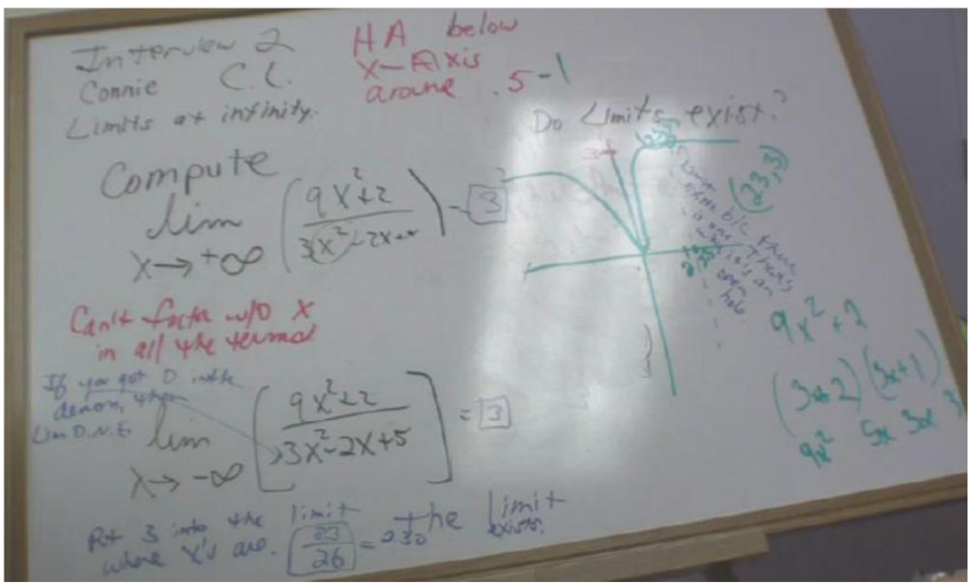


Figure D.10.58: Carrie’s work on Task 9.

Analysis:

Inference: Carrie selected the second problem given, and immediately used the calculator to sketch the graph. She thought there was a way to factor but had no success. So she ended up dividing 9/3 to get 3 since the x-squares divided out. Even though she had found the limit, she didn't realize it and so what she continues to do with the 3 is remarkable. She plugged 3 back into the original function separately in the denominator and then the numerator, and got an improper fraction 83/26 as the limit. There was evidence of knowing how to do some basic math including order of operations, but she incorrectly divided and got 2.3 instead of 3.19. She did not check her result, either. Meanwhile, she drew a vertical line at x=2.3 to represent this limit or restraint. Then she explained that if she had gotten a 0 in the denominator, then the limit would not exist which contracts previous statements where she says the limit would exist because the 0 would constitute a restraint or barrier.

Hypothesis: Students might not know there is a relationship between the limit and the horizontal asymptote.

Study the graph below and describe the behavior of the function values for large x in the positive and negative directions then explain how the function values behave near x=2.

$$f(x) = ? \quad \lim_{x \to \infty} f(x) = ? \quad \lim_{x \to -\infty} f(x) = ? \quad \lim_{x \to 2} f(x) = ?$$

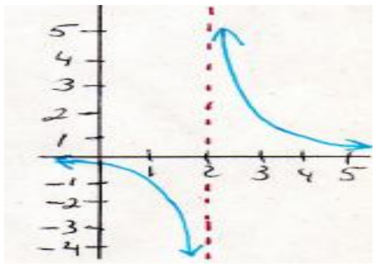


Figure D.1.59: Task 10 Limit at infinity versus at a point for f(x)= $\frac{1}{(x-2)}$

C: The function is that 1/x thing from the first interview. I think it's 1/(x+2). I can put it in the calculator to be sure. No wait, it's 1/(x-2) I always get that confused. For these, what is under the "lim" thing tells you what the limit is. As x goes to infinity, the limit is infinity. As x goes to minus infinity, the limit is minus infinity. As x goes to 2, there is a restraint at 2, or a brick wall it can't go past so the limit as x goes to 2 is just 2. See you are coming from the left going right towards 2, that's what the arrow says to do under the 'lim' notation. There is that brick wall at 2, so that's the limit. But on the brick wall at 2 itself the limit doesn't exist because you can go up and down the brick wall without topping. The limit exists and is infinity at x=2. That's how I would explain it. *(Turn 200, 38:50)*

Analysis:

Inference: Carrie needed the calculator to come up with the graph. Carrie infers that the arrow under the "lim" notation tells her in which direction to go. Since it points to the right, she goes right and stops at 2 which is her brick wall. Then she explains what happens at the brick wall, just as she did earlier inside of holes, which is that the limit does not exist because you can keep going up and down the brick wall without stopping. Carrie seems to perceive two related events occurring with a limit at a point. She thinks the limit exists at x=2 because that's where the arrow points under the "lim" notation but then she explains that the limit is x=2. Since it's a barrier or brick wall, she thinks the limit doesn't exist because she can go up and down the brick wall without stopping.

Hypothesis: Students cannot find the function due to lack of algebra proficiency and the limit notation is a source of confusion.

Discuss any possible relationship limits at infinity of rational functions have with horizontal asymptotes.

Figure D.1.60: Task 11 Problem Statement

C: Well, I guess I'd say that some of these fraction kinds of problems are funky and you have to draw horizontal asymptotes on the graph to see what the function is doing. Sometimes the calculator gives you something that looks horizontal but it's not. They aren't exactly horizontal because they are bent and cross over. Usually I just draw dotted lines across somewhere as horizontal asymptotes and then see if the graph crosses it or not. *(Turn 198, 37:40, SDV_0010.mp4)*

Analysis:

Inference: Carrie draws in her own horizontal asymptotes and does not see the relationship between limits at infinity and horizontal asymptotes. She does not understand the relationship because she hasn't learned the rules for finding asymptotes and their relevance to limiting behavior.

Hypothesis: Students may not know that if the degree of the numerator and denominator are the same in the leading term, then the ratio will be the limit and the horizontal line with that value will be the horizontal asymptote.

Summary

Carrie has significant misconceptions about limits at infinity. The "lim" notation is a major source of misunderstanding with limits because Carrie could essentially cover up "lim", look at the $x \to \infty$ part below it and say where x was going as the limit. Carrie thinks limits are about x, not y, so she continued drawing limits in the form of vertical lines on the x-axis. Carrie presents deficits with basic math and lacks algebra proficiency, including an understanding of rational functions evidenced by not knowing different rules to find asymptotes. Carrie did not demonstrate a mathematical understanding of what limits are but instead, imposed every-day interpretations evidenced by terms such as barriers or restraints. At time, she confuses where x and y are on the graph and often follows points on a graph going in a certain direction instead of studying just the behavior of the second coordinates. By following points on the graph, for instance, she decided that as x goes to infinity for the reciprocal function 1/x, the limit was infinity instead of 0. Moreover, Carrie did not seem to know that function values decreasing in the negative direction were approaching negative infinity, not positive infinity. When the function values got larger without bound in either the positive or negative directions, Carrie concluded that the limit existed because it was equal to positive infinity only. Most of Carrie's responses to the tasks appear to be at a level of pre-instrumental understanding.

Limits that Do Not Exist: Carrie

This interview explored limits that do not exist, as this topic presents significant ambiguity. Limits do not exist in three situations: infinite limits in which the limit is equal to infinity; piecewise functions with jump discontinuities and trigonometric functions with oscillations or periodicities. Infinite limits are confused with limits at infinity, so students don't know that an infinite limit is the result, or the behavior of the function values. Also, when a limit approaches infinity, students think the limit exists because it equals infinity. It's seldom understood that the limit does not exist because function values get larger without bound. In particular, the function values do not get closer to a number as required by the definition of limit. There are two other cases in which limits do not exist: piecewise and trigonometric functions. Limits at a point of piecewise functions with jump discontinuities involve comparing the left-hand limit to the right-hand limit. Limits do not exist for trigonometric functions due to oscillations or periodicities. Twelve tasks were presented to acquire a more in-depth understanding of under what conditions students think limits exist or don't exist, and to assess their understanding of infinite limits. In addition to rational functions, limits of piecewise and trigonometric functions were analyzed.

What are limits at infinity? Provide examples. What are infinite limits? Describe the difference between an "infinite limit" and a "limit at infinity". Give examples of limits that do not exist.

Figure D.1.61: Task 1 Problem Statement

C: Infinite limits are the same thing as limits at infinity. They are limits going to infinity. I'll draw some examples. You look at what's under the limit symbol, and see the infinity thing by the arrow. That tells you it's an infinite limit. So if it equals 3 or if it equals infinity, you just look at what is says x is approaching. Like I said before, limits are about x and have nothing to do with y. *(Turn 6,00:29, SDV_0011.mp4)*

C: Infinite limits, to me, means more the type of problem it is, not the solution. I would see hundreds of different limits, multiple limits, that is what infinite limits means.The answer has multiple limits. The word infinite means many, forever, you know, so I think it's a bad name to call them infinite limits and the books should change it to something else. *(Turn 74, 36:30, SDV_0011.mp4)*

C: A line with a solid dot is a limit that does not exist because you can walk across it without falling into some hole. If a limit doesn't exist, that means there is no restraint like a hole or a brick wall. If you get 1/x and put a zero in the denominator then there's no limit either. *(Turn 8, 1:00, SDV_0011.mp4)*

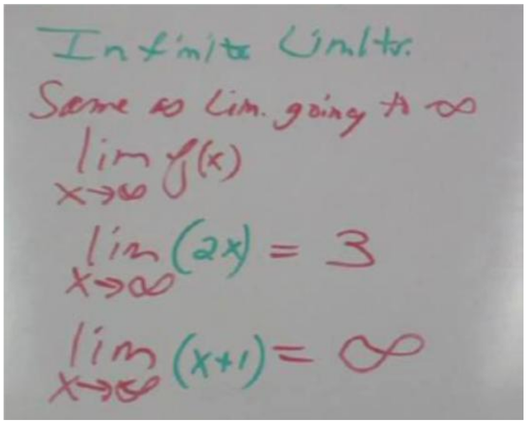


Figure D.1.62: Carrie's work on Task 1.

Analysis:

Inference: Carrie thinks infinite limits and limits at infinity are exactly the same. She also thinks it means multiple limits. She does not recognize that infinite limits pertain to the result, which is that that the limit does not exist because it is equal to infinity.

Hypothesis: Students don't know the difference between infinite limits and limits at infinity, and might not know what an infinite limit means.

Explain the behavior of the function value as x gets larger in the positive and negative directions. Graph the function and explain whether or not the limits exist.

$$\lim_{x \to \pm\infty} \frac{2x^2 + x - 1}{2 - x}$$

Figure D.1.63: Task 2 Problem Statement

C: Oh man, I hate that kind with the number first in the denominator. I can't do them unless the letter is first. So first I have to switch it somehow to get the x in there first. I'll try to factor the numerator. (paused). It doesn't look like it factors. So all I can do is put it in the calculator the way it's up there and see graph. *(Turn 85, 42:05, SDV_0011.mp4)*

R: Explain what you see as x is approaching positive infinity and negative infinity. Does the limit exist? *(Turn 86, 44:20, SDV_0011.mp4)*

C: There is a restraint at 2 because it's undefined there. At 2 there is no limit. On the left though the limit's going up to plus infinity and on the right it's going out to positive infinity. So yeah, the limits exist because they equal infinity on both sides. *(Turn 87, 44:40, SDV_0011.mp4)*

R: How come you labeled them both plus infinity, and how does this relate to what's under the "lim" notation? *(Turn 88, 44:55, SDV_0011.mp4)*

C: What's under the "lim" notation tells you what direction x is going, so I look to the left, and then look to the right. The answer is plus infinity for both of them because on the bottom it's going out to the right on the x-axis and on the left, on maybe it should be negative infinity. But I don't think so. I think it is positive infinity because it's above the x-axis shooting up. *(Turn 89, 45:35, SDV_0011.mp4)*

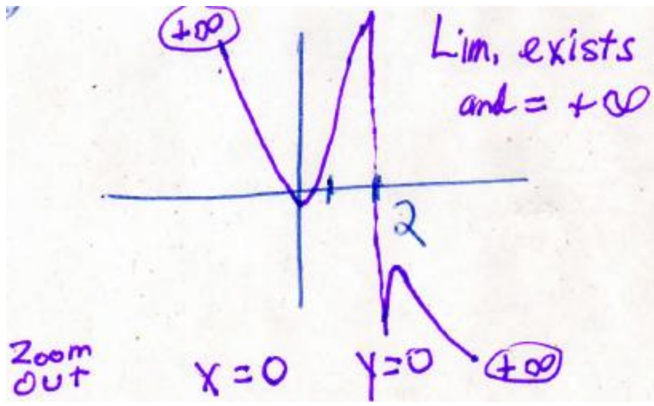


Figure D.1.64: Carrie’s work on Task 2

Analysis:

Inference: Carrie could not work with the denominator because the number preceded the variable, so she used the calculator to see the graph. She decided there was no limit at x=2 because the function was not defined there. She said the limit existed at plus infinity. She said "there is no limit" in place of "the limit does not exist." Carrie wrote $+\infty$ on the lower right side (fourth quadrant) probably because she is following points in the positive direction of the x-axis. On the other hand, she wrote $+\infty$ up in the second quadrant because she perceives this line as belonging with the y-axis. Carrie does not articulate her thoughts with mathematical terminology, such as saying "the limit does not exist" because she has not made the transition from the use of every day language.

Hypothesis: Students don’t consider the quadrant when labeling infinity as positive or negative, and may be following both x and y for each piece of the graph.

Compare these problems. Explain the behavior of the function values as x gets larger in the positive and negative directions. Graph the functions and explain whether or not the limits exist.

$$\lim_{x \to \pm\infty} x^2 \quad \lim_{x \to 0} \frac{1}{x^2} \text{ and } \lim_{x \to \infty} 2x+1$$

Figure D.1.65: Task 3 Problem Statement

C: For x-squared, the limit is 0 because there's nothing below it. For 1/x-squared there is no limit because it's not defined at 0 so you can't get to the brick wall to climb up or down. The 2x+1 if I put in different numbers it keeps going without stopping so it goes up to infinity. The limit exists and equals infinity. *(Turn 78, 38:55)*

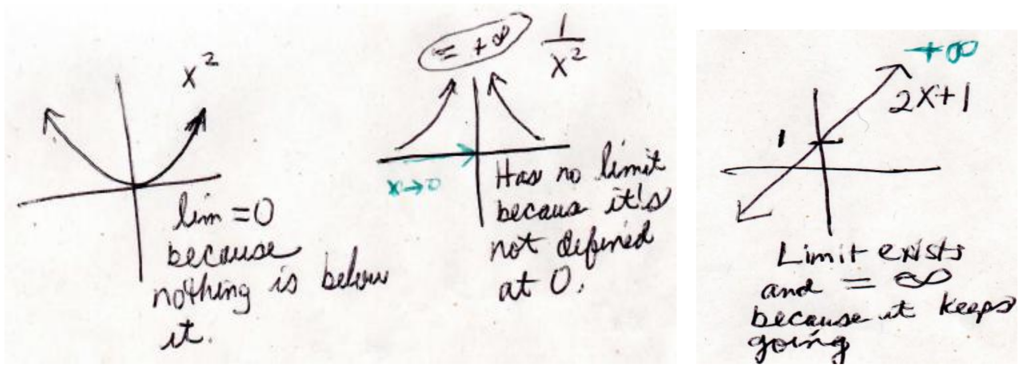


Figure D.1.66: Carrie’s work on Task 3.

Analysis:

Inference: Carrie thinks that for x-squared, the limit is 0 because there is nothing below it. She distinguishes this from the second graph, 1/x-squared, which she says "has no limit" which really means the limit does not exist. Her reason is that the function is not defined when x=0. For the linear function, she says the limit equals infinity because it keeps going. Carrie looked at the vertical position of the x-squared function, as she had previously done with the function $5x^2+2$ at which time she decided that the limit was 2 because nothing could go below it. In both cases, she is looking at the distance from the x-axis. In the second graph of 1/x-squared, she thinks there is no limit, which seems to be the same as saying "the limit does not exist." The reason the limit doesn't exist is because it never touches the wall on either side. In order for a limit to exist, there must be a restraint or a wall that can be reached, such as with limits at a point.

Hypothesis: Instead of saying that the limit does not exist because the function values are increasing without bound, students may think the line keeps going and doesn't stop, so the limit exists and equals infinity. The reason for this is because of the infinity symbol itself, which makes students think that infinity is a number when it is not.

Compare and describe the limiting behaviors of the functions. Sketch the graphs. Explain if the limit exists or not.

$$\lim_{x\to 0}\frac{1}{x^2},\ \lim_{x\to 0}\frac{1}{x}\ \text{and}\ \lim_{x\to\pm\infty} x^3$$

Figure D.1.67: Task 4 Problem Statement

C: The first two have no limit because it's undefined at 0. The limit doesn't exist because nothing hits the brick wall. The cubic one keeps going for a while but stops. I think there is one at 3, so as x goes to infinity the limit is 3. *(Turn 82, 40:45, SDV_0011.mp4)*

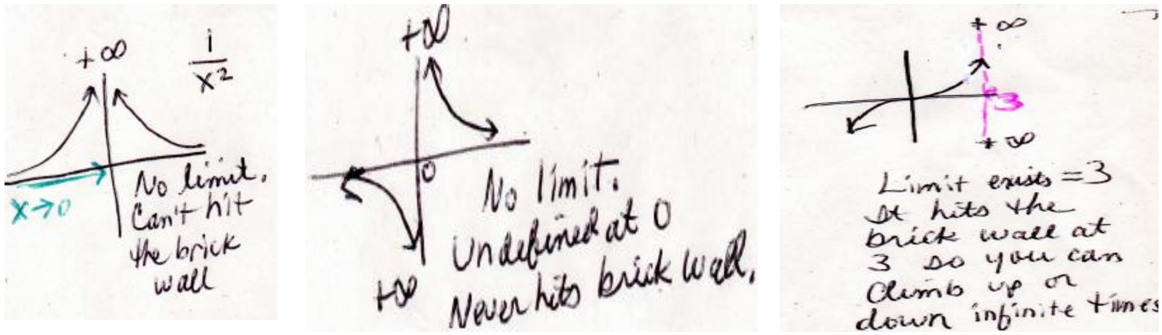


Figure D.1.68: Carrie's work on Task 4.

Explain the behavior of the function as x approaches 0. Graph the function and explain whether or not the limit exists.

$$\lim_{x \to 0} \cos \frac{1}{x}$$

Figure D.1.69: Task 5 Problem Statement

C: I have to put it in the calculator. Getting graphs are easier. I'd say the limit would be 1 and -1 so you can't go past it. As it comes to zero, it comes in here. The limit is also 0 because that's what it says under the "lim" thing. *(Turn 52, 21:31, SDV_0011.mp4)*

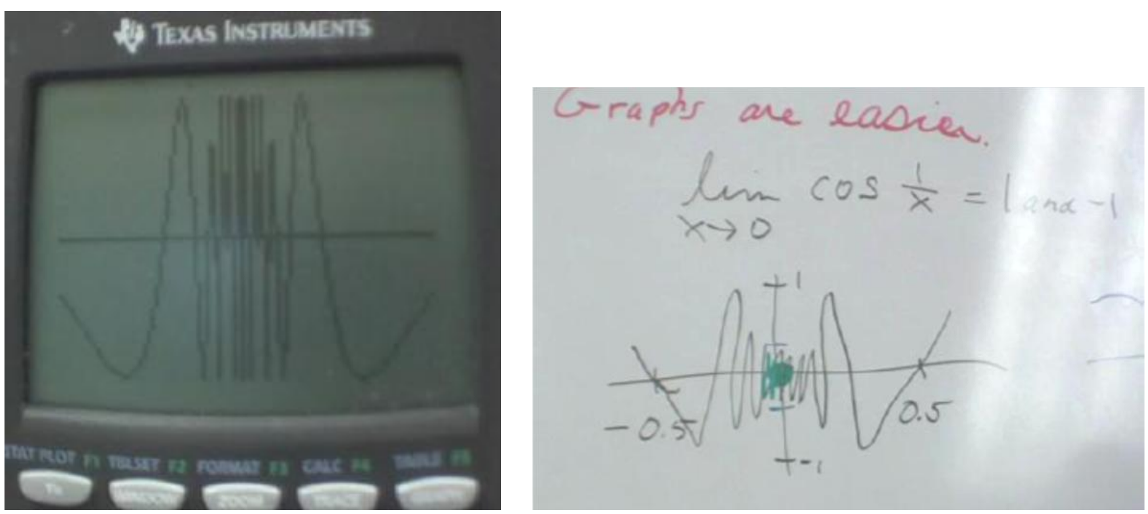


Figure D.1.70: Carrie's work on Task 5

Analysis:

Inference: Carrie relied on the calculator to get the graph and had to adjust the window to get an accurate view, but she did not identify that the limit did not exist due to oscillations near 0. Instead, similar to the function sin(x), she looked at the boundaries on the top and bottom and decided those were the limits. She decided that y=-1 and y=1 were the restraints that the graph couldn't go past In addition, she thought the limit was 0 because that's what appeared under the "lim" notation.

Hypothesis: Students lack understanding 0 in the denominator and the behavior of the function values near 0, which don't settle down so the limit doesn't exist.

Sketch the graph of this function. Compute the limit as x approaches 10 and explain if the limit exists at x=10.

$$\text{f(x)}=\begin{cases} 3 & \text{if } x>10 \\ \text{x-4} & \text{if } x\le 10 \end{cases} \qquad \lim_{x\to 10} f(x) =$$

Figure D.1.71: Task 6 Problem Statement

C: I think the y part is the y=3 and y=x-4. So, I have to think how to draw this because the calculator can't do it. (long pauses). Well there is a barrier or brick wall at 10, because that's the limit. It says that under the "lim" notation so I know it's gonna stop there. Then there is a horizontal asymptote at 3, so you get a line there. Then another one of them at y= -4. I don't think that x can ever be greater than 10 because the limit only lets you go up to 10, because it stops. So the limit exists because it stops at 10. That's what it says under the "lim" notation. So I would say the graph looks like this. *(Turn 76, 37:20)*

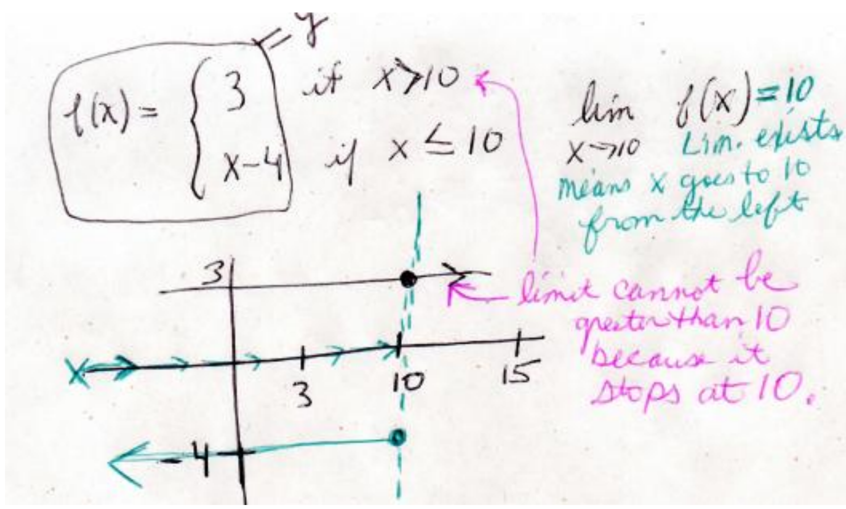


Figure D.1.72: Carrie's work with Task 6.

Analysis:

Inference: Carrie has difficulty with translating the piecewise function's formula to a graph. She knows that f(x)=y, as denoted in Figure 6, but plotted the 3 in both places because there was uncertainty about if it represented the x or the y. Then she used the x-4 and correctly knew there was a y-intercept at -4, but drew a horizontal line instead of a line with slope equal to 1. Carrie drew a vertical line at 10 to represent a barrier that the limit can't go past. Then she stated the limit exists at 10, and only approaching from the left since the arrow under the "lim" points in that direction. She stated the limit cannot exceed 10 because the limit stops there.

Hypothesis: Students get confused easily by this type of function because they probably look at the whole formula instead of looking at each component separately.

Explain the behavior of the function values as x approaches zero. Graph the function and explain whether or not the limit exists. $\lim_{x \to 0} \frac{1}{x^4}$

Figure D.1.73: Task 7 Problem Statement

C: I can do this with the calculator. The limit doesn't exist as x goes to 0 because you plug 0 into the x value and it is gonna be undefined. The lines could cross the y-axis though way up there. *(Turn 62, 28:16, SDV_0010.mp4)*

C: I guess the limit exists since it's not going to cross. It'll get close to 0, so the limit does exist. *(Turn 62, 28:16, SDV_0010.mp4)*

R: What's the reason it exists? *(Turn 65, 30:00, SDV_0010.mp4)*

C: No it doesn't exist. The points don't line up on here on the graph anywhere. They're all over the place. Points from the table don't line up. *(Turn 66, 30:05, SDV_0010.mp4)*

R: As x goes to zero, where are the function values going? *(Turn 67, 30:15)*

C: They go off to infinity. There is no limit. They just keep going and going. The lines keep going and going. There is no limit up here so up here it does not exist. The table says they don't exist either. *(Turn 70, 31:00, SDV_0010.mp4)*

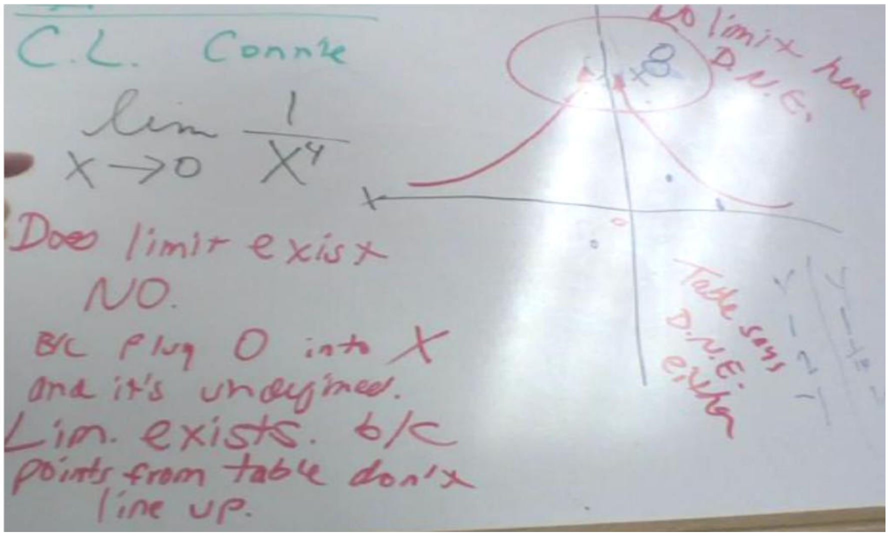


Figure D.1.74: Carrie’s work with Task 7.

Analysis:

Inference: Carrie decided that there was no limit at 0 due to the fact that the function was undefined at 0. She said the limit was infinity but did not ascertain that the limit did not exist. She did not describe the function values getting larger without bound on both sides.

Hypothesis: Students think infinity is a number and the limit exists.

Explain the behavior of the function values as x gets larger in the positive and negative directions and then as x approaches 0. Graph the function and explain whether or not the limits exist for both cases. $\lim_{x \to \pm\infty} \sin(x)$ $\lim_{x \to 0} \sin(x)$

Figure D.1.75: Task 8 Problem Statement.

C: Would the limit not be 1 and -1? I'm thinking you're not going past that restraint, those 2 numbers. *(Turn 36, 12:00, SDV_0011.mp4*)

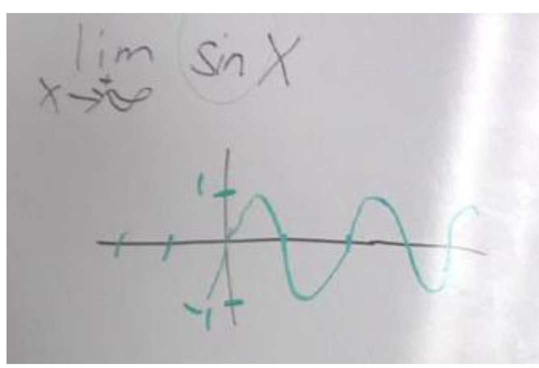


Figure D.1.76: Carrie’s work on Task 8.

Analysis:

Inference: Carrie switched to thinking the limits were on the y-axis instead. She thinks sin (x) has 2 limits, at y=1 and at y=-1. She does not know that the limit does not exist because the function oscillates between these 2 numbers and hence the function values do not get close to a number.

Hypothesis: Students might think there are horizontal asymptotes at y=1 and y=-1 which are the limits and might follow the line on the graph going to infinity without understanding that the limit does not exist due to oscillations.

Explain the behavior of the function values as x gets larger in the positive and negative directions. Graph the function and explain whether or not the limits exist.

$\lim_{x\to\infty} \arctan(x) \quad \lim_{x\to -\infty} \arctan(x)$

Figure D.1.77: Task 9 Problem Statement

C: The limit would be pi/2 as x goes to plus infinity and negative pi/2 as x goes to minus infinity. It can't go past that. (Turn 42, 15:55, SDV_0011.mp4)

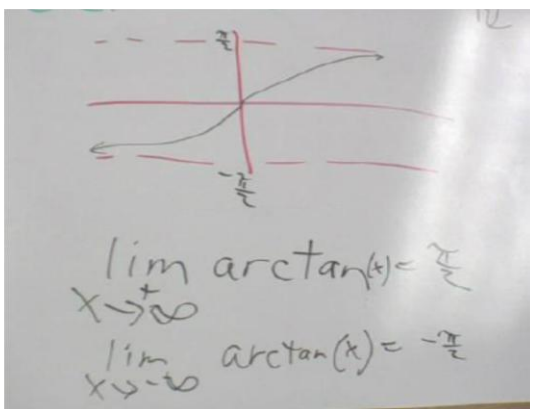


Figure D.1.78: Carrie's work on Task 9

Analysis:

Inference: She performed this task successfully but still views the limit as a barrier of sorts because of the horizontal asymptotes. Any line suggestive of a horizontal asymptote constitutes a restraint, and therefore a limit.

Hypothesis: Maybe a graph like this, that appears opposite to 1/x, is visually easier to identify the limit and that the function values need only get close to $\pi/2$ in order for $\pi/2$ to be the limit.

Explain the behavior of the function values as x gets close to $\pi / 2$ on the left and on the right. Graph the function and explain whether or not the limits exist.

$$\lim_{x \to \frac{\pi^-}{2}} \tan(\mathrm{x}) \qquad \lim_{x \to \frac{\pi^+}{2}} \tan(x)$$

Figure D.1.79: Task 10 Problem Statement.

C: Yes. You'd have limits between this and this, this and this, and this and that. The lines are the limit. 0 and A, pi/2, and A, b and pi/2, so 0 is a limit, A is a limit, pi/2 is a limit and B is a limit. Each one separately is a limit because it's a restraint you can't go past. *(Turn 40, 15:30,SDV_0011.mp4)*

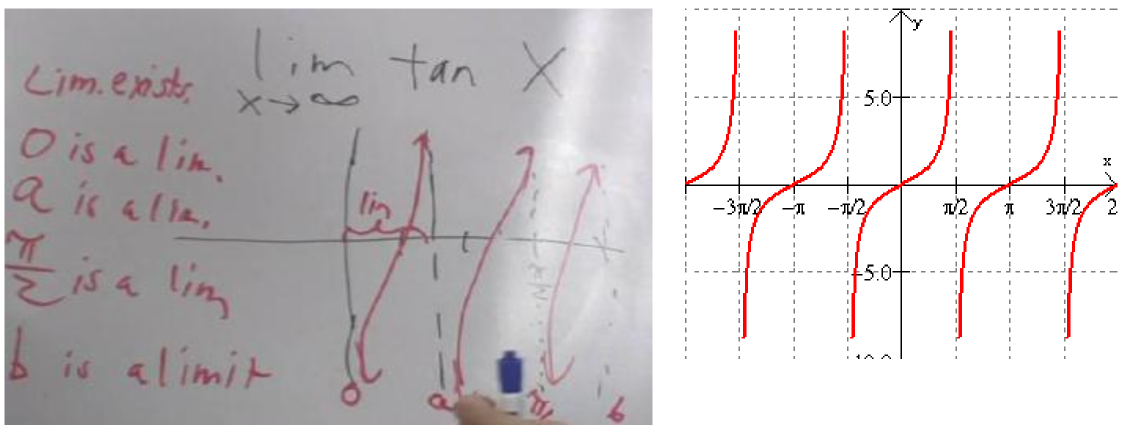


Figure D.1.80: Carrie's work with Task 10.

Analysis:

Inference: Carrie does not know that the limit does not exist as the function values get larger without bound in each direction. Once again, she identifies limits as vertical lines on the x-axis, and tried to label each one separately so it can be seen that these are each separate limits. She associates 0, a, pi/2 and b with numbers, because these letters she wrote represent negative infinity. Carrie did not acknowledge that as x approaches pi/2 from the left, the limit does not exist because the function values get larger without bound, and that as x approaches pi/2 from the right, the function values get smaller without bound. Though she used the calculator to get the graph, she did not reconstruct her graph correctly.

Hypothesis: Students think infinity is a number, so the limit exists.

Explain the behavior of the function values as x gets larger in the positive and negative directions. Graph the function and explain whether or not the limits exist.

$$\lim_{x \to \pm\infty} \frac{x^2 - x - 6}{x - 3}$$

Figure D.1.81: Task 11 Problem Statement

C: I think you can factor the top but I don't know for sure. Yeah, it's (x-3)(x+2) divided by (x-3) so these cancel out and you get (x+2) left on top. I don't know what kind of shift thing that is, if it goes up or down. But I can find out from the calculator, ha ha ha ha. Oh, good thing I checked because I thought it had a y-intercept at 2. *(Turn 91, 46:50)*

R: How do you know you would not get an y-intercept at 2? *(Turn 92, 47:55)*

C: Because you can't. If you plug in 2 you get 4, so your y-intercept would be 4. It can't be the same as the x-intercept. *(Turn 93, 48:20, SDV_0011.mp4)*

R: Do the limits exist as x approaches plus or minus infinity? *(Turn 94, 48:35)*

C: Yes, here on the graph on the right the limit exists at plus infinity and on the left it equals negative infinity. *(Turn 95, 48:40, SDV_0011.mp4)*

R: Out of curiosity, how come in a previous problem you said that if the line is on the other side, going down in the 4th quadrant, that the limit is positive infinity. *(Turn 96, 48:55, SDV_0011.mp4)*

C: Oh, you just have to look at what's under the "lim" thing. The other one didn't have my graph pointing toward negative x's, they were pointing toward positive x's so even though the arrow went down, it was positive infinity. But see in this problem, under the "lim" it says as x approaching negative infinity and it has something on the graph going in that direction, left in the 3rd quadrant, so that's why you can say the limit exists and equals negative infinity. Like I said, limits are all about what x is doing and all you gotta do is look at what's under the "lim"thing to see what they want. *(Turn 97, 49:30, SDV_0011. mp4)*

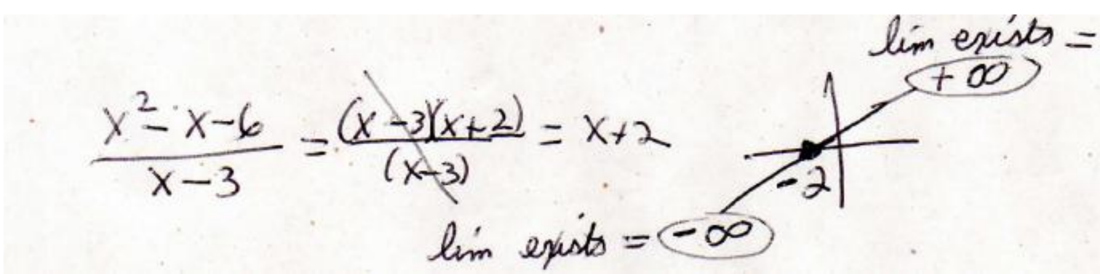


Figure D.1.82: Carrie’s work with Task 11.

Analysis:

Inference: Carrie correctly factors but that's about it. She uses every day language such as "top" instead of "numerator". When left with (x+2) in the numerator she did not know if this would be a horizontal or vertical shift so she put it into the calculator to find out. She concluded that the x-intercept was 2, and that the y-intercept was equal to 4 after plugging in the 2 for the x-intercept. In fact, she thinks it's not possible to the same number for the x-intercept and the y-intercept. She said the limit exists and equal plus and minus infinity, but for the wrong reason which is that she looked at what was under the "lim" notation instead of explored the behavior of the function values as x increased or decreased.

Hypothesis: Students might look at the x-axis and then follow the points on the graph to see if they go right (positive) or left (negative), instead of looking at what is occurring up and down on the y-axis with respect to the behavior of the function values. Students may also think a limit exists at infinity because infinity is a place or number represented by a symbol. This might explain why they can’t consider if the limit equals infinity, it's because the function values are getting larger without bound.

Construct a possible function from the graph below and explain if the limit exists as x approaches 2 from the left and as x approaches 2 from the right.
Note: expect piecewise

$\lim_{x\to 2^-} f(x) = ?$ $\lim_{x\to 2^+} f(x) = ?$

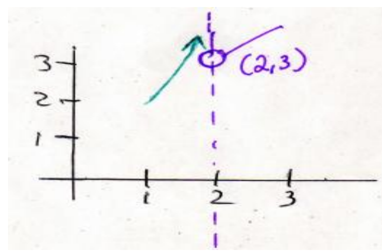


Figure D.1.83: Task 12 Problem Statement

C: There is a restraint at 2 with a hole. So if you climb up the brick wall there is a hole you can fall into. You want a function for this? (long pauses) I think that approaching 2 from the left, it can't hit the brick wall but maybe it can. I think on the left the function is 2x because at 2 it's undefined. Then on the right it's 2x+3. So as x approaches 2 from the right, the limit exists and is 2 because there is both a brick wall and a hole you can fall into. Actually, it has a restraint and a hole so it has 2 limits at 2. The way I know, too, is just by looking under the "lim" thing. It tells you what the limit is gonna be without even doing anything. *(Turn 99, 52:20)*

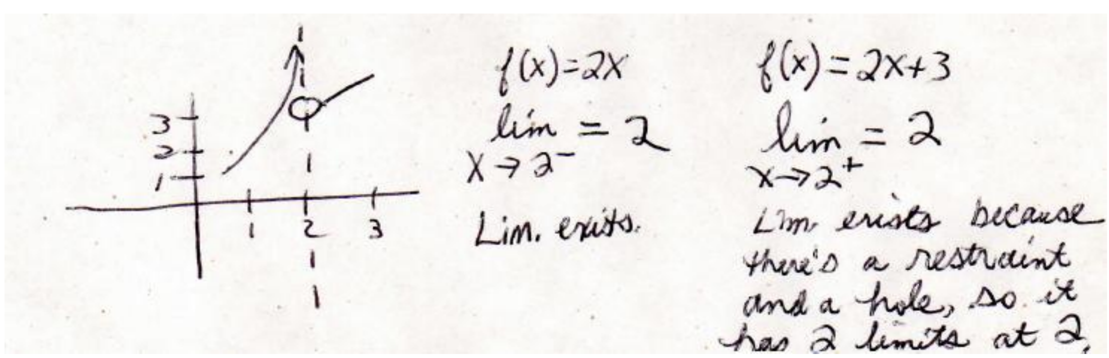


Figure D.1.84: Carrie’s work on Task 12.

Analysis:

Inference: Carrie does not elicit evidence of knowing this function is piecewise and she was unable to construct a piecewise function from the graph. She did not pick up on information from previous problems, in which she saw that $\frac{1}{x^2}$ shifted right 2 units would give her the left piece as x approaches 2 from the left, and that a linear function with $x \neq 3$ would give her the right hand piece, as x approaches 2 from the right. She was unable to produce the correct piecewise components because she lacks the ability to look at graphs and ascertain their functions. Also, she could not write the traditional format for a piecewise function that contains a bracket, followed by the function values and their domains. It was hoped she would be able to recognize some information from the previous piecewise tasks she worked on in this interview and in the study. As for limits, she concluded that the limit existed at 2.

Hypothesis: Students can't work backwards with this problem due to lack of proficiency with domains and piecewise functions.

Summary

Carrie does not know what it means for a limit not to exist or what the textbook term "infinite limits" means. Infinite limits involve multiple limits. No evidence was shown for understanding that infinite limits are the result of what's computed and include limits that do not exist because the function values increase or decrease with bound. Her examples of limits that do not exist were opposite to what they should be. For example, a solid dot on a line to Carrie meant that the limit does not exist because there is no hole to fall into to prevent one from getting to the other side. She lacks basic math skills and has deficits with algebra, to the extent that she cannot factor when there is a negative number involved, such as in the 2-x denominator example. She has a great deal of trouble working with piecewise functions in any context, and cannot develop possible formula from such given graphs. At best, she can plug a function into a calculator to get a graph, but cannot start with the graph under any circumstances to extract information about it. Entering data for a function into a calculator requires no math skill whatsoever and so this is why she seems to be at the pre-instrumental level of understanding. Her interpretations of limits are esoteric but she articulates them quite well and most of all, very consistently across tasks. She thinks that limits are only about what x is doing, not y, and she thinks you can find a limit simply by looking beneath the "lim" notation. Meanwhile, she does not refer to the behavior of the function values as x approaches plus or minus infinity, but instead, follows points on a graph along the x-axis in either direction. Overall, her understanding of limits that do not exist are minimal and her deficits in mathematics are quite remarkable. Therefore, she is best characterized at the instrumental level of understanding of Skemp's model.

Analysis of Interview about Functions: Nicole

A narrative summary of how Nicole explored function tasks is presented. Transcript evidence from her problem solving activities is presented to highlight her underlying ideas about functions. This summary serves as a foundation for discussing her ideas about limits. Overall, Nicole had more mastery and proficiency with mathematics.

Define function. Write or draw examples of functions. What is a function value? What's the difference between a function and a function value? How are functions and function values related?

Figure D.2.1. Task 1 Problem Statement

R: Can you tell me what a function is and give some examples?

N: A function is an equation that maps an independent variable x in the reals R to a single value y in R. So f mapping R to R. Has to be 1-1 and an example f(x) = x squared, f(x)=5x cubed +3. Other types are exponential, logs, quadratic, rational. *Turn 6, 1:09, SDV_0049.mp4.*

N: Um, the function value is what a function equals at a specific point so it's any point in the form (x,y). For example, (3,7) is a function value. A function is the line, the graph, defined on the entire domain. A function is the entire behavior from the entire domain. It's hard to explain this. A function value a point on the graph. Whereas a function is everything you see like entire graph line. You can't define anything at just one point. You need an equation to cover its entire domain. (*Turn 85, 23:30, SDV_0051)*

Analysis:

Inference: Nicole satisfactorily described what a function was but did not provide a definition involving a relation. Examples such as quadratic, linear, rational, exponential, logarithmic and trigonometric functions were elicited. Like other students, she used the vertical line test to determine if a curve in the plane was the graph of a function but incorrectly thought that the vertical line test was to see if a function was 1-1. (Determining if a function has an inverse is done with the horizontal line test to see if the function is 1-1). She also said a function had to be 1-1 which is not true, as not all functions have inverses. She also incorrectly equated 1-1 with the vertical line test. Nicole used mathematical terminology, "reals, mapping and 1-1" and revealed her knowledge of different types of functions. However, she elicited an incorrect statement about a function having to be 1-1.

Hypothesis: Students do not know the definition of function and do not know what it means for a function to be 1-1.

R: What are function values?

N: They are values that functions have. So it's the "x,y" or any coordinate on the graph. *Turn 8, 2:00, SDV_0049.mp4*

R: Do y-values have anything to do with function values? Are they the same thing or 2 different things?

N: No, y-values are the "y" part on a point. Function values are the x and the y because functions have 2 kinds of values, x's and y's. A function is what you get out. *Turn 10, 23, SDV_0049.mp4*

Analysis:

Inference: Nicole said a function value referred to both x and y. She does not know that "function value" refers only to the 2nd coordinate on a graph. She seems to associate the word "value" with a number, which is why she says "x is a value and y is a value, and plug in the value of x." Since points on the graph have to do with the function, Nicole sees both x and y are the function values.

Hypothesis: Students do not know the difference between a function and a function value so they think these refer to the same thing.

R: Now can you tell me what a limit is. Not necessarily a formal definition, just something intuitive in your own words.

N: A limit is the "value a function approaches as x approaches the limit value", i.e., or for example as x squared approaches 2, f(x) approaches 4. You know what x is approaching so you want to find out what f(x) is approaching. (*Turn 12, 3:15)*

Analysis:

Inference: Nicole could not articulate precisely what a limit was but gave a good example but conceptually she understood limits involved the 2nd coordinate. The term "limit value" is not traditional and appeared to be something she made up. Calling it "the value of the limit" would have been a better statement. She used the words "function, function value and value of a function" inconsistently. She thinks a function and function value mean the same thing which is why she refers to the second coordinate as the "value a function approaches."

Hypothesis: Students might not know that limits are about what happens with the second coordinate on a graph.

R: Can you explain how functions and limits are related?

N: A limit can basically help you estimate or determine what a function equals at a certain x. For example with f(x)=e to the x, we don't evaluate what x is at 100 but if you know the graph and know your sense of limits you know that the limit of f(x) as x goes to 100 it goes to infinity. You know what x is approaching so you want to find out what f(x) is approaching. (*Turn 14& 16, 4:00, SDV_0049.mp4)*

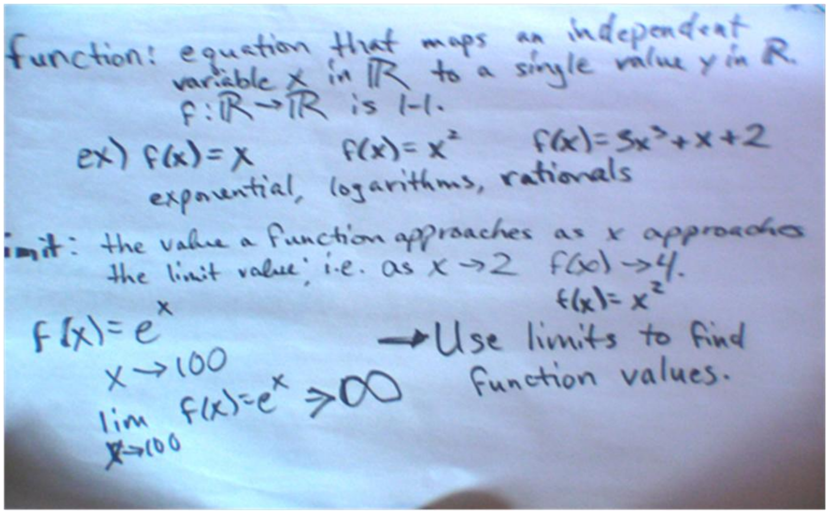


Figure D.2.2: Nicole's definition of function and limit.

Analysis:

Inference: Again, Nicole thinks "function" and "function values" mean the same as seen where she says "determine what a function equals at a certain x." She also claims "you use limits to find function values." She claimed that x was not being evaluated at 100, but essentially it was. What she did not explain or articulate very well is was that for large x, the function values get larger and larger without bound and go to infinity, in which case the limit would not exist.

Hypothesis: Students do not have a good definition of function in place, don't know the difference between functions and function values and don't know the relationship between functions and limits. Students also lack mathematical literacy as a result of not knowing the definitions.

Explore the circle, and the top and bottom separately, to decide which are graphs of functions.

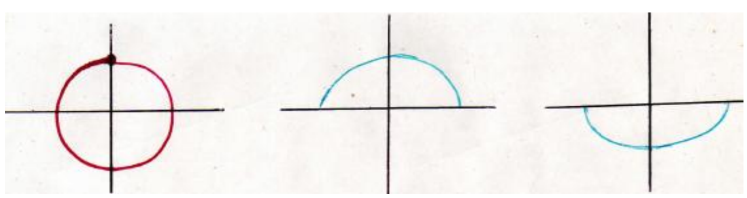


Figure D.2.3: Task 2 Problem Statement

N: The upper and lower halves are graphs of functions but the whole circle is not because it's not 1-1. The circle has more than one function value for any given value of x. *Turn 46, 46:10.*

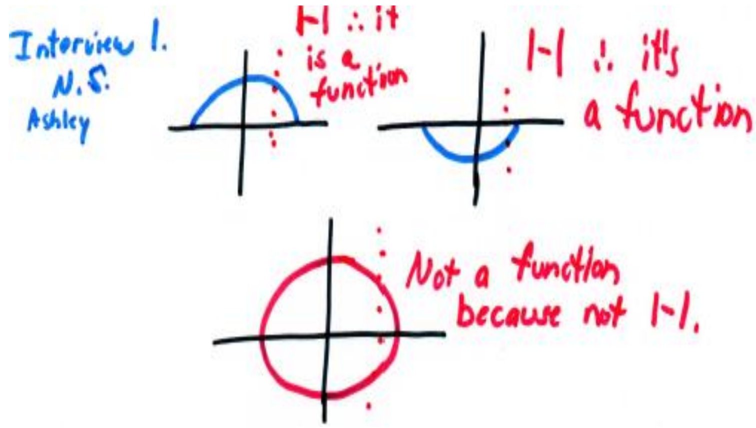


Figure D.2.4: Nicole's work on Task 2.

Analysis:

Inference: Nicole conceptually understood what a function was and drew the vertical line test but referred to this as being 1-1. In fact, these functions in the first 2 drawings are not 1-1 because they do not have inverses. 1-1 means that there can only be one x for each y-value.

Hypothesis: Students confuse the meaning of 1-1. Instead of 1 y for each x for a function, they might think 1 x for each y.

Explores the graph of a discontinuous function with an isolated point on the same graph.

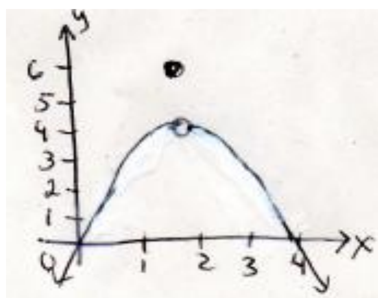


Figure D.2.5: Task 3 Problem Statement

R: I want you to tell me if the point is on the graph of the function, and also name the function we're talking about here.

N: Yes, the point is on the graph of the function and the point is part of the function itself. It's piecewise. The bottom piece looks like it's a negative quadratic because it is concave down, and the point above the open hole is the function value, but there's a discontinuity at 2 so x cannot equal 2 for the quadratic but can equal 6 when x is equal to 2. (*Turn 47, 42:44)*

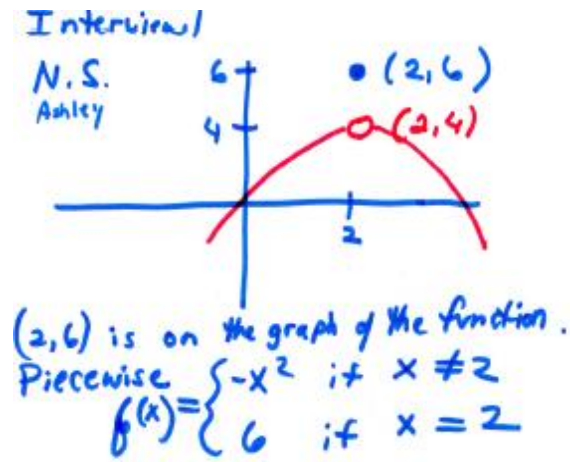


Figure D.2.6: Nicole’s work on Task 3

Analysis:

Inference: Nicole knew this was the graph of a piecewise function and was able to explain there were two parts or pieces to the graph, when x did ad did not equal 2. Most students would say this function is quadratic and that there is a random point sitting above it.

Hypothesis: Algebra proficiency and understanding of domain is necessary for understanding piecewise functions.

Follow up questions about the graph of a discontinuous function with an isolated point. A point was added to the graph of a piecewise function so that it was now not the graph of a function.

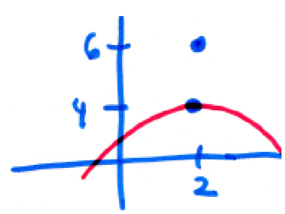


Figure D.2.7: Task 4 Problem Statement

R: Compare this graph to the one you just saw, where it is the same except (2,6) is a solid dot.

N: This one is not a function because it is not 1-1. So it's not piecewise like this other one. Here for x=2, there are 2 y-values which you can't have for 1-1 functions.

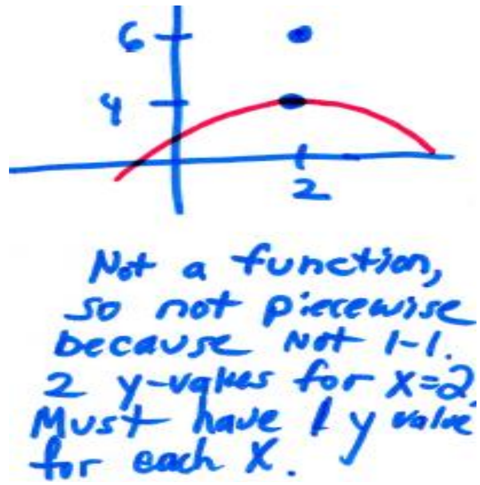


Figure D.2.8: Nicole's work on Task 4.

Analysis:

Inference: Nicole knew immediately that this was not the graph of a function but again, gave the wrong reason stating that it's not 1-1. This is a consistent finding. She should have instead said it failed the vertical line test which is used to determine the graph of a function. She knew this graph was not a piecewise function, unlike the one given prior to this. She clearly knows that for each x, there can only be one value of y.

Hypothesis: Students confuse the definition of function with 1-1 (reserved for determining if a function has an inverse).

Given VA at x=$-\frac{3}{2}$, x=$\frac{3}{2}$ *and* HA at y=$\frac{1}{2}$ find which matches below.

$a.\ f(x)=\dfrac{x^2}{x^2-\frac{9}{4}}$ b. $f(x)=\dfrac{x^2}{2x^2-\frac{9}{2}}$

c. $f(x)=\dfrac{1}{2x^2-\frac{9}{4}}$ d. $f(x)=\dfrac{x^2}{2(x-\frac{3}{2})^2}$

Figure D.2.9: Task 5 Problem Statement.

N: First and foremost for the vertical asymptotes I would test each of these x values in each function. For choice A, I'd plug in 3/2 and get 9/4. So this is undefined and does have a VA at x=3/2. A is a possibility. Now I will try B. f(3/2) gives you 9/4 over zero, so this is another possibility. Now for C, we would not have it undefined. Because if you plug in 3/2 you don't get 0 in the denominator, you end up with 4/9, so that's defined at 3/2 which means there is no VA. For D, that's 9/4-6/4 which all you care about is that's not zero so it is also defined, with no VA so D is not a possibility. When you check for VA's, all you want to see is if the denominator is zero at the x-value. And if it isn't as in the case of C and D, which means the function is defined, then neither of these functions match the equation that we want.

R: How would you use the information y=1/2 to decide what the function could be?

N: For that one, let's set the function equal to 1/2. This is not possible. It doesn't have a horizontal asymptote because it can never equal 1/2. It doesn't have the horizontal asymptote for solution A so it's not the right one.

I then asked Nicole if she new about rules for finding horizontal asymptotes comparing the degrees of the numerator and denominator, and she admitted that she forgot those. She claimed that the horizontal asymptote had to be 1/2 in this problem. She went off track when she associated y=1/2 with setting the whole function equal to 1/2.

N: You set this function equal to 1/2. All I know is when you have the graph, at the horizontal asymptote the function tends to cross it but can follow it depending on the function. For example, e to the x, it has a HA at y=0 so it never crosses it but I can't explain why you would look at the coefficients. It wasn't the method I was taught. So B is the answer because y is 1/2, because A doesn't work. VA's is what a function can't cross. So it can equal at some point x can equal 1/2 like with e tothe x and it can't ever equal zero. Functions can cross a HA. but VA's strictly a function cannot cross. But it's true than a x-value can equal 1/2. *Turn 24, 14:47*.

Even though she got the correct answer B by visual inspection of the 1/2 in the answer choices, Nicole discovered that her method of setting the f(x)=1/2 was not working, so I asked her why she thought this method did not work.

N: Functions can cross horizontal asymptotes but not vertical asymptotes. It's still true than an x value can equal 1/2 but in both of these cases it never does. So like with e to the x the function can never equal zero. The only other method is choose values near 1/2, and see what the function does. If it tends to approach a value... but wait, that won't work. If you can get function values that get near 1/2 in either the negative or positive directions towards positive or negative infinity then you know it has a horizontal asymptotes if goes towards this value then you can tie that idea into limits. *Turn 26, 17:00*.

Analysis:

Inference: Nicole still refers to "function" instead of "function values" and uses these terms interchangeably, incorrectly set the function equal to the horizontal asymptote y=1/2 hoping to solve for x and get x=3/2 and x=-3/2 in the denominator. She refers to how "functions" can cross a horizontal asymptote but not a vertical asymptote. A better way to state this would be "the graph of the function can cross..." She set the whole function equal to 1/2 because she was thinking f(x)=y. Nicole conceptually recognized the connection of horizontal asymptotes with limits at infinity.

Hypothesis: Students lack proficiency with algebra, rules for finding asymptotes, understanding asymptotic behavior, and mathematical terminology. Students may not see the connection between horizontal asymptotes and limits at infinity.

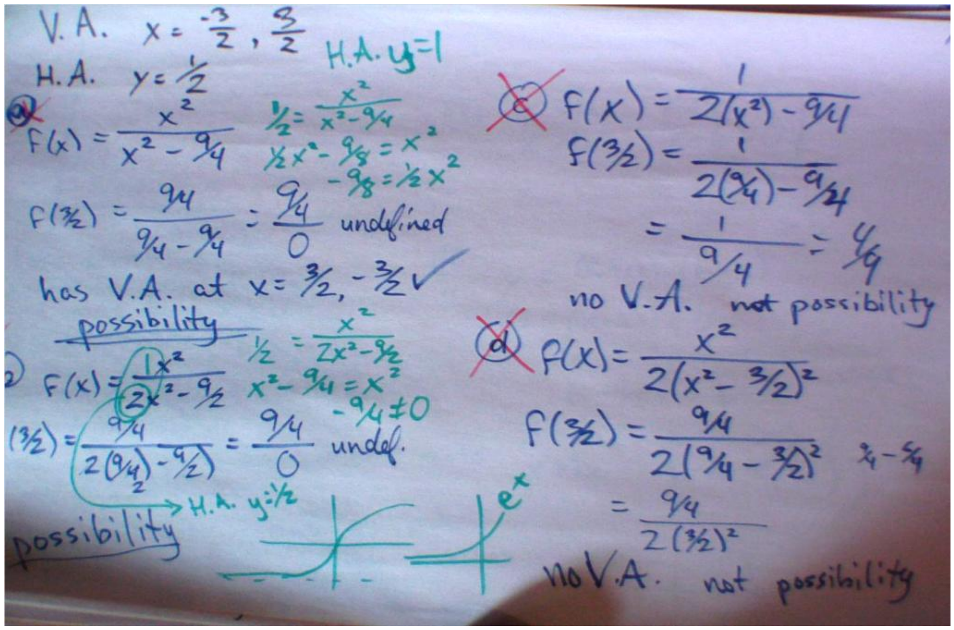


Figure D.2.10: Nicole’s work on Task 5.

Given this graph of a function and five answer choices, can you tell which equation would match the graph?

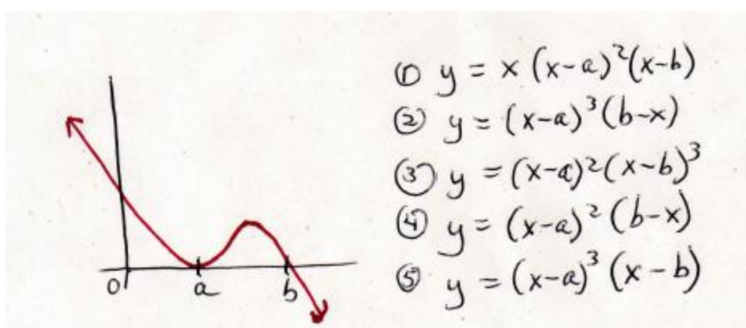


Figure D.2.11: Task 6 Problem Statement.

N: At A and B the function could be zero so that's something to test for. If you plug in A and B could you get zero out of these, um, for all these graphs you get zero. Can you take derivatives? Well if so, I'd see what would give the minimum at A and the maximum between A and B. This is a hard way to go about it with all these terms. *Turn 30, 18:43.*

N: I would say an easier way would be to get roots, so it's 2 roots but here it bounces off, doesn't go below the x-axis. Here at A it bounces it off the x axis so there is a root at A but it is squared. So you want to look for a function that will have a root at A but that's squared. So whenever, for example with #3, it wouldn't be #4 b/c you'd get x is neg. "b". No,.... wait. (long pauses) For (x-a) squared=0, since it's squared it will never be negative after passing "a", because when you square a negative it is always positive so it bounces back off the x-axis. For (b-x)=0 if you test values larger than b, then it's negative. So that should work, compared to #3. (*Turn 33, 18:43)*

N: That task was very tricky because the choices were all so similar but writing down x is greater or less than "b" was a good way to test them. You normally don't get answer choices that are all this similar. When I was first learning these, the choices would be quite different and all we had to really look at was the number of roots. I would find which equations have only 2 roots to give you answers number 3 and 4. You would test the behavior of the graph for #3 and #4 for x<b and x>b which matches the graph. *(Turn 35, 25:52)*

Analysis:

Inference: She explored answer choice #4 as a possibility and explained why it would or would not be correct. Nicole has good algebra skills, knowing about how at point "a" there was a root of multiplicity with an even power and at "b" there was a root with an odd power. She worked with each piece separately and knew under what conditions function values would be positive or negative, and how to arrange the factors in the last term to get the negative slope.

Hypothesis: Students need algebra proficiency including understanding of polynomials, factors and roots.

Can you match each of the graphs with the functions?

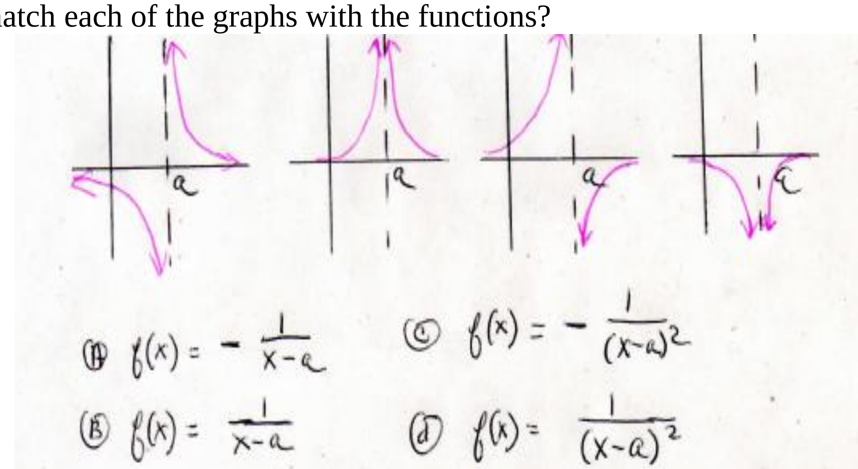


Figure D.2.12: Task 7 Problem Statement.

N: The factor (x-a) puts us at the vertical line at x=a compared to the y axis that is 1/x. The first is the graph of 1/x. So 1/(x-a) is the first graph. Another way to determine is to test values on both sides. Values x>a is positive, so values of the function are up here for x>a, and when x<a the function values are below the x-axis. So this 1/(x-a) is the only one that matches. The only one that has function values positive is the second

one. But we determined that when the function is less than "a", the function values are negative so this 2nd graph wouldn't be right. So x>a, f(x)<0 which is B. So the first graph goes with B. *Turn 37, 29:45.*

R: For the last 2, can you predict what graphs would match?

N: Yeah. The 3rd graph is the opposite of the 1st. So everything is flipped. It's the negative of choice B, so it's A. For the last problem it's the negative of the 2nd graph, which would be answer choice C.

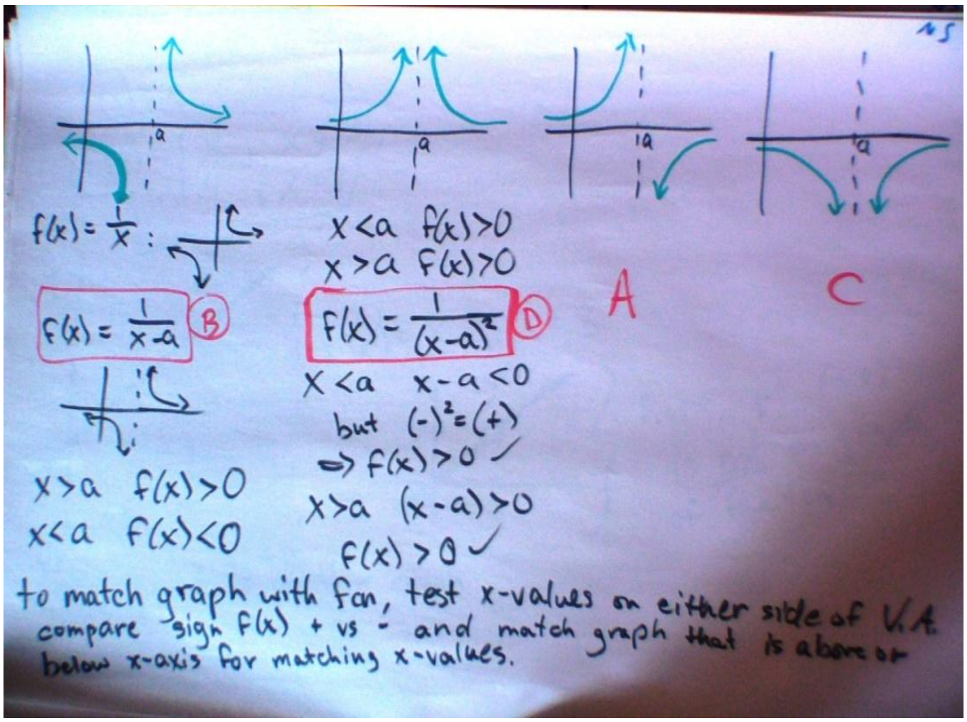


Figure D.2.13: Nicole’s work on Task 7.

Analysis:

Inference: Nicole uses x=a as a point of reference prior to examining the factor (x-a). She did not explicitly state the horizontal shift right "a" units and why a is subtracted from x, as she seemed to focus on the problem more visually using test points for values greater and less than "a". She connected the graph's features with the function values, for example, the function values were always positive pointing upwards when the denominator was squared, and negative pointing downward when there was a negative sign in front of the equation. Nicole knows about the properties of elementary functions, and also has facility with knowing how function values behave when the exponent in the denominator is raised to an odd versus an even power.

Construct a rational function for the graph with a hole at (2,7).

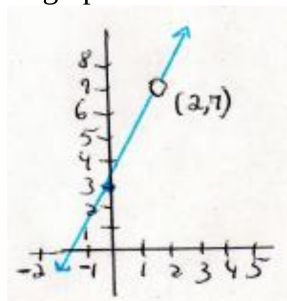


Figure D.2.14: Task 8 Problem Statement

N: It has positive slope with y intercept at 3 with a discontinuity at (2,7). It's piecewise because from (minus infinity,2) and then from (2, positive infinity). But we need a rational function so the piecewise x>2 and x<2 you end up with. We can't have an x value at 2, so (x-2) has to be a factor in the denominator. To get a hole you need the exact same factor in the numerator. One factors till left in the numerator to find out to describe the positive slope with y-intercept at 3. I would look at the slope and take (2,7) and (0,3).)y2-y1)/(x2-x1) so that is (7-3)/(2-0) so that's 4/2 which is 2. The slope equals 2. Then y=mx+b, and I just plug in the 2 and 3 for y intercept, giving y=2x+3 as the remaining factor in the numerator which gives me the straight line with the hole in it when I cancel out the factors (x-2) in the numerator and denominator. So the rational function is (2x+3)(x-2)/(x-2). (*Turn 44, 36:45. SDV_0050A.mp4)*

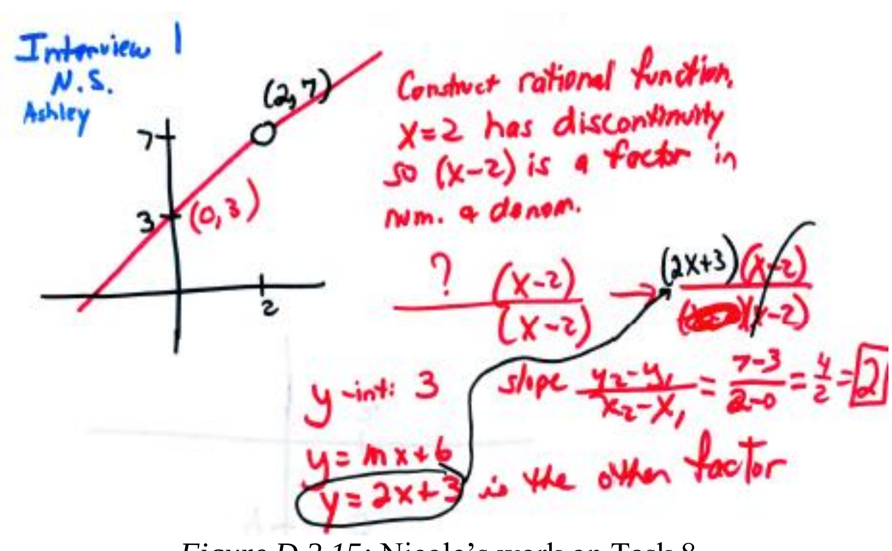


Figure D.2.15: Nicole's work on Task 8.

Analysis:

Inference: Nicole systematically takes pieces of the problem and analyzes them separately. She started with the fact that the slope is positive then divided the graph in to two separate pieces, when x>2 and x<2, though she did not explicitly write that part down on her sheet, as she was focused on getting a rational function. She correctly determined that a common factor was needed in the numerator and denominator to divide out, leaving only the numerator which would contain the slope and y-intercept. She knows some specific features about rational functions as well, such as that a common factor was involved in the numerator and denominator.

Hypothesis: Students need algebraic proficiency and skills to construct a possible rational function for this problem.

Summary

Eight tasks were presented on functions. Some deficits were observed with mathematical terminology. She thought "1-1" referred to the vertical line test and she exhibited an inconsistent use of the terms "function and function values." Her mastery with understanding piecewise functions shows later on to be fundamental to understanding limits in calculus.

Nicole demonstrated above average instrumental understanding and average conceptual understanding because she performed the necessary algebraic steps in the problems with few errors. On occasion, she was able to explain her reasons why, but her primary focus was more on clearly articulating algorithmic steps. She has had a lot of math courses, including differential equations, and is ahead of most students who often cannot provide steps or explain why they performed certain operations. Although Nicole does not verbalize much about what her results mean, she articulated what she was doing very well and she elicited sufficient evidence through written work that she possesses some forms of conceptual understanding.

Analysis of Interview on Limits at a Point: Nicole

Limits at a point involved tasks in which a student looked at graphs and described whether or not limits existed. Other tasks involved computing limits, constructing graphs and describing the limiting behavior of the function near a point. These problems were presented at the end of interview 1 and beginning of interview 2.

What is a limit? Provide a definition or an intuitive explanation. Explain whatever comes to mind when you think of finding a limit. Write or draw examples of problems involving limits, both that exist and do not exist. Do limits refer to either the first or second coordinate on the graph, or do they refer to both? How are functions and limits related?

Figure D.2.16: Task 1 Problem Statement

R: What is a limit?

N: A limit is the value a function approaches as x approaches the limit value. For example as x approaches 2, f(x) approaches 4. *Turn 8, 3:00, SDV_0049.mp4*.

R: Can you explain how functions and limits related?

N: A limit can basically help you estimate or determine what a function equals at a certain x. For example with f(x)= e^x we don't evaluate what x is at 100 but if you know the graph and sense of limits you know that as x goes to 100, f(x) goes to infinity. Limits that don't exist would be like a piecewise function.

R: What do you look for with limits, is it the x or the function values?

N: The function values. You know what x is approaching so you want to find out what f(x) is approaching. *(Turn 13, 5:19, SDV_0049.mp4)*

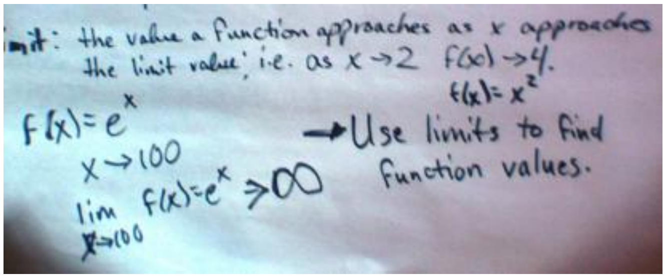


Figure D.2.17: Nicole's work on Task 1.

Analysis:

Inference: Nicole correctly articulated what a limit was, even though she did not provide a definition.

Hypothesis: She appears to understand that the second coordinate on the graph involves nearness and the limiting behavior of function values. Since she was able to give an example of f(x)= e^x this suggests that she understands the behavior of function values for large x.

Compute the limit if it exists as x approaches 3, then sketch a graph of this function.

$$\lim_{x \to 3}(2x+1) \quad \text{or} \quad \lim_{x \to 3}(5x+2)$$

Figure D.2.18: Task 2 Problem Statement.

R: Can you compute the limit of 5x+2 as x approaches 3 and explain what the limit means and say if the limit exists? Draw a sketch.

N: The x is 3, y is 17. Just plug in 3 and see what the function does. So as you approach 3 for x, you head toward 17 for y and have slope of 5. Each of these values get close to 17 from either direction. Limit means what is it approaching not what is it equal and limit exists at (3,17) . *Turn 37,1:07:15, SDV_0050.mp4.*

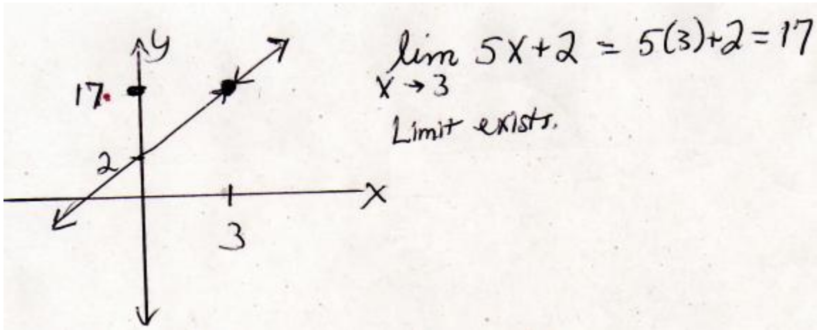


Figure D.2.19: Nicole's work on Task 2.

Analysis:

Inference: Nicole interchanges the terms "function value" with "function". She correctly articulated how first she looked at what the x was approaching, then looked at what y was approaching. She noted that the notion of nearness is different from what a function value equals but in this case she knew to plug the x value in to compute f(x) and knew that a solid dot would occur on the straight line at (3,17).

Hypothesis: Nicole has the right idea in general about what a limit is and correctly knew to plug in x into the function.

Compute the limit if it exists. Explain the behavior of the function values near x=2.

$$\lim_{x\to 2}\frac{x^2+4}{x-2}$$

Figure D.2.20: Task 3 Problem Statement

N: Negative for values less than 2 and greater for values greater than 2. If I plugged in 2.1 I'd get about 4 for the square. 4+4 over.... it's a large number so it's going to infinity. It would be approx. 8 over a very small number. So with other values you get about 8 over infinity, so numbers closer to 2, these get closer to 4, wow, that's way wrong. Why didn't I try to factor. I think it's in my head wrong. Let me try this again. I was picturing this in y head the wrong way. These cancel so you get a hole. So this is just a line x+2 you get. Looking at the graph, the limit exists. Let's see. So now I can look at from the positive direction, the limit is 4 and that equals 4 and as x goes to 2 from the negative direction it goes to 4. You get the same here, so this is small, this cancels, so for these limits of this function, you can simplify down this one and can test form either direction. From positive it's 4, from negative it's 4, so since limits are equal the limit exists at 4. *(Turn 28, 56:40, SDV_0050.mp4)*

R: Where would the limit be on the graph of this line x+2?

N: At the hole. *Turn 30, 1:01:15, SDV_0050.mp4.*

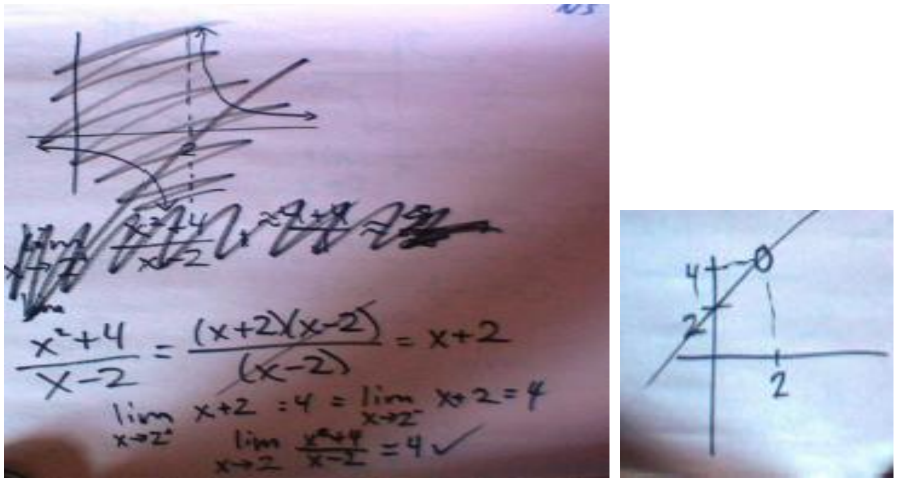


Figure D.2.21: Nicole's work on Task 3.

Analysis:

Inference: Nicole incorrectly factored the numerator as the difference of squares so her graph was wrong. However, she did know that common factors in the numerator and denominator could generate a hole rather than a vertical asymptote. After she divided out the common factors, she plugged in 2 and got a limit of 4, so correctly re-drew the graph with a hole. After the initial error of incorrectly factoring the numerator, the rest of her steps and strategies were correct.

Hypothesis: Nicole didn't apparently acknowledge the plus sign in the numerator. Thinking it was a minus sign, she factored it. With more careful inspection of the task at hand, she could have done a simple procedure to check if the numerator could be factored, so she failed to utilize algebraic tools to demonstrate an understanding the relationship between roots and factors.

Compute the limit if it exists. Explain the behavior of the function values near x= -3.

$$\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$$

Figure D.2.22: Task 4 Problem Statement

R: Can you compute the limit and sketch the graph as x approaches -3.

N: There's a hole here at (0, -3). When you cancel you get a hole. Want limit as x approaches -3. So there is a hole but darn it, I am brain frozen. Hole at -3.

R: Does the limit exist as x approaches -3?

N: It's at y=-6. *Turn 35, 1:06:25, SDV_0050.mp4*.

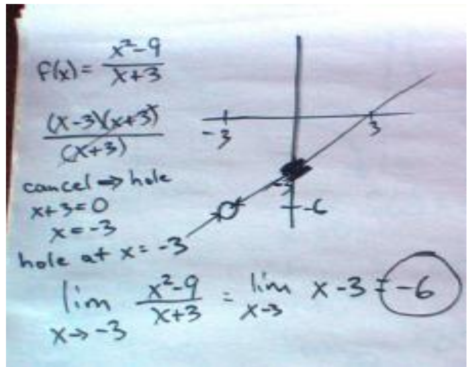


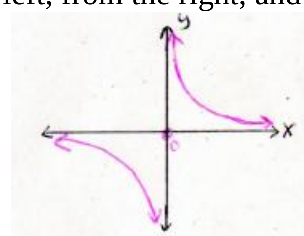
Figure D.2.23: Nicole's work on Task 4.

Analysis:

Inference: Nicole uses the layman's term "cancel" instead of the mathematical term, "divide". Confusion occurred with this problem at first when she thought the -3 was on the y-axis instead of the on the x-axis, as can be seen by the darkened dot above that she ultimately tried to correct. However, she knew that dividing out common factors could generate a hole, rather than a vertical asymptote in this case. After she caught her mistake she plugged in the -3 for x and correctly computed -6 for y, and then fixed this result on the graph.

Hypothesis: Translating the x and y information from the task to the graph can be a source of difficulty, but Nicole demonstrates an understanding of when common factors divide out and generate holes instead of solid dots in the graph or vertical asymptotes.

Compute the limit. Explain how the function values behave near 0. Does the limit exist as x approaches 0 from the left, from the right, and from both sides together?



$$\lim_{x\to 0}\frac{1}{x}$$

Figure D.2.24: Task 5 Problem Statement

N: As x approaches 0, the limits do not exist because on the right it goes to positive infinity, and left it goes to negative infinity and since positive infinity is not equal to negative infinity, then we say the limits don't exist. Looking at each one separately from the left and right, the limits do exist and equal infinity. If the arrows both went up in the same direction like with x^2, then we'd say it the limit would exist because it equals infinity on both sides. *Turn 2, 1:00, SDV_0051.mp4*

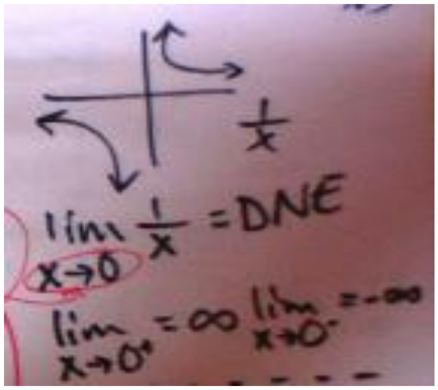


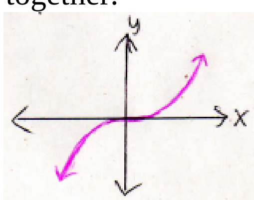
Figure D.2.25: Nicole's work on Task 4.

Analysis:

Inference: Nicole correctly said the limit does not exist for the entire problem as x approaches 0, but for the wrong reason. She incorrectly compared the left hand side to the right hand side or $-\infty \neq +\infty$. Later she wrongly stated that the limit exists separately in each direction and that the limit equals infinity. She seems to think that infinity is a number.

Hypothesis: Students do not focus on the behavior of the function values getting larger in either the positive or negative directions; otherwise, they would say the limit does not exist because the function values were getting larger. It might be that seeing an equal sign next to infinity makes students conclude that $=\infty$ is synonymous with being equal to a number, which would be incorrect. Incorrect instruction on the topic of when limits exist and do not exist might be an underlying casue.

First, can you tell what kind of function this is, then explain how the function values behave near 0. Explain if the limit exists as x approaches 0 from the left, from the right, and from both sides together.



$f(x) = ?$ $\lim_{x \to 0} f(x) = ?$

Figure D.2.26: Problem Statement Task 6

R: Look at this graph. Identify the type of function this is and explain if the limit exists as x approaches 0.

N: As x goes to zero from both sides, the limit is zero because the left side equals the right side and it's also equal to the value of the function there. *Turn 42, 01:40, SDV_0051.mp4*

Analysis:

Inference: Nicole identified this as a cubic function, which suggests knowledge of the elementary functions. She looked at the behavior of function values as x approached 0 from the left and right and ended up at the exact same location on the graph and correctly determined the limit as being equal to 0.

Hypothesis: Students will know the limit if they compare the left hand limit to the right hand limit, being x-cubed is continuous.

Look at the following graphs and explain if the limits exist as x approaches 3 from the left and the right.

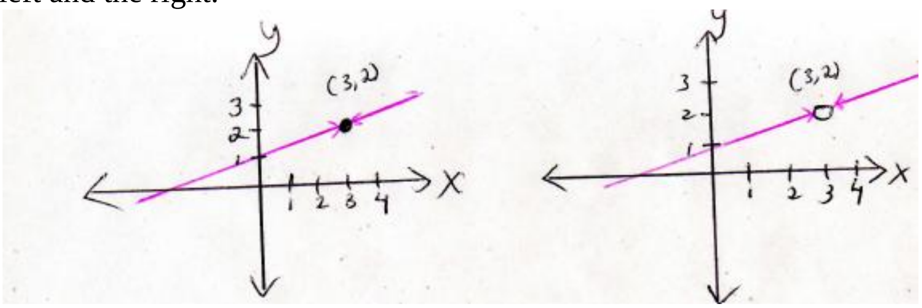


Figure D.2.27: Task 7 Problem Statement.

N: We look at continuity. In the first case, the limit exists. As x approaches 3 from the left and right is equal to 2 because the left side equals the right side. In the second case, the limit also exists for the same nearness reason, the left side and right side are approaching 2 for y. *Turn 139, 32:40, SDV_0051.mp4*

Analysis:

Inference: Nicole correctly identified the limits existing at a point in both cases and compared the left hand side to the right hand side.

Hypothesis: Some students may understand the relationship between limits and continuity, that a limit can exist even though a function is discontinuous at a point, as long as the left and right hand sides are approaching the same number.

What kind of function is this? Explain if the limit exists as x approaches 2 from the left and from the right.

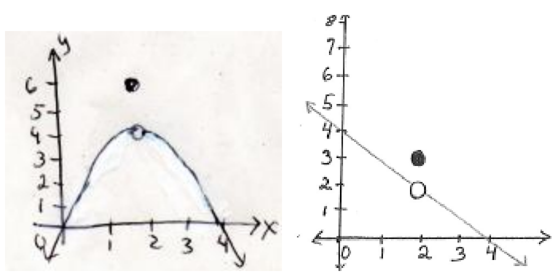


Figure D.2.28: Task 8 Problem Statement

N: They're both piecewise. The limit is 4 for the one on the left because the left side equals the right side at y=4 because as you head toward 2 for x, you head toward 4 for y. The limit is 2 on the second one because the left side equals the right side at y=2. This points on the graphs (2,6) and (2,3) are both on the graphs of these functions. The limit tells what the function is approaching not what it equals. The point tells you what the function equals. *Turn 39, 1:09:45, SDV_0050.mp4*

Analysis:

Inference: Nicole correctly identifies these graphs as piecewise and knew that the limit is not necessarily equal to the value of the function.

Hypothesis: She knows that limits are about what happens near a point. She seems to understand that holes mean discontinuities at that point and so the solid dot above or below a point is the value of the function.

What type of function is this? Explain if the limit exists as x approaches 2.

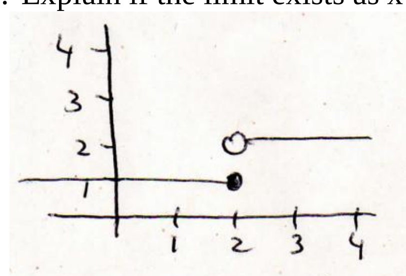


Figure D.2.29: Task 9 Problem Statement.

N: This is a piecewise function. The limit does not exist at x=2, and that's due to thefact that it is discontinuous, there is a jump here and as we are taught to check specifically the limit as x approaches 2 from negative side is 1, and limit as x approaches 2 from positive direction is 2, and since they're not equal, it verifies that this is undefined. So jump in graph, it's discontinuous, and when it's discontinuous the limit does not exist. *Turn 8, 42:25, SDV_0050.mp4*

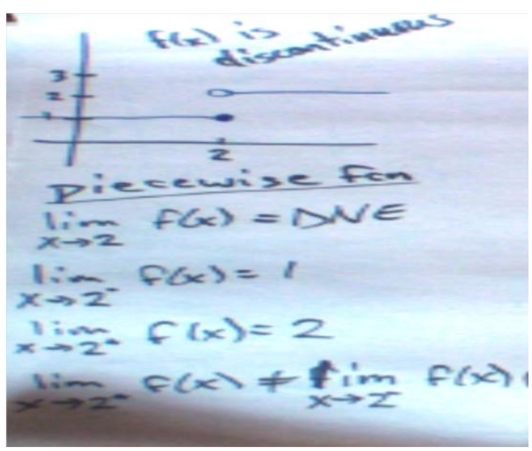


Figure D.2.30: Details exhibited of the limits of the piecewise function.

Analysis:

Inference: Nicole correctly identified the function as piecewise and concluded that the limit did not exist because the left hand side did not equal the right hand side. She acknowledges the discontinuity at x=2. She even sited what the limits were in each case, from the left and right. However, her last statement is not correct "when it's discontinuous, the limit doesn't exist." Task 7 shows this whereby the limit could exist but is not equal to the value of the function.

Hypothesis: Nicole appears to possess content knowledge and mastery of piecewise functions which facilitates her ability to identify discontinuities and her subsequent reasons for why the limit does not exist for the given function. Her last statement was incorrect because she probably did not think long enough about finding a counter example to refute her claim. If she had, then she could have recalled Task 7 in which the limit was not equal to the value of the function. Perhaps limits involve a lot of information that can potentially become confusing.

Given a restricted domain, explain if the one-sided limit exists as x approaches 2 from the left.

$$D:\{x \mid 0 \le x < 2\} \qquad \lim_{x \to 2^-} f(x)$$

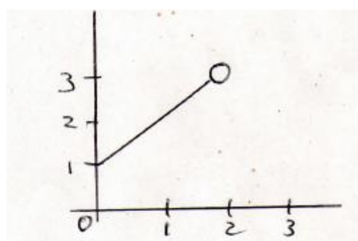


Figure D.2.31: Task 10 Problem Statement.

N: I never saw one like this before. As x approaches 2 from the left, I would say that yes, it does exist because of the domain restriction. We don't care what's going on for x greater than 2. The only thing we care about for this function since it is defined only up until 2, then can it possibly approach at 2 and that f(x)=3, so I would say that it does exist. (*Turn 26, 54:20, SDV_0050.mp4)*

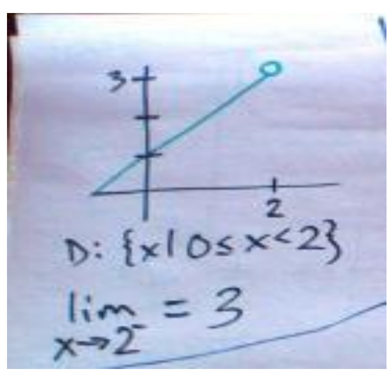


Figure D.2.32: Nicole’s work on Task 10.

Analysis:

Inference: Nicole was correct with saying he limit existed.

Hypothesis: If students understand domains, then they will see that because of this restricted domain, the limit of the function as x approaches 2 from the left occurs.

Nicole was presented with a constant function. She had to determine what the limit was of the constant function 5 as x approached 3.

What is the limit as x approaches 3? Study the following answer choices and explain your reasoning.

$$\lim_{x \to 3}(5) = ? \quad a.\ 3 \quad b.\ 5 \quad c.\ 0 \quad d.\ \infty$$

Figure D.2.33: Task 11 Problem Statement.

R: What is the limit as x approaches 3? Explain your reasoning.

N: The 5 is basically the f(x) and f(x) is really the y, so y is 5. It is a straight horizontal line drawn at 5. So the limit as x approaches 3 or any number for x would just be 5. So it is answer choice B. *Turn 58, 50:15, SDV_0049.mp4.*

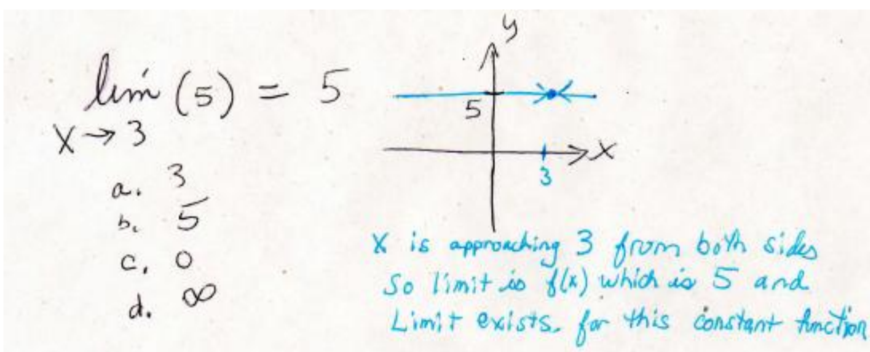


Figure D.2.34: Nicole's work on Task 11.

Analysis:

Inference: Nicole was able to work with this problem even though there were no variables. She associated f(x) with the y-value and identified it as a constant function. This is typically a confusing question for most students because there is nothing to plug in or compute.

Hypothesis: She knows that the task involves a limit of a constant function and didn't get confused because she knows to first look at what x is doing, and then study the behavior of the function values. She drew a horizontal line y=5 because that is the graph. The constant function's behavior was the same as x approached 3 from the left and from the right. She generalized for any value of x, the value of y would remain constant at 5.

What type of function is this? Construct its graph. Explain if the limit exists as x approaches 3.

$$f(x)=\begin{cases}5x+2 & \text{if x} \neq 3\\ 20 & \text{if x=3}\end{cases} \qquad \lim_{x\to 3} f(x)=?$$

Figure D.2.35: Task 12 Problem Statement.

N: This x is 3, y is 17. Common mistake to put the function value at 3. You want the behavior of the function as it approaches 3, not at 3. Here is goes what the line would have equaled at 17. So as you approach 3 for x, you head toward 17 for y, Each of these values get close to 17 from either direction. There is nothing else going to 20. Limit means what is it approaching not what is it equal. There are no other points around 20 defined by the function. This point is on the graph of this function (3,20) which makes this piecewise but limit tells what the function is approaching not what it equals. (*Turn 37, 1:07:15, SDV_0050*)

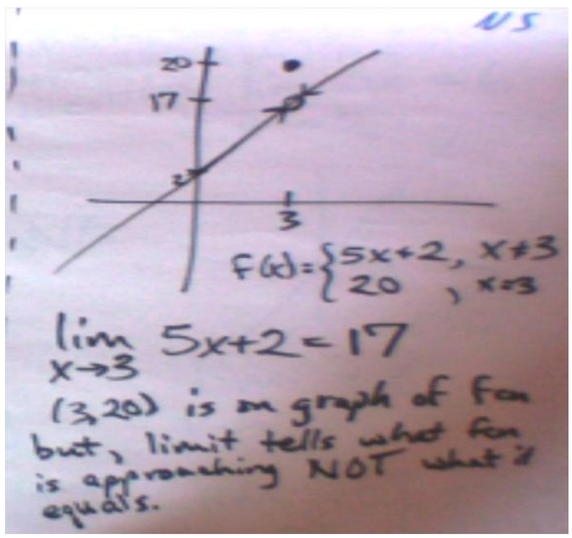


Figure D.2.36: Nicole’s work on Task 12.

Analysis:

Inference: Nicole correctly indicated that the point (3,20) was on the graph of the function and that the limit was 17, but was not equal to the value of the function, 20.

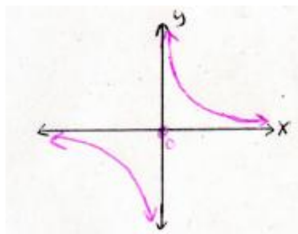
Hypothesis: Students must possess proficiency with understanding the graphs of piecewise functions. They must understand what it means when there is a discontinuity, which is that the limit is not equal to the value of the function.

Summary

Twelve tasks were presented on limits at a point. She consistently demonstrated facility with various piecewise functions which might explain her overall good understanding of limits at a point. Nicole has most of the necessary algebraic skills, despite a few errors such as trying to factor a sum of two squares. One notable mistake she makes is thinking that infinity is a number, which is why she thinks a limit exists when it equals infinity. This particular situation of limits being equal to infinity are taken up separately in subsequent interviews to explore that misunderstanding more in depth. Although she does not articulate much about what her results mean, she thoroughly explains her thinking with detailed steps.

Analysis of Interview on Limits at Infinity: Nicole

Compute the limit as x gets larger in the positive and negative directions. Explain whether or not the limit exists as x approaches positive and negative infinity.



$$\lim_{x\to\infty}\frac{1}{x} \qquad \lim_{x\to -\infty}\frac{1}{x}$$

Figure D.2.37: Task 1 Problem Statement.

R: Look at the graph below of the function 1/x and tell me if the limits exist.

N: As x goes to positive infinity, I look at the graph and see it goes to 0. One divided by a very large number is a very small number so 1/x keeps shrinking. The whole fraction shrinks. So the limit exists and it is equal to zero as x goes to both positive and negative infinity. (*Turn 40, 00:30, SDV_0051.mp4)*

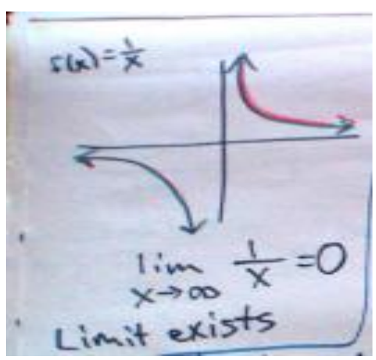


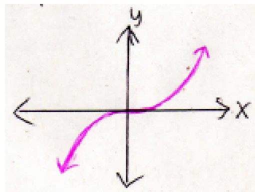
Figure D.2.38: Nicole's work on Task 1.

Analysis:

Inference: Nicole visually inspected the graph but also used mathematical rationale for her explanation. She referred to the quotient as a fraction.

Hypothesis: Students must consider the behavior of the function values rather than watch the points on the graph, in order to see the function values are getting smaller tending to zero.

Recall this type of function. Describe the limiting behavior as x gets larger in the positive and negative directions. Explain whether or not the limit exists as x approaches positive and negative infinity.



$\lim_{x\to\infty} f(x) \qquad \lim_{x\to -\infty} f(x)$

Figure D.2.39: Task 2 Problem Statement.

R: Look at this graph. Identify the type of function this is and explain if the limit exists as x goes to both positive infinity and negative infinity.

N: This looks like x cubed. Anyway it's an odd function. The limit as x goes to infinity of x cubed goes to plus and minus infinity. It is defined for every x value so if you plug ones larger and larger in both directions, the function values keep getting larger in both directions so the limit does exist separately in each direction and it's positive infinity on the right and negative infinity on the left. On the left, the cube of a negative number is a negative number and so the function value will become a smaller and smaller number. So since the function values go to negative infinity, then the limit exists and it is negative infinity. Any positive number cubed is positive. The limit exists and is positive infinity. So the function goes to positive infinity on the right side.(*Turn 42, 01:40, SDV_0051.mp4)*

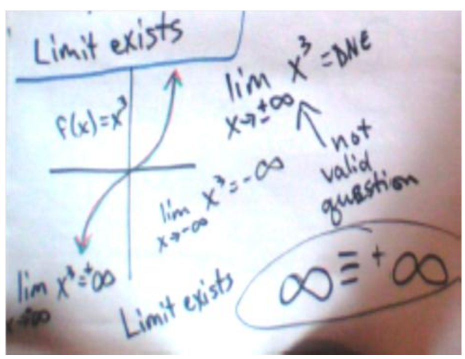


Figure D.2.40: Nicole's work on Task 2.

R: What if we re-write this as the "limit of x cubed as x goes to plus or minus infinity**,** then would the limit exist?

N: No, it wouldn't exist because you are looking for the limits at 2 different x values in 2 opposite directions so they would not exist. One arrow goes right to plus infinity, and the other arrow goes left to negative infinity, so left doesn't equal right so limit doesn't exist. But if the function was $\lim_{x \to 0} \frac{1}{x^2}$ 1/x squared, and both the arrows went up to infinity in the same direction, they would be approaching the same number. So in that case then the limits would equal infinity and so they WOULD exist. (*Turn 44)*

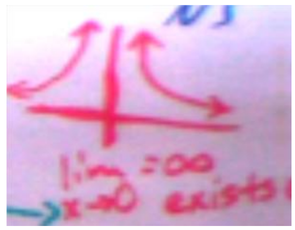


Figure D.2.41: Nicole's work on Task 2 with incorrect conclusion for $\lim_{x \to 0} \frac{1}{x^2}$.

N: In my opinion, professors shouldn't write (plus/minus) infinity in front of the infinity symbol. They should be specific and break it up where it's clear you get 2 different answers, in fact, answers which are opposite. See with 1/x-squared if x goes to positive or negative infinity the function values go to zero, the same answer. But as x goes to zero, the values both are going to the same place, up to infinity so the limit would exist and equal infinity. As a student, I shouldn't be required to think about splitting these up into 2 separate problems, but even the books use this notation with the plus/minus in front of infinity and I think it's wrong. I see this all the time. (*Turn 45, 04:36, SDV_0051.mp4*)

Analysis:

Inference: Nicole interchanges the terms "function" and "function values" and uses the word "different values" vaguely. She also uses the word "values" with a sense of uncertainty, so it seems she may have a vague understanding of mathematical language. Her responses were consistently incorrect, that the limits exist separately in each direction. The limit did not exist because the function values kept getting larger in both the positive and negative directions.

Nicole stated that the limit would not exist if x approached plus/minus infinity, but gave the wrong reason. She incorrectly thinks that the limit exists when the limit equals infinity because she thinks infinity is a number being compared. She incorrectly compared the left hand side to the right hand side, which is only done for limits at a point, to determine if the limit exists. Since she determined that the left hand side did not equal the right hand side, or $-\infty \neq +\infty$, she concluded the limit does not exist. When evaluating problems involving limits at infinity, the left hand side should not be compared with the right hand side. As a point of clarification, the limit as x goes to plus infinity, where x moves to the right, is not the same as a right-hand limit where x moves to the left. In any event, comparing the two sides is totally irrelevant to these kinds of problems. Nicole clearly confuses "smaller" with "lesser." Apparently she thinks that the more negative a number is, the smaller it is.

Nicole expressed concerns about writing the (plus/minus) in front of the infinity symbol and does not think students should be responsible on their own to split these up. In some cases, plus/minus is legitimate and exists throughout the literature and in textbooks. This should not be done for a function like $f(x)=e^x$ because you get two different results in each direction. The proposed misconception was not knowing that each side is to be considered separately and that +/- infinity means, by definition, to separate the problem into two problems. So in this problem, as x approaches plus or minus infinity, she concluded that the limit does not exist.

Nicole showed evidence of not knowing the meaning of the +/- notation in front of the infinity symbol. It means the logical "exclusive or", so it is perfectly acceptable to use this notation if it is explained in context.

Hypothesis: Students may not understand the difference between a function, "what you do" and function values "what you get." Many may compare the left hand side with the right hand side regardless if the problem involves a limit at a point or a limit at infinity. Some students may also perceive infinity as a number of place where something goes and might not understand the logic that understand what $\pm$ means in terms of "or" not "and", which is why some might say $\pm$ should not be written together, though this is standard textbook notation.

Describe the limiting behavior as x approaches plus or minus infinity and explain whether or not the limit exists. $\lim_{x\to\infty}\frac{3x}{2x^2+1}$ $\lim_{x\to-\infty}\frac{3x}{2x^2+1}$

Figure D.2.42: Task 3 Problem Statement:

N: We can see the denominator is raised to a larger power than numerator, so this limit should be zero. This is the easiest one you do when you test infinite limits because you just compare the powers of the x's. If you have a larger power in the denominator then the limit is zero. The numerator will be infinity. If the powers are equal, then you take the fractional coefficients to be the limit. *Turn 54, 13:00.*

R: Do the limits at infinity have anything to do with the horizontal asymptotes then? *(Turn 55, 13:53, SDV_0051)*

N: It's similar. In fact, you apply the same rules. You do $2x^2+1>3x$ and divide both sides by 2x squared plus 1 you switch the sign and get numbers very small less than 1, and so the limit of this function has to be 0. The double less than signs << means the fraction on the left of it is going to zero. About asymptotic behavior, if the power is higher in the denominator then the limit of the function values go to zero. If the powers are the same, the asymptote or limit is the fractional coefficient. Say limit of $\frac{ax^3}{bx^3}$ it is going to be $\frac{a}{b}$. (*Turns 56 & 57, 13:56)*

R: What happens if you get a larger power in the numerator?

N: The limit goes to infinity. An example is 3x cubed/5x squared. Look at 5. The numerator when x is 5 and you get 125 over 25, so the numerator as x goes to infinity of 3x cubed is greater than denominator so values get larger and larger which means the whole function or the limit is going to infinity. *Turn 59, 16:20.*

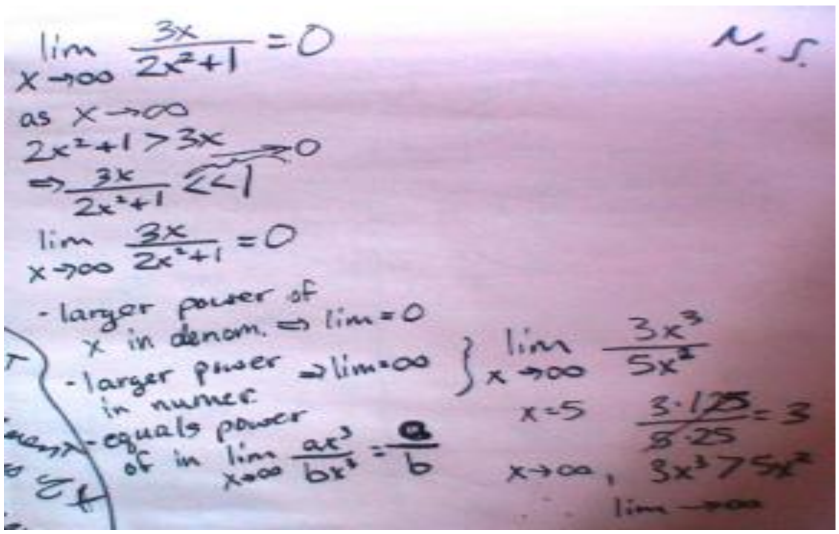


Figure D.2.43: Nicole's work on Task 3.

Analysis:

Inference: Nicole demonstrated proficiency and mastery and understood why the limit would be equal to zero. She made the connection between the horizontal asymptote and the limit under the conditions in which the powers of the leading terms were the same on the top and on the bottom. She was able to articulate the rules as well as show numerically how to determine the limit of this function.

Hypothesis: Proficiency with algebra facilitates success with this task.

Describe the limiting behavior as x approaches plus or minus infinity and explain whether or not the limit exists.

$$\lim_{x\to\infty}(1+\tfrac{1}{x})^x \qquad \lim_{x\to -\infty}(1+\tfrac{1}{x})^x \text{ or } \lim_{x\to\pm\infty}(1+\frac{1}{x})^x$$

Figure D.2.44: Task 4 Problem Statement.

N: As x gets large, this goes to 0 so you essentially get 1 here. As x goes to infinity, 1/x gets negligible and so 1+0 raised to a large power is simply 1. But lets double check with the calculator. Maybe I'm way off. It goes to 3 on the calculator. I'll draw the graph. It approaches 3 as x goes to infinity. The HA is at 3 and it never crosses this line so as values get larger and larger but never equals 3 but approaches it. Same as x approaches negative infinity. It approaches 3 but never crosses 3. On the calculator, I graphed the function and it to see what value the HA is at I plugged in a large number and you can do it also without the graph. So (1+1/10,000) raised to the 10,000 and it's about "e" as a matter of fact. The 3 is wrong. So the answer is the constant e not 3 but it was close. e is pretty close to 3. For infinite limits, you just plug in a large number and see what you get. *Turn 81, 17:15, SDV_0051*

R: Can you describe the behavior of the function values as x gets larger for $\lim_{x\to\pm\infty}(1+\frac{1}{x})^x$?

N: The behavior of the function values? As x increases, they approach e. And as it goes to negative infinity, they also approach e, I am fairly certain but I'll check on the calculator. It's true on the other side as well. (*Turn 83, 22:05, SDV_0051)*

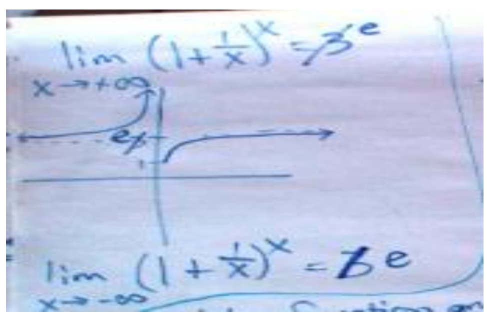


Figure D.2.45: Nicole's work on Task 4, asymptotic behavior of the function.

Analysis:

Inference: Nicole demonstrated the ability to work with this function both by plugging in large value of x and then by checking her results with a calculator. Her prediction was correct that the limiting behavior would be the same on both sides and when there were doubts she proceeded to further check her results.

Hypothesis: A student can successfully navigate through this task because she seems to have the necessary basic mathematical and algebra tools.

Particle Problem: Given s(t) represents the position of a particle at time "t", sketch a graph and describe the limiting behavior as t approaches positive infinity for s(t)=$1-e^{-t}$.

$$\lim_{x\to\infty}(1-e^{-t})$$

Figure D.2.46: Task 5 Problem Statement.

N: For very large values of t, we are raising e to very large values of t to a negative power. This essentially means the same as $\frac{1}{e^t}$ so take large values of t, you get a very small fraction and is essentially therefore negligible so this is approximately the limit as t approaches positive infinity of $(1-e^{-t})$ which is the limit as t goes to positive infinity equals 1. This is because 1 minus a negligible number or very small number near zero, so 1-0=1. So 1 is your answer. It approaches 1 mile but never reaches it so the limit exists and is 1. *Turn 94, 1:00, SDV_0055.mp4.*

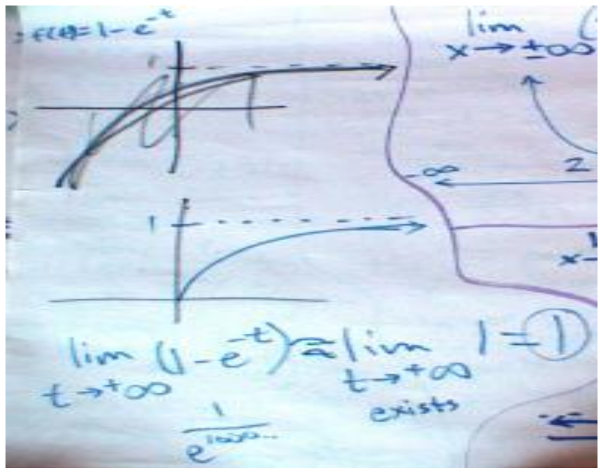


Figure D.2.47: Nicole's work on Task 5 Particle Problem.

Analysis:

Inference: Nicole made the negative exponent positive by moving the exponential from the numerator to the denominator, in which case the negative exponent becomes positive. This expedited her ability to work on the problem. She revealed an understanding that as the denominator got larger, the second term would tend to zero which is why when zero was subtracted, the result was 1. Nicole did not explain what the problem meant after she derived the correct result. She uses the word "negligible" which appears to mean zero. Nicole articulated how to solve the problem and had number sense, but did not interpret or explain how the particle was slowing down as time increased, probably because her focus is on getting the answer than on what the answer means.

Hypothesis: Students need number sense and an understanding of reciprocal functions.

Describe the limiting behavior of the function below as x gets larger in the positive and negative directions. Sketch a graph of the result.

$$\lim_{x\to\infty}(5x^2+2) \qquad \lim_{x\to -\infty}(5x^2+2)$$

Figure D.2.48: Problem Statement Task 6.

N: As x goes to positive and negative infinity, the limit exists and it is positive infinity. The y intercept is at 2 and first term's squared so you shift up vertically 2 units. You split them up into 2 parts. I'd be annoyed with the teacher for writing x approaches +/- infinity but it is permissible since the limit at positive or negative infinity is the same value, positive infinity. In this problem, the limit exists and is equal to positive infinity. (*Turn 99, 3:09, SDV_0055.mp4)*

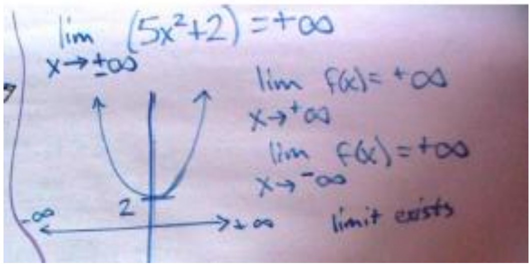


Figure D.2.49: Nicole's work on Task 6.

Analysis:

Inference: Nicole revealed how she could easily identify the y-intercept and explain the vertical shift. She constructed the graph from scratch and described the end behaviors, but thought it was not necessary to split the problem into two parts. She was clearly irritated with how professors write the +/- notation. Her interpretation was wrong about the limit existing and being equal to infinity.

Hypothesis: Students need mastery and proficiency with algebra including basic tools to identify the y-intercept and describe shifts and translations. This also includes understanding the shape of a quadratic's graph with a positive first term.

Describe the limiting behavior of the functions. Find any horizontal asymptotes, vertical asymptotes, holes and limits. Graph the function and explain the limiting behavior as x approaches infinity and as x approaches 1. $\lim_{x\to\infty}\frac{2x+1}{x-1}$ $\lim_{x\to 1}\frac{2x+1}{x-1}$

Figure D.2.50: Task 7 Problem Statement.

N: For large x as x approaches infinity, you have infinity over infinity and the powers are the same and cancel so you get 2 as a horizontal asymptote. I would say 2 is the limit and that it exists. For the limit as x goes to 1, you can't plug in b/c you get undefined in the denominator. But for numbers close to 1, the denominator gets small and the numerator gets close to 3, so that's possible. To find a VA you set the denominator equal to zero so you get a VA at x=1. The question is where it shoots of. For values greater than 1, and values less than 1, I'd have to find out. Might have a negative over a negative, so it's still positive infinity so I'll have to check the graph of that. For a horizontal asymptote, as x goes to infinity but it approaches a value, but in this case you either evaluate it or graph it. Nothing cancels in numerator or denominator so there are no holes. Now I put it in the calculator. *Turn 102, 5:00, SDV_0055.mp4*

R: Can you explain more about what you said with the numerator and denominator doesn't cancel. When would something cancel and why?

N: If you had (x-1) in the numerator and (x-1) in the denominator when you have things that can cancel like this, it is suggestive of a hole. But we don't have a hole in this particular problem, obviously, because nothing cancelled. Here is what the graph should look like for this with a HA at 2 and VA at 1. It turns out that as x approaches 1, the function approaches negative infinity. *Turn 104, 7:42, SDV_0055.mp4*

R: How did you know to draw a horizontal asymptote at 2?

N: When you have equal powers in the denominator and numerator you compare their coefficients so at y at 2 there is a HA. So the value as x approaches infinity is 2 and so the limit exists at 2**,** but as x approaches 1 the limit does not exist because the left side goes in a different direction than the right side which means that positive infinity is a different value than negative infinity. From left $-\infty \neq \infty$ from right. *Turn 106, 9:03, SDV_0055.mp4*

R: So the limit does not exist because the left side which approaches negative infinity does not equal the right side which approaches positive infinity.

N: Yes, that's it. *Turn 108, 10:28, SDV_0055.mp4*.

R: Say the denominator was squared and now both arrows or limits at x=1 are positive infinity. Then you'd say that the limit exists?

N: Yes, because when they go in the same direction, they approach the same val positive infinity and that's why the limit would exist and would be positive infinity. Say the denominator was squared and now both arrows went toward the y-axis in the same direction, then you'd say that the limit exists? *Turn 110, 10:28, SDV_0055.mp4.*

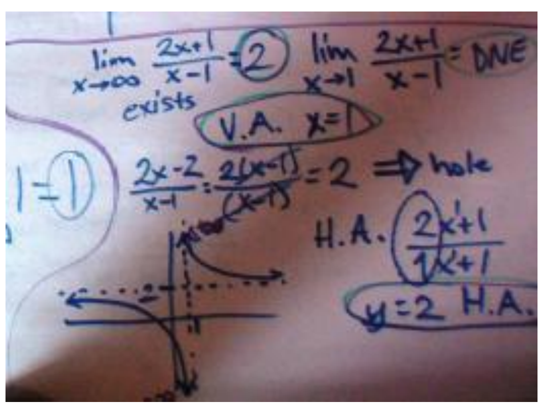


Figure D.2.51: Nicole's work on Task 7.

Analysis:

Inference: Nicole divided infinity over infinity to get a number for the horizontal asymptote. The procedure itself for dividing out the highest powers of x from the leading terms was correct. However, division cannot be done with infinity because infinity is not a number. She uses the everyday word "cancel" instead of the mathematical operation "divide." She said this initially about canceling infinity then again when looking for holes in the graph at which time she said nothing canceled in the numerator and denominator. She knew what to look for to find holes, but what she did not articulate well was that she was looking for common factors to divide out because sometimes dividing out common factors in the limit removes the zero in the denominator and leaves holes in the graph.

She has good number sense to know how the function behaves when numbers close to 1 get plugged into the denominator and articulates how the function might behave for values less than and greater than 1. Next, she correctly determined that the horizontal asymptote and limit was equal to 2 as x approached plus or minus infinity, and also that the limit did not exist for the limit as x approached 1. However, she provided the wrong reason for the latter, stating that the left hand side, minus infinity) did not equal the right hand side (plus infinity), so the limit did not exist, but that for each piece the limit existed and was equal to either plus infinity or minus infinity. Evidence shows consistency with this misunderstood notion that the left hand side has to equal the right hand side even if the limits don't exist. Even more skilled students and instructors make this mistake. When asked about a problem in which the denominator was squared making the arrows go up toward infinity, she stated that the limit exists because the arrows go in the same direction and approach the "same positive value, infinity."

Hypothesis: Other students may think infinity is a number, and will therefore attempt to do mathematical operations with it.

Describe the limiting behaviors. Sketch the graph and explain how the function behaves for large x in the positive and negative directions. Do the limits exist?

$$\lim_{x\to\infty}\frac{3x^3-x^2-3}{2x+3} \qquad \lim_{x\to-\infty}\frac{3x^3-x^2-3}{2x+3}$$

Figure D.2.52: Task 8 Problem Statement.

N: I'm glad these are split up this time into approaching positive and negative infinity. In the denominator you have 2 infinity, and 3 infinity cubed over 2 infinity. I would say the limit is gonna be infinity since you have a higher power in the numerator. And in the 2nd one, it's negative infinity. I'll draw the graph though using the calculator. So for the first one, it's positive infinity as x approaches positive infinity. For the second one as x goes to negative infinity, it goes to positive infinity. I don't know if it loops or shoots back up. If you set the denom. equal to 0 gives a VA at 3/2 so those just shoot down there but these curve up. On the left, it continues to go up and outwards but starts out very large and grows very fast. -3/2 is -1.5 so it jumps from positive to negative values. If you try to calculate -1.5 it is undefined, so it must be a vertical asymptote there after all. There is no hole though because nothing cancels. It's defined for values greater than 1.5. *Turn 116:16:00, SDV_0055.*

R: Are there any horizontal asymptotes with this?

N: No because there's a higher power in the numerator.

R: Is there vertical asymptote at negative 3/2?

N: No. *Turn 115,15:30,SDV_0055*.

R: Are there any other kinds of asymptotes other than vertical or horizontal that could be associated with this graph?

N: It doesn't look like there are any. *Turn 124:18:20, SDV_0055.*

R: Do any limits exist then as x approaches +/- infinity?

N: The limits separately exist for both functions. For the first, the limit exists and positive infinity and for the second problem, the limit exists and is positive infinity. So together the limit exists and equals positive infinity. (*Turn 126)*

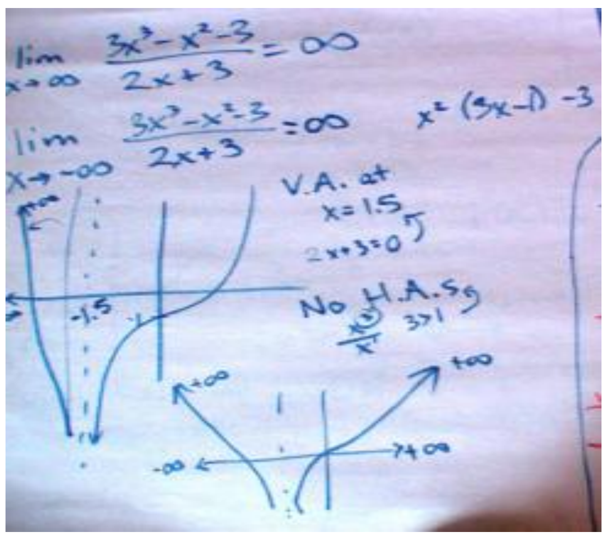


Figure D.2.53: Nicole's work on Task 8.

Analysis:

Inference: Nicole prefers limits being split up into two separate problems. She multiplied and divided with infinity as a short cut to get the answer. A computation error occurred with the denominator where she got a vertical asymptote of 3/2 instead of -3/2, but she corrected her mistake. Although she correctly indicated there would be no horizontal asymptotes and there would be a vertical asymptote at x=-3/2, she did not mention the parabolic asymptote with this function. Nicole incorrectly compared the left side of the graph with the right side of the graph to determine if the limit existed. Then she correctly stated that the limit equals infinity but incorrectly stated that the limit exists. The reason the limit does not exist is because the function values keep increasing without bound in as x approaches both positive and negative infinity.

Hypothesis: Students might know to use long division of polynomials to determine the asymptotic behavior. Algebra proficiency with factoring out the highest power of x separately from the numerator and denominator would facilitate success with this problem, as would not comparing the left hand side of the graph with the right hand side to determine if the limit exists.

Describe the limiting behaviors. Sketch the graph and explain how the function behaves for large x in the positive and negative directions. Do the limits exist?

$$\lim_{x\to\pm\infty}\frac{3x^2+2}{9x^2-2x+5} \quad \text{or} \quad \lim_{x\to\pm\infty}\frac{9x^2+2}{3x^2-2x+5}$$

Figure D.2.54: Task 9 Problem Statement.

N: I'm glad these are split up again into 2 problems**.** For this one, the HA is 1/3 since you are raising infinity to the same power. Same for negative infinity since you are squaring negative values they come out positive. (Uses calculator) There is a HA at 1/3, comparing the degrees of the numerator and denominator, you take the fractional coefficients. *Turn 128, 21:00, SDV_0055.*

R: So is the HA of 1/3 the limit?

N: No, not really b/c horizontal asymptotes are unlike VA's because the graph can cross them (HA) and in this one, the limit seems to be going to zero. (*Turn 135,18:15)*

R: So can graphs cross horizontal asymptotes?

N: Yes. Graphs can cross HA's but not VA's. Yeah, it looks like the limit as x approaches positive infinity is going to zero. It keeps shrinking. It still looks like it wants to be 1/3 but it crosses and keeps going toward zero. It follows a HA for a length of time but eventually passes it so it goes to zero. Wait, I forgot to square the numerator. I did this wrong. It really does approach 1/3 so the limit exists and is 1/3. I thought it was weird going to zero. In general though, graphs can still cross HA's and so therefore you cannot assume that the HA is going to be the limit even though the powers are the same. *Turns 138 & 139, 30:25, SDV_0055*

R: Is there an association between limits at infinity and horizontal asymptotes?

N: No not always. Teachers don't tend to spend much time on infinite limits so that's why most people don't know about connections between limits and horizontal asymptotes, and under what conditions the horizontal asymptote turns out to also the limit and when it doesn't. In high school they spend about a month and in college a day or so, then the topic is hardly revisited. Once they get through limits, they don't come back. Many students don't ask why their answer is wrong, just how do you get the right answer and then forget about it. (*Turn 141)*

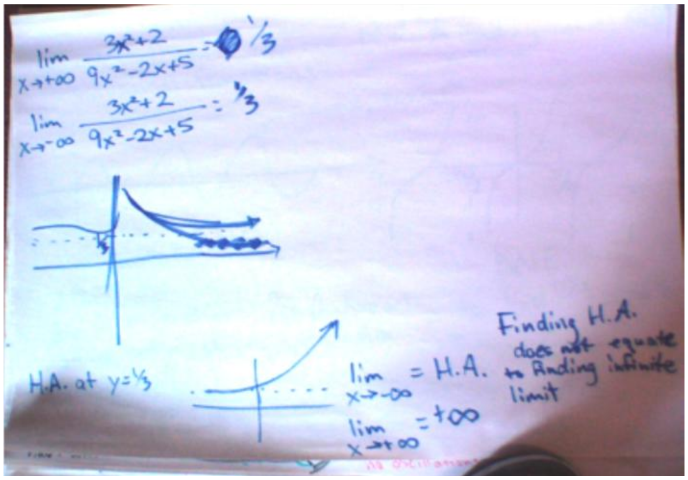


Figure D.2.55: Nicole's work on Task 9.

Analysis:

Inference: Nicole correctly identified the horizontal asymptote as also being the limit, knowing the degrees of the leading terms are the same in the numerator and denominator. After catching the mistake with squaring the numerator, she found the horizontal asymptote 1/3 which was also the limit. She knows other types of functions, such as exponential, can yield asymptotic behavior only on one side, given experience working with a lot of different types of functions. Nicole knows when a limit is the horizontal asymptote and when it is not.

Hypothesis: Knowing the rules about finding asymptotes would facilitate identifying when the limit is equal to the horizontal asymptote.

Study the graph below and describe the behavior of the function values for large x in the positive and negative directions. Explain how the function values behave near x=2.

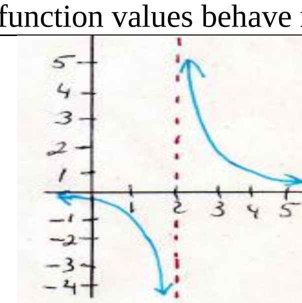
$f(x) = ?$ $\lim_{x\to\infty} f(x) = ?$ $\lim_{x\to -\infty} f(x) = ?$ $\lim_{x\to 2} f(x) = ?$	

Figure D.2.56: Task 10 Problem Statement.

N: This looks like 1/x translated 2 units to the right, so it's 1/(x-2). As x approaches plus infinity, the values get smaller and smaller so the limit exists and is 0. As x approaches negative infinity going left, the values also get smaller so they go to 0. There is a horizontal asymptote at 0. And you can see this b/c the degree of the numerator is x to the 0 power, which is less than the degree of the denominator x to the 1st power. So that's how you can check. But as x approaches 2 from both sides, the limit does not exist. Separately though they exist as x approaches 2 from the right the limit exists and equals positive infinity and as x approaches 2 from the left, the limit exists which is negative infinity. (*Turn 145, 34:50)*

Analysis:

Inference: Nicole was able to correctly extract the information from the graph but once again stated incorrect conclusions about the limits existing when, in fact, they didn't exist.

Hypothesis: Nicole has proficiency with horizontal shifts, so she was able to identify the function correctly. Because she thinks that infinity is a number, she continues to claim that the limit exists at infinity. She also incorrectly concludes that the limit doesn't exist at x=2 but for the wrong reason. She thinks that since the left hand side, negative infinity, does not equal the right hand side, positive infinity so therefore the limit doesn't exist as x approaches 2. The reason is because she incorrectly compares the left hand side of the graph with the right hand side. The misconception she has is thinking the limit exists when it's infinity.

Discuss any possible relationship limits at infinity of rational functions have with horizontal asymptotes.

Figure D.2.57: Task 11 Problem Statement.

R: Does a horizontal asymptote with rational functions have anything to do with limits at infinity? Is there a connection? (*Turn 129:23:22, SDV_0055)*

N: Most people probably see these as 2 different things. You can't always be sure that the horizontal asymptote is going to be a limit. I am trying to think of a counter example where it wouldn't be. I don't know, but if you had a HA like this, and you wanted to know for say e to the x, that shoots in the sky but it still has a HA for when the function goes left as x approaches negative infinity. It depends where it shoots off into, if it goes to positive or negative infinity. Sometimes it will be the limit, but other times it won't. So you should never assume a HA is a limit. The limit as x goes to negative infinity is the HA but the limit as x goes to positive infinity is positive infinity. (*Turn 130:23:22, SDV_0055)*

Analysis:

Inference: She sees the connection between horizontal asymptotes and limits for rational functions but gave a counter example with an exponential function which was a good idea but not a valid comparison since it's not another rational function. She did not distinguish the horizontal asymptote, which is a line, from the limit, which is a number.

Hypothesis: Algebra proficiency and rules of horizontal asymptotes with rational functions facilitate success with this problem. Students may not know that horizontal asymptotes are lines and limits are numbers.

Summary:

Nicole is an above average student majoring in math. Despite minor mathematical mistakes and attempts to do invalid mathematical operations with the infinity symbol, she exhibits mastery and proficiency with algebra. Her algebra skills, including those involving rational functions, facilitated her ability to solve tasks presented. However, Nicole displayed consistent misconceptions about limits at infinity just as she did with limits at a point, both of these in the infinite limit case. She incorrectly thinks a limit exists when it equals infinity and perceives infinity as a number. For limits at infinity, she will correctly conclude that the limit doesn't exist but gives the wrong reason, incorrectly comparing the left hand side of the graph with the right hand side and then claim that since minus infinity on the left does not equal plus infinity on the right, the limit does not exist. In this case, one misconception she presents is that each side is a completely separate problem but instead compares the right hand side to the left hand side. Other misconceptions are that she thinks a limit exists if it equals infinity because she thinks infinity is a number. It is not. By definition, it is a behavior of numbers.

Analysis of Interview on Limits that Do Not Exist: Nicole

This interview explored notions students have about limits that do not exist. Infinite limits are of significance because of the misconception that a limit exists if it is equal to infinity. Rather, the function values increase or decrease without bound so the limit does not exist. There are two other cases of limits that do not exist: piecewise and trigonometric functions. Limits at a point of piecewise functions with jump discontinuities involve comparing the left-hand limit to the right-hand limit, and it is standard to write "does not exist" instead of a symbol when the two are not equal. Limits at infinity do not exist for trigonometric functions due to their periodicities. The limit does not exist for the tangent function as x approaches pi/2 and -pi/2 (and other values) because the function values increase or decrease without bound. Twelve tasks were presented including some from previous interviews in order to acquire a more in-depth understanding of when and why students think limits exist or don't exist. In addition to limits of rational functions, limits of piecewise and various trigonometric functions were analyzed.

What are limits at infinity? Provide examples. What are infinite limits? Describe the difference between an "infinite limit" and a "limit at infinity". Give examples of limits that do not exist.

Figure D.2.58: Task 1 Problem Statement.

N: Sinusoidal functions. At a point you can have a limit but at infinity, absolutely not. This applies to sine x and cosine of x, due to the oscillations the function never settles on but only approaches a single value. Tangent function too. *Turn 2,00:42, SDV_0056.*

N: For tangent, the limits don't exist as x goes to pi/2 and -pi/2 and in other multiples because it is still repeating or oscillating at a specific pattern. As x goes to infinity, the function jumps between positive and negative infinity so the limits do not exist. As x goes to 0, the limit exists b/c the left hand side equals the right hand side. But for arctangent, the limits do exist as x approaches infinity. For arctangent, the limit exists as x approaches infinity. (*Turn 4, 02:00, SDV_0056)*

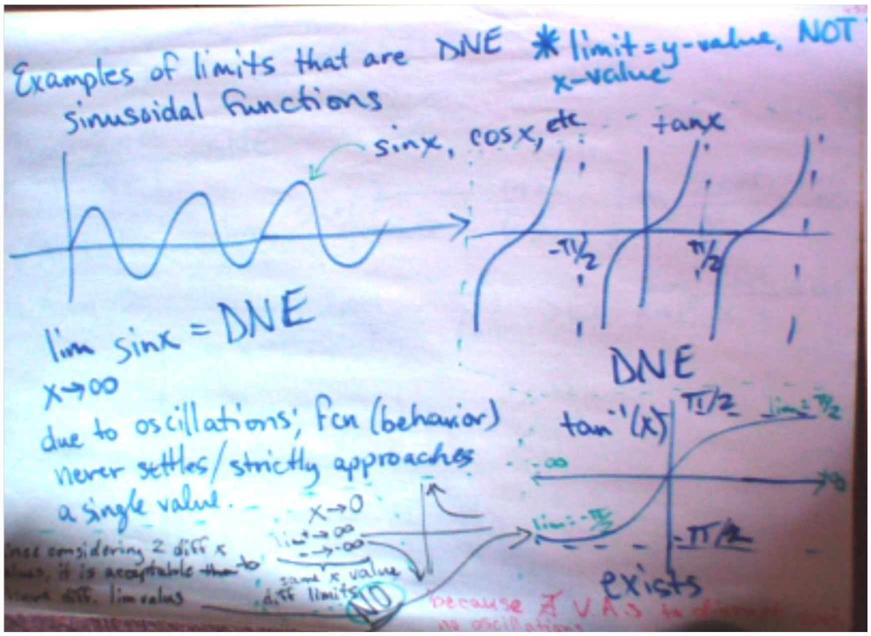


Figure D.2.59: Nicole's work on Task 1.

R: Can you explain the difference between limits at infinity and infinite limits.

N: Limits at infinity are values that functions approach as x goes to infinity. For example, limit x goes to infinity of 1/x. Infinite limits are those that equal infinity or go to infinity. Infinite limits are the end result of what you get, what y is doing. Infinite limits, for example is limit as x approaches zero from the left of 1/x.Limits at infinity are about where x is going. I want to pick an example that doesn't involve going to infinity. But still, maybe pick one with x approaching 0, but it's not defined from both directions, so I'll say the limit as x approaches 0 from the right of 1/x. The big difference is where the infinity is. With limits at infinity, the infinity sign is with the x. With infinite limits, infinity is the answer.*Turn 88, 33:10, SDV_0056.*

R: Can you give me some more examples of limits that do not exist.

N: Piecewise functions. With these limits do not exist when the function does not approach the same y value or function (I'll say y-value or the function because people get confused with x value) from both the positive and negative from the left and right directions. For example, limit as x goes to zero of 1/x the limit is positive infinity and as x goes to zero from the left the limit is negative infinity. So these are 2 examples of infinite limits because the answer is infinity. *Turn 90,35:59, SDV_0056.*

Analysis:

Inference: Nicole uses the word "function" or phrase "values functions approach" when she is referring to "function values." She gave good examples of limits that don't exist. It is correct to say the limit does not exist at a point for piecewise functions with jump discontinuities. She switched focus to 1/x, though, and decided not only that the left-hand and right-hand limits existed but that these were both infinite limits. For piecewise functions where there is a jump discontinuity, the limit just does not exist. Also, as x goes to 0 on either side, the limit of 1/x, the limit does not exist but she thinks it does.

Hypothesis: Students confuse "functions" and "function values" and confuse infinite limits with limits that do not exist. When students think a limit equals infinity and that the limit exists, they lose sight that the function values are either increasing or decreasing without bound and hence the limit does not exist.

Explain the behavior of the function value as x gets larger in the positive and negative directions. Graph the function and explain whether or not the limits exist.

$$\lim_{x\to\pm\infty}\frac{2x^2+x-1}{2-x}$$

Figure D.2.60: Task 2 Problem Statement.

N: OK, this one is supposed to be split up as x goes to plus infinity and as x goes to minus infinity. The plus sign is not necessary in front of infinity. Same for negative infinity. I would try to factor the numerator. Nothing cancels so there is no hole. VA at x=2. As for HA, larger power in numerator meaning we have none. Comparing powers of numerator and denominator. When numerator is smaller than denominator, then the HA is zero. *Turn 43, 10:15, SDV_0056*

N: For x going to positive infinity, with the calculator we see it is going to negative infinity because you have large numbers on top and smaller ones in the denominator. But the numerator is definitely going to positive infinity. As x goes to negative infinity, the numerator gives very large positive numbers but in the denominator you get large negative numbers. So I can kind of sketch this roughly without the calculator but immediately checks with calculator and has to zoom out to be able to see the whole thing. So this is what it is doing on the right side. Nothing canceled in numerator and denominator, so there's no hole. *Turn 52, 15:33, SDV_0056*

N: So obviously in this problem for x going to positive infinity, this graph is going to negative infinity for very large numbers on top but it is negative due to the denominator so the limit is going to minus infinity. The limit exists and equals minus infinity. *Turn 59, 18:58, SDV_0056.*

N: For the other half as x goes to negative infinity, the numerator is positive when you plug in large negative x because of the x squared in the first term, then in the denominator you subtract a large negative value so the denominator is also positive.

So the limit exists and is positive infinity, and that can be seen from the graph. So at either end of the spectrum the limit exists at positive or negative infinity. *Turn 62, 19:50, SDV_0056.*

R: Now, can you incorporate the slant asymptote showing how each part of the graph goes into different directions, using the vertical asymptote at x=1 and the slant or diagonal asymptote?

N: The function's behavior falls in between the vertical and the slant so you need this information to construct the graph. *Turn 64, 21:09, SDV_0056.*

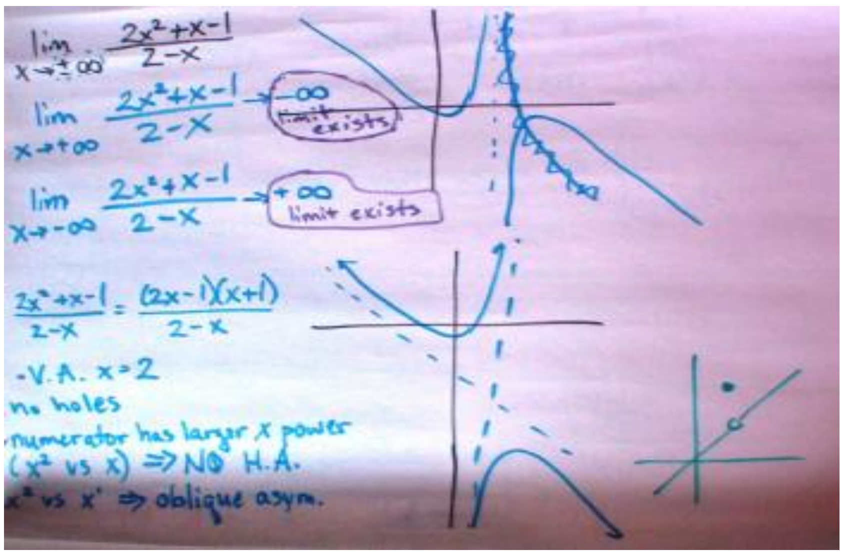


Figure D.2.61. Nicole’s work on Task 2.

Analysis:

Inference: Nicole correctly identified the vertical asymptote but refers to function values as "the function". She also determined that the limit equals plus infinity on the left and minus infinity on the right, but incorrectly said that these limits exist. There would not be any horizontal asymptotes with this function, as she notes, but there would in fact be a slant asymptote which she ultimately found and successfully graphed.

Hypothesis: Students need algebra proficiency to articulate thinking and construct the proper corresponding graphs. Students don’t know that a limit does not exist when it equals infinity.

Compare these problems. Explain the behavior of the function values as x gets larger in the positive and negative directions. Graph the functions and explain whether or not the limits exist. $\lim_{x\to\pm\infty} x^2$ $\lim_{x\to 0}\frac{1}{x^2}$ and $\lim_{x\to\infty} 2x+1$

Figure D.2.62: Task 3 Problem Statement.

N: It's not valid to write plus/minus infinity for x-squared but anyway, the limit exists because on the left side the function values go to plus infinity and on the right side they go to plus infinity and so the limit exists and equals infinity. Then as x approaches 0 for 1/x-squared, the left side goes to plus infinity and right side also to plus infinity so the limit exists and equals infinity. That's because the left side equals the right side, they're both infinity. *Turn 140, 1:28:30, SDV_0058.*

N: With 2x+1, it's headed on a straight path, no oscillations. It's headed toward a steady value so that's why we say the limit exists. It keeps going to infinity so it exists. *Turn 107, 54:40 SDV_0058.*

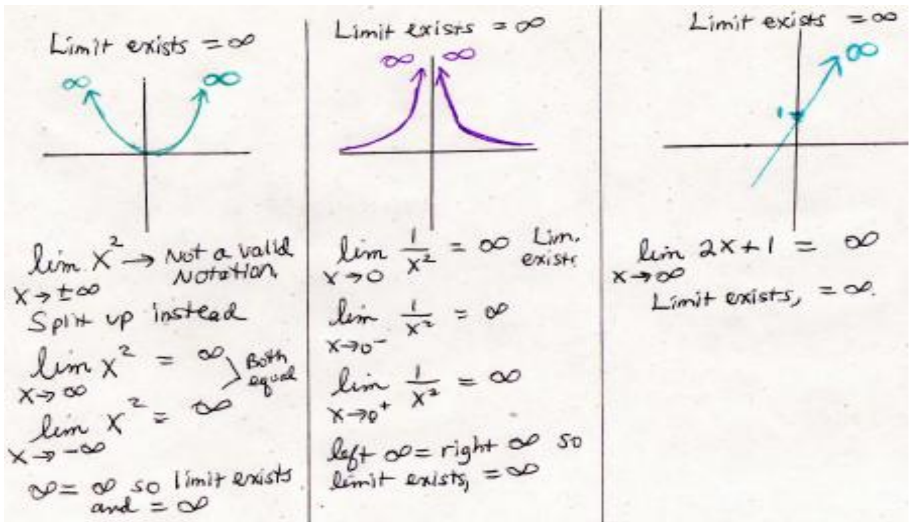


Figure D.2.63. Nicole's work on Task 3.

Analysis:

Inference: Nicole consistently does not accept the +/- notation and incorrectly compares the left side with the right side to decide if the limit exists. In the first two cases the function values go to positive infinity and therefore do not exist. However, Nicole concludes that the limits exist because the function values go to infinity on both sides of each graph. The last limit problem as x approaches infinity of 2x+1 confirmed her misconception, because she describes how it's "headed on a straight path, headed to a steady value" infinity.

Hypothesis: Students incorrectly compare the left side with the right side, rather than understand that the function values keep increasing without bound and therefore, the limits do not exist.

Compare and describe the limiting behaviors of the functions. Sketch the graphs. Explain if the limit exists or not. $\lim_{x\to 0}\frac{1}{x^2}$, $\lim_{x\to 0}\frac{1}{x}$ and $\lim_{x\to\pm\infty} x^3$

Figure D.2.64: Task 4 Problem Statement.

R: Lets compare the graphs of 1/x squared, 1/x and x-cubed. To reiterate what you said, for 1/x-squared as x goes to zero, the arrows both to up toward the y-axis in the same positive direction so the limit exists and equals infinity. Then to compare with 1/x, you said the limits exist separately but doesn't exist as x approaches 0 because the arrows (function values) point in 2 different directions. Then for x cubed as x approaches plus or minus infinity, you said the limits do exist separately but not for the whole function because those arrows also go in 2 different directions, positive on the right and negative on the left. So can you explain the difference about limits existing for these 3 functions and explain why the limits exist separately and equal infinity but don't exist for the whole function? *Turn 48, 5:57, SDV_0051.mp4*

N: For 1/x squared, the limit equals infinity as x goes to zero because they are going in the same direction to the same number, infinity. So the limit exists and equals infinity. With 1/x, if you're asking about as x goes to 0, you'd get "different values" plus and minus infinity and so then you'd say the limit doesn't exist. For 1/x as x goes to 0, the limit exists also because it is going to infinity. As x approaches 0 from the left, the y-values get more negative so the limit exists and is negative infinity. For x-cubed as the limit goes to plus or minus infinity, again you are looking for "2 different x-values" so you again get 2 different answers, plus and minus infinity and since the left side doesn't equal the right side the limit doesn't exist. This is why the plus/minus don't make it a valid question. You should only be asked what does it approach at one x-value not at two x-values at the same time. They should be split up when asking the question about if the limit exists. Teachers need to stop confusing everyone with this stuff, too, and should never write the plus and minus signs together with any limit problems. (*Turn 49, 6:00, SDV_0051.mp4)*

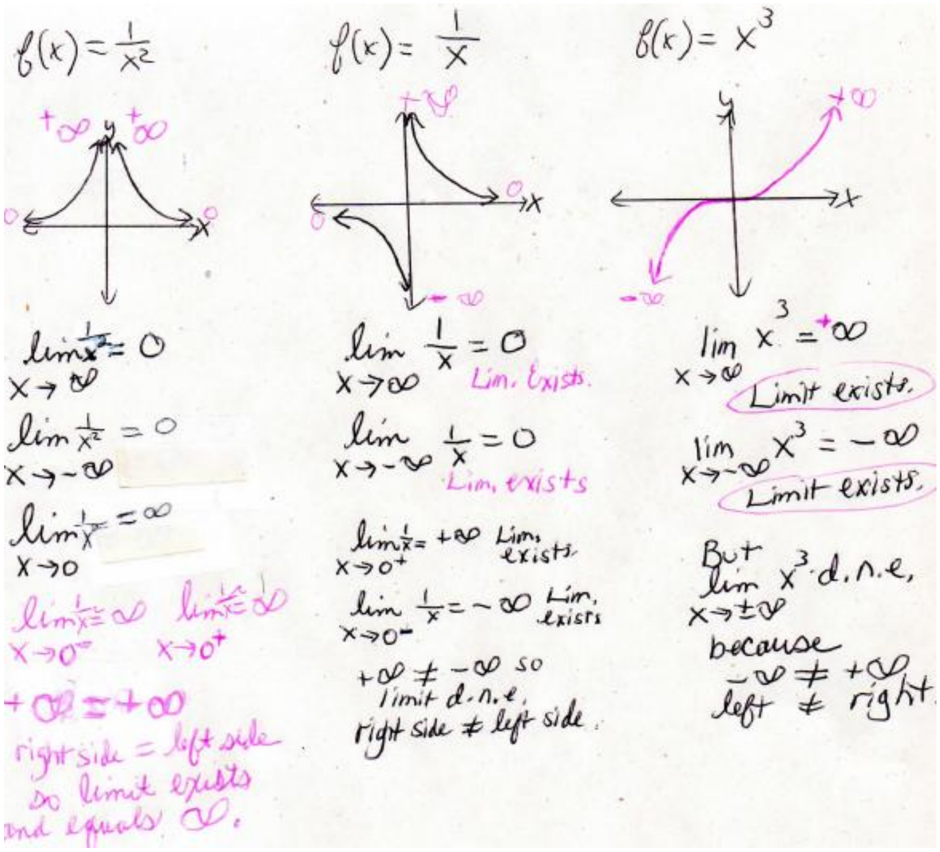


Figure D.2.65: Nicole's work on Task 4.

Analysis:

Inference: Nicole misunderstands when limits exist for limits at infinity. She stated that the limit would not exist if x approached plus/minus infinity, but gave the wrong reason. She incorrectly compared the left hand side to the right hand side, as would be done only with limits at a point, to determine if the limit exists. Since she determined that the left hand side does not equal the right hand side, $-\infty \neq +\infty$, she concluded the limit does not exist. When evaluating problems involving limits at infinity, the left hand side should not be compared with the right hand side.

The plus/minus notation is misunderstood. It means the logical "exclusive or", so it is perfectly acceptable to use this notation if it is explained in context. Nicole is taking the $\pm\infty$ as a single entity, hence she sees one limit problem not two. She seems to think $\pm$ means "+ and -" not the "exclusive or". She apparently thinks of x going to $+\infty$ and $-\infty$ simultaneously instead of separately. In general, she seems to have questionable mastery with definitions in mathematics. The lack of definitions being internalized by students seems quite widespread

Hypothesis: Infinite limit problems present problems for students who compare the left hand side with the right hand side regardless if the problem involves a limit at a point or a limit at infinity.

Explain the behavior of the function as x approaches 0. Graph the function and explain whether or not the limit exists. $\lim_{x \to 0} \cos \frac{1}{x}$

Figure D.2.66: Task 5 Problem Statement.

N: I think its undefined b/c as x goes to zero, oh my God. Assume x goes to 0 for positive values of x, this 1/x is approaching infinity. And we just saw in the previous example that any sinusoidal function, it doesn't have a limit at infinity so I would guess this is undefined but I think I'll have to graph it. It stops up at y=1 as x goes to zero. I use -0.1 and 0.1 as the range. I put in a smaller number and it's doing something different. I don't know what it's doing. It is oscillating perhaps. Yup, it is oscillating like mad. This problem is really equal to limit as x goes to infinity of cosine x because the limit as x goes to 0 from right of 1/x = infinity, and as x goes to 0 from left, 1/x = -infinity it oscillates, which is exactly what this graph is doing here, oscillating. (*Turn 102: 50:40, SDV_005)*

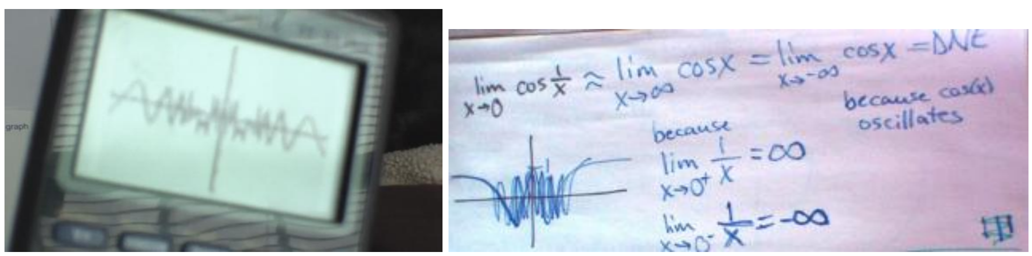


Figure D.2.67: Nicole's evidence limit that does not exist due to oscillation near 0.

Analysis:

Inference: Nicole attempted to divide by zero, which can't be done because it's undefined but she correctly determined that the limit doesn't exist because the function values oscillate near 0. She checked her result by adjusting the window on the calculator to get a good view of what was happening near 0. She knew the limit does not exist due to oscillations near 0 and kept in mind that the function was not defined at 0.

Hypothesis: Proficiency with the graphing calculator, trigonometric functions, oscillatory behavior and number sense with the reciprocal function facilitates the ability to determine the limiting behavior of this function as x approaches 0.

Sketch the graph of this function. Compute the limit as x approaches 10 and explain if the limit exists at x=10.

$$f(x)=\begin{cases} 3 & \text{if } x > 10 \\ x-4 & \text{if } x \leq 10 \end{cases} \qquad \lim_{x \to 10} f(x) =$$

Figure D.2.68: Problem Statement Task 6.

N: Solid dot at (10,6) and hole at (10,3). So the limit doesn't exist because of the function is undefined where the jump discontinuity is. *Turn 132, 1:21:55, SDV_0057.mp4*.

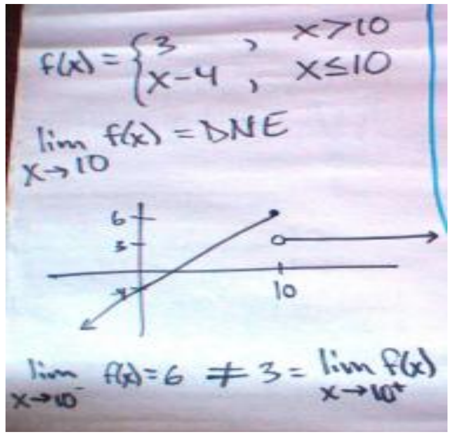


Figure D.2.69: Nicole’s work on Task 6.

Analysis:

Inference: Nicole exhibits proficiency with piecewise functions of this type, and was able to correctly determine that since the left-hand limit did not equal the right-hand limit, the limit did not exist. Since she demonstrated that 6 was not equal to 3, she concluded the limit did not exist as x approached 10.

Hypothesis: Algebra proficiency facilitates success with this task.

Explain the behavior of the function values as x approaches zero. Graph the function and explain whether or not the limit exists. $\lim_{x \to 0} \frac{1}{x^4}$

Figure D.2.70: Task 7 Problem Statement.

N: Limit as x approaches 0 of 1/x^4 equals infinity. Infinity here is positive. Here is the graph and it looks like the x squared function. Since you raise it to an even power, all of the y values will be positive. The limit exists as the function is approaching the same value infinity form both the left and the right sides. The limits also exist separately for the right side and for the left side and are equal to positive infinity. Separately they both exist and approach infinity, and together the limit also exists and is positive infinity. (*Turn 134, 1:23:45)*

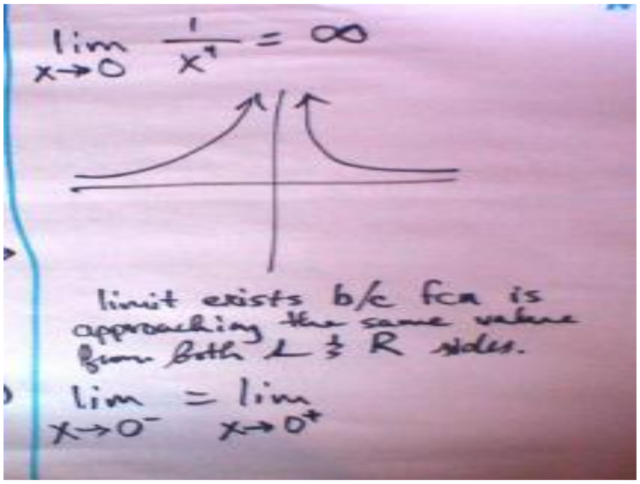


Figure D.2.71: Nicole's work on Task 7.

Analysis:

Inference: Nicole understands the function values are positive for this function but incorrectly thinks the limit exists and equals infinity.

Hypothesis: Students need algebra proficiency as well as knowledge of function values increasing without bound on either side to know the limit does not exist.

Explain the behavior of the function values as x gets larger in the positive and negative directions and then as x approaches 0. Graph the function and explain whether or not the limits exist for both cases. $\lim_{x\to\pm\infty} \sin(x) \qquad \lim_{x\to 0} \sin(x)$

Figure D.2.72. Task 8 Problem Statement.

N: The limit for sin(x) as x goes to infinity doesn't exist because it doesn't head to or settle down at any particular value. We haven't defined any specific x value. For this, we only have a concept or an idea about what infinity is. So we can't plug it in and we can't get a defined number. We don't know if the large x value we are talking about is close to 2, pi/2, 3pi/2 or 0. If we did, we'd get a definite number. We don't know which one of these it is a multiple of so we can't define the limit. It keeps oscillating, jumping between the y values -1 and 1, so it oscillates infinitely the limit doesn't exist. It doesn't stop at a function value that we could call a limit because we haven't specified what large number for x we want. We don't know if it'll be closer to 1, -1 or 0. The limit for sin(x) though equals 0 because left side = right side as it gets close to 0. *Turn 107, 54:40, SDV_0057.*

N: Limits either go to infinity, or they do not exist. They can't do both at the same time. It is either one or the other. People who are not familiar with the sine function might think this limit goes to infinity, but it doesn't. It just doesn't exist. The range is always -1 to 1. *Turn 119, 1:01:45, SDV_0057.*

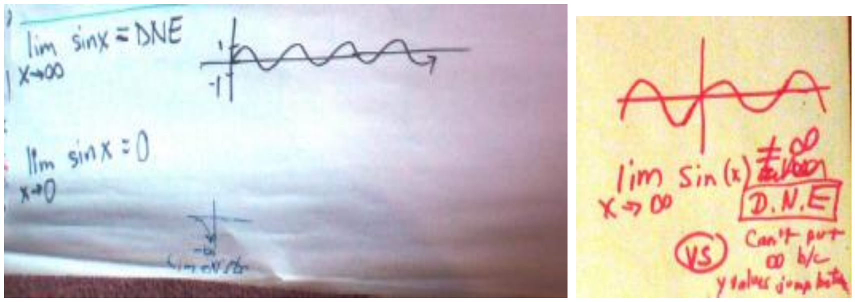


Figure D.2.73: Nicole's work on Task 8.

Analysis:

Inference: Nicole seems to know the difference between a limit that equals infinity and a limit that does not exist due to oscillations in this problem. She stated that limits either go to infinity or don't exist, which is not true. Limits can also exist at a point, and she clearly articulated this previously, including with sin(x) as x approached 0. Meanwhile, she also revealed the range for this function.

Hypothesis: Students may not know that with sin(x) the limit does not exist because of the oscillatory behavior which doesn't settle down to any one function value.

Explain the behavior of the function values as x gets larger in the positive and negative directions. Graph the function and explain whether or not the limits exist.

$$\lim_{x\to\infty}\arctan(x) \qquad \lim_{x\to-\infty}\arctan(x)$$

Figure D.2.74: Task 9 Problem Statement.

N: The arctangent's range comes from the inverse tangent's interval of x, over a particular interval. As x goes to infinity the limit is pi/2 and for x approaches minus infinity it's minus pi/2. The limits exist. (*Turn 121, 1:05:20, SDV_0057)*

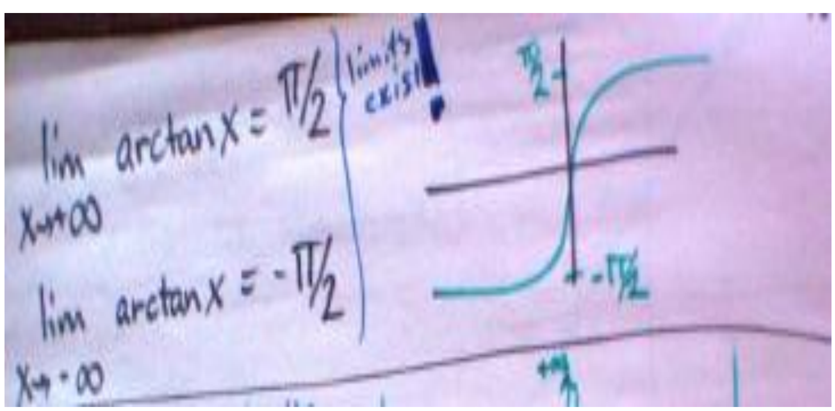


Figure D.2.75: Limit as $x \to \infty$ of arctangent(x).

Analysis:

Inference: Nicole knows that the arctangent is the inverse of the tangent, so when the truncated interval from the tangent becomes the range for the arctangent. She correctly determined the limits existing at the horizontal asymptotes y= $\frac{\pi}{2}$ and -$\frac{\pi}{2}$.

Hypothesis: A knowledge of the behavior of trigonometric functions and their inverses could facilitate seeing the limiting behavior in both directions as x approaches positive and negative infinity.

Explain the behavior of the function values as x gets larger in the positive and negative directions. Graph the function and explain whether or not the limits exist.

$$\lim_{x \to \frac{\pi}{2}^-} \tan(\mathrm{x}) \quad \lim_{x \to \frac{\pi}{2}^+} \tan(x)$$

Figure D.2.76: Task 10 Problem Statement.

N: The first one, approaching from the left, the limit is positive infinity where as from the right, the limit is negative infinity. I used Zoom 7 on the calculator, and I see it repeats around -pi/2. When you specify a direction from the left or from the right, then yes, the limits exist which is infinity. If you don't specify a direction, then no--the left side doesn't equal the right side so the limits do not exist. For tangent when x approaches pi/2 without any directions specified then the limit DNE. In that case, it would NOT equal infinity either. The limit just DNE because it repeats, it has a repeating pattern so for a large x value you don't know what the limit is ever is, if it's going to be close to 0 or close to pi/2. If it's pi/2 you have to know what side the function is on. If it's 0, then the limit is 0. If the function is near pi, then the limit is going to 0. (*Turn 123, 1:07:00 SDV_0057.mp4)*

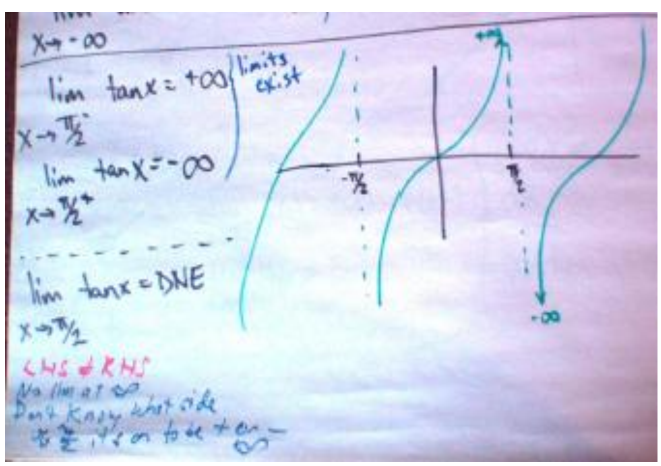


Figure D.2.77: Nicole’s work on Task 10.

Analysis:

Inference: Nicole constructed the graph of the tangent from memory and could easily explain how the function values behaved in each direction. However, she incorrectly concluded that the limit exists and equals infinity for the top and bottom sections of the graph. She correctly concluded that the limit doesn't exist, but for the wrong reason once again because she compared the left hand side, minus infinity, to the right hand side, plus infinity. She also talked about limits at infinity even though it was not part of this task.

Hypothesis: Proficient algebra skills and knowledge of the tangent function can facilitate success with this problem. Students may still think the limit does not exist because the left hand side doesn't equal the right hand side.

Explain the behavior of the function values as x gets larger in the positive and negative directions. Graph the function and explain whether or not the limits exist.

$$\lim_{x \to \pm\infty} \frac{x^2 - x - 6}{x - 3}$$

Figure D.2.78: Task 11 Problem Statement.

N: I'll factor the numerator to get (x-3)(x+2). The (x-3) cancel in numerator and denominator so you are left with x+2. You cancel so you get a hole at x=3. Here is the graph. It's essentially just this line, the line goes through y=2 and hole at (3,2). You got a straight line with a hole, so since you have 2 different x values approaching the same y-value, the limit exists and is 2 but as x goes to infinity on both sides, the left side goes to negative infinity and right side goes to positive infinity. So both of these limits exist, separately. *Turn 128, 1:12:40, SDV_0057.mp4.*

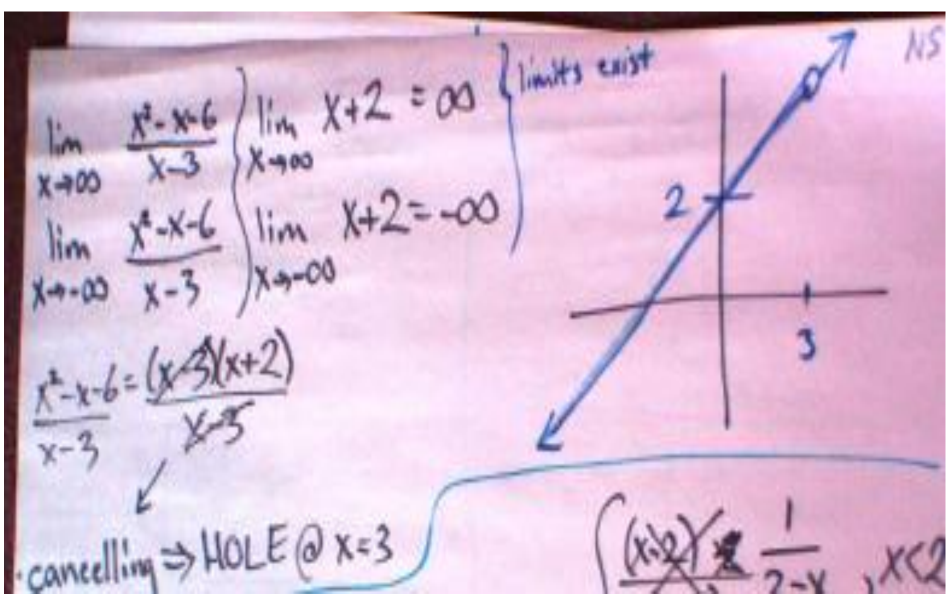


Figure D.2.79: Limit at infinity of rational function that's also an infinite limit.

Analysis:

Inference: Nicole correctly factored the numerator and divided out common factors. When zero in the denominator occurs, there is a hole in the graph as she correctly illustrated. Her problem is a limit at infinity, so the zero in the denominator has nothing to do with this limit.

Hypothesis: Most students don't know that this function could either be piecewise, or it could be a rational function with common factors that divide out thereby generating a hole and leaving a factor of (x+2) in the numerator.

Construct a possible function from the graph below and explain if the limit exists as x approaches 2 from the left and as x approaches 2 from the right.

Note: expect piecewise (one piece linear, other rational)

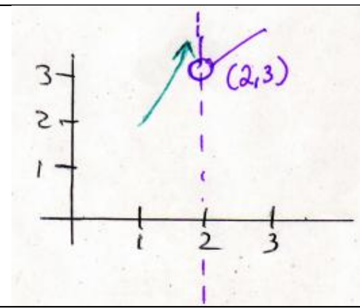
$\lim_{x\to 2^-} f(x) = ?$ $\lim_{x\to 2^+} f(x) = ?$	

Figure D.2.80: Task 12 Problem Statement.

N: As x goes to 2 from the left, f(x) goes to positive infinity. As x approaches 2 from the right, f(x) approaches 3. The function looks linear after but exponential before? I'd like to break it into 2 different parts. For x less than 2 you have (x-2)/(x-2), going to positive infinity. The 2nd piece is x+1 for x>2. The slope might be 1. So it's piecewise. Gosh, darn it. I feel like this should be easy. Top piece might be 1/(2-x) for x<2 would be more like it and you get the hole. For the linear piece, I just didn't know what to do with the x, to make it shoot up. Then you need to divide by a really small number, so you pick numbers real close to 2 for the denominator. *Turn 130, 1:16:00, SDV_0057.*

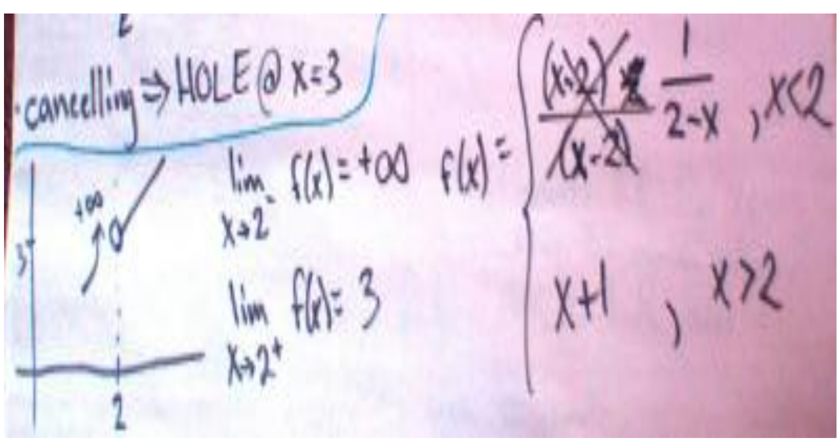


Figure D.2.81: Nicole's work on Task 12.

Analysis:

Inference: This problem was more difficult as it involved working backwards to find the original function. She broke it down into each interval on the graph to study each part separately. She knew she must have a zero in the denominator at x=2. So she took the reciprocal of (x-2) and then adjusted the sign to get the correct representation. It took her some time and thought, but she figured out what functions might represent each piece.

Hypothesis: Students can construct the piecewise function if they have proficiency with algebra and knowledge of piecewise and rational functions.

Summary:

Nicole exhibits proficiency with number sense, algebra, trigonometric functions and the graphing calculator. Throughout the tasks she uses the word "function" when referring to the function values. She is knowledgeable of limits that do not exist with piecewise and trigonometric functions; however, her knowledge and understanding of infinite limits is problematic. She thinks infinity is a number and so a limit can equal infinity. Therefore, she incorrectly concludes that the limit exists when it equals infinity. She compares the left-hand limit to the right-hand limit for limits at infinity to decide if the limit exists, which is incorrect. Instead, her focus should be on the behavior of the function values as x tends toward plus or minus infinity separately, noticing that the function values increase or decrease without bound. Perhaps textbook notation could be changed from "lim f(x) = ∞"to "lim f(x) $\rightarrow \infty$" so that the emphasis is on approaching infinity rather than being equal to infinity. This might prevent Nicole's and other students' misunderstandings with the notation. Overall, she shows good instrumental understanding and some conceptual understanding evidenced by her ability to precisely articulate her thoughts and produce written drawings of the tasks presented.

APPENDIX E: PHASE I SUMMARY WITH TABLES

The two original research questions of this study were: In what ways do students think about limits, and how do their understandings about limits manifest in their task solutions?

The first phase of this study collected exploratory pilot data consisting of functions, limits at a point, limits at infinity and limits that do not exist. Over the course of three interviews that lasted about two hours each, students were given graphs to look at as well as some computations, and while they thought aloud, they drew their responses on a white board and on 18” x 24” sheets of paper. The rationale for collecting initial data was to identify common themes of misconceptions that emerged between and within subjects, as well as to select particular areas in which deeper probing into the thinking process could occur.

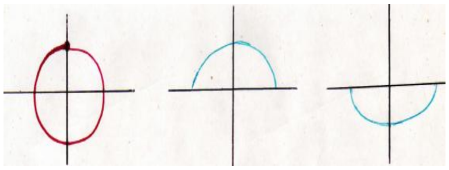
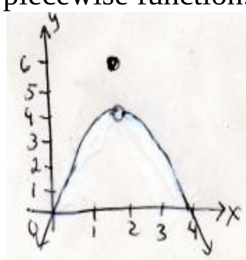
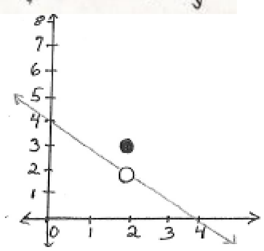
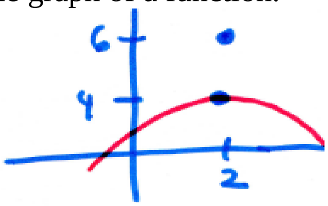
In order to provide an in-depth analysis of how they think about limits, 41 mathematical problem solving activities were given. In addition to interview transcripts, snapshots of their written solutions were analyzed. According to Cobb (1988), analyses that employ clinical interviews are most effective when the interviews document examples of task-involved activity. The nature of the research questions considered the validity of their mathematical correctness as further probing of students’ understanding evolved. Given the nature of this evolving model of understanding, new hypotheses and research questions were formulated in response to this pilot data and were subsequently explored with an additional nine students.

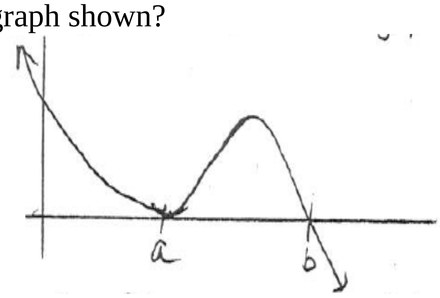
Two students in Calculus III with contrasting ability levels were selected for the initial pilot cases with results summarized in the four consecutive tables below. First, these students were selected based on demonstrating high motivation, engagement and the ability to think aloud. They also had to articulate well in the tasks presented, thereby providing a measure of authenticity in their actions. Nicholls (1984) and Cobb (1988) discussed different levels of engagement during clinical interviews. In particular, subjects were said to be "task involved" if their verbal and non-verbal actions indicate that they are fully engaged and motivated. Second, they were selected as a subset from a larger group of 15 students according to differences in their mathematical content knowledge. CL and NS each demonstrated high level motivation, engagement and articulation ability; were very generous in explaining their thinking aloud, and demonstrated a range of activity in their individual problem solving. CL had minimal algebra proficiency opposed to NS, who had a more comprehensive background in mathematics.

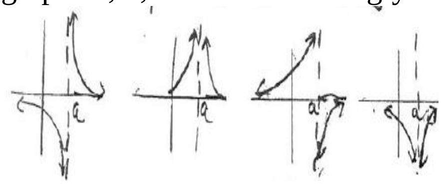
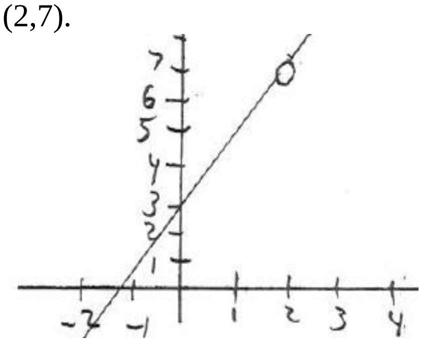
Appendix E-1

Table E.1: Results for Functions

Tasks	CL's Results	NS's Results
Task 1: Define a function or explain what a function is. Write or draw examples of functions. What is a function value? What's the difference between a function and a function value? How are functions and function values related?	CL could determine when relations were functions by the vertical line test. Erroneously referred to functions having to be 1-1, 1 y for each x, thereby confusing this with the definition of a function. Identified characteristics of a relation "restraints" that prevented the relation from being a function.	Determined if relations were functions via. vertical line test, erroneously referred this relationship as being 1-1. Confused 1-1 with definition of a function. Cited examples that were not functions. Terms function and function value used interchangeably.

Task 2: Explores the circle, and the top and bottom separately, to decide which are graphs of functions. 	CL knew that only the top or bottom half could be graphs of functions, but not the whole circle via the vertical line test.	NS used vertical line test to show only the top or bottom halves were graphs of functions, but not the whole circle because of 2 y's for each x. Confuses definition of function and V.L.T. with 1-1. Does not know 1-1 is reserved to determine if a function has an inverse. She'd need a horizontal line and get 1 x for each y in the range. These are NOT 1-1.
Task 3: Explores the graph of a piecewise function.  	CL distinguished when a point is on the graph versus when it is on the function. Did not know it was piecewise. Thought point did not belong with the graph.	NS correctly recognized this as a piecewise function. One piece was quadratic, the other piece a solid dot which is the value of the function. Knows function value not defined at 4 when x=2. Results important to later tasks involving discontinuities involving limits at a point.
Task 4: Follow up questions about the graph of a discontinuous function with an isolated point. A point was added to the graph of a piecewise function so that it was no longer the graph of a function. 	CL could correctly determine that this was not the graph of a function.	NS correctly identified this as not being the graph of a function but for wrong reason, that it's not 1-1. Confused 1-1 with the vertical line test and definition of function. Did not report 1-1 pertains to a function having an inverse.

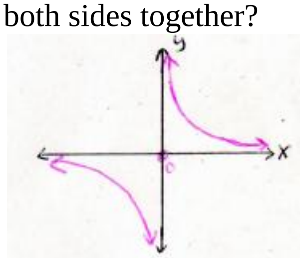
<u>Task 5</u>: Given VA at x=$-\frac{3}{2}$, x=$\frac{3}{2}$ *and* HA at y=$\frac{1}{2}$ find the function below. $a. f(x)=\frac{x^2}{x^2-\frac{9}{4}}$ b. $f(x)=\frac{x^2}{2x^2-\frac{9}{2}}$ c. $f(x)=\frac{1}{2x^2-\frac{9}{4}}$ d. $f(x)=\frac{x^2}{2(x-\frac{3}{2})^2}$	CL worked backwards starting with answer choices, relying on calculator. Used trial and error method but got it correct that way. She could not solve the problem starting with given properties. She lacked algebraic tools such as rules for finding HA's that would have expedited the problem solving process.	Set the function equal to y=1/2 (the horizontal asymptote), but found the mistake and reason it would not work. Reported a "function" cannot cross a VA instead of "function values can't cross". Inconsistent mathematical terminology at times. Demonstrated algebra skill and proficiency.
<u>Task 6</u>: Which polynomial function might represent the graph shown?  *a.* y = x(x-a)$^2(x-b)$ b. y = (x-a)$^3(b-x)$ *c.* y = (x-a)$^2(x-b)^3$ d. y = (x-a)$^2(b-x)$ *e.* y = (x-a)$^3(x-b)$	CL worked backwards starting with answer choices. Used trial and error method with calculator. She lacked the necessary algebraic skills to construct the original function.	NS started with the graph then eliminated answer choices. She did not need calculator; had sufficient algebraic skills and proficiency and used her skills to derive the correct function to match the graph.

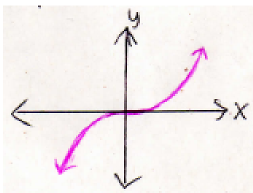
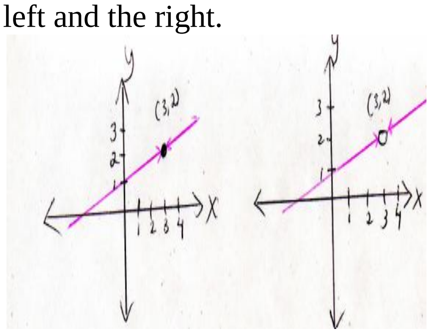
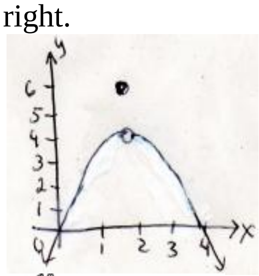
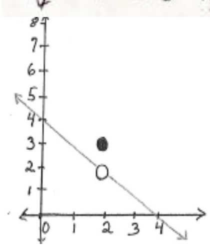
Task 7: Match the graphs with the correct functions and label the graphs a, b, c and d accordingly.  *a.* $f(x) = -\frac{1}{x-a}$ b. $f(x) = \frac{1}{x-a}$ *c.* $f(x) = -\frac{1}{(x-a)^2}$ d. $f(x) = \frac{1}{(x-a)^2}$	CL worked backwards starting with answer choices. Used trial and error method with calculator. She lacked the necessary algebraic skills to construct the original function. .	NS started with graphs. Recognized the elementary function $f(x)=\frac{1}{x}$. Computed function values for each answer choice. Correctly reported the horizontal translation 2 units to the right. Stated function values squared go in same direction, positive pointing up or negative pointing down with a negative sign in front of the function. Algebra proficient.
Task 8: Construct a rational function for this graph with a hole at (2,7). 	CL reported that holes only appear in limit problems not in functions, and that because this was a straight line, it could only be a linear function such as f(x)=x+3. This suggests deficits with knowledge of rational functions, number sense, and algebraic skills.	NS reported common factors were involved in the numerator and denominator and what was left in the numerator would be the slope and y-intercept, with the denominator not equal to zero. Evidence of algebra proficiency.

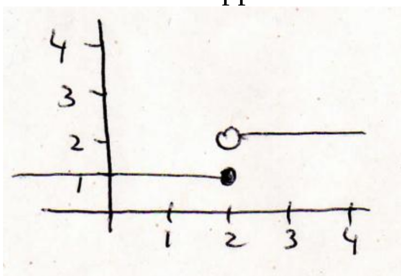
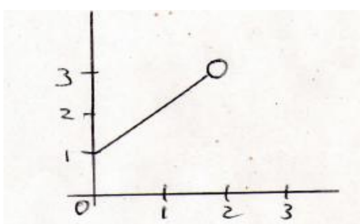
Appendix E-2

Table E.2: Results for Limits at a Point

Tasks	CL's Results	NS's Results
Task 1: What is a limit? Provide a definition or an intuitive explanation. Explain whatever comes to mind when you think of finding a limit. Write or draw examples of limits that exist and do not exist. Do limits refer to either the first or second coordinate on the graph, or do they refer to both? How are functions and limits related?	CL used everyday language to describe limits: brick walls, barriers, restraints, constraints. Limits are vertical asymptotes and holes because they're barriers. Limits are about x only. There's no relation between functions and limits.	NS correctly identified a limit as the 2nd coordinate on a graph of a function and gave an intuitive, not formal, definition. She could articulate how to find a limit at a point. Reported that piecewise functions are examples of limits that don't exist.

Task 2: Compute the limit if it exists as x approaches 3, then sketch a graph of this function. $\lim_{x\to 3}(2x+1)$ or $\lim_{x\to 3}(5x+2)$	Computed the limit of the first task and correctly got 7 but incorrectly put it on the x-axis. She reported 3 limits: x=3, x=7 and region between 3 and 7.	NS plugged 3 into the function and computed the limit, then correctly graphed it as a point (3,17).
Task 3: Compute the limit if it exists. Explain the behavior of the function values near x=2. $\lim_{x\to 2}\frac{x^2+4}{x-2}$	Incorrectly factored the numerator as a difference of squares. Insufficient algebra skills. Erroneously reported limits at x=2 and at x=0. No mention of roots and factors.	Incorrectly factored the numerator as difference of squares, so drew the wrong graph. Didn't check the numerator separately to see if it factored. Questionable knowledge of roots and factors.
Task 4: Compute the limit if it exists. Explain the behavior of the function values near x= -3. $\lim_{x\to -3}\frac{x^2-9}{x+3}$	Correctly factored as a difference of squares. Drew a vertical line at x= 3 as the limit or restraint. Incorrectly computed the limit as being equal to 6, so the graph was wrong. Lack of algebra proficiency.	Correctly factored difference of squares. Graph drawn incorrectly at first, hole at (0,-6). Knew common factors meant a hole would occur. Uses word "cancel" instead of "divide".
Task 5: Compute the limit. Explain how the function values behave near 0. Does the limit exist as x approaches 0 from the left, from the right, and from both sides together?  $\lim_{x\to 0}\frac{1}{x}$	Erroneously reported the limit was the y-axis, which was a brick wall where the points on the graph kept going up to infinity; incorrect conclusion that the limit existed at infinity.	Correctly stated limit d.n.e. for whole problem, but for wrong reason. Used $-\infty \neq +\infty$ but only one side going to infinity is needed to say limit d.n.e. Reported infinity is a number, therefore, the limit exists on top and bottom separately. Incorrect instruction could be the underlying source.

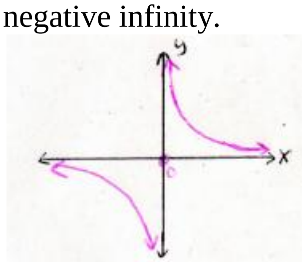
Task 6: First, tell what kind of function this is, then explain how the function values behave near 0. Explain if the limit exists as x approaches 0 from the left, from the right, and from both sides together.  $f(x)=?$ $\lim_{x\to 0} f(x)=?$	Identified a cubic but used visual inspection to report 2 limits existed as x $\to$0: x=3 and x= -3. Reported another limit existed on the x-axis between -3 and 3. Incorrectly wrote $+\infty$ in 3rd quadrant for behavior of function values. Lacks graphing ability, basic arithmetic skills and was unable to label the y-axis correctly.	Correctly identified this as a cubic function and that limit existed at 0 by visual inspection because left side = right side.
Task 7: Look at the following graphs and explain if the limits exist as x approaches 3 from the left and the right. 	First graph: limit did not exist because it was a solid dot one could walk right over, so no restraint or limit for getting to the other side. Second graph: limit existed near the hole because holes are limits one can fall into. At the hole itself, the limit did not exist because once you fall in, you keep going down.	Correctly acknowledged the limit exists at 2 as x approaches 3 in both cases. The hole and solid dot were not a source of confusion. Seems to understand the notion of "nearness" by her dialogue of describing how the function values approach 2 from both the left and from the right.
Task 8: Describe what kinds of functions these are. Explain if the limit exists as x approaches 2 from the left and from the right.  	First graph: called x-squared. The dot at (2,6) was random and didn't belong there. She correctly said the limit existed at (2,4) but gave wrong reason--one could fall into the hole. “The limit does not exist at 6 because there was no line you could use as a path to walk over to it”.	Correctly identified both graphs as piecewise functions. For first graph, stated the limit exists at 4 but is not equal to the value of the function, 6. Reported limits are about nearness or approaching and that limits can be different from function values.

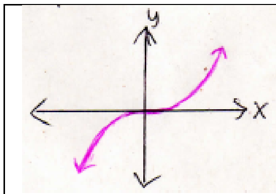
<u>Task 9</u>: Describe what type of function this is. Explain if the limit exists as x approaches 2. 	Correctly recognized this graph as a piecewise function. Incorrect about it having 3 limits that exist: (2,1), at (2,2) and area between y=1 and y=2. Correctly reported the limit does not exist because the left limit doesn't equal the right limit.	Correctly identified this function as piecewise and noted the discontinuity at x=2. She correctly stated the limit does not exist because the left hand side does not equal the right hand side.
<u>Task 10</u>: Given a restricted domain, explain if the one-sided limit exists as x approaches 2 from the left. $D:\{x \mid 0 \le x < 2\}$ $\lim_{x \to 2^-} f(x)$ 	Reported the limit existed at (2,3) but gave wrong reason—"because there is a hole one could fall into". If it was a solid dot instead, then "the limit would not exist because one could walk across the solid dot and keep going to the other side".	Correctly identified the limit existing for this one-sided limit, and being equal to 3. Was not confused by the hole and did not claim it the limit does not exist because it never reaches 2. She seems able to understand problems with domain restrictions.
<u>Task 11:</u> What is the limit as x approaches 3? Study the following answer choices and explain your reasoning. $\lim_{x \to 3}(5) = ?$ a. 3 b. 5 c. 0 d. ∞	Did not identify this constant function. Reported 3 limits: x=3 that came with the "lim" notation; 5 because that was in the problem; and 0 to represent the area on the x-axis where y is 0. Also the $x \to 3$ notation was perceived as directional, approaching from the left. Changed mind to say limit $= \infty$ because nothing stops the vertical line at x=5 from going up and down.	Recognized this rather untypical problem as a constant function, was able to find the limit, and explained the limit exists and is 5 for all values of x, not just 3.

Task 12: Describe what type of function this is. Construct its graph. Explain if the limit exists as x approaches 3. $f(x)=\begin{cases}5x+2 & \text{if } x\neq 3\\ 20 & \text{if } x=3\end{cases}$ $\lim_{x\to 3} f(x)=?$	Unable to report what type of function this (piecewise) is but said it had 2 pieces. Originally drew 2 separate graphs, then added the 2nd to the first. Claims "the limit exists at 3 because one could fall into the hole" but it didn't exist at 20 "because there was no line going to it". Limit exists at 3 because "that's what was written under the "lim" notation".	Correctly identified the limit being equal to 17, but not being equal to the value of the function at (3,20). Demonstrates proficiency with piecewise functions.

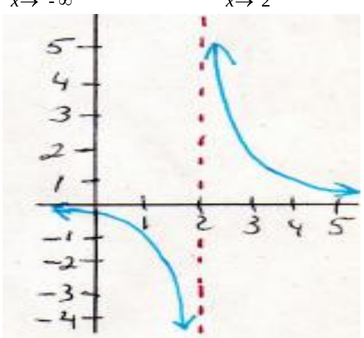
Appendix E-3

Table E.3: Results of Limits at Infinity

Tasks	CL's Results	NS's Results
Task 1: Compute the limit as x gets larger in the positive and negative directions. Explain whether or not the limit exists as x approaches positive and negative infinity.  $\lim_{x\to\infty}\frac{1}{x}$ $\lim_{x\to -\infty}\frac{1}{x}$	CL followed points on the graph and erroneously concluded the limit was equal to infinity instead of 0, since the points on the graph were "going to infinity". She also decides the limit will be whatever appears beneath the "lim" notation.	NS correctly determined the limit exists and equals 0. Gave the correct reason, a large number in the denominator makes the whole number smaller and smaller.
Task 2: Name this function best you can. Describe the limiting behavior as x gets larger in the positive and negative directions. Explain whether or not the limit exists as x approaches positive and negative infinity.	CL decided by visual inspection alone that there was a limit at 3, and symmetric about the y-axis. She arbitrarily considered the graph to be on the x-axis between	NS correctly identified this function as cubic (or odd). She incorrectly thought the limit exists as x approaches plus infinity and as x approaches minus

 $\lim_{x\to\infty} f(x)$ $\lim_{x\to -\infty} f(x)$	x=-3 and x=3 but refers to both x-values as positive 3 due to its symmetry.	infinity, referring to infinity as a large number. Incorrectly compares left hand side $(-\infty)$ with right hand side $(+\infty)$ to determine the two limits are not equal and therefore, the limit does not exist. Does not report the limit does not exist because function values increase or decrease without bound.
<u>Task 3</u>: Describe the limiting behavior as x approaches plus or minus infinity and explain whether or not the limit exists. $\lim_{x\to\infty}\frac{3x}{2x^2+1}$ $\lim_{x\to-\infty}\frac{3x}{2x^2+1}$	Used a horizontal line test to decide this wasn't a function because "the limit crossed it twice". The limit equals infinity because of what appears under the "lim" notation.	Correctly determined the limit is 0, which is also the horizontal asymptote. Shows evidence of number sense, algebra proficiency with rational functions.
<u>Task 4</u>: Describe the limiting behavior as x approaches plus or minus infinity and explain whether or not the limit exists. $\lim_{x\to\infty}(1+\frac{1}{x})^x$ $\lim_{x\to -\infty}(1+\frac{1}{x})^x$	Originally thought the limit was 1. Plugged in values of x but plotted function values on x-axis and reported infinitely many limits, considering every function value as a limit.	Exhibits number sense with reciprocal functions. Used calculator to check result. Adding a small number to one then raising it to that power tends toward the constant "e" in both positive and negative directions.
<u>Task 5</u>: Particle Problem: Given s(t) represents the position of a particle at time "t", sketch a graph and describe the limiting behavior as t approaches positive infinity for s(t)=$1-e^{-t}$. $\lim_{x\to\infty}(1-e^{-t})$	Dislikes word problems. Confuses x with y and follows points along the graph, so points going to infinity equates to limit equals infinity. Infinity symbol under the "lim" notation specifies the limit.	Shows evidence of number sense, experience working with exponentials and algebra proficiency. Correctly determined limit=1 but could not articulate interpret the meaning of the word problem.

Task 6: Describe the limiting behavior of the function below as x gets larger in the positive and negative directions. Sketch a graph of the result. $\lim_{x\to\infty}(5x^2+2)$ $\lim_{x\to -\infty}(5x^2+2)$	Reported function values went to infinity and that the limit existed; the limit was equal to 2 because there were no function values less than 2 so this was a barrier, and hence, a limit.	Demonstrates algebra proficiency, including y-intercept and shape of graph, quadratic. Wrong interpretation. Claimed limit exists and equals infinity when it does not exist. Claims infinity is a number.
Task 7: Describe the limiting behavior of the functions. Find any horizontal asymptotes, vertical asymptotes, holes and limits. Graph the function and explain the limiting behavior as x approaches infinity and as x approaches 1. $\lim_{x\to\infty}\frac{2x+1}{x-1}$ $\lim_{x\to 1}\frac{2x+1}{x-1}$	She determined the limit according to what's written beneath the "lim" notation.. As $x\to 1$ there's a restraint at 1, so the limit doesn't exist there where the wall is because one could climb up and down this wall x=1 without stopping. Wall is a limit, but there is no limit to the number of times one could climb up and down. Has misconceptions with limiting behavior of function values.	Demonstrated algebra proficiency including knowledge of when holes occur due to common factors in the numerator and denominator in the graph of a rational function. Reported 3 cases for asymptotes, so correctly found horizontal asymptote y=2 as x approached infinity. Incorrect interpretation with limit at a point claiming it exists and equals plus or minus infinity when it does not exist. Reports infinity is a number.
Task 8: Describe the limiting behaviors. Sketch the graph and explain how the function behaves for large x in the positive and negative directions. Do the limits exist? $\lim_{x\to\infty}\frac{3x^3-x^2-3}{2x+3}$ $\lim_{x\to -\infty}\frac{3x^3-x^2-}{2x+3}$	Difficulty occurred inputting function on the calculator and interpreting the graph correctly. Could not determine parabolic asymptote. Reported the limits are directional on the x-axis, approaching infinity going right, but stops at x=-1. Said the limit existes because of the brick wall restraint at x= -1. Could not describe behavior of function values going to infinity, in which case the limit does not exist.	Attempted invalid arithmetic operations with infinity. Good algebra skills with rational functions. Determined there is no horizontal asymptote but lacked knowledge of parabolic asymptotic behavior, even with the graphing calculator. Incorrect interpretation that the limit exists because both sides equals ∞.

<u>Task 9</u>: Describe the limiting behaviors. Sketch the graph and explain how the function behaves for large x in the positive and negative directions. Do the limits exist? $\lim_{x \to \pm\infty} \frac{3x^2+2}{9x^2-2x+5}$ or $\lim_{x \to \pm\infty} \frac{9x^2+2}{3x^2-2x+5}$	Selected the second problem. Guessed the limit was 3, but then plugged 3 back into the function to get 83/26 and divided wrong to get 2.3. Plotted this result on x-axis and said the limit existed at 2.3. Reported the limit would not exist if she got 0 in the denominator. Deficits with basic arithmetic, algebra skills, rational functions and knowledge of limits.	Good algebra skills with rational functions and knows 3 cases for asymptotes so knew the horizontal asymptote was y=1/3 and that limit existed at 1/3. Reported correct relationship between horizontal asymptote and limit at infinity.
<u>Task 10</u>: Study the graph below and describe the behavior of the function values for large x in the positive and negative directions then explain how the function values behave near x=2. $f(x) = ?$ $\lim_{x \to \infty} f(x) = ?$ $\lim_{x \to -\infty} f(x) = ?$ $\lim_{x \to 2} f(x) = ?$ 	What's beneath "lim" notation reveals what the limit is going to be without doing any math. Difficulty determining the original function, experimented with the calculator. As $x \to 2$ the limit exists because it's a restraint or brick wall but does not exist once on the wall because one could climb up and down without stopping. Did not report that $x \to 2$, the limit d.n.e. because the function values increase without bound in the positive or negative directions. Uses every-day language to explain what limits are. Lacks algebra proficiency.	Algebra proficiency demonstrated with identifying reciprocal function shifted 2 units right. Correct interpretation of horizontal asymptote at y =0 as x approached +/- ∞ and limit is 0, but incorrectly said limit exists as x $\to \pm\infty$. Reported infinity is a number.

Task 11: Discuss any possible relationship limits at infinity of rational functions have with horizontal asymptotes.	Reported no relationship. She did not articulate that when the leading term's powers are equal in the numerator and denominator, the limit is the ratio of the coefficients. Lacks mastery and proficiency with algebra of rational functions.	Shows algebra proficiency with rational functions so knows relationship between horizontal asymptote and limit--take the coefficient of the leading terms when the powers are the same. She gave counter examples of $f(x) = e^x$ to show horizontal asymptote only on one side and positive infinity on the other.

Appendix E-4

Table E.4 Results of Limits that Do Not Exist

Tasks	CL's Results	NS's Results
Task 1: What are limits at infinity? Provide examples. What are infinite limits? Describe the difference between an "infinite limit" and a "limit at infinity". Give examples of limits that do not exist.	Reported limits at infinity and infinite limits were the same. Did not report that infinite limits are limits that do not exist because the function values get larger or smaller without bound. Her example of a limit that does not exist was incorrect. She described a solid dot on a line as a limit not existing because one could walk across it without falling in a hole. Uses everyday language to articulate ideas.	Provided good examples including trigonometric, piecewise functions and infinite limits. Her method used to determine infinite limits, though, is incorrect as she compares the right-hand limit with the left-hand limit instead of focusing on the behavior of the function values increasing or decreasing without bound, and thereby determined that the limit in fact does not exist.

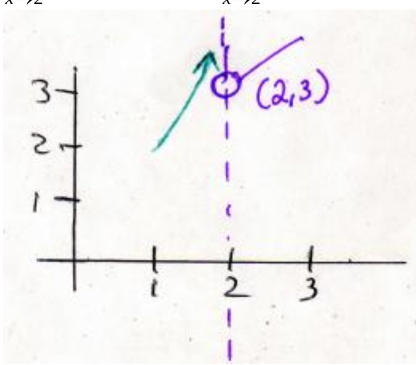
Task 2: Explain the behavior of the function value as x gets larger in the positive and negative directions. Graph the function and explain whether or not the limits exist. $\lim_{x \to \pm\infty} \frac{2x^2 + x - 1}{2 - x}$	Confused when denominator had a number followed by a variable. She acknowledged a restraint at 2 because that's where the function is undefined, but said the limit exists and equals $+\infty$, even in the 4th quadrant because she followed points along the x-axis instead of seeing that the function values were increasing in the negative direction. Difficulties exist with the coordinate system and with the ∞ symbol.	Possesses algebraic skills and proficiency with rational functions to determine that the limits equal infinity. Refers to function values as "functions." Her conclusion was wrong about the limit existing, being equal to infinity.
Task 3: Compare these problems. Explain the behavior of the function values as x gets larger in the positive and negative directions. Graph the functions and explain whether or not the limits exist. $\lim_{x \to \pm\infty} x^2$ $\lim_{x \to 0} \frac{1}{x^2}$ and $\lim_{x \to \infty} 2x + 1$	Used the x-axis as a barrier and decided it was the limit. Reported this as a limit as at a point problem instead of limit at infinity. 2nd and 3rd tasks, she said limit equals infinity. Did not say a limit does not exist if function values go to infinity.	Incorrectly said the limits exist and equal infinity because the function values were positive in both cases and kept increasing in without bound, rather than report that the limits do not exist.

Task 4: Compare and describe the limiting behaviors of the functions. Sketch the graphs. Explain if the limit exists or not. $\lim_{x\to 0}\frac{1}{x^2}$, $\lim_{x\to 0}\frac{1}{x}$ and $\lim_{x\to\pm\infty} x^3$	Reported the limit existed and was equal to infinity for the first and last task. Reported middle one had no limit because function was undefined at 0, so one could never reach the brick wall. Uses everyday language to explain what limits are.	Incorrectly said the limits exist and equal infinity separately in each direction. She incorrectly compares the left and right limits then concludes the limits as x approaches plus or minus infinity instead of studying the behavior of the function values to see they increase or decrease without bound, concluding the limits do not exist.
Task 5: Explain the behavior of the function as x approaches 0. Graph the function and explain whether or not the limit exists. $\lim_{x\to 0}\cos\frac{1}{x}$	Reported the limits were y=1 and y=-1 by visual interpretation because that's as high up the graph went on the calculator. Said it has another limit at 0 because of what appears under the "lim" notation. She did not report how function oscillated and did not settle down near 0.	She reported the function is undefined at 0 then correctly determined the limit did not exist due to oscillations near 0, which suggests understanding the notion of nearness for limits at a point.

Task 6: Sketch the graph of this function. Compute the limit as x approaches 10 and explain if the limit exists at x=10. $f(x)=\begin{cases}3 & \text{if } x>10\\ x-4 & \text{if } x\le 10\end{cases}$ $\lim_{x\to 10} f(x) =$	Possesses a prior schematic representation of what piecewise functions should look like, so she forced her result to appear as such. $x \to 10$ means the limit approaches 10 from the left and stops there. Drew the graph wrong. Lacks proficiency with piecewise functions. Could not break down the function into each component; did not mention domain; or order vs. magnitude of numbers.	She saw that this piecewise function represented and knew that given a jump discontinuity, the limit at a point did not exist.
Task 7: Explain the behavior of the function values as x approaches zero. Graph the function and explain whether or not the limit exists. $\lim_{x\to 0}\frac{1}{x^4}$	Was uncertain if the limit existed but concluded the limit does not exist and confirmed her result with the table, reason being the function is undefined at 0. Did not mention function values get larger without bound. Reported limit does not exist, drew $+\infty$ symbol, said that the limit existed and $=\infty$. Conclusions and interpretations inconsistent.	She made the same mistake as earlier, stating the limits exist and equal infinity because the function values were positive in both cases. The fact is the function values increase without bound, so the limit does not exist.

Task 8: Explain the behavior of the function values as x gets larger in the positive and negative directions and then as x approaches 0. Graph the function and explain whether or not the limits exist for both cases. $\lim_{x\to \pm\infty} \sin(x)$ $\lim_{x\to 0} \sin(x)$	Acknowledged the y-values this time, but said there are 2 limits because the function can't go past them. The y=-1 and y=1 are the restraints, drew horizontal lines there and called them limits. No report the limit didn't exist as function values oscillate and do not settle down.	Demonstrated proficiency with trigonometric functions. Reported the limit did not exist due to oscillations as x approached infinity. Reported the limit existed as x approached 0 and was equal to 0.
Task 9: Explain the behavior of the function values as x gets larger in the positive and negative directions. Graph the function and explain whether or not the limits exist. $\lim_{x\to\infty} \arctan(x)$ $\lim_{x\to -\infty} \arctan(x)$	Drew valid horizontal asymptotes, not imaginary ones from just drawing them in and so she made the same conclusion, that the there were 2 limits at pi/2 and negative pi/2.	Demonstrated proficiency with trigonometric functions. Reported that the arctangent is the inverse tangent, so the domain and range are interchanged. The limit existed as x approached plus/minus infinity given the horizontal asymptotes at pi/2 and -pi/2.

Task 10: Explain the behavior of the function values as x gets larger in the positive and negative directions. Graph the function and explain whether or not the limits exist. $\lim_{x \to \frac{\pi}{2}^-} \tan(x)$ $\lim_{x \to \frac{\pi}{2}^+} \tan(x)$	Used the calculator but did not reconstruct the graph correctly. Gave labels to areas of the graph that should have been labeled with $-\infty$. Did not report the limit does not exist because the function values get larger without bound in each direction.	Demonstrated proficiency with trigonometric functions and reported the limit did not exist due to periodicities as x approached infinity.
Task 11: Explain the behavior of the function values as x gets larger in the positive and negative directions. Graph the function and explain whether or not the limits exist. $\lim_{x \to \pm\infty} \frac{x^2 - x - 6}{x - 3}$	Factored correctly but said the limit exists and equals infinity in both directions. Should be $-\infty$ on the left and $+\infty$ on the right, and she should have said the limit does not exist in each case, but did not do so. She judges what the limit will be by what appears under the "lim" notation, and focuses on the direction the points follow along the x-axis when looking for a limit.	Demonstrated algebra proficiency with rational functions given she factored the numerator, then divided out common factors in numerator and denominator to get zero in the denominator and hence a hole in the graph.

Task 12: Construct a possible function from the graph below and explain if the limit exists as x approaches 2 from the left and as x approaches 2 from the right. Note: expect piecewise (one piece linear, other rational) $\lim_{x \to 2^-} f(x) = ?$ $\lim_{x \to 2^+} f(x) = ?$ 	Difficulty working with this task. Her first function was 2x because of the vertical line at 2 which she multiplied by x, then for the second function got 2x+3 to include the y-intercept where the hole is. She concluded the limit exists and equals 2 because that is what appears beneath the "lim" notation. She noted that the vertical asymptote is not only a restraint mentioned there are 2 limits at x=2 because there is a restraint and also a hole. Lacks skills with piecewise functions and lacks the mathematical skills to work from graphs to finding functions.	Though this problem was nontraditional since she had to work backwards, she possessed the algebra proficiency to determine that one piece of the graph would be linear and the other rational.

Summary

The table below summarizes different ways that two students with quite opposite levels of proficiency think about functions and limits and how their understandings manifest through problem solving tasks. It should be noted that neither student correctly interpreted the meaning of 1-1, a term reserved for inverse functions. Both students had vague ideas about what the difference was between a function and a function value and had vague ideas about what a limit was. Neither student could pinpoint a limit as being a number. Interestingly, CL thought limits are exclusively about x, not y whereas NS acknowledged that limits were about the behavior of the y-coordinate.

Both students had misconceptions about infinity in terms of thinking infinity is a very large number, and erroneously thought that a limit exists if it equals infinity, most likely because of the infinity symbol representing a large number. The limit notation was confusing for CL, who perceived the arrow beneath "lim" to be directional in nature, pointing to the right only and she also revealed that whatever appears below the "lim" notation is what the limit is going to be. So if $\lim_{x \to \infty} \frac{1}{x}$, CL thinks the limit will be ∞ because of $x \to \infty$ appearing beneath "lim".

NS recognized a piecewise function when she saw one, but CL did not and she did not think the isolated point above a piece involving a discontinuity even belonged to the graph of the function. Rational functions for limit at a point revealed that both students erroneously compared the left and right hand limits, which cannot be done with limits that tend toward infinity. NS erroneously used mathematical operations with infinity though infinity is not a number. CL's case was more pathological in nature, but she was also very consistent in her responses which suggested that her ideas about functions and limits were quite fixed, and that she was not guessing. For instance, CL repeatedly revealed that given a discontinuity with a hole, the limit exists because there is a hole to fall into whereas NS referred to the right hand limit being equal to the right hand limit for the limit to exist. Interestingly for a continuous function, CL thought the limit does not exist because one could walk right over the mark on a golf course, with no hole to fall into. CL associated limits with being able to fall into a hole, whereas NS referred to the function's behavior near the point.

Table E.1: Summary of Results from Pilot Study

CL

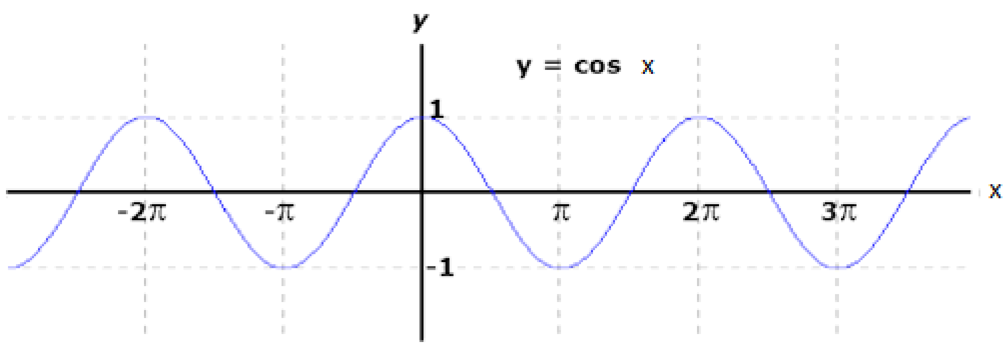
- A function is f(x)= x^2 .
- A function must be 1-1 (1 y for each x).
- Function values refer to a set of coordinates, x and y.
- Functions and function values are synonymous terms.
- Limits are physical barriers, “restraints” in the form of vertical asymptotes (brick walls) and holes one can fall into.
- Limits are about x, not y.
- Limit notation implies direction from left to right. Ex: $\lim\limits_{x \to 3}$ means approach 3 from the left.
- The limit will be whatever appears beneath the “lim” notation.
- Left and right hand limits are not compared for limits at a point.
- Limits exist where there are holes in discontinuous functions.
- Limits do not exist for continuous functions because there is no hole to fall into.
- Infinity is a large number.
- A limit exists if it equals infinity.
- The $\lim\limits_{x \to \infty} \frac{1}{x} = \infty$ because it never stops or touches the x-axis.
- The limit exists for $\lim\limits_{x \to 0} = \frac{1}{x}$ because of the brick wall the function can’t go past.

NS

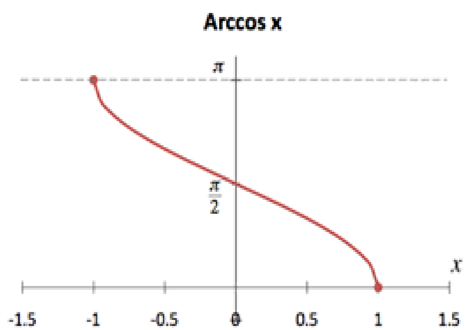
- A function is an equation that maps x an independent variable x into a single y.
- A function must be 1-1 (1y for each x)
- Function values are what functions equal at specific points in the form (x,y).
- Functions and function values are synonymous terms.
- A limit is a value a function approaches as x approaches a limit value.
- Limit notation not confusing.
- Limits are about y, not x.
- Left and right hand limits are compared for limits at a point, including rational functions.
- Limits exist when there are holes in discontinuous functions because the left hand limit equals the right hand limit.
- Limits and function values are different given a piecewise function with a discontinuity and a function value.
- Infinity is an indefinable large number.
- A limit exists if it equals infinity.
- The $\lim\limits_{x \to \infty} \frac{1}{x} = 0$ because values get smaller.
- $\frac{2x^3}{x^2} = \frac{2\infty}{\infty} = \infty$ so limit exists and equals infinity.
- The limit does not exist for $\lim\limits_{x \to 0} = \frac{1}{x}$ because the left hand limit $-\infty \neq$ the right hand limit ∞.

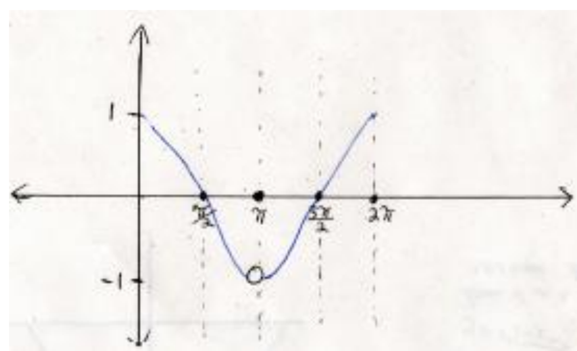
APPENDIX F: TASKS USED IN PHASE II WITH DESRIPTIONS

Task 1: Explore Cosine, Arccosine and Piecewise Component



Graph A: cos x





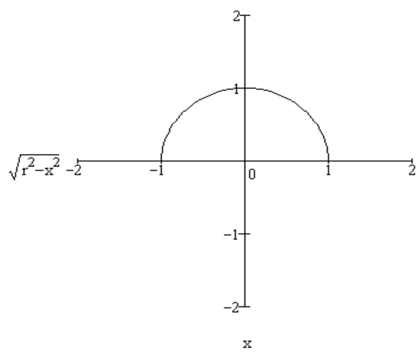
Graph B: Arccos x

Graph C: Piecewise component

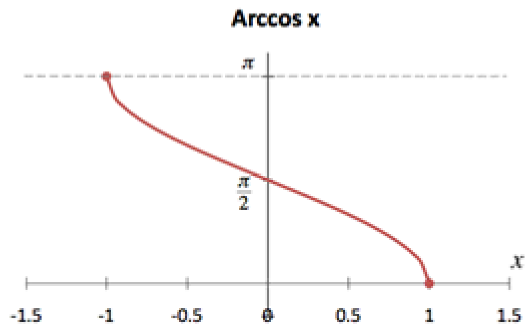
Define function. What are function values? Is it x, y, or both coordinates? What is the difference between a function and a function value?
Describe what 1-1 means and when it's used. Is domain part of the definition of a function?
What is a limit? Define limit at a point and limit at infinity.
Describe any association you see between domains and finding limits.
What is an infinite limit? Explain or draw a graph with an example.
Describe the domain and ranges. Compute these limits and explain if the limits exist.
Graph A: $\lim_{x\to 0}\cos x$ $\lim_{x\to \pi}\cos x$ $\lim_{x\to \infty}\cos x$ $\lim_{x\to -\infty}\cos x$
Describe the domain and ranges. Compute these limits and explain if the limits exist.
Graph B: $\lim_{x\to 0}\arccos x$ $\lim_{x\to 1}\arccos x$ $\lim_{x\to -1}\arccos x$ $\lim_{x\to \infty}\arccos x$ $\lim_{x\to -\infty}\arccos x$
Study the graph and explain if the limit exists as x approaches pi.
Graph C: What's the following limit: $\lim_{x\to \pi} f(x)$? Does the limit exist? Explain why or why not. Explain if $(\pi, 0)$ is on the graph of the function.

Task 2: Domain investigation with functions and limits

Half Circle

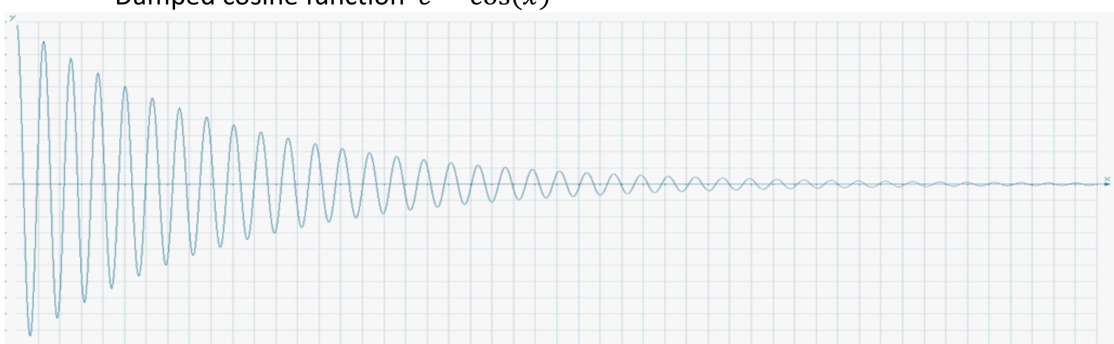


Graph A



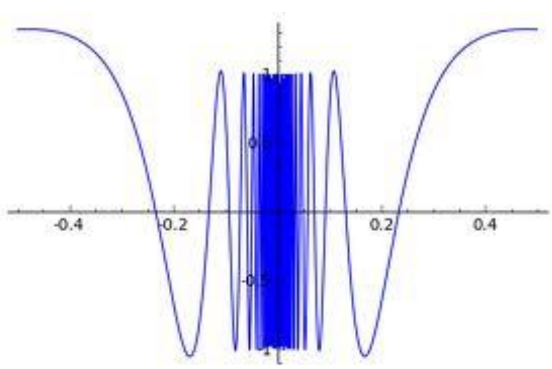
Graph B

Damped cosine function $e^{-x}\cos(x)$

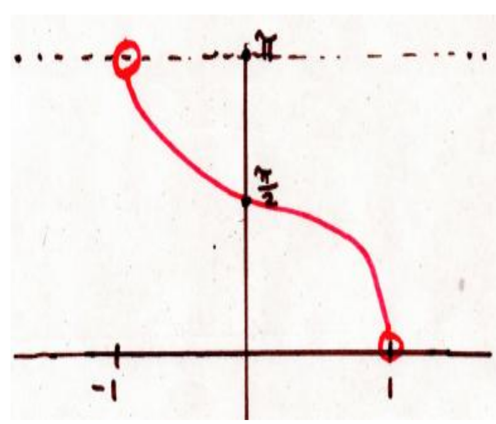


Graph C

$\cos\frac{1}{x}$



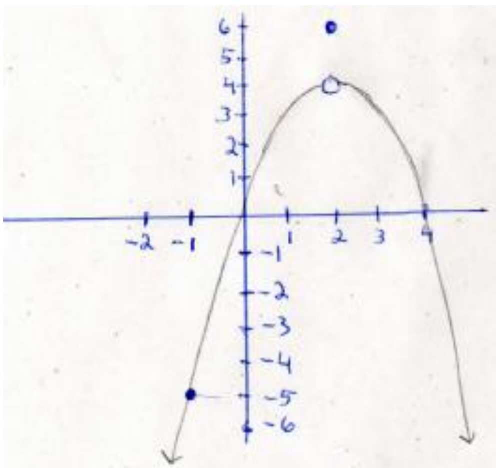
Graph D



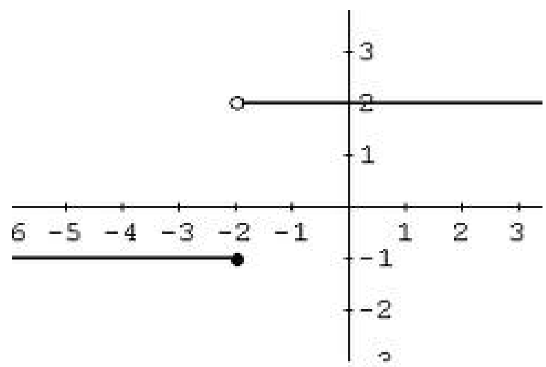
Graph E

State the domains and ranges for each function.
What does the function tell you about the domain?
How does changing the domain change the function?
Compute the given limits, at a point and at infinity. Explain if these limits exist or not.
Explain what the “lim” notation means. Does it imply direction as x approaches a number from the left only? Is the limit revealed underneath the “lim”notation, $\lim_{x\to\infty}$ or $\lim_{x\to 3}$?
Are limits about finding out what x is doing or what y is doing?
What does the ∞ symbol mean? Can a limit exist if it equals infinity? Why or why not?

Task 3: Piecewise Functions



Graph A



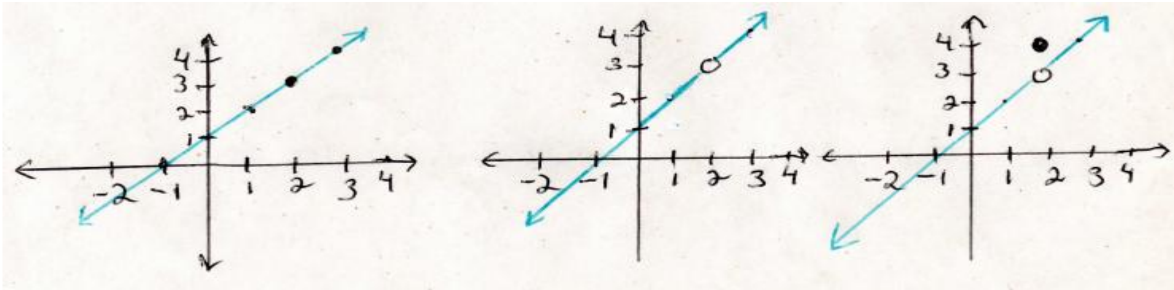
Graph B

Name what type of functions these are. State the domain and ranges. Write the formulas for each function. State any similarities or differences between the 2 graphs. Is the point (2,6) on the graph of the function for Graph A? Explain why or why not. How does knowing the domain assist with knowing if the point is on the graph of the function? How does the domain assist with writing the formulas for these graphs?
Compute the limits and explain if the limits exist in each case?

Graph A: $\lim_{x\to 2} f(x)$ $\quad \lim_{x\to\infty} f(x)$ $\quad \lim_{x\to -\infty} f(x)$

Graph B: $\lim_{x\to -2} f(x)$ $\quad \lim_{x\to\infty} f(x)$ $\quad \lim_{x\to -\infty} f(x)$

Task 4: Progression from Linear to Piecewise Functions



Graph A Graph B Graph C

Compare the 3 graphs. State the domains and ranges for each, then compute the limits and explain if they exist.

Graph A: $\lim_{x\to 2} f(x)$ $\lim_{x\to\infty} f(x)$ $\lim_{x\to-\infty} f(x)$

Graph B: $\lim_{x\to 2} f(x)$ $\lim_{x\to\infty} f(x)$ $\lim_{x\to-\infty} f(x)$

Graph C: $\lim_{x\to 2} f(x)$ $\lim_{x\to\infty} f(x)$ $\lim_{x\to-\infty} f(x)$

Explain if the point (2.4) is on the graph of the function.
Explain how the domain changes for the graphs and how that is associated with the limiting behavior near 2. What does the function tell you about the domain?
How does changing the domain change the function?
Explain or give an example of how different functions could have the same domain?

Task 5: Linear Function

Sketch a graph of this function and answer the following questions.

$f(x) = 2x+1$

$\lim_{x\to 3} 2x+1$ $\lim_{x\to 2} 2x+1$ $\lim_{x\to\infty} 2x+1$ $\lim_{x\to-\infty} 2x+1$

Explain if these limits exist? Is (3,8) on the graph of the function? Explain why or why not.

Task 6: Rational Function with Sum of Squares

$\lim_{x\to 2}\frac{x^2+4}{x-2}$ $\lim_{x\to\pm\infty}\frac{x^2+4}{x-2}$

Limit 1 Limit 2

Describe the domain. Compute the limits if they exist. Explain the behavior of the function values.

Task 7: Rational Function with Difference of Squares, Hole in Graph

$$\lim_{x \to -3} \frac{x^2 - 9}{x + 3} \qquad \lim_{x \to \pm\infty} \frac{x^2 - 9}{x + 3}$$

Limit 1 Limit 2

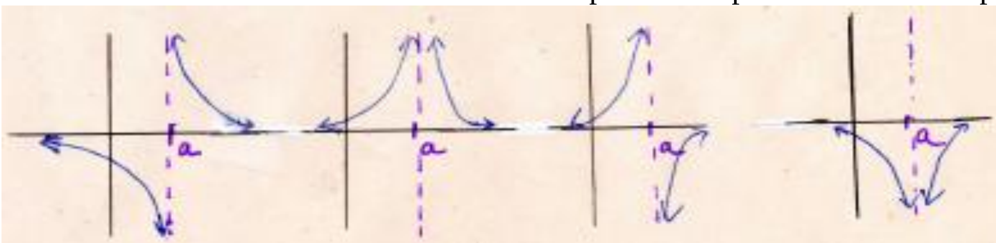
Describe the domain. Compute the limits if they exist. Explain the behavior of the function values.

Task 8: Rational Function with Horizontal Asymptote as the Limit

$$\lim_{x \to \pm\infty} \frac{9x^2 + 2}{3x^2 - 2x + 5}$$

Describe the domain and the limiting behaviors. Sketch the graph and explain how the function behaves for large x in the positive and negative directions. Do the limits exist? What's the relationship between horizontal asymptotes and limits, if any. Describe how knowing the domain is associated with knowing how to find the limits.

Task 9: Rational Function Match Graphs and Explore Limits of Graph 1



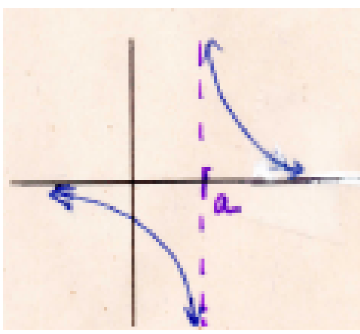
Graph 1 Graph 2 Graph 3 Graph 4

a. $f(x) = -\dfrac{1}{x-a}$ b. $f(x) = \dfrac{1}{x-a}$

c. $f(x) = -\dfrac{1}{(x-a)^2}$ d. $f(x) = \dfrac{1}{(x-a)^2}$

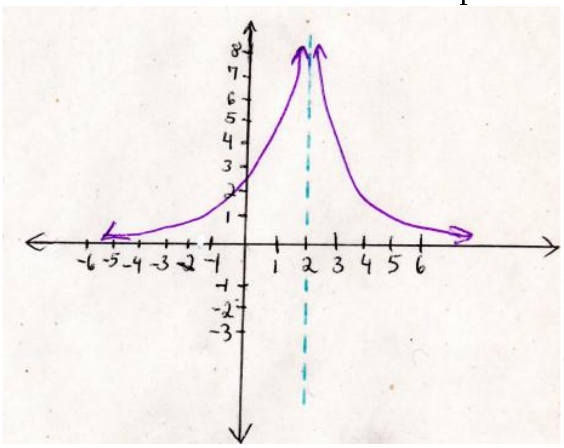
Match the graphs with the correct functions and label the graphs a, b, c and d accordingly.

Describe the domain and explore the limits of the first function whose graph is repeated below. Explain if these limits exist. Is it necessary to split up the first limit into $\lim_{x \to a-}$ and $\lim_{x \to a+}$ to decide if the limit exists at "a"? By splitting them up, do they exist separately? Explain.

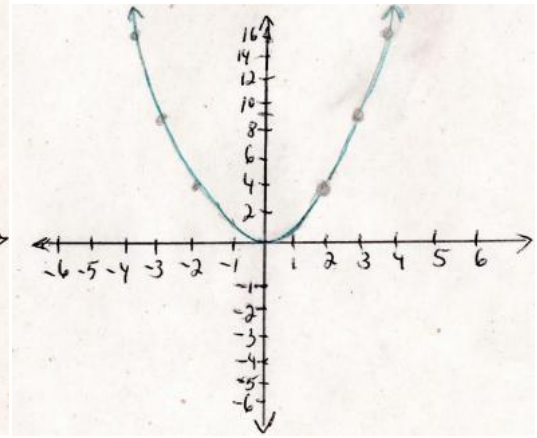


$$\lim_{x \to a} \frac{1}{(x-a)} \text{ and } \lim_{x \to \infty} \frac{1}{(x-a)}$$

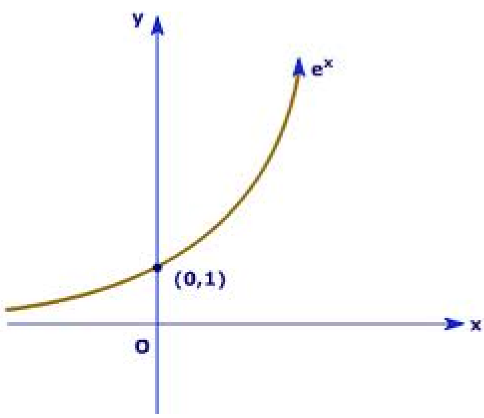
Task 10: Compare End Behaviors of 3 Graphs



Graph A



Graph B



Graph C

Describe the domain and range for each of the functions and then find their limits. Explain how knowing about the domain is associated with finding limits. Describe the end behaviors for each one.

Graph A: $\lim_{x \to 2} \frac{1}{(x-2)^2}$ $\lim_{x \to \pm\infty} \frac{1}{(x-2)^2}$

Does the limit exist as x approaches 2? Does it exist as x approaches 2 from the left only? Approaches 2 from the right only? Both together from the left and right? Explain. Does the limit exist as x approaches +/- infinity? Explain. Explain the behavior of the function values as x increases without bound in both directions. If the limit equals infinity on one side alone, does the limit exist? What about if the limit equals infinity on both sides, then does the limit exist? Does it exist because the arrows both point upwards in the same direction moving toward the line x=2? Do you compare the left hand limit with the right hand limit? Explain.

What is the difference between a limit that does not exist versus a limit that is equal to infinity? Are there any instances in which a limit does not exist other than it equals infinity? If a limit equals infinity, does it exist? Why or why not. Explain what the ∞ symbol means. Describe some graphs or problems where the limit does not exist for various reasons.

Graph B**:** $\lim_{x \to 2} x^2$ $\lim_{x \to \pm\infty} x^2$

Does the limit exist as x approaches 2? Does the limit exist as x approaches +/- infinity? Does the limit exist as x approaches plus infinity only? As x approaches minus infinity only? Explain the behavior of the function values as x increases without bound in both directions. If the limit equals infinity on one side alone, does the limit exist? What about if both sides or arrows point up outwards and go to infinity, then does the limit exist? Does it exist because the arrows both point upwards in the same direction and move away from the y-axis? Do you compare the left hand limit with the right hand limit? Explain.

Are there any similarities or differences between Graphs A and B in terms of deciding if limits exist as the function values increase without bound? Notice the arrows on Graph A converging towards the y-axis, but in Graph B, the arrows head out in opposite positive directions. Do limits exist in one case but not in the other? Explain.

Graph C: What's the limit as x approaches 0? $\lim_{x \to 0} e^x$ Does this limit exist?

Identify the limiting behavior of the function values for $\lim_{x \to \infty} e^x$ and $\lim_{x \to -\infty} e^x$

Explain if these limits exist. Do you compare the left hand limit to the right hand limit, or do you only consider one side at a time to decide if each limit exists? Explain.

APPENDIX G: PHASE II DETAILED RESULTS

Introduction

The analysis that follows presents in-depth ways that students think about limits. The impact of domains was a focus as well as to what extent their various definitions of limit and infinity consistently manifested in their work on various tasks. Following these results, summaries of all nine students' work on the tasks are presented.

Research Questions

The original research questions were: How do students think about limits and how do their understandings manifest through various tasks? Given the results of the pilot study in Phase I which showed revealed different ways students think about limits at a point, limits at infinity and limits that do not exist, the following research questions were developed:

Limits at a Point

Research Question 1A

In what ways do students apply their definition of limit to their task solutions?

Research Question 1B

In what ways do students perceive the domain to be involved when deciding if a limit exists at a point, i.e., for $\lim_{x \to a} f(x)$ does "a" have to be in the domain?

- Functions defined for all x.
- Functions defined on finite intervals: closed and open endpoints.

Research Question 1C

In what ways do algebra skills affect task solutions?

Limits at Infinity

Research Question 2A

Given $\lim_{x \to \infty} f(x)$, in what ways do students think the domain is involved given limits of functions with finite interval domains?

Research Question 2B

Do students recognize a relationship between a computed limit at infinity of a rational function and the horizontal asymptote?

Research Question 2C

In what ways do algebra skills affect task solutions?

Limits that Do Not Exist

Research Question 3A

In what ways do students apply their definition of infinity to infinite limits and other limits that do not exist in their task solutions, including the comparison of left versus right infinite limits with rational functions?

Research Question 3B

In what ways do students think about limits that do not exist for various functions, particularly, under what conditions do students write "d.n.e." versus $= \infty$?

Research Question 3C

How do students distinguish between a limit at infinity and an infinite limit?

How New Tasks Help Answer Research Questions

A total of ten tasks that appear in Appendix F are presented in this study to address the hypotheses and answer the more focused research questions. These tasks constitute a combination of those previously asked as well as some new ones that better address several unanswered questions in the initial two cases. In the previous study, several discrete tasks were asked on functions, then on limits. The presentation style has been revised so that each task will now ask everything across the spectrum from functions to limits, capturing more information with fewer tasks. In the last study, a lot of time was spent on computations but with the current tasks, computations are minimized because the focus is mostly on the limiting behavior seen on graphs of functions; however, computations of rational functions have not changed as this captures difficulties with algebra that could cause the wrong limit to be computed.

More emphasis is placed on definitions of limit and infinity and domains of functions, given this is central to understanding limiting behavior of all types of functions. In fact, results from the initial cases gave rise to further probing into their understanding of finite domains versus infinite domains and so a new task has been designed exclusively for capturing the students understanding about these domains. The initial cases revealed that students don’t realize when one changes the function, one has also changed the domain. Students seem to disconnect the relationship between functions and domains, and so the new questions help address a main focus of the study which is to identify how a student must first understand functions in order to understand limits. Therefore, questions about domains are presented with all ten tasks. Unlike the tasks for the initial cases, more graphs will be presented with the function already given. By doing this, students can easily explain characteristics about functions such as the definition, domain, inverse and meaning of 1-1, and differentiate between functions and function values. This also facilitates describing limiting behaviors of function values across tasks for limits at a point and for limits at infinity.

Further probing into the limit notation occurs to further explore how it is interpreted, given the initial cases revealed misconceptions about this. A specific look at if students perceive what occurs under the “lim” notation is what reveals the limit, as well as whether or not the arrow implies direction from the left, as previously seen. A further investigation of students understanding of infinity will also occur given results of the initial cases, to identify why students think that a limit can exist if it equals infinity, and doing mathematical operations with infinity. A specific look at how limiting behavior is determined will occur by them pointing to the graph and pointing with their finger to explain what they are thinking, and this will help determine if they focus on the function values increasing or decreasing without bound, or if they follow points on the graph. A further investigation into the understanding of limits that don’t exist is also new to the study because it will gather specific information about if students can differentiate between a limit that don’t exist because the function values tend toward infinity, versus those limits that don’t exist due to jump discontinuity or oscillatory behavior.

One new goal of this data collection process is to maximize verbal articulation and minimize time spent drawing figures and responses because reconstructing figures from the original booklet consumed too much time. Therefore, these better designed tasks utilize less interview time and easier transcription since every turn or episode number is across subjects. In order to capture significant aspects of the tasks presented, students will be given tasks on large sheets of paper prepared ahead of time in order to minimize their time spent with drawing. They only add remarks to the large sheets including definitions, computations, as well as constructing and labeling graphs.

Breakdown of Specific Tasks to Answer Refined Research Questions

Task 1 is a new task that explores the function cos x, arccosine x and piecewise cosine and addresses most all research questions and hypotheses that evolved from the initial two cases. All specific interview questions are cross referenced with each research question that evolved from each new hypothesis. In the past, sin(x) was used and generated a wealth of information, but had limitations due to the fact that both the limit was 0 as x approached 0 for limit at a point. However, cos(x) fixes this problem because as x approaches 0, the limit is 1, so the researcher can clearly distinguish which coordinates to which the students make reference.

This is a continuous function which explores a full spectrum, in-depth understanding of functions as well as the limiting behavior. It can extract knowledge about the definition of function, domains, inverses, and with distinguishing between functions and function values. It can also pinpoint the student's understanding about nearness, the notion of nearness, the limit notation and the meaning of infinity. A discontinuity with a hole in the graph, followed by a jump discontinuity in which the limit exists but is not equal to the value of the function occur subsequently, so this ties in with their understanding of the definition of function and what it means for a point to be on the graph of a function. Meanwhile, as a limit at infinity, it will be important to study what students think happen as x increases without bound. Since the end behavior oscillates, never settling down to any specific point, this is an interesting case in which a limit does not exist as x increases without bound. Given the results of the initial cases, students confuse what happens with the end behaviors and will think the limit equals infinity in both directions rather than realize that the limit does not exist. This presents the opportunity to probe into their understanding of infinity as it pertains to the x behavior. Therefore, this task can reveal their understanding of the end behaviors.

Task 2 is a new task that requires comparing 4 different graphs the half a circle, arccos(x) with closed and open endpoints, the damped cosine function $f(x) = e^{-x}\cos x$ and $f(x) = \cos\frac{1}{x}$. Given finite interval domains, students are asked about how the limit does not exist as x approaches infinity because it doesn't even make sense to ask that question. This is tightly coupled to understanding domains as the first two graphs consist of finite domains. The last graph of the damped cosine has an unrestricted domain and constitutes an example of a limit that does not exist as x goes to minus infinity because the amplitude increases at an exponential rate without bound, whereas at plus infinity, the amplitude decreases to zero. An example of this function's application is the ringing of a bell. The sound eventually dies out, providing us with a physical application example of limit at plus infinity. This is also an example of an infinite limit (as x goes to minus infinity) because the waveforms keep getting larger, oscillating without bound, and so due to deficits in understanding of limits that do not exist in the initial cases, new students will be probed about what it means in this context for a limit not to exist. In the last graph of $f(x) = \cos\frac{1}{x}$ students must recognize the limit does not exist as $x \to 0$ due to the oscillatory nature of the function rather than to the function not being defined with 0 in the denominator. The purpose of this task is to explore how students describe the behavior of function values near 0 and to what extent they acknowledge the limit does not exist due to the oscillatory behavior.

Task 3 combines two tasks of piecewise functions previously used with the initial cases. One is a piecewise quadratic, the other is a stepwise function. The purpose of these is to explore piecewise functions with jump discontinuities as these are common functions seen in calculus, and also pose great difficulty to students. It answers most research questions and hypotheses that evolved from the initial two cases. All specific interview questions are cross referenced with each research question that evolved from each new hypothesis. Two graphs are given, one with jump discontinuity in which the limit exists, and the other with jump discontinuity in which the limit does not exist so further probing into their understanding of functions will occur. Results of the initial cases suggest a need for continued probing into their understanding of the limit notation and the meaning of infinity, which should reveal what they think about nearness, limits at infinity and infinite limits versus limits that do not exist. This task addresses important questions about functions and knowledge about domains, as well as reasons why they don't recognize the first graph as being piecewise, while probing into their understanding of the behavior of function values near a point. Knowing about the domains of the function here is nicely linked to their understanding of the limit not existing for the second graph in which case the limit does not exist due to jump discontinuity whereas in the first graph, the limit exists but is not equal to the value of the function. The end behaviors are also considered. This is not the primary focus, but will help reveal their understanding about infinity as it pertains to both x and y, and hopefully, to replicate prior responses.

Task 4 is an extension of a previously used task in the pilot study. It is composed of 3 graphs that start out being a straight line for a linear function, then a straight line with a hole in it, and finally a straight line with a hole and a dot (point) above it. Previously, only the first 2 graphs were compared. The purpose of this task is to consecutively explore their understanding of domain and how it relates to understanding the limiting behavior of function values near a point and what it means for a point to be on the graph of a function. All specific interview questions are cross referenced with each research question that evolved from each new hypothesis. It answers most research questions and hypotheses that evolved from the initial two cases. In the first graph, the limit exists and is equal to the value of the function. In the second graph, the limit exists but results of the initial cases show that students sometimes get confused with the hole and might say that the limit doesn't exist because of the hole. Students typically won't see there is no function value at that point. In the 3rd graph, there is an opportunity to replicate the above findings with piecewise functions as it explores if the student understands that the change in domain changes the function and that the limit exists but does not equal the value of the function. Repeated probing into their understanding of the limit notation and the meaning of infinity also occurs. The end behaviors are also considered though not the primary focus, but will reveal their understanding about infinity, and about the limit not existing if it equals infinity.

Task 5 was previously used with the initial cases. It consists of a straight-forward single linear function in which they have to first draw the graph of the function. It answers most research questions and hypotheses that evolved from the initial two cases. All specific interview questions are cross referenced with each research question that evolved from each new hypothesis. Questions about definition of function, the meaning of 1-1, and domains are asked to probe deeper into their understanding about later questions that follow on limits. One purpose of this task for limit at a point is to see if they can work with a simple function to compute and graph. Also it is used to extract any possible idiosyncrasies similar to what CL did, such as thinking limits are all about x and drawing vertical asymptotes where the limit is and thinking the limit doesn't exist because there is no hole to fall into. Further probing into their understanding of the limit notation and the meaning of infinity also will also occur. The end behaviors are also explored because they go in opposite directions and so results from the initial cases suggest students might erroneously compare the left hand limit $-\infty$ to the right hand limit $+\infty$ and conclude the limit does not exist because both sides are not equal. Finally, further probing into their understanding about infinity occurs, to replicate prior responses about the limit existing if it equals infinity.

Task 6 was previously used in the initial study. It is a rational function is computational in nature, in which there is a sum of squares in the numerator, so no factoring can be done though it is expected they will try as seen in the initial cases. It answers most research questions and hypotheses that evolved from the initial two cases including those specific to rational functions not used with other tasks previously seen. All specific interview questions are cross referenced with each research question that evolved from each new hypothesis. This task gives an opportunity to explore their algebra skills including knowledge about the domain of this function and knowledge about using rules for asymptotes or factoring out the highest power of x separately from the numerator and denominator to compute the limit. It's an opportunity to find out if they know the difference between the techniques used for limit at a point versus limit at infinity, as these are quite different. Questions now get more specific to the behavior of the function values near a point and at infinity and address if they think a limit exists or not if it equals infinity and what it means for a limit not to exist.

Task 7 was previously used in the initial study. It is a rational function, also computational, involving a difference of squares in the numerator which can be factored. It answers most research questions and hypotheses that evolved from the initial two cases including those specific to rational functions. All specific interview questions are cross referenced with each research question that evolved from each new hypothesis. There's an opportunity to explore their algebra skills and their understanding of domain since there are common factors which divide out, hence, changing the function. For limit at a point, the function will have a hole in the graph so further questioning can be asked if the limit exists and what the hole means since results from the initial cases revealed major difficulty with this. For limit at infinity, there's an opportunity to further explore their thinking about the limit not existing as seen with task 4 with the linear function, because results of the initial cases suggest students think the left hand limit does not equal the right hand limit, e.g. $-\infty \text{ on left} \neq +\infty \text{ on right}$. One goal is to find out if they know that the techniques for finding a limit at a point differs from those for limits at infinity, and whether or not they know about the rules of asymptotes to find limits at infinity, comparing the degree of the numerator with the degree of the denominator. This is because of algebra deficits and lack of knowledge about the rules for finding asymptotes occurred in the initial two cases. One sided limits are explored to determine if those limits exist and why they do not just consider one side exclusively when deciding whether or not the limit exists for the entire function, since only once side is needed. This was a very common source of difficulty in the initial two cases.

Task 8 was previously used in the initial study. It is a rational function that's specific to generating a horizontal asymptote which is also the limit. Further probing into their understanding of functions and domains occurs, as well as their knowledge of rules for finding asymptotes, as there was evidence of deficiencies in the initial two cases. All specific interview questions are cross referenced with each research question that evolved from each new hypothesis. The focus is on exclusively studying the students understanding of limit at infinity and the objective is to explore the student's solution technique. This task helps answer most research questions and hypotheses that evolved from the initial two cases including those specific to rational functions. We determine if they use the algorithm that compares the degree of the numerator with the degree of the denominator (in this case they are the same) or the more comprehensive technique of producing reciprocal powers of x by separately factoring out the highest power of x in the numerator and the denominator. Of course we would very much like to know if they associate limits at infinity with horizontal asymptotes, and if they acknowledge that the limit exists rather tends toward infinity near the horizontal asymptote as done in the initial cases. This task gives rise to understanding what students perceive going on as x tends toward infinity, whether they focus on the behavior of the function values in which case they would determine the correct horizontal asymptote as the limit, or if they simply follow points on the graph and determine that the line keeps going to infinity.

Task 9 was previously used in the initial study. This task involves matching 4 graphs to 4 corresponding rational functions. Upon doing this, the function 1/x is focused upon with similar in-depth questions about functions and domains. This task helps answer most research questions and hypotheses that evolved from the initial two cases including those specific to rational functions. All specific interview questions are cross referenced with each research question that evolved from each new hypothesis. The limit at a point should reveal that the limit does not exist but instead students might think the limit does exist and equals infinity as occurred in the initial cases. So there is opportunity to explore the reasons why students think this is the case. Also as x approaches infinity, the limit is 0 but students might think it equals infinity, so the reasons why will be explored since this also occurred with the initial cases. This is also an example of an infinite limit and so they will be probed about what it means in this context for a limit to exist or not to exist. We also probe into what students how students think about limits that do not exist in general, if the know that writing “equals infinity” is appropriate for some limits that don’t exist, but for other cases, such as oscillations or jump discontinuities, that the limit just does not exist for reasons unrelated to being equal to infinity.

Task 10 was previously used in the initial study. It involves comparing graphs of 2 rational functions, $\frac{1}{(x-2)^2}$. x^2 and e^x. This task answers most research questions and hypotheses that evolved from the initial cases, thereby exploring infinite limits for both limit at a point and limit at infinity. Questions about functions and their domains are still asked. All specific interview questions are cross referenced with each research question that evolved from each new hypothesis. The primary concern is whether the students think limits exist when we write the limit is equal to infinity and how they perceive the direction of the arrows, since in the first case the arrows converge but in the second case, the arrows diverge and so they might say the left side of x^2 goes to minus infinity, as previously seen in the initial cases. Further probing will reveal reasons why they think so, such as if they follow points on the graph along the x-axis rather than looking at the behavior of the function values. This task presents an opportunity to explore previous observations of confusing x and y with these tasks.

How New Data Extends Results in Pilot

Ten tasks in Appendix F were used as a means of addressing the new hypotheses and research questions that emerged from the two initial pilot cases. These tasks combined those previously used as well as new ones to address new or unanswered questions in the initial two cases. The summarized work from two selected students reveals further evidence of how students think about functions and limits. The responses from the new tasks shows how the students' work extends beyond the pilot students and how it enables research inferences to be made that goes beyond the initial two pilot cases.

The way this work extends beyond the pilot students and how it enables making research inferences is accomplished with the tasks themselves. The tasks start off by asking about domains and ranges, then go into questions about limits at a point, limits at infinity, limits that do not exist and infinite limits. This evolving flow of questioning across the board from functions to limits for each tasks revealed how students think in detail about particular problems. Spending enough time on such tasks facilitates probing deeper into learning how students consider the relationship of domains to finding limits. This new data has an emphasis on domains throughout to see the extent to which understanding limits depends on understanding functions. Looking at domains including those with restricted intervals was not previously done and by focusing on domains, this gives meaningful answers to address in what ways students think about functions and limits.

In the initial study, both CL and NS articulated different ideas about what a function was, and did not know that 1-1 was not part of the definition of function. This seemed unusual since functions are taught throughout the middle/secondary curriculum. Neither CL nor NS knew what the term "function value" was. CL thought that function values involved both the x and y coordinates, not just the y-coordinate while NS thought that functions and function values were synonymous terms. Given these misunderstandings, the new interviews probed further by having them verbally articulate what function values were and to physically identify locations on the graph that represented functions and function values.

The new data replicates previous findings in which YJ and EB's understanding of functions stemmed mostly from knowing that a function has two variables which included the standard input-output representation of a function, x in, y out. They both knew y is the function value.

What the new data shows that was not previously addressed is that these students lacked a mathematical connection of functions with domains and of the more general notion of a relation. This is evidenced by how they did not connect the domain of the function with the function itself which manifested the inability to recognize piecewise functions. They had a good grasp on the graphical representation of a function as long as the function was continuous in which case where the domain was not particularly relevant in which case it was easy for them not to consider domains, but in the other cases involving piecewise functions, not considering domains caused difficulties.

When a point was missing from the domain or there was a jump discontinuity, the graph did not fit their internalized graphical representation. The new results showed that students think limits do not exist where there are open holes in graphs because the domain is not defined at the point, and they also think that in the case of the piecewise function that has a solid dot above the hole, that the limit is the function value in the solid dot. YJ did not think that limits could exist at a point if the domain was not defined there, but EB did. Further probing into finding limits, such as with Task 4, revealed that students did not recognize that when the domain changes, the function also changes evidenced by the fact that they could not come up with a new function to represent the change in the graph. Also, there was difficulty seeing that there could be two different functions for the same domain in which case the limit is also the same.

When presented with finite interval domains, they seemed to understand that the limit cannot exist as x approaches infinity outside the interval, but there were issues about including the endpoints when finding the limits. With finite intervals YJ thought it only made sense to ask about limit approaching 1 from the left and approaching -1 from the right because x approaching 1 by itself was invalid. There was confusion over whether the limit could exist at endpoints because of what the function values were only approaching the endpoints from one side. So they did not consider the domain when finding the limit because they did not acknowledge that the point was included in the domain. What was found this time, unlike with the pilot study, is that she did not consider the endpoints being in the domain so this new data answers the research questions by showing how knowing about the domain is key to understanding functions.

Since domains and ranges were not previously addressed in the prior study, what was found with the new results is that students struggle with identifying domains and ranges of piecewise and rational functions. The results answer the research questions about how students think about domains and ranges, and support the hypotheses that understanding functions is prerequisite to understanding limits.

Neither CL nor NS gave a definition of limit, but just a simple description, and further probing with the new students resulted in the same responses. EB thought the limit was “looking at the behavior at a certain point” whereas YJ thought a limit was “a place where a function is defined because there is no constraint.” Constraint meant no hole or vertical asymptote. These responses answer the research questions by comparing the description of limits the students described in the initial cases to the new ones.

CL and NS had different interpretations of the notation. CL thought that the arrows below the “lim” $x \to 2$ notation implied direction “approaching 2 from the left., but NS did not. Also, CL thought that limits were only about x, not y. NS did not have these problems, but thought it was wrong to write $\lim_{x \to \pm\infty} f(x)$ with the plus and minus together, and thought this was not a valid way to write them. Additional data was collected to see what other students think and both YJ and EB were more like NS in the sense that they understood that the limits involved the function value, not the x-value. Neither one thought that the notation under the “lim” $x \to \pm\infty$ was problematic, and even said that just means they can be split up into two separate limits at infinity. These responses answer the research questions by replicating the task and by comparing the responses of the students in the initial cases to the new ones.

Before collecting this new data, CL showed evidence that she did not know when a limit at a point existed or did not exist whereas NS knew quite well. CL thought that the limit existed where a hole was because one could fall into it and the limit did not exist when there was a solid dot on the line because one could walk across it without stopping. So the new data attempted to replicate these findings and answered the research question by looking at how other students think about limits at a point. Specifically, when given a straight line with a solid dot, both YJ and EB said the limit existed but when the dot changed to an open hole, YJ said “the limit does not exist because the function was undefine there”. While EB seemed to know that the domain changed going from a solid dot to a hole, she did not think the domain really had anything to do with the limit. All she had to do was see that if x is in the domain, then there is a limit. On the other hand, YJ thought that the limit existed where there a solid dot but did not exist if there was a hole because the domain did not include the point. In the old study, domains were not addressed with these problems so the new data addresses how domains are involved with understanding limits. So these new findings extend beyond the initial study and the results answer the research question by showing how students think about domains as they pertain to limits.

In the initial study, both CL and NS had different notions about what the infinity symbol meant and showed misunderstandings with infinity in general. So the new study expanded up on this. More research was done on how students perceive the infinity symbol. They way they articulated and wrote their responses helped to answer this research question because YJ revealed that infinity was “undefined” whereas EB thought of infinity as a very large number. These responses were proven to be important later on with understanding limits that do not exist because since YJ thought infinity was undefined, then the limit could not exist if function values went to infinity whereas EB thought that the limit would exist since infinity was a number.

In the initial study, CL was following points on line instead of looking at the behavior of the function values, so she saw the line going to infinity rather than going to 0. NS, though, did not have this problem. This new data helped answer questions on limits at infinity because with the deeper probing it was seen how YJ sometimes confused x and y when finding limits at infinity because she was also following points on the line whereas EB clearly focused on the behavior of the function values.

The new data helped answer the research question about algebra deficiencies by looking further into what these deficiencies were. They had to do with factoring, rules for asymptotes, when to get holes in graphs, and knowledge about domains. As an extension from the last study showed that in this study, it was observed that neither student could describe what to do differently for a limit at a point versus a limit at infinity, and did not know when there would be a vertical asymptote or a hole. Attempting to factor a sum of squares in the numerator was seen with YJ as well as not considering the domains of these functions correctly. If YJ knew the domain, she did not use that information for finding limits at infinity for the half circle or arccos(x) that had restricted interval domains. She thought the limit would be equal to infinity but it did not exist because for x’s outside of the interval. EB had better understanding, evidenced by the fact that she knew that a point had to be in the domain in order for it to have a limit, so when there were holes within lines, she knew that the limits existed where the hole was whereas YJ did not. YJ's conception of a limit is based on the graphical representation of a function.

She disconnected the notion of behavior of the function values and looks at points on the graph. No point, no limit. She replaces the behavior of the function values with the function value at the limit point. So her understanding of limits is strongly tied to points on the graph, not either x or y, but the point (x,y). The fact that she used the function terminology of the function being undefined at a point, i.e. the point is not in the domain of the function, and applied that to limit, indicates again that she has dissociated the domain of the function from the function itself.

NS thought the infinity symbol represented a number that was too large to measure, but because she thought it was a number she performed mathematical operations with it. She also said when function values approached infinity, the limit existed. CL had similar problems, and like NS, compared the left hand infinite limit to the right hand infinite limit instead of just considering one side to determine the limit did not exist. The new data helped answer the research question by probing further to replicate these findings, which was successful because results showed that EB compared the left hand limit with the right hand limit, both at infinity and at a point when it was not necessary to do so and this is because she thought infinity was a number. So she was comparing numbers on the left and on the right. YJ, on the other hand, only considered one side but kept saying that the limit was “undefined” rather than say it did not exist because it the function values went to infinity.

Unlike the first study, the new data collect extends the exploration of domains of various functions, with emphasis on piecewise and those with finite intervals. There were also issues with transferring the information from the computation to the graph, and improper procedures were used to find limits at infinity such as using L-Hospital’s Rule.

So this new data not only replicates some previous findings but also extends beyond the pilot study and answers the research question by having looked more closely at domains of different functions including those that oscillate, piecewise, rational and those with restricted intervals domains.

While CL thought that “infinite limits” meant “multiple limits”, NS knew what infinite limits were about. Further probing with the new students involved having them explain and draw graphs of what they thought an infinite limit was, and they also had to distinguish between that and limits at infinity. YJ knew the difference, but EB did not evidenced by the words were synonymous. This research question extended beyond the pilot, involving further probing, which compared the responses of the initial cases to the current ones, confirming the finding that there is confusion about what is meant by an “infinite limit.”

This topic is an extension from the last study because limits that do not exist were not looked at as a separate entity. Given the results of the pilot though, doing this seemed important. In the pilot, CL had esoteric notions about when limits did not exist. For instance, if there was a solid dot on a line, she thought the limit did not exist because she could walk right over it without stopping. Both CL and NS thought that if function values went toward infinity, then the limit existed which was incorrect. So new data was collected to see if this same kinds of responses would be replicated or if new students had other ideas about limits that exist or do not exist. So new tasks were designed and implemented to better extract this information and to see if students understood under what conditions a limit would equal infinity of just plain did not exist with nothing to do with infinity. So the cosine, damped cosine and cosine (1/) functions all contained limits that do not exist due to oscillations. Also, tasks with finite domains and piecewise functions contained limits that do not exist, as well as rational functions, all of which elicited important information about how students disconnect knowledge of the domain when considering the functions. Unlike the pilot study, this time their knowledge of domain and range were also considered.

In the current study, YJ disconnected the behavior of function values and limits from the domain. YJ was thinking visually and applied everyday notions of infinity being someplace very far off so instead of seeing oscillatory behavior she thought the limits did not exist because function values were going to infinity. The same occurred with rational functions in the form 1/x. Since the points on the graph were moving far away from the origin, she perceived that as meaning that the limit is equal to infinity. Her notion of the limit does not exist comes from everyday experience with something not being there. So in the case of the finite interval domain functions, since the function values "aren't there" for large x, the limit does not exist. YJ thought infinity was part of the domain whereas EB didn’t. In fact, EB had a better handle of infinity, but not when it came to deciding if the limit exists or not if it is equal to infinity and she said that when the limit equals infinity, it exists. Moreover, she compared the left hand limit to the right hand limit and would do exactly what NS did with both limits at a point as well as limits at infinity, taking 1/x as an example. The reason she does this is because she transfers the procedure to what one does for limits at a point, comparing the left limit with the right limit. She knew that domains were important when considering limits especially with finite intervals because she said it made no sense to ask what the limits at infinity were beyond those intervals. One reason for thinking limits exist if they are equal to infinity is the equal sign in the notation, according to EB. If something is "equal" to something else, then it must exist otherwise it could not be equal. So the notion is "equality" implies "existence". So this new data not only replicates some previous findings but extends beyond the pilot study and answers the research question by having looked more closely at the notion of infinity and the notation. Also, the new data answered the research questions by looking more in-depth at the students' knowledge of domains of different functions including those that oscillate, piecewise, rational and those with restricted intervals domains.

Limits at a Point

Research Question 1A: In what ways do students apply their definition of limit to their task solutions?

Students' Definition of Limit

Determining how students define the word "limit" is necessary to understand how they apply it to their task solutions. Defining a function is easier for them than defining a limit, evidenced by the fact that they could often articulate what a function was and also draw a graph using a vertical line test, explaining that there would be exactly one y-value for each x-value.

Hesitancy occurred when trying to explain what a limit is, as the word "limit" has vague meaning to most students. The words "approaching" (what x approaches) versus "nearness" (what f(x) is near) are used interchangeably. Students don't identify a limit as being a number that describes the behavior of function values near a point, but refer to it as "something approached". Given a continuous function, some state that a limit cannot be both what the function values are approaching and what the function value equals.

Other students erroneously think that a point above a discontinuity constitutes both the limit and function value. Below is evidence from Turn 10.

Evidence

EB: A limit is how a function is behaving at a certain point or at a certain section.
KB: A limit is where a hole is but it doesn't include the hole. It is how far you go to a certain point as you approach it.
LA: A limit is when you see if the left side is equal to the right side. There is no limit if there is a hole.
YJ: A limit generalizes the output of a function; it is a constraint on your input. When you find a limit, you want to find what the maximum value is going to reach so then you evaluate the function at that maximum value.
BK: It's the y-value approached. As the number gets closer and closer to the given value, it's the output it approaches. $\forall \varepsilon > 0\ \exists \delta > 0 \ni 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$. F(x) is within epsilon of L, for every x near a. Then you come up with a number for delta. That tells you the meaning of nearness to a. It's like the goal is to get f(x) within epsilon of L by keeping "x"
JY: A limit is what the function is approaching. The value of the function is what the function equals for a particular x, so they are two different things. Given x, a number approaches a number but does not equal it.

The evidence of how students define "limit" is summarized below.

- A limit is how a function behaves near a point (n=6).
- A limit generalizes output of a function, finding a maximum value (n=1)
- Given an epsilon, find a delta. It's the y-value approached (n=1).
- A limit is what a function approaches but cannot equal (n=2).

Analysis

Many students cannot provide an accurate informal definition of limit as they do not mention that a limit is a number or that a limit describes the behavior of function values. Instead, their descriptions of limit appear vague without reference to a limit being a number.

Functions versus Function Values

In the pilot study, CL and NS did not differentiate between functions and function values. In order to understand limits and be able to describe limits as being about the behavior of function values, is was important to find out how other students thought about functions versus function values.

Evidence

LA: A function is the thing itself and the function value is what you get for y.
YJ: A function is the mapping. The value is the output.
KB: A function has an equation over all the points. A function value is just one point or value.
JY: A function is the line or the curve, like x-squared. The function value is the output, the y.
AK: When 2 points come together like that, if there is a dot or a hole, I think.
EB: A function is where it has to be 1-1, 1 x for 1 y. I have not heard of that term function value before, but I might guess that it is a specific value at a point of x or a point of y.

The evidence is summarized below.

- A function is an operation, and a function value is the output.
- Two points coming together.
- A function is an equation; a function value is just one point or value.
- A function is 1-1 (1x for 1y); "function value" not heard before, so either x or y.

Analysis

This is important because in order to understand limiting behavior, a student must first be able to associate limits with function values. They must able to ultimately be able to describe limiting behavior as behavior of the function values. Interestingly, EB claimed to have never heard the term "function value" before, whereas others had vague ideas. In teaching, this has significant implications as math instructors must instill the difference between input and output, and the difference between the operation and the output. If a student does not know what a function value is, they might have difficulty identifying one as such when studying graphs involving limits.

Relationship between Function Values and Limits

Function values can be limits when a function is continuous, but not all limits are function values, as with points of discontinuity. Function values and limits both involve the second or y-coordinate. So when finding a limit, the second coordinate in the plane or the behavior of the function values is being determined. Limits are about how function values behave as x approaches some number “a” or increases without bound. CL thought limits were only about the x-coordinate and had nothing to do at all with the y-coordinate, so CL never made a connection between limits and function values

In Figure G.1, function values are limits when a function is continuous or for one-sided limits such as those on finite interval domains. The half circle or arccosine are examples from Task 2. However, limits are not always function values. The graph of the piecewise cosine function from Task 1 shows the case given a discontinuity. There is a hole in the graph at $(\pi, -1)$, but the limit is not equal to the function value $(\pi, 0)$.

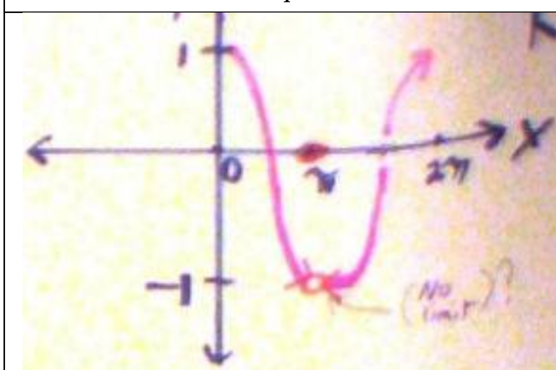
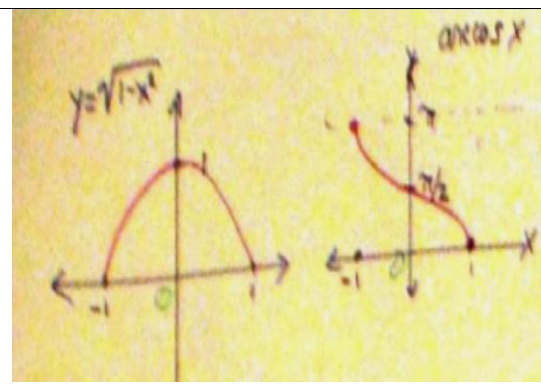
Task 1: Discontinuity in graph at $x = \pi$. The limit does not equal the function value.	Task 2: Graphs with finite interval domains with limit equals the function value.
	

Figure G.1: Comparisons of limit with jump discontinuity versus at endpoints.

Most students reported that with continuous functions as $x \to a$, the limit was equal to the function value $f(x) = f(a)$. However, JY reported that limits and function values cannot be the same for continuous functions the reason being limits are only about what is approached (Figure G.2). Hence, limits cannot equal function values. This reasoning is consistent with her definition of limit which focuses on the word "approaching". On the other hand, YJ reported that function values and limits can be the same, but if there is a point located above the hole, she thought the point (function value) is the limit (Figure G.3). This is a case in which the definition of limit given was not applied to her task solution. LA reported the limit does not exist because of the hole, but said the limit is the function value. In Turn 12, students answered whether or not limits and function values can be the same.

Evidence

JY: No. You either have a function value, or you have a limit. So with holes from discontinuity, there is a limit and no value. Straight lines or curves with dots, the limit does not exist. There is no limit but there is a value. If you compute you get a value for the limit but the limit does not exist.

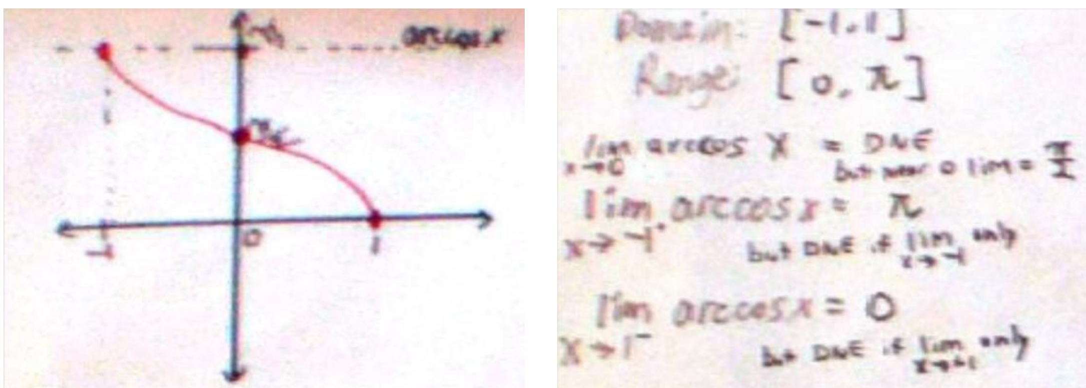


Figure G.2. JY's incorrect interpretation for Task 2 arccosine; limit does not exist as $x \to 0$.

YJ: Yes, for 1/x as x goes to infinity then both the function value is infinity and the limit is infinity.[contradicts this later-says infinity is undefined yet uses it here as a number.] For piecewise cosine with the hole in the curve and dot on top, the limit is not going to cross at the hole. The value of the function is above the hole at the solid dot. At this particular point x=pi, the limit does not exist at the hole. I would say the limit is the same as the function value (pi,0).

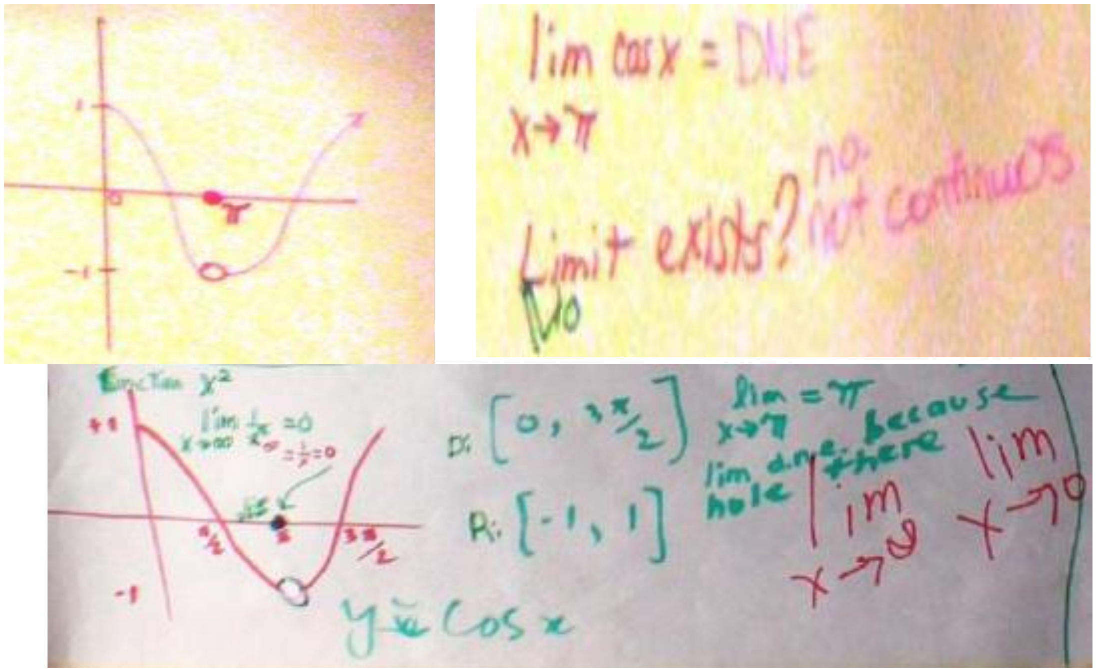


Figure G.3. YJ incorrectly identifies $(0, \pi)$ as the limit.

EB: Yeah when you have a continuous function at any point, then the function exists so the limit is the same as the value. So that could be on a straight line or on a parabola.

BK: Yeah, like when you have a line or some continuous function. So a limit at a point that exists is also a function value. Well the main thing is the function value definitely equals a number but a limit is about what the y-values are like near a number on the x-axis.

LA: Yes, I think so. If there is a dot there, not a hole. It's the same when the left equals the right only if there is a dot there (Figure G.3).

The evidence is summarized below.

- As $x \to a$, limits can equal function values when a function is continuous.
- As $x \to a$, limits cannot equal function values because limits are only about what is "approached" not "equals".
- Given a function with a discontinuity and point above it, the limit can only be the function value.

Analysis

Graphs of continuous and discontinuous functions are subject to multiple interpretations, given one's understanding of the definition of limit. In JY's case, she does not think a limit can equal a function value because she is fixated on the notion of "approaches". However what she does not realize is that "approaches" includes "equals" when the function is continuous. In fact, given continuous functions, the limit can equal the function value for any given point in the domain. The reason is the definition of limit is not understood. The limit is not only about what the function values are near, but it includes what the limit actually equals.

Students sometimes are confused thinking that the limit must equal a function value, even though the limit exists at $(\pi, -1)$. YJ does this, evidenced in Figure G.3, with the piecewise cosine function.

Piecewise functions are important and of interest is because these show where the confusion emerges between knowing how to identify the limit separately from the function value. Students were consistent in applying their definition of function values to explaining its relationship to limits.

A proper procedure to facilitate understanding how "approaching includes equals" would first be to look at the x-value as it approaches some number, "a", and then watching the output value as f(x) to see if it approaches some number, L. In this case, "L" represents a limit. These results show that students generally equate the limit with the second coordinate, unlike CL who reported that limits were only about how the x-values behaved. Although some of CL's responses were not seen in the new students, this does not mean her perceptions were isolated.

Understanding Limit Notation

Understanding the limit notation and being able to break it down into meaningful components is helpful to understanding that limits are about finding y given x. There is also a relationship between the limit notation and the definition of limit. The relationship is what the y-coordinate f(a) will be near as $x \to a$. First one looks at the notation and considers what happens with x, then takes the limit by studying the limiting behavior of the y-values. CL reported that limits were only about x-values because of the notation, that the arrow beneath "lim" implied direction from the left only. Moreover, given $\lim_{x\to\infty}$ or $\lim_{x\to 0}$, CL noted that the subscript beneath the "lim" notation ($x \to \infty$ or $x \to 0$) reveals the limit. Hence, no math was involved in computing limits.

Unlike CL, the new students seemed to understand the notation, as they consistently looked to find y-values. After acknowledging what x is approaching, limits were about nearness to second coordinate. What appears beneath "lim" informs one to first approach the x-coordinate, then to approach the y-coordinate. BK applies his formal definition of limit, showing consistency from the definition to an application and emphasizes that with epsilon-delta, given epsilon (y) one finds delta (x). In turn 69 they were asked: Is it true with the limit notation that whatever appears below "lim" tells you what the limit is going to be, so that you don't have to do any math at all? Turn 70 summarizes their responses.

Evidence

JY: No. $x \to a$ means approaching from both sides. It only gives the direction of x and then you have to still find the limit y-value if one exists.
YJ: No, under "lim" notation it tells where x is approaching. You still have to find the output value if it exists.
EB: No. Underneath tells you only what x is approaching, not the whole limit.

BK: No, it only tells you what x is tending toward, not y. That is the formula that specifies finding a precise y-value or a y-value that is in the neighborhood of x.
LA: No, the notation says both in a way. It's what the x is going to be when you are looking for the y- value. So it tells you the first part of what you need to know to later find the y.

Below is a summary of how students perceived the limit notation.

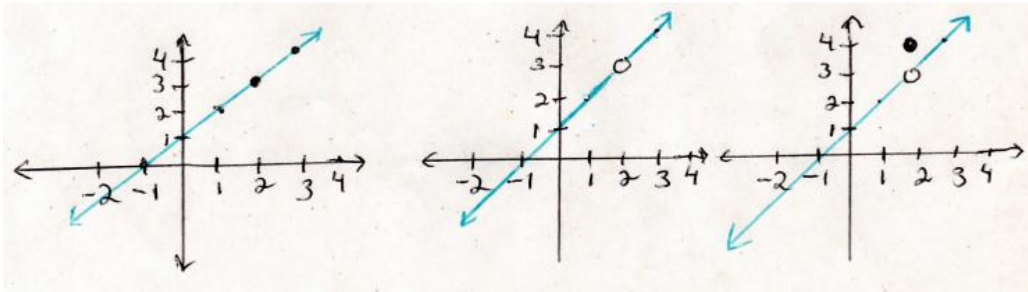
- What appears beneath the "lim" notation does not imply direction from one side only, so $x \to a$ means approaching "a" from both sides.
- The "lim" notation refers to both x and y because x must be approached before finding y.
- The "lim" notation refers to the y-coordinate.
- The relationship between the definition of limit and notation is the specification of what x is approaching, and what y-value will be subsequently approached, either equal or near.

Analysis

Although CL's reasoning was not observed with the new group of students, there may still be others who think as she does. What is important here is that students must understand the meaning of the limit notation in order to decipher the problem. Students should be taught that a limit is a number and describes the behavior of the function values. The notation "$\lim_{x \to a} f(x) =$" is read from left to right, recited "the limit as x approaches "a" of the function f(x) equals. The order of what occurs within the notation matters. Next, limits are about functions, and one takes limits of functions to describe nearness, so in the case of limits at a point, the behavior of the function values are studied near the point $x \to a$.

How Definition of Limit is Applied

Given the evolving nature of this study, the consistency of applying the definition of limit was explored. One way that individual students' definitions of limits are manifested in their work was through a task that involved three graphs. The graphs changed from continuous to discontinuous (Figure G.4).



Graph A Graph B Graph C

Figure G.4: Task 4 graphs.

Continuous Functions

Students were asked about Graph A and if a limit can equal the value of the function. One can refer back to the initial definition of limit each student reported.

Results from Turn 100 appear below. JY does not think the limit can exist if it equals a function at x=2, which is consistent with her definition of limit. BK incorporates the epsilon-delta definition of limit in his answer, whereby he consistently applies the definition in his work. The other students give a general description in their work about what the limit is as x is approaching 2, which is consistent with their descriptions of "approaching and nearness" in their definitions of limit.

Evidence

JY: Domain and range is everything. Limit as x approaches 2 does not exist because the limit is about what it is near, not what it equals.

YJ: Domain and range $(-\infty, \infty)$.The limit as x approaches 2 from both sides is 3 so it exists.

EB: Here in the first graph the domain is $(-\infty, \infty)$ The limit is 3 as x approaches 2, left equals right by definition of limit at a point, so it exists.

BK: Here in the first graph the domain is $(-\infty, \infty)$ because it has a value at the hole but it not continuous. For any epsilon band around 3 that I get closer to, I can tighten the band and get close to 2 from the left and from the right, so given an epsilon near 3, I can find a delta as x approaches 2.The limit is 3 as x approaches 2, so it exists.

LA: The domain is $(-\infty, \infty)$ and range is also $(-\infty, \infty)$. The limit as x goes to 2 from the right is 3 and from the left it is also 3.

Below is a summary of findings referring only to Graph A.

- A limit can equal a function value (n=8).
- A limit cannot equal a function value (n=1).

Analysis

Only one student did not think a limit could equal a function value, whereas the others did. Most students apply their definition of limit to their task solutions. They compare the left hand limit with the right hand limit to determine if the limit exists at a point. They also use their definition of limit and notion of approaching when deciding if the limit is also the function value. When the definition of limit is not correct, it manifests in task solutions. In the case of the continuous function, JY reports limits are "a certain number or amount" yet the limit does not exist because it can only approach and not equal the function value. An oxymoron occurs because the student reports the limit exists since a number was computed, but yet it does not exist on the graph because a limit can only "approach".

Discontinuous Functions

In contrast to the continuous case, more difficulties emerged with discontinuous functions when the focus switched from Graph A to Graph C, which both happen to share the same domain. Deeper probing into understanding the domains is reported separately. Students were probed specifically to describe the limiting behavior of the function as $x \to 2$ and if the limit exists (Turn 104). Below are responses from seven students who manifest their definitions of limit through discussing the limiting behavior of the discontinuous function in Graph C.

Evidence

JY: Domain is everything. Range is everything. As x goes to 2, the limit is 3.
EB: As x goes to 2, the limit exists and is still 3. The left side equals the right side at y=3.
BK: Domain: $(-\infty,\infty)$ because it has a value at the hole but it not continuous. For any epsilon band around 3 that I get closer to, I can tighten the band and get close to 2. So the limit exists when x=2. The limit exists, because it's about what happens in the neighborhood of x, not at x.
YJ: The limit does not exist at the hole because there is no value there but the limit exists at 3 because that is where there's a value.
LA: The domain and range are $(-\infty,\infty)$ As x approaches 2, the limit does not exist because of the hole.
KB: Domain is $(-\infty,3)\cup(3,\infty)$. As x approaches 2, the limit does not exist either because of the hole.
AK: The domain is gonna be $(-\infty,2)\cup(2,\infty)$. The limit as x approaches 2 is 4, where the solid dot is above the hole. There is no limit at the hole.

Below is a summary of how students applied their definition of limit.

- If there is a hole, the limit exists (n=4).
- If there is a hole, the limit does not exist (n=5).
- If there is a hole with function value, the limit is the function value. (n=4)

Analysis

The first three students, JY, EB and BK consistently apply their definition of limit in their work, and it appears that they connect the definition with the notion of the limit existing in spite of the hole because limits are about nearness, and nearness also includes equals. In JY's case, her definition of limit being "what the function approaches, but not equals", is consistent in her work as well since the limit only approaches a number here but is not the function value. BK consistently uses his epsilon-delta definition of limit in his explanation below as well. KB and LA report the limit does not exist if there is a hole, which is consistent with their definitions of limit. LA reported a limit is a "y-value" and the left side must equal the right side, whereas KB reported earlier a limit approaches a hole but does not include the hole.

Another issue emerges with confusing the limit with the function value. YJ reports that a limit cannot exist at a hole, but only where there is a function value. This is consistent with her definition of limit being “when you find a limit, you want to find what the maximum value is going to reach.” By selecting the function value above the limit, this appears to represent the “maximum value”.

Overall, students consistently use their definition of limit tasks they work on. If they report that the limit is what is being “approached”, they don’t refer to the “what” as a number, and think that a limit cannot equal a function value. On the other hand, other students think that a limit can only be a function value, so even though the limit exists where there is a hole at the site of the discontinuity, they instead look for the isolated point because it is a function value and report that the limit exists and is equal to the function value. What they must understand is that a limit can exist whether or not the point is in the domain, as the limit is about the behavior of the function values near both sides of 2. Therefore, a limit can exist where they see a hole at the point of discontinuity.

Graphs with Discontinuities

Next, the goal was to find out if students perceived the point (2,4) in Graph C to be on the graph of the function. Results are in turn 106:

Evidence

JY: Yes. Because 4 is the value. Limit is 2.
YJ: Yes it’s on the graph because it has the value.
EB: Yes because the point has the function’s value when x is equal to 2.
BK: Yes because it is still part of the function.
LA: Yes because it’s on the graph but just not on the function.
KB: No because it’s not slid down onto the function.
AK: No. Because it’s not on or within the line.

Two types of responses were elicited:

- the point was on the graph of the function with various reasons given (n=7).
- the point was not on the graph of the function since the point was not sitting directly on the function (n=2).

Analysis

The results suggest some students are confused about the role of the point on the graph. This suggests that they might not recognize the graph as piecewise with 2 parts to the formula, with the bottom part being discontinuous at x=2 and having a function value of 4 when x=2. Instead, they see this as one function and think the point is supposed to be where it would make the function continuous. Therefore, if they do not understand the nature of the function they are exploring, then they have difficulty finding the limits or determining the behavior of the function values when x=2.

Identifying Piecewise Functions

Difficulties occurred with identifying functions as piecewise (Figure G.6). Students were asked about the graphs of two functions in task 3, what kind of functions they were, and what the formula would be.

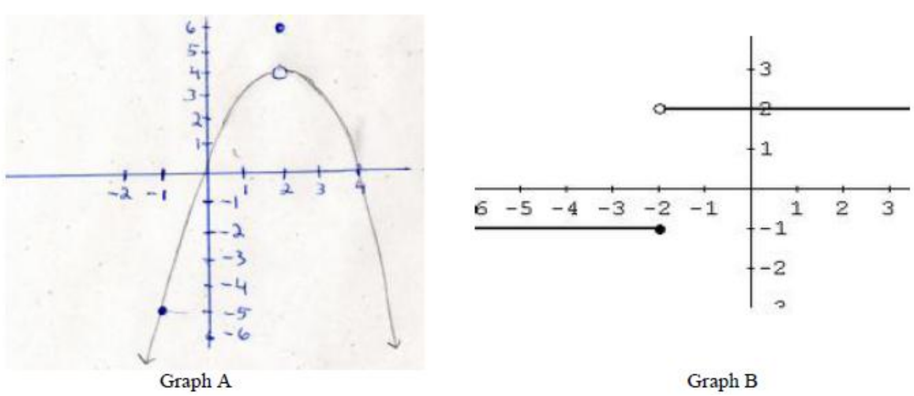


Figure G.5: Task 3.

Results showed that very few students identify the graphs as piecewise and fewer wrote formulas for them. The most common mistake for Graph A was to focus only on the quadratic piece. The discontinuity was often ignored, and many students thought the point (2,6) was not on the graph of the function because it "was not on the function." In Graph B, very few students could name the function as piecewise, and had produced errors reporting if the limit exists as x approached -2. In this graph, $x \to -2$ the left hand limit of -1 does not equal the right hand limit of 2, so the limit does not exist at x= -2.

Some students thought the limit only existed at -1. For instance, YJ erroneously thought the limit exists at x= -2 and that the function value was equal to -1. KB reached the correct conclusion that the limit does not exist, but for the wrong reason, that the bottom line stops at x=-2. Neither student compared the left hand limit with the right hand limit whereas the others did and correctly concluded that the limit did not exist. Other students got the answer right but articulated it with some difficulty. The types of responses that occurred by the students come from Turn 92.

Evidence

JY: No it doesn't exist. You are trying to approach this but cannot. With parabola, it was approaching. With here, it is not. From the left, the limit is approaching -1 and from the right it is approaching 2 and only the limit exists for this, not from the left.

YJ: Yes, limit exists as x approaches -2 because the value of the function is defined at – 2. The limit is -1.

EB: Limit is -1 from the left and limit is 2 from the right, so since left does not equal the right, the limit does not exist. For limits to exist, the right and left have to equal each other. The function doesn't exist on top where the hole is but the function does exist at the bottom because there's a solid dot but the function has a value there.

BK: This graph is piecewise. As x goes to 2 the limit does not exist. As x goes to 2 from the left is -1 and as x approaches -2 from the right the limit is 2.

LA: Well it is linear. As x goes to 2 the limit does not exist because as you go to- 2 from the left the limit is -1 and as you go to -2 from the right the limit is 2.

KB: It's exponential I think. As x approaches -2, the limit does not exist because it stops at the bottom point (-2,-1).

Below is a summary of findings.

- Piecewise function perceived as linear and exponential (n=2).
- The limit does not exist (n=3).
- As $x \to 2$ the limit exists (n=6)
- The limit exists on the bottom piece, endpoint represents the function value (n=2).

Analysis

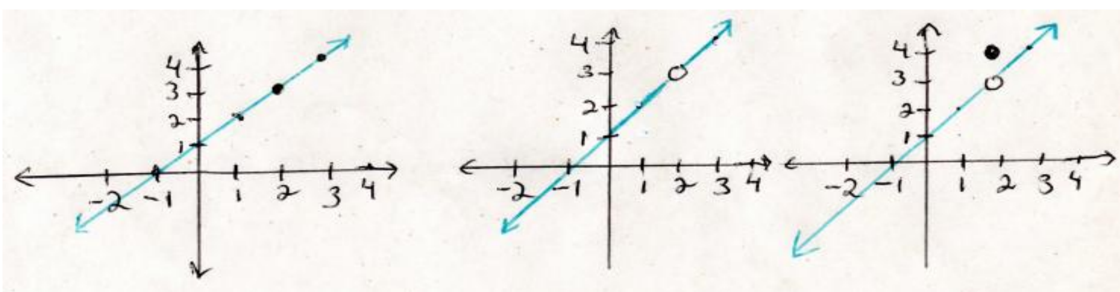
These results suggest that piecewise functions impose a source of confusion. If students cannot identify a function as piecewise, then this makes it very difficult for them to find limits of these functions. It is important to first understand why a function is piecewise and acknowledge its properties before attempting to describe its limiting behavior.

Functions with jump discontinuities or piecewise functions in general are a major source of confusion for as students do not consider the function's behavior over different parts of the domain. Instead, they try to force the point located above the discontinuity to slide down onto the quadratic component, which was seen with the first graph. This could be because they do not know think a single point can be part of the function. They might also not consider how the domain is defined. Given this type of function and graph is common to introductory calculus courses, instructional practices should include more focus on domains of functions, and functions with discontinuities.

Research Question 1B: In what ways do students perceive the domain to be involved when deciding if a limit exists at a point; as $x \to a$, does "a" have to be in the domain?

Understanding Domains of Functions

Given the significant confusion students have with identifying functions as piecewise, deeper probing was done into what students understand about domains of functions. It was of particular interest to explore how students thought about domains of piecewise functions, as well as to see if they understood how the domains can change by simply removing one point. Students compared three graphs in Task 4 (Figure). After comparing the domains of all three graphs, many students reported the domains were the same for all three, whereas only a few students noticed that the domain was different only for Graph B. Some students reported that if the domain changes, the function would not change, while others reported the function does indeed change.



Graph A Graph B Graph C

Figure G.6: Task 4

The answers to the question about if the domain changes and if the functions are the same or different in the graphs above in Figure G.6 are summarized in Turn 116.

Evidence

JY: The domains are $(-\infty, \infty)$ for the first and last graph. If I start with straight line in graph 1 and then have a hole at some point like (4,5) now in graph 2, the graph is not defined at 4 so I can write the new function with two parts from minus infinity to 4 and from 4 to infinity, so in that case the function changed.

YJ: Yes, of course it's the same domain and function in all 3 graphs, but they just look different.

EB: The 2nd graph has a different domain from the first and last, which is $(-\infty, \infty)$. The graphs may look similar but no, you would get a different function for each one.

BK: The first and last graph has the same domain, $(-\infty,\infty)$. With rational functions, things cross out and you could end up with a linear function and that could be the case of the second graph. Or if you change the domain you could have a piecewise one by just taking out the value of the function and having just a hole.

LA: The domains are the same in the first and last graph $(-\infty,\infty)$. From looking at these 3 graphs, you have the same function because it's still linear. All you are doing with the second one though is getting rid of a point but yeah, I'd say it is still linear so the function does not change if you change the domain.

Analysis

Understanding what happens to functions of the domain changes has important implications with limits. Even though they identify the correct domains, some students do not make a connection that once the domain changes, so has the function. Some students might not even typically consider the domain, so when the function is changed from continuous to discontinuous, they do not realize they have a new function. It is important to study if they recognize a change in the domain, particularly when discontinuities are involved because this later helps with understanding limits of functions with discontinuities. They can't find the limit of a function when they don't know what kind of function they have in front of them. If the domain changes and the function changes from continuous to discontinuous, then their problem solving actions fall apart. They will say the limit does not exist where there is a discontinuity because the point is no longer in the domain which suggests that they confuse limits with continuity.

Continuous Functions

As $x \to a$, does "a" have to be in the domain? Students do not consider the domains of functions on their own, unless specifically prompted to report what the domain is. Instead, they tend to focus on doing a computation and getting an answer with minimal effort. Domains are important to understanding limits because particularly in the case of discontinuities, the point “a” does not have to be in the domain for the limit to exist as $x \to a$; however, students are not aware of this and in fact, think that “a” has to be in the domain. Turn 112 captures responses to the question, “ $x \to a$ does the “a” have to be in the domain?”

Evidence

JY: Yes, for limit at a point, if ‘a’ is in the domain then there is no limit unless you split it up from approaching from the left and approaching from the right.
YJ: Yes the ‘a’ has to be in the domain of course otherwise the function is not defined. In order for a limit to exist at the point there must be a solid dot which means that ‘a’ is in the domain. When there is a hole, then the function is not defined and so the limit cannot exist.
EB: No, ‘a’ does not have to be in the domain. The limit is about what happens near ‘a’, not at ‘a’.
BK: No. The limit is about what happens near ‘a’, not at ‘a’.

A summary below of the three types of responses generated referring to Graph A for the continuous function appears below.

- “a” has to be in the domain for a limit to exist (n=4).
- “a” does not have to be in the domain for a limit to exist (n=4).
- If “a” is in the domain, then the limit does not exist (n=1).

Analysis

Students do not appear to understand the role of the domain with limits. Some students reported that the “a” has to be in the domain for a limit to exist, whereas others did not. This could be because they confuse limits with continuity. In the case of continuous functions, there was a problem previously undetected. Unlike her counterparts, JY reported that if “a” is in the domain then the limit doesn’t exist. This is an interesting finding because it is consistent with her definition of limit, in which case f(x) can approach, but not equal. Results suggest students do not understand the definition of limit, otherwise they would know that as $x \to a$, “a” does not have to be in the domain in order for the limit to exist. Only the behavior of the function values near "a" matters. Therefore, the "a" need not be in the domain.

Discontinuous Functions

As $x \to a$ does "a" have to be in the domain? Limits at a point of discontinuous functions and understanding domains were also explored. Using Graph B from Task 4 (Figure G.7), it could be studied how students perceived the transition from Graph A to Graph B, in which case the domain changed by one point, thereby changing the function. Using Graph B from Task 4, students stated the domain was and if the limit existed as $x \to 2$.

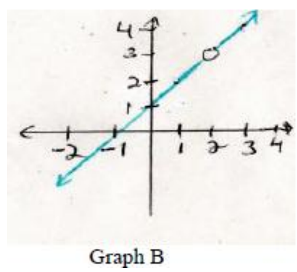


Figure G.7. Task 4, Graph B.

YJ and LA claimed that the limit did not exist at x=2 where there was a hole in the graph because x=2 was not in the domain. They think that x=2 must be in the domain in order for the limit to exist. This is not the case because the limit can exist whether or not the point is in the domain. JY, EB and BK reported that the limit exists because of the definition of limit being about what is being approached, and that x=2 did not have to be in the domain. Below are statements from Turn 102 pertaining to Graph B in Task 4 (Figure G.8):

Evidence

JY: Domain is everything except 2. Limit is 3. Left side and right side are equal.
YJ: The limit as x approaches 2 does not exist because there is no value.
EB: The domain would have to exclude the point x=2. As x approaches 2, the limit

exists and equals 3. For a limit to exist it does not mean the limit has to exist at the point but just near the point.

BK: Domain is $x \neq 2$ $(-\infty, 2) \cup (2, \infty)$ Range: $(-\infty, 3) \cup (3, \infty)$. As x approaches 3, the limit exists and equals 2 because 2 is in the neighborhood of 3. The left hand limit of 2 equals the right hand limit of 2.

LA: The domain is $x \neq 2$ The limit as x goes to 2 does not exist because there is nothing there. So I will put DNE because of the hole. I think if there is a hole then it doesn't exist.

Results revealed two types of responses for a limit at a point given a hole representing a discontinuity in a function:

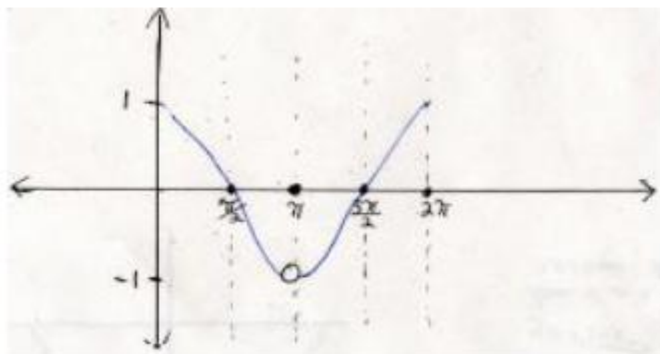
- If there is a hole representing a discontinuity in the graph, the limit exists (n=4).
- If there is a hole representing a discontinuity in the graph, the limit does not exist (n=5).

Analysis

These results suggest reasons why students don't think a limit exists when there is a hole. The hole looks empty as if nothing is there, therefore, the limit does not exist. This seems to suggest confusion between limits and continuity. Students do not typically consider the domain when deciding if limits exist or not; they just use visual inspection. Better instructional practices including the role of the domain might prevent this misconception of "a" having to be in the domain from occurring.

The next goal was to explore how students perceived the transition from Graph B to Graph C in task 4, in which case there was now a function value above the discontinuity. By asking students what the domains were for each function, that gave rise to a deeper understanding of why they may or may not think the limit exists where there is a hole. So with the Graph C, it was possible to explore how their perception of the graph changed given the change of domain and given a function value above the hole.

Earlier in the study, they were shown the graph of cosine, and then a truncated portion of the graph that was deliberately designed to be piecewise (Figure G.8). Without even studying how they perceived the domain at that point, the purpose was to see if they thought the point $(\pi, 0)$ was on the graph of the function and then as $x \to \pi$, if the limit exists at $(\pi, -1)$.



Graph C: Piecewise component

Figure G.8: Task 1 piece of cosine with discontinuity

The responses were recorded in Turn 40, students had to report whether or not the limit exists as $x \to \pi$. Overall, students appeared confused about the hole in the graph and appeared uncertain. Even students with more skill typically forgot what happens when they see a hole. EB made an interesting comment that the function value does not exist at π versus the limit exists at π.

Evidence

JY: Yes it exists because when x is pi, the limit equals -1, because left side and right side are equal.

YJ: No, limit does not exist at the hole. The limit exists at the value of the function $(\pi, 0)$ E: Yeah. It's -1. It exists because as x approaches pi from the left, it is approaching -1 and from the right it's also approaching -1. The function does not exist at $(\pi, 0)$ but the limit exists at $(\pi, -1)$ because the left limit equals the right limit.

LA: No. The limit does not exist because it is not continuous.

BK: Yes. The limit does exist and equals pi, but it is not continuous. I would say the limit exists because it is an actual number where the closed dot is. I think that's what it was. It's hard to remember.
KB: Yes because it is on the x-axis.
AK: Yes, it exists everywhere up and down, if I drew a vertical line.

Four types of responses generated included the following.

- The limit exists as $x \to \pi$ and equals the function value, π (n=).
- The limit exists at $x = \pi$ but the function value does not exist (n=1).
- The limit exists where the hole is and equals -1 (n=).
- The limit does not exist because of the hole at $(\pi, -1)$ (n=).
- The limit exists everywhere up and down on a vertical line (n=).

In order to further probe further to find out if they knew what type of function this was (piecewise), they were asked in turn 42 to state if the point $(\pi, 0)$ was on the graph of the function. Some students thought it was which made it seem they recognized the function as piecewise, whereas others did not. This indicated that they only focused on the lower piece of the graph, and not the point. Their responses on this task lead to further probing in Task 4 specifically about the domains.

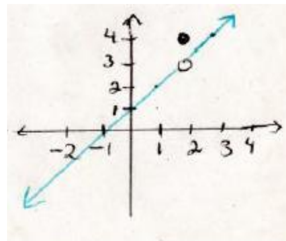
Evidence

JY: Yes because 0 is the value when x=pi.
YJ: Yes because it's the value. It has a value when x is pi.
EB: I never heard that phrase before "on the graph of the function". I would consider it to be part of the function because technically if you don't have this point here at (pi,0) and you don't have a y-value for the x where the hole is at -1, then having this point (pi,0) filled in covers that. So I'm not sure. If you took that point away it would be a graph of a function? I think it goes back to the 1-1 idea where you have a function but with a discontinuity. If you drew the vertical line test through it, it's still covered and works so it's still a function.
BB: Yes, depending on what the function is assuming it is one function, then yes.
LA: No because it is not on the function.
AK: Yes because it is on the x-axis.

Three types of responses were generated:

- No, it is not on the graph of the function because it is not on the function.
- Yes, it is on the graph of the function because it is on the graph, on the x-axis.
- Yes, it is on the graph of the function because it is part of the piecewise function.

Similar questions were asked again later on about Graph C in Task 4 (Figure G.9), to explore their thinking about domains, as well as for the purpose of reliability and validity of the results found in Task 1 using the cosine piecewise function.



Graph C

Figure G.9. Task 4, Graph C Piecewise Function.

In turn 104, they considered the discontinuity and reported if the limit existed as x approaches 2, then they were asked if the point (2,4) was on the graph of the function. YJ and LA incorrectly identified that the limit did not exist because of the hole; however, YJ identified a limit as being the function value which was a dot on the graph located above the discontinuity. Other students correctly reported that the limit was 3, disregarding the function value.

Evidence

JY: Domain is everything. Range is everything. As x goes to 2, the limit is 3. Since x= 2 is in the domain so that's why the limit exists.

YJ: The limit does not exist at the hole because there is no value there but the limit exist at 6 because that is where there's a value. The x=2 must be in the domain.

EB: The domain is just negative infinity to positive infinity, but the point is still covered with the dot. As x goes to 2, the limit is still 3. The left side equals the right side at y=3. The x=2 does not have to be in the domain.

BK: Domain: $(-\infty,\infty)$ since it has a value at the hole but it not continuous. Range: I'm not sure. Since there is no y-value at that point, I'd say it's the same as this other on $(-\infty,2)\cup(2,\infty)$. For any epsilon band around 3 that I get closer to, I can tighten the band and get close to 2 from the left and from the right, so given epsilon near 3, I can find a delta as x approaches 2. By definition, as x approaches a, "a" does not have to be in the domain.

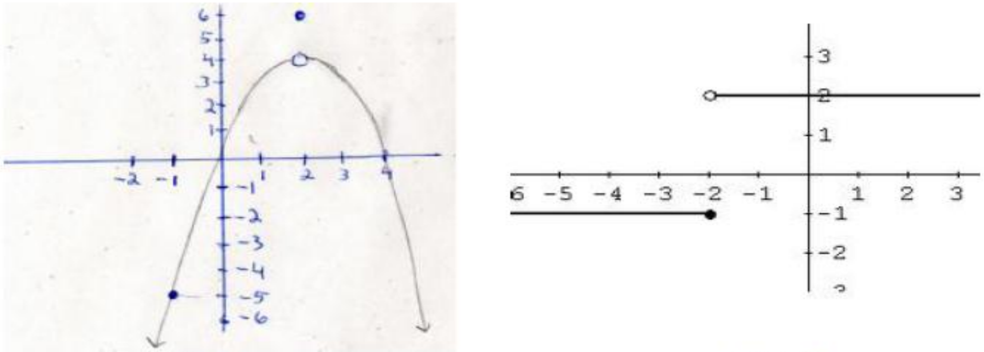
LA: The domain is $(-\infty,\infty)$ and range is $(-\infty,\infty)$ As x approaches 2, the limit does not exist because 2 is not in the domain.

Analysis

Students do not identify functions with two parts as piecewise, and often ignore the solid dot (function value) above a discontinuity. Students do not tend to think about domains when solving limit problems, so doing so in the graphs of Task 4 was a novel idea. When asked if the limit exists considering whether or not "a" is in the domain as $x \to a.$, they were consistent with their responses across tasks. If they think that "a" has to be in the domain, then they report the limit does not exist because that point is not in the domain. Meanwhile, they disregard the function value.

In order for a function to be continuous, the point "a" must be in the domain by definition. So students think if the point is removed making the function discontinuous, then the limit does not exist. However, when they correctly report that "a" does not have to be in the domain, this is evidence of understanding that the limit is independent of whether or not "a" is in the domain. Most students do not consider domains with limit problems but if they did, they might recognize that "a" need not be in the domain for a limit to exist.

Students were further probed about the role of the domain with other functions involving jump discontinuities (Figure G.10). Earlier, Task 3 was used to see if students could identify the function as piecewise and if they could write a formula, but now the focus is on the domain. In turn 95 they were asked about any potential relationship between domains and limits.



Graph A Graph B

Figure G.10. Task 3.

Evidence

JY: If the x value is not in the domain, then the limit exists for graph 1 where the hole. Same for the 2nd graph if you look at each line separately, and so for the top one the limit exists where the hole is because x is not in the domain. If x is in the domain, then the limit does not exist. So on the second graph, where the line is on the bottom, it ends with a solid dot. So there is no limit there because x is in the domain. That is how I think about this.

YJ: If x=2 is not in the domain, then the limit is undefined because there is no value.

EB: In the first problem, the domain includes everything. The x value does not have to be in the domain for the limit to exist because the limit is about how near x is to the point, not at the point.

BK: Well the domain on the first one is continuous, so the limit can be anywhere in the domain. Even if there was no closed dot and you just had a hole, the limit would still exist even though the point at x=2 was not in the domain. In the second graph, it's a similar situation. The domain is all x but the limit does not exist at 2 because the left doesn't equal the right sides so in this case the limit does not depend on the what's in the domain.

LA: Well for the first graph, x is not in the domain so the limit does not exist where the hole is. For the second graph, the limit does not exist even though x is in the domain only in the lower part of the graph. So you have to look at the domain like when you decide if a limit exists or not.

A summary of the findings include:

- Graphs were not identified as piecewise.
- Graph A was perceived as quadratic; graph B as exponential or linear.
- Formulas could not be derived given graphs not identified as piecewise.
- As $x \rightarrow a$ "a" does not have to be in the domain.
- As $x \rightarrow a$ "a" must be in the domain for the limit to exist.
- Some students do not differentiate between the limit and the function value.

Analysis

The overall relationship that emerges between functions and limits comes about with piecewise functions because students do not differentiate between the limit and the function value evidenced by students who do not think any limit exists or reports that the limit is equal to the function value instead of at the point of discontinuity.

As seen earlier, functions with jump discontinuities were seldom identified as piecewise. Graph A was erroneously referred to as quadratic. Nobody identified the second graph as “piecewise” but referred to it linear or exponential. Two different responses were: as $x \to a$ the “a” does not have to be in the domain versus “a” must be in the domain for the limit to exist. JY reported that the limit does not exist if it is equal to the function value. This is consistent with what she reported in her definition of limit, being about what x is approaching, not what x equals and applied that reasoning to y.

Limits of Functions with Finite Interval Domains

Case of Closed Endpoints

Understanding functions with finite interval domains (restricted intervals) is also associated with understanding limits at a point. Some students do not think limits can exist for such one-sided limits because the other side is missing.

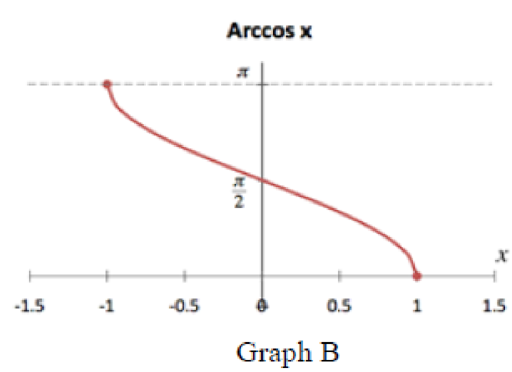


Figure G.11: Graph 3, Task 2.

The arccosine with closed endpoints generated two types of responses for limits at a point (Figure G.11). The limit exists where the dot was at (1,0) and (-1.0) versus the limit does not exist because there must be two sides to the endpoint for a limit to exist. JY was consistent with applying her definition of limit because she reported here that the limit is about what value is approached, not what it equals. As a result, she erroneously reported that the limits do not exist. The other students did not exhibit difficulty evidenced by having gotten the limits correct and reaching the correct conclusions. Interestingly, LA stated an incorrect domain and did not connect the domain at all to her interpretation of the limit existing or not. Similar results occurred for half circle. At the endpoints, some students reported that the limits on both sides existed because they were one-sided limits, while others reported the limit did not exist because the limit cannot equal the value at the endpoint or because limits cannot be one sided.

Evidence

JY: For the half circle, the domain is [-1,1]. Range is [0,1]. Limit as x goes to 1 does not exist, as x approaches 0 the value is 1 but the limit does not exist because the limit is about what it is approaching not what it is at. When x is zero, the value is exactly 1. It IS 1, it is not approaching 1. So the limit does not exist. As x approaches -1, the limit does not exist. The limit cannot exist and equal the value. For arccosine, the domain is [-1,1] and range is [0,pi]. As x goes to 0, the limit is pi/2 because both sides equal the same but the limit does not exist for the same reason, the value IS pi/2. It is not approaching pi/2. As x goes to -1, it’s a one sided limit so the limit does not exist because the left side does not equal the right side. As x goes to 1, it’s a one sided limit approaching from the left so the limit does not exist either because the left side doesn’t equal the right side. The left is 0 but the right there is no x-value. When it’s continuous, you need the little superscripts to show limit as x approaches from the left only or from the right only.

YJ: For half circle, the domain is [-1,1]. Range [0,1]. Limit as x goes to 1 or to -1 is just going to be 0. Arccosine domain is [-1,1]. Range [0,pi]. Limit as x approaches 0 is pi/2, as x goes to -1 limit is pi, and as x goes to 1 limit is 0.

EB: Half circle: domain is [-1,1]. Range is [0,1]. The limits as x approaches -1 is 0 and as x approaches 1 the limit is 0. Arccosine, the domain is [-1,1]. Range is [0,pi]. The limit as x goes to -1 is pi. The limit as x goes to 0 is pi/2. The limit as x goes to 1 is 0.

BK: Half Circle: Limit as x goes to 1 is 0, as x approaches 0 the limit is 1, and as x approaches -1, the limit is 0. Arccosine; the domain is [-1,1] and range is [0,pi]. As x goes to 0, the limit is pi/2 because both sides equal the same. As x goes to -1, it's a one sided limit so the limit pi. As x goes to 1, it's a one sided limit approaching from the left so the limit is 0. When it's continuous like this you don't need the little superscripts to show limit as x approaches from the left only or from the right only, because we are not evaluating continuity here, only the limit. I'm thinking about continuity at the same time, but the limit would still be pi on the left.

LA: Half Circle: The domain is $x^2 - 1 \geq 0, x \geq \sqrt{1}$. The range is [0,1]. As x goes to 0 the limit is 1. As x goes to 1 the limit is 0 and as x goes to -1 the limit is also 0. As x goes to infinity or minus infinity, the limits do not exist. You can't go past -1 or past 1. The limit as x is going to infinity doesn't exist and same for minus infinity, it doesn't exist. Arccosine: The domain is [-1,1] and range is (0,pi). Limit as x goes to 0 is pi/2. Limit as x goes to -1 is pi, and limit as x goes to 1 is 0. Limit as x goes to infinity or minus infinity does not exist.

Responses from Turn 52 yielded the following results:

- The limits exist at closed endpoints because x is in the domain (n=).
- The limits do not exist at closed endpoints because even though the endpoint is in the domain, the function terminates (n=).

Analysis

What these results mean is that students are don't use the correct definition of limit in their task solutions, evidenced by reports of "a" having to be in the domain for the limits to exist. Limits are not contingent upon point "a" being in the domain. Moreover, they may not be very familiar with one-sided limits and that limits can exist at endpoints on a finite domain. Another explanation is they don't distinguish between the point being on the graph of the function in which case the point is part of the function, versus with the limit, the point can be in the plane (hole) but not on the graph of the function. If they use the domain information and claim that the endpoint is in the domain, they conclude that the limit exists because the function value is in the domain.

The next step was to see what students did with graphs involving open endpoints. Further information was gathered about what the knowledge of domains does to understand what the limits are.

Limits of Functions with Finite Interval Domains

Case of Open Endpoints

This problem helps to probe further into the relationship between limits and understanding finite interval domains. Changing the closed endpoint to an open one using the same function caused difficulty because of the previous notion of thinking that a limit does not exist where there is a hole (Figure G.12). The one-sided limit with a hole at the endpoint was problematic for some students, particularly for those who thought as $x \to a$ that "a" must be in the domain for the limit to exist. For these students, they reported the limit does not exist. Other students who struggle with one-sided limits in general reported the limit does not exist because the other side is missing.

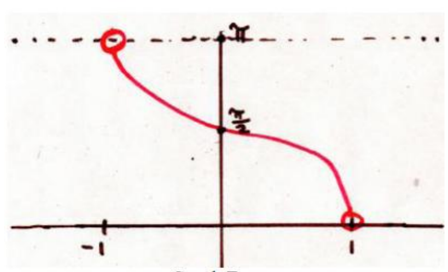


Figure G.12: Task 2, Graph E.

JY continues to apply her definition of limit to this task, reporting that the limit exists because the open hole endpoint signifies what the function value is approaching, not what it equals. So her definition of limit and application to this task are correct and consistent. EB and BK reported similar interpretations, and so all three students reach the correct conclusion that the point does not have to be in the domain for the endpoint for the limit to exist. LA and YJ think the limits do not exist at the endpoints because there are holes there (no function value) and because as $x \to a^-$ and x $\to$-a$^+$ the points “a” and “-a” are not in the domain. A summary of results from Turn 52 include:

Evidence

JY: The domain is (-1,1) and range is (0,pi). It does not include the endpoints. So the limit as x approaches 1 is 0 and so it exists and as x approaches -1 the limit is pi and so that limit exists. Even if you change a little bit, everything will be far apart. The limits exist even though there are open dots because limits are about what happens near the x value, not at x per se. So even though x is not in the domain on either endpoint, the limit still exists.

YJ: The limit does not exist as x goes to -1 or to 1 because of the holes and this is because now -1 and 1 are not in the domain. So the limit cannot exist if x is not in the domain. As x goes to infinity, the limit stops near 1 so the limit is 0.99999 or so and as x goes to -1, the limit would be -0.99999 or so because those are the last values on the domain.

EB: Last graph like arccosine but with open end points, the limits are the same as with the arccosine, pi and 0, because even though the function doesn’t exist at the endpoints, the limits do and that’s because the x’s do not have to be in the domain for the limits to exist.

BK: The domain is now (-1,1) and range is (0,pi). It does not include the endpoints. So the limit as x approaches 1 is 0, and as x approaches -1 the limit is pi. The limits exist even though there are open dots because limits are about what happens near the x value, not at x per se. So even though x is not in the domain on either endpoint, the limit still exists.

LA: The domain is (-1,1) and the limit as x goes to minus infinity doesn’t exist because of the hole and neither does the limit as x goes to positive infinity. There is no limit because of the hole I think. The x is not in the domain, so it doesn’t exist.

Here is a summary of the two types of responses generated by students.

- The limits exist at open endpoints because the endpoint need not be in the domain.
- The limits do not exist at open endpoints because the endpoint is not in the domain.

Analysis

If students think that the domain has nothing to do with whether or not the limit exists, then they correctly identify the one-sided limit as such and are not confused or side-tracked by the hole. On the other hand, if they think that the “a” has to be in the domain as $x \to a$, then they see nothing there where the hole is and decide that the limit does not exist because of the hole. The hole is there because “a” is missing, and so it appears that a limit has to be a solid dot, not a hole. What might be happening in this case is they are confusing limits with continuity. With this function, the correct way to perceive the endpoints is that even though the endpoints are not in the domain, the limit still exists.

Research Question 1C: In what ways do algebra skills affect task solutions?

Algebra Skills

Being able to factor correctly is an important step to understanding limits of rational functions. Algebra skills are important for understanding limits at a point, particularly with rational functions. The reason is that if one does not know how to factor a difference of squares or try to factor a sum of squares, then the correct limit will not be computed. In the pilot study, NS initially factored a sum of squares thinking it was a difference of squares, but later caught the error. Some new students tried the same, but did not correct the error and computed an incorrect limit.

Moreover, students are not able to determine when there will be a hole or a vertical asymptote in a graph. Many assume that whenever there is a rational function, there will automatically be a vertical asymptote on the graph, found by setting the denominator equal to zero and solving for x. They do not typically start by first plugging in whatever x is approaching to see if the denominator is equal to 0, and then doing the same for the numerator. Different results occur if the denominator equals zero but the numerator does not; if the numerator equals zero but the denominator does not, and if both numerator and denominator equal zero. Moreover, students who do not know that when there are common factors in the numerator and denominator that divide out also do not know when there could be a discontinuity on the graph in the form of a hole.

In this study it was discovered that students often use the same factoring techniques for limits at infinity as for limits at a point, when doing so is not correct. While factoring and plugging in for x is typically done for limits at a point, factoring out the highest power of x separately in denominator and numerator is done for limits at infinity, as is comparing the degrees of the numerator to the degrees of the denominator to study asymptotic behavior. Different techniques are used in each case, and so knowing which techniques to use are important to understanding limits.

One interesting revelation from the study, initially seen with the pilot cases, is that students extend the technique of comparing the left and right hand limits at a point to the rational functions in the form of $\lim_{x \to 0} \frac{1}{x}$. What is seen in this case is students will find that the limit as x approaches 0 from the left is $-\infty$, and then compare that to the limit as x approaches 0 from the right which gives them ∞, then erroneously comparing the left hand limit with the right hand limit and conclude that the limit does not exist because $-\infty \neq \infty$. In order to understand limits, one must know that is not correct to extend the notion of comparing the left hand side with the right hand side for rational functions, and in fact, it only takes one side going to infinity to declare that the limit does not exist in this case.

Two tasks were given with rational functions to explore their understanding of limits at a point. Task 6 involved a sum of squares in the numerator while the one in Task 7 included a difference of squares in the numerator.

Rational Functions with Sum of Squares in Numerator

In this problem, students were asked study the limiting behavior of function values when there is a sum of squares in the numerator. They computed the limit at a point as $x \to 2$ and graph their results for $\lim_{x \to 2} \frac{x^2+4}{x-2}$. If x=2 is plugged into the denominator, the difference is 0 while in the numerator, the sum is 8. Given the 0 in the denominator, there is a vertical asymptote at x=2.

Students often try to factor the numerator, divide out common factors of (x-2) in the numerator and denominator and then think they are left with a straight line of (x+2) and when they plug in x=2, they compute an incorrect limit of 4. Here it is helpful to present some actual transcript evidence from turn 127 and sketches along with their work.

Given the level of detail for each person with the tasks involving rational functions, there is a summary following the evidence for each person reported in this discussion, and one analysis at the end.

Evidence

YJ: $\lim_{x \to 2} \frac{x^2+4}{x-2}$ As x approaches 2, the limit is infinity because I plugged in the value and get 8/0. On the graph you get a hole on the x-axis at x=2 because that is where it is undefined, but everywhere else it is a continuous function. (YJ pointed to x=2 on the x-axis to show the circle means the function is not defined at that point.)

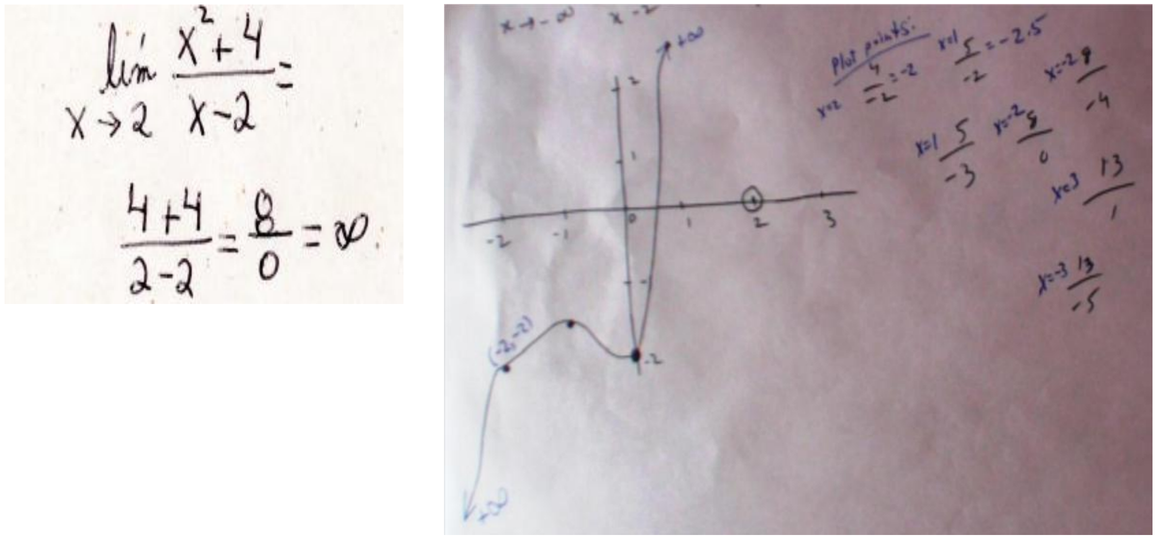


Figure G.13: YJ's solutions for Task 6.

In Figure G.14, YJ elicited the correct conclusion that the limit did not exist, but offered incorrect reasons. She did not report there would be a vertical asymptote and instead, reported there was a hole instead right on the x-axis at x=2. This statement appears disconnect from the graph she drew, because she computed several points corrected and plotted them correctly on the graph, but drew them increasing in the 1st quadrant in a region to the left of x=2. As a result, this positively increasing line had no relationship to the hole at x=2 she drew on the x=axis. YJ presents an unusual case. The graph she drew contained the correct points as she plugged in values close to x=2, but having drawn a hole at x=2 directly on the x-axis revealed confusion. The hole was put in the wrong place and appeared to not be associated with the function. It seemed as if she guessed where to put the hole, and did not think about whether a hole or a vertical asymptote would occur. So she did not draw the graph correctly.

JY: $\lim_{x \to 2} \frac{x^2+4}{x-2}$ Domain is $x \neq 2$. First I draw a vertical asymptote at x=2, then tail ends of the function. The limit as x approaches 2 does not exist anywhere. It's just a rough graph. We are taught to draw graphs in high school to get a rough estimate.

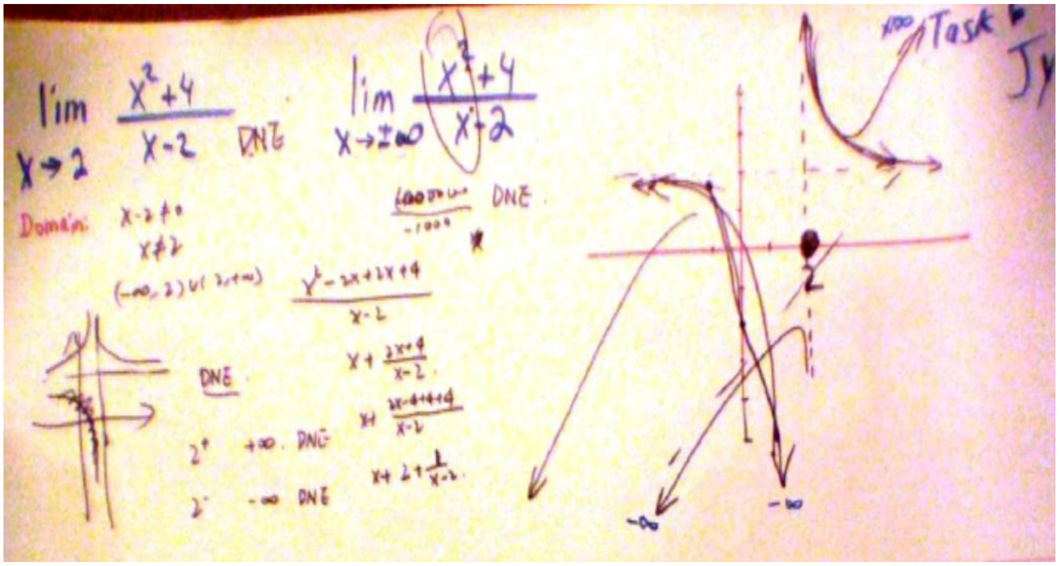


Figure G.14: JY's solution for Task 6.

In Figure G.14, JY presented a unique factoring technique and made the correct determination that the limit as x approached 2 the limit did not exist. She also looked at how x approached 2 separately from the right and from the left in her work, and concludes in each ease that in each direction where the limit equals infinity, the limit does not exist. This response is in contrast to other students who think the limit exists if it is equal to infinity. JY did not use the graphing calculator. All graphing was constructed manually.

EB: $\lim_{x \to 2} \frac{x^2+4}{x-2}$ I'm not sure what the domain of this is, maybe a vertical asymptote at 2. You cannot factor the numerator because it would have to have a minus in it. You can't have a 0 in the denominator so the function does not exist at x=2. I think you use L-Hospital's rule. Since you get 0/0 you take the derivatives of the top and of the bottom. I think the limit would be found with L-Hospital's rule, so you get 2x/0, so if you plug in you get 2(2)=4. If it has the VA at 2, then this doesn't work. I don't know why L H rule doesn't work yet it's supposed to be something you can always fall back on. You're supposed to be able to keep taking derivatives until you can't take them anymore, and sometimes you get the answer on the first try, so I think I'm just doing something wrong. Anyway, I would say the limit does not exist because of the graph, since the function does not exist at x=2.

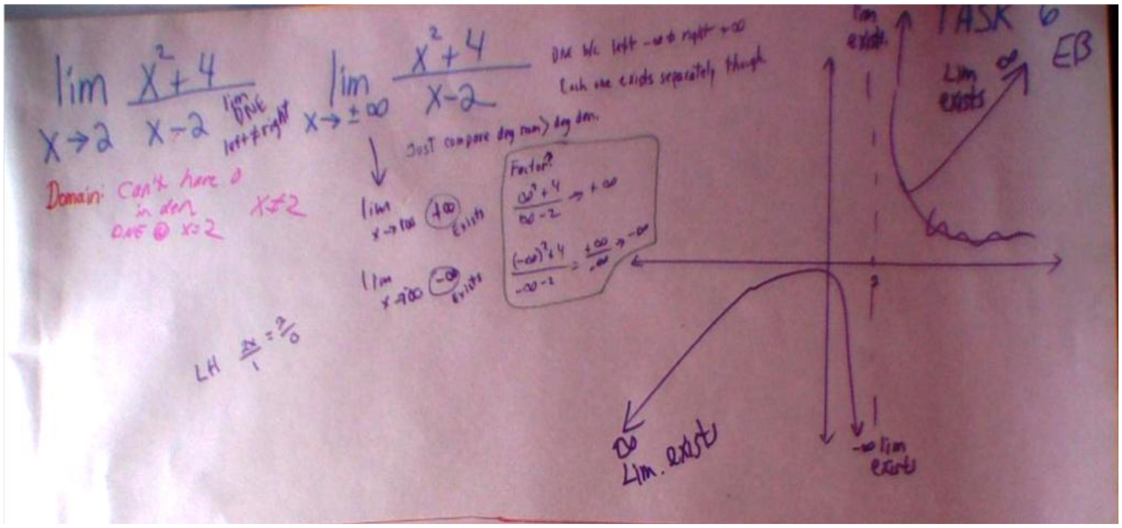


Figure G.15: EB’s solutions for Task 6.

In Figure G.15, EB acknowledges that the numerator cannot be factored, but incorrectly determines that she would end up with $\frac{0}{0}$ an indeterminate form and be able to use L’Hospital’s Rule. This was obviously due to a mistake, as she did not look at the numerator carefully to see that she would not get 0 if she plugged in 2. Yet, she seems to catch her mistake and decides to apply L’Hospital’s rule to the numerator and denominator and end up with $\frac{2x}{0}$. Here, she disregarded the 0 in the denominator and plugged in x=2 getting 2(2)=4 for the numerator. It appeared EB was expecting this 4 to match the vertical asymptote at x=2, and when it did not reconcile, she decided to look at the graph in order to conclude that the limit did not exist. Although this was the correct determination, when she was probed further as to why the limit did not exist, it was revealed that the left hand limit did not equal the right hand limit, $-\infty \neq \infty$.

When then further probed about each particular piece of the graph, she said that as x approached 2 from the left, the limit existed and was equal to minus infinity and when x approached 2 from the right, the limit existed and was equal to plus infinity. This was an important finding because these statements about the limit exists if it equals infinity comes directly from her definition of limit being how a function is behaving at a certain point. In turn 145, I ask her if the limit exists if it equals infinity, and in turn 146, she reports “yes, the limit exists because it is equal to something.” The topic of infinity comes subsequent to this analysis, but it will be seen that students, including EB, think infinity is a place or represents a very large number.

BB: This is where I get confused. I know you set the denominator equal to 0. Do I set the numerator equal to 0 as well? I think it’s the denominator so $x \neq 0$. As x goes to 2, the limit does not exist because if you plug in 2, you get 8/0 and that is undefined.

BB gave a straightforward response but since her written work is similar to those above, it is omitted here. She started off with the right idea but it was not about setting the denominator equal to zero, but was supposed to be about plugging x=2 into the denominator to see if she would get 0. Then she was supposed to do the same procedure for the numerator. In any event, she got the computation correct, $\frac{8}{0}$ and made the correct determination that the limit did not exist.

AK: $\lim_{x\to 2}\frac{x^2+4}{x-2}$ The domain is $(-\infty,2)\cup(2,\infty)$. You can factor the numerator but it doesn’t work. So you just plug in 2. When you do, you get 4/0 so the limit does not exist.

AK, like EB, took a close look at the numerator. She knew not to factor the numerator but added wrong and got an answer of 4 instead of 8 in the numerator. She determined that the limit did not exist because of the 0 in the denominator which made the function undefined. Her conclusion was similar to that of BB.

KB: $\lim_{x \to 2} \frac{x^2+4}{x-2}$ The domain is $x \neq 2$ otherwise the function is undefined. I was taught for this to just plug in numbers close to 2 on both sides. Like 1.9999 and 2.0001. If they're similar, it's what the limit is. If you plug 2 into x, you get 0 in the denominator so the limit does not exist if x equals 2. You can't factor the numerator because it doesn't work with (x+2)(x-2).

BK: $\lim_{x \to 2} \frac{x^2+4}{x-2}$ Domain is $(-\infty, 2) \cup (2, \infty)$. The limit as x goes to 2. You cannot plug in directly. I wish it was a minus in the numerator so it would cancel. As x goes to plus infinity you plug in values greater than 2, you get a smaller number on the bottom so you get infinity. If you pick numbers smaller than 2 on the left you get negative infinity. So pick 1 on the left it's -5 and if you pick 3 on the right you get 13. So since -5 does not equal 13, the limit does not exist.

KB also considered that plugging in x=2 would make the function undefined, so she used the approach of plugging in values close to 2 on both sides to test her idea. She was determined to make sure the numerator could not be factored, and meanwhile, made the correct determination that the limit did not exist. Similar to KB, BK picked numbers to the left and right of x=2 to plug in and made the correct conclusion that the limit did not exist because the left hand limit did not equal the right hand limit.

Analysis

The results show that proficiency with algebra and arithmetic is necessary for computing limits at a point of rational functions. Even though NS in the initial study was the only person who tried to factor the sum of squares in the numerator, this may not be an isolated event. In fact, EB and AK acknowledged the numerator not being factorable, so students do at least think about factoring the sum of squares. In JY's case, there was a unique way of factoring presented, but she ultimately computed the correct limit.

Arithmetic skills are important because if one plugs in an x-value into the numerator and adds wrong, then this could affect the answer. AK did this, but got the limit correct by coincidence because the denominator was equal to 0, in which case she correctly concluded the limit did not exist.

Students perceive rational functions as a fractions which can be problematic. Referencing Task 6 once more, students realize the denominator cannot be equal to 0. By plugging in x=2, then 2-2=0 would occur in the denominator. In this case the function is undefined at 2. Students report the limit does not exist because the function is undefined rather than because the limiting behavior of the function values increases without bound in the positive and negative directions. This is an important distinction because with limits, one should focus not on the denominator, but on how the function values are behaving near a point.

Later on, a separate section is devoted to studying infinity more closely and discussing limits that do not exist, but for this particular section, it is worth noting that students apply the technique of comparing the left hand side to the right hand side with rational functions, when this is not supposed to be done. They are told in general to compare the left and right hand sides for limit at a point, so they generalize doing this for rational functions. Instead of correctly determining that one side alone goes to infinity and therefore, the limit does not exist, instead they compare the left hand limit with the right hand limit, $-\infty \neq \infty$ and conclude that because they are not equal, the limit does not exist. This is not the correct way to think about limits at a point for rational functions, as only one side needs to go to infinity to decide that the limit does not exist because infinity is not a number.

Difference of Squares in Numerator

Students can typically factor a difference of squares. However, they sometimes think all rational functions have vertical asymptotes and do not know under what conditions a hole occurs. Students factor the binomial correctly, divide out the common factors, plug in for "x" but then do not know how to transfer the information to the graph and do not know if there will be a hole or a vertical asymptote. Results from turn 143 appear below:

Evidence

YJ: $\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$ The domain is everything, minus infinity to infinity. Limit as x approaches minus 3 is 0/0 so it's 0. When x equals -3, the limit is 0 and that's because -3 is in the domain.

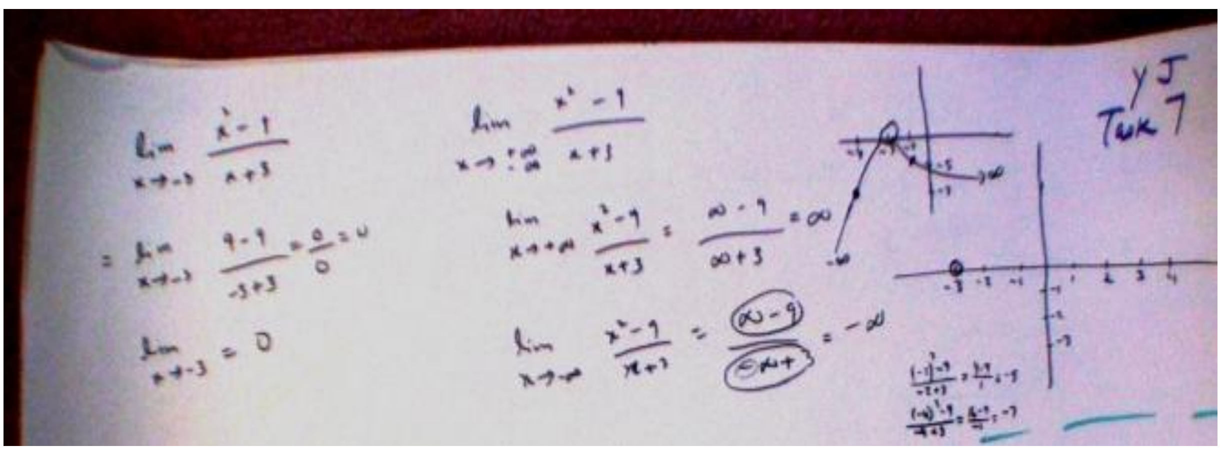


Figure G.16: YJ's Task 7 Solutions.

In Figure G.16, YJ cited the wrong domain and computed an incorrect limit. She plugged 3 into the function and got an indeterminate form. When she got 0, she considered that to be a function value so she drew a hole on the x-axis at x=-3. This is similar to what she did in the other problem involving the sum of squares. Moreover, by not factoring, it can be seen that the graph is drawn incorrectly looking nowhere like a linear function with a discontinuity, and the wrong limit was computed. Finally, she states that the limit exists because x=-3 is in the domain. This is consistent with what she reports earlier about “a” having to be in the domain for the limit to exist.

EB: $\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$ Domain is all x, except x not equal to 3. The y intercept is at -3 after you factor the numerator because you only have (x-3) left in the numerator. You plug in –3 and get -6. On the graph, the -6 is the y value so on the graph you go over left -3, then down -6 units and you put a solid dot I think, but I am not sure. I wonder if it’s a hole because of if you plug -3 into the original function. It does not look like there is a point at (-3,-6). You get holes when the function does not exist. You have to get a hole where x= -3. You look at the domain which is where x cannot equal -3, so you get a hole in the graph. When you cross things out you get a hole. The limit exists even though -3 is not in the domain.

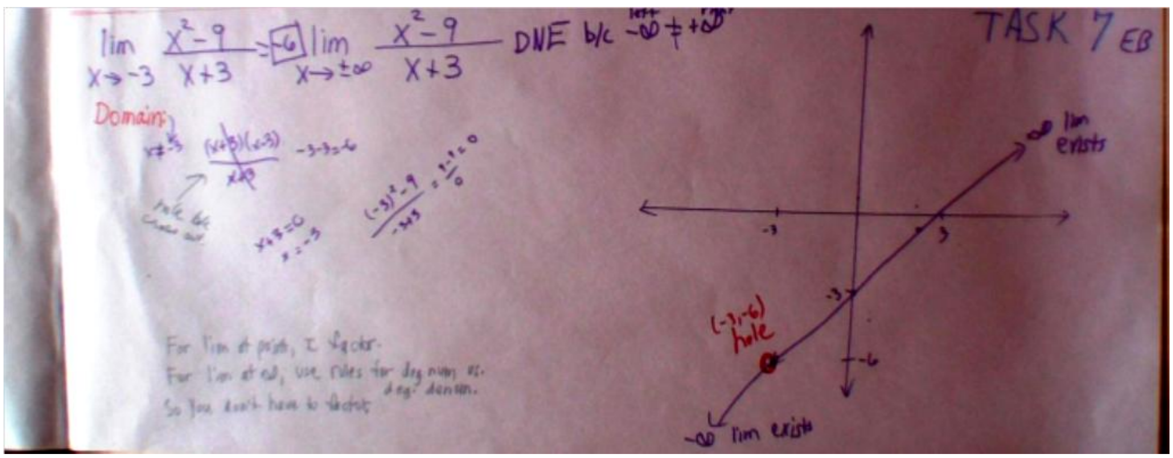


Figure G.17: EB Task 7’s solution.

EB cited the correct domain, computed the correct limit and drew the graph correctly. She correctly reported that the x=-3 does not have to be in the domain for the limit to exist (Figure G.17).

JY: $\lim_{x \to -3} \frac{x^2 - 9}{x+3}$ Domain: $(-\infty, 3) \cup (3, \infty)$ The limit doesn't exist because it is equal to -6.

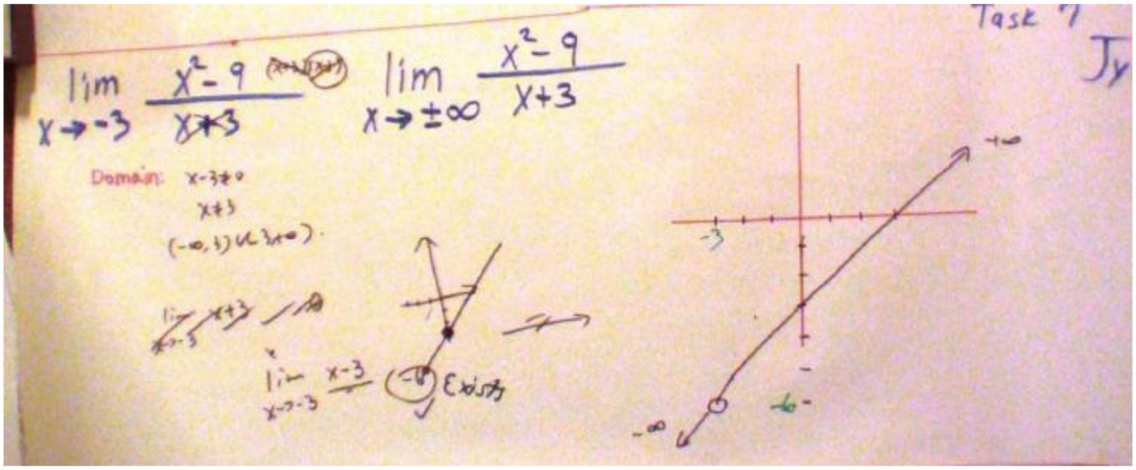


Figure G.18: JY's Task 7 Solution.

In Figure G.18, JY gave the wrong domain, using $x = 3$ instead of $x \neq -3$. This was probably an oversight, but important because other students do this. Consistent with her definition of limit, a number that can only be approached and not equal to the function value, this is another example of how her definition of limit manifests in her solution. She computed the correct limit, -6, but then stated that this limit does not exist. The reason is consistent with her definition of limit in which the limit can only be near f(a) not equal to f(a). At first, she drew the graph incorrectly, with the point placed at (0, -3), which has been seen with the students in the pilot study.

BK: $\lim_{x \to -3} \frac{x^2 - 9}{x+3}$ This one is easier because it factors and so you are left with (x-3). So the limit is going to be -6. On the graph, you might have a hole because two of them cancel when they don't cancel you get a vertical asymptote. It cancels into a linear function in this case. Even though $x \neq -3$ is the domain, the -3 does not have to be in the domain for the limit to exist because the limit is about what happens as x approaches 3, not just at when x equals 3.

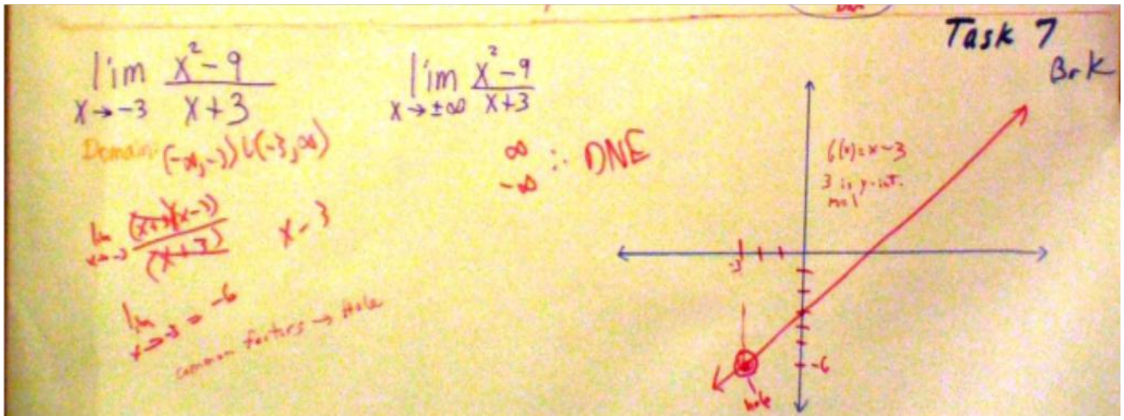


Figure G.19: BK's Task 7 Solution.

In Figure G.19, BK computed the limit correctly, reported there would be a hole in the graph, and then drew then drew the function correctly on the graph, placing a hole at (-3,-6). He used knowledge about the domain to correctly report that x=-3 does not have to be in the domain for the limit to exist.

KB: $\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$ Domain: $x \neq -3$ I think the numerator factors, so the top and bottom factors out and then you are left with (x-3). Then you plug in -3 and end up with -6. What you do with it on the graph, though, I don't know. I don't know if it's a dot there or a hole. I forgot.

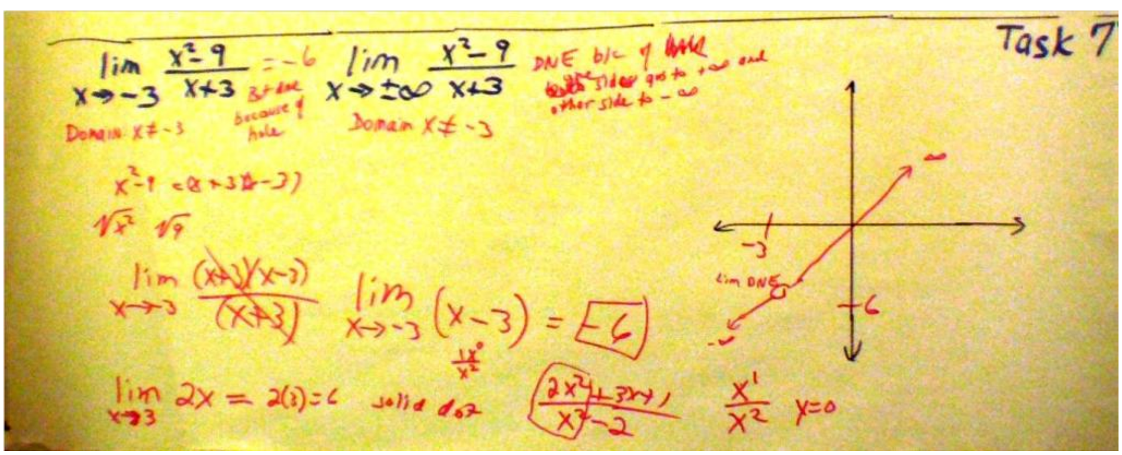


Figure G.20: KB's Task 7 Solution.

In Figure G.20, KB got as far as she could with the algebra, and was able to divide out common factors, but once she got the limit, she did not know how to transfer this information to the graph. Later in the process she eventually guessed and then reported that the limit did not exist because of the hole.

LA: $\lim\limits_{x \to -3} \dfrac{x^2 - 9}{x + 3}$ Domain $(-\infty, \infty)$. This one factors and the top and bottom cancels out, so we have left (x-3). If you plug in -3 you get -6 which is the limit. X=-3 is in the domain so that's why I'd say that the limit exists (Figure G.21).

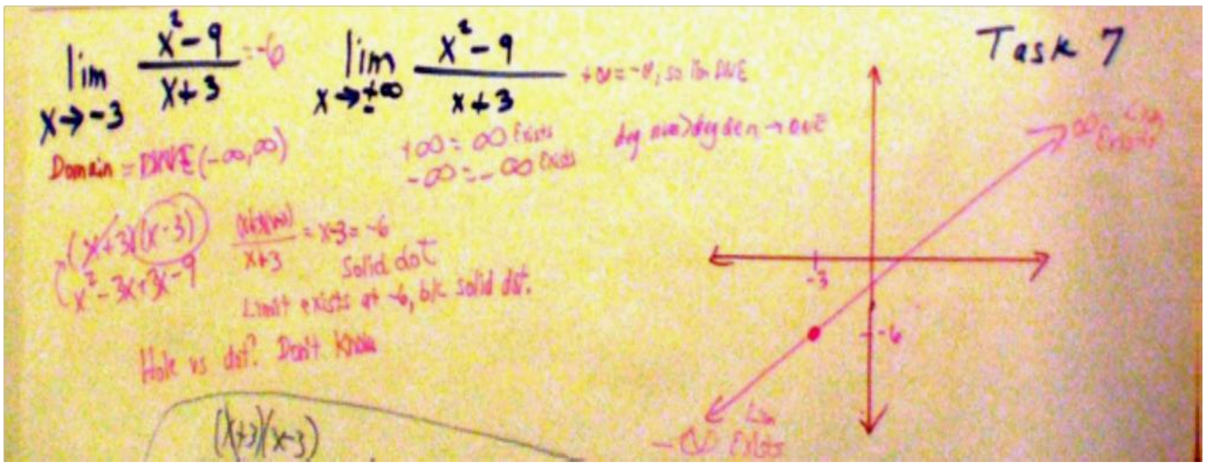


Figure G.21: LA's Task 7 Solution.

This is a case where the numerator and denominator are factored correctly, but there was uncertainty about whether to put a hole or a dot. So LA put a dot at (-3,-6). Even thought the correct limit was computed, the information was not translated to the graph accurately. She concluded that since x=-3 is in the domain, the limit exists.

Analysis

All the students recognized the correct domain for this function, but there were some problems with deciding how to graph the information because they did not have the knowledge of when there would be a vertical asymptote versus a hole. The reason could be overall unfamiliarity with rational functions. The problem remains about whether or not as $x \to a$, if "a" has to be in the domain for the limit to exist. It seems there continues to be two types of responses here, and so the correct way to think about this is that "a" does not have to be in the domain for the limit to exist because by definition of limit, one studies how the function values behave near "a" not just at "a".

Summary

Table G.2 below summarizes the ways in which students should think about limits versus ways that students do not think correctly about limits at a point. Overall, there are several identifiable factors that facilitate understanding limits at a point. First, it is important to know the definition of limit and internalize what this definition means. This includes knowing that limits involve the y-coordinate. Sometimes the y-coordinate is on the graph of the function in the case of a point, and in other cases it is not on the graph of the function but only in the plane, in the case of the hole. Understanding this will then facilitate understanding that limits can approach as well as equal a function value, in the case of a continuous function. Most of all, students must know that limits are about the behavior of the function values near some value of "a" as $x \to a$.

In the case of piecewise functions or functions with any kind of discontinuities, understanding the definition of limit will facilitate understanding that limits can exist where there are "holes" in graphs. When students say that the limit does not exist due to a hole in the graph, they appear to be confusing limits with continuity. When they see an empty hole, they tend to say nothing exists there so therefore the limit does not exist.

Also, limits can exist in the case of discontinuous functions with function values above holes, as there can be a limit and a function value on the same graph. Understanding the definition of limit facilitates understanding rational functions as well because where there is a vertical asymptote, as x approaches that asymptote from the left or from the right approaching minus infinity or infinity, the limit does not exist due to just one case, either from the right or from the left. Also in the case where the numerator and denominator share common factors, there could be a hole in the graph rather than a vertical asymptote. Students need to have the algebra proficiency to determine when a graph will have a vertical asymptote versus a hole.

Limits can also exist over domains with finite intervals, both at when the endpoints are included in the domain and when they are not. It was seen that some students reported that the endpoints must be in the domain for the limit to exist. In another case, one student reported the limit only existed if there was a hole because of her definition of limit referring to “approaching” only. Other students were not familiar with one-sided limits and so they thought that limits did not exist if they were only coming from one direction, both where the endpoints were closed and open.

Knowing the definition of infinity and understanding that infinity is not a number will help students understand the difference between a limit which is a number, and infinity, which is not a number. Therefore, they would see that a limit can only exist if it is a number, and cannot exist if it equals infinity because infinity is not a number.

Considering the domains of functions are essential to understanding limits. There are three conditions which must be satisfied for a limit to be continuous at a point, a. First, as $x \to a$, the function value exists at “a”, i.e. f(a) exists. So “a” has to be in the domain of the function. Also, the limit must exist, i.e., limit=L for some number L. Last, the limit must be equal to the function value at “a”, i.e., L=f(a). Students recognize the point being in the domain for continuous functions, but when a discontinuity occurs, students need to acknowledge that “a” does not have to be in the domain for a limit to exist because limits are about the behavior of the function values near a point, as well as "at" the point.

Table G.2: Summary of How Students Think about Limits at a Point

Correct Notions	Incorrect Notions
• Function values are perceived as being the y-coordinate. • The terms "input and output" are applicable to functions. • A limit can approach as well as equal a function value. • Limits and function values are sometimes different for piecewise functions. • Piecewise functions were the limit is not equal to the function value still has a limit. • Limits do not exist for piecewise functions, e.g. a step function, when the left-hand limit is not equal to the right-hand limit. • Piecewise functions with jump discontinuity, i.e. step function: the limit does not exist because the left hand limit does not equal the right hand limit. • Notation unambiguous. First consider $x \to a$, then consider f(x) for each x. • The limit can exist where there is a hole representing a discontinuity. • The limit is independent of the function value at the limit point. • The function value is independent of the limit in piecewise functions where the function value occurs above or below the hole. • The limit exists at endpoints of intervals with finite domains, regardless if "a" is in the domain. • The left and right hand limits can be compared regardless of the function being continuous or discontinuous.	• Function values have ambiguous meaning, perceived as x, y, or as both (x,y) as points on the graph. • Limits are about x only, not about y (initial study). • A limit can only equal a function value (current study). • A limit can approach a function value, but cannot equal it (current study). • Limits and function values are the same with piecewise functions. • Piecewise functions: The limit does not exist where the hole appears. • Piecewise functions with jump discontinuity, e.g. step functions, only have one-sided limits that exist. • Left and right-hand limits can't be compared for piecewise functions. • Notation ambiguous. What appears beneath "lim" notation reveals the limit so no math is required to find the limit (initial study). • The limit cannot exist where there is a hole in a discontinuous graph. • If a point exists above a hole in a discontinuous piecewise function, the limit exists and is equal to the function value. • The limit does not exist at endpoints of intervals with finite domains unless "a" is in the domain. • The left hand limit and right hand limits are not consistently compared for continuous or discontinuous functions.

- As $x \to a$, "a" does not have to be in the domain for the limit to exist.
- Once a domain is changed, so has the function.
- Limits of rational functions in the form $\lim_{x \to 0} \frac{1}{x}$ require algebra and arithmetic proficiency.
- Limits do not exist if they equal infinity because infinity is not a number.
- Limits of rational functions only require one side to not exist in order for the whole function's limit to not exist.

- As $x \to a$, "a" has to be in the domain for the limit to exist.
- If the domain is changed, the function stays the same.
- As $x \to 0$ of rational functions in the form $\lim_{x \to 0} \frac{1}{x}$, the limit exists because the left hand limit $-\infty \neq \infty$, the right hand limit.

Limits at Infinity

Research Question 2A: In what ways do students perceive the domain to be involved with limits at infinity with functions that have finite interval domains?

Meaning of the Infinity Symbol

First, knowing how students think about the infinity symbol gives rise to how they later manifest their understandings in their task solutions. In turn 192, students present vague notions about the meaning of the infinity symbol.

Evidence

LA: It could be a number really large that you can think of like exponential.
EB: It's an infinitely large number.
JY: It goes to an amount that you cannot count. It is even more than a large number because it is so large you cannot even tell.
BK: It represents an unreachable number.
YJ: It represents something undefined.

A summary of the definitions of infinity are as follows:

- Undefined
- Very large number

Analysis

Some students who report that the infinity symbol is number sometimes also erroneously state the limit exists when it equals infinity. Later on this has implications when exploring how they perceive end behaviors. As function values increase or decrease without bound, some students say a limit exists if it equals infinity. The perception of the infinity symbol as a large number is later consistent with their use of the infinity symbol in arithmetical operations. So the definition sets the stage for other investigations.

Limits at Infinity and Role of Domain

Finite Interval Domains

Case of Open Endpoints

Given a continuous linear or quadratic function, students would agree that each point is in the domain as x approaches infinity. Domains with finite restricted intervals proved more challenging, though. For instance in Task 2, given a closed endpoint with the half circle or arccosine, students reported that the domain is [-1,1] but when asked if the limit existed as x approached infinity, they reported the limit existed at -1 or 1 since that is where the function ended. A few students correctly reported that $x \to \infty$ did not make sense because there were no x-values for x<-1 or x>1.

In Turn 53, students were asked to explain precisely the behavior of the function values as x approaches plus or minus infinity for each the half circle and arccos(x), talk about if these limits exist as x approaches plus or minus infinity. Turn 54 reveals their responses.

Evidence

LA: The limits do not exist. You can't go past -1 or past 1 because it's outside of the domain.

JY: The limits don't exist because there is no x in the domain. So the question going to infinity makes no sense (Figure G.22).

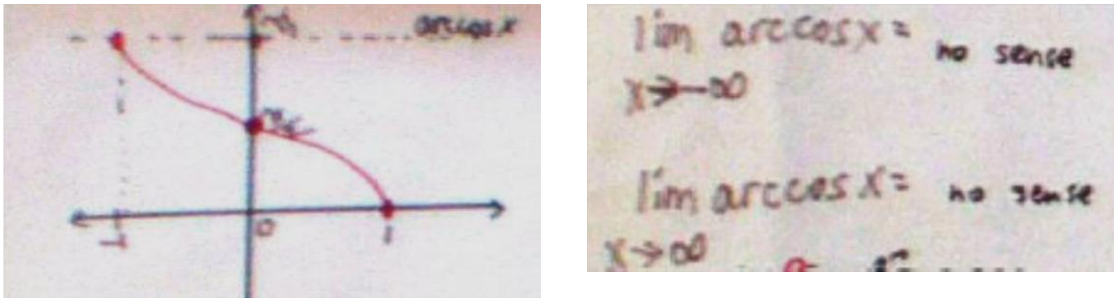


Figure G.22: JY's solution to Task 2.

AK: For arccosine, the domain is [-1,1]. The range is [0,pi]. As x goes to infinity or minus infinity the limit is 1 and -1 (Figure G.23).

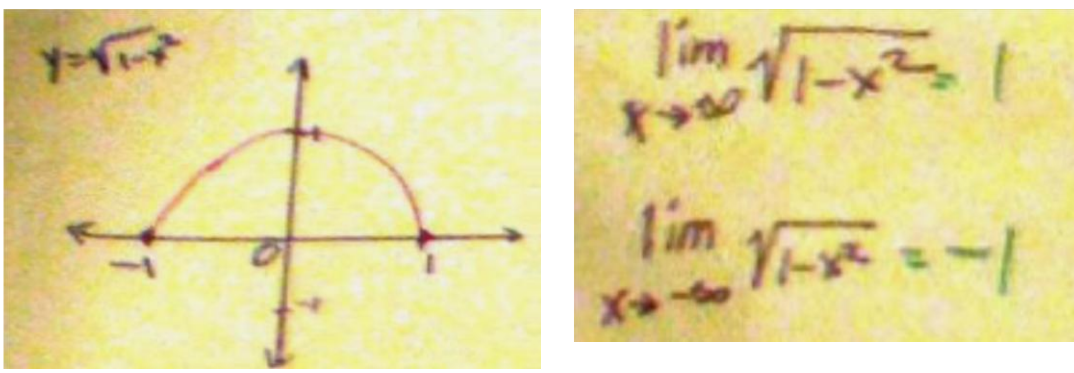


Figure G.23: AK's solution to Task 2 half circle.

YJ: For half circle, as x goes to -1 the limit is 0. As x goes to plus 1, limit is 0. These limits exist. As x approaches 0, the limit is 1 because it's in the range. For arccosine, limit as x goes to -1 is pi. As x goes to positive 1 the limit is 0 (Figure G.24).

Figure G.24: YJ's solution to Task 2 half circle.

Below is a summary of the results for closed endpoints:

- The $\lim_{x\to\pm\infty} f(x)$ exists at the endpoints, the x-values -1 and 1, rather than the y-values.

- As $\lim_{x \to \pm\infty} f(x)$ the limits equal 0, the y-values.
- The $\lim_{x \to \infty} f(x)$ does not exist because x<-1 or x>1 are not in the domain which is the correct way of thinking.

Case of Open Endpoints

In the case of the open endpoints, the responses were consistent with how they perceived the limit at a point earlier, in which case they reported the limit at the endpoint did not exist because of the hole and as $x \to a$, "a" was not in the domain. Otherwise, they reported that the limit only existed from one side, or that the limit existed because the "a" did not have to be in the domain. Students applied these same understanding to what happens as $x \to \pm\infty$ with the open endpoint in Turn 52.

LA: As x goes to infinity or minus infinity, the limits do not exist. You can't go past -1 or past 1.

JY: As x approaches infinity, the limit does not exist because there is no x value so problem makes no sense. Same thing with minus infinity so the limit does not exist (Figure G.25).

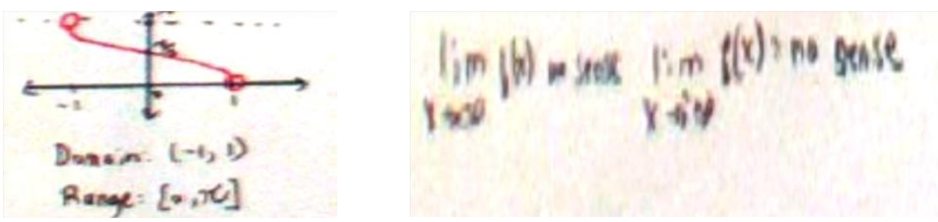


Figure G.25: JY's solution to Task 2 with open endpoints.

YJ: Half circle, limit as x goes to infinity is 1 and as it goes to negative infinity it is -1 because that is as far as the domain goes and because these are in the domain. For arccosine, as x goes to minus infinity, the limit is pi. As x goes to infinity the limit is 0. That's because pi and 0 are in the domain.

AK: As x approaches infinity and minus infinity, the limit won't exist either because it's not in the domain.

Below is a summary of the results for open endpoints.

- The $\lim_{x \to \pm\infty} f(x)$ exists at the endpoints.

- The $\lim_{x \to \pm\infty} f(x)$ does not exist because x<-1 or x>1 are not in the domain.

Analysis

These tasks drew out the fact that students erroneously think that as $x \to a$, that “a” has to be in the domain for limits to exist. Limits exist at endpoints independent of the domain being defined. Meanwhile, the essential point of this task is that since the domain is not defined for all x as $x \to \infty$, so it makes no sense to ask how function values behave for large x in either direction.

When the half circle and arccosine tasks were modified to have open endpoints over the interval (-1,1), AK applied consistent reasoning that a limit does not exist because of a hole. LA, though, did not apply that previous reasoning but instead, decided the limit obviously exists at the endpoints where the holes are but the function is not defined past the finite interval domain. JY reported that the limit could not exist at x=-1 or x=1 because those two points were not in the domain. In JY’s case, when asked if the limit existed as x approached infinity for the arccosine with open endpoints, she report the limit does not exist because the function only went near x=-1 or x=1. This is consistent with her definition of limit meaning “nearness, not equal”. Other students successfully recognized that it made no sense to ask if the limit existed as x approached infinity because large x’s were not included in the domain.

Thinking that a one-sided limit could only exist if there were closed endpoints suggests that students think the point has to be in the domain. So when the task was modified to include an open endpoint, that gave evidence of their erroneous ideas that endpoints had to be included for the limit to exist.

Research Question 2B: Do students recognize the relationship between a computed limit at infinity of a rational function and the horizontal asymptote?

Relationship of Limit at Infinity to Horizontal Asymptote

Task 8 was $\lim_{x \to \pm\infty} \frac{9x^2+2}{3x^2-2x+5}$ in which they had to compute the limit and sketch a graph, then decide if the limit they computed had any relationship to the horizontal asymptote. In most cases, there was a disconnection of the computed limit from the graph sketched. They could have applied rules for asymptotes, in which case the power of the leading term in the numerator was equal to the power of the leading term in the denominator so the ratio of the coefficients is the limit. Turn 154 reveals how students computed the correct limit and even used test points to construct the graph from scratch, but did not see that the limit they computed had any relationship to the graph they were sketching. In another instance, YJ computed the wrong limit altogether and got a completely different result.

Evidence

LA: The exponents are the same so you take the number out front and then the limit is 3 because the number is 9/3. As x goes to infinity, the function values keep going to infinity. Same thing as x goes to negative infinity the function values are going to negative infinity. No wait. It keeps going to 3.

KB: The domain is probably $[0,3]$. I think since the exponents are the same you get what is out front, which is 9/3 which is 3. So the limit is 3? As x goes to plus infinity the limit is 3, and as x goes to minus infinity the limit is also 3. I don’t if this could be graphed though.

EB: The limit is 3 because of the ratio of the leading coefficients you use with limits at infinity. The limit though is something, I guess it’s 3 (Figure G.26).

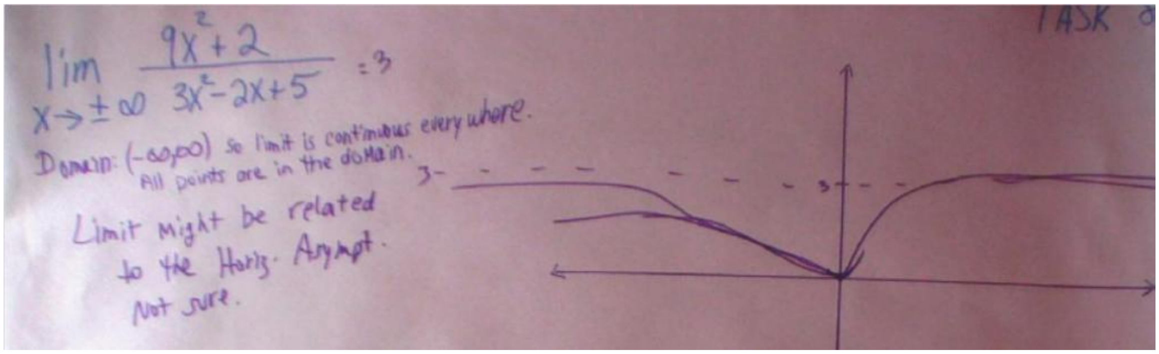


Figure G.26: EB’s solution for Task 8.

Next, students were asked in Turn 160 if there was a relationship between the limit and the horizontal asymptote.

LA: The limit turns out to be the horizontal asymptote. It’s the same thing even though I never thought about it like this before. You learn the asymptote thing in algebra but in calculus they show you the limit but they change the wording.

EB: I never thought about it that way, but yeah. They are the same. It’s the first time I looked at it this way.

KB: They are the same. I don’t know why, it just is. But in this problem it might just be a coincidence because none of these limit problems ever give you an easy answer.

YJ: There is no horizontal asymptote in this problem so it does not have a relationship with the limit.

Below in Figure G.27 (Task 8), YJ computes the limit manually. The graph is constructed incorrectly. This work also contains arithmetical operations with infinity.

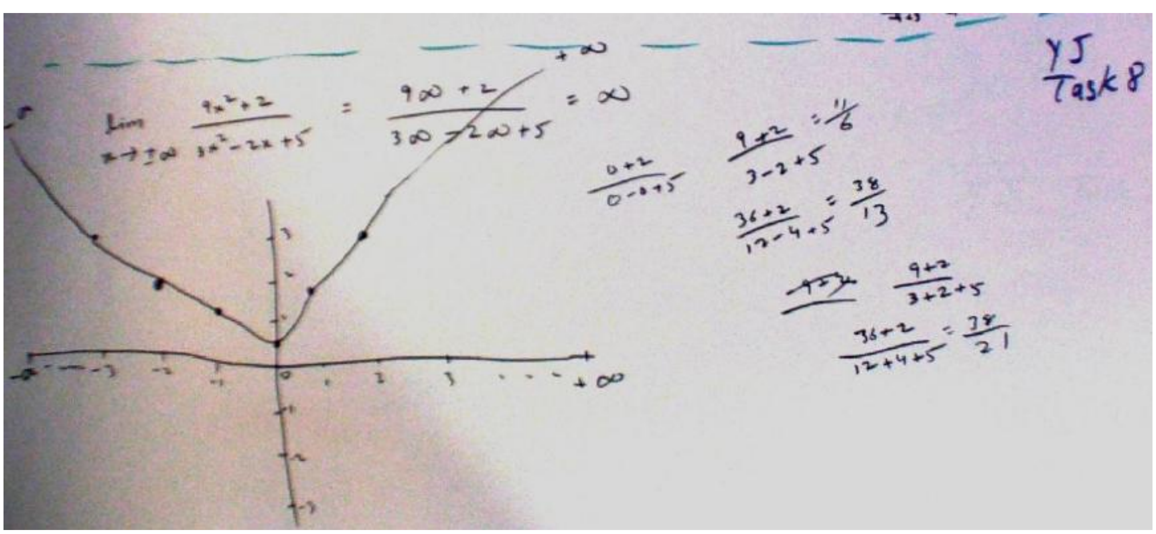


Figure G.27: YJ’s solution for Task 8.

There are two types of responses seen with this problem in Task 8.

- Given $\lim_{x \to \pm\infty} \frac{9x^2+2}{3x^2-2x+5}$ =3 and is the horizontal asymptote, but the reason why is not know.
- Given $\lim_{x \to \pm\infty} \frac{9x^2+2}{3x^2-2x+5} = \infty$ and so there is no horizontal asymptote.

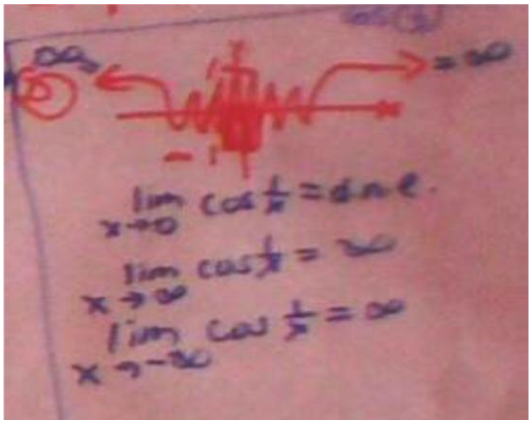
Analysis

In this case, no relationship between the horizontal asymptote and limit were found because the limit was incorrectly computed. YJ did not identify the horizontal asymptote as being y=3 because the algebra done afterwards distracted her. This also means she did not rely on the algebraic rules for finding asymptotes because if she had, she would have identified the ratio of the leading term's coefficient $\frac{9}{3} = 3$.

Since the focus is on limits at infinity and horizontal asymptotes, another type of example from Task 2 and Task 9 are presented. Looking at the graph of $\lim_{x \to \infty} \cos \frac{1}{x}$ from Task 2 as well as with the graph of $\lim_{x \to \infty} \frac{1}{(x-a)}$ from Task 9 it can be seen with YJ's work in Figure G.28 that some students can easily confuse interpreting the behavior of the function values for large x in either direction. YJ reported the limit is $\pm\infty$ instead of 1.

YJ: (Task 2, Turn 52) The limit as x goes to infinity and to minus infinity is infinity so it does not exist on either side.

YJ: (Task 9, Turn 164) As x is approaching "a" the limit is infinity so it does not exist. Both top and bottom go to infinity so on bottom as x goes to"a" the limit does not exist because it is also going to plus infinity. Then as x approaches plus infinity, the limit is 0. As x approaches minus infinity, the limit is going to still be 0.



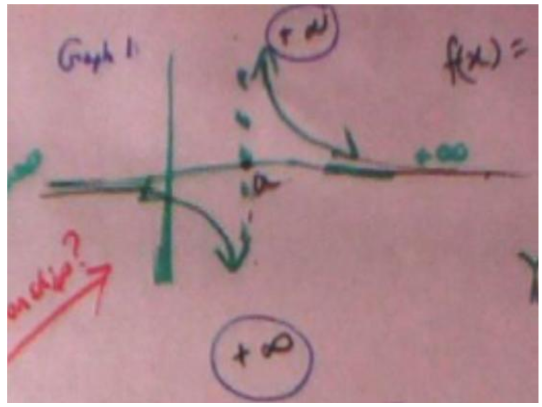


Figure G.28: YJ’s solutions for Task 2 and Task 9, respectively.

Analysis

Students don’t connect the relationship between the limit and the graph. They don’t connect that the limit is about the behavior of the function values or y-values, and so if the limit is equal to 3, then there should be asymptotic behavior around y=3.

Research Question 2C: In what ways do algebra skills affect task solutions?

Algebra Skills and Replacing Infinity Symbol for Variable

As it can be seen above with the work on Task 9, algebra becomes significant in particular with rational functions. Using Task 6, most students such as EB, BB and AK plug 2 into the numerator and denominator, concluding the limit does not as $x \to 2$ because $\lim_{x\to 2}\frac{x^2+4}{x-2}=\lim_{x\to 2}\frac{8}{0}=\varnothing$. When the second part of the problem, though, requires them to compute the limit at infinity, in Figure G.29 EB plugs in ∞.

Evidence

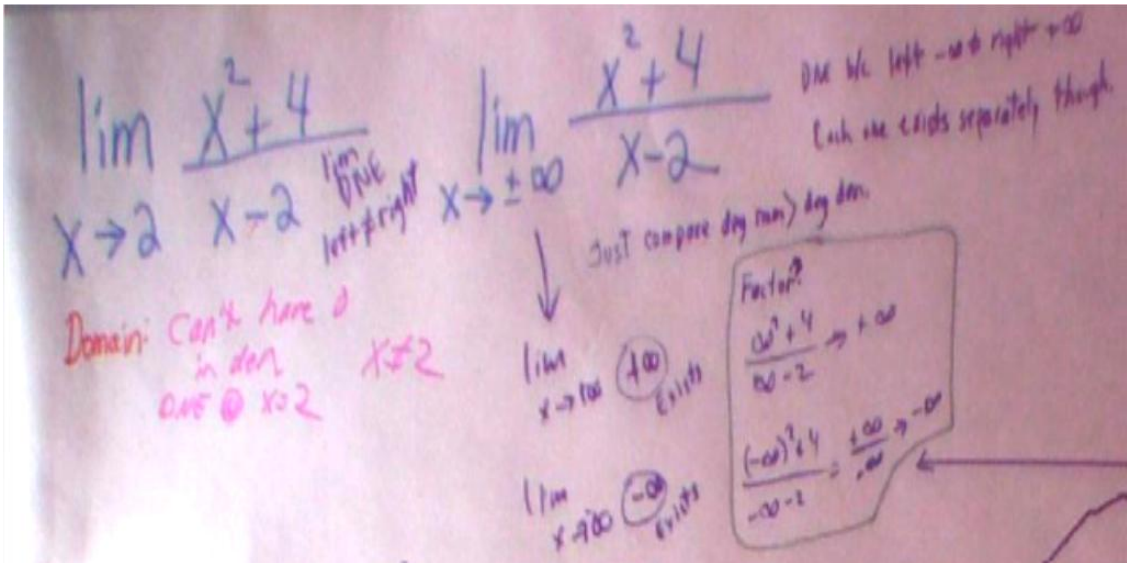


Figure G.29: EB's solution for Task 6

Meanwhile in Figure G.30, BB solved the limit at a point as $x \to 2$ two different ways. She first plugged in 2 which yielded a result being undefined. Next she tried to factor the sum of squares and plug in 2, which yielded an incorrect limit of 4. When she moved onto the next part of the task, to evaluate the limit as $x \to \pm\infty$, she used L'Hospital's Rule. Since the function was not indeterminate, this procedure was not usable. She also reported the limit is equal to 2, which is not correct.

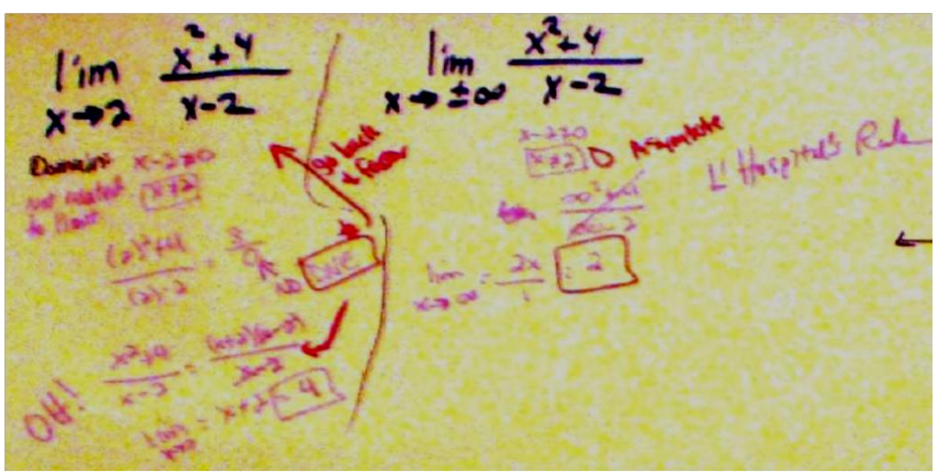


Figure G.30: BB's solution for Task 6.

In Figure G.31, AK tries to initially factor the sum of squares and plugs the ∞ symbol into x, getting the correct answers, but clearly uses the algebra rules for finding asymptotes.

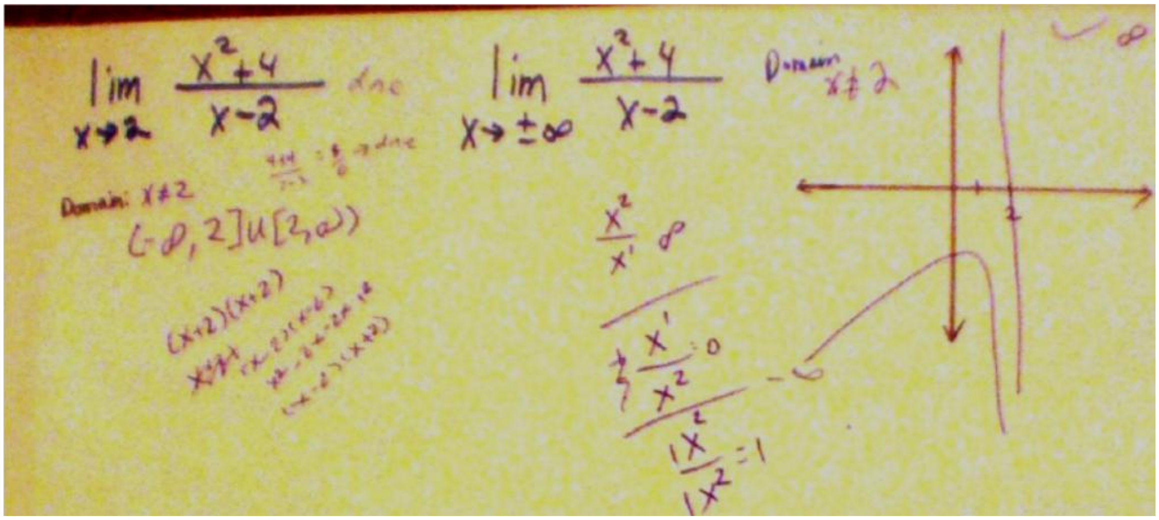


Figure G.31: AK's solution to Task 6

Analysis

Thinking of the infinity symbol as a large number can manifest through a students' work when they plug in the symbol to replace a numerical value, and perform mathematical operations. This is not a mathematically sound procedure, dividing with infinity reveals that a student thinks of infinity as a substitute for a large number.

A few students used L'Hospital's Rule to solve tasks involving limits at infinity for rational functions, though this procedure is reserved for limits of functions that are indeterminate. BB computed the derivative of the numerator, then the derivative of the denominator which yielded an incorrect limit, i.e. a constant times a variable,

$2x=2(\infty)=2$. What these results imply is that students use faulty generalizations. They learn about a particular procedure then erroneously apply it to a situation where it does not apply.

Arithmetical Operations with Infinity

During the interviews, some students performed arithmetical operations with the ∞ symbol when working on rational functions. Students who took shortcuts when solving tasks sometimes multiplied or divided with infinity. It can be seen in Task 6, by incorrectly using L'Hospital's rule, plugging in the infinity symbol into the variable yielded an incorrect limit of 2. Here, YJ (Figure G.32) and EB (Figure G.33) divide with infinity in Task 8. YJ substitutes the infinity symbol into the exponent (Figure G.34). Evidence

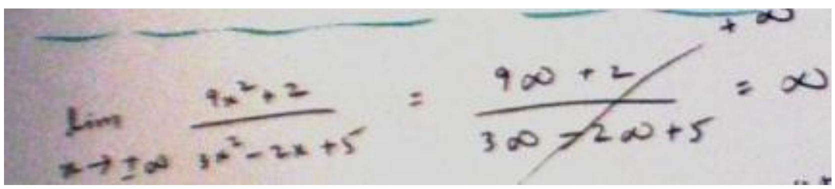


Figure G.32: YJ divides with infinity

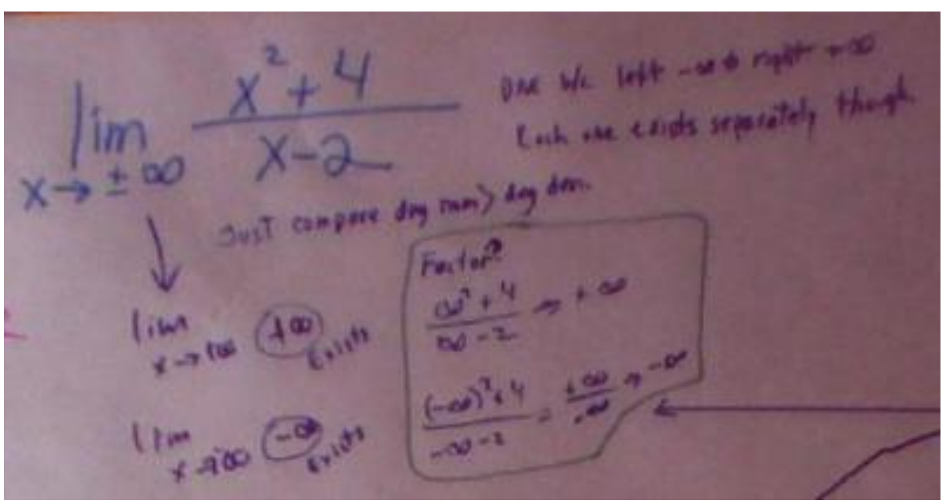


Figure G.33: EB divides with the infinity symbol

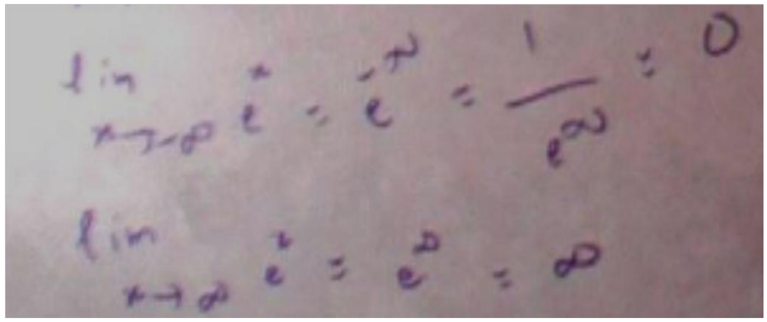


Figure G.34: YJ substitutes infinity into the exponent

Analysis

The importance is that students are performing arithmetic with infinity despite that the infinity symbol is not a number. Although they would get the final answer correct in many cases, they should only be substituting real numbers into x.

Asymptotic Behavior and Rational Functions

Results for a limit at infinity of rational functions in the form $\lim_{x\to\infty}\frac{1}{x-a}$ also revealed an important finding. In Task 9, many students correctly reported that as x approached infinity, the function values were getting smaller and getting close to 0, so therefore, the limit was 0. An example is EB’s work in Figure G.35. EB and others sometimes used test values or set up a table as well to verify their result, though many could just explain this verbally given their level of expertise with the content. There were other students, though, who erroneously thought as x approached infinity, the function values also approached infinity because function (line) would keep going and never touch the x-axis. An example is KB’s work in Figure G.36. These students did not use test values or construct tables with function values, and so they appeared to base their decisions by visual inspection of the graphs. The results are summarized below.

- Given $\lim_{x\to\infty}\frac{1}{x-a}$ the limit is equal to 0.
- Given $\lim_{x\to\infty}\frac{1}{x-a}$ the limit is equal to ∞.

Evidence

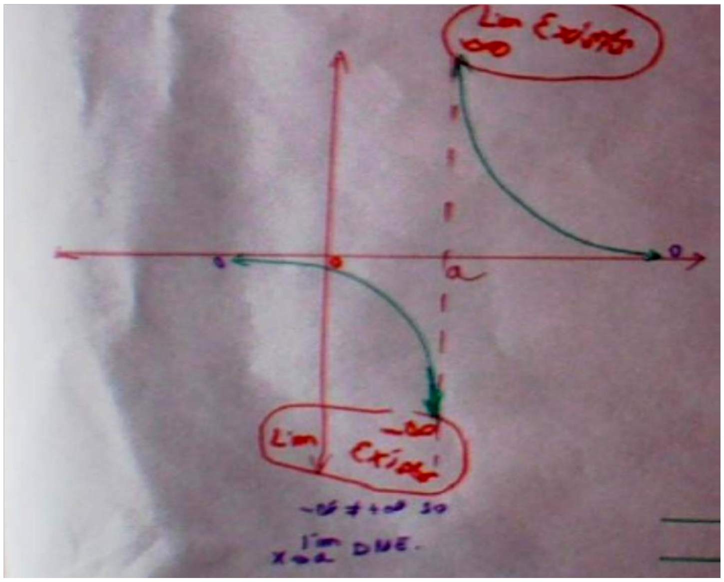


Figure G.35: EB’s solution to Task 9.

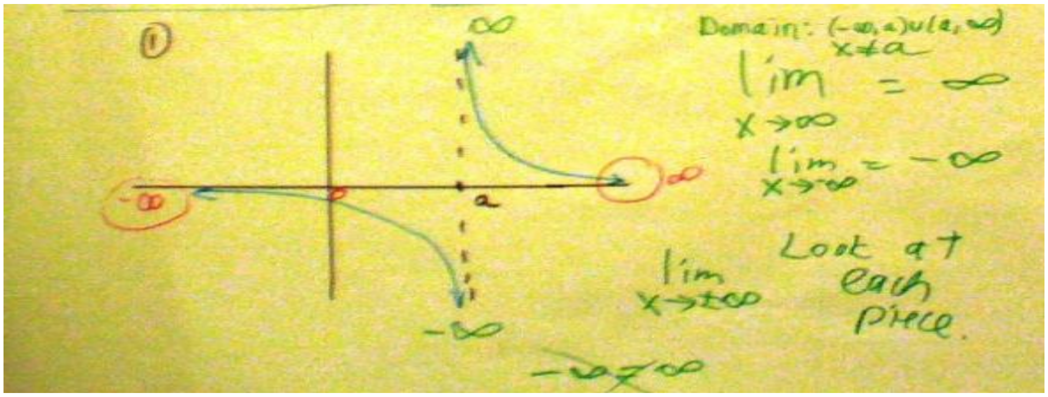


Figure G.36: KB’s solution to Task 9.

Analysis

One reason for this occurring is that some students are not considering how the function values are getting smaller and smaller, eventually going to 0. This would be the correct way to think about this problem and to check their work, construct tables of values by plugging in large x-values into the function. By studying the graph strictly from a visual perspective without performing any mathematics, it seems that the student is likely to follow points along the line and end up deciding that the line keeps on going and does not stop. As a result, the line keeps going to infinity. The two examples above show this distinction of how two students can perceive the solution quite differently.

Asymptotic Behavior and Oscillatory Trigonometric Functions

Graphs with oscillatory behavior revealed some interesting findings as well. The cosine function in both directions, and the damped cosine function going only in the left hand direction, revealed that either students correctly reported the limit did not exist because the function oscillated without settling down to any particular function value, or they just moved their fingers along the graph and said that the limit was equal to infinity because the function kept going and did not stop. So there were two opposing ways students looked at these functions.

In Task 2 with the damped cosine function, $\lim_{x\to\infty} e^{-x}\cos x$, many students such as YJ erroneously reported the limit exists and is equal to infinity because it keeps going. Other students such as LA reported the limit tended toward 0 (Turn 54). Figure G.37 shows LA's and YJ's solutions.

LA: As x goes to plus infinity the limit is 0. As x goes to negative infinity the limit does not exist because it keeps oscillating.

JY: The limit as x goes to infinity diverges to 0 because the values keep bouncing up and down and get smaller and smaller. It is becoming less and less but is approaching 0.As x goes to minus infinity then the limit doesn't exist because it's like the cosine on the left side of the graph it keeps bouncing up and down but gets larger and larger.

KB: As x goes to infinity the limit does not exist because the limit is going to infinity. As x goes to negative infinity, the thing keeps going up and down so the limit is infinity.

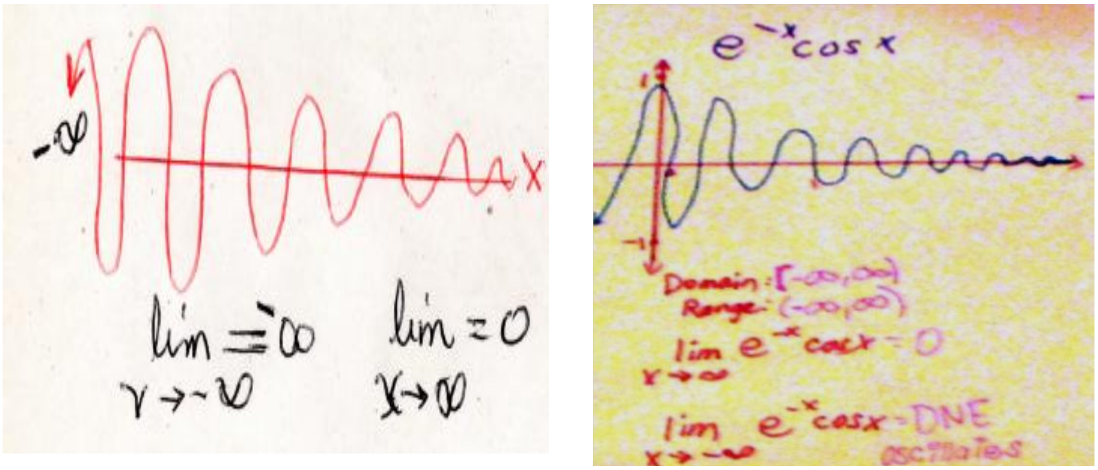


Figure G.37: YJ's solution (left) and LA's solution (right) to Task 2.

A summary of the two types of responses are as follows:

- Given $\lim_{x\to\infty} e^{-x}\cos x$ as $x \to \infty$ the limit is equal to 0.
- Given $\lim_{x\to\infty} e^{-x}\cos x$ as $x \to \infty$ the limit is equal to ∞.

Analysis

The results for the damped cosine function $\lim_{x\to\infty} e^{-x}\cos x$ in Task 2 show that while many students correctly report the function values get smaller and tend toward 0, other students perceive the end behavior tending toward infinity. In the latter case, the students might perceive the points to be going to infinity so could be focusing on the visual aspects of the line. This is the same phenomenon that occurs with the rational function $\lim_{x\to\infty} \frac{1}{x}$. With the damped cosine function, the students were even told the function represents the mathematical model of a bell ringing and that after the bell stops ringing, the graph would look gradually level off. In this case, there was a disconnect knowing that if something stops, then it is no longer occurring. Yet, they could not transfer this mathematically to say that the function values diverged to 0. Hence, the limit was 0.

Results for Task 10, the exponential function, revealed the same type of result as x approached negative infinity. Some students such as KB erroneously reported that the limit would be negative infinity since the function kept going and never touched the x-axis. Again, they did not consider the behavior of the function values, construct a table, or consider the definition of limit. KB's work on Task 10 appears in Figure G.38.

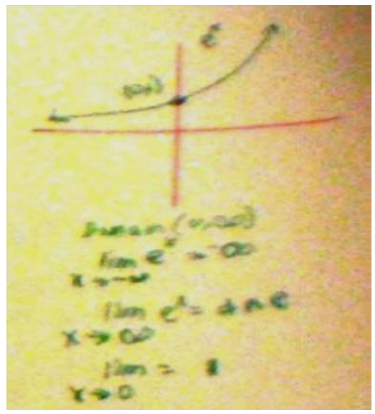


Figure G.38: KB's solution to Task 10.

These results replicate findings in the pilot study with CL, showing students can be confused when they just look at graphs without attempting any mathematical analysis. Results suggest that students with less expertise might not keep in mind that limits are about the behavior of function values, and so if they are solely concentrating on the behavior of x going to infinity, then they might not be paying attention to the y-axis and the behavior of the function values which are getting smaller. Students with more expertise often already demonstrated knowledge of elementary functions, but also performed computations to check and explain their results. If students erroneously think that $\lim_{x \to -\infty} e^x = -\infty$ instead of 0, they might experience difficulty later understanding what it means for a series to diverge to infinity versus converge to a number.

Summary

Table G.3 provides a summary of how students think about limits at infinity correctly and incorrectly. Students try to use the same factoring procedures for limits at infinity that they use for limits at a point, and also apply faulty generalizations such as using L'Hospital's Rule when the problem at hand is not indeterminate. With limits at infinity, factoring out the highest power of x separately are done in the numerator and denominator. Students can check their results with algebraic rules for asymptotes. It is important to distinguish between factoring techniques used with limits of rational functions for limits at a point versus limits at infinity. When the leading term's powers are the same in the numerator and denominator, the relationship between the limit and asymptotic behavior is not clear to some students because they compute the limit but then don't see how the limit appears on the graph. L'Hospital's Rule is reserved for rational functions that are considered to be indeterminate only, and that this procedure should not be used any time one happens to encounter a rational function.

Table G.3: Summary of How Students Think about Limits at Infinity

Correct Notions	Incorrect Notions
• Infinity symbol represents "undefined". • The infinity symbol cannot be used in arithmetical operations. • Limits at a point of rational functions involve factoring and plugging in vs. limits at infinity of rational functions require factoring out the highest power of x separately from the numerator and denominator. • When powers equal on leading terms of rational functions, the limit is the horizontal asymptote.	• Infinity is a very large number. • The infinity symbol can be plugged into the numerator or denominator. • Limits at a point can be solved just by plugging in. • Limits at infinity can be solved with L'Hospital's Rule even when the rational function is not indeterminate. • Limits at infinity and horizontal asymptotes are not related.

Limits that Do Not Exist

Research Question 3A: In what ways do students apply their definition of infinity to infinite limits and other limits that do not exist in their task solutions?

How Definition of Infinity Manifests in Task Solutions

Students were asked to give a definition of infinity at the beginning of the interview and again later on for reliability. If they initially reported that the infinity symbol represented a big number, they did not change their minds.

In the case of the infinite limits with many continuous functions, students often report that as $x \to \infty$ the limit exists because it equals infinity, regardless of their prior definition of infinity. An example is EB's work in Figure G.39. Meanwhile, other students correctly stated that the limit does not exist if it goes to infinity, often not consistent with how they defined it earlier. An example of a correct response comes from KB in Figure G.40.

Responses from Turn 194 reveals their responses to "explain why or why not a limit exists if it is equal to infinity".

Evidence

YJ: If you get a finite number, then you say the limit exists. But infinity is not a finite number so the limit does not exist. Infinity is not itself a number.
KB: No. You can't ever reach the number, because the number keeps going on and on.
LA: Yeah. The limit exists because it keeps approaching infinity.
EB: Yes because the limit has to equal something and infinity is a very large number.

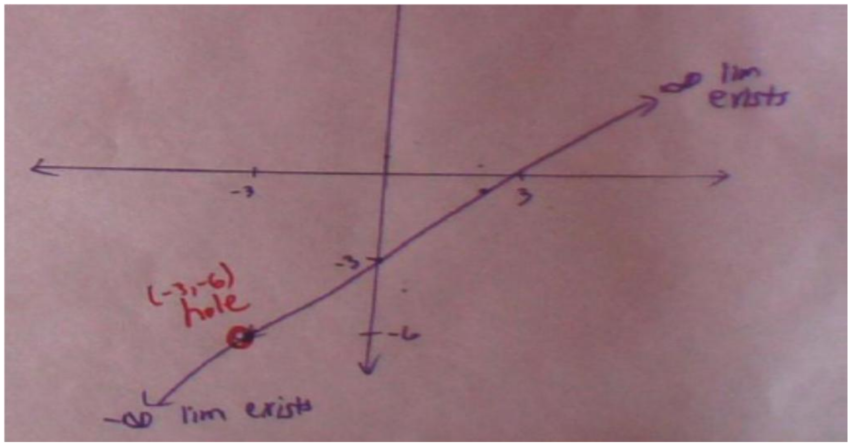


Figure G.39: EB's solution to Task 7.

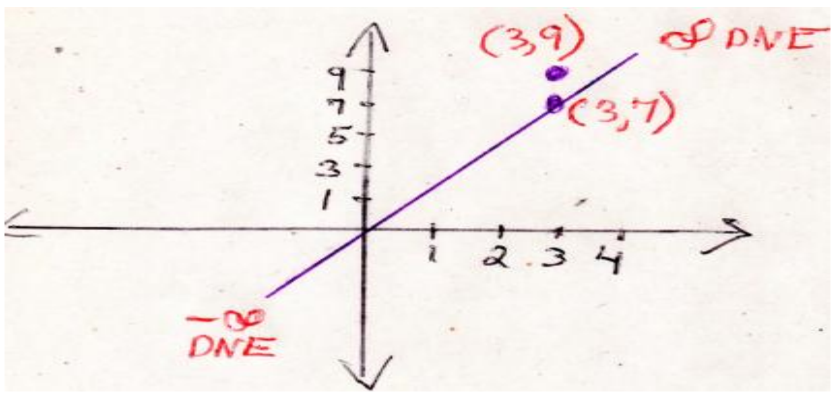


Figure G.40: KB's solution for Task 5

Analysis

Considering infinity as a large number might explain why students think a limit exists if it equals infinity. EB clearly states that the "limit has to equal something" so therefore, something has got to exist there on the ends of lines. Therefore, since she perceives the infinity symbol as representing a large number, it seems that her initial definition of the infinity symbol manifests in her work on the task. On the other hand, thinking that the infinity symbol is not a number might explain why some students correctly report that the limit does not exist when function values increase or decrease without bound. Instructors need to make it clear that infinity is not a number; therefore, if function values tend toward infinity, the limit does not exist.

The cosine function, given its continuous nature, was essential to this study because of the oscillatory behavior it generates. There were some students such as EB in Figure G.41 who correctly reported that as $x \to \pm\infty$, the limit does not exist because the function values oscillate, not settling down at any given function value. Similar to CL in the pilot case, KB reported there being 2 limits at y= -1 and y=1, with the reason given there were 2 y-values the function went between (Figure G.42).

Evidence

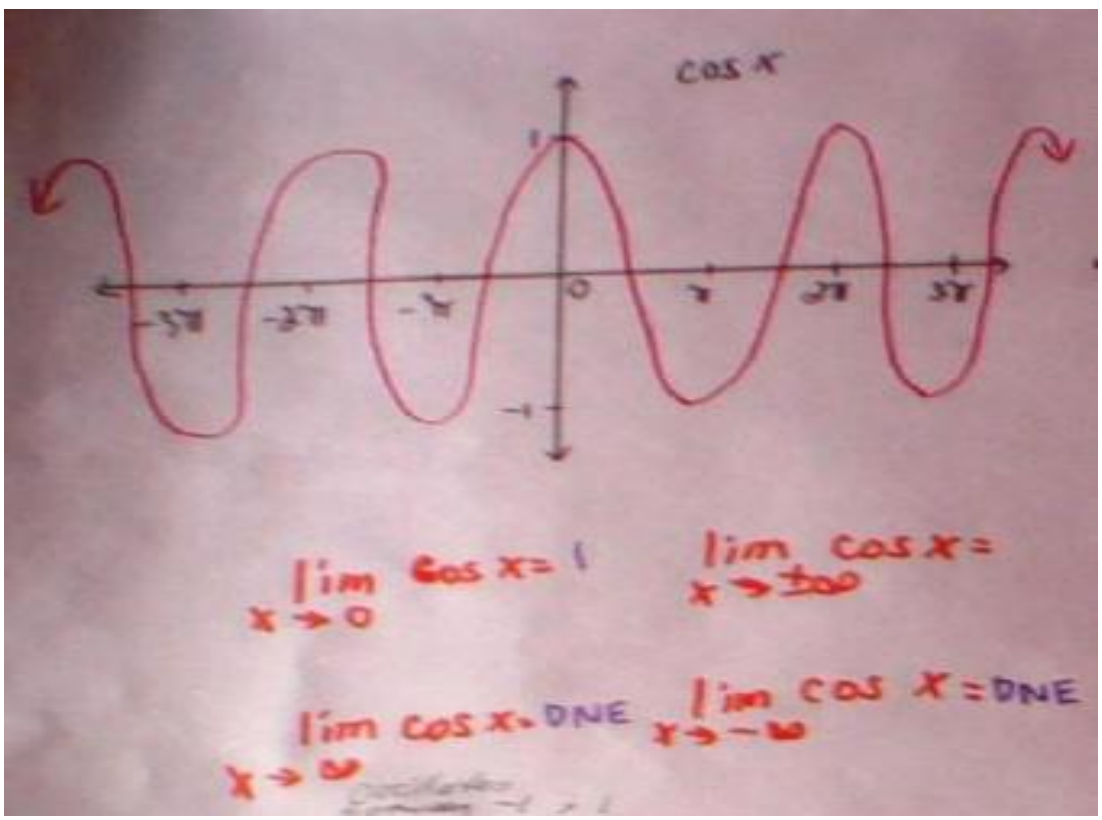


Figure G.41: EB’s correct solutions to Task 1.

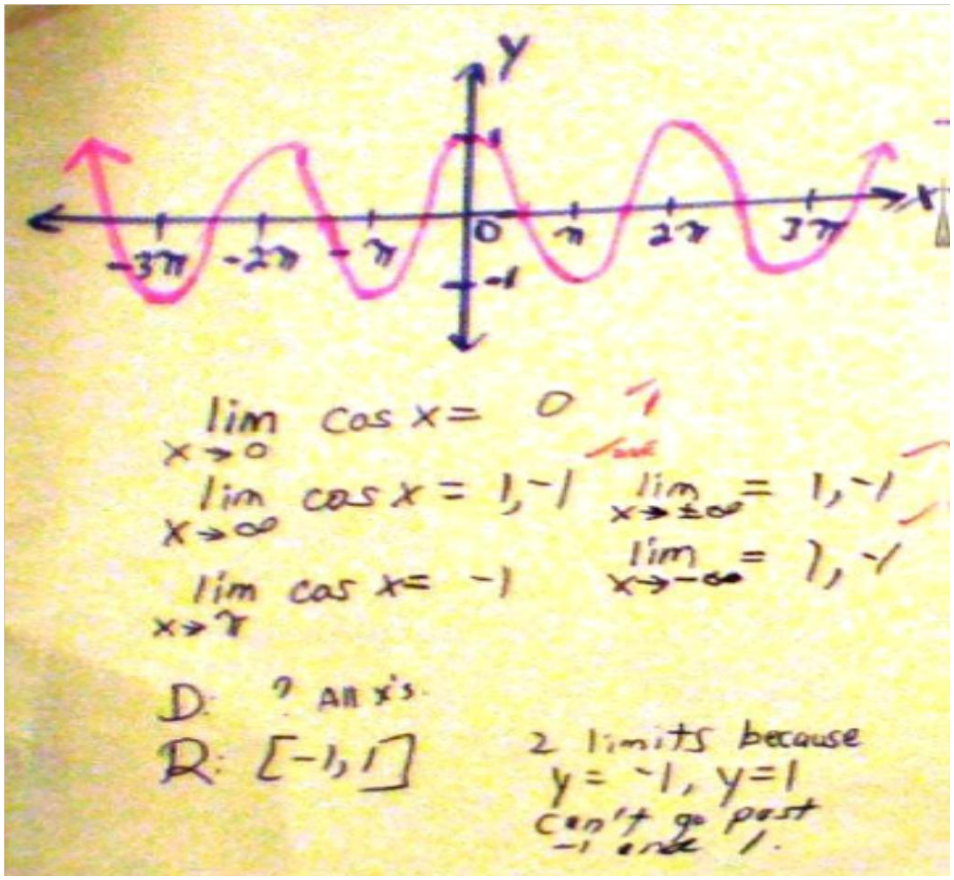


Figure G.42: KB's incorrect solutions for Task 1 as $x \to \infty$ and $x \to -\infty$.

Two types of responses are summarized:

- $\lim_{x \to \infty} \cos x = d.n.e.$, which is correct.
- $\lim_{x \to \infty} \cos x = \infty$, which is incorrect.

Analysis

Some students erroneously think $\lim_{x \to \infty} \cos x = \infty$ because the function keeps going on indefinitely. The same case occurs for $\lim_{x \to -\infty} \cos x = -\infty$. The reason could be not correctly thinking about how the function values are behaving, which in this case, happens to be the function values do not settle down at any particular point. Identifying such potential misunderstandings a prior and by implementing instructional enrichment can prevent misunderstandings from occuring. Students must be directly instructed that with oscillatory functions such as sine and cosine, the limit does not exist because the function values never settle down. Also, the graph cannot be labeled “$=\infty$ or $=-\infty$” on either end because the function values are not increasing or decreasing without bound, but rather, are not settling down.

Limits at a Point of Rational Functions

Results of rational functions in the form $\lim_{x \to a} \frac{1}{x-a}$ revealed a few important findings. First, students often acknowledged that the function values increased or decreased without bound, and labeled those areas of the graph as “ $=\infty$ or = $-\infty$”, respectively. Several students reported the limit exists if it equals infinity. These were the same students who reported that the infinity symbol represented a very large number, and so their responses are quite consistent with their definitions. Meanwhile, those students who correctly reported that the limit does not exist if it equals infinity since the function values increase or decrease without bound reported earlier that the infinity symbol represented “undefined”, and so their responses are consistent with their definition.

Research Question 3B: In what ways do students think about limits that do not exist, particularly, under what conditions do students write “d.n.e.” versus $=\infty$?

Limit “Does Not Exist” versus “Equals Infinity”

The main focus here was on whether or not students knew when it was appropriate to write “does not exist or d.n.e.” versus “$=\infty$”. Given limits at a point for piecewise functions with jump discontinuities, if the left hand side does not equal the right hand side, only “d.n.e.” would be the correct answer.

Graphs with Oscillations

In Task 2 students studied the oscillatory function $\lim_{x \to 0} \cos \frac{1}{x}$. This was an interesting endeavor revealing that most students initially thought that the limit did not exist because if 0 were to be plugged into the denominator, the function would be undefined. However, the purpose was to explore the limiting behavior near 0, not at 0, and so although their intuitions were correct, deeper probing explored how function values behaved near 0. A few students set up a table of values and others plugged the function into the calculator to explore it further, seeing it was undefined when at x=0. What many discovered was that the function oscillated between y=-1 and y=1 near 0 and correctly concluded that the limit did not exist due to oscillatory behavior. Below is evidence of two students reporting different reasons for the limit not existing (Turn 54, Figure G43).

Evidence

BK: As x approaches 0, the limit does not exist because it keeps fluctuating getting infinitely smaller.

KB: As x goes to 0, the limit does not exist because you can’t have zero in the denominator.

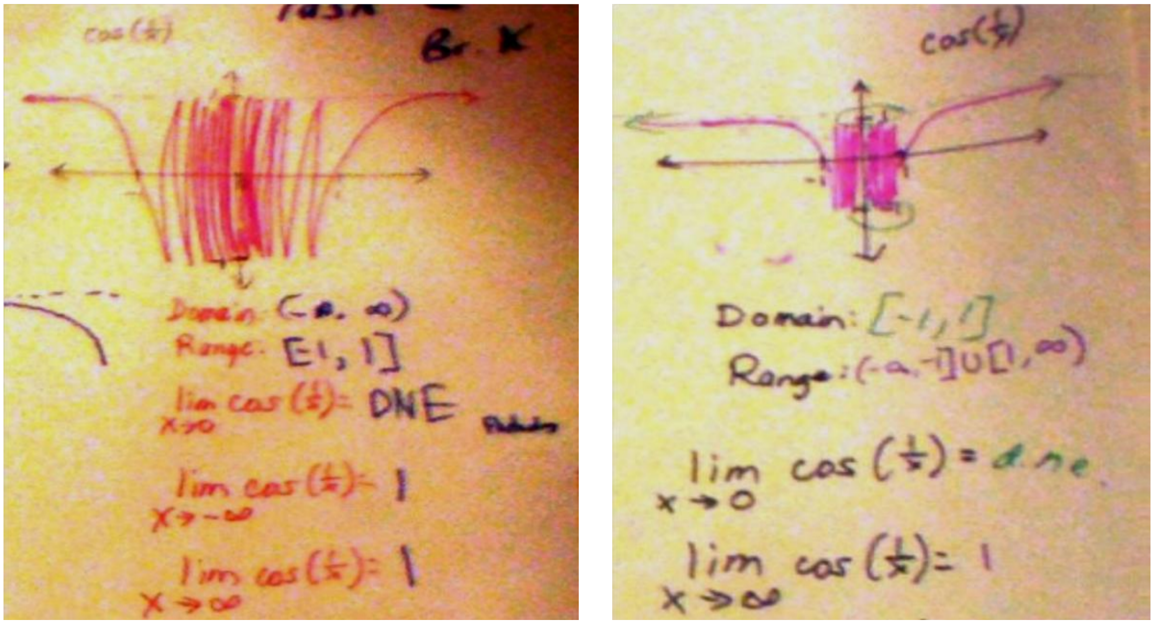


Figure G.43: Task 2 solutions with BK on left and KB on right

Analysis

The results show that students consider what happens near the point as they should, but others consider what happens at the point, evidenced by plugging 0 directly in the denominator. By plugging in 0, they are not extracting the essence of the task, which is to describe the behavior of the function values near 0 on both sides. BB describes the function as oscillating new 0, so therefore, the function values do not settle down to any one point, whereas KB writes the incorrect domain and range, and claimed the limit did not exist because when she plugged in 0 into the calculator, she got an error.

Next are results from limits at infinity for the cosine function or damped cosine function. As $x \to \infty$ some students correctly write d.n.e. but others write $= \infty$ which is incorrect (Turn 28). Given infinite limits, both $= \infty$ or "d.n.e." are considered correct. In this case, the problem occurs when students think if a limit exists if it equals infinity.

Perceptions of Oscillatory Function, Cosine

JY: Limit as x goes to infinity or minus infinity keeps bouncing up and down so the limit does not exist.

AK: As x approaches infinity, the limit exists because it equals infinity. As x goes to negative infinity the limit exists because it is negative infinity which keeps going and doesn't stop.

Perceptions of Infinite Limits

JY: As x goes to infinity, the limit is infinity so does not exist and as it goes to minus infinity the limit is minus infinity so does not exist either. (Turn 120).
LA: As x goes to infinity, the limit exists because it equals positive infinity (Turn 182).

In Task 10, the exponential function $\lim_{x \to \pm\infty} e^x$ reveals how KB uses "d.n.e." to describe end behavior for function values tending toward infinity on the right side of the graph, but then uses "$= -\infty$" on the left hand side (Figure G.44). She does not appear to differentiate between the use of these labels. In fact, on the left hand side, the function values tend toward 0.

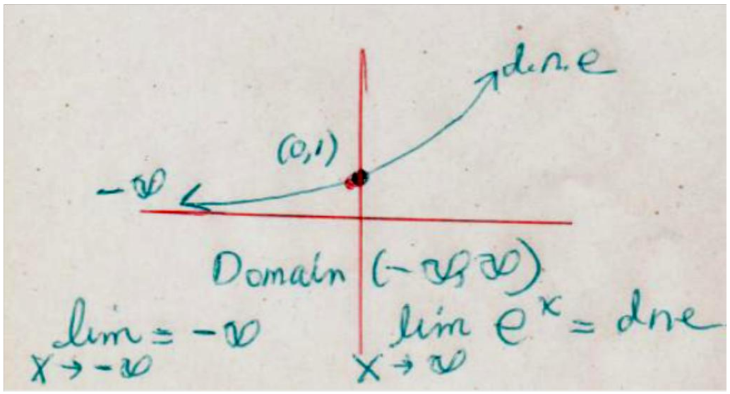


Figure G.44: KB's solution for Task 10.

Analysis

In the case of cosine, JY seems to understand that if function values tend toward infinity, the limit does not exist. An interpretation for AK is she appeared to follow points on the graph in the direction of x. Also, since she defines infinity as a large number, she concludes that the limit exists. In the case of infinite limits, JY was consistent in explaining that if the limit equals infinity, it does not exist whereas LA associated $= \infty$ with meaning the limit exists.

One problem with cosine might be a student thinks d.n.e and $= \infty$ have the same meaning, when in fact only "d.n.e." is specific to end behaviors that oscillate.

It appears d.n.e and $= \infty$ can be perceived to have different meanings. With infinite limits, "$= \infty$" is typically interpreted by students as meaning a large number or undefined. In KB's graph above with $\lim_{x \to \pm\infty} e^x$, function values on the left side approaching the x-axis are labeled with $= -\infty$ whereas function values tending toward infinity are expressed specifically with d.n.e. With both cosine and the exponential, it seems "$= \infty$" is used when the student perceives the limit to exist. Back in turn 24, KB reports infinity is "a number too large to even measure'. So this suggests that her above interpretations are consistent with her definition.

Piecewise Functions with Jump Discontinuities

When presented with a piecewise step function, the abbreviation "d.n.e." can be written as answers to limits at a point where there are such jump discontinuities. The step function in Task 3B was unfamiliar to most students and students had a hard time reporting if the limit existed at the point of discontinuity.

Results from Task 3A reveal how students either do not identify this graph as piecewise, or identify it as piecewise but for the wrong reason—that the quadratic has two separate pieces on either side of the hole. The results also revealed how some students focused on the piece of the graph in which the limit was equal to the function value and said that is where the limit exists. Given the type of function with the quadratic below the point, some students reported that the limit did not exist as $x \to 2$ because of the hole, or that the limit existed where the isolated function value appeared.

Results from Task 3B that used the stepwise-piecewise function reveal that some students knew that the limits did not exist because the left hand limit did not equal the right hand limit, and correctly denoted this with "d.n.e." Other students drew a vertical line through the hole and endpoint, then concluded that the limit existed. In Figure G.45, KB thought the limit did not exist but for the wrong reason--that the bottom piece stops at (-2,-1). KB identified this function not as piecewise, but exponential.

Evidence

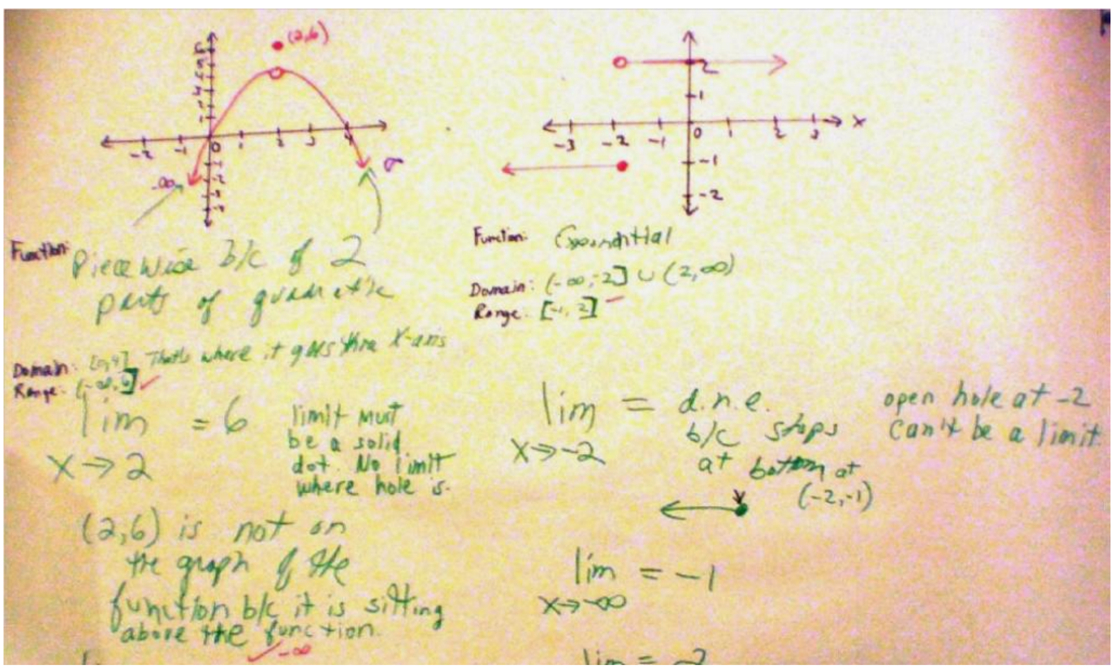


Figure G.45: KB's work on Task 3.

Analysis

By not being able to correctly identify a function as being piecewise, students will have trouble trying to determine the limiting behavior near a point. However, those who clearly identify the functions correctly tend to make the correct conclusions about when a limit exists or does not exist. In Task 3A, the limit exists at x=2 but does not equal the value of the function, whereas with Task 3B, the limit does not exist when the left hand limit does not equal the right hand limit.

Rational Functions

Another important finding was with limits at a point of rational functions such as $\lim_{x \to 2} \frac{1}{x-2}$. Students often erroneously compared the left hand limit with the right hand limit, and decided that since $-\infty \neq +\infty$, the limit does not exist. They compared the left hand limit $-\infty$ with the left hand limit $+\infty$ to make this determination which is not the correct procedure. Only limits can be compared and since the ∞ symbol is not a number, it cannot be compared. Meanwhile, the fact of the matter is that only one side needs to be considered when deciding if the limit exists or not for this type of rational function. Although comparing the left and right hand limits is typically done for limits at a point of most other functions, it is not done with rational functions.

Another way students perceive rational functions such as $\lim_{x \to 0} \frac{1}{x}$ is by looking at the limits separately from each side. Then they erroneously conclude the limit exists if it equals infinity. Students were asked, "If a limit is equal to infinity, does that limit exist? (Turn 22).

Evidence

BK and JY: No.
YJ: No because infinity is not a number.
KB: Yes, I say the limit exists. I'm not sure because I confuse limits with infinity.
LA: Yes.

In Figure G.46 below, there are other difficulties that LA exhibited. Note that the end behaviors are labeled correctly with $-\infty$ but in each case these limits are reported to exist. So the work on the task coincides with her original definition provided.

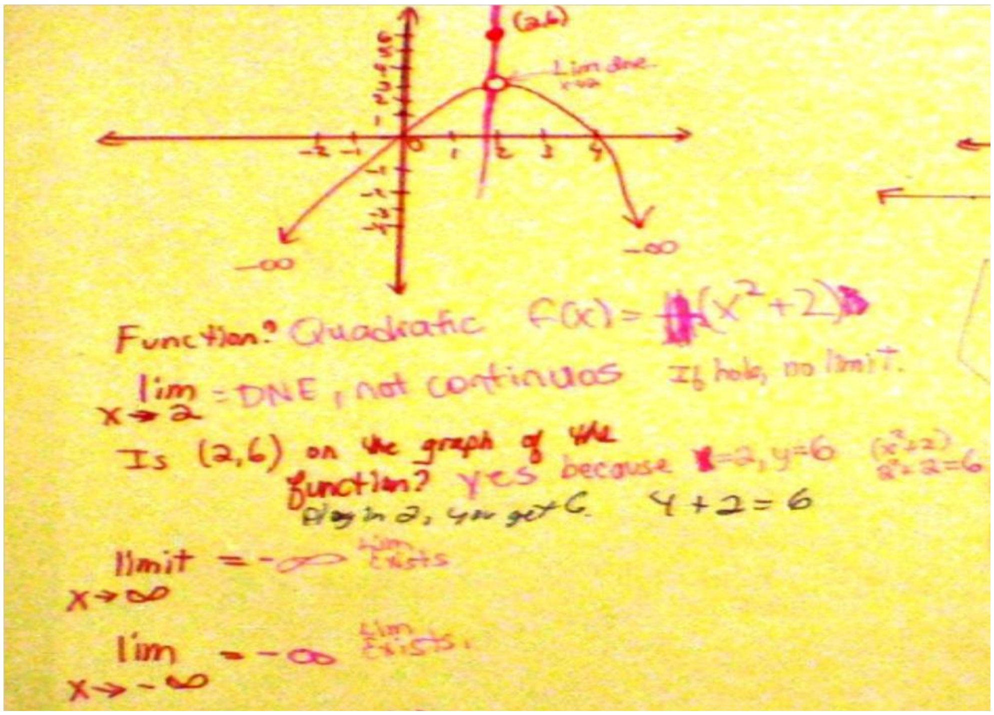


Figure G.46: LA’s solutions for Task 3.

Next it will be seen the trend continues for rational functions. LA then compares the end behaviors, and concluded that since $-\infty \neq +\infty$, the limit does not exist (Figure G.47). So even though each end individually is reported to exist, once the two ends are compared near the point as $x \to 2$, LA concluded that the limit did not exist because each side was different. Had each limit been the same, then LA would say the limit exists, e.g., $\infty = \infty$ or $-\infty = -\infty$. Turn 140 reveals the response to “For limit at a point, here as x approaches 2 do you compare the left hand limit with the right hand limit to decide if the limit exists, or do you just look at one side to decide”?

Evidence

LA: Yes, you must compare both sides. If they're opposite then the limit doesn't exist. If they are the same, like both positive, then the limit exists.

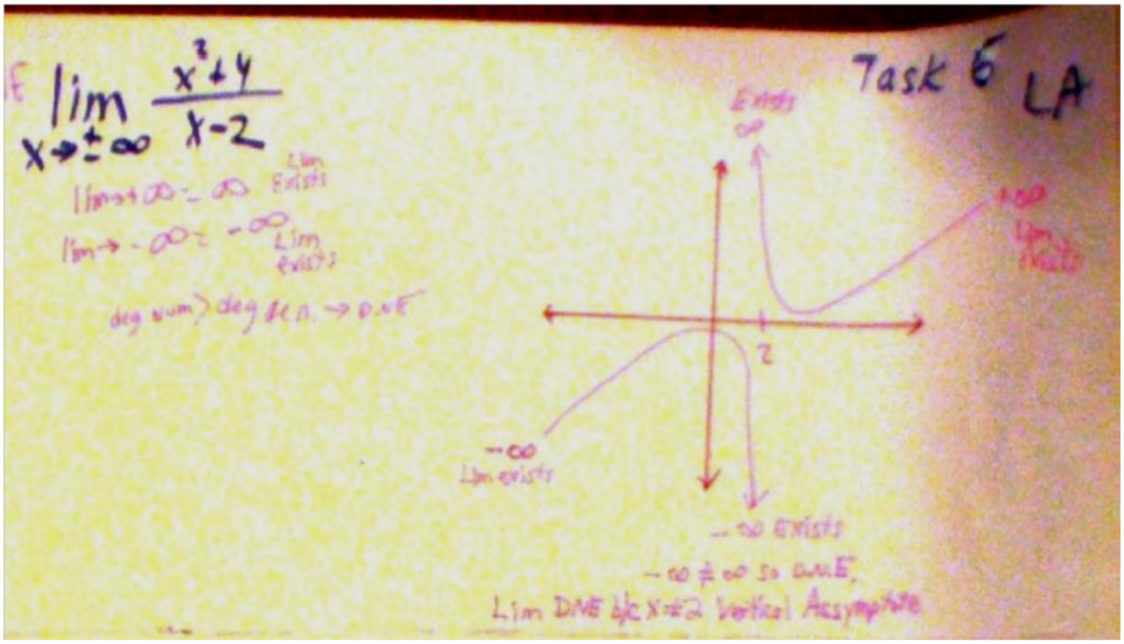


Figure G.47: LA's solution with Task 6.

Analysis

These are important findings because students are using the wrong procedure, comparing the left and right hand limits for this rational function. They appear to be generalizing this technique they learned to compare the left hand side with the right hand side, and transferring their knowledge to the rational function. The reason this is not correct is that infinity is not a number, and so it makes no sense to compare $+\infty$ with $-\infty$ and decide that they are not equal. Limits must be numbers in order to be compared, best illustrated with a continuous function whose limit is either a point of discontinuity or happens to be equal to the function value, in which case numbers or function values are being compared on the left hand side to the right hand side. It could be argued that for rational functions, students could pick some very small x-values near 0 on both sides, but the fact of the matter is that the function values increase or decrease without bound, and so it is not mathematically correct to compare $+\infty$ with $-\infty$. The reason so many students make this mistake could be due to insufficient content knowledge of instructors or questionable instructional practices.

End Behaviors Involving Infinite Limits

Task 10 (Figure G.48) contains different functions that capture an interesting phenomenon that occurred in the pilot study, which is that students compare end behaviors not just with rational functions, but with other types such as quadratic and exponential when determining if a limit exist. In Graphs A and B, they compare the left hand side ($+\infty$) with the right hand side ($+\infty$) and erroneously conclude that the limit exists because both sides are equal, i.e. $\infty = \infty$. In Graph C, with the exponential function, KB and AK erroneously report that since the left side ($-\infty$) does not equal the right side ($+\infty$) the limit does not exist, i.e. $-\infty \neq +\infty$. Many students compare both ends of the graph. They indicate the limit exists on the left side since $\lim_{x \to -\infty} e^x = 0$ whereas on the right side the limit is either infinity or d.n.e. However, in Turn 189 students compare end behaviors on the left and right and some students such as LA conclude that since $0 \neq \infty$ the limit does not exist for the whole function. Comparing end behaviors is common but not mathematically correct.

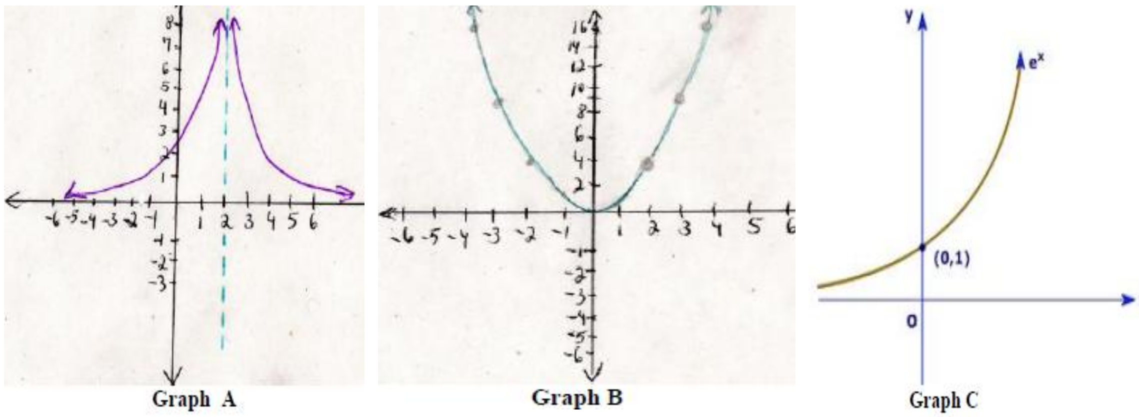


Figure G.48: Graphs from Task 10.

Evidence Graph A (Turn 185)

LA: As x goes to 2, the limit does not exist because of the asymptote. But also they are both limits are going to infinity because of the asymptote so it doesn't make sense. As x goes to infinity, the limits though exist because both equal positive infinity. But the limits really don't exist because the function is not continuous. You don't have to even look at the ends of the arrows though when you have a vertical asymptote because the asymptote tells you the function is not continuous.

AK: You plug in 2 and get 0, so you do 1.999 and 2.0001 to see what it's doing. Do 1.9 – 2 and square it. Do 1.99 – 2. Then do 1.999-2. You are going to positive infinity. Under 2, you do 2.1, 2.01, so take 2.1-1 and then 2.01-2 and you also get plus infinity. The limit exists and equals positive infinity. Since both sides are infinity, the limit exists.

Evidence for Graph B (Turn 187)

LA: As x approaches infinity, the limit exists and equals infinity. Then you compare the end arrows. Since infinity on the left equals infinity on the right, then the limit exists.

JY: Right side goes to positive infinity and left to positive infinity, so neither one exists.

BK: You don't have to compare the ends since they are going outwards toward infinity. You only compare the ends for limit a point.

EB: You don't have to compare the left and right limits at infinity with these but each limit exists separately.

Evidence for Graph C (Turn 189)

KB: Domain: $(0,\infty)$ As x approaches infinity, the limit is infinity so it doesn't exist? As x goes to minus infinity, the limit is $-\infty$ because it keeps on going and doesn't stop.

AK. As x goes to infinity, the limit is infinity and it exists. As x goes to minus infinity, the limit is minus infinity and so it exists.

LA: As x goes to minus infinity the limit is 0 and as x goes to plus infinity the limit is plus infinity. Since 0 does not equal infinity, the limit does not exist. Or it only exists on one side, 0. You have to compare the two sides though I think.

JY: As x approaches 0, the limit does not exist. As x approaches minus infinity, the limit is 0 because the values get smaller and smaller. As x approaches plus infinity the limit is infinity. You don't compare the sides.

Analysis

As with the case of the rational functions, here is evidence that students use faulty generalizations with techniques for limits at a point. There is no need to compare sides in these graphs. As $x \to \infty$ or as $x \to -\infty$ only one side needs to be considered and if the limit increases or decreases without bound, then the limit does not exist.

Referring to Graph A, LA elicits the wrong conclusion. She claims the limit does not exist due to the asymptote, but the reason the limit does not exist is because the function values increase without bound on the left hand side and from the right hand side. Because her focus is on the asymptote breaking up the graph, making the function discontinuous, LA does is not seeing that her focus instead should be on the behavior of the function values near 2 as $x \to 2$. Her understanding of limiting behavior is compromised due to focusing on the wrong aspect of the graph.

AK, on the other hand, plugs in test values for Graph A and decides the limit equals infinity separately on both sides, and the limit also exists because both sides are equal, i.e. $\infty = \infty$. It seems here that if the infinity symbols have the same sign, then the limit exists. AK also reported in turn 198 that "infinity doesn't exist but it's still a limit" while back in Turn 26 she stated that the infinity symbol means "numbers go on forever so there is no particular number for the answer". This suggests she perceives the infinity symbol as a place holder for a number, and so she is inclined to report that limits exist if they are equal to infinity.

In Graph B, while some students know it is not necessary to compare end behaviors for a quadratic function, other students compare them regardless. Students might conclude the limit exists either on one side or both because of the infinity symbol representing a place holder for a number, and also because they are focusing on the infinity symbol and not on the behavior of the function values.

In Graph C, while some students can tell immediately that the left side of the exponential function tends to 0 while the right side increases without bound and so the limit does not exist, others are inclined to compare the left end with the right end to erroneously determine that the limit does not exist. Again, faulty generalizations might be occurring, in which case they take the procedure used for limits at a point and generalize the notion of "comparing sides" to most every situation encountered. Also, they claim limits exist if the limit equals infinity because the infinity symbol serves as a place holder for very large numbers. The infinity symbol might be a distraction in this case given they focus on the symbol rather than on the behavior of the function values for large x in either the positive or negative directions.

Research Question 3C: How do students distinguish between a limit at infinity and an infinite limit?

Limits at Infinity versus Infinite Limits

Since textbooks do not typically distinguish between a limit at infinity (what x is approaching) and an infinite limit (the result computed), this was an exploratory question to see what students understood about these two terms. Results showed that several students knew that an infinite limit involved the result of y-values increasing without bound. However, there were a few students who reported there was no difference between the two terms, and even suggested that math books use these terms interchangeably. In Turn 48, they respond to the proposed question above, citing the difference between a limit at infinity and an infinite limit.

Evidence

LA: I think they're the same thing.
JY: Limit at infinity is x. Infinite limit is y. It doesn't exist because it goes to infinity.
AK: The limit at infinity is where x goes. The infinite limit is where y goes.
BK: The infinite limit is when the limit does not exist because it is going to infinity. The limit at infinity is about what x is tending toward, not y.

Analysis

Most students understand this distinction in this particular subset of students. In the pilot study, CL thought the words were used interchangeably. Textbooks often mention infinite limits without actually defining what they are, which explains why there could be confusion with understanding the difference between a limit at infinity and an infinite limit.

Summary

Students present difficulties with limits that do not exist. Some students correctly report that a limit does not exist if it equals infinity, others report that the limit does exist. Moreover, some students do not write "d.n.e." instead of $=\infty$ because they view these as opposite. If a limit exists and equals infinity, then they cannot say the limit does not exist. Table G.4 below summarizes some of the findings that occur when students have correct versus incorrect notions about limits that do not exist.

Students have trouble recognizing graphs of piecewise functions. They do not identify two parts of the graph in the case of the discontinuous cosine and quadratic piecewise, and do not identify a solid dot above a hole as being a piece of the graph. As a result, they get confused with the dot above the hole and conclude that the limit does not exist for the function. These problems result from not understanding the definition of limit and the meaning of limiting behavior, because if they did, then they would see the connection between the definition of limit and the limit existing where there is a hole. The connection is that a limit does not have to equal a function value, but can be equal to a point in the domain that is not on the graph of the function.

With rational functions for limit at a point, students sometimes apply incorrect techniques for computing limits and for perceiving various features on graphs when deciding if the limits exist. There is a tendency to think limits do not exist because of vertical asymptotes, rather than because function values increase or decrease without bound. Also, students erroneously compare end behaviors for graphs of functions such as linear, quadratic and exponential to decide if limits exist. Many students claim that if the left hand side equals the right hand side, $\infty = \infty$, then the limit exists. The underlying problem here could be that they do not use the definition of limit, being a number that describes the behavior of function values and think about the infinity symbol being a place holder for a number, in which case they say that a limit exists if it equals infinity.

Table G.4: Summary of How Students Think about Limits that Do Not Exist

Correct notions	Incorrect notions
• A limit exists if it equals infinity. • Infinite limits are about y-values that tend toward infinity while limits at infinity describes the behavior of an x-value. • Given rational functions in the form $\lim_{x\to 0} = \frac{1}{x}$ only one side is required to go to infinity to conclude the limit d.n.e. • End behaviors are not compared for limits at a point of rational functions or for infinite limits of linear, quadratic, exponential or other continuous functions. • Oscillatory functions such as sine and cosine, as $x \to \infty$ the limit does not exist due to not settling down at any particular point. • As $\lim_{x\to a} f(x)$ for stepwise functions, if the left hand limit does not equal the right hand limit, the limit does not exist.	• A limit does not exist if it equals infinity. • Infinite limits and limits at infinity mean the same. • Given rational functions in the form $\lim_{x\to 0} = \frac{1}{x}$ one must compare the left side with the right side. Since $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = \infty$, the limit d.n.e. because $-\infty \neq \infty$. • End behaviors are compared for limits at a point of rational functions or for infinite limits of linear, quadratic, exponential or other continuous functions. • Oscillatory functions such as sine and cosine, as $x \to \infty$ the limit exists and equals infinity because it keeps going and never stops. • As $\lim_{x\to a} f(x)$ for a stepwise function, the limit exists because of one piece having a function value and because a vertical line can be drawn connecting the dot and hole.

APPENDIX H: SUMMARIES OF NINE CASES FROM PHASE II

Synopsis of Work on Tasks from Nine New Students

APPENDIX H-1: Interview with AK

Introduction

AK (Amanda) is a freshman systems engineering major enrolled in calculus III who passed the AP calculus exam in high school. She was enthusiastic about the interview and articulated quite well. Overall, her understanding of functions and limits was mediocre.

Functions

Similar to CL in Phase I, her knowledge of functions was somewhat limited. Her definition of function was “a mathematical function that takes an input and gives an output. To find out if something is a function, she said to use the vertical line test and if there are 2 y’s then it’s not a function. “ To illustrate, she drew half a circle in the 1st and 4th quadrants to represent what was not a function, then drew a parabola and a constant function to demonstrate what was. When asked what a function value was, she said it referred to what the x value equals, not the y-value. Yet, she was quite knowledgeable of domains and ranges.

Limits at a Point

Her definition of limit was very weak, being “when 2 points come together” and made reference to a limit being the y-value. Next, she drew a line with a solid dot to demonstrate that the limit existed, then replaced it with a hole to show that now the limit did not exist, which is a significant misconception seen by a lot of students, as they confuse limits with continuity. Pointed to the graphs to show that where there is a hole within two lines, the limit would not exist but where there was a dot connecting 2 lines, then the limit would exist because the dot had a y-value. She reported as $x \to a$ that “a” must be in the domain for the limit to exist, which was different from what she did in her work where she compared the right and left hand limits with discontinuities and said the limit exists. This is in spite of the point not being in the domain.

AK sometimes gave conflicting responses, which suggested she did not have much understanding with identifying limits from graphs. For instance, she identified the limits at a point correctly for the arccos x, but for cosine x, she thought the limit as $x \to 0$ did not exist. Interestingly, for the cosine piecewise component graphed in Task 1, that contained a hole at $(\pi, -1)$ and solid dot above it at $(\pi, 0)$, she did something CL would do, which is drew a vertical asymptote thought the dot and hole and said that the limit is the vertical asymptote. She explained that the limit for $\lim_{x \to \pi} f(x)$ existed everywhere up and down on this vertical asymptote. She also said that the point $(\pi, 0)$ was not on the graph of the function because it was above the function. In contrast, when she looked at the piecewise functions in Task 3, she focused on the quadratic piece of the first graph and said the limit did not exist because of the hole but did not draw the vertical asymptote this time. She also said the point (2,6) was not on the graph of the function because it was above the function, so this finding was consistent with the previous task. Moreover, the domain and range were wrong and she was not able to connect knowledge of the domains to finding the limit. Later in Task 4 with the linear function that becomes progressively piecewise, she said the limit does not exist in the second graph because of the hole within the line, and when she got to the last graph that had the hole with solid dot above at (2,4), now she said that for $\lim_{x \to 2} f(x) = 4$, and concluded that this limit existed. In this case, she took the function value to be the limit, and this is because she thought the limit did not exist where the hole was. Earlier on, she said the limit would not exist for this type of piecewise function. This suggests she lacks understanding of this type of piecewise functions. She then said the point (2.4) was not on the graph of the function because it was not on the straight line. She said it has to be on the line to be on the on the graph of the function. Meanwhile, she knew the limit did not exist for the graph on the right in Task 3 because the left hand limit did not equal the right hand limit, but could not figure out the domain or range. The different permutations of responses suggest that her understanding of limits at a point is minimal.

As for domains, AK knew that domains pertained to the x-axis, and the range, the y-axis, but did not make a connection with limits. In most cases she wrote the domains correctly except for piecewise functions. The reason might be because knowing the domains facilitates writing the formulas for piecewise functions.

Given tasks with finite domains such as the half circle and arccosine x, she looked at the graphs and correctly identified the domains as well as their corresponding limits at a point, but did not make any connection between the two. This is later confirmed with limits at infinity, because for those particular tasks, if she considered the finite interval domains, she would not have gotten those wrong. Overall, she did not use information about domains when finding limits, and thought that finding domains and limits were two different things, having nothing to do with each other.

As for the endpoints for these functions, she did not think the limits would have to be split up specifically as x approaching -1 from the right and x approaching +1 from the left. She said the notation $x \to 1 \text{ or } x \to \text{-1}$ was sufficient because she understood what it meant but she did not make reference to the domain and its restriction.

Given the rational function $\lim_{x \to 2} \frac{1}{(x-2)}$, AK knew that each sided was computed separately and that the limit did not exist as x approaches 2 from the left, and as x approached 2 from the right. The notation $x \to 2$ did not make her think the limit had to be split up into 2 parts, as x approached 2 from the left and from the right separately in order to get the limits, and the arrow did not imply direction from the left, either. Unlike NS and CL, she did not erroneously conclude that the limit did not exist because the left hand limit did not equal the right hand limit.

What appeared beneath the limit notation was not of concern for AK as she knew it did not involve direction and was not indicative of what the limit would be. She knew limits were about what the y-value was doing. However, like NS, AK perceived the notation $x \to \pm\infty$ beneath the “lim” to be problematic and invalid. She said professors are supposed to write it as $x \to -\infty$ and $\mathrm{x} \to +\infty$.

As for the linear function in Task 5, she recognized the function in the form y=mx+b and used that to construct a graph first, then used the graph to decide if the limits existed instead. Next she plugged x-values into the function to check the graph drawn. She got the limits correct but erroneously concluded that the point (3,8) was on the graph of the function because “it is on the graph but not on the function.” She did not use the vertical line test.

When given rational functions, AK’s algebra skills were questionable. She tried to factor the sum of squares in the numerator with Task 7, gave up, plugged in 2 into the numerator and denominator and when she got $\frac{8}{0}$, she concluded the limit did not exist. In Task 7, she factored correctly and knew since the numerator and denominator had common factors, a hole would occur. Since she could not construct the graph by hand, like CL, she relied on the calculator. In spite of this, she drew the hole in the wrong place, at (0,-3) instead of at (-3,-6) which a lot of people do. So she got the limit correct but could not translate that information to the graph. As for the domains, she said for Task 6, the domain had nothing to do with the limit but in Task 7, she said the domain was not defined at x=-3 and that's where the hole was.

Limits at Infinity

When asked what the infinity symbol meant, she said it was "not a number" and that the symbol represents "forever". When a limit goes to infinity, she said it means the limit "either exists or does not exist" depending on which calculus course being taken.

AK rarely considered the domains for limits at infinity. When given problems that involved a finite interval domain such as arccosine x or the half circle, she did not know it did not make sense to ask about the limit as x approaches plus or minus infinity because those large x-values were not in the domain. She said if the interval stopped at -1 on the left, then -1 would be the limit as x went to infinity, and said the limit would be 1 as x approached positive infinity. Yet she appeared to disconnect information about the domain when she determined what the limits were, as she did not get the limits correct claiming limits exist if they equal infinity. Interestingly for the rational function in task 8 which generated the horizontal asymptote at y=3, she said there was no relationship of limits to domain, the reason being the limit is 3 but the domain being $(-\infty, \infty)$.

Given the rational function $\lim_{x \to \infty} \frac{1}{(x-a)}$, AK thought the limit equaled infinity, not 0. The said the limit cannot go to 0, and it does not exist if it goes to 0. This is an important finding as a lot of people do this, and probably follow points on the graph instead of considering the behavior of the function values, which get smaller. With rational functions in Tasks 6 and 7, AK did not do any factoring, but just used the rules for finding asymptotes and the graphing calculator to see the graph. In Task 8 where the rational function yielded a horizontal asymptote at y=3, she exclusively relied on the rules for finding asymptotes for this and got the correct limit. Afterwards, she used the calculator to graph the function before sketching a horizontal asymptote at y=3. However, she did not make the connection that the limit she found was also the horizontal asymptote, as these were perceived as two different things to do.

Limits that Do Not Exist

AK did not know that for the cosine function in Task 1 the limit did not exist as x approached plus or minus infinity. Instead, she said the limit was equal to infinity "it keeps oscillating to infinity." So she knew how the function behaved but did not connect this with the function's end behavior. Later with the damped cosine function, she drew the same wrong conclusion, saying as $x \to -\infty$ the limit was equal to $-\infty$. This could be because she focused on the x-values, or also because she did not know the difference between when to write ∞ versus "d.n.e." Yet she knew for the damped cosine function that the limit does exist as $x \to \infty$ and equals 0.

With piecewise functions in Task 3, she knew that the limit at a point did not exist due to jump discontinuity, in which case the left hand limit did not equal the right hand limit. She did not find it necessary to split those tasks up into the limit as x approached a number from the left separately from x approaching from the right.

Given $\cos\frac{1}{x}$ in Task 2, she knew that the limit would not exist as x approached 0 because the function was not defined at 0. Given problems with finite interval domains, though, she did not ascertain that limits did not exist beyond the interval specified. So near a point she knew the limit does not exist, but not in the other direction as $x \to \infty$.

Further probing into her understanding of domains revealed how he considered there to be a relationship between domains and limits. Given $\lim_{x \to 2} \frac{1}{x-2}$ she said the limit "does not exist" near 2 because each side goes off in separate directions to plus or minus infinity. Also, x=2 cannot be in the domain because the denominator would be 0 making the function undefined.

Quite consistently, when a limit approached infinity, she correctly said that the limit does not exist because the function values increased without stopping. There addressed research question 6B. Also, she did not attempt to ever compare the left hand limit with the right hand limit for limits at infinity, as he said only one side was needed to make the determination that the limit does not exist. She said some professors do this, though, and so it's taught wrong.

Infinite Limits

AK had an interesting interpretation. Using the example $\lim_{x \to \infty} 2x = \infty$ she said the limit goes to infinity so it does not exist. However, she said "it does exist because it is there." She said the infinity symbol exists, but the limit doesn't exist because it keeps going. She distinguished between the $\lim_{x \to \infty}$ as the limit at infinity and the $= \infty$ as the infinite limit. This suggested she understood the differences between these.

Summary

Overall, the research questions were addressed and satisfactorily answered in there interview. AK articulated very well about her understanding of functions and limits. Her understanding of infinity was very good and knew that a limit did not exist when it tended toward infinity. However, she often gave inconsistent responses with limits at a point. Her algebra and knowledge of domains was limited, and she had difficulty citing how understanding domains is related to finding limits. One misconception she had was with the limit at infinity of a rational function in the form of $\lim_{x\to\infty}\frac{1}{x-a}$ as she thought the limit did not exist because it went to infinity. She drew similar conclusions with the cosine and damped cosine functions, saying the limit went to infinity rather than did not exist. In summary, AK has some limited algebra proficiency and knowledge of limits.

APPENDIX H-2: Interview with BK

Introduction

BK (Brendon) is a mechanical engineering major with a strong background in mathematics and mathematics-based engineering courses. He attended high school in Mooresville, where he took AP calculus and passed the AP exam. He said his calculus teacher had some misconceptions, teaching students that a limit exists if it equals infinity. He is currently enrolled in calculus III and differential equations. Overall, his understanding of functions and limits was excellent and his responses quite remarkable, suggesting what prerequisite knowledge seems to be important for understanding limits.

Functions

His knowledge of functions was excellent. Though he didn't provide a formal definition of a function, he could explain it very well. He said a function was a "mathematical function that takes an input and gives an output, and that it's a group of values that differ off a given equation." He also knew that function values referred to the y-values, "what the value is at the specific given input number." He also said if a function is 1-1 then it has an inverse, like exponential and log. He remembered this from high school. He was quite knowledgeable of domains and ranges of most functions, including piecewise, and was close to constructing the correct formulas for each.

Limits at a Point

His definition of limit was both intuitive and formal. He said a" limit is a y-value that is approached by a given a function, and the limit can equal the function value." After correctly writing down the formal definition of limit,

$\forall \varepsilon > 0\ \exists \delta > 0 \ni 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$, he drew a graph explained that for any number epsilon greater than 0, there is a number delta greater than 0, and that for any value of x-a in the band between 0 and delta, there is a number within the band f(x)-L that's less than epsilon and then described how that band gets smaller and smaller in smaller neighborhoods. He said he learned this in high school and used it all the time.

BK had no trouble with any of the tasks, stating domains and ranges, or with finding limits at a point. He correctly stated that as $x \to a$, that "a" does not have to be in the domain for the limit to exist because limits are about what x is approaching and what f(x) is near, not necessarily equal to. He knew that for piecewise functions, to compare the left hand limit to the right hand limit, and knew that the limit could exist where there was a hole or dot if both sides were approaching the same y-value. For the trigonometric functions, he identified all the limits at a point without hesitation.

As for domains, BK thought that limits are dependent on the domain because the domain provides the inputs and this is required to find the output values to determine the limit. When asked about the domains of the piecewise functions in Task 3, he could state the domain but had a little trouble with the algebra involving the horizontal shift. He knew, for instance, the quadratic part was shifted right 2 units and that came first, then shifted up 4 units which was written last. He squared the x instead of squaring the quantity (x-2). Although he did not readily identify them as "piecewise functions", he correctly articulate characteristics about the function and knew that (2,6) was on the graph of the function because it was defined in the domain. What was also interesting was he thought the first graph with the quadratic component and dot above was just a quadratic function, but yet he articulated his accurate perception of the graph as being piecewise. With the second graph, he forgot what it was called, but said it resembled a step function which was correct. Of course it does not matter what he calls graphs as long as he knows the underlying properties they have, which were all correct. In both cases, he identified the domain first before constructing the pieces of the graph in the formulas.

The tasks with finite interval domains such as arccosine x and the half circle were interesting and he knew immediately that limits would exist at a point anywhere in the interval because these functions are continuous on their domains. Likewise, he knew that a limit would not exist at a point outside the specified domain.

As for the endpoints for these functions having finite interval domains, he did not think the limits would have to be split up specifically as x approaching -1 from the right and x approaching +1 from the left. He said the notation $x \rightarrow 1$ or $x \rightarrow -1$ was sufficient because these points were included in the domain in which case the limits would exist.

One misconception BK had with limit at a point occurred with the rational function, $\lim_{x \to 2} \frac{1}{(x-2)}$. The problem BK had was once he computed the limits separately from the left and right and stated each limit did not exist, he then erroneously concluded that the limit does not exist because he compared the left hand limit $-\infty$ to the right hand limit $+\infty$. When asked why he did this, he reported the lady calculus teacher at Lake Norman high school taught it like that.

What appeared beneath the limit notation was not of concern for BK as he knew it did not involve direction and was not indicative of what the limit would be. The notation $x \rightarrow 2$ did not mean to him that the limit had to be split up into 2 parts, as x approached 2 from the left and from the right separately in order to get the limits, but he split them up anyway. The arrow did not imply direction from the left, either. He clearly knew limits were about what the function value was doing, and that limits were about nearness. Also, Unlike NS, BK did not perceive the notation $x \to \pm\infty$ beneath the “lim” to be problematic or invalid. He said from there, it is no big deal to just split up into $x \to -\infty$ and $\mathrm{x} \to +\infty$.

As for the linear function in Task 4, he immediately recognized the form y=mx+b and used that to construct the graph. He then used his graph to decide if the limits exist instead of plugging x into the function, but later did plug in and got the same result. He also concluded that the point (3,8) was not on the graph of the function because if 3 got plugged into the function, the result would be 7, not 8. He relied on algebra instead of the graph or vertical line test.

When given rational functions, BK possessed great algebra skills. He knew not to factor a sum of squares in the numerator, and that when the numerator and denominator had common factors, a hole would occur because the domain changed. He cited this as another reason why the domain is associated with finding limits. Instead of plugging in for x, he picked values close to 2 from both sides and plugged those in to study the behavior of the function values. He relied on concepts, not short cuts that NS used, such as L-Hospital’s Rule. Most all graphing was constructed by hand rather than with the use of a graphing calculator, but sometimes the calculator was used to check for accuracy.

Limits at Infinity

BK considered the domains for limits at infinity as well. When given problems that involved a finite interval domain such as arccosine x or the half circle, he knew it made no sense to ask about the limit as x approaches plus or minus infinity because those large x-values were not in the domain. He said if the interval stopped at -1 on the left, then there would be no answer for a limit at a point past that or after +1 infinity because the domains in those two tasks were restricted to a finite interval.

When asked what the infinity symbol meant, he said "it is not a reachable number, therefore by definition, infinity it is not a number". Since he knows that infinity is not a number and that limits must be numbers, this explains why he says that as function values approach infinity, the limits do not exist.

Given rational functions, BK knew there were rules for finding asymptotes but didn't rely on them. In Tasks 7 and 8, he did not do any factoring but just plugged in very large numbers for x to decide if the limits went to either plus or minus infinity. In Task 8 where the rational function yielded a horizontal asymptote at y=3, he did not determine this by the rules for finding asymptotes but simply by plugging in 100 and -100 into the first terms in the numerator and denominator which were squared. In both cases, $\frac{9000}{3000}=3$. He did not focus on the exact values but only the approximate values by ignoring lower degree terms. He then graphed the result and said that the limit in this case is also the horizontal asymptote, thereby making the connection between limits at infinity and horizontal asymptotes. He did not use a calculator to do any graphing but rather, did all of his graphing by hand.

Limits that Do Not Exist

BK knew from the cosine function in Task 1 that the limit did not exist as x approached plus or minus infinity because it kept "fluctuating" for large x and as a result, he wrote "d.n.e." Fluctuating is his way of describing oscillatory behavior that does not settle down to value. This was a good observation on his part and noteworthy because the goal was to find out if he might say the limit did not exist because it approached infinity by watching points on the graph, or worse, say the limit exists and is equal to infinity. This is the case in which students might follow a point on the line. His responses addressed research questions 4B and 6A.

With piecewise functions in Task 3, he knew that the limit at a point did not exist due to the jump discontinuity, in which case the left hand limit did not equal the right hand limit. Just like before, he would split those tasks up into the limit as x approached a number from the left separately from x approaching from the right, to make the correct determination.

When looking at the $\cos\frac{1}{x}$ in Task 2, he knew that the limit would not exist as x approached 0 because the function values "fluctuated near 0." Given problems with finite interval domains, he knew that the limit did not exist as x increased in either direction without bound because of the finiteness of the domain.

Further probing into his understanding of domains revealed how he considered there to be a relationship between domains and limits. Given $\lim_{x\to 2}\frac{1}{x-2}$ he said this function with domain $(-\infty,2)\cup(2,\infty)$ has a limit that "does not exist" near 2 because each side goes off in separate directions to plus or minus infinity. He said it's clear that 2 cannot be in the domain because if x were 2, the denominator would be 0 and hence, the function would be undefined. What went wrong at the end of this task, though, was he said though each side did not exist, he did what NS did which was compared both sides to decide that the limit for the whole function did not exist. He said the calculus teacher in high school said to do it this way.

The damped cosine function in Task 2 revealed he knew the limit did not exist due on the left side to oscillatory behavior whereas on the right, the limit exists and is equal to

0. This demonstrated good understanding of the end behaviors, as many students like CL would say the limits would approach minus infinity on the left and plus infinity on the right for the damped cosine function. He knew when to appropriately write "d.n.e." for the damped cosine function.

Quite consistently, when a limit was infinity, he correctly said that the limit does not exist because the function values increased without bound. This addressed research question 6B. He did not ever compare the left hand limit with the right hand limit for limits at infinity. He said only one side was needed to make the determination that the limit does not exist. He did note, though, that some professors he's had in high school said just the opposite, that the limit exists if it equals infinity. Overall, BK demonstrated good understanding of cases in which limits do not exist and in spite of what the lady high school teacher told him at Lake Normal High School, he stands his ground on knowing that a limit does not exist when the function values increase without bound, and that this draws from the formal definition of limit.

Infinite Limits

BK knew that infinite limits referred to the result of getting a limit that did not exist because the function values approached infinity and drew graphs denoting this. For instance, he sketched a quadratic and linear function and referred to the end behaviors. He then compared that with the cosine function and said that the end behaviors on that was not an infinite limit. This suggested he understood the difference between an infinite limit and a limit at infinity.

Summary

Overall, all of the research questions were addressed and satisfactorily answered in this interview. BK articulated very well and was quite knowledgeable of functions and limits. Unlike NS, CL and other students, he knew the formal definition of limit and referenced it at times in the interview. His knowledge of domains was outstanding and he presents an understanding of the relationship of domains to limits. He correctly stated that as $x \to a$, that “a” does not have to be in the domain for the limit to exist because limits are about what x is approaching, not what x is equal to at a certain point. The only main misconception he had was with the limit at a point of a rational function in the form of $\lim_{x \to a} \frac{1}{x-a}$. He knew that as x approaches a from the left, the limit does not exist because it goes to minus infinity and that from the left it does not exist because it goes to plus infinity, but erroneously decided that he had to compare the left hand limit to the right hand limit to make the determination that the limit does not exist. This is so, even though he stated that the limit does not exist if only one of the one-sided limits does not exist. BK thinks that limits involving discontinuities are the types of problems that cause trouble for most engineering majors.

APPENDIX H-3: Interview with BB

Introduction

Originally from New Jersey where she grew up and attended high school, BB is a junior meteorology major enrolled in calculus III. She struggles with math in general so she is currently repeating this course for the third time. In spite of fears she might not know much about limits, she was enthusiastic and did well in the interview, completing all tasks within 90 minutes. Overall, her understanding of functions and limits was mediocre.

Functions

Interestingly, she said a function "describes a physical system and how it changes", then gave some examples such as x-squared being a function and a semi-circle in the 1st and 4th quadrants not being one. She emphasized that a function represents a system. When asked what a function value was, she said it was a point on the graph, then said it was only the y-value. She was quite knowledgeable of domains and ranges.

Limits at a Point

Though she could not define "limit", she said a limit was "the maximum value a function reaches on the graph. It gets close to a point but never touches it." This was remarkable about the notion of "never touching". She made reference to a limit being the y-value. Moreover, she drew a line with a solid dot to demonstrate that the limit existed, then replaced it with a hole to show that now the limit did not exist, which is a significant misconception seen by a lot of students because they confuse limits with continuity. In tasks all tasks that involved a discontinuity with a solid dot above the hole, she said that the limit existed referencing the function value on the solid dot. She erroneously thought that as $x \to a$ that "a" has to be in the domain in order for the limit to exist, so she doesn't seem understand the definition of limit being about nearness and about the behavior of the function values near a point.

BB gave consistent correct responses when identifying limits at a point as long as they did not have discontinuities involved. Interestingly, for the cosine piecewise component graphed in Task 1 that contained a hole at $(\pi,-1)$ and solid dot above it at $(\pi,0)$, first she said that $\lim_{x\to\pi} f(x)=0$, referencing the function value in the point $(\pi,0)$. In spite of getting a limit, she said the limit did not exist where the hole was but said "I don't know why though." She also said that the point $(\pi,0)$ was not on the graph of the function because it was above the function.

In Task 3 with piecewise functions, she focused on the quadratic piece of the first graph and referred to this as a quadratic function, not piecewise. She said the limit did not exist because of the hole, then said the limit was equal to 6, the function value and stated the point (2,6) was not on the graph of the function because it was above the function. Moreover, she wrote the the domain and range wrong and was not able to connect knowledge of the domains to finding the limit. Later in Task 4 with the linear function that becomes progressively piecewise, she said the limit does not exist in the second graph because of the hole within the line, and when she got to the last graph that had the hole with solid dot above at (2,4), now she said that for $\lim_{x\to 2} f(x)=4$, and erroneously concluded that this limit existed, again taking the function value to be the limit. She did not think limits could exist if there was a hole. This suggests she lacks understanding of this type of piecewise functions. Just like before, she said the point (2.4) was not on the graph of the function because it was "not on the line and has to be on the line to be on the on the graph of the function". Meanwhile, with the second graph in Task 3, she called this a “discontinuous” instead of piecewise function, which is acceptable and she knew the limit did not exist because the left hand limit did not equal the right hand limit. She did not get the limit correct, as she said the $\lim_{x \to -2} f(x) = -1$ when in fact the limit did not exist. The reason she said the limit equals -1 was because there was a function value at (-2,-1). This confirms the notion that she perceives limits can only exist when there are solid dots with function values, not open holes. Quite notably, she could not figure out the range.

As for domains, BB knew that domains pertained to the x-axis, and the range, the y-axis, but did not make a connection with limits. In most cases she wrote the domains correctly except for piecewise functions. The reason might be because knowing the domains facilitates writing the formulas for piecewise functions.

Given tasks with finite domains such as the half circle and arccosine x, she looked at the graphs and correctly identified the domains as well as their corresponding limits at a point, but did not make any connection between the two. This is later confirmed with limits at infinity, because for those particular tasks, if she considered the finite interval domains, she would not have gotten those wrong. Overall, she did not use information about domains when finding limits, and thought that finding domains and limits were two different things, having nothing to do with each other.

As for the endpoints for these functions, she did not think the limits would have to be split up specifically as x approaching -1 from the right and x approaching +1 from the left. She said the notation $x \to 1$ or $x \to -1$ was sufficient because she understood what it meant but she did not make reference to the domain and its restriction. This addresses research question 3A.

Given the rational function $\lim_{x \to 2} \frac{1}{(x-2)}$, BB knew that each sided was computed separately and that the limit did not exist as x approaches 2 from the left, and as x approached 2 from the right. The notation $x \to 2$ did not make her think the limit had to be split up into 2 parts, as x approached 2 from the left and from the right separately in order to get the limits, and the arrow did not imply direction from the left, either. Unlike NS and CL, she did not erroneously conclude that the limit did not exist because the left hand limit did not equal the right hand limit.

What appeared beneath the limit notation was not of concern for BB as she knew it did not involve direction and was not indicative of what the limit would be. She knew limits were about what the y-value was doing. Unlike NS, BB did not perceived the notation $x \to \pm\infty$ beneath the "lim" to be problematic and invalid, and said it's easy to just split them up into $x \to -\infty$ and $\mathrm{x} \to +\infty$.

As for the linear function in Task 5, she recognized the function in the form y=mx+b and used that to construct a graph first, then used the graph to decide if the limits existed instead. She liked this task saying how it's fun to plot points and draw the graph because it gives a complete picture. Next she plugged x-values into the function to check the graph drawn. Interestingly, she stumbled when she computed a limit equaling 7 and questioned herself over whether or not if a limit approaches 7, if the limit really exists. The hesitation there was of concern. She got the limits correct then said point (3,8) was not on the graph of the function because the vertical line test would yield 2 values of x for the same y, and said the function "would not be 1-1", so the point is not on the graph of the function.

When given rational functions, BB's algebra skills were questionable. She tried to factor the sum of squares in the numerator with Task 6 and after plugging in x=2 into the numerator (x+2), she got a limit of 4 and decided she made a mistake. She then plugged in 2 into the numerator and denominator and when she got $\frac{8}{0}$, she correctly concluded the limit did not exist. In Task 7, she factored correctly and knew since the numerator and denominator had common factors, a hole would occur. Since she could not construct the graph by hand, like CL, she relied on the calculator and correctly drew a hole at (-3,-6). As for the domains, she said for Task 6, the domain $x \neq 2$ had nothing to do with the limit but in Task 7, she said the domain was not defined at x=-3 and that's where the hole was.

Limits at Infinity

When asked what the infinity symbol meant, she said it was "a number that is too large to exist, so the number is not defined. When a limit goes to infinity, she said it means the limit does not exist. Her responses addressed research question 4A.

As for domains, when given problems that involved a finite interval domain such as arccosine x or the half circle, she knew it did not make sense to ask about the limit as x approaches plus or minus infinity because those large x-values were not in the domain, so she correctly concluded the limits did not exist. Meanwhile, she appeared to disconnect information about the domain when she determined what some limits were, as she did not get some of the limits correct. For instance, in Task 3's piecewise function with the quadratic component, she incorrectly thought the domain was (-2,4) as she looked at the where the arrows pointing downward appeared to go, which seemed to be heading away from -2 and 4 on the x-axis. Here, she got the domain wrong, then incorrectly thought the $\lim_{x\to\infty} = 4$ and $\lim_{x\to-\infty} = 2.$ Noteworthy is that for the rational function in task 8 which generated the horizontal asymptote at y=3, she acknowledged the relationship of the limit to the horizontal asymptote. She also said since the domain is $(-\infty, \infty)$ then the limit exists everywhere at a point and at y=3 for large x.

Given the rational function, $\lim_{x \to \infty} \frac{1}{(x-a)}$, BB correctly knew the limit equaled 0. However with rational functions in Tasks 6 and 7, interesting things occurred. BB did not do any factoring. Instead, she used L'Hospital's rule to find the limit, which was not the correct thing to do and major mistakes occurred. For instance, with Task 6, she got an incorrect limit of 2 and drew a vertical asymptote, concluding that the limit did not exist at 2; yet, what she did had was not at all connected to x getting larger in either direction. She also tried to do mathematical operations with infinity to demonstrate how to use L'Hospital's Rule. In Task 7, she did this again and incorrectly got another limit at 2, producing a vertical asymptote at x=2. In both cases, she confused x with y, among other things. Amazingly, she drew a vertical asymptote on the graph at x=2 and then plugged 2 back into the original limit, and computed a function value at -1. As a result, she got a hole at (-2,1) and none of what she did had anything to do with limit at infinity.

In Task 10, she kept taking derivatives of x^2 and erroneously concluded that $\lim_{x \to \pm\infty} x^2 = 2$. When asked if there were different procedures for finding limits at a point from finding limits at infinity, she said L'Hospitals rule is used for limits at infinity, and for limits at a point, factoring and plugging in is done.

Limits that Do Not Exist

BB knew that the limit did not exist in Task 1 involving the cosine as x approached plus or minus infinity. Later with the damped cosine function, she drew the same correct conclusion, saying as $x \to -\infty$ the limit did not exist. Meanwhile for the damped cosine function she correctly knew the limit does exist as $x \to \infty$ and equals 0.

With the second piecewise function in Task 3, she correctly reported that the limit at a point did not exist due to jump discontinuity, in which case the left hand limit did not equal the right hand limit. She did not find it necessary to split those tasks up into the limit as x approached a number from the left separately from x approaching from the right. However, with Task 3's first graph, she erroneously said the limit did not exist at the hole in the quadratic piece, but that it existed above where the dot was, and equaled the function value.

Given $\cos\frac{1}{x}$ in Task 2, she knew that the limit would not exist as x approached 0 because the function oscillates near 0 and never reaches 0. With the damped cosine function, she aso correctly said the limit did not exist as $x \to -\infty$ due to oscillations. Given problems with finite interval domains, she correctly reported that limits did not exist beyond the interval specified.

Further probing into her understanding of domains revealed how she considered relationships between domains and limits. Given $\lim_{x\to 2}\frac{1}{x-2}$ she said the limit "does not exist" near 2 because each side goes off in separate directions to plus or minus infinity. Also, x=2 cannot be in the domain because the denominator would be 0 making the function undefined.

Quite consistently, when a limit approached infinity, she correctly said that the limit does not exist because the function values increased without stopping. Also, she did not attempt to ever compare the left hand limit with the right hand limit for limits at infinity, as he said only one side was needed to make the determination that the limit does not exist. She said some professors do this, though, and so it's taught wrong.

Infinite Limits

BB said infinite limits were those that went to infinity. She drew an example of something that resembled the sine function, and labeled the end behavior as going to plus infinity on the right and minus infinity on the left, and also drew a parabola pointing upwards labeling both end behaviors. However, in the first quadrant the end behavior pointed to infinity, and in the second quadrant, she incorrectly labeled the arrow going off to negative infinity. In any event, she distinguished between the $\lim_{x \to \infty}$ as the limit at infinity and the $= \infty$ as the infinite limit which suggested she understood the differences between these.

Summary

Overall, BB articulated very well about her understanding of functions and limits. Her understanding of infinity was very good and knew that a limit did not exist when it tended toward infinity. However, she often gave inconsistent responses with limits at a point and was certain that when a domain did not include a particular point, then the limit did not exist. Her algebra and knowledge of domains was limited, and she had difficulty citing how understanding domains is related to finding limits. Her use of L'Hospital's Rule was incorrect for limits at infinity of rational functions, and like CL, appeared to come up with her own technique involving plugging in the new x-value into the original function, generating a vertical asymptote where there really wasn't any.

APPENDIX H-4: Interview with EB

Introduction

EB is an engineering major enrolled in Calculus III who previously took AP Calculus in high school. She articulates very well and seems to be quite knowledgeable of functions and limits. This interview took 90 minutes and all tasks were completed. EB has a good background in mathematics.

Functions

EB did not define function, but simply said it was “1 x for each y.” The term “function value” was not previously heard before, but she suspected it was a specific value for x or y. When referring to the behavior of function values, she would simply slide her finger along the graph, x-squared for instance, and say “when the function stops at 2, then y is 4.

The trigonometric and transcendental functions were familiar and useful. She knew that the cosine function was periodic and continuous, versus the arccosine function, which had a discrete domain. As for domains, she knew domains involved x-values, and correctly recognized finite domains with the arccosine x and half circle. With piecewise functions involving jump discontinuity, she also correctly identified these domains and was not thrown off when the function value appeared above a hole.

Limits at a Point

EB neither provided a formal nor intuitive definition, but described a limit as “when you look at the behavior at a certain point.” She said a limit can equal a function value, such as with a continuous function, or not equal a function value in the case of a piecewise function. She did not perceive the limit notation as directional, $x \to a$ x approaching “a” from the left only, but made the distinction that the limit can be broken down further “as x approaches ‘a’ from the left, and as x approaches ‘a’ from the right”. As $x \to a$, she said “a” did not have to be in the domain for the limit to exist because limits were about behavior near “a” not necessarily at “a”. She also knew that limits only involved the y-coordinate, and involved the notion of nearness so the left hand limit was equal to the right hand limit, whether there was a hole or a solid dot, then the limit existed.

When exploring the piecewise functions, EB again compared the left hand limit to the right hand limit and correctly knew when the limit was not equal to the value of the function. Although she knew what the domains were of the piecewise functions, she was unable to identify the graphs of functions in Task 3 as piecewise. Specifically, she had difficulty with the function having a quadratic component involving a hole and a solid dot above it. Even though she knew what the domain was, she seemed to disconnect the domain when trying to come up with a formula for each piece of that function. This could be because she did not make the connection between the domain and the need to split it up into 2 parts to get the formula. Aside from that, for continuous functions such as cosine x, she knew the limit at a point exists everywhere because the domain ranged from minus infinity to plus infinity. Given the piecewise function with the quadratic component, she knew the limit would exist where the hole was because of the domain being continuous for all x, which made her consider if the left side was equal to the right side. With the step function type, she knew the domain involved all x but because it was not continuous due to jump discontinuity, she said the limits did not exist since the right hand limit did not equal the right hand limit.

Limits at Infinity

At first glance, EB seemed to correctly know how to describe the behavior of the function values as x was increasing in either direction without bound. When the function values kept getting larger or smaller, she would write “=infinity or =minus infinity”. She did not have any problems confusing x and y, and wrote the correct infinity symbols on the graph with the correct prefix, +/-. She did not think the “lim” notation implied direction from the left, as CL had thought. However, she thought that writing $\lim_{x \to \pm\infty}$ was invalid even though the limit would be the same answer if split up into 2 separate problems.

EB knew not to factor a sum of squares in the numerator of a rational function, and where there was a difference of squares, she factored correctly and knew she would get a hole in the graph instead of a vertical asymptote, because of the common factors cancelling out. Moreover, she did not factor limits at infinity of rational functions. In fact, she said the only technique needed is comparing the degrees of the numerator to denominator, then figuring out what’s negative. This addressed research question 5B. When probed further into how she would decide this, she used mathematical operations to figure it out, such as $\lim_{x \to \infty} \frac{x^2+4}{x-2} = \lim_{x \to \infty} \frac{\infty^2+4}{\infty-2}$. She said the infinities cancel out and the answer is infinity.

Infinite Limits and Limits that Do Not Exist

EB was doing pretty well up until this point because it was determined that she has misconceptions about infinity. Probing further about what this meant revealed that she thought the limit existed when it was equal to infinity or minus infinity. This coincides with what both CL and NS thought. Furthermore when looking at rational functions such as $\lim_{x \to a} \frac{1}{(x-a)}$, she compared the left hand limit $-\infty$ to the right hand limit $+\infty$ and determined that the limit did not exist because $-\infty \neq +\infty$. She made the mistake of using this technique which pertains to limits at a point. In the example above with $\lim_{x \to \infty} \frac{x^2+4}{x-2}$ she found that the limit is equal to infinity and thought this limit exists. So when asked what infinity meant, she said it is "a very large number." This seems to suggest that if the limit is a very large number, then by definition of a limit being a number, then in her mind the limit should exist.

One task involved a rational function with the same power of x in the numerator and denominator. EB identified this limit immediately as being equal to 3, by taking the leading term ratio of the coefficients. She then put the function into the graphing calculator to sketch the graph. After she sketched the graph with the horizontal asymptote at y=3, I asked her if there was any relationship between the limit and the horizontal asymptote. She paused and studied this for a while. She saw the 3 for the limit and the y=3 for the horizontal asymptote but claimed she did not see the connection, but thought there probably was one. She knew that the domain was all x for that rational function, because no value of x would ever make the denominator zero and therefore knew there were no holes or vertical asymptotes. She also thought the domain being all x suggested that the limit at infinity would exist everywhere at y=3. However, she did not see connect the limit with being the horizontal asymptote.

Given functions with restricted domains such as the half circle or arccosine x, she knew that it did not make sense to ask about a limit approaching infinity because of the domain being finite and concluded that the limit did not exist. With the cosine function and damped cosine function that the limit did not exist as x approached minus infinity and that it was appropriate to write “d.n.e.” because of the oscillatory behavior in which case the function values never settle down to any one number. With the damped cosine function as x approached plus infinity, she knew the limit existed and was equal to 0 because the function values were getting smaller and getting near to 0. She was able to refer to the piecewise functions with jump discontinuity and explain that the limit did not exist when the limits at a point were different on the right hand side versus the left hand side. However, she incorrectly perceived infinite limits as meaning the limits exist for one-sided limits, when they, in fact, did not.

Infinite Limits

EB did not know what this term referred to, and thought that it might be the same as a limit at infinity. She saw no difference but understood it once she was debriefed. Throughout her work on the tasks, she correctly identified limits as being equal to plus or minus infinity, but did not know that these results were referred to as infinite limits.

Summary

Overall, all research questions were addressed and answered satisfactorily. EB has a good understanding of limits and limiting behavior, evidenced by her understanding of functions, domains and algebra skills. Her understanding of limits at a point was very good but her understanding of limits that do not exist when a limit is equal to infinity was noteworthy. She shares the same misunderstandings that both CL and NS demonstrated, thinking the limit exists when it equals infinity, as well as comparing the left hand limit to the right hand limit for limits that converge near a point for rational functions, such as

$\lim_{x \to a} \frac{1}{x}$. Since EB perceives the infinity symbol to represent a very large number, then that is why the left hand negative number is not equal to the right hand positive number in this case.

APPENDIX H-5: Interview with JW

Introduction

JW is a math major enrolled in calculus III and is planning to uprsue a master's degree in Mathematical Finance. He attended high school in the west claims he always did good with math. Last semester he completed differential equations and plans to do his senior project on differential forms. Overall, his understanding of functions and limits was excellent and his responses quite remarkable, suggesting what prerequisite knowledge seems to be important for understanding limits.

Functions

His knowledge of functions was excellent. Though he didn't provide a formal definition of function, he could explain it very well. He said a function was a "mapping of one set to another set." He also knew that function values referred to the y-values, "what the value is at the specific given input number." Unlike peers, he did not think it was about mapping 1 x into the range resulting in 1 x, but knew that 1-1 was "the same y-value for all x" and graphed the log and exponential functions to illustrate this. He also said if a function is 1-1 then it has an inverse, like exponential and log. He remembered this from high school. He was quite knowledgeable of domains and ranges of most functions, including piecewise, and was close to composing the correct formulas for each.

Limits at a Point

His definition of limit was strictly intuitive. and formal. He said a" limit is the value that a function tends to when it is arbitrarily close to a point in the domain of the function." Function values, he said, were "values you get in R2, which are y-values. Overall, he demonstrated a very good knowledge of functions.

JW had very little trouble with any of the tasks, stating domains and ranges, or with finding limits at a point. He used the correct brackets and open parentheses for stating domains and ranges, and easily identified limits at a point for limits at a point, explaining he had to compare the left hand limit to the right hand limit. He also knew to do this for piecewise functions, and knew that the limit could exist where there was an open hole or dot if both sides were approaching the same y-value. For the trigonometric functions, he identified all the limits at a point without hesitation.

As for domains, JW thought that limits existing depend on the domain. As $x \to a$, he thought "a" has to be in the domain for the limit to exist, which is correct. The reason was if an x-value was not in the domain, then there could not be a limit or y-value there. When shown Task 3, he immediately recognized the graphs as piecewise, easily stated the domains and correctly wrote the formulas for both graphs. When asked if the point (2,6) was on the graph of the first function, he said it was because it was "a filled in dot" and is defined in the domain. With the second graph, he identified the domain first before correctly constructing the formula.

The tasks with finite interval domains such as arccosine x and the half circle were interesting and he knew immediately that limits would exist at a point anywhere in the interval because that's how the domain was specified. Likewise, he knew that a limit would not exist at a point outside the specified domain.

As for the endpoints for these functions having restricted interval domains, he gave two very different responses though the answers would have been identical. Given the graph of the half circle, he knew that the limit would be 0 as x approached 1 and -1. However, he did not generalize this to the second graph, the arccosine x. With that, he said that the limit did not exist at 1 and -1, and said he had to split them up into approaching 1 from the left and approaching 1 from the right. In that case, the limit existed as was x approaching -1 from the right and x approaching +1 from the left. He said the notation $x \to 1$ or $x \to -1$ was insufficient because of the finite interval domain. This appears to be a misconception about the limit not exiting at x=1 and at x=-1, when in fact, the limits do exist there at those end points.

One misconception JW had with limit at a point occurred with the rational function, $\lim_{x \to 2} \frac{1}{(x-2)}$. The notation $x \to 2$ did not make him think the limit had to be split up into 2 parts, as x approached 2 from the left and from the right separately in order to get the limits, and the arrow did not imply direction from the left, either. Once JW computed the limits separately from the left and right and stated each limit did not exist, he then correctly concluded that the limit did not exist because it only needed one side to not exist. Unlike NS, he did not think the end behaviors had to be compared to conclude the limit does not exist such that the left hand limit $-\infty$ to the right hand limit $+\infty$.

The $x \to \pm\infty$ notation that appeared beneath the limit notation concerned JW. Unlike CL, he didn't think it involved direction but like NS, said the 'plus/minus" was invalid and should not be written that way. He said it proper to write it split up as $x \to -\infty$ and $x \to +\infty$.

As for the linear function in Task 5, he immediately recognized the form y=mx+b and used that to construct a graph. He then used the graph he drew to decide if the limits existed instead of just plugging x into the function, but later did plug in and got the same result. He also concluded that the point (3,8) was not on the graph of the function because "it did not follow the rule for 3." When asked what that meant, he said the rule is the function, and at x=3 the output is 7, so the output cannot also be 8. He said it is impossible to get 2 outputs for 1 input.

When given rational functions, JW showed algebra competence but got stuck on how to use L'Hospital's rule with Task 6.He knew not to factor a sum of squares in the numerator, but after taking the derivative separately in the numerator and denominator, he ended up with $\lim_{x\to 2}\frac{2x}{1}=4$. He was certain this result was correct but could not reconcile this with the graph, which had a vertical asymptote at 2 and limits that did not exist near 2. As a result, he doubted himself whether he used L'Hospital's rule correctly and if he was supposed to use it at all. In Task7, he did not use this method. Instead, he factored the numerator and knew that when the numerator and denominator had common factors, a hole would occur. He said this was because the domain changed and claimed this was another reason why the domain is associated with finding limits. He was able to correctly construct the graph by hand with the hole. Most all graphing of rational functions was done by hand instead of using the calculator, but sometimes the calculator was used to check for accuracy.

Limits at Infinity

JW knew that the domains played an important role for limits at infinity as well. When given problems that involved a finite interval domain such as arccosine x or the half circle, he knew it made no sense to ask about the limit as x approaches plus or minus infinity because those large x-values were not in the domain. He said if the interval stopped at -1 on the left, there would be no answer for a limit at a point past that or after +1 infinity because the domains in those two tasks were restricted over a finite interval.

Also, with respect to domains, he added that “if a function is only defined on an interval (0,1), it’s not possible to take a limit at 2 or infinity. so he made that connection.”

When asked what the infinity symbol meant, he simply stated that “ it was not any particular number so when seen, limits can’t exist ”. This why he would consistently say that as limits approached infinity, they did not exist.

Given rational functions, JW knew there were rules for finding asymptotes but didn’t rely on them. In Tasks 6 and 7, he still relied on L-Hospital’s Rule, so when he got the function down to the last derivatives, he concluded the limits did not exist for either case in Task 6 $\lim_{x \to \infty} \frac{2x}{1} = \infty$ and $\lim_{x \to -\infty} \frac{2x}{1} = -\infty$. The same technique was used with Task 7. He did not do any factoring but just plugged in very large numbers for x to decide if the limits went to either plus or minus infinity. When asked about what techniques are used for limits at a point versus limits at infinity, he claimed L’Hospital’s rule can be used for both cases do he doesn’t think about doing anything different for limits at a point versus limits at infinity. At no time did he attempt to factor out the highest powers of x from the numerator or denominator separately, or do long division of polynomials.

In Task 8 he was reluctant to put his work down on paper because he saw the answer in his head, that the result was a horizontal asymptote at y=3. He did not plug in any test values, either, and later used the graphing calculator to sketch the correct shape of the graph but stressing that for large x in either direction, that the answer was y=3.

When asked if there’s a relationship of domains to limits, he said “the domain lets you know where it is appropriate to take a limit, so with this rational function, you can take the limit everywhere because the domain is defined and that’s because there’s no 0 in the denominator." He did not use the calculator to do any graphing but rather, did all of her graphing by hand.

Limits that Do Not Exist

JW knew from the cosine function in Task 1 that the limit did not exist as x approached plus or minus infinity because it kept "oscillating" for large x. He consistently wrote "d.n.e." for such instances, to describe how the function values don't settle down to one point. This finding differs from weaker students who say the limit did not exist because it approached infinity and or worse, say the limit existed and was equal to infinity. In this instance, students might follow points on the line. One interesting observation was for arccosine x, JW thought that the limit did not exist as $x \to -1$ and as x $\to$1, the reason being in each case that the left hand limit did not equal the right hand limit, in spite of the domain being [-1,1]. Therefore he appeared to disconnect knowledge of the domain from interpreting the limits correctly.

With piecewise functions in Task 3, he knew that the limit at a point did not exist due to jump discontinuity, in which case the left hand limit did not equal the right hand limit. Just like before, he would split those tasks up into the limit as x approached a number from the left separately from x approaching from the right, to make the correct determination.

When looking at the $\cos\frac{1}{x}$ in Task 2, he knew that the limit would not exist as x approached 0 because the function was not defined at 0, and that the function values "oscillated near 0." Given problems with finite interval domains, he knew that the limit did not exist as x increased in either direction without bound because of the restriction on the domain

Further probing into his understanding of domains revealed how he considered there to be a relationship between domains and limits. Given $\lim_{x \to 2} \frac{1}{x-2}$ he said this function with domain $(-\infty, 2) \cup (2, \infty)$ has a limit that "does not exist" near 2 because each side goes off in separate directions to plus or minus infinity. He said it's clear that 2 cannot be in the domain because if x were 2, the denominator would be 0 and hence, the function would be undefined.

The damped cosine function in Task 2 revealed he knew the limit did not exist due on the left side to oscillatory behavior whereas on the right, the limit existed and approached 0. This demonstrated good understanding of the end behaviors, as many students like CL would say the limits would approach minus infinity on the left and plus infinity on the right for the damped cosine function. He knew when to appropriately write "d.n.e." on the left side of the damped cosine function.

Quite consistently, when a limit approached infinity, he correctly said that the limit does not exist because the function values increased without bound. This addressed research question 6B. He did not attempt to ever compare the left hand limit with the right hand limit for limits at infinity, as he said only one side was needed to make the determination that the limit does not exist. Overall, JW demonstrated good understanding of cases in which limits do not exist.

Infinite Limits

JW knew that infinite limits referred to the result of getting a limit that did not exist because it approached infinity and drew graphs denoting this, such as $\lim_{x \to 0} \frac{1}{x^2}$ .He then compared that with the cosine function, stating the end behaviors did not constitute an infinite limit. This suggested he understood the difference between an infinite limit and a limit at infinity

Summary

Overall, all of the research questions were satisfactorily answered in this interview. JW articulated well, was knowledgeable of functions and limits, had good understanding of domains and could explain the relationship of domains to limits. He has good understanding of infinity. His misconceptions with finite interval domains are minimal.

APPENDIX H-6: Interview with JY

Introduction

JY (Jean) is a double major in Math and English currently enrolled in Calculus III. She currently works as a grader in the Math Department but wants to be an elementary school teacher. Her background in math is extensive, as she is originally from China and took many math courses there before pursuing her undergraduate degree here. JY possessed the ability to do all computations and graphing by hand, and unlike most students, she rarely relied on the assistance of a calculator. She was very enthusiastic about being in the study and has ambitious goals about improving student learning.

Functions

Her knowledge of functions was good; except she could not give a definition other than explain "given x, it's how to get y—the output." She knew that function values referred to "y" or f(xShe appeared to be knowledgeable of domains and ranges of most functions, including piecewise, and was even able to construct the correct formulas for each.

Limits at a Point

Her definition of limit was basically "given x, a certain number approaches 0 or some number but does not equal it." This turned out to have quite significant meaning later on when exploring graphs because when she saw a solid dot on a line and was asked if the limit existed, she said "no." Her reason the limit did not exist was due to what was happening strictly near the point and not at the point. Yet, interestingly enough, she contradicted herself with the first few tasks for continuous functions at which time she clearly knew that the limits actually did exist at the point. One problem here seems to be the difference of seeing a solid dot on the line versus just a continuous line. When she saw a solid dot on the curve, she said the limit does not exist but when she saw a continuous curve with no dots and was asked about a particular limit, she stated the limit exists. Returning to her definition of a limit, she considers limits to be about what is being approached, not equal to. Therefore, she has not internalized the definition of limit and does not really understand what it means.

Knowing about domains seemed to help with finding limits, except when a limit was equal to the function value. When asked about the domains of the piecewise functions in Task 3, she thought it was important to know about the domains in order to construct the formulas for the functions. She stressed that in her country, teachers emphasized domains with every single problem, so it was natural to consider the domain. She said the domain specifies what x values to include. When given a curve with a hole and dot above it and finding the limit, one has to consider what's near, not at, the x-value. She reported if there is a hole with no dot above it, then one just eliminates that value of x from the domain. She reported as $x \to a$, "a" does not have to be in the domain for a limit to exist and that students probably confuse limits with function values in piecewise functions.

The problems with finite interval domains for continuous functions such as arccosine x and the half circle were interesting. She knew immediately that limits would exist at a point anywhere in the interval. No solid dots appeared on these graphs, so she deviated from her incorrect definition and knew the correct responses intuitively. She knew that the limits at a point would exist for every point in the interval. Likewise, she knew that a limit would not exist at a point outside the specified domain.

Also, given the restricted domains for arccosine x and the half circle, she said the limit could not exist at the interval endpoints because it did not make sense to ask what the limit was as x approached 1 or -1 in both cases. She said if it only asked if x approached -1 from the right and approached positive 1 from the left, then that was included in the domain and therefore, the limit would only exist in such a case.

Interestingly, she generalized this notion later on with the rational function, $\lim_{x\to 2}\frac{1}{(x-2)}$. She said writing $x \to 2$ made no sense because it had to be split up as x approached 2 from the left and from the right separately in order to get the limits. When probed further to find out why she has to split up the limits into approaching from the left and right separately, she said it's because that's how the domain is defined and explained that limits are about what happens near the function only, not at the function. This of course misses the point about nearness. Therefore, she says, the notation written as $x \to 2$ implies what happens at the function and from both sides simultaneously which she thought can't be done, again missing the point about nearness.

What appeared beneath the limit notation was not of concern for JY because she knew it did not involve direction and was not indicative of what the limit would be. She clearly knew limits were about what the y-value was doing, and she basically knew limits were about nearness around some values of x or for large x. Meanwhile, other things about the limit notation were remarkable. For instance, it was problematic for JY when $x \to \pm\infty$ appears beneath the "lim". Like NS, she took this to mean plus "and" minus infinity and thought this notation was invalid because, obviously, you cannot go in two different directions at once.

When given rational functions, JY set the denominators equal to zero and excluded that point from the domain. She factored only where possible, such as with a difference of squares. She also recognized that when common factors divide out, the limit would be represented with a hole in the graph rather than a vertical asymptote All of her graphing was done by hand rather than with the use of a calculator.

Limits at Infinity

JY continued to consider the domains for limits at infinity as well. When given problems that involved a finite interval domain such as arccosine x or the half circle, she said it made no sense to ask about the limit as x approaches plus or minus infinity because those large x-values were not in the domain. She said if the interval stopped at -1 on the left, then there would be no limit at a point for x=2 or for limit at negative infinity because the domain was restricted.

When asked what the infinity symbol meant, she said it is “an amount you cannot count. It is more than a large number.” This seemed to suggest that she did not consider the symbol to be a number. This might help to explain how she determined that as function values get larger without bound, the limit did not exist

Given rational functions, JY did not know about the three useful rules for finding asymptotes. She determined the limits by plugging in very large numbers for x, She also attempted to factor out powers of x from only the numerator to determine the behavior of the function values. In these instances, she correctly determined when a limit is positive or negative infinity. In Task 8 where the rational function yielded the horizontal asymptote y=3, JY did not determine this by the rules for finding asymptotes but rather used an interesting new technique I have not seen. She added four extra “ imaginary” terms to the numerator with a common factor of 3, so she could factor out a 3 from all of the terms in the trinomial. By doing so, she got the horizontal asymptote by only factoring the numerator. Moreover, she plugged in large positive and large negative values for x and got positive 2.99999 on the right and 3.00002 on the left which confirmed her result that y=3 is a horizontal asymptote. She did not use the calculator to do any graphing but rather, did all of her graphing by hand.

Limits that Do Not Exist

JY knew from the cosine function in Task 1 that the limit did not exist as x approached plus or minus infinity because it kept "bouncing" for large x. As a result, she wrote "d.n.e." That's her way of describing oscillatory behavior that does not settle down to one point. This was a good observation on her part and noteworthy because the goal was to find out if she might say the limit did not exist because it approached infinity. Or worse, she might have said the limit exists and was equal to infinity, in which instance students might follow a point on the line.

With piecewise functions in Task 3, she knew when a limit at a point did not exist due to a jump discontinuity, in which case the left hand limit did not equal the right hand limit. Just like before, she would split those tasks up into the limit as x approached a number from the left separately from x approaching from the right, to make the correct determination.

When looking at the $\cos\frac{1}{x}$ in Task 2, she knew that the limit would not exist as x approached 0 because the function values "went crazy up and down" near 0. Given problems with finite interval domains, she knew that the limit did not exist as x increased in either direction without bound because there were no function values outside of the domain.

Further probing into her understanding of domains revealed that she thought there was a relationship between domains and limits. Given $\lim_{x \to 2} \frac{1}{x-2}$ she said this function with domain $(-\infty, 2) \cup (2, \infty)$ has no limit when x is near 2 because each side goes off in separate directions to plus or minus infinity. She compared this to a made-up example of a finite interval domain [-3,19] and indicated this function had a lower chance of the function values approaching plus or minus infinity given the restriction on the domain.

The damped cosine function in Task 2 revealed she knew the limit at minus infinity did not exist due to the oscillatory behavior whereas for the limit at plus infinity, the limit exists and is equal to 0. This demonstrated good understanding of the end behaviors, as many students like CL would say the limits would approach minus infinity on the left and plus infinity on the right for the damped cosine function. She knew when to appropriately write "d.n.e." for the damped cosine function.

One major misunderstanding exhibited was with limits at a point, when JY erroneously determined that the limit does not exist where as the limit does exists and equals the value of the function at the point. This was a rather unique and interesting finding. This happened early in the interview with Task 2 involving the half circle, then with the linear functions in Task 4, and again with Task 10 which involved looking at $\lim_{x \to 2} x^2$. She thought the limit does not exist because "4 IS that number, but it's not approaching the number 4 and for a limit to exist it can only approach, not actually BE the number."

Quite consistently, when a limit was infinity, she correctly said that the limit does not exist because the function values increased without bound. This addressed research question 6B. She did not ever attempt to compare the left hand limit with the right hand limit for limits at infinity. She knew only one side was needed to make the determination that the limit does not exist. Her responses are correct but unfortunately, seldom seen amongst her peers. Overall, JY has a good understanding of the various conditions in which limits do not exist.

Infinite Limits

JY knew that infinite limits referred to the result of getting a limit that did not exist because the function values approached infinity. She knew the difference between an infinite limit and a limit at infinity.

Summary

Overall, all of the research questions were addressed and satisfactorily answered in this interview. JY articulated very well and was quite knowledgeable of limits. The only main misconception she had was that a limit cannot exist at a point when there is a dot on a line, because limits are about what’s near, not at, a point. This explanation coincides with the definition she gave at the beginning of the interview.

APPENDIX H-7: Interview with KB

Introduction

KB is a sophomore secondary education major enrolled in calculus III. Overall, her understanding of functions and limits was mediocre. She had difficulties with limits at a point, but essentially no problems with infinity.

Functions

Similar to CL, her knowledge of functions was somewhat limited. Her definition of function was "it describes a relationship of x and y, and it can only have 1 point on the x-axis. " Like CL, reported a function value referred to both x and y coordinates, not just y. She drew two graphs to identify her understanding of what a function was. So she drew a linear function and then a circle explaining the first was a function, the circle wasn't and used the vertical line test to illustrate there. She had some limitations with domains and ranges. For instance with the cosine x, she could not identify the domain but got the range correct [-1,1]. Later, she associated ranges with limits. In many instances, she confused the x and y coordinates on the graph.

Limits at a Point

Her definition of limit was very weak, being "a value you get for x, it's the y." She pointed to the graphs to show that where there is a hole within two lines, the limit would not exist. However, where there was a dot connecting 2 lines, then the limit would exist because the dot had a y-value.

KB had trouble with several tasks, as she often confused x and y on graphs and seemed to follow points on the graph instead of considering the behavior of function values. She reported that for piecewise functions she had, to compare the left hand limit to the right hand limit, but she did not know when a limit existed or not. In Task 1's third graph which is a truncated piece of the cosine function, a hole was drawn at $(\pi, -1)$ with a point above it at $(\pi, 0)$. She said the limit did not exist at $(\pi, -1)$ because of the hole. She also said that the point $(\pi, 0)$ was not on the graph of the function but was on the x-axis. Yet, in a similar task that appears in Task 3, where there is a hole on the quadratic part with a solid dot above it at (2,6), she reported that the limit existed and was equal to the value of the function at (2,6). In other tasks such as looking at graphs of the cosine, arccosine, and half circle she often had long pauses while thinking. Even though the limits were very easy to identify, she still got several of them wrong. In some cases she confused x with y, sometimes focused on the x-value, and at other times, looked at the maximum point on the graph. That seems to suggest her understanding of limits at a point is very weak.

As for domains, KB reported that limits had something to do with the range, and not much with the domain. Her reason was that since limits are about the y-value, that's why the range is involved. An example is the cosine function where she reported the limits are -1 and 1, which is the range. When asked about the domains of the piecewise functions in Task 3, she said the domain was between [0,4] because that's where the quadratic went through the x-axis. Moreover she said the point (2,6) was not on the graph of the function because it was "sitting above the function." She said that $\lim_{x \to 2} f(x) = 6$ which was not correct, but she said a limit has cannot be an open hole with two adjoining lines but instead, has to be a solid dot, and the only solid dot was sitting above the quadratic function. The limit for this problem exists at 4, but just is not equal to the value of the function so she was unable to correctly interpret this graph.

Given the second piecewise function in Task 3, amazingly, she reported this function was exponential of all things. Moreover, she said the limit did not exist as $x \to -2$ but gave an unusual reason, being the limit stops where the solid dot is. She said nothing about the left hand limit not equaling the right hand limit. In spite of knowing the domain in the second graph, she could not produce the formulas for either of them, and that seems to be because she disconnected information about the domains from the functions as if the domain had nothing to do with the function.

The tasks with finite interval domains such as arccosine x and the half circle revealed that KB tends to follow the behavior of the x-value and after long pauses, seems to guess what the limits were, often getting them wrong. Apparently she confused x with y. As for the endpoints for these functions, she did not think the limits had to be split up specifically as x approaching -1 from the right and x approaching +1 from the left. She said the notation $x \to 1$ or $x \to -1$ was sufficient because she understood what it meant but she did not make reference to the domain and it's restriction.

Given the rational function $\lim_{x \to 2} \frac{1}{(x-2)}$, KB reported that each sided was computed separately and that the limit did not exist as x approaches 2 from the left, and as x approached 2 from the right. The notation $x \to 2$ did not make her think the limit had to be split up into 2 parts, as x approached 2 from the left and from the right separately in order to get the limits, and the arrow did not imply direction from the left, either. Unlike NS and CL, she did not erroneously conclude that the limit did not exist because the left hand limit did not equal the right hand limit. In fact, she said doing that is wrong.

What appeared beneath the limit notation was not of concern for KB as she reported it did not involve direction and was not indicative of what the limit would be. She clearly reported limits were about what the y-value was doing, and that limits were about nearness. Also, Unlike NS, KB did not perceive the notation $x \to \pm\infty$ beneath the "lim" to be problematic or invalid. She said she's seen it before and it means to split them up into $x \to -\infty$ and $\text{x} \to +\infty$.

As for the linear function in Task 5, she recognized the function in the form y=mx+b and used that to construct a graph first, then used the graph to decide if the limits existed instead. Next she plugged x-values into the function to check the graph drawn. She also concluded that the point (3,9) was not on the graph of the function because "it was not on the line" and referenced the vertical line test.

When given rational functions, KB possessed great algebra skills. She reported not to factor a sum of squares in the numerator for Task 6, and just plugged in numerical values close to 2 on both the left and right, like 1.999 and 2.001 to correctly determine that the limit did not exist. In Task 7, she reported since the numerator and denominator had common factors, a hole would occur. Unlike NS, she did not use short cuts such as L-Hospital's rule, but used the calculator to graph functions for accuracy.

Limits at Infinity

KB considered the domains for limits at infinity for tasks involving finite interval domains such as arccosine x or the half circle. She reported it made no sense to ask about the limit as x approaches plus or minus infinity because those large x-values were not in the domain. She said if the interval stopped at -1 on the left, then there would be no answer for a limit at a point past that or after +1 infinity because the domains were restricted. Yet she appeared to disconnect information about the domain when she determined what the limits were, as she did not get the limits correct. When asked what the infinity symbol meant, she said it was “a number too large to measure.” She said that when a limit goes to infinity, that means the limit does not exist.

Given rational functions in Tasks 6 and 7, KB did not do any factoring. She used both the rules for finding asymptotes and also the graphing calculator to see the graph. In Task 8 where the rational function yielded a horizontal asymptote at y=3, she exclusively relied on the rules for finding asymptotes for this. Once she got the limit, she used the calculator to graph the function, then sketched a horizontal asymptote at y=3. However, she did not make the connection that this was also the horizontal asymptote, as these were perceived as two different things to do.

It was remarkable to find that like CL, when she was given the task $\lim_{x \to \pm\infty} \frac{1}{(x-2)}$, she erroneously concluded that the limit did not exist because it went to infinity instead of 0. That seems to suggest she was following points on the graph instead of thinking about the behavior of the function values.

Limits that Do Not Exist

Remarkably, KB reported that when a limit went to infinity, that the limit did not exist and that it was appropriate to write “d.n.e. or $=\infty$. She said the limit doesn’t exist when function values keep getting larger or going up. She was very consistent with making this determination, and at no time did she compare the left hand infinite limit with the right hand infinite limit.

However, there were some problems noted with other things. KB did not realize that the limits did not exist for the cosine function in Task 1 as x approached plus or minus infinity. Like CL, she focused on the range and said there are 2 limits at -1 and 1. The reason given was limits are y-values and that this function kept going in between these two y-values. Moreover, when presented with any piecewise function where the limit is not equal to the value of the function, e.g., the case of the solid dot above the hole, she was wrong with her conclusion that the limit did not exist because of the hole. She also reported that the limit existed where the solid dot was because it was above the hole and could be slid down to the function. She apparently confuses limits with continuity. With the second piecewise function in Task 3, she reported that the limit at a point did not exist but not due to jump discontinuity. Instead, she said it was because the line stopped at the solid dot (-2,-1).

When looking at the $\cos\frac{1}{x}$ in Task 2, she reported that the limit would not exist as x approached 0 because the function was not defined at 0. Given problems with finite interval domains, she reported that the limit did not exist as x increased in either direction without bound because of the restriction on the domain.

Further probing into her understanding of domains revealed how she considered there to be a relationship between domains and limits. Given $\lim_{x\to 2}\frac{1}{x-2}$ she said there function with domain $(-\infty,2)\cup(2,\infty)$ has a limit that "does not exist" near 2 because each side goes off in separate directions to plus or minus infinity and that getting 0 in the denominator makes the function undefined. She also said that when there is a hole joining 2 lines, the domain excludes the point so the limit does not exist. This was wrong, but it was her perception. Also with the rational function with a common factor of (x+3) in the numerator and denominator, she said the limit cannot exist at x= -3 since the domain was not defined there, and that's because one cannot have a 0 in the denominator. For a linear function, she said the domain was all x so the limit was everywhere in between along the line, which was correct. Finally for the piecewise function that's quadratic with the solid dot above at x=2, she said the limit cannot exist at 2 because the domain is not defined at 2. But, she said there is a limit at 4 which "jives' with the range. This was incorrect, but nevertheless, it was her interpretation and sheds light on her misconceptions.

The damped cosine function in Task 2 revealed she thought the limit did not exist due on the left side to oscillatory behavior, whereas she did not do this with the previously seen cosine x function. In any event, she got the left side correct but on the right side, she incorrectly perceived this limit to equal infinity rather than 0, just like she did with $\lim_{x\to\infty}\frac{1}{x-2}$ where she said the limit was also equal to infinity. Once again, she seemed to follow points on the graph instead of consider the behavior of the function values. So, she seemed to know when to appropriately write "d.n.e." on the left side of the damped cosine function.

Infinite Limits

KB reported that infinite limits referred to the result of getting a limit that did not exist because it approached infinity and drew the graph of a linear function denoting there. She then compared there with the notation for a limit at infinity showing that $\lim_{x\to\infty} x+1$ was a limit at infinity and the result was the infinite limit, ∞. This suggested she understood the difference between an infinite limit and a limit at infinity.

Summary

Overall, all of the research questions were addressed and satisfactorily answered in this interview. KB articulated very well but had restricted knowledge of functions and limits. It was not as restricted, though, as CL's. Like CL, she sometimes gave unusual, but interesting, responses to questions, mostly because she lacked content knowledge and understanding. Her knowledge of domains was minimal and she did not present an understanding of the relationship of domains to limits. In fact, she misunderstood this relationship by focusing on the range, or for giving erroneous reasons why the domain was important and this was because she did not make the connection. Domains were also problematic as she thought that as $x \to a$ that "a" has to be in the domain for a limit to exist. The main misconceptions she had were with limits at a point, thinking the limit did not exist when she saw a hole within a line, and with identifying limits with restricted interval domains. With rational functions, she avoided algebra whenever possible, instead relying on the rules for asymptotes or by using the graphing calculator to see the correct graphs. However, she consistently reported that when a limit was equal to infinity, that the limit did not exist. Overall, an underlying problem is that she does not really understand the definition of limit being about the behavior of the function values near a point of for large x as x is increasing.

APPENDIX H-8: Interview with LA

Introduction

LA (Linsey) is a math major enrolled in calculus III. She passed the AP calculus exam in high school and enrolled directly into Calculus III at UNCC but admitted that she usually had trouble with limits and calculus courses in general. She was enthusiastic about the interview, completed all tasks within 90 minutes. Overall, her understanding of functions and limits was satisfactory, but had misconceptions about infinity.

Functions

Similar to CL, her knowledge of functions was somewhat limited. Her definition of function was “1 x for each y, like with x-squared, and the vertical line test can decide if 2 values are not the same.” To illustrate, she drew a parabola saying that was a function and a circle saying that was not a function. When asked what a function value was, she said “it’s when you plug in numbers for x and get f(x), so it is just the y-value.”

Limits at a Point

Her definition of limit was minimal, being “it’s about nearness, when you look at the behavior at a certain point when the left side equals the right side. It can be the same as a function value” and made reference to a limit about the second coordinate. Next, she drew a line with a solid dot to demonstrate that the limit existed, then replaced it with a hole to show that now the limit did not exist, which is a significant misconception seen by a lot of students, as they confuse limits with continuity.

LA gave inconsistent responses for limits at a point. One main misconception was when she saw an open hole within 2 lines, she said the limit does not exist because the function is not continuous. In Task 1, when asked if the point $(\pi, 0)$ was on the graph of the function, she did not think so because the function was not continuous. This happened again in Tasks 4, and in each case, she said the point (2,4) was not on the graph of the function and that whenever there is a hole, that means there is no limit. Limits were confused with continuity. However with the piecewise function in Task 3 that had the quadratic piece, she said the limit did not exist a 2 because of the hole, but said the point (2, 6) actually was on the graph of the function because when she plugged 2 into the incorrect formula she wrote, she happened to get 6. The incorrect formula she wrote was $f(x) = x^2 + 2$ so by coincidence she got a result of 6.

For this same task , the domain was right but the range was wrong because she referenced x-values, such as $x \neq 3$. She did not seem able to connect knowledge of the domains to finding the limits evidenced by writing the wrong formula for this piecewise function. As for other domains, most were correct but one was quite interesting and noteworthy. It involved the domain of the half circle, $\sqrt{x^2 - 1}$. She eliminated the radical and got this as the domain: $x^2 - 1 \geq 0, x \geq \sqrt{1}$. In general, she reported that domains pertained to the x-axis, and the range, the y-axis, but did not make a connection with limits.

Given tasks with finite domains such as the half circle and arccosine x, she looked at the graphs and correctly identified the domains as well as their corresponding limits at a point and said there the limits existed because they were within the domain. This was later seen with limits at infinity, because for those particular tasks, she reported the limit could not exist outside of the domain. if she considered the finite interval domains, she would not have gotten those wrong. Overall, she did not use information about domains when finding limits, and had a very hard time using knowledge of domains to write formulas for piecewise functions.

Regarding endpoints for these functions, she did not think the limits would have to be split up specifically as x approaching -1 from the right and x approaching +1 from the left. She said the notation $x \to 1$ or $x \to -1$ was sufficient because she understood what it meant but she did not make reference to the domain and its restriction.

Given the rational function , $\lim_{x \to 2} \frac{1}{(x-2)}$, LA reported that each sided was computed separately and that the limit did not exist as x approaches 2 from the left, and as x approached 2 from the right. The notation $x \to 2$ did not make her think the limit had to be split up into 2 parts, as x approached 2 from the left and from the right separately in order to get the limits, and the arrow did not imply direction from the left, either. However, like NS and CL, she said the limit did not exist because the left hand limit did not equal the right hand limit.

What appeared beneath the limit notation was not of concern for LA as she reported it did not involve direction and was not indicative of what the limit would be. She reported limits were about what the y-value was doing. Like NS, she reported the notation $x \to \pm\infty$ beneath the "lim" was problematic and invalid and said professors do this but they are supposed to split them up into $x \to -\infty$ and $x \to +\infty$.

As for the linear function in Task 5, she recognized the function in the form y=mx+b and used that to construct a graph first, then used the graph to decide if the limits existed instead. Next she plugged x-values into the function to check the graph drawn. She got the limits correct but erroneously concluded that the point (3,8) was not on the graph of the function because "there were 2 y-values for the same x" and used the vertical line test for this.

When given rational functions, LA's algebra skills were questionable. She tried to factor the sum of squares in the numerator with Task 7, gave up, plugged in 2 into the numerator and denominator and when she got $\frac{8}{0}$, she concluded the limit did not exist. With the rational function in Tasks 6, LA tried to factor the sum of squares but after she distributed the terms she saw the mistake because nothing added up. She then plugged in x=2 into the function and determined that the limit did not exist, which was correct. In Task 7, LA correctly factored the numerator being a difference of squares, and got a correct limit of -6. However, she did not know if she would get a hole or a dot, she said she always got confused with this. So she drew a solid dot at (-3,-6) and said the limit existed at that point. Like CL, since she could not construct the graph by hand, she relied on the calculator. So she got the limit correct but limited algebra proficiency prevented drawing an accurate graph. As for the domains, she got the correct domain for Task 6 and said there cannot be a limit where x=-2 because of the vertical asymptote there. In Task 7, she wrote the wrong domain, thinking it was all x. In fact, x could not equal -3 to be defined. Interestingly, she reported thee limit existed at (-3,-6) because of the solid dot, and this is because she reported the domain included all x.

Limits at Infinity

When asked what the infinity symbol meant, she said it was "any large number that is like being exponential because it keeps going" and that the symbol means that "the limit exists". When a limit goes to infinity, she consistently said the limit exists.

LA considered the domains for limits at infinity. For instance, when given problems that involved a finite interval domain such as arccosine x or the half circle, she reported did not make sense to ask about the limit as x approaches plus or minus infinity because those large x-values were not in the domain. Given continuous functions such as cosine x, she reported that since there were no domain restrictions, that the limit would "keep going" and not exist due to oscillations. However with other continuous functions where the function values tended toward infinity, she consistently said that the limit existed.

Given the rational function $\lim_{x\to\infty}\frac{1}{(x-a)}$, LA reported the limit equaled 0. Also when she had tasks with infinite limits, she wrote the correct plus or minus infinity symbols on the graph. Students sometimes don't know when to write $+\infty$ *or* $-\infty$ next to the ends of functions on graphs because they confuse x and y, but this was not an issue here. She did not do any factoring, but just used the rules for finding asymptotes and the graphing calculator to see the graph. In Task 8 where the rational function yielded a horizontal asymptote at y=3, she said there was no need to factor anything and just used the rules for finding asymptotes, getting the correct limit. Afterwards, she used the calculator to graph the function before sketching a horizontal asymptote at y=3. She noticed that the horizontal asymptote she got just happened to be the limit, but she was curious about why that was the case.

When asked more what the relationship is of domains to limits, she gave three examples. She drew a parabola with a hole in it at x=2 and said the domain was $x \neq 2$ so the limit did not exist at 2, which is wrong. Then for $f(x)=\frac{1}{x-2}$ she said the domain was also

$x \neq 2$. Since there was a vertical asymptote there, that meant the limit does not exist at x=2 due to zero in the denominator making the function undefined at 2, which was correct. Lastly, she gave a linear function and said the domain was $(-\infty, \infty)$ and so the limit could be anywhere along the line, which is correct. This validates previous explanations she gave about the cosine x function.

Limits that Do Not Exist

LA reported that the limit did not exist for the cosine function in Task 1as x approached plus or minus infinity because of periodicity and oscillations, which was correct. Later with the damped cosine function, she drew the same correct conclusion as $x \to -\infty$, that the limit did not exist. She seemed to know the difference between when to write ∞ versus "d.n.e." Meanwhile, she also reported for the damped cosine function that the limit does exist as $x \to \infty$ and equals 0. The main problem she had was deciding that the limit existed if went to either plus or minus infinity. Like NS, she also compared the left hand limit to the right hand limit for limits at infinity and whenever infinity went off into two different directions, she would then say that the limit did not exist. So from her perspective, a limit could exist could separately on each side if it went to infinity, or if they both went to the same direction such as with $f(x) = \frac{1}{x^2}$ or x^2 but the limit did not exist if infinity went off into two different directions such as with f(x)=x+1.

With piecewise functions in Task 3, she reported that the limit at a point did not exist due to jump discontinuity, in which case the left hand limit did not equal the right hand limit. Moreover, she did not find it necessary to split those tasks up into the limit as x approached a number from the left separately from x approaching from the right. In Task

9 when shown $\lim_{x \to 2} \frac{1}{x-2}$ she said the limit "does not exist" at 2 because each side goes off in separate directions to plus or minus infinity. Moreover, she said in that case she must compare the left side to the right side near the point, and since $-\infty \neq +\infty$ then the limit did not exist. Given $\cos\frac{1}{x}$ in Task 2, she reported that the limit would not exist as x approached 0 because the function oscillated near 0 but was not defined at 0. She also reported that for cosine x, the limit does did not exist as $x \to \pm\infty$. When given problems with finite interval domains, she reported limits did not exist beyond the interval specified so overall she demonstrated good understanding of when limits do not exist in the case when it's appropriate to write "d.n.e." These responses address research question 4B and 6A.

Quite consistently, when a limit approached infinity, she reported the limit existed. This stems from the fact that she thinks infinity is "any large number." Moreover, she compared the left hand limit with the right hand limit for limits at infinity and infinite limits to determine if limits existed. She did not think one side was needed to make the determination that the limit does not exist. When asked further about why she reported limits exist if they equal infinity and why she compares the right and left limits, she said some professor she grades for says so.

Infinite Limits

LA did not know the difference between a limit at infinity and an infinite limit. She reported these both that had the same meaning and referred to $\lim_{x \to \infty}$.

Summary

Overall, all of the research questions were addressed and satisfactorily answered in this interview. LA articulated very well about her understanding of functions and limits. Her understanding of when limits exist or not was good and she cited valid reasons, such as oscillations for when a limit doesn’t exist. However, she reported limits exist if they equal infinity and like NS, erroneously compared the left and right hand limits, both near a point and at infinity. Her algebra skills had some limitations, such as just using rules for finding asymptotes instead of factoring, and in other cases, she usually found mistakes if she tried factoring. Her knowledge of domain and range was good but she did not always correctly explain how domains are related to finding limits.

APPENDIX H-9: Interview with YJ

Introduction

YJ is an undergraduate math major, a student enrolled in Calculus III who plans to pursue doctoral studies in information systems. In spite of her extensive background in mathematics, most of which was learned abroad, her knowledge of limits was very limited. This interview took 90 minutes.

Functions

YJ gave a definition of function which included one y for each x that x was the input and y was the output. YJ correctly distinguished between functions and function values being the y-value, and knew that the cosine function was continuous and even due to its symmetry. She had a very difficult time with functions involving discontinuities. She struggled with identifying the domains of these functions whenever she saw a hole with a dot above it. She was not able to construct any piecewise function formulas as she did not correctly identify these functions as piecewise. Without the function value present above the discontinuity, she could identify the domain and construct a function. Discontinuities are significant sources of confusion and quite typical of many calculus students.

Limits at a Point

YJ's definition or description of a limit was essentially “a place where a function is defined”, because there was no “constraint.” Here, constraint means no hole or vertical asymptote. This is similar to CL's forcing everyday language in to the mathematical framework.

As for the limit notation, she knew that the arrow did not imply direction and that it meant "approaching from both sides." CL thought the arrow meant approaching from the left only. YJ thought that limit and continuity went together because for a limit to exist, the function had to be continuous. This suggests she equates limits with continuity.

Given a linear function with a dot at a particular point on the line, she knew the limit existed and said it was the function value. When the domain changed with the dot becoming a hole, further probing revealed she disconnected the change in domain with discontinuity of the function. Further probing revealed when there is a hole within a straight line, the limit does not exist there because the hole is a "constraint." YJ misses the fact that the limit existing or not depends only on the behavior of the function values near the point but not at the point. The underlying problem is she does not understand the definition of limit.

CL and YJ use the word "constraint" in opposite ways. CL would say the limit does not exist when there is a straight continuous line with a dot because there is no hole to fall into, whereas YJ thinks the limit does exist there. YJ thought the limit did not exist at the hole, whereas CL did. YJ thought the function was not defined where the hole or constraint was, whereas CL did not consider the function being defined or not and simply thought the limit exists where the hole is because the hole prevents one from getting to the other side. CL thinks the limit exists where the hole is, but YJ thinks the hole means the limit does not exist.

YJ made no reference to comparing the right hand limit to the left hand limit in any of the problems and could not explain the notion of nearness, because limits did not appear to be about nearness but about functions being defined. This posed more trouble when a point appeared directly above the hole in which case she said the limit did not exist because it did not equal the function value.

With rational functions, no factoring was attempted in any of the problems. Instead, she simply plugged in the x-value and often computed the wrong result, such as in Task 6 which involved a sum of squares in the numerator. She did not know when to plug the number in, versus when to try to factor.

YJ had erroneous notions about domains. She said as $x \to a$ that "a" had to be in the domain for the limit to exist. Meanwhile, she did not seem to consider the domains of rational functions. Given the rational function $\lim_{x \to 0} \frac{1}{x}$, she knew the limit did not exist at x=0 because the limit tended toward infinity, and knew infinity was not a number. She correctly stated that in order for a limit to exist it must be a number but incorrectly said that the limit is undefined instead of saying the limit does not exist.

Limits at Infinity

YJ showed algebra difficulties with rational functions. She did no factoring where needed, nor did she use any rules for asymptotes. She looked at the numerator and denominator separately and decided what went to either plus or minus infinity.

Similar to NS, she performed mathematical operations with infinity with division of rational functions. Then, concluded the limit was equal to plus or minus infinity. Her intuition was fairly good but not her computations. When asked for the domain of rational functions, she was often incorrect because she said it was all x not accounting for zeros in the denominator. She did not consider how the denominator is associated with the domain of these functions. As a result of not understanding domains of rational functions, she drew the wrong conclusions about the behavior of the function values.

Very often, she confused x with y when labeling an infinite limit on the graph. When graphs pointed upward in the 2nd or downward in the 3rd or 4th quadrants she said the limit went toward positive infinity seemingly watching the points on the graph of the function. However, she correctly determined that the limit did not exist, though, because the limit had to be a number in order for it to exist. For the rational function 1/x, she knew that the limit at infinity was 0 because the function values were getting smaller. This was also seen with other functions where x increased without bound in either direction and the limit was 0. In this case, she did not follow points on the graph.

When given Task 8, the rational function involving the same powers of x in the denominator and numerator, she did not consider the domain at all could not determine that the limit was equal to the horizontal asymptote. In fact, she incorrectly determined that the limit was infinity because she plugged in infinity into the numerator and denominator. Then she concluded that since she ended up with $\frac{9\infty}{3\infty}$ that this was in fact going to infinity in both directions. Interestingly, she was not consistent with "cancelling" out infinity this time which would have given her the horizontal asymptote, at y=3, and thus, the limit being equal to 3.

YJ did not know the difference between techniques to use for limits at a point versus limits at infinity. She basically plugged in the point or plugged in infinity for both without any factoring, which ended up with incorrect limits and incorrect graphs.

Limits that Do Not Exist

She correctly reported that a limit did not exist when it tended toward infinity. She did not have to compare the left hand limit with the right hand limit for infinite limits as she correctly only took one side at a time into consideration. Functions with oscillatory behavior such as cosine x or the damped cosine function revealed that she did distinguish between when a limit did not exist versus when it was equal to infinity. In fact, she followed the points on the graph and decided the limit was equal to infinity for functions which oscillated. Functions with restricted domains such as the half circle or arccosine revealed that she did not consider the domain as she claimed the domain was all x. In these two examples, she reported the limit was equal to infinity. With most graphs, she labeled minus infinity incorrectly, writing down infinity instead but insisted she wrote it correctly. So she does not seem to know where plus and minus infinity appear on the graph, though she can make the distinction with computations. Moreover, she computed limits to be both plus or minus infinity but referred to the graph labels as plus infinity only. She does not seem to consider that the function has very large positive values on one side and very large negative values on the other. So she has disconnected somewhat from the limit being about the behavior of the function values.

Infinite Limits

She knew that infinite limits referred to the limit being equal to infinity, and not "multiple limits" as CL thought. She also knew the distinction between an infinite limit and a limit at infinity, hence the difference between x and y behavior.

Summary

YJ has extensive background in mathematics, but lacks proficiency with functions and limits. Her responses model both CL and NS at times. She struggles with limits existing at a point with piecewise functions or those with jump discontinuities because she thinks limits are not defined where there are holes. She reported that as $x \to a$, "a" must be in the domain for the limit to exist, which is not correct. The reason could be that she does not really understand the definition of limit being about behavior of function value as $x \to a$. Given oscillating functions, she follow points on the graph instead of looking at the behavior of the function values and therefore erroneously concludes the limit equals infinity. She does not consider the oscillatory behavior being the reason and that the function does not settle down at any particular point. Moreover, she has very limited algebra skills with rational functions, but has some fairly good intuition with trying to graph functions manually. The rational functions were useful to identify weak algebra proficiency which resulted in computing and with graphing limits incorrectly. She knows that infinity is not a number and that a limit cannot exist if it equals infinity. In spite of this, she was seen performing mathematical operations with infinity as if it were a number.

APPENDIX I: TRANSCRIPT EVIDENCE

APPENDIX I-1: Transcript Evidence of BK

APPENDIX I-2: Transcript Evidence of JY

APPENDIX I-3: Transcript Evidence of LA

APPENDIX I-4: Transcript Evidence of AK

APPENDIX I-1: Transcript Evidence of BK

Video file name: brendon.doc   Running Head: 90 Minute Interview

Interviewer: MA   Interviewee: Brendon   HDSC References:

Turn	Time	who	Utterance
1.	00:01	MA	Begin Task 1: Define or describe what a function is and give examples.
2.		BK	A set of values that are based off a given equation.
3.		MA	What is a function value?
4.		BK	What you get out, the output. That phrase "function values" isn't used much. They usually say just say output. It kind of sounds like it means both x and y.
5.		MA	What is the difference between a function and a function value?
6.		BK	A function is what happens with the input, like you plug in some number for x into a function. Cube it or something, and the function value is the output. Note: The phrase function values can have different synonyms such as "output" or "values".
7.		MA	What does mean that a function is 1-1? Is 1-1 part of the definition of function?
8.		BK	It's part of the definition for a function that has an inverse but not for a regular function. I'm pretty sure it means you get the same y-value for all x's. So when a function has an inverse like log or exponential, then you can draw a horizontal line through it. If a function is 1-1 then it has an inverse so if the original function mapped one x from the domain into the codomain, and it has an inverse then now you can see that for every value of y, there are multiple x values. If you draw a vertical line up and down, you are seeing if it's a function. If you draw a horizontal line, you are seeing if it has an inverse. Note: Knew definition of 1-1 and use of horizontal line test for inverses vs. vertical line test for functions
9.		MA	Define or describe what a limit is and give examples.
10.		BK	It's the y-value that is approached. As the number gets closer and closer to the given value, it's the output that it approaches. A value the function approaches. If you want a formal definition, it would be $\forall \varepsilon > 0 \; \exists \delta > 0 \ni 0 < \lvert x-a \rvert < \delta \Rightarrow \lvert f(x) - L \rvert < \varepsilon$ Here you have to show that f(x) is within epsilon of L, for every x near a. Then you come up with a number for delta. That tells you the meaning of nearness to a. It's like the goal is to get f(x) within epsilon of L by keeping "x" within delta of "a". If you can find a delta for every epsilon, then you know that the limit is L. I know this

			is for limit at a point and there is also a definition for limit at infinity like this that you use with sequences. I think that involves convergence. You have this strip for L. So only a finite number of a's can be outside the strip around L. Meanwhile I'll try to remember how to draw a graph but we didn't do much graphing with this in high school. Note: Wrote the formal definition of limit and drew a corresponding graph.
11.		MA	Can limits and function values be the same?
12.		BK	Yeah like when you have a line or some continuous function. So a limit at a point that exists is also a function value.
13.		MA	What's the difference between the limit and the value of the function?
14.		BK	Well the main thing is the function value definitely equals a number but a limit is about what the y-values are like near a number on the x-axis.
15.		MA	Do limits pertain to the x-coordinate, y-coordinate or both when you find a limit?
16.		BK	It's both but first you are given what x is approaching then you have to find the y-value.
17.		MA	Can a limit be a number?
18.		BK	Yes it is either a number or it is infinity.
19.		MA	Do you say: "The limit equals infinity", or do you say "the function values are approaching infinity"? Explain.
20.		BK	When a limit goes to infinity, I just say the limit does not exist but on graphs I'll write "$=\infty$" but it means d.n.e.
21.		MA	If a limit is equal to infinity, then does that limit exist?
22.		BK	No.
23.		MA	What exactly is infinity?
24.		BK	An abstract notion of endlessness, not reachable.
25.		MA	What does the infinity symbol mean?
26.		BK	It's not a reachable number. I would not say it is a very large number because that would signify that it's a number that is reachable but that would defeat the meaning of infinity. Note: Infinity is not a reachable number. I ask this later on for reliability
27.	6:00	MA	Graph A: Study the graphs in Task 1 for cosine, arccosine and for the last graph. State the domain and range for each and then compute the limits at a point and at infinity. Explain what you are doing and write down and explain if these limits exist. Focus on explaining the end behaviors for each function and the differences in their domains.
28.		BK	Domain for cosine is $(-\infty,\infty)$ which are not reachable numbers. Range is [-1,1].

			Well the limit as x goes to 0 for cosine, since it's the same from the left and from the right the limit is 1. It goes to the same point, there are no discontinuities. The limit as x goes to infinity has no set value it settles down at so since it keeps oscillating the limit does not exist. As x goes to pi, the limit is -1 because the left equals the right. You can like move your finger along the x-axis and watch the y-values they range between 1 and -1 and don't stop at any point so that is why the limit does not exist.
29.		MA	For cosine(x), can there be 2 limits, at -1 and 1? Explain.
30.		BK	No. That's just the range. The limit is going to be between these 2 values but you can never anchor down one fixed number.
31.		MA	Graph B: For arccos(x), what's the limit as x approaches plus or minus infinity? Can a limit exist to the right of 1 or to the left of -1? Explain.
32.		BK	As goes to plus or minus infinity I think it keeps continuing on below the arccosine, but based off this piece the domain is fixed between [-1,1]. So I would say since there is no fixed value outside of that as you approach infinity, these limits do not exist. As x goes to infinity there's no value for y in either direction.
33.		MA	What is the limit as x approaches 0, 1 and then -1?
34.		BK	As x goes to 0, the limit is pi/2 because both sides equal the same. If you have an open dot you would not have a limit there, if I remember right. I am not sure though. As x goes to -1, it's a one sided limit so the limit pi. As x goes to 1, it's a one sided limit approaching from the left so the limit is 0. When it's continuous like this you don't need the little superscripts to show limit as x approaches from the left only or from the right only, because we are not evaluating continuity here, only the limit. I'm thinking about continuity at the same time, but the limit would still be pi on the left. The endpoint doesn't have to be in the domain for the limit to exist. Note: <u>Open dot</u> might mean there's not limit there.
35.		MA	Describe the domain and range of arccosine and explain any relationship between the limits at the endpoints and the domain.
36.		BK	Domain x values range filled in from [-1,1] and Range is the y-values and that's filled in [0,pi]. The domain fluctuates between -1 and 1.
37.		MA	Graph C: Can you describe what type of graph the last graph is on the end and describe the domain and limiting behavior of this function as x approaches pi?
38.		BK	The domain is between [0, 2pi] and range is (-1,1] I think. The limit is pi. Note: Wrong limit. Limit is -1, not pi. Confused x and y with this one.
39.		MA	Does the limit exist at a point as x approaches pi?

40.		BK	Yes. The limit does exist, and it equals pi, but it is not continuous. I would say the limit exists because it is an actual number where the closed dot is. I think that's what it was. It's hard to remember. Note: Closed dot meaning "dot". Different words for dots and holes.
41.		MA	Is (pi,0) on the graph of the function? Why or why not?
42.		BK	Yes, depending on what the function is assuming it is one function, then yes because it's a function value.
43.		MA	What's the limit as x goes to infinity for the first and second graphs?
44.		BK	They do not exist because they're not in the domain.
45.		MA	What is an infinite limit? Give examples.
46.		BK	The graph goes off to infinity and so with 1/x as you go to 0 the limit goes to positive infinity. Note: Drew illustrations.
47.		MA	What is the difference between a limit at infinity and an infinite limit?
48.		BK	The infinite limit is when the limit does not exist because it is going to infinity. The limit at infinity is about what x is tending toward, not y.
49.		MA	What might be some relationships between domains and limits? Explain and give examples.
50.		BK	Domain is the input so let's see… the limit is the y value you are approaching. The domain is the given input of the limit, to find the value as the limit approaches is dependent on the domain what is near the x value. The domain is the input. The limit is the value that x approaches given the input in the domain. The limit is dependent upon the domain. Note: Limit depends on the domain.
51.		MA	Begin Task 2: Study the 5 graphs given. State the domain and range for each, compute the limits and explain the behavior of the function values and if the limits exist of not. Explain what you are doing.
52.		BK	For the half circle, the domain is [-1,1]. Range is [0,1]. I plugged in 0 and I got square root of 1 which is 1. I wanted to make sure that was the y value for the range to make sure it actually reaches the 1 value or whether it approaches it. But it looks like I didn't have to think about this so much. Limit as x goes to 1 is 0, as x approaches 0 the limit is 1, and as x approaches -1, the limit is 0. These limits all exist. As x approaches infinity, the limit does not exist because it's an imaginary number. Same thing with minus infinity, the value drops so you get an imaginary number. So the limit does not exist. For arccosine, the domain is [-1,1] and range is [0,pi]. As x goes to 0, the limit is pi/2 because both sides equal the same. If you have an open dot you would not have a limit there, if I remember right. I am

			not sure though. As x goes to -1, it's a one sided limit so the limit pi. As x goes to 1, it's a one sided limit approaching from the left so the limit is 0. When it's continuous like this you don't need the little superscripts to show limit as x approaches from the left only or from the right only, because we are not evaluating continuity here, only the limit. I'm thinking about continuity at the same time, but the limit would still be pi on the left. As x goes to infinity the limit doesn't exist because if it's out here past the domain then it's not going to have an actual y-value. <u>Damped cosine</u>: Domain is all x and the range also looks like it's all y or minus infinity to infinity. The limit as x goes to infinity converges to 0 because the values keep oscillating and get smaller and smaller. I feel like you can turn this into a squeeze theorem value problem and prove that it went to infinity. It depends if it allows you to get to infinity. We never really messed with the squeeze theorem. No matter what, though, the limit is going to be 0. Because as it continues on the gap gets smaller and smaller and it gets closer and closer to 0 each time. If it progresses and goes to infinity where it would stop fluctuating, the average value is still 0 and that is obviously where it is going to stop. As x goes to minus infinity then the limit doesn't exist because it's like the cosine on the left side of the graph it keeps oscillating but gets larger and larger. We had to write a program for this function once and one for a clock that oscillates with the pendulum swinging back and forth then make visual images of it. This reminds me of a problem with springs and dampers in systems dynamics—values of chasse and shock systems on the computer and on paper. We drew it a little differently though. Note: Damped cosine: considered squeeze theorem. <u>For cosine 1/x</u>: the domain is all x-values and the range is [-1,1]. It levels off at 1 when the x's go to infinity either direction, plus or minus. As x approaches 0, with zero in the denominator, the limit does not exist because it keeps fluctuating and oscillating, getting infinitely smaller, not settling down to any particular number. The function may not be defined at zero but the limit is about what is going on near zero, so nobody cares what's in the denominator. As x goes to minus infinity and plus infinity the limit is 1. <u>For arccosine with holes at endpoints</u>: The domain is now (-1,1) and range is (0,pi). It does not include the endpoints. So the limit as x approaches 1 is 0, and as x approaches -1 the limit is pi. The limits exist even though there are open dots because limits are about what happens near the x value, not at x per se. So even though x is not in the domain on either endpoint, the limit still exists. For limits the point doesn't have to be in the domain for the limit to exist, so that's why when there is a hole at the endpoint the limit exists.

53.		MA	Explain precisely the behavior of the function values as x approaches plus or minus infinity for the half circle and arccos(x).
54.		BK	With the half circle, as x goes to plus or minus infinity the limit does not exist because there's not going to be an actual y-value past where the domain stops. With arccosine, the limits don't exist either for the same reasons. As x goes to infinity the limit doesn't exist because if it's out here past the domain then it's not going to have an actual y-value.
55.		MA	Is there any relationship of limits to function values?
56.		BK	Yeah, if there is a continuous function then the limit is the value of the function but if it's discontinuous with a closed dot above the hole then the limit is where the hole is but the value of the function is at the dot.
57.		MA	Look at the damped cosine function and compare the end behaviors to that of the cosine function. Do any of the function values tend toward infinity? Do they tend toward 0? Which limits exist and do not exist as you compare them.
58.		BK	The left side is similar in both problems because as x goes to minus infinity the limit does not exist because the function oscillates and doesn't stop at any specific number. But as x approaches infinity on the right, the cosine function's limit does not exist for the same reason but the damped cosine's limit is 0. If you plugged in input values for x the output values get smaller and smaller.
59.		MA	What relationship do you see between the domains and limits in this problem?
60.		BK	Well these are continuous functions so basically whatever is in the domain is also going to have a limit so the domain doesn't have anything much going on with limits in this one. Note: Interesting. Continuous functions don't yield interesting information about limits.
61.		MA	What is the difference between the arccos x with the closed dots at the endpoints and its counterpart that has holes at the endpoints instead in terms of the domain and the limits? Explain the behavior of the function values and why the limits exist or not.
62.		BK	For arccosine, when there are closed dots at endpoints, the limits exist where x is defined in the domain. However when there are holes at the endpoints, the domain is now (-1,1) and range is (0,pi). It does not include the endpoints. The limits exist even though there are open dots because limits rely on the domain to the extent that you consider what happens near the point x in the domain, not at the point itself. So even though x is not in the domain on either endpoint, the limit still exists.

63.		MA	Give examples of cases in which limits do not exist.
64.		BK	Piecewise ones with the jump discontinuity where there is an open dot above with a horizontal line and a closed dot below with another line. I'm thinking about continuity and I keep mixing it up with continuity. There are 3 types like 1/x where x goes to a point so the limit is infinity. For cosine it fluctuates so the limit doesn't exist and the piecewise ones where it is not continuous.
65.		MA	Limit notation. Explain what the notation means. Does the arrow under the "lim" imply direction from the left?
66.		BK	It means finding the limit from both directions if it's at a point. Or find the limit as x goes to infinity. Maybe that way it's from the left but the arrow doesn't really mean direction here.
67.		MA	Is it invalid to write $x \to \pm\infty$ together or must that be split up?
68.		BK	Yes I've seen it that way so it means you just split them up. . My teachers used plenty of shorthand, yes, it's fine.
69.		MA	Is it true with the limit notation that whatever appears below "lim" tells you what the limit is going to be, so that you don't have to do any math at all? Explain.
70.		BK	No, it only tells you what x is tending toward, not y. That is the formula that specifies finding a precise y-value or a y-value that is in the neighborhood of x.
71.		MA	If you gave somebody a problem $\lim_{x \to 2} 3x+1$ and they computed the limit and got 7, then they said there were 2 limits which are vertical asymptotes at x=2 and x=7, what would you say they are doing wrong here? Anything?
72.		BK	This is insane! Who the hell did this? Man, this is really something. I'd say they have no idea what a limit is and they surely are not trying to find any y-values or what y-values are in the neighborhood of x.
73.		MA	If you have a straight line with a hole in it, would you say that limits exist because they are holes you can fall into and that vertical asymptotes are like brick walls acting like restraints which you cannot go past?
74.		BK	What? Say this again? No, holes don't mean limits exist and vertical asymptotes are not restraints. This person must have a very creative visual interpretation or something.
75.		MA	Begin Task 3: Study the two graphs. First, what kinds of functions are these, are the linear, quadratic, cubic, etc.
76.		BK	The first graph is quadratic, based on the looks of it. There is a hole at 4 and a dot at 6. I know the y=shift is outside and the x-shift is outside for the function. And a negative outside flips the graph down. It has a value of 6 at 2. I know it's not right but pretty close. Note: Wrote the function close to correct, as a piecewise, but called it quadratic. Didn't connect the formula with the graph.
77.		MA	What's the domain and range for the first graph?

78.		BK	Domain: $(-\infty,\infty)$ Range: $(-\infty,4)\cup[6]$ Note: Looks correct.
79.		MA	For the first graph, compute the limits the behavior of the function values. Is (2,6) on the graph of the function? Explain why or why not.
80.		BK	Yes because 6 is the value of the function. There is no value at 4, so 6 is going to be the limit. Note: Incorrect
81.		MA	Explain the behavior of the function values as x approaches 2, and does that limit exist?
82.		BK	The limit as x goes to 2 is 6. It is either 6 or does not exist. If I compare the left with the right and it approaches 4 but the only place there is a value is at 6. Since the left hand equals the right hand I'd say the limit is 6. Something's not right because this graph really got me. Actually now that I think about it, I think the limit might be 4 because the values approach it but don't actually reach it. As they get infinitely close to it, the limit is 4. The function can't actually equal the 4 at that point but equals 6 because it is discontinuous. Note: Confused with continuity. Got the limit wrong but then thinks about it further and corrects himself. Gets the limit correct, limit=4.
83.		MA	In order for the limit to exist, does x have to be in the domain? Explain.
84.		BK	No, it doesn't have to be in general but in this problem, because the function is continuous, x happens to be in the domain.
85.		MA	Explain the behavior of the function values as x approaches plus or minus infinity and explain if these limits exist or not.
86.		BK	As x goes plus or minus infinity, the limit is minus infinity for both.
87.		MA	Can a limit ever be equal to the value of the function? Explain why or why not?
88.		BK	Yes, in this example the limit is 6 and is equal to the value of the function which is also 6. On continuous functions without the holes, too, the limit would equal the value of the function. Note: Limit is not 6, only function value is 6.
89.		MA	What's the domain and range for the second graph?
90.		BK	Domain: (minus infinity, infinity) Range: [-1] U [2] Note: Wrote the correct function.
91.		MA	For the second graph, compute the limits. Does the limit exist as x approaches -2? Explain.
92.		BK	This graph is piecewise. As x goes to 2 the limit does not exist. As x

			goes to 2 from the left is -1 and as x approaches -2 from the right the limit is 2.
93.		MA	Explain the behavior of the function values as x approaches plus or minus infinity, do these limits exist?
94.		BK	As x goes to plus infinity the limit is 2 because the y-value is the same across all x. As x goes to minus infinity the y-value is -1 for the same reason. -1 for all x.
95.		MA	What does the domain have to do with the limits in these two problems, if anything?
96.		BK	Well the domain on the first one is continuous, so the limit can be anywhere in the domain. Even if there was no closed dot and you just had a hole, the limit would still exist even though the point at x=2 was not in the domain. In the second graph, it's a similar situation. The domain is all x but the limit does not exist at 2 because the left doesn't equal the right sides so in this case the limit does not depend on the what's in the domain. Note: I think this interpretation is correct. Not sure about last sentence though.
97.		MA	Begin Task 4: Study these 3 graphs. They all look linear but have some differences as they progress from one to the next. State the domains and ranges for each one of these and compute the limits. Explain the behavior of the function values.
98.			
99.		MA	For the first graph, do the limits exist? Explain.
100		BK	The domain is $(-\infty,\infty)$. Range: $(-\infty,\infty)$ Limit as x approaches 3 is 2, since 2 is in the neighborhood of 3. That's because from the left the limit exists and equals 2 and from the right the limit exists and equals 2 and so since the left equals the right the limit is 2. The limit as x approaches plus infinity is positive infinity and the limit as x approaches minus infinity is negative infinity. Those limits do not exist since infinity cannot be reached. It depends on who the teacher is. The high school teacher said it equals infinity so it exists, that the value exists even though infinity is technically not a real number. The teacher said it depends on if you consider infinity existing or not. But the college professors said the limits don't exist if they go to infinity, and I tend to believe them.
101	56:00	MA	For the second graph, do the limits exist? Explain. Since the limit approaches the same value of 2 from the left and from the right, I'd say the limit is 3. The value exists at 3 but the limit? I think it exists. I have to think about the difference between when a limit exists versus when the value exists.
102	47:00	BK	Domain is $x \neq 2$ $(-\infty,2)\cup(2,\infty)$ Range: $(-\infty,3)\cup(3,\infty)$ As x approaches 3, the limit exists and equals 2 because 2 is in the neighborhood of 3 and the left and limit of 2 equals the right hand limit of 2.

103		MA	For the third graph, do the limits exist? Explain. As x approaches 2, the limit is 3 since it approaches 3 from the left and from the right but the actual value of the function is 4.
104		BK	Domain: $(-\infty,\infty)$ since it has a value at the hole but it not continuous. Range: I'm not sure. Since there is no y-value at that point, I'd say it's the same as this other one $(-\infty,2)\cup(2,\infty)$. For any epsilon band around 3 that I get closer to, I can tighten the band and get close to 2 from the left and from the right, so given epsilon near 3, I can find a delta as x approaches 2. Note: D and R took some time to think about.
105		MA	Is the point (2,4) on the graph of the function? Explain why or why not and mention if the domain plays a role.
106	58:30	BK	Yes because it is still part of the function.
107		MA	Can a limit be equal to the value of the function? Explain.
108		BK	Yeah, just like with the first graph, the limit and the value are the same.
109		MA	Give an example of when the limit and the function value and the function value are not the same.
110		BK	In the third graph or that quadratic one before, the value is above the limit.
111		MA	In order for the limit to exist, does 'a' have to be in the domain? Explain.
112		BK	No. The limit is about what happens near 'a', not at 'a'.
113		MA	Explain how the domains change among the graphs and any relationship you see between domains and functions.
114		BK	The first and last graph has the same domains but the second one doesn't. Even though the domain is different in the second one, the limit still exists at the point because the x-value does not have to be in the domain for the limits to exist. Then in graphs 1 and 3 the limits just happen to exist with x is in the domain. But the limit exists, because it's about what happens in the neighborhood of x, not at x.
115		MA	If you have a function and the domain changes, do you still have the same function? Explain.
116		BK	No. with rational functions, things cross out and you could end up with a linear function and that could be the case of the second graph. Or if you change the domain you could have a piecewise one by just taking out the value of the function and having just a hole.
117		MA	Explain if different functions can have the same domains.
118		BK	Yeah, something like 2x+1, x^2 , x^3 , e^x , knock out x=2 and they all have the same domains. Going to infinity, 2x+1, x^2 , x^3 would have the same domain $(-\infty,\infty)$.
119		MA	Begin Task 5: Look at the function f(x)=2x+1. Compute the limits at a point as shown and at infinity, sketch a graph, explain how you

			graph this, Explain the behavior of the function values and explain if these limits exist.
120		BK	If you look at y=mx+b so you get the slope from that and the y-intercept. . If I don't know this much then I would fail at being an engiheer. You just plug in 3 into the function and then the limit is 7 so I just plug it in. The limit as x approaches 2 is 5. Limit as x approaches infinity is infinity. If you plug in 2 times infinity plus 1 you still get infinity. Same in the negative direction, plug in negative infinity and you get negative infinity. Note: Did a mathematical operation, plugged in infinity into the function. $f(\infty)=2(\infty)+1=\infty$ $f(-\infty)=2(-\infty)+1=-\infty$
121		MA	Is the point (3,8) on the graph of the function? Explain.
122			It is not, because if you plug in 3, you do not get 8. You can actually see it's not on the line of the graph. 2(3)+1 is not equal to 8 so that's why it cannot be on the function. I am not sure what you mean by graph of the function, but I think it means the same as being on the line of the graph. In any event you cannot have 2 y values for the same x value.
123		MA	Does the domain have any involvement with this point (3,8) being on the graph of the function? Explain.
124		BK	No. The (3,8) has to do with this still being a function because there would be 2 values of y for the same x.
125		MA	Begin Task 6: State the domain of the following problems and then compute the limits: $\lim_{x\to 2}\frac{x^2+4}{x-2}$ and $\lim_{x\to\pm\infty}\frac{x^2+4}{x-2}$. Explain what procedures you use for doing each one, if the procedures are the same or different. Explain if you must split up +/- infinity and then explain if these limits exist. Explain the behavior of the function values. Graph your result and label everything.
126			
127		BK	$\lim_{x\to 2}\frac{x^2+4}{x-2}$ Domain is $(-\infty,2)\cup(2,\infty)$. The limit as x goes to 2. You cannot plug in directly. I wish it was a minus in the numerator so it would cancel. As x goes to plus infinity you plug in values greater than 2, you get a smaller number on the bottom so you get infinity. If you pick numbers smaller than 2 on the left you get negative infinity. So pick 1 on the left it's -5 and if you pick 3 on the right you get 13. So since -5 does not equal 13, the limit does not exist. Note: Natural response is still compare the left hand limit with the right hand limits for limits at infinity.
128		BK	$\lim_{x\to\pm\infty}\frac{x^2+4}{x-2}$ As you plug in values, evaluating from the left and right. From the left I plug in a number less than 2, like 1.999.Then

			from the right like 2.0001 and you get a positive on one side and a negative on the other. There is some asymptote at 2 so the limit does not exist as x goes to 2. Then since the left hand side near 2 does not equal the right hand limit near 2, the limit does not exist. As it approaches from the left, it's neg. infinity and from the right it's positive infinity. From the right it is slightly greater than 2, so it will be a very tiny number in the denominator. Then a slightly smaller number on the left and this gives you a slightly smaller number than 2. Note: Did this wrong. Did for limit at a point what should have been done for limit at infinity and vice versa. Switched these up. Graph in calculator done right. This one was a mess.
129		MA	Are you familiar with any rules for finding asymptotes? If so, when do you use them, for limits at a point or for limits at infinity?
130		BK	I remember there were ones like that, where you compare degrees. You use those rules for limits at infinity but I forgot those rules for exponents. But that's easy to find out. If I take 1/x then the numerator is x^0 so there is a horizontal asymptote at 0. OK. I remember. Degrees less in numerator is horizontal asymptote at y=0. Degrees same you use the leading coefficients. If greater in the denominator, it's infinity. So as x goes to negative infinity, the limit is going to be negative infinity and as x goes to positive infinity, the limit is going to be plus infinity but in both cases these limits do not exist so on the graph I will put DNE. Going back to as x approaches 2, those limits don't exist either. As x approaches 2 from the right the limit's plus infinity and from the left it's minus infinity. I am trying to remember the plug-in convention though. Because the numerator has a plus sign you cannot factor it in to (x+2)(x-2). So that's $\frac{\infty^2}{\infty}$. As x approaches 2 from the right, the numerator goes up exponentially the denominator is slower, but the limit goes to plus infinity. For negative infinity you have a negative infinity on the bottom so it's $\frac{(-\infty)^2}{-\infty} = -\infty$ so you get negative infinity as x goes to 2 from the left. If you could factor you'd be left with (x+2) and the ends would be exactly the same. The reason I do math with infinity is not to do math, per se, but just to get an idea what it would look like before I would attempt any math. Instead of just saying "does not exist" without knowing what direction it goes into, positive or negative. Note: Apparently he got confused earlier and now corrected the problem, pointing out limits at a point involved taking values close to 2. Now he is correct in his interpretation of limits at infinity.

			Used algebra to remember the rules for asymptotes. Mathematical operations with infinity.
131		MA	In general, do you use the same techniques for limits at a point and limits at infinity, or are they different? Explain.
132		BK	All I do is plug in values close to the point or pick some further away from the point for infinity. I only factor when it is possible things cancel out. If I remember those rules for exponents, I use those too for limits at infinity. For limits at a point you can just pretty much try to plug in values close to the number or you can try to factor.
133		MA	If the limit equals infinity, does the limit exist?
134	31:30	BK	It's completely opinion based on the person who is asking it. I know a lot of people who would say it doesn't exist and then others that would say it does, my high school calculus teacher for instance said the limit exists if it equals infinity but the college professors said it doesn't. Personally, I think the professors are right and it doesn't exist.
135		MA	For limit at infinity, do you have to compare any left hand limits with right hand limits to decide if the limit exists?
136		BK	No, not for these, only for limits at a point like with 1/x when you get from the left going up and from the right going down you compare minus infinity to plus infinity and then say the limit doesn't exist because they're not equal.
137		MA	If one side tends toward plus infinity and the other side toward minus infinity, do you just use one side to determine if the limit exists or not or do you have to compare $-\infty$ with $+\infty$ and then conclude the limit does not exist?
138		BK	You're supposed to just look at one side when you decide. Like with x-cubed as x goes to plus or minus infinity you only look at one side at a time and decide if the limit exists or not.
139		MA	For limit at a point, here as x approaches 2 do you compare the left hand limit with the right hand limit to decide if the limit exists, or do you just look at one side to decide?
140		BK	You have to look at both sides unless it is a one-sided limit like the arcsine or arccosine.
141		MA	Begin Task 7: State the domain of the following problems and then compute the limits: $\lim_{x\to -3}\frac{x^2-9}{x+3}$ and $\lim_{x\to \pm\infty}\frac{x^2-9}{x+3}$. Explain what procedures you use for doing each one, if the procedures are the same or different. Explain if you must split up +/- infinity and then explain if these limits exist. Explain the behavior of the function values. Graph your result and label everything.
142		BK	Domain: $x \neq 3$
143		BK	$\lim_{x\to -3}\frac{x^2-9}{x+3}$ This one is easier because it factors and so you are left

			with (x-3). So the limit is going to be -6. On the graph, you might have a hole because two of them cancel when they don't cancel you get a vertical asymptote. It cancels into a linear function in this case. Note: Drew graph correctly with the hole at (-3,-6).
144		BK	$\lim_{x\to\pm\infty}\frac{x^2-9}{x+3}$ As x goes to plus infinity the limit is plus infinity and at negative infinity the limit is negative infinity because you have a linear function.
145		MA	If the limit equals infinity, does the limit exist?
146		BK	No. They go to infinity so they don't exist.
147		MA	Do you have to compare any left hand limits with right hand limits to decide if the limit exists at a point or at infinity for this problem? Explain.
148		BK	No, not for limits at infinity.
149		MA	How do you know when you get a vertical asymptote versus when you get a hole in the graph?
150		BK	When things cancel out you get a hole. Vertical asymptotes when you can't factor and you have something in the denominator of a rational function. Maybe with piecewise functions too with discontinuities.
151		MA	How do you know when you would get a hole in a graph? What kind of function might you see and what would occur to get the hole?
152		BK	When things cancel you get a hole. Rational functions where you can cancel things out or piecewise functions have holes.
153	1:21:00	MA	Begin Task 8: $\lim_{x\to\pm\infty}\frac{9x^2+2}{3x^2-2x+5}$ State the domain, explain how you compute the limit explain the behavior of the function values, and sketch a graph of this result.
154		BK	Domain is all x. You can't factor so you can just try plugging in big values or just represent abstractly by plugging in $+\infty$ or $-\infty$ to get a general idea instead of computing with actual numbers. But with this the exponents are the same so you can use those rules and in this case it's the ratio of the leading coefficients so the limit is just going to be 3. Note: Mathematical operations with infinity to get a quick abstract representation.
155		MA	Can any factoring be done with this problem? Explain.
156		BK	No. Nothing factors in the numerator or denominator.
157		MA	Explain if there is any relationship between the domain and the limit?
158		BK	Well the limit can be anywhere in the domain because the function is continuous.

159		MA	Explain if there is any relationship between the limit and the horizontal asymptote.
160		BK	The horizontal asymptote is the limit.
161		MA	Begin Task 9: Study the 4 graphs and the 4 formulas below them. Match the graph to the formulas by just writing down "a, b, c, or d" next to the graphs above them. Explain the behavior of the function values for each. Explain how you select your answer choices.
162		BK	The y-values are always positive so I look at the squared ones, same with those y-values which are negative. So I think graph 3 is A and graph 1 is B. Then I look at the last 2 graphs and try a few x-values, so I think graph 2 is D and graph 4 is C. I just plug in large values of x for 1/x basically and then it switches when you multiply by a negative. Note: Got these correct.
163		MA	Now let's study the first graph, shifted over a units to the right. First state the domain, then compute the limits at a point and at infinity. Explain the behavior of the function values for limit at a point and at infinity. As you do this, explain if any of these limits exist and why.
164		BK	As x goes to infinity and minus infinity the limit is 0. As x approaches a the limit does not exist because the left hand limit does not equal the right hand limit. Note: Limit d.n.e. b/c left does not equal the right for limit at a point.
165		MA	As $x \to a$ do you have to split these up separately from the left and from the right, or is this implied in the notation?
166		BK	No, it's implied in the notation to split them up. You don't have to officially do it though, not unless you have a one-sided limit you have to find.
167		MA	As $x \to a$ do you have to compare the left hand limit with the right hand limit to decide if the limit exists?
168			Yes.
169		MA	As $x \to \infty$ how do you know if the function values go to infinity or if they go to 0? Do they go to both? Explain.
170		BK	The function values go to 0 and that's why the limit is 0. The values get smaller and smaller as x increases.
171		MA	In general, if I draw a line with a solid dot on it, does the limit exist there at the dot?
172		BK	Yes because the function is continuous.
173		MA	If I draw a line with a hole in it, does the limit exist there at the hole?
174		BK	Yes because the limit is about what occurs near x, and that's from the formal definition I gave earlier about limits.
175		MA	Now look at 2 smaller graphs over to the right. The graph from the previous problem in Task 8, versus the graph of 1/x. Explain the

			behavior of the function values and the differences in the end behaviors. Does one go to 3 and the other go to 0 or to plus/minus infinity? Explain.
176		BK	The top one goes to 3, and the other one goes to 0. Neither one goes to infinity. It's the x-values in both of these that go to infinity, not the y-values.
177			
178			
179		MA	Begin Task 10: Study the 3 graphs and we are going to be comparing them in various ways. First state the domain for each, explain the behavior of the function values for each graph, compute the limits and then explain if they exist.
180		BK	Graph 1: Domain: $(-\infty, 2) \cup (2, \infty)$
181		MA	Explain the behavior of the function values as x approaches 0 and if you have to compare the right hand limit to the left hand limit.
182		BK	The limit does not exist because as x approaches 2 from the left the limit is minus infinity and from the right the limit is plus infinity. Since they are not the same the limit does not exist. As x goes to infinity, the limit is 0. The infinity symbol tells you the value you are trying to approach. It's not a number though so it can't exist. Note: Compares left with right end behaviors going to infinity for limit at a point.
183			
184		BK	Graph 2: Domain: $(-\infty, \infty)$
185		MA	Explain the behavior of the function values as x approaches 0 and 2, and as x approaches infinity and if you have to compare the right hand limit to the left hand limit.
186		BK	As x approaches 0, the limit is 0. As x approaches 2 the limit is 4. As x approaches infinity, it goes off to infinity in both cases so it does not exist. They both approach infinity but infinity means it does not exist. You don't have to compare the ends since they are going outwards toward infinity. You only compare the ends for limit a point. Note: Compare ends for limit at a point, whether tending toward a point or away from the point.
187		BK	Graph 3: Domain: $(-\infty, \infty)$
188		MA	Describe the behavior of the function values as x approaches 0 and as x approaches infinity and if you have to compare the right hand limit to the left hand limit.
189		BK	As x approaches 0 the limit is 1. Anything raised to the 0 is 1. As x approaches plus infinity the limit is infinity therefore it does not exist. As x goes to negative infinity is 0. It can't go past 0 because e^x can't ever be negative.

			Note: Knows why e^x can't be negative. Plugs values in to find the limits in addition to looking at the graph.
190			
191		MA	Infinity Symbol: Explain what the infinity symbol means.
192		BK	It represents an unreachable number.
193		MA	Can a limit exist if it equals infinity? Explain why or why not.
194		BK	It depends on who the teacher is. The high school teachers said the limit exists if it equals infinity because it has to be equal to something but the college professors said the limit does not exist and I tend to believe them because infinity is not a precise number and it keeps going without stopping.
195		MA	What are some reasons limits do not exist?
196		BK	The limits equal infinity, the function oscillates like with sine, cosine, tangent. In those cases the limits don't exist as x goes to plus or minus infinity. Those with finite domains like sinx or cosx the limits don't exist either because there are no x-values outside of the domain. I would say functions that are piecewise or have discontinuities would have limits that don't exist at a point.
197		MA	When do you know to write d.n.e. versus equals infinity? What are some examples?
198		BK	d.n.e. for limits that go to infinity, and also for sinx or cosx because they oscillate. For infinity, you can either write "d.n.e." or you can write " $=\infty$ ". For the piecewise ones where there is a hole maybe in the bottom and dot on top you write d.n.e. because that's the only thing that makes sense. Note: Knows when to write d.n.e. exclusively.
199	1:35:00	MA	Thank you Brendon for coming to today's interview.
200		BK	No problem. It was fun.

APPENDIX I-2: Transcript Evidence of JY

Video file name: jean.doc   Running Head: Limit Study   90 Minute Interview
Interviewer: MA   Interviewee: Jean   HDSC References: 0006

Turn	Time	Person	Utterance
1.	00:01	MA	Begin Task 1: Define or describe what a function is and give examples.
2.		JY	Given x you get y. So input and output.
3.		MA	What is a function value?
4.		JY	How it's on the graph, just the y or f(x).
5.		MA	What is the difference between a function and a function value?
6.		JY	A function is the line or the curve, like x-squared. The function value is the output, the y.
7.		MA	What does it mean that a function is 1-1? Is 1-1 part of the definition of function?
8.		JY	For 1 x, you only got 1 y. If given one x, and 2 y's then it is not. So parabola is 1-1 and half circle sideways is not 1-1. Note: Does not know meaning of 1-1.
9.		MA	Define or describe what a limit is and give examples.
10.	2:00	JY	A certain number or amount is approaching (0 or number) but never equals (0 or that number). It approaches but never equals it.
11.		MA	Can limits and function values be the same?
12.		JY	No. You either have a function value, or you have a limit. So with holes from discontinuity, there is a limit and no value. Straight lines or curves with dots, no limit but there is a value.
13.		MA	What's the difference between the limit and the value of the function?
14.		JY	A limit is what the function is approaching. The value of the function is what the function equals for a particular x, so they are two different things.
15.		MA	Do limits pertain to the x-coordinate, y-coordinate or both when you find a limit?
16.		JY	I don't know what you mean, coordinate. I say it is y.
17.		MA	Can a limit be a number?
18.		JY	Yes. A number or infinity sometimes.
19.		MA	Do you say: "The limit equals infinity", or do you say "the function values are approaching infinity"? Explain.
20.		JY	I say the limit equals infinity.
21.		MA	If a limit is equal to infinity, then does that limit exist?
22.		JY	no

23.		MA	What exactly is infinity?
24.		JY	It is too large to count.
25.		MA	What does the infinity symbol mean?
26.		JY	Just the way to represent something too large to count.
27.		MA	Graph A: Study the graphs in Task 1 for cosine, arccosine and for the last graph. State the domain and range for each and then compute the limits at a point and at infinity. Explain what you are doing and write down and explain if these limits exist. Focus on explaining the end behaviors for each function and the differences in their domains.
28.		JY	Domain for cosine is $(-\infty, \infty)$ which are not reachable numbers. Range is [-1,1]. Well the limit as x goes to 0 for cosine, since it's the same from the left and from the right the limit is 1. Because there is no dot there. If it has a dot then I would say the limit does not exist because the limit approaches 1 but can't equal it. As x goes to pi on this graph the limit is -1 because it's continuous, with no dot. Limit as x goes to infinity or minus infinity keeps bouncing up and down so the limit does not exist. As x goes to pi, the limit is -1 because the left equals the right. You can like move your finger along the x-axis and watch the y-values they range between 1 and -1 and don't stop at any point so that is why the limit does not exist. Note: If no dot on contin. function, limit exists. When there is a dot, then the limit does not exist because limits are about what is near, not at, x.
29.		MA	For cosine(x), can there be 2 limits, at -1 and 1? Explain.
30.		JY	No.
31.		MA	Graph B: For arccos(x), what's the limit as x approaches plus or minus infinity? Can a limit exist to the right of 1 or to the left of -1? Explain.
32.		JY	So I would say since there is no fixed value outside of that as you approach infinity, the limit either does not exist or it is pi. As x approaches negative infinity, the limit does not exist because it's not in the domain so this one doesn't have meaning. I say does not exist for both because it has a certain value of y, so it is fixed in this interval and so it is not approaching anything.
33.		MA	What is the limit as x approaches 0, 1 and then -1?
34.		JY	As x approaches 0, the limit is pi/2. As x goes to positive 1, the limit is converging to 0 and as x goes to -1 the limit is converging to pi but these limits do not exist.
35.		MA	Describe the domain and range of arccosine and explain any relationship between the limits at the endpoints and the domain.
36.		JY	The domain is fixed between [-1,1]. Range is [0,pi]. The limit is

			about what x is approaching, not equal to, in the domain. The limits do not exist at the endpoints.
37.		MA	Graph C: Can you describe what type of graph the last graph is on the end and describe the domain and limiting behavior of this function as x approaches pi?
38.		JY	It is part of cosine.
39.		MA	Does the limit exist at a point as x approaches pi?
40.		JY	It exists because when x is pi, and equals -1, because left side and right side are equal.
41.		MA	Is (pi,0) on the graph of the function? Why or why not?
42.		JY	Yes because 0 is the value when x=pi.
43.		MA	What's the limit as x goes to infinity for the first and second graphs?
44.		JY	The limits do not exist.
45.		MA	What is an infinite limit? Give examples.
46.		JY	A very simple one is it goes to infinity at the ends. The limit does not exist so it goes to infinity.
47.		MA	What is the difference between a limit at infinity and an infinite limit?
48.	11:00	JY	Limit at infinity is x. Infinite limit is y. It doesn't exist because it goes to infinity.
49.		MA	What might be some relationships between domains and limits? Explain and give examples.
50.		JY	For arccosine, the domain is [-1,1] so the limit exists because it is about nearness to x—1 and x=1. But the limit does not exist AT x=1 or x=-1 because solid dot means in general the point cannot be included. Also it is not valid to just ask about x approach 1 or x approach -1 because limit does not exist on the sides of the dot at $x \to -1^-$ and $x \to 1^+$
51.		MA	Begin Task 2: Study the 5 graphs given. State the domain and range for each, compute the limits and explain the behavior of the function values and if the limits exist of not. Explain what you are doing.
52.		JY	For the half circle, the domain is [-1,1]. Range is [0,1Limit as x goes to 1 does not exist, as x approaches 0 the value is 1 but the limit does not exist because the limit is about what it is approaching not what it is at. When x is zero, the value is exactly 1. It IS 1, it is not approaching 1. So the limit does not exist. As x approaches -1, the limit does not exist As x approaches infinity, the limit does not exist because there is no x value so problem makes no sense. Same thing with minus infinity so the limit does not exist. Note: Used calculator to get the domain and range to see if range went past -1 and 1. For arccosine, the domain is [-1,1] and range is [0,pi]. As x goes to 0, the limit is pi/2 because both sides equal the same. If you

			have an open dot you would not have a limit there, if I remember right. I am not sure though. As x goes to -1, it's a one sided limit so the limit pi but this limit doesn't exist. As x goes to 1, it's a one sided limit approaching from the left so the limit is 0 but it doesn't exist. Because function values can't be limits. When it's continuous like this you don't need the little superscripts to show limit as x approaches from the left only or from the right only, because we are not evaluating continuity here, only the limit. I'm thinking about continuity at the same time, but the limit would still be pi on the left. As x goes to infinity the limit doesn't exist because if it's out here past the domain then it's not going to have an actual y-value. <u>Damped cosine</u>: Domain is all x and the range also looks like it's minus infinity to infinity. I must use the calculator to plug in values for –x to see where it goes for the domain and range. The limit as x goes to infinity diverges to 0 because the values keep bouncing up and down on left and get smaller and smaller on right. It is becoming less and less but is approaching 0.As x goes to minus infinity then the limit doesn't exist because it's like the cosine on the left side of the graph it keeps bouncing up and down but gets larger and larger. <u>Cosine 1/x</u>: the domain is all x-values and the range is [-1,1]. It levels off at 1. Cosine (0) is 1 so the answer cannot be 0. So as x approaches 0, the limit does not exist because values keep bounding. Limit is about what is near not at 0. As x goes to minus infinity and plus infinity the limit is 1. <u>For arccosine with holes at endpoints</u>: The domain is (-1,1) and range is (0,pi). It does not include the endpoints. So the limit as x approaches 1 is 0 and so it exists and as x approaches -1 the limit is pi and so that limit exists. Even if you change a little bit, everything will be far apart. The limits exist even though there are open dots because limits are about what happens near the x value, not at x per se. So even though x is not in the domain on either endpoint, the limit still exists. The point can't be in the domain. As x approaches negative infinity, the limit does not exist because it's not in the domain so this one doesn't have meaning. I say the limit does not exist for both at plus or minus infinity, because it is fixed in the interval and so it can't approach anything.
53.		MA	Explain precisely the behavior of the function values as x approaches plus or minus infinity for the half circle and arccos(x).
54.		JY	The limits don't exist because there is no x in the domain. So the question makes no sense.

55.		MA	Is there any relationship of limits to function values?
56.		JY	No. If you have a value, the value cannot also be the limit. You either get a value or you get a limit.
57.		MA	Look at the damped cosine function and compare the end behaviors to that of the cosine function. Do any of the function values tend toward infinity? Do they tend toward 0? Which limits exist and do not exist as you compare them.
58.		v	Cosine bounces up and down so both ends don't exist. Damped cosine on left is the same. But on right damped cosine goes to 0 because the values get smaller and smaller. Neither problem goes to infinity at the ends.
59.		MA	What relationship do you see between the domains and limits in this problem?
60.		JY	The limit exists everywhere in the domain where domain is $(-\infty, \infty)$. Outside of domain of [-1,1] for half circle and arccosine there is no domain so no limit.
61.		MA	What is the difference between the arccos x with the closed dots at the endpoints and its counterpart that has holes at the endpoints instead in terms of the domain and the limits? Explain the behavior of the function values and why the limits exist or not. Note: If hole, then limit exists. If solid dot, limit does not exist because limit is about "near" not "at".
62.		JY	Arccosine limit does not exist at ends because limit is what it is approaching, not what it is equal to. But arccosine, the limit exists at endpoints because it's about what it is near.
63.		MA	Give examples of cases in which limits do not exist.
64.		JY	With those going to infinity I would say sine, cosine, tangent because for each period limit is infinity and minus infinity. Step function at each point too because left side doesn't equal right side.
65.		MA	Limit notation. Explain what the notation means. Does the arrow under the "lim" imply direction from the left?
66.		JY	No, it is either, both or none.
67.	14:00	MA	Is it invalid to write $x \to \pm\infty$ together or must that be split up?
68.		JY	No I would never write it that way because I was never taught this way. It is very confusing. Because the graph could be in a way so you can't see plus and minus infinity at the same time so each side could have a different answer.
69.		MA	Is it true with the limit notation that whatever appears below "lim" tells you what the limit is going to be, so that you don't have to do any math at all? Explain.
70.		JY	No. It only gives the direction of x and then you have to still find the limit if one exists.
71.		MA	If you gave somebody a problem $\lim_{x \to 2} 3x+1$ and they computed

			the limit and got 7, then they said there were 2 limits which are vertical asymptotes at x=2 and x=7, what would you say they are doing wrong here? Anything?
72.		JY	They don't know limits are about y-values.
73.		MA	If you have a straight line with a hole in it, would you say that limits exist because they are holes you can fall into and that vertical asymptotes are like brick walls acting like restraints which you cannot go past?
74.		JY	The limit is what it's approaching so with the solid dot, the limit does not exist but with the hole, the limit exists because it is not equal to the value.
75.		MA	Begin Task 3: Study the two graphs. First, what kinds of functions are these, are the linear, quadratic, cubic, etc.
76.		JY	The first one is quadratic with two pieces. The second one is very easy. It is a line with a line. The formula for the first one though is $y = \begin{cases} 6, & x-2 \\ -(x-2)^2, & x \neq 2 \end{cases}$ Note: Wrote correct formula.
77.		MA	What's the domain and rang e for the first graph?
78.		JY	Domain: $(-\infty, \infty)$ everything. Range: $(-\infty, 4) \cup \{6\}$ Note: Curley brackets
79.		MA	For the first graph, compute the limits the behavior of the function values. Is (2,6) on the graph of the function? Explain why or why not.
80.		JY	Yes because 2 is in the domain.
81.		MA	Explain the behavior of the function values as x approaches 2, and does that limit exist?
82.		JY	Limit is 4 and it exists because left side goes to 4 like the right side.
83.		MA	In order for the limit to exist, does x have to be in the domain? Explain.
84.		JY	No. If not defined, limit exists so x cannot be in the domain. That is inside a continuous function with a hole and with arccosine with end points that are holes. So x cannot be in the domain for the limit to exist there. Note: If hole, limit exists and x cannot be in domain for limit to exist.
85.		MA	Explain the behavior of the function values as x approaches plus or minus infinity and explain if these limits exist or not.
86.		JY	They do not exist.
87.		MA	Can a limit ever be equal to the value of the function? Explain why or why not?
88.		JY	Yes if a function is continuous they can be equal for every point.

89.		MA	What's the domain and range for the second graph?
90.		JY	Domain: $(-\infty,\infty)$ Range: {-1} $\cup$ {-2} The formula is $y=\begin{cases}2 & x>-2\\ -1 & x\leq -2\end{cases}$ Note: Curley brackets. Wrote correct formula.
91.		MA	For the second graph, compute the limits. Does the limit exist as x approaches -2? Explain.
92.		JY	No it doesn't exist. You are trying to approach this but cannot. With parabola, it was approaching. With here, it is not. From the left, the limit is approaching -1 and from the right it is approaching 2 and only the limit exists for this, not from the left. Note: This is a mess.
93.		MA	Explain the behavior of the function values as x approaches plus or minus infinity, do these limits exist?
94.			No because it keeps going without stopping. Note: Confused x with y.
95.		MA	What does the domain have to do with the limits in these two problems, if anything?
96.		JY	If the x value is not in the domain, then the limit exists for graph 1 where the hole. Same for the 2nd graph if you look at each line separately, and so for the top one the limit exists where the hole is because x is not in the domain. If x is in the domain, then the limit does not exist. So on the second graph, where the line is on the bottom, it ends with a solid dot. So there is no limit there because x is in the domain. That is how I think about this. Note: Limits and solid dots don't mix.
97.		MA	Begin Task 4: Study these 3 graphs. They all look linear but have some differences as they progress from one to the next. State the domains and ranges for each one of these and compute the limits. Explain the behavior of the function values.
98.			Domain and range is $(-\infty,\infty)$
99.		MA	For the first graph, do the limits exist? Explain.
100.		JY	Domain and range is everything. Limit as x approaches 2 does not exist. As x goes to plus infinity the limit is infinity and for x goes to minus infinity, the limit is minus infinity so they don't exist.
101.		MA	For the second graph, do the limits exist? Explain.
102.		JY	Domain is everything except 2. Limit is 3.
103.		MA	For the third graph, do the limits exist? Explain.
104.		JY	Domain is everything. Range is everything. As x goes to 2, the limit is 3.
105.		MA	Is the point (2,4) on the graph of the function? Explain why or why not and mention if the domain plays a role.

106.		JY	Yes. Because 4 is the value. Limit is 3.
107.		MA	Can a limit be equal to the value of the function? Explain.
108.		JY	No. In first graph, limit does not exist but value is 3. For last graph, limit is 3, value is 4.
109.		MA	Give an example of when the limit and the function value and the function value are not the same.
110.		JY	Arccosine as x approaches 0. The value is pi/2 but the limit does not exist unless you split them up separately from the left and right. And for this last graph here, the limit is 3 but the value is 4 so they are not the same.
111.		MA	In order for the limit to exist, does x have to be in the domain? Explain.
112.		JY	Yes because for limit at infinity for half circle or arccosine, there is no domain past [-1,1] so no limit. For limit at a point, if x is in the domain then there is no limit unless you split it up from approaching from the left and approaching from the right.
113.		MA	Explain how the domains change among the graphs and any relationship you see between domains and functions.
114.		JY	First and last graph domain and range the same. Middle graph domain changes and does not include 2. Relationship between domains and functions? Well with two pieces like in task 3 the first graph, you see that when x is not equal to 2, the function is squared and when x is equal to 2, the function value is 6. So to have a function, the domain must be defined.
115.		MA	If you have a function and the domain changes, do you still have the same function? Explain.
116.		JY	If I start with straight line and then have a hole at some point like (4,5) now the graph is not defined at 4 so I can write the new function with two parts from minus infinity to 4 and from 4 to infinity, so in that case the function changed.
117.		MA	Explain if different functions can have the same domains.
118.		JY	Yes. Continuous functions have same domains, many from minus infinity to infinity like y=x or cosine. Also from 0 to infinity, like log. The last 2 graphs in task 4 have the same domains, too, but they are different functions because the last one has a value and the middle one doesn't have any value.
119.		MA	Begin Task 5: Look at the function f(x)=2x+1. Compute the limits at a point as shown and at infinity, sketch a graph, explain how you graph this, Explain the behavior of the function values and explain if these limits exist.
120.		JY	I graph this by first drawing a graph to show for every rise of 1 unit, I have a slope of 2 units. If slope is 1, I use one box and it looks like this. If slope is 2, I go through 2 boxes and slope is steeper so it looks like this. Every time you do plus 1, you get 2y. With this diagram it is easier for me to solve this problem. Limit as x goes to 3 is 7 but the limit does not exist. As x goes to 2, also limit does not exist. It is not approaching something. As

			x goes to infinity, the limit is infinity and as it goes to minus infinity the limit is minus infinity. Note: Limit does not exist for points on continuous functions.
121.		MA	Is the point (3,8) on the graph of the function? Explain.
122.		JY	No. The line is the function. Since the point is not on the line, it is not part of the function. Note: Line is the function.
123.		MA	Does the domain have any involvement with this point (3,8) being on the graph of the function? Explain.
124.		JY	No. It doesn't matter because the point isn't even on the function.
125.		MA	Begin Task 6: State the domain of the following problems and then compute the limits: $\lim_{x\to 2}\frac{x^2+4}{x-2}$ and $\lim_{x\to\pm\infty}\frac{x^2+4}{x-2}$. Explain what procedures you use for doing each one, if the procedures are the same or different. Explain if you must split up +/- infinity and then explain if these limits exist. Explain the behavior of the function values. Graph your result and label everything.
126.			
127.		JY	$\lim_{x\to 2}\frac{x^2+4}{x-2}$ Domain is $x\neq 2$. First I draw a vertical asymptote at x=2, then tail ends of the function. The limit as x approaches 2, does not exist anywhere. It's just a rough graph. We are taught to draw graphs in high school to get a rough estimate. Asians don't like using calculators so much so I take time doing it by hand. Note: Unique way to think about the problem. Gets it correct, too.
128.	1:03:0 0	JY	$\lim_{x\to\pm\infty}\frac{x^2+4}{x-2}$ As x approaches minus infinity, the limit does not exist and as x approaches plus infinity, the limit does not exist.
129.		MA	Are you familiar with any rules for finding asymptotes? If so, when do you use them, for limits at a point or for limits at infinity?
130.		JY	No, we Asians don't do tricks or mess with calculators. We work it out by hand.
131.		MA	In general, do you use the same techniques for limits at a point and limits at infinity, or are they different? Explain.
132.		JY	For limits at a point, you try to factor. Otherwise you just plug it in. For limits at infinity, you get those from the graph you drew for limit at a point, otherwise you just plug in real large values for x as test points to see.
133.		MA	If the limit equals infinity, does the limit exist?

134.		JY	no
135.		MA	For limit at infinity, do you have to compare any left hand limits with right hand limits to decide if the limit exists?
136.		JY	no
137.		MA	If one side tends toward plus infinity and the other side toward minus infinity, do you just use one side to determine if the limit exists or not or do you have to compare $-\infty$ with $+\infty$ and then conclude the limit does not exist?
138.		JY	You look at one side at a time. You don't compare these.
139.		MA	For limit at a point, here as x approaches 2 do you compare the left hand limit with the right hand limit to decide if the limit exists, or do you just look at one side to decide?
140.		JY	Yes, you must compare the left and right sides to see if it is approaching the same number, not equaling the same number.
141.		MA	Begin Task 7: State the domain of the following problems and then compute the limits: $\lim_{x\to -3}\frac{x^2-9}{x+3}$ and $\lim_{x\to \pm\infty}\frac{x^2-9}{x+3}$. Explain what procedures you use for doing each one, if the procedures are the same or different. Explain if you must split up +/- infinity and then explain if these limits exist. Explain the behavior of the function values. Graph your result and label everything.
142.		JY	Domain: $(-\infty,3)\cup(3,\infty)$
143.		JY	$\lim_{x\to -3}\frac{x^2-9}{x+3}$ The limit doesn't exist because it is equal to -6.
144.			$\lim_{x\to \pm\infty}\frac{x^2-9}{x+3}$
145.		MA	If the limit equals infinity, does the limit exist?
146.		JY	No.
147.		MA	Do you have to compare any left hand limits with right hand limits to decide if the limit exists at a point or at infinity for this problem? Explain.
148.		JY	No. Why compare it? I never thought about doing it that way because you can't see two sides at a time.
149.		MA	How do you know when you get a vertical asymptote versus when you get a hole in the graph?
150.		JY	I know this because of the domain, the denominator cannot be 0.
151.		MA	How do you know when you would get a hole in a graph? What kind of function might you see and what would occur to get the hole?
152.		JY	A hole happens when top and bottom cancel out. Those are in rational functions.
153.		MA	Begin Task 8: $\lim_{x\to \pm\infty}\frac{9x^2+2}{3x^2-2x+5}$ State the domain, explain how

			you compute the limit explain the behavior of the function values, and sketch a graph of this result.
154.		JY	Domain: $(-\infty,\infty)$
155.		MA	Can any factoring be done with this problem? Explain.
156.		JY	No. It is very hard to draw a picture like this, too. It takes time. I can try factoring out an x step by step to see if it factors. I know I will get a horizontal line at 3 because the smaller terms all go to 0 and you are left with 3. Square divided by square is 1. But I must check to how the left goes to 3 and how the right goes to 3. If it is -1000 it is gonna be more than 3. If it is 1000 it is gonna be less than 3. But both are around 3.
157.		MA	Explain if there is any relationship between the domain and the limit?
158.		JY	The limit can exist everywhere because the domain is everything. A function with domain $(-\infty,2)\cup(2,\infty)$ has no limit when x is near 2 for 1/(x-2). If a function has a comain 9-3,19) it has fewer choices to have y approaching plus/minus infinity. For half circle, it makes no sense to ask about x<-3 or x>19 because the x is not in the domain for that.
159.		MA	Explain if there is any relationship between the limit and the horizontal asymptote.
160.		JY	The limit in this problem is also a horizontal asymptote.
161.	1:25:00	MA	Begin Task 9: Study the 4 graphs and the 4 formulas below them. Match the graph to the formulas by just writing down "a, b, c, or d" next to the graphs above them. Explain the behavior of the function values for each. Explain how you select your answer choices.
162.		JY	1B, 2D 3A 4C. The first thing we do is 1/x. If negative it goes like this, flipped. You just remember one so you can change the minus sign.
163.		MA	Now let's study the first graph, shifted over a units to the right. First state the domain, then compute the limits at a point and at infinity. Explain the behavior of the function values for limit at a point and at infinity. As you do this, explain if any of these limits exist and why.
164.		JY	As x goes to infinity the limit is 0. Same for minus infinity, the limit is 0. Those limits exist. As x goes to a, the limit does not exist. Domain is $x \neq a$
165.		MA	As $x \to a$ do you have to split these up separately from the left and from the right, or is this implied in the notation?
166.		JY	Yes. As x approaches a from the left, the limit is minus infinity so it doesn't exist. Same from the left, it goes to plus infinity so it does not exist.
167.		MA	As $x \to a$ do you have to compare the left hand limit with the right hand limit to decide if the limit exists?
168.		JY	Yes.
169.		MA	As $x \to \infty$ how do you know if the function values go to infinity

			or if they go to 0? Do they go to both? Explain.
170.		JY	They get smaller so they go to 0.
171.		MA	In general, if I draw a line with a solid dot on it, does the limit exist there at the dot?
172.		JY	No, because the limit only exists separately from the left and from the right, but not together because the limit is about what single value it is approaching, not what it equals. Note: Noteworthy
173.		MA	If I draw a line with a hole in it, does the limit exist there at the hole?
174.		JY	Yes. The limit exists separately from the left and from the right, and together because the limit is about what single value it is approaching, so it is approaching the same value from both sides. Note: Limits don't exist for continuous functions, only for discontinuous ones.
175.		MA	Now look at 2 smaller graphs over to the right. The graph from the previous problem in Task 8, versus the graph of 1/x. Explain the behavior of the function values and the differences in the end behaviors. Does one go to 3 and the other go to 0 or to plus/minus infinity? Explain.
176.		JY	One goes to 3 and the other goes to 0 because the values keep getting smaller. I think about the values, not just looking at the graph. Note: Not so with the piecewise function though.
177.			
178.			
179.		MA	Begin Task 10: Study the 3 graphs and we are going to be comparing them in various ways. First state the domain for each, explain the behavior of the function values for each graph, compute the limits and then explain if they exist.
180.		JY	Graph 1: Domain: $x \neq 2$
181.		MA	Explain the behavior of the function values as x approaches 2 and if you have to compare the right hand limit to the left hand limit.
182.		JY	As x approaches 2, limit equals infinity so it does not exist. I don't compare left hand side with right hand side for this. Just one side at a time.
183.			
184.		JY	Graph 2: Domain: $(-\infty, \infty)$
185.		MA	Explain the behavior of the function values as x approaches 0 and 2, and as x approaches infinity and if you have to compare the right hand limit to the left hand limit.
186.		JY	Right side goes to positive infinity and left to positive infinity, so

			neither one exists. As x approaches 2, the limit is 4 but it does not exist. 1.9999 on left, 2.00001 on right so I only think open hole is a limit, so not here. If there is a solid dot, there is no limit. Note: Limit equals 4 but does not exist.
187.		JY	Graph 3: Domain: $(-\infty,\infty)$
188.		MA	Describe the behavior of the function values as x approaches 0 and as x approaches infinity and if you have to compare the right hand limit to the left hand limit.
189.		JY	As x approaches 0, the limit does not exist. As x approaches minus infinity, the limit is 0 because the values get smaller and smaller.
190.			
191.		MA	Infinity Symbol: Explain what the infinity symbol means.
192.		JY	It goes to an amount that you cannot count. It is even more than a large number because it is so large you cannot even tell.
193.		MA	Can a limit exist if it equals infinity? Explain why or why not.
194.		JY	No.
195.		MA	What are some reasons limits do not exist?
196.		JY	Left side and right side approach separately like approaching a solid dot. Then don't exist because keep bouncing up and down like sine and cosine, and damped cosine. If limits are infinity, they don't exist either.
197.		MA	When do you know to write d.n.e. versus equals infinity? What are some examples?
198.		JY	D.N.E. when there is a solid dot on a line because x is approaching separately from the left and right sides. Also when there is jump discontinuity like in Task 3 the 2nd graph. Also for cosine and for sine as x goes to infinity or minus infinity, and tangent as x goes to plus or minus infinity for each period.
199.		MA	Thank you for coming to today's interview.
200.		JY	I like this. Let me know if you need more help.

APPENDIX I-3: Transcript Evidence of LA

Video file name: linsey.doc   Running Head: 90 Minute Interview

Interviewer: MA   Interviewee: Linsey   HDSC References: 0008

Turn	Time	who	Utterance
1.	00:01	MA	Begin Task 1: Define or describe what a function is and give examples.
2.		LA	F(x)=x^2. I just think of a graph.
3.		MA	What is a function value?
4.		LA	When you plug in numbers for x and you get one out. It's the y.
5.		MA	What is the difference between a function and a function value?
6.		LA	A function is the thing itself and the function value is what you get for y.
7.		MA	What does mean that a function is 1-1? Is 1-1 part of the definition of function?
8.		LA	I've heard this. That there is only 1 x that can equal the y. Maybe a point on a graph is an example.
9.		MA	Define or describe what a limit is and give examples.
10.		LA	A limit is when you see if the left side is equal to the right side. I don't know how to explain it other than that. There is no limit if there is a hole.
11.		MA	Can limits and function values be the same?
12.		LA	Yes, I think so. If there is a dot there, not a hole.
13.		MA	What's the difference between the limit and the value of the function?
14.		LA	It's the same when the left equals the right only if there is a dot there. If there is a hole, there is no limit.
15.		MA	Do limits pertain to the x-coordinate, y-coordinate or both when you find a limit?
16.		LA	Both kind of. You start with x and end up with y.
17.		MA	Can a limit be a number?
18.		LA	Yes.
19.		MA	Do you say: "The limit equals infinity", or do you say "the function values are approaching infinity"? Explain.
20.		LA	I say the limit equals infinity. It takes too long with function values. I would get confused saying all that and since you are finding a limit not a function value, I would just say the limit equals infinity.
21.		MA	If a limit is equal to infinity, then does that limit exist?
22.		LA	Yes.
23.		MA	What exactly is infinity?
24.		LA	Infinity is a place that has no end to it. So if a limit equals infinity then infinity is where the limit would exist.
25.		MA	What does the infinity symbol mean?
26.		LA	It just means it keeps going and there is no end to it. There is no

			assigned number, it just keeps going.
27.		MA	Graph A: Study the graphs in Task 1 for cosine, arccosine and for the last graph. State the domain and range for each and then compute the limits at a point and at infinity. Explain what you are doing and write down and explain if these limits exist. Focus on explaining the end behaviors for each function and the differences in their domains.
28.		LA	Domain $(-\infty, \infty)$ Range [-1,1] Limit of cosine as x goes to 0 is 1. As x goes to plus infinity the limit doesn't exist because it could be anything it's a whole bunch of points and it keeps going forever so it's hard to determine where it is going when it goes to infinity. If I plugged in 3 pi it's that number -1 but for 4 pi it is up there at 1 so the left doesn't equal the right because it keeps going. The same thing as x goes to minus infinity. As x goes to pi, the limit is -1.
29.		MA	For cosine(x), can there be 2 limits, at -1 and 1? Explain.
30.		LA	Well no, you can only have 1 limit at a time. There would be 2 limits if you look at 2 problems like limit as x goes to 0 and limit as x goes to pi, then you have 2 limits.
31.		MA	Graph B: For arccos(x), what's the limit as x approaches plus or minus infinity? Can a limit exist to the right of 1 or to the left of -1? Explain.
32.		LA	Domain is [-1,1] Range is [0,pi]. No the limit does not exist outside of 1 and -1 because there aren't any x-values out there to put into the function to find the limit.
33.		MA	What is the limit as x approaches 0, 1 and then -1?
34.		LA	As x goes to 0 the limit is pi/2. As x goes to -1 the limit is pi. As x goes to 1, the limit is 0.
35.		MA	Describe the domain and range of arccosine and explain any relationship between the limits at the endpoints and the domain.
36.			
37.		MA	Graph C: Can you describe what type of graph the last graph is on the end and describe the domain and limiting behavior of this function as x approaches pi?
38.		LA	Domain is [0,pi/2].
39.		MA	Does the limit exist at a point as x approaches pi?
40.		LA	I kind of want to say the limit does not exist because it is not continuous. Note: Incorrect conclusion that the limit does not exist.
41.		MA	Is (pi,0) on the graph of the function? Why or why not?
42.		LA	No because it is not on the function. Note: Incorrect.
43.		MA	What's the limit as x goes to infinity for the first and second graphs?
44.		LA	They don't exist.

45.		MA	What is an infinite limit? Give examples.
46.		LA	I don't remember this term at all. Note: Infinite limit unknown to her.
47.		MA	What is the difference between a limit at infinity and an infinite limit?
48.		LA	I think they're the same thing.
49.		MA	What might be some relationships between domains and limits? Explain and give examples.
50.		LA	If x is not in the domain then the function is not continuous therefore the limit does not exist there. For something like a parabola where all x's are in the domain, then the limits exist everywhere. Note: Reasons are not quite true.
51.		MA	Begin Task 2: Study the 5 graphs given. State the domain and range for each, compute the limits and explain the behavior of the function values and if the limits exist of not. Explain what you are doing.
52.	8:50	LA	<u>Half Circle</u>: The domain is $x^2 - 1 \geq 0, x \geq \sqrt{1}$. The range is [0,1]. As x goes to 0 the limit is 1. As x goes to 1 the limit is 0 and as x goes to -1 the limit is also 0. As x goes to infinity or minus infinity, the limits do not exist. You can't go past -1 or past 1. The limit as x is going to infinity doesn't exist and same for minus infinity, it doesn't exist. Note: Transcribed the formula wrong so came up with wrong domain. Did not check by looking at the domain visually. <u>Arccosine</u>: The domain is [-1,1] and range is (0,pi). Limit as x goes to 0 is pi/2. Limit as x goes to -1 is pi, and limit as x goes to 1 is 0. Limit as x goes to infinity or minus infinity does not exist. <u>Damped cosine</u>: The domain is $(-\infty, \infty)$ Range is also $(-\infty, \infty)$ As x goes to plus infinity the limit is 0. As x goes to negative infinity the limit does not exist because it keeps oscillating. <u>Cosine 1/x</u>: the domain $(-\infty, \infty)$ Range [-1,1] As x goes to minus infinity and to infinity the limit is 1. As x goes to 0 the limit doesn't exist. Because you can't pinpoint any point on there. It goes crazy by 0. If you plug in 0 it's undefined but close to 0 there are all different numbers. <u>Arccosine with holes on the ends</u>: the domain is (-1,1) and the limit as x goes to minus infinity doesn't exist because of the hole and neither does the limit as x goes to positive infinity. There is no limit because of the hole I think.
53.		MA	Explain precisely the behavior of the function values as x approaches plus or minus infinity for the half circle and arccos(x).

54.		LA	The limits do not exist. You can't go past -1 or past 1 because it's outside of the domain.
55.		MA	Is there any relationship of limits to function values?
56.		LA	They're the same when you have a dot in the graph like with the half circle as x goes to 0, the limit is 1 and so is the function value.
57.		MA	Look at the damped cosine function and compare the end behaviors to that of the cosine function. Do any of the function values tend toward infinity? Do they tend toward 0? Which limits exist and do not exist as you compare them.
58.		LA	The domain is $(-\infty,\infty)$ Range is also $(-\infty,\infty)$ As x goes to plus infinity the limit is 0. As x goes to negative infinity the limit does not exist because it keeps oscillating.
59.		MA	What relationship do you see between the domains and limits in this problem?
60.		LA	The x values have to be in the domain for the limit to exist. Note: Wrong---x does not have to be in the domain for a limit to exist.
61.		MA	What is the difference between the arccos x with the closed dots at the endpoints and its counterpart that has holes at the endpoints instead in terms of the domain and the limits? Explain the behavior of the function values and why the limits exist or not.
62.		LA	Since -1 and 1 are not in the domain, the limit doesn't exist for the arccosine with the holes. When there are solid dots for endpoints the limits exist because -1 and 1 are in the domain. Note: Interesting.
63.		MA	Give examples of cases in which limits do not exist.
64.			Cosine and sine.
65.		MA	Limit notation. Explain what the notation means. Does the arrow under the "lim" imply direction from the left?
66.		LA	It means find the limit of some function. No, the arrow does not mean from the left. It just means approaching some number x.
67.		MA	Is it invalid to write $x \to \pm\infty$ together or must that be split up?
68.		LA	I have never seen it like that.
69.		MA	Is it true with the limit notation that whatever appears below "lim" tells you what the limit is going to be, so that you don't have to do any math at all? Explain.
70.		LA	No, it tells you what the x is going to be when you are looking for the y-value. So it tells you the first part of what you need to know to later find the y.
71.		MA	If you gave somebody a problem $\lim_{x\to 2} 3x+1$ and they computed the limit and got 7, then they said there were 2 limits which are vertical asymptotes at x=2 and x=7, what would you say they are doing wrong here? Anything?
72.		LA	They don't think any y-values are in this.
73.		MA	If you have a straight line with a hole in it, would you say that

			limits exist because they are holes you can fall into and that vertical asymptotes are like brick walls acting like restraints which you cannot go past?
74.		LA	No. Limits don't exist where there are holes so there is nothing to fall into. Vertical asymptotes actually are like brick walls, though, because the function can't go past it and that's why the function is not continuous. Note: Brick wall idea for vertical asymptote resemble's Carrie's reasoning.
75.		MA	Begin Task 3: Study the two graphs. First, what kinds of functions are these, are the linear, quadratic, cubic, etc.
76.		LA	The first graph is quadratic. I think it's got an x-squared in it for the function but I don't know. I'll say x^2 + 2, and it is moved up 2, so I think you have to multiply it by 2. It's something like that. The second graph is linear maybe. Note: Has a hard time with the domain and range, and with coming up with the function.
77.		MA	What's the domain and range for the first graph?
78.	18:25	LA	Domain: $(-\infty,\infty)$ Range: $(-\infty,6)$, $x \neq 3$ I think for the range you have to do 2 of them. If x=2, you have $(-\infty,6)$ and then you do something separately for if x is greater than or less than 2 where the curve is. So if x=2, and if x>2 and x<2. I don't know what to do. I'm kind of thinking maybe it's not a function. I mean because it is just a point. I think it's not a function because if you draw a vertical line through it there is nothing at the hole. It's not like it hits something, so that one makes it OK. I looks like it passes the vertical line test because it's not hitting anything at the hole. Maybe range has something to do with x not being equal to 3. Note: X not being equal to 3 refers to the y-value of 3 not being on the graph. Interesting way to write the range.
79.		MA	For the first graph, compute the limits the behavior of the function values. Is (2,6) on the graph of the function? Explain why or why not.
80.		LA	The limit as x goes to 2 does not exist because it is not continuous. (2,6) is on the graph of the function because if I plugged in 2, you should get 6 and that is definitely not right. So maybe the function is really x^2 + 3 because if you plug in x=2, you get 6. Ha ha ha…that works! Note: Incorrect
81.		MA	Explain the behavior of the function values as x approaches 2, and

			does that limit exist?
82.		LA	The limit does not exist as x approaches 2 because of the hole.
83.		MA	In order for the limit to exist, does x have to be in the domain? Explain.
84.		LA	Yes, it has to be in the domain otherwise if there is a hole there it's discontinuous and so the limit doesn't exist.
85.		MA	Explain the behavior of the function values as x approaches plus or minus infinity and explain if these limits exist or not.
86.		LA	I think as x goes to plus infinity the limit exists and is negative infinity. As x goes to negative infinity the limit exists because it's negative infinity.
87.		MA	Can a limit ever be equal to the value of the function? Explain why or why not?
88.		LA	Yes it can when you have a straight line or any function like the arccosine where you ask what's the limit as x approaches -1 the limit is pi and pi is also a function value.
89.		MA	What's the domain and range for the second graph?
90.		LA	Domain: $(-\infty,\infty)$ Range: [-1,2] The graph is linear. Note: Range not accurate for this piecewise function.
91.		MA	For the second graph, compute the limits. Does the limit exist as x approaches -2? Explain.
92.		LA	Well it is linear. As x goes to 2 the limit does not exist because as you go to- 2 from the left the limit is -1 and as you go to -2 from the right the limit is 2.
93.		MA	Explain the behavior of the function values as x approaches plus or minus infinity, do these limits exist?
94.		LA	As x goes to infinity the limit is 2 as x goes to minus infinity the limit is 1.
95.		MA	What does the domain have to do with the limits in these two problems, if anything?
96.		LA	Well for the first graph, x is not in the domain so the limit does not exist where the hole is. For the second graph, the limit does not exist even though x is in the domain only in the lower part of the graph. So sometimes you have to look at the domain like when you decide if a limit exists or not. Note: Thinks x has to be in the domain for limits to exist.
97.		MA	Begin Task 4: Study these 3 graphs. They all look linear but have some differences as they progress from one to the next. State the domains and ranges for each one of these and compute the limits. Explain the behavior of the function values.
98.		LA	All three graphs are linear I would say.
99.		MA	For the first graph, do the limits exist? Explain.
100.		LA	The domain is $(-\infty,\infty)$ and range is also $(-\infty,\infty)$ The limit as x

			goes to 2 from the left is 3 and also from the right is 3. As x goes to plus infinity the limit exists and is plus infinity and as x goes to minus infinity the limit exists and is minus infinity.
101.		MA	For the second graph, do the limits exist? Explain.
102.		LA	The domain is $x \neq 2$ The limit as x goes to 2 does not exist because there is nothing there. The point x=2 is not the domain. So I will put DNE because of the hole. I think if there is a hole then it doesn't exist.
103.		MA	For the third graph, do the limits exist? Is (2,4) on the graph of the function? Can you write the formula for the function? Explain.
104.		LA	The domain is $(-\infty, \infty)$ and range is $(-\infty, \infty)$ As x approaches 2, the limit does not exist because of the hole. (2,4) on the graph but it is not on the function or graph of the function. The formula is f(x)=(x-1)+1 I think. I confuse what goes first, the shift up or down.
105.		MA	Is the point (2,4) on the graph of the function? Explain why or why not and mention if the domain plays a role.
106.		LA	Yes because it's on the graph but just not on the function. It's just some random point. It's got nothing to do with the linear function since it's above the function. If you slid it down or whatever, then it would be part of the function. Note: Interesting.
107.		MA	Can a limit be equal to the value of the function? Explain.
108.		LA	Yes, like in the first graph where x is 2, both the function's value is 3 and the limit is also 3.
109.		MA	Give an example of when the limit and the function value and the function value are not the same.
110.		LA	I don't know. I think they have to be the same all the time. Note: Interesting.
111.		MA	In order for the limit to exist, does x have to be in the domain? Explain.
112.		LA	Yes because otherwise the limit won't exist if there is a hole there. Yeah, x=2 has to be in the domain. If there is a hole in the line, that means nothing's there so there is nothing in the domain and no limit.
113.		MA	Explain how the domains change among the graphs and any relationship you see between domains and functions.
114.		LA	Well looking at these, I would say the domains change and so by getting rid of x in the 2nd graph, the domain changed so that means that x has to be in the domain for a limit to exist. Note: Incorrect
115.		MA	If you have a function and the domain changes, do you still have the same function? Explain.
116.		LA	Yeah, pretty much. From looking at these 3 graphs, you have the same function because it's still linear. All you are doing with the second one though is getting rid of a point but yeah, I'd say it is still linear so the function does not change if you change the

			domain. Note: Incorrect.
117.		MA	Explain if different functions can have the same domains.
118.		LA	Well sine and cosine have the same domains $(-\infty,\infty)$ and so does this linear function because it's also $(-\infty,\infty)$. I don't know if at a point different functions can have the same domain. That's hard to kind of think about. Note: Interesting.
119.		MA	Begin Task 5: Look at the function f(x)=2x+1. Compute the limits at a point as shown and at infinity, sketch a graph, explain how you graph this, Explain the behavior of the function values and explain if these limits exist.
120.		LA	First I want to graph this so I can answer the limits. If you plug in 0, you get 1. Rise over run for slope. The y intercept is 1. If I plug in 2, I get (2,5) as a point. If I plug in -1, I get -1. As x approaches 3, the limit is 7 because you plug 3 in. As x approaches 2, the limit is 5. As x approaches infinity the limit is infinity, and as x approaches minus infinity the limit is minus infinity.
121.		MA	Is the point (3,8) on the graph of the function? Explain.
122.		LA	No because if you plug in 3 you get 7, not 8.
123.		MA	Does the domain have any involvement with this point (3,8) being on the graph of the function? Explain.
124.		LA	No, I don't think so. 3 has to be in the domain and if it is, you can only have one y-value.
125.		MA	Begin Task 6: State the domain of the following problems and then compute the limits: $\lim_{x\to 2}\frac{x^2+4}{x-2}$ and $\lim_{x\to\pm\infty}\frac{x^2+4}{x-2}$. Explain what procedures you use for doing each one, if the procedures are the same or different. Explain if you must split up +/- infinity and then explain if these limits exist. Explain the behavior of the function values. Graph your result and label everything.
126.			
127.	30:35	LA	$\lim_{x\to 2}\frac{x^2+4}{x-2}$ The domain for this one is $x\neq 2$. I can try to factor it . It doesn't work because nothing is going to cancel. So I'd say that the limit just does not exist
128.		LA	$\lim_{x\to\pm\infty}\frac{x^2+4}{x-2}$. As x goes to plus infinity it's going to infinity because if you plug in large numbers, you get infinity/infinity. As x goes to negative infinity, you get negative infinity also because you can plug in minus infinity. You just plug infinity in and you get an idea without computing anything. Note: Mathematical operations with infinity.
129.		MA	Are you familiar with any rules for finding asymptotes? If so, when do you use them, for limits at a point or for limits at infinity?

130.		LA	I know there is something you do with the first number, I think that's asymptotes.
131.		MA	In general, do you use the same techniques for limits at a point and limits at infinity, or are they different? Explain.
132.		LA	For the limits as a point you have to factor but you can't with limits at infinity. Instead you just plug in big numbers or just plug infinity in to see what it's doing.
133.		MA	If the limit equals infinity, does the limit exist?
134.		LA	Yes.
135.		MA	For limit at infinity, do you have to compare any left hand limits with right hand limits to decide if the limit exists?
136.		LA	No. You only do that with infinities for limits at a point. Note: You compare left with right infinities for limits at a point. Very common trend.
137.		MA	If one side tends toward plus infinity and the other side toward minus infinity, do you just use one side to determine if the limit exists or not or do you have to compare $-\infty$ with $+\infty$ and then conclude the limit does not exist?
138.		LA	Well if they equal infinity on either side, the limit exists and for limits at infinity, I don't compare left and right. I only do that for things like 1/x at a point where the infinities go off into two different directions. So I guess that's the thing here, the limits actually do exist if they equal infinity. Note: Helpful comment which distinguishes between when she compares left with right for infinity problems.
139.		MA	For limit at a point, here as x approaches 2 do you compare the left hand limit with the right hand limit to decide if the limit exists, or do you just look at one side to decide?
140.		LA	Yes, you must compare both sides. If they're opposite then the limit doesn't exist. If they are the same, like both positive, then the limit exists. Note: Compares left with right infinities.
141.		MA	Begin Task 7: State the domain of the following problems and then compute the limits: $\lim_{x\to -3}\frac{x^2-9}{x+3}$ and $\lim_{x\to \pm\infty}\frac{x^2-9}{x+3}$. Explain what procedures you use for doing each one, if the procedures are the same or different. Explain if you must split up +/- infinity and then explain if these limits exist. Explain the behavior of the function values. Graph your result and label everything.
142.		LA	Domain: d.n.e. no, wait… it is $(-\infty,\infty)$
143.		LA	$\lim_{x\to -3}\frac{x^2-9}{x+3}$ This one factors and the top and bottom cancels out,

			so we have left (x-3). If you plug in -3 you get -6 which is the limit.
144.		LA	$\lim_{x \to \pm\infty} \frac{x^2 - 9}{x+3}$ As x goes to plus infinity the limit is infinity and if x goes to negative infinity the limit is negative infinity. On the graph, here is -3 and -6 and you put a dot there.
145.		MA	If the limit equals infinity, does the limit exist?
146.		LA	Yes. Note: Important.
147.		MA	Do you have to compare any left hand limits with right hand limits to decide if the limit exists at a point or at infinity for this problem? Explain.
148.		LA	For limit at a point, yes you compare the left with right and you get a limit of -6. At infinity, you don't have to compare them because you do each one separately since they're out in the other direction. Only when it's infinity at a point that you worry about comparing the left infinity with right infinity to see if they are the same.
149.		MA	How do you know when you get a vertical asymptote versus when you get a hole in the graph?
150.	35:2 3	LA	A vertical asymptote is when there is nothing there. You draw a line up and down because of the denominator. I don't know about the dots versus the holes. Note: Does not know when to get dots or holes.
151.		MA	How do you know when you would get a hole in a graph? What kind of function might you see and what would occur to get the hole?
152.		LA	I have no idea.
153.		MA	Begin Task 8: $\lim_{x \to \pm\infty} \frac{9x^2 + 2}{3x^2 - 2x + 5}$ State the domain, explain how you compute the limit explain the behavior of the function values, and sketch a graph of this result.
154.		LA	The domain is $(-\infty, \infty)$
155.		MA	Can any factoring be done with this problem? Explain.
156.		LA	No. The exponents are the same so you take the number out front and then the limit is 3 because the number is 9/3. As x goes to infinity, the function values keep going to infinity. Same thing as x goes to negative infinity the function values are going to negative infinity. No wait. It keeps going to 3. Note: Take the number out front. Not connecting the limit with the horizontal asymptote at all. Must be following x, not y. She caught her mistake.
157.		MA	Explain if there is any relationship between the domain and the

			limit?
158.	40:04	LA	The domain is $(-\infty, \infty)$ and you get the same limit of 3 across the whole domain.
159.		MA	Explain if there is any relationship between the limit and the horizontal asymptote.
160.		LA	The limit turns out to be the horizontal asymptote. It's the same thing even though I never thought about it like this before. You learn the asymptote thing in algebra but in calculus they show you the limit but they change the wording. Note: Algebra and calculus use different wording.
161.		MA	Begin Task 9: Study the 4 graphs and the 4 formulas below them. Match the graph to the formulas by just writing down "a, b, c, or d" next to the graphs above them. Explain the behavior of the function values for each. Explain how you select your answer choices.
162.		LA	Graph 1 is B, graph 2 is D, graph 3 is A, and graph 4 is C. Note: Correct.
163.		MA	Now let's study the first graph, shifted over a units to the right. First state the domain, then compute the limits at a point and at infinity. Explain the behavior of the function values for limit at a point and at infinity. As you do this, explain if any of these limits exist and why.
164.		LA	The domain is $x \neq a$. As x goes to infinity the limit is 0 because the lines get closer to 0. Same for minus infinity, the limit is 0. As x approaches a, the limit does not exist though.
165.		MA	As $x \to a$ do you have to split these up separately from the left and from the right, or is this implied in the notation?
166.		LA	Yeah.
167.		MA	As $x \to a$ do you have to compare the left hand limit with the right hand limit to decide if the limit exists?
168.		LA	Yeah. The left limit equals minus infinity an it exists, but the right limit equals plus infinity and that exists so you have to compare the left with the right and since minus infinity does not equal plus infinity the whole limit as x approaches a does not exist. Back up there with Graph 2, the 1/x^2 graph you also would have to compare the right with the left since it's a limit at a point and since plus infinity on the left is the same as plus infinity on the right, then you say that the limit exist. Note: Compares left with right limits for limit at a point even though each limit is going to infinity.
169.		MA	As $x \to \infty$ how do you know if the function values go to infinity or if they go to 0? Do they go to both? Explain.
170.		LA	The limit is going to be 0 because the numbers on the y-axis are

			getting smaller.
171.		MA	In general, if I draw a line with a solid dot on it, does the limit exist there at the dot?
172.		LA	Yes.
173.		MA	If I draw a line with a hole in it, does the limit exist there at the hole?
174.		LA	No.
175.		MA	Now look at 2 smaller graphs over to the right. The graph from the previous problem in Task 8, versus the graph of 1/x. Explain the behavior of the function values and the differences in the end behaviors. Does one go to 3 and the other go to 0 or to plus/minus infinity? Explain.
176.		LA	I think people get x and y confused because they look at it and think it keeps going to infinity in the bottom one instead of seeing that is it going to 0. That's how I was thinking of the 1/x graph at first, that the limit was going to infinity but I had to keep looking at it to see the limit was 0. It's easy to look at the graph and see the arrow keep going to infinity.
177.			
178.			
179.		MA	Begin Task 10: Study the 3 graphs and we are going to be comparing them in various ways. First state the domain for each, explain the behavior of the function values for each graph, compute the limits and then explain if they exist.
180.		LA	Graph 1: Domain: $x \neq 2$
181.		MA	Explain the behavior of the function values as x approaches 2 and if you have to compare the right hand limit to the left hand limit.
182.		LA	As x goes to 2, the limit does not exist because of the asymptote. But also they are both limits are going to infinity because of the asymptote so it doesn't make sense. As x goes to infinity, the limits though exist because both equal positive infinity. but the limits really don't exist because the function is not continuous. You don't have to even look at the ends of the arrows though when you have a vertical asymptote because the asymptote tells you the function is not continuous. Note: Problems here with deciding how to decide if limits exist or not. Changes mind.
183.			
184.		LA	Graph 2: Domain: $(-\infty, \infty)$
185.		MA	Explain the behavior of the function values as x approaches 0 and 2, and as x approaches infinity and if you have to compare the right hand limit to the left hand limit.
186.		LA	As x approaches 0 the limit is 0. As x approaches 2 the limit is 4. As x approaches infinity, the limit exists and equals infinity. Then you compare the end arrows. Since infinity on the left equals infinity on the right, then the limit exists.

187.		LA	Graph 3: Domain: $(-\infty, \infty)$
188.		MA	Describe the behavior of the function values as x approaches 0 and as x approaches infinity and if you have to compare the right hand limit to the left hand limit.
189.		LA	The limit as x approaches 0 is 1. As x goes to minus infinity the limit is 0 and as x goes to plus infinity the limit is plus infinity. Since 0 does not equal infinity, the limit does not exist. Or it only exists on one side, 0. You have to compare the two sides though I think.
190.			
191.		MA	Infinity Symbol: Explain what the infinity symbol means.
192.			It just means it keeps going and there is no end to it. It could be a number really large that you can think of like exponential. There is no assigned number, it just keeps going. Infinity doesn't equal a number.
193.	50:45	MA	Can a limit exist if it equals infinity? Explain why or why not.
194.		LA	Yeah. The limit exists. He he he!! If you take one half of the graph 1/x and look at x approaches a, I say that the limit exists because it keeps approaching infinity.
195.		MA	What are some reasons limits do not exist?
196.		LA	When infinity on the left does not equal infinity on the right for a limit at a point. Also for sine and cosine functions as x goes to infinity because they keep oscillating.
197.		MA	When do you know to write d.n.e. versus equals infinity? What are some examples?
198.		LA	You write d.n.e. for sine and cosine functions as x is going to infinity and then you can write it for a linear function like in task 4 where x goes to plus infinity vs minus infinity.
199.		MA	Well this concludes the interview, Linsey. Thanks for your time and help!
200.	1:05:20	LA	You are welcome. It was fun!

APPENDIX I-4: Transcript Evidence of AK

Video file name: Amanda.doc Running Head: 90 Minute Interview

Interviewer: MA Interviewee: Amanda HDSC References: 0004

Turn	Time	who	Utterance
1.	00:01	MA	Begin Task 1: Define or describe what a function is and give examples.
2.		AK	When you have a graph, and you can do the vertical line test. When you have x=2 and x=3 and that's not a function because there are 2 x's. A drawing of a function is this, a sideway half circle, this a parabola and this, a horizontal line. Note: Said 2 x's so not a function but drew graphs right.
3.		MA	What is a function value?
4.		AK	What x equals. Note: About x
5.		MA	What is the difference between a function and a function value?
6.		AK	When 2 points come together like that, if there is a dot or a hole, I think.
7.		MA	What does mean that a function is 1-1? Is 1-1 part of the definition of function?
8.		AK	It's a 1-1 ratio, I don't remember.
9.		MA	Define or describe what a limit is and give examples.
10.		AK	A limit is when 2 points come together.
11.		MA	Can limits and function values be the same?
12.		AK	No. A limit is what you get if something approaches a value but the function value is what x is approaching. Note: Doesn't understand meaning of function value.
13.		MA	What's the difference between the limit and the value of the function?
14.		AK	A limit is y. A function value is x.
15.		MA	Do limits pertain to the x-coordinate, y-coordinate or both when you find a limit?
16.		AK	I think it's x. No, maybe it's y? Isn't it about as x approaches then f(x) is something and that's really y?
17.		MA	Can a limit be a number? Note: Incorr. Def of lim
18.		AK	Yes sometimes. Other times it's infinity. Note: Incorr. work

19.		MA	Do you say: “The limit equals infinity”, or do you say “the function values are approaching infinity”? Explain.
20.		AK	The limit equals infinity.
21.		MA	If a limit is equal to infinity, then does that limit exist?
22.		AK	Usually, yes. John T said no in the first chapter but later said yes it exists later on in the course. Note: References J.T.
23.		MA	What exactly is infinity?
24.		AK	It’s not a reachable number, it is just a symbol. It represents “forever”.
25.		MA	What does the infinity symbol mean?
26.		AK	It means the numbers go on forever so there is no particular number for the answer.
27.		MA	Graph A: Study the graphs in Task 1 for cosine, arccosine and for the last graph. State the domain and range for each and then compute the limits at a point and at infinity. Explain what you are doing and write down and explain if these limits exist. Focus on explaining the end behaviors for each function and the differences in their domains.
28.		AK	The domain is gonna be, well the range is [1,-1]. The domain is $(-\infty, \infty)$ The limit as x approaches 0 for the equation cosine x I would guess does not exist. As x goes to pi, the limit does not exist but maybe it’s -1. As x approaches infinity, the limit exists because it equals infinity. As x goes to negative infinity the limit exists which is negative infinity because it keeps going and doesn’t stop.
29.		MA	For cosine(x), can there be 2 limits, at -1 and 1? Explain.
30.		AK	Yes because it bounces off -1, then bounces off 1, then bounces again off -1 so it keeps bouncing off these 2 numbers so I think these would both be limits. Note: Similar to Carrie.
31.		MA	Graph B: For arccos(x), what’s the limit as x approaches plus or minus infinity? Can a limit exist to the right of 1 or to the left of -1? Explain.
32.		AK	As x approaches positive infinity, the limit is 1. As x goes to minus infinity, the limit is -1.
33.		MA	What is the limit as x approaches 0, 1 and then -1?
34.		AK	As x goes to 0, the limit is pi/2. As x goes to -1, the limit is pi. And as x goes to 1, the limit is 0.
35.		MA	Describe the domain and range of arccosine and explain any relationship between the limits at the endpoints and the domain.
36.		AK	The domain is [-1,1]. The range is [0,pi].
37.		MA	Graph C: Can you describe what type of graph the last graph is on the end and describe the domain and limiting behavior of this function as x approaches pi?

38.		AK	It's discontinuous.
39.		MA	Does the limit exist at a point as x approaches pi?
40.		AK	Yes, it exists everywhere up and down, if I drew a vertical line. Note: Used vertical line to show limit exists at x=pi.
41.		MA	Is (pi,0) on the graph of the function? Why or why not?
42.		AK	Yes because it is on the x-axis.
43.		MA	What's the limit as x goes to infinity for the first and second graphs?
44.		AK	Positive one for both of them.
45.		MA	What is an infinite limit? Give examples.
46.		AK	Maybe it's one that goes to infinity? A picture would be this, a straight line with an arrow going off to infinity.
47.		MA	What is the difference between a limit at infinity and an infinite limit?
48.		AK	The limit at infinity is where x goes. The infinite limit is where y goes.
49.		MA	What might be some relationships between domains and limits? Explain and give examples.
50.		AK	If you are getting a limit, then what you are getting has to be in the domain otherwise you won't find it.
51.		MA	Begin Task 2: Study the 5 graphs given. State the domain and range for each, compute the limits and explain the behavior of the function values and if the limits exist of not. Explain what you are doing.
52.		AK	<u>Half Circle</u>: [-1.1] Range [0, 1]. The limit as x goes to -1 and to 1 is 0. As x goes to infinity or minus infinity the limit is 1 and -1. As x goes to 0, the limit is 1. <u>Arccosine</u>: The domain is [-1,1]. The range is [0,pi]. As x goes to minus 1, the limit is pi. As x goes to 1, the limit is 0. As x goes to minus infinity, the limit is -1 and to positive infinity the limit is 1. <u>Damped cosine</u>: The domain and range is $(-\infty, \infty)$ As x approaches infinity the limit is gonna be closer to 0 so the limit is 0. As x goes to minus infinity, the limit is gonna be negative infinity. Note: Confused x with y for damped cosine function, and also got the domain wrong. <u>Cosine 1/x</u>: The domain and range is [-1,1] As x approaches 0, the limit is gonna be hard because I don't understand 1/x. I guess the limit does not exist because of 0 in the denominator. As x goes to infinity, the limit is infinity. As x

			goes to minus infinity, the limit is gonna be infinity. Well, what is it going to, 2 here on the x-axis? Wait. As x goes to infinity, the limit is 1 and the other way it's -1because I am just looking at where it stops on the x-axis and can't go any further. Arccosine with holes: The domain is (-1,1) and the limit does not exist as x approaches 1 and -1 because it is not in the domain. As x approaches infinity and minus infinity, the limit won't exist either because of the hole, it's not in the domain.
53.		MA	Explain precisely the behavior of the function values as x approaches plus or minus infinity for the half circle and arccos(x).
54.		AK	As x goes to infinity or minus infinity the limit is 1 and -1.
55.		MA	Is there any relationship of limits to function values?
56.		AK	Yes. The function value is where x is going, and the limit is the y value that you get. Note: Does not understand meaning of "function value".
57.		MA	Look at the damped cosine function and compare the end behaviors to that of the cosine function. Do any of the function values tend toward infinity? Do they tend toward 0? Which limits exist and do not exist as you compare them.
58.		AK	As x approaches infinity the limit is gonna be closer to 0 so the limit is 0. As x goes to minus infinity, the limit is gonna be negative infinity. No, the limit does not tend toward 0 because it keeps going and doesn't stop.
59.		MA	What relationship do you see between the domains and limits in this problem?
60.		AK	Well the function is continuous so there are limits existing everywhere in the domain. Note: Interesting.
61.		MA	What is the difference between the arccos x with the closed dots at the endpoints and its counterpart that has holes at the endpoints instead in terms of the domain and the limits? Explain the behavior of the function values and why the limits exist or not.
62.		AK	Since they only approach one side, then the limit does not exist because when there is a hole you have to use the finger test to compare the negative side with the positive side and since it's missing a side, then the limit can't exist there. The domain basically helps tell you if your limit will be included, so for arccosine the domain is [-1,1] and so the limit is included in it. When you have a missing hole at the ends, it tells you the domain can't be there. Note: Interesting.

63.		MA	Give examples of cases in which limits do not exist.
64.		AK	When you use the finger test and the two fingers don't meet then it doesn't exist and with that arccosine one with the ends missing the limit won't exist either. Note: Interesting.
65.		MA	Limit notation. Explain what the notation means. Does the arrow under the "lim" imply direction from the left?
66.		AK	Well, no. Unless it has a little minus sign thing then it's from the left.
67.		MA	Is it invalid to write $x \to \pm\infty$ together or must that be split up?
68.		AK	I haven't seen it like that before. It's not legal.
69.		MA	Is it true with the limit notation that whatever appears below "lim" tells you what the limit is going to be, so that you don't have to do any math at all? Explain.
70.		AK	No, it just tells you what x is going to be from either direction but you usually still have to do something with math otherwise there wouldn't be any reason to make us learn this stuff.
71.		MA	If you gave somebody a problem $\lim_{x \to 2} 3x+1$ and they computed the limit and got 7, then they said there were 2 limits which are vertical asymptotes at x=2 and x=7, what would you say they are doing wrong here? Anything?
72.		AK	I would say they started off right but they ended up wrong because the 7 is supposed to be on the y-axis.
73.		MA	If you have a straight line with a hole in it, would you say that limits exist because they are holes you can fall into and that vertical asymptotes are like brick walls acting like restraints which you cannot go past?
74.		AK	It depends but no. If there's a hole then the limit can't exist there. When there is a dot and the two sides are equal then it does.
75.	23:30	MA	Begin Task 3: Study the two graphs. First, what kinds of functions are these, are the linear, quadratic, cubic, etc.
76.		AK	The 2^{nd} graph is piecewise. The first one is I don't know it's name. It's quadratic because it is like parabola. The dot's there for some unknown reason.
77.		MA	What's the domain and range for the first graph?
78.		AK	Domain: $(-\infty, 2) \cup (2, \infty)$ Range: $(-\infty, 6]$
79.		MA	For the first graph, compute the limits the behavior of the function values. Is (2,6) on the graph of the function? Explain why or why not.
80.		AK	No. Because it's up in the air. I feel like they did not push it together down to where the hole is. It's like a hole in the graph messes things up..
81.		MA	Explain the behavior of the function values as x approaches 2, and does that limit exist?

82.		AK	No, because there is a hole.
83.		MA	In order for the limit to exist, does x have to be in the domain? Explain.
84.		AK	Yes or there is no limit.
85.		MA	Explain the behavior of the function values as x approaches plus or minus infinity and explain if these limits exist or not.
86.		AK	I stretch the graph out and I don't know if it exists or not. I stretch out the graph for a visual aid. It doesn't exist because it's not going anywhere.
87.		MA	Can a limit ever be equal to the value of the function? Explain why or why not?
88.		AK	No. a limit is a limit and a value of the function is the value of the function. They're two different things.
89.		MA	What's the domain and range for the second graph?
90.		AK	Domain: not sure Range: not sure
91.		MA	For the second graph, compute the limits. Does the limit exist as x approaches -2? Explain.
92.		AK	If I look at -2 coming from the left, and look at -2 coming from the right. So It does not exist because they're not the same.
93.		MA	Explain the behavior of the function values as x approaches plus or minus infinity, do these limits exist?
94.		AK	It's negative something and to the right it's positive something. Whether or not it exists depends what chapter we were in. In algebra, it's no. In calculus, the answer is yes that it exist. But they don't teach you anything so I guess it's yes, the limit exists depending on the teacher error. Note: Important
95.		MA	What does the domain have to do with the limits in these two problems, if anything?
96.		AK	If you don't know the domain, you can still get the limit so the domain probably doesn't mean anything. Note: Interesting
97.		MA	Begin Task 4: Study these 3 graphs. They all look linear but have some differences as they progress from one to the next. State the domains and ranges for each one of these and compute the limits. Explain the behavior of the function values.
98.		AK	The domain is $(-\infty, \infty)$
99.		MA	For the first graph, do the limits exist? Explain.
100		AK	As x goes to 2 the limit is 3. As x goes to infinity, the limit exists and is infinity and as it goes to negative infinity the limit exists is gonna be negative infinity.
101		MA	For the second graph, do the limits exist? Explain.
102		AK	The domain is gonna change because there is a hole. Um, it's

			gonna be $(-\infty, 2) \cup (2, \infty)$. The limit doesn't exist because of the hole. As x goes to positive infinity the limit exists because it equals infinity.
103		MA	For the third graph, do the limits exist? Can you write the formula for the function? Explain.
104		AK	The domain is gonna be $(-\infty, 2) \cup (2, \infty)$. The limit as x approaches 2 is 4, where the solid dot is above the hole. There is no limit at the hole but there is one where the dot is because the dot is not a hole. The formula is f(x)=x-1.
105		MA	Is the point (2,4) on the graph of the function? Explain why or why not and mention if the domain plays a role.
106		AK	No. Because it's not within the line. Not on the line.
107		MA	Can a limit be equal to the value of the function? Explain.
108		AK	No because a limit is y and the value of the function is x.
109		MA	Give an example of when the limit and the function value and the function value are not the same.
110		AK	If you have a parabola and x=2 for x^2. The function value is 2 and the limit is 4.
111		MA	In order for the limit to exist, does x have to be in the domain? Explain.
112		AK	Yes because you have to have an x to compute y.
113		MA	Explain how the domains change among the graphs and any relationship you see between domains and functions.
114		AK	The domains of the last 2 were the same because the hole meant that a point is missing, it didn't include 2 so 2 was not in the domain. But in the first graph, the domain was across the board all the x values.
115		MA	If you have a function and the domain changes, do you still have the same function? Explain.
116		AK	Yes, in this case the function is still linear even with holes or no holes.
117		MA	Explain if different functions can have the same domains.
118		AK	Yes. Cosine and x-squared and 2x+1 all have the same domains.
119		MA	Begin Task 5: Look at the function f(x)=2x+1. Compute the limits at a point as shown and at infinity, sketch a graph, explain how you graph this, Explain the behavior of the function values and explain if these limits exist.
120		AK	As x approaches 3, the limit is you plug it in and get 7. As x goes to 2, the limit is gonna be 5. As x goes to infinity, the limit is infinity, and for the negative infinity one it's negative infinity. To draw a graph, I just plug in 3 across 7 up. Then 2 across, 5 up and draw the line. The formula probably says something, like it's positive just like the graph, shifted left 1 unit because of the y-intercept. So slope is 2 and the y-intercept is 1 just like what I drew. Note: Wrong conclusion about y-intercept as it's not shifted left.

121		MA	Is the point (3,8) on the graph of the function? Explain.
122		AK	Yes because it is right above the other one.
123	40:30	MA	Does the domain have any involvement with this point (3,8) being on the graph of the function? Explain.
124		AK	Yes. It is in the domain because it has an x-value.
125		MA	Begin Task 6: State the domain of the following problems and then compute the limits: $\lim_{x\to 2}\frac{x^2+4}{x-2}$ and $\lim_{x\to\pm\infty}\frac{x^2+4}{x-2}$. Explain what procedures you use for doing each one, if the procedures are the same or different. Explain if you must split up +/- infinity and then explain if these limits exist. Explain the behavior of the function values. Graph your result and label everything.
126			
127		AK	$\lim_{x\to 2}\frac{x^2+4}{x-2}$ The domain is $(-\infty,2)\cup(2,\infty)$. You can factor the numerator but it doesn't work. So you just plug in 2. When you do, you get 4/0 so the limit does not exist.
128		AK	$\lim_{x\to\pm\infty}\frac{x^2+4}{x-2}$ The limit is gonna be plus infinity as x goes to plus infinity, and minus infinity as x goes to minus infinity. Note: Calculator only.
129		MA	Are you familiar with any rules for finding asymptotes? If so, when do you use them, for limits at a point or for limits at infinity?
130		AK	I forgot, I don't remember what to do really. When the exponents are the same it's the leading coefficient. That's all I remember. But looking at the graph, as x goes to infinity, the limit is infinity. As x goes to negative infinity, the limit is negative infinity.
131		MA	In general, do you use the same techniques for limits at a point and limits at infinity, or are they different? Explain.
132		AK	I have no idea. I always hated limits.
133		MA	If the limit equals infinity, does the limit exist?
134		AK	It depends on the teacher and what chapter we're learning. If a limit equals infinity on one side, it exists.
135		MA	For limit at infinity, do you have to compare any left hand limits with right hand limits to decide if the limit exists?
136		AK	Yes because if they are the same, you say the limit exists and equals either one or the other, like plus infinity. If they are different, you just say left doesn't equal the right.
137		MA	If one side tends toward plus infinity and the other side toward minus infinity, do you just use one side to determine if the limit exists or not or do you have to compare $-\infty$ with $+\infty$ and then conclude the limit does not exist?
138		AK	You have to compare both ends and if they are different then you say the left side is different from the right side and so the limit

			does not exist.
139		MA	For limit at a point, here as x approaches 2 do you compare the left hand limit with the right hand limit to decide if the limit exists, or do you just look at one side to decide?
140		AK	If they are different then you say the left side is different from the right side and so the limit does not exist. One by itself exists.
141		MA	Begin Task 7: State the domain of the following problems and then compute the limits: $\lim_{x\to -3}\frac{x^2-9}{x+3}$ and $\lim_{x\to \pm\infty}\frac{x^2-9}{x+3}$. Explain what procedures you use for doing each one, if the procedures are the same or different. Explain if you must split up +/- infinity and then explain if these limits exist. Explain the behavior of the function values. Graph your result and label everything.
142		AK	Domain: $(-\infty,3)\cup(3,\infty)$
143			$\lim_{x\to -3}\frac{x^2-9}{x+3}$ You can factor this one So the limit is gonna be -6. There is probably a hole or some crap. When stuff cancels out you get a hole. There is probably some baby hole you can't see. I was taught how to compute, not do the dumb graph. Note: Uses calculator to graph. Drew hole in wrong place at (0,-3)
144	50:00	AK	$\lim_{x\to \pm\infty}\frac{x^2-9}{x+3}$ The limit is plus infinity and minus infinity. Note: Calculator.
145		MA	If the limit equals infinity, does the limit exist?
146			It depends on what the teacher's mood is, but I think the last thing we learned in calculus was yes, the limit exists.
147		MA	Do you have to compare any left hand limits with right hand limits to decide if the limit exists at a point or at infinity for this problem? Explain.
148		AK	Yes you compare right and left hand limits with both types, at a point and at infinity.
149		MA	How do you know when you get a vertical asymptote versus when you get a hole in the graph?
150		AK	You get a hole when things cancel out. You get a vertical asymptote when nothing cancels out and you are stuck with something easy in the denominator that clearly is undefined if you can make it equal 0.
151		MA	How do you know when you would get a hole in a graph? What kind of function might you see and what would occur to get the hole?
152		AK	When stuff cancels out you get a hole. That's what they said in

			algebra.
153		MA	Begin Task 8: $\lim_{x\to\pm\infty}\frac{9x^2+2}{3x^2-2x+5}$ State the domain, explain how you compute the limit explain the behavior of the function values, and sketch a graph of this result.
154		AK	Domain: $(-\infty,\infty)$ Note: Calculator to graph
155		MA	Can any factoring be done with this problem? Explain.
156		AK	I don't know. It's too hard to find out. But here is the graph and it levels off at 3.
157		MA	Explain if there is any relationship between the domain and the limit?
158		AK	The limit's everywhere because the domain is everywhere.
159		MA	Explain if there is any relationship between the limit and the horizontal asymptote.
160		AK	There probably is. The limit is 3 and so is the horizontal asymptote.
161		MA	Begin Task 9: Study the 4 graphs and the 4 formulas below them. Match the graph to the formulas by just writing down "a, b, c, or d" next to the graphs above them. Explain the behavior of the function values for each. Explain how you select your answer choices.
162		AK	3 is A. 1 is B. 2 is D. and 4 is C because it is positive and negative. Note: Put each answer choice in the calculator to graph.
163		MA	Now let's study the first graph, shifted over a units to the right. First state the domain, then compute the limits at a point and at infinity. Explain the behavior of the function values for limit at a point and at infinity. As you do this, explain if any of these limits exist and why.
164		AK	The domain: As x goes to infinity, the limit is infinity and so it exists, and the limit exists and is minus infinity the other way. Note: Perceives it different than end behave of cosine; said appr. 0.
165		MA	As $x \to a$ do you have to split these up separately from the left and from the right, or is this implied in the notation?
166		AK	It is implied in the notation what you're supposed to do.
167		MA	As $x \to a$ do you have to compare the left hand limit with the right hand limit to decide if the limit exists?
168		AK	Yes because they go in 2 different directions so in each direction the limit exists since it equals infinity, but comparing them together it doesn't exist because one is plus infinity and the other is minus infinity. You compare both ends to see if the limit exists.
169		MA	As $x \to \infty$ how do you know if the function values go to infinity

			or if they go to 0? Do they go to both? Explain.
170		AK	They kind of go to both, but it mostly keeps going to infinity because it never reaches 0. Note: Lim=inf, incorrect
171		MA	In general, if I draw a line with a solid dot on it, does the limit exist there at the dot? Note: Hole vs. dot.
172		AK	Yes.
173		MA	If I draw a line with a hole in it, does the limit exist there at the hole?
174		AK	No.
175		MA	Now look at 2 smaller graphs over to the right. The graph from the previous problem in Task 8, versus the graph of 1/x. Explain the behavior of the function values and the differences in the end behaviors. Does one go to 3 and the other go to 0 or to plus/minus infinity? Explain.
176		AK	Yeah, the top graph the graph is going to 3, but down below here for 1/x the graph is going to infinity. Maybe it's infinity at the ends. It can't go to 0 because it doesn't exist at 0. Note: Limit does not exist because the function doesn't exist at 0.
177			
178			
179		MA	Begin Task 10: Study the 3 graphs and we are going to be comparing them in various ways. First state the domain for each, explain the behavior of the function values for each graph, compute the limits and then explain if they exist.
180		AK	Graph 1: Domain: x not equal to 2. It's an exponential function. As x approaches 2, the limit equals infinity on both sides and so it exists. The limit exists as x approaches infinity and is equal to infinity and as it approaches negative infinity, the limit exists and is also negative infinity.
181		MA	Explain the behavior of the function values as x approaches 2 and if you have to compare the right hand limit to the left hand limit.
182		AK	You plug in 2 and get 0, so you do 1.999 and 2.0001 to see what it's doing. Do 1.9 – 2 and square it. Do 1.99 – 2. Then do 1.999-2. You are going to positive infinity. Under 2, you do 2.1, 2.01, so take 2.1-1 and then 2.01-2 and you also get plus infinity. The limit exists and equals positive infinity. Since both sides are infinity, the limit exists
183			
184		AK	Graph 2: Domain: $(-\infty, \infty)$ It's a parabola. Note: Graph is not exponential

185		MA	Explain the behavior of the function values as x approaches 0 and 2, and as x approaches infinity and if you have to compare the right hand limit to the left hand limit.
186		AK	As x goes to 0, the limit is 0. As x goes to 2, the limit is 4. As x goes to infinity, the limit is infinity and it exists. As it goes to minus infinity, the limit is also gonna be infinity. I would say since the limits equal infinity, the limits exist. Note: Now the limit exists and = ∞
187		AK	Graph 3: Domain: $(-\infty, \infty)$
188		MA	Describe the behavior of the function values as x approaches 0 and as x approaches infinity and if you have to compare the right hand limit to the left hand limit.
189		AK	As x goes to infinity, the limit is infinity and it exists. As x goes to minus infinity, the limit is minus infinity and so it exists.
190			
191		MA	Infinity Symbol: Explain what the infinity symbol means.
192		AK	There is no number because you can always add 1 to it and still get infinity. It's not a number, it is just a symbol. It represents "forever". If you 2x^2 divided by 2, you can plug in infinity for x to get a good idea. But you really can't cancel them. Note: Some arithmetic with infinity.
193		MA	Can a limit exist if it equals infinity? Explain why or why not.
194		AK	Yes and no. It depends on what chapter we are learning from. John Taylor said sometimes it exists, and sometimes it doesn't.
195		MA	What are some reasons limits do not exist?
196		AK	If you use the finger test if the negative side does not equal the positive side.
197		MA	When do you know to write d.n.e. versus equals infinity? What are some examples?
198	1:20:08	AK	I was taught by John Taylor and he taught this finger thing, where you would take the negative side and the positive side and see if it would equal the same number. If it did not equal the same number, then it didn't exist. If something goes to infinity out in the air, see I don't know if it exists or not I don't know. He only taught us how to find out by equations not from graphs. He taught us that if the limit goes to infinity, then it doesn't exist but it does. It doesn't exist but it's a limit so not always. In the first chapter Taylor would say the limit did not exist, but later on he said it did. It just depends what chapter we were learning from.
199		MA	Thanks Amanda for coming to today's study.
200	1:20:10	AK	You're welcome.

APPENDIX J: IMAGES OF STUDENTS WORK

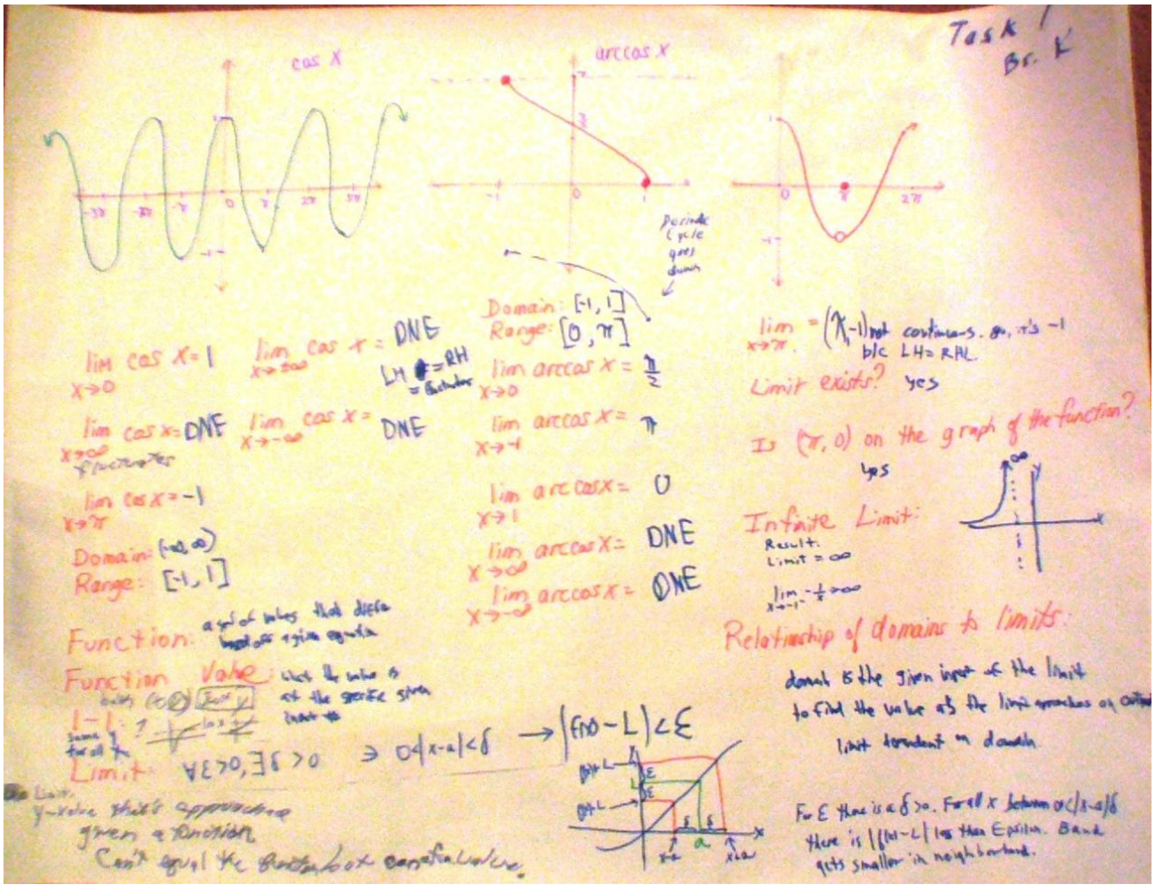
APPENDIX J-1: Images of BK'S Work

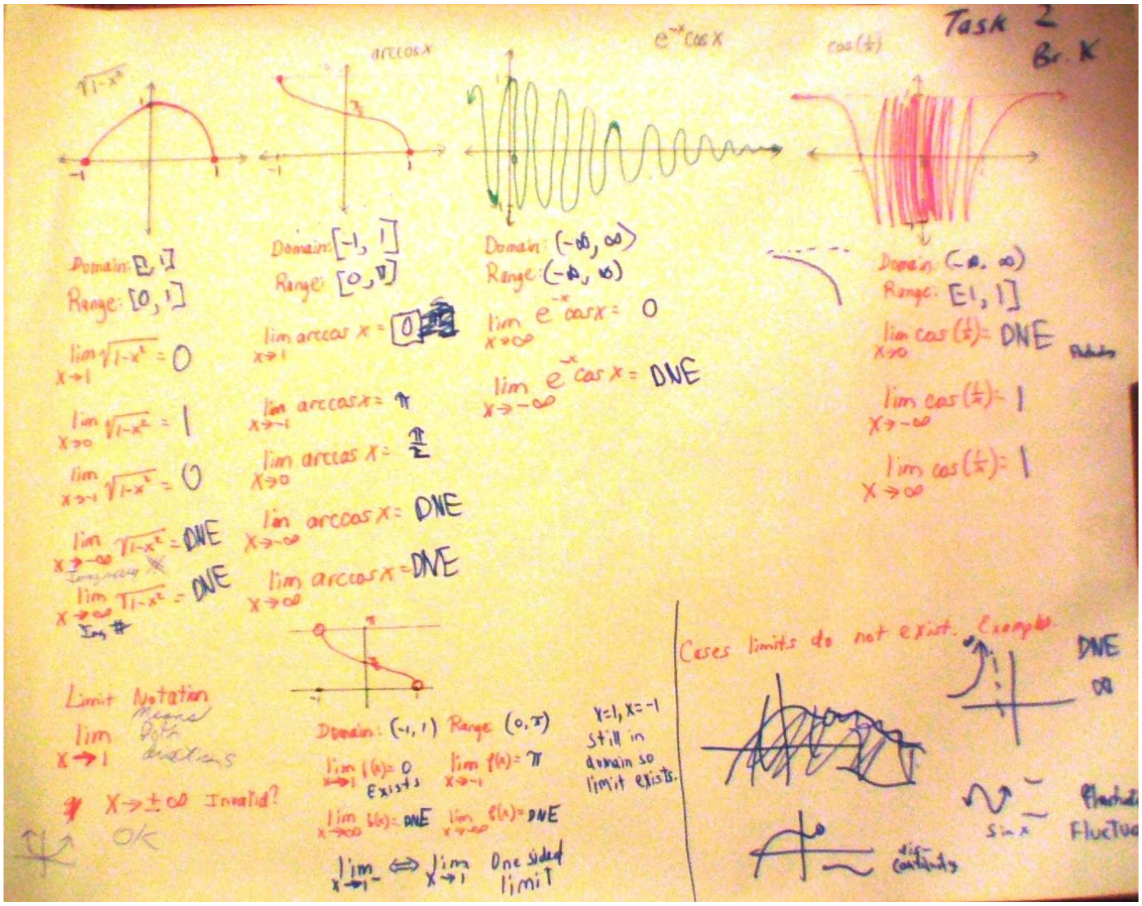
APPENDIX J-2: Images of JY'S Work

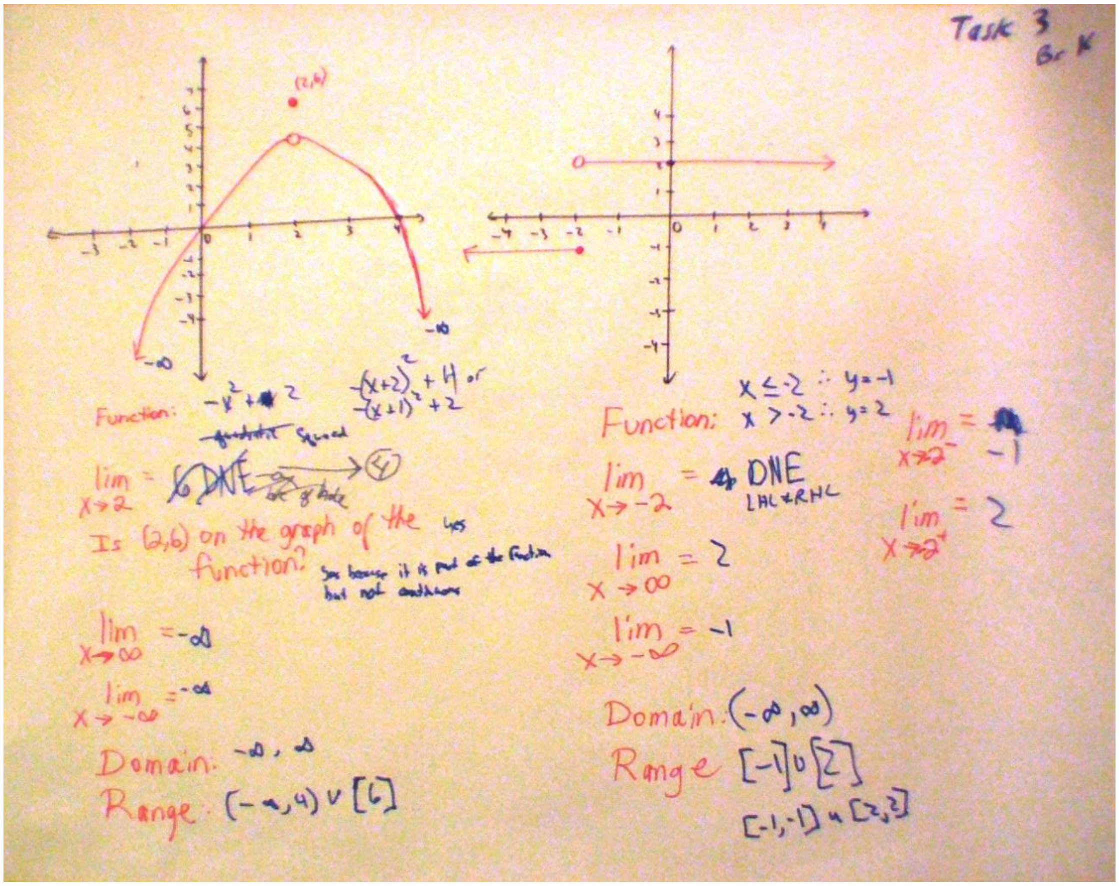
APPENDIX J-3: Images of LA'S Work

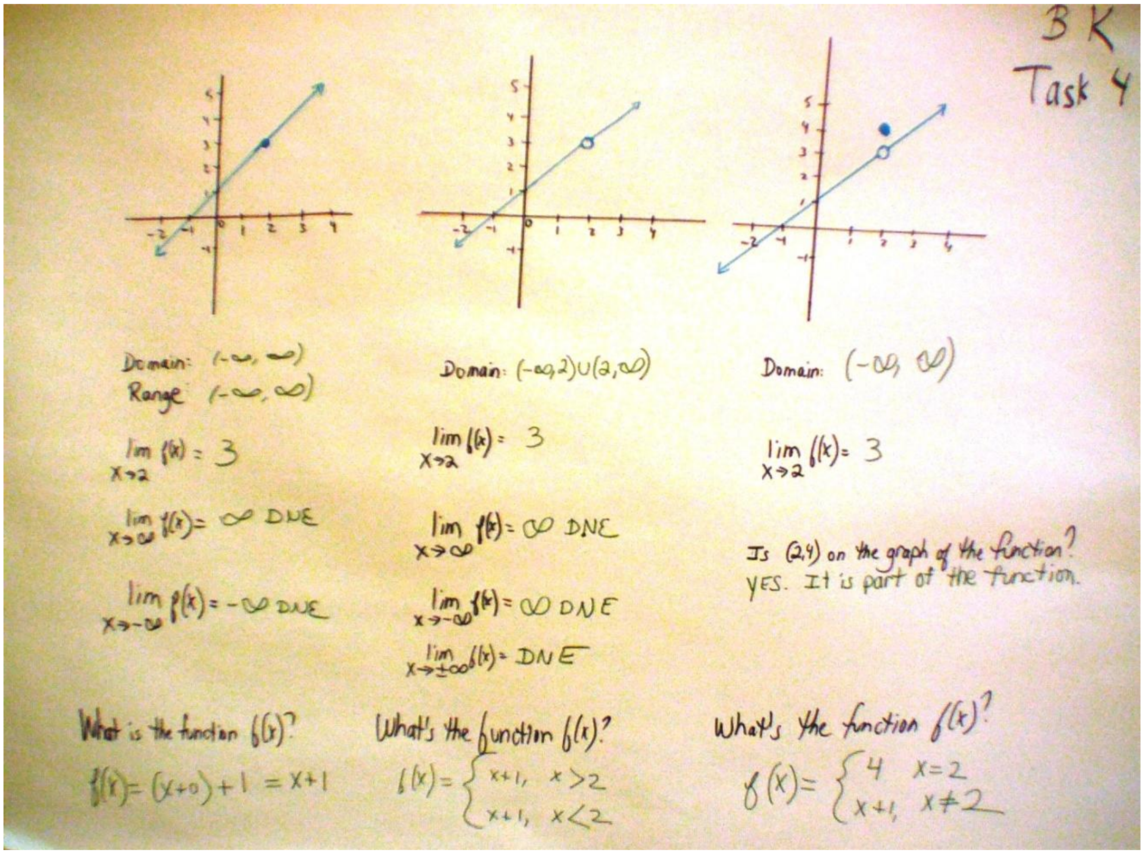
APPENDIX J-4: Images of AK'S Work

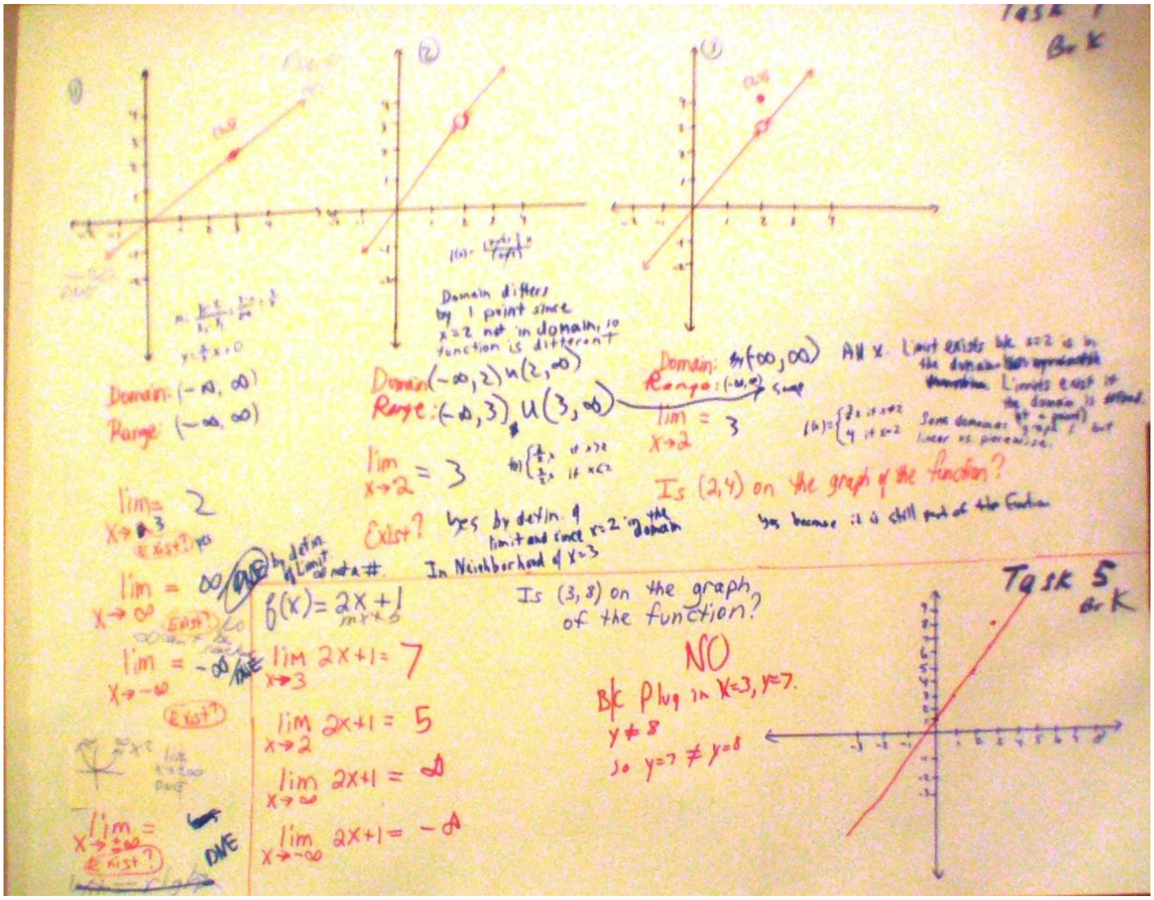
APPENDIX J-1: Images of BK'S Work

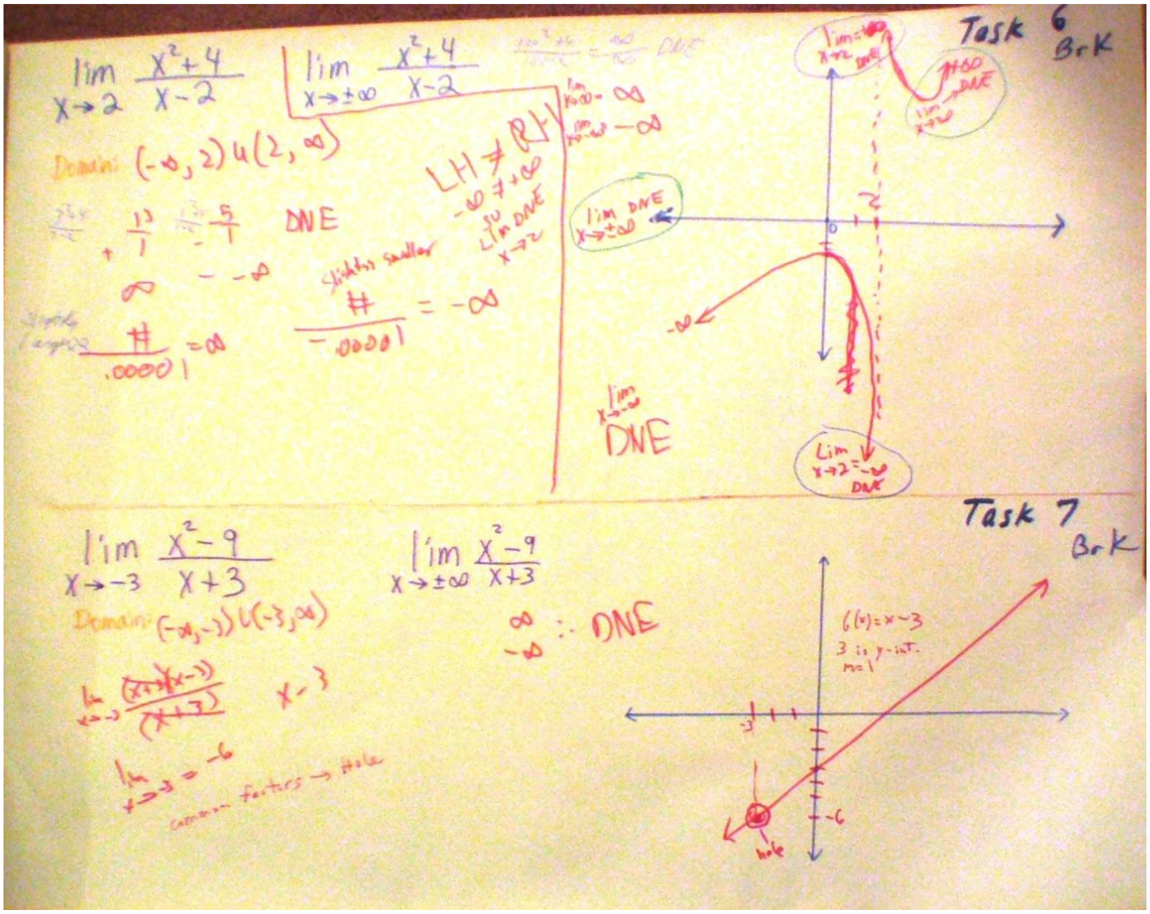


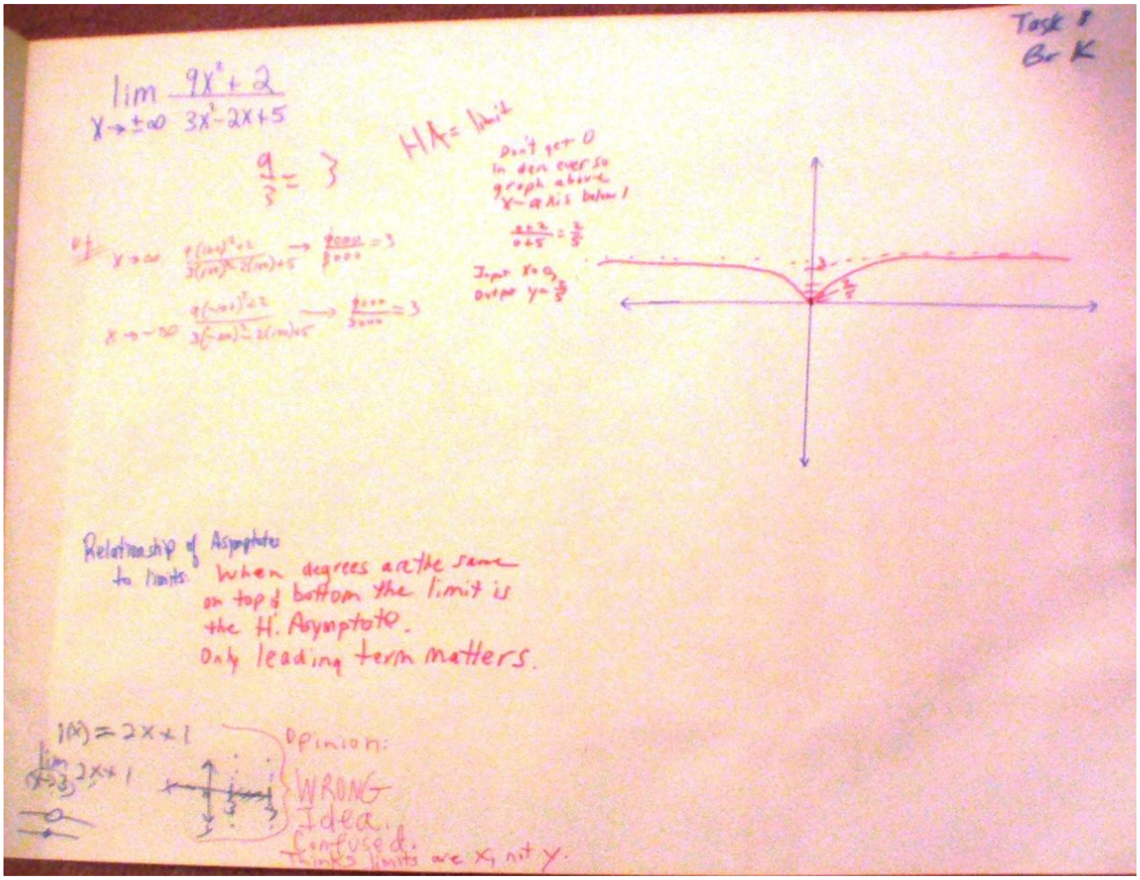

Task 2
arccos x
Domain: [-1, 1]
Range: [0, 1]
Domain: [-1, 1]
Range: [0, π]
Domain: (-∞, ∞)
Range: (-∞, ∞)
Domain: (-∞, ∞)
Range: [-1, 1]
lim cos(1/x) = DNE
lim cos(1/x) = 1
lim arccos x = π
lim arccos x = DNE
lim e^-x cos x = 0
lim e^-x cos x = DNE
Limit Notation
X → ± ∞ Invalid?
OK
Domain: (-1, 1) Range (0, π)
Exists
One sided limit
still in domain so limit exists.
Cases limits do not exist. Examples
DNE
∞
discontinuity
sin x
Fluctua

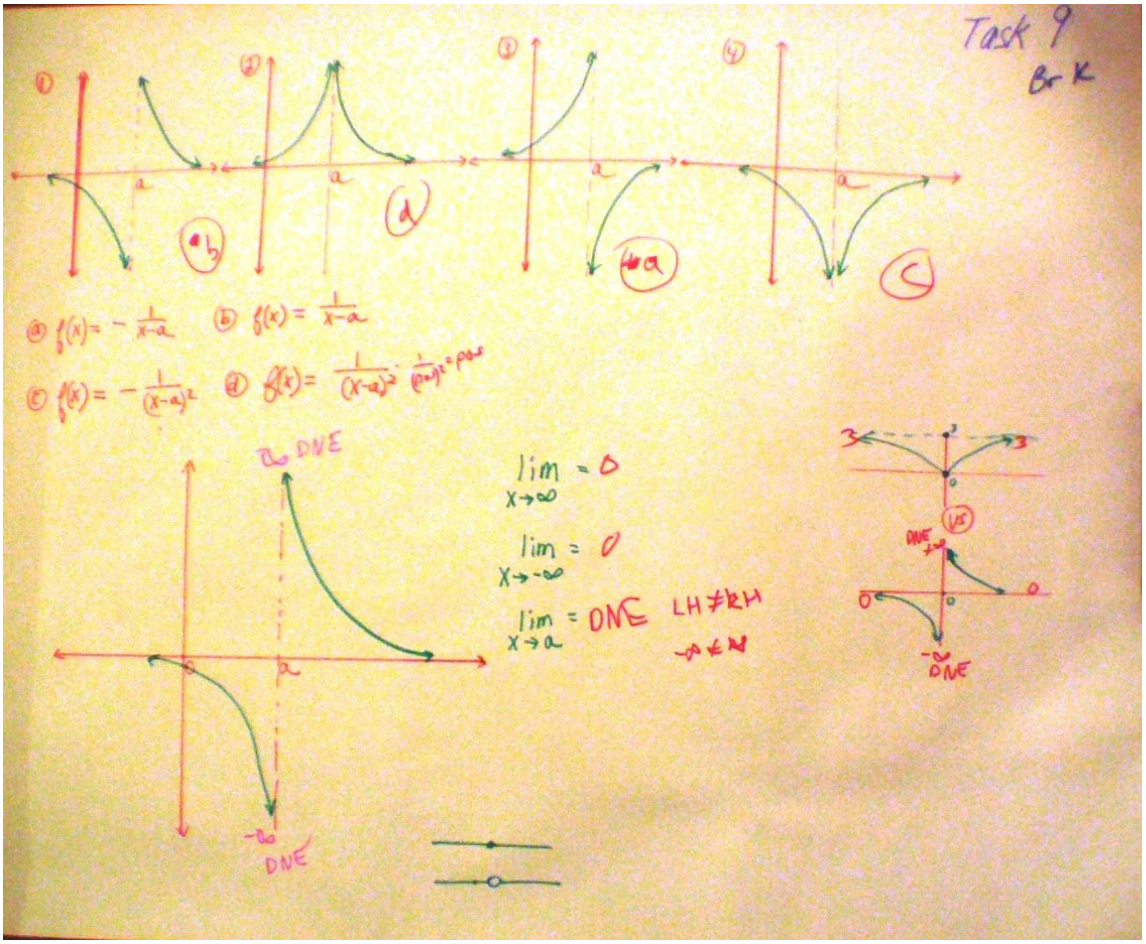

Task 3
Function:
lim = DNE
x→2
Is (2,6) on the graph of the function?
lim = -∞
x→∞
lim = -∞
x→-∞
Domain: -∞, ∞
Range: (-∞,4) ∪ [6]
x ≤ -2 ∴ y = -1
x > -2 ∴ y = 2
Function:
lim = DNE
x→-2
LHL≠RHL
lim = 2
x→∞
lim = -1
x→-∞
Domain: (-∞,∞)
Range [-1] ∪ [2]
[-1,-1] ∪ [2,2]
lim = -1
x→-2⁻
lim = 2
x→-2⁺

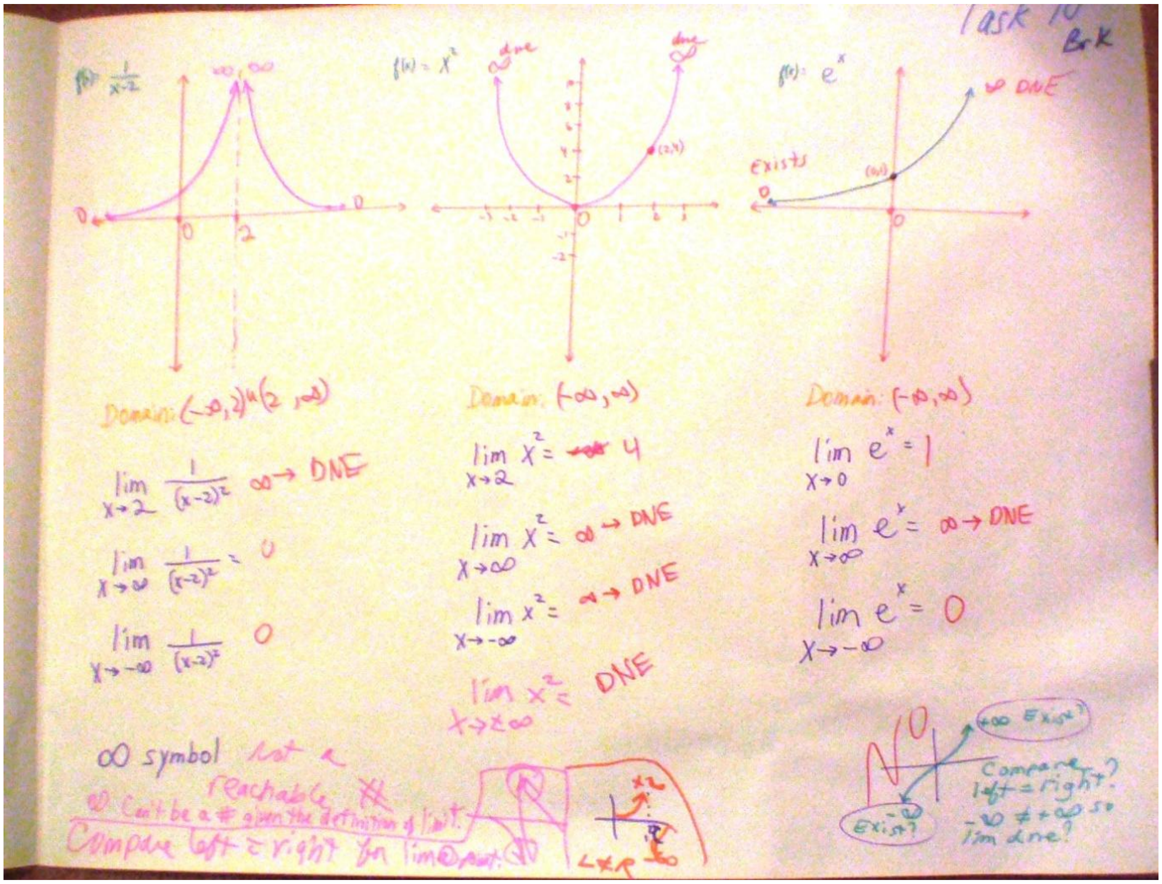

BK
Task 4
Domain: (-∞, ∞)
Range: (-∞, ∞)
lim x→2 f(x) = 3
lim x→∞ f(x) = ∞ DNE
lim x→-∞ f(x) = -∞ DNE
Domain: (-∞,2)∪(2,∞)
lim x→2 f(x) = 3
lim x→∞ f(x) = ∞ DNE
lim x→-∞ f(x) = ∞ DNE
lim x→±∞ f(x) = DNE
Domain: (-∞, ∞)
lim x→2 f(x) = 3
Is (2,4) on the graph of the function?
YES. It is part of the function.
What is the function f(x)?
f(x) = (x+0)+1 = x+1
What's the function f(x)?
f(x) = { x+1, x>2 ; x+1, x<2
What's the function f(x)?
f(x) = { 4 x=2 ; x+1, x≠2


Task 5
Domain: (-∞, ∞)
Range: (-∞, ∞)
Domain: (-∞, 2) ∪ (2, ∞)
Range: (-∞, 3) ∪ (3, ∞)
Is (2,4) on the graph of the function?
f(x) = 2x+1
Is (3,8) on the graph of the function?
NO
B/c Plug in X=3, y=7.
y ≠ 8
so y=7 ≠ y=8
lim 2x+1 = 7
lim 2x+1 = 5

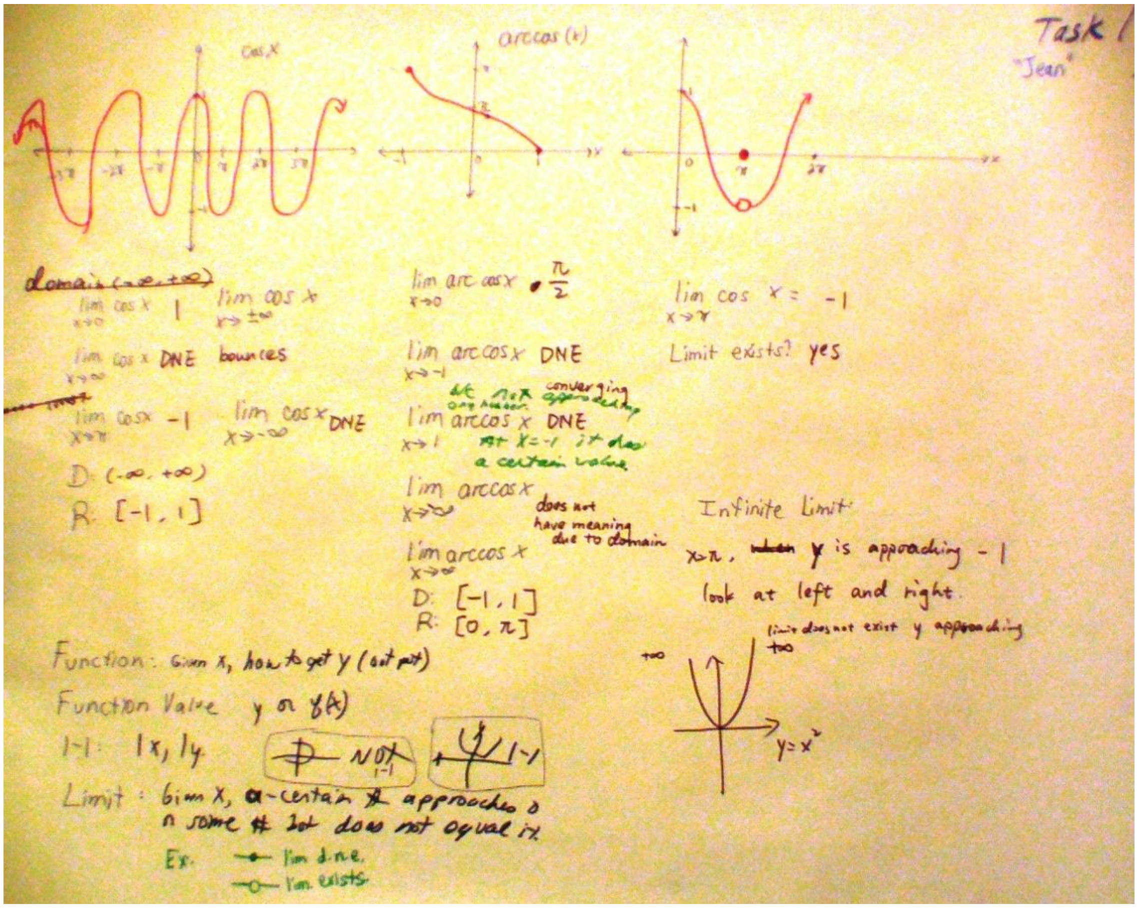

Task 6
BrK
lim
Domain: (-∞, 2) ∪ (2, ∞)
DNE
LH ≠ RH
.00001
-.00001
= -∞
Task 7
BrK
Domain: (-∞,-3) ∪ (-3,∞)
∴ DNE
x-3
common factors → Hole
hole
-6

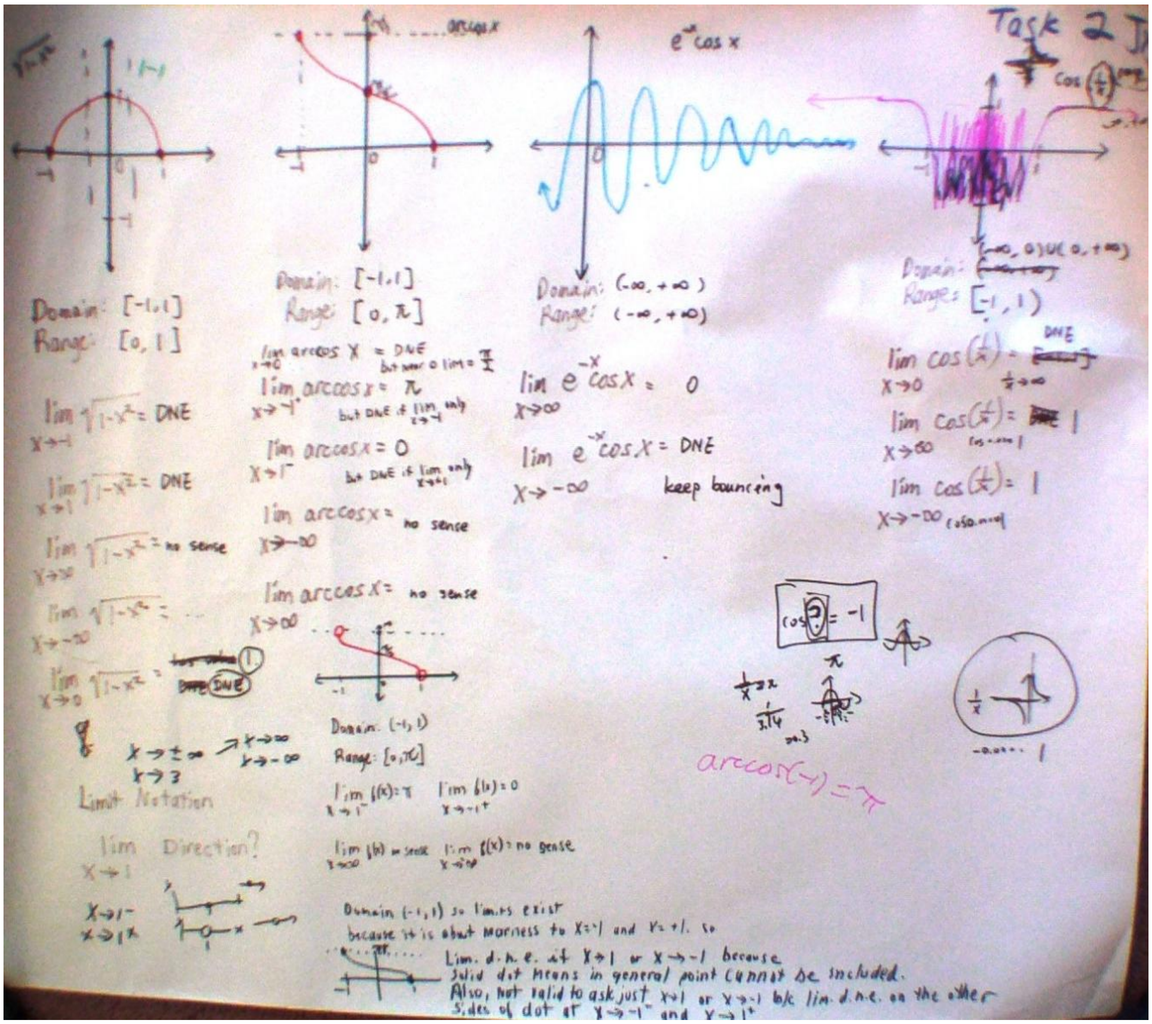

Task 8
Gr K
HA = limit
Don't get 0 in den ever so graph above x-axis below!
Input x=0, Output y=2/5
Relationship of Asymptotes to limits:
When degrees are the same on top & bottom the limit is the H. Asymptote.
Only leading term matters.
Opinion:
WRONG Idea.
confused.
thinks limits are x, not y.


Task 9
Br K
ⓐ f(x) = − 1/(x−a)
ⓑ f(x) = 1/(x−a)
ⓒ f(x) = − 1/(x−a)²
ⓓ f(x) = 1/(x−a)²
lim x→∞ = 0
lim x→−∞ = 0
lim x→a = DNE
LH ≠ RH
DNE
vs

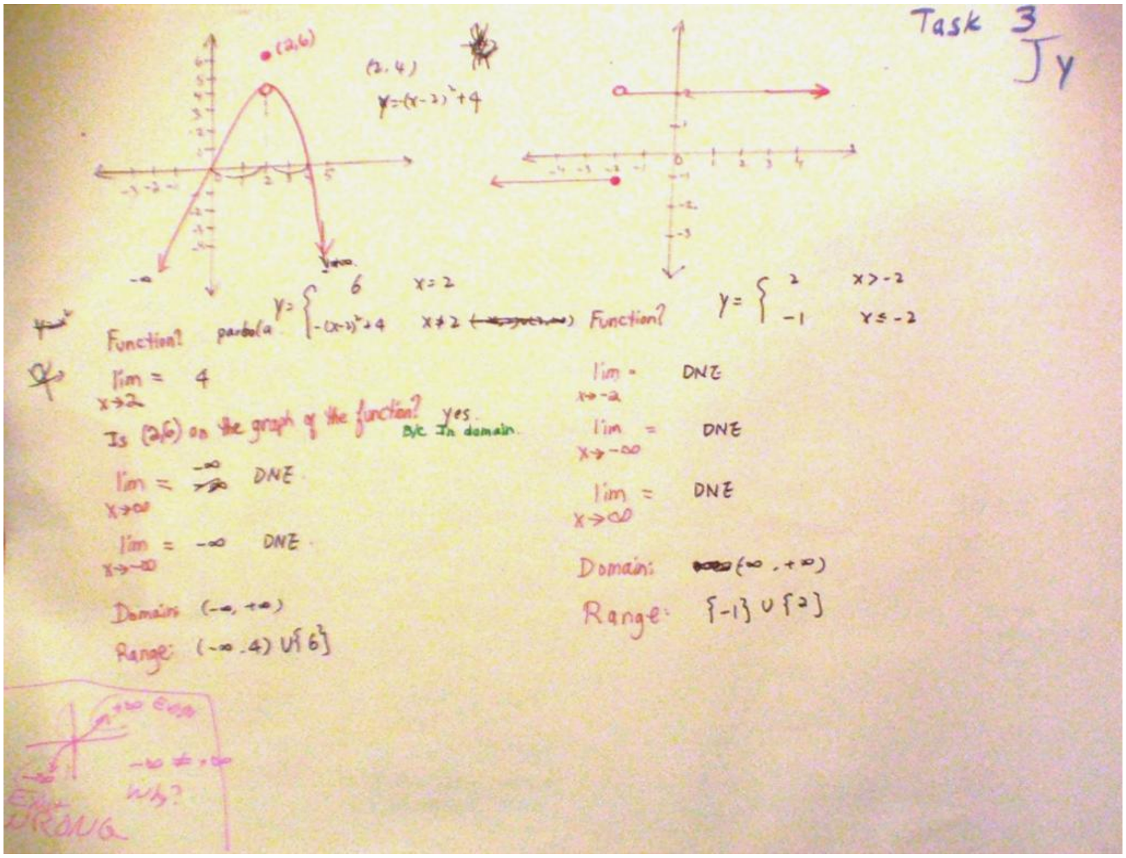

Task 10
Domain: (-∞, 2) ∪ (2, ∞)
Domain: (-∞, ∞)
Domain: (-∞, ∞)
Exists
∞ DNE
∞ symbol not a reachable #
Compare left = right for lim @ point
Exist?
Compare left = right?

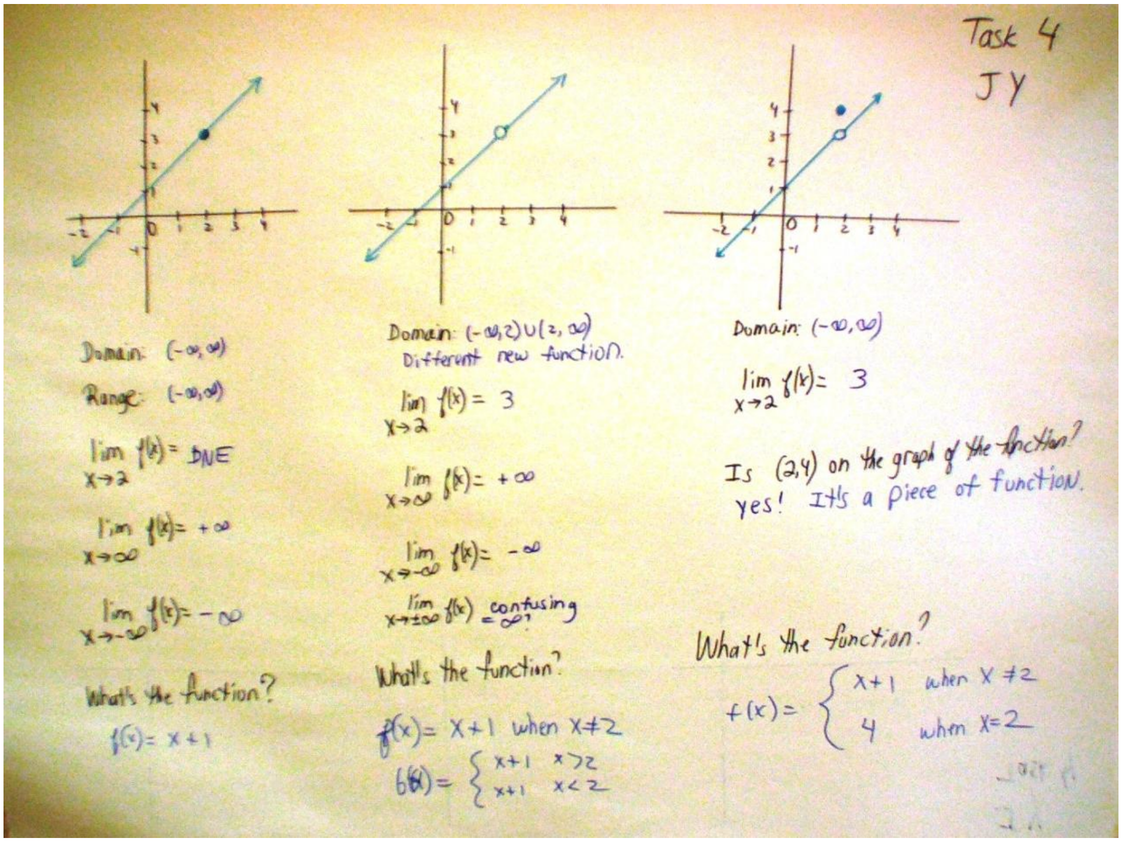
APPENDIX J-2: Images of JY’S Work

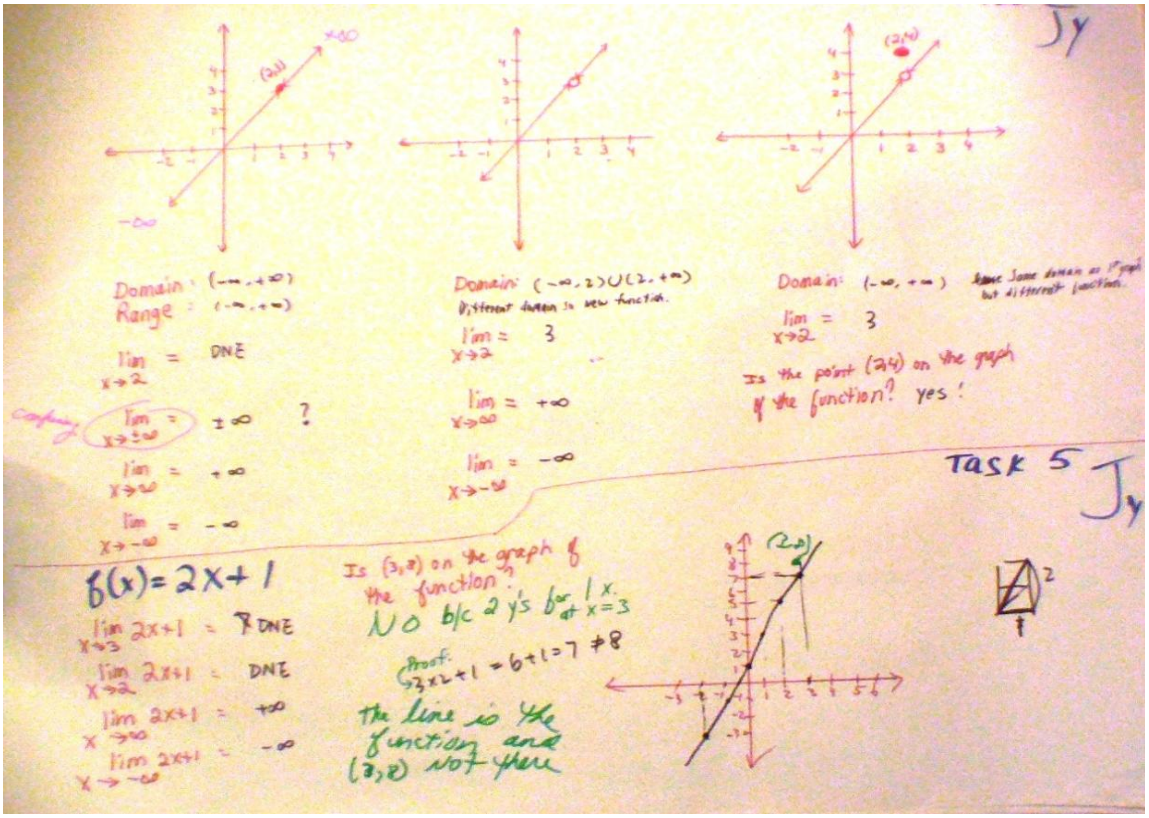




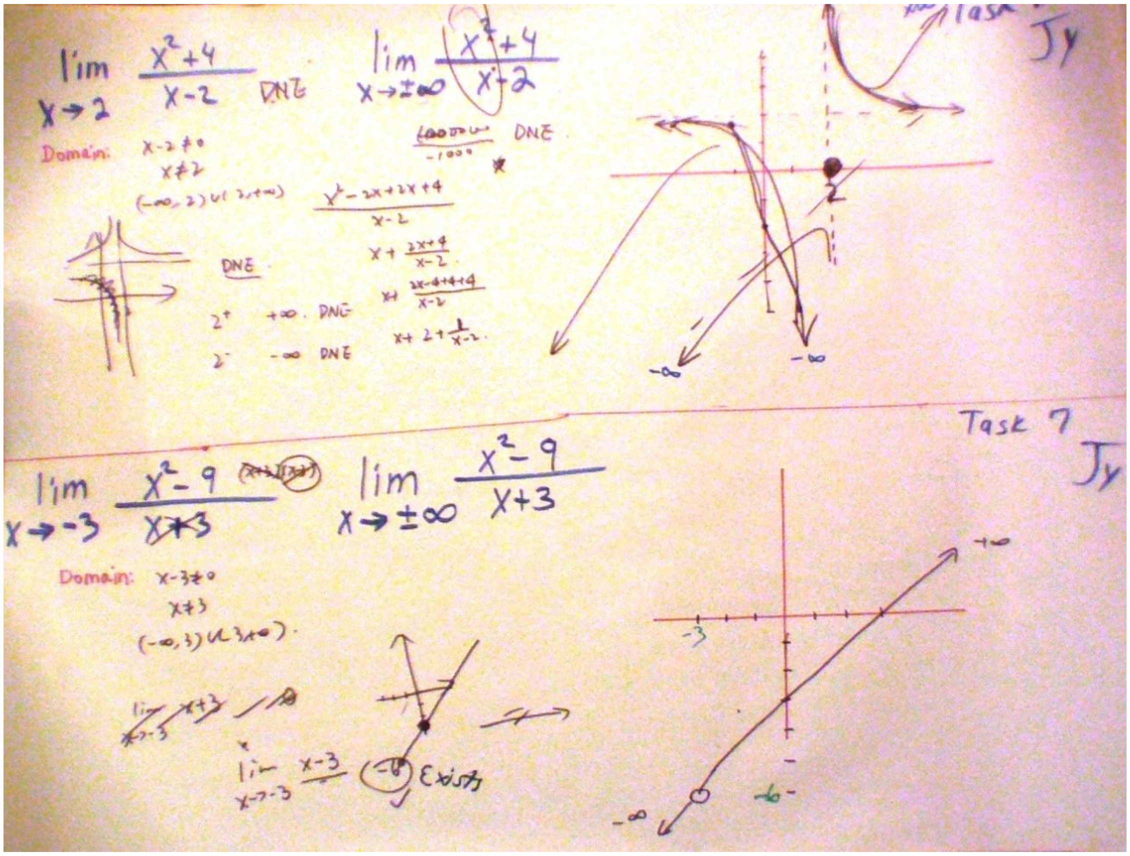
Task 2
arccos x
$e^{-x}\cos x$
Domain: [-1,1]
Range: [0, 1]
Domain: [-1,1]
Range: [0, π]
Domain: (-∞, +∞)
Range: (-∞, +∞)
Range: [-1, 1]
lim √(1-x²) = DNE
x→-1
lim √(1-x²) = DNE
x→1
lim √(1-x²) = no sense
x→∞
lim arccos x = π
x→-1⁺
lim arccos x = 0
x→1⁻
lim arccos x = no sense
x→-∞
lim arccos x = no sense
x→∞
lim e^{-x} cos x = 0
x→∞
lim e^{-x} cos x = DNE
x→-∞
keep bouncing
lim cos(1/x) = DNE
x→0
lim cos(1/x) = 1
x→∞
lim cos(1/x) = 1
x→-∞
cos(π) = -1
arccos(-1) = π
x→±∞
x→3
Limit Notation
lim Direction?
x→1
Domain: (-1, 1)
Range: [0, π]
lim f(x) = π
x→1⁻
lim f(x) = 0
x→-1⁺
Domain (-1, 1) so limits exist
Lim. d.n.e. at x→1 or x→-1 because
Solid dot means in general point cannot be included.

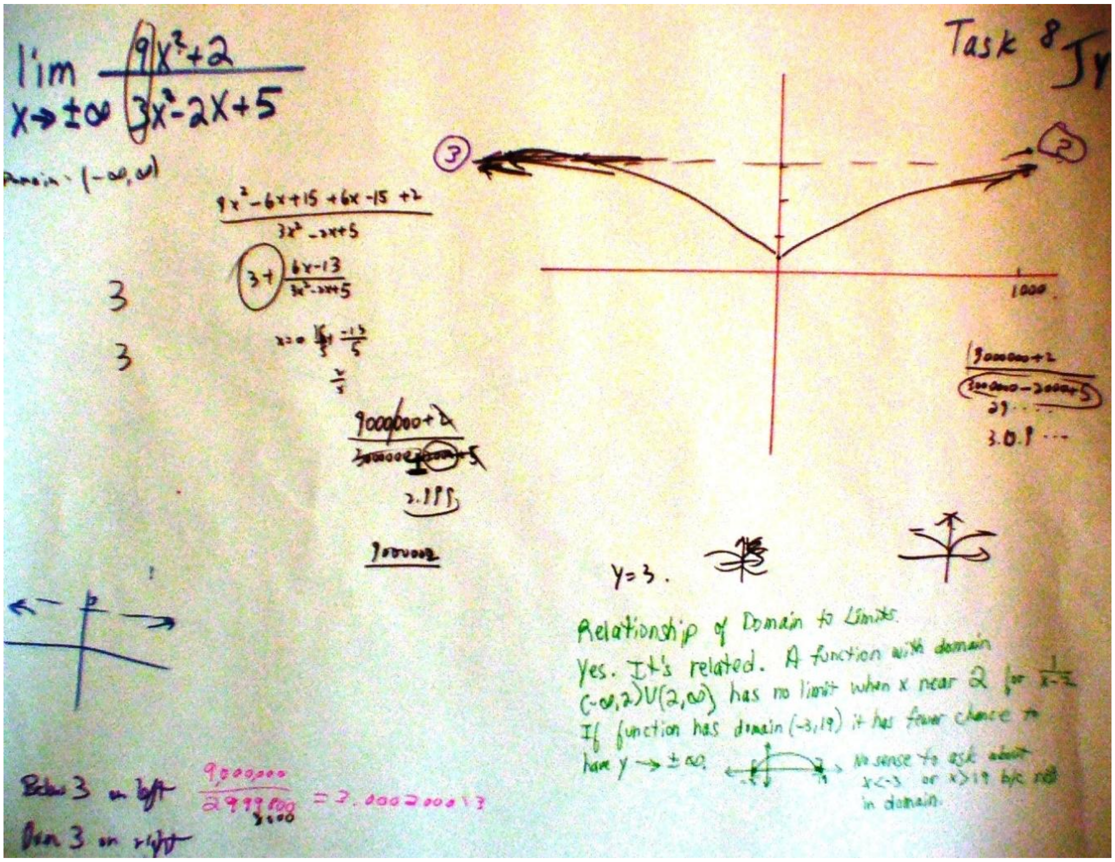

Task 3
Jy
(2,6)
(2,4)
y=-(x-2)^2+4
y= { 6 x=2 ; -(x-2)^2+4 x≠2
Function? parabola
lim = 4
x→2
Is (2,6) on the graph of the function? yes.
B/C In domain.
lim = -∞ DNE
x→∞
lim = -∞ DNE
x→-∞
Domain: (-∞, +∞)
Range: (-∞,4) U{6}
Function?
y= { 2 x>-2 ; -1 x≤-2
lim = DNE
x→-2
lim = DNE
x→-∞
lim = DNE
x→∞
Domain: (-∞, +∞)
Range: {-1} U {2}
-∞ ≠ +∞
Why?
WRONG

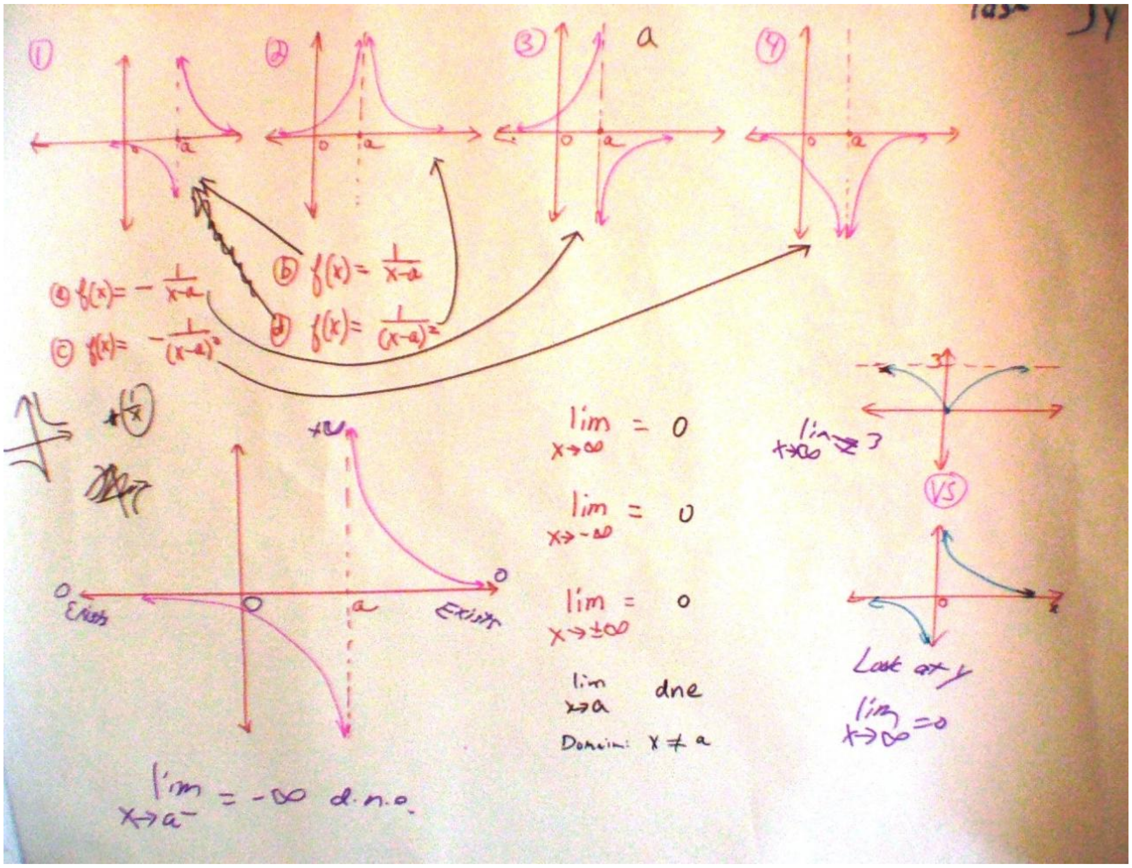

Task 4
JY
Domain: (-∞,∞)
Range: (-∞,∞)
lim f(x) = DNE
x→2
lim f(x) = +∞
x→∞
lim f(x) = -∞
x→-∞
What's the function?
f(x) = x + 1
Domain: (-∞,2)∪(2,∞)
Different new function.
lim f(x) = 3
x→2
lim f(x) = +∞
x→∞
lim f(x) = -∞
x→-∞
lim f(x) confusing
x→±∞
What's the function?
f(x) = x+1 when x≠2
x+1 x>2
x+1 x<2
Domain: (-∞,∞)
lim f(x) = 3
x→2
Is (2,4) on the graph of the function?
yes! It's a piece of function.
What's the function?
f(x) =
x+1 when x≠2
4 when x=2



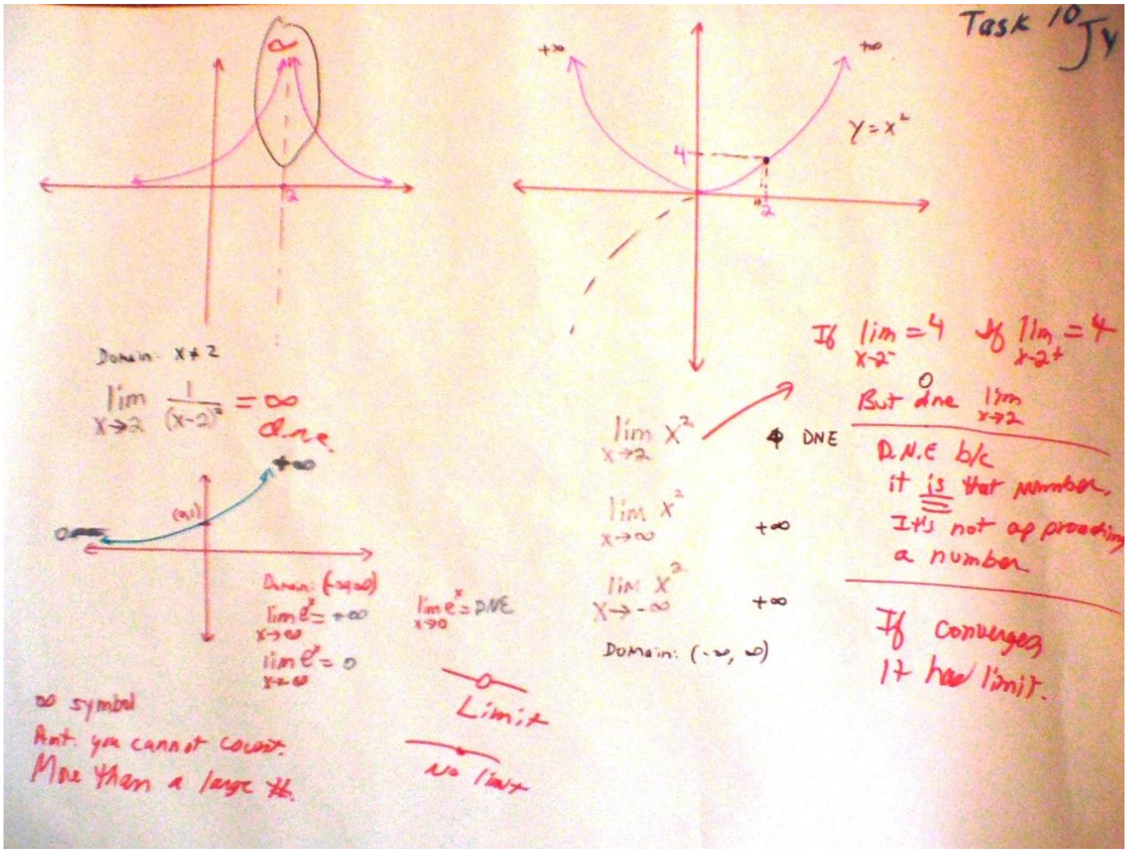
Domain: (−∞, +∞)
Range: (−∞, +∞)
lim x→2 = DNE
lim x→±∞ = ±∞ ?
lim x→∞ = +∞
lim x→−∞ = −∞
Domain: (−∞, 2)∪(2, +∞)
lim x→2 = 3
lim x→∞ = +∞
lim x→−∞ = −∞
Domain: (−∞, +∞)
lim x→2 = 3
Is the point (2,4) on the graph of the function? yes!
Task 5
f(x) = 2x + 1
lim x→3 2x+1 = DNE
lim x→2 2x+1 = DNE
lim x→∞ 2x+1 = +∞
lim x→−∞ 2x+1 = −∞
Is (3,8) on the graph of the function?
NO b/c 2 y's for 1 x. at x = 3
3x2+1 = 6+1 = 7 ≠ 8
The line is the function and (3,8) not there



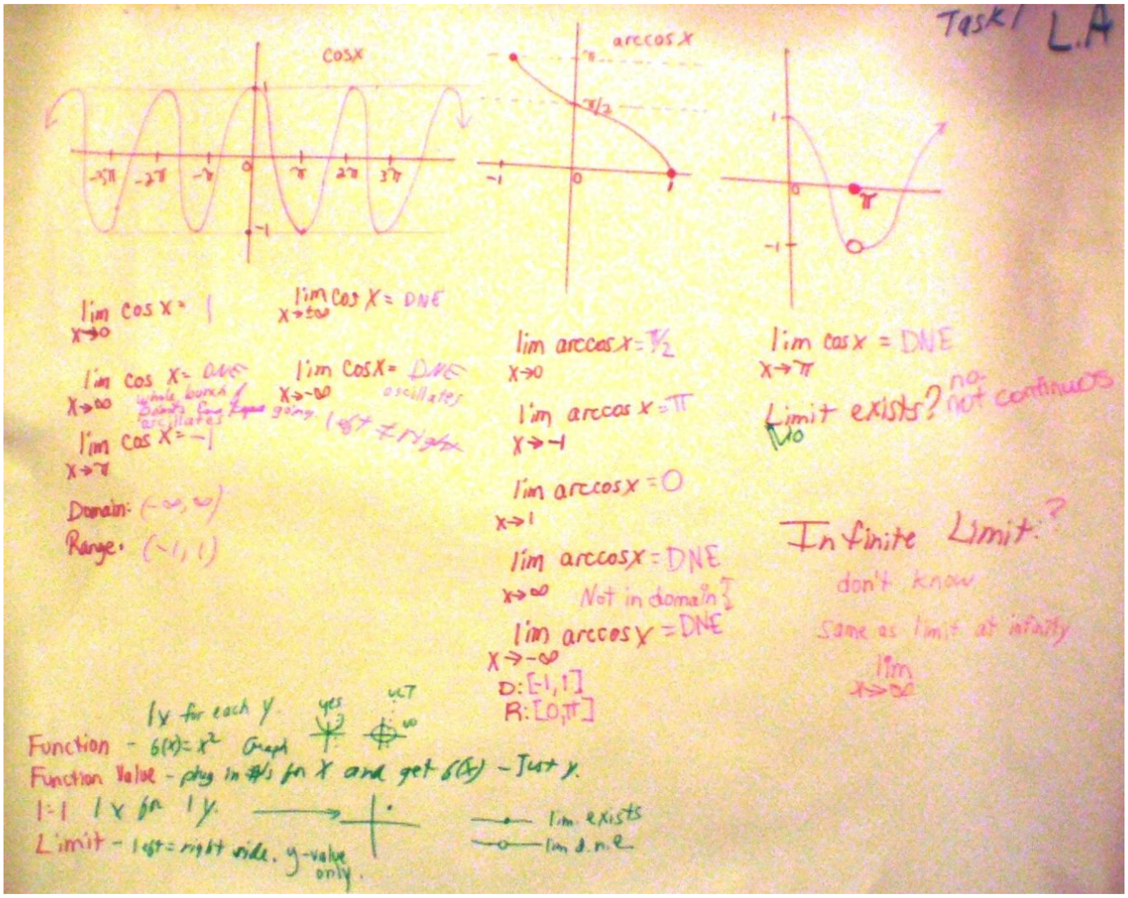

Task 8
$\lim_{x \to \pm\infty} \frac{9x^2+2}{3x^2-2x+5}$
y=3.
Relationship of Domain to Limits.
Yes. It's related. A function with domain
(-∞,2)U(2,∞) has no limit when x near 2 for 1/(x-2)
If function has domain (-3,19) it has fewer chance to
have y → ±∞.
No sense to ask about x<-3 or x>19 b/c not in domain.
Below 3 on left
Over 3 on right

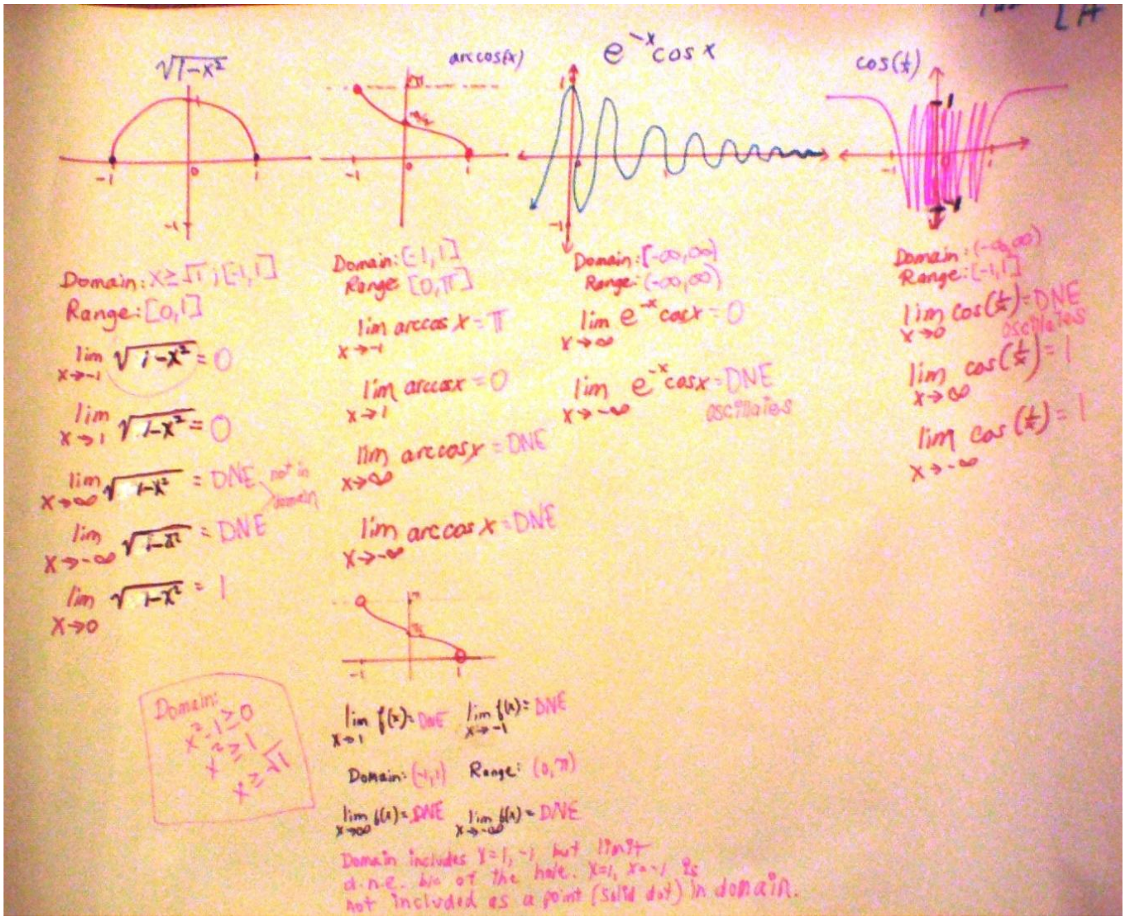


lim = 0
x→∞
lim = 0
x→−∞
lim = 0
x→±∞
lim dne
x→a
Domain: x ≠ a
lim = −∞ d.n.e.
x→a⁻
Look at y
lim = 0
x→∞

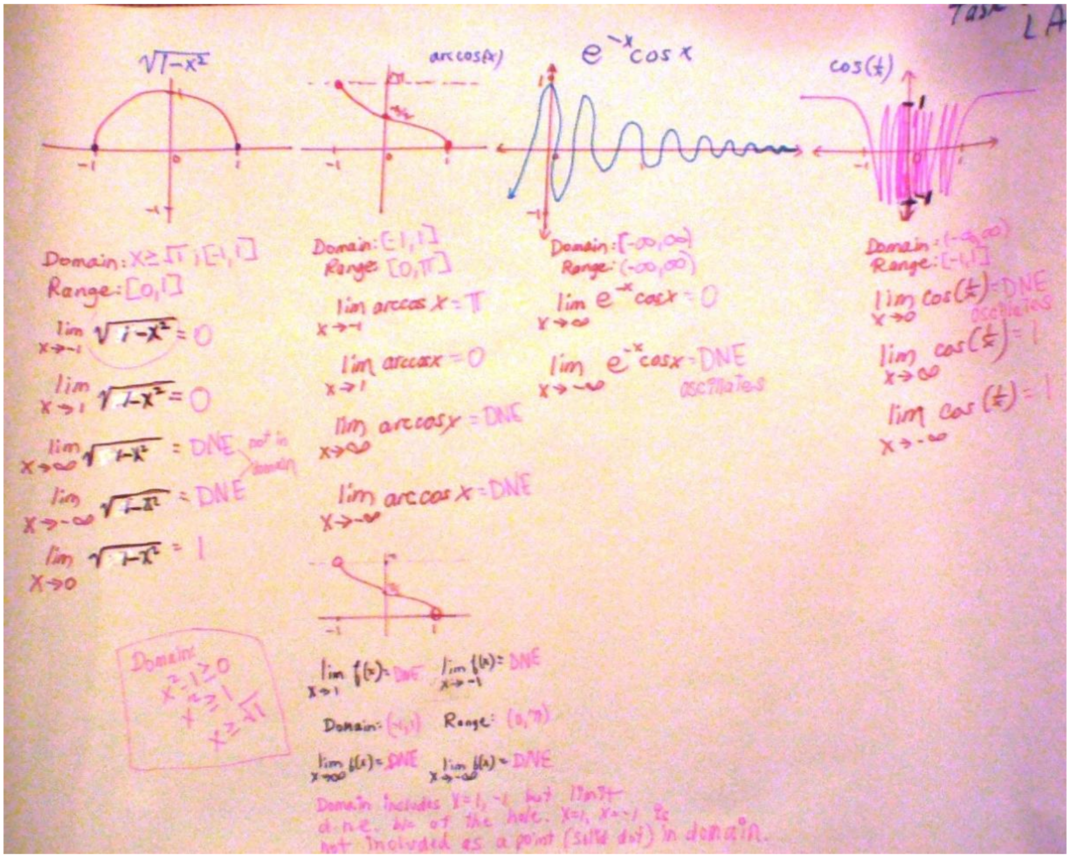

Task 10 JY
Domain: x ≠ 2
lim x→2 1/(x-2)² = ∞
d.n.e.
Domain: (-∞,∞)
lim x→∞ e^x = +∞
lim x→-∞ e^x = 0
lim x→0 e^x = DNE
∞ symbol
Amt. you cannot count.
More than a large #.
Limit
No limit
y = x²
lim x→2 x² DNE
lim x→∞ x² +∞
lim x→-∞ x² +∞
Domain: (-∞, ∞)
If lim x→2⁻ = 4 If lim x→2⁺ = 4
But dne lim x→2
D.N.E b/c it is that number.
It's not approaching a number.
If converges it has limit.

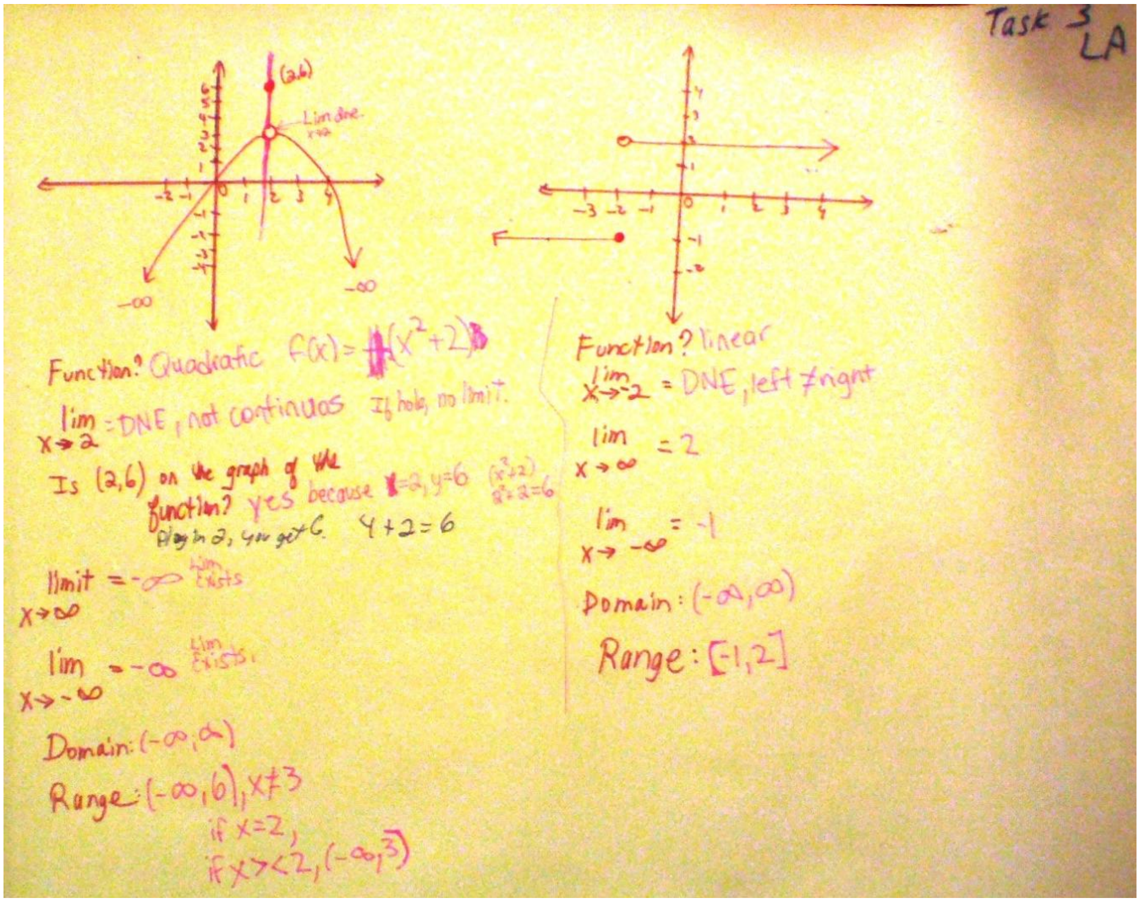
APPENDIX J-3: Images of LA'S Work

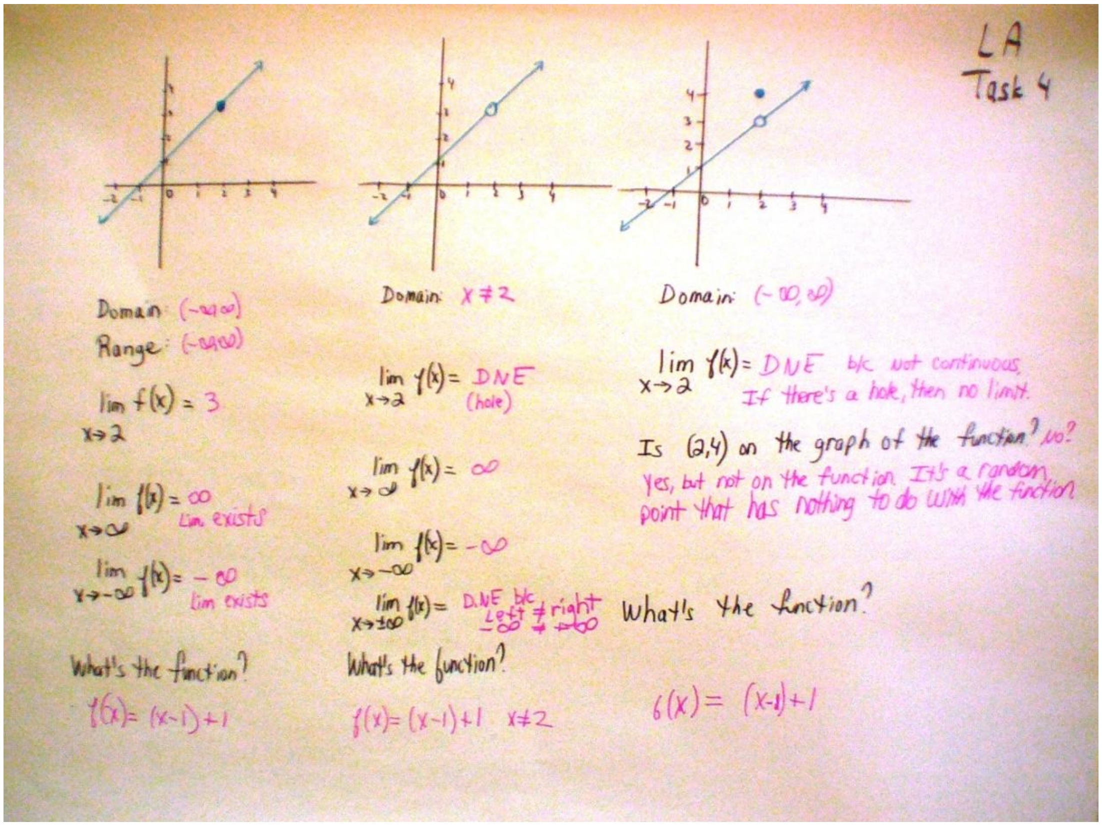


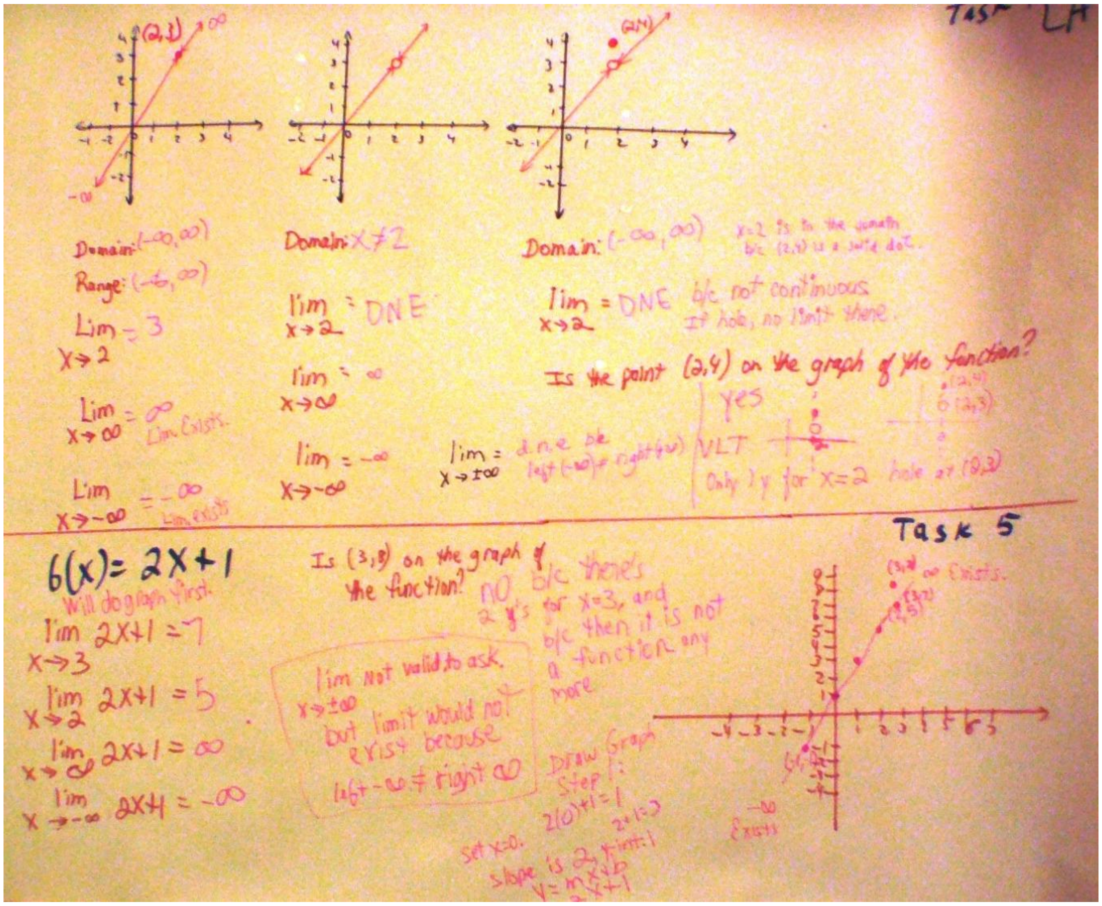


arccos(x)
e^-x cos x
cos(1/x)
Domain: x ≥ √1 ; [-1, 1]
Range: [0, 1]
lim x→-1 √(1-x²) = 0
lim x→1 √(1-x²) = 0
lim x→∞ √(1-x²) = DNE
not in domain
lim x→-∞ √(1-x²) = DNE
lim x→0 √(1-x²) = 1
Domain: x²-1 ≥ 0
x² ≥ 1
x ≥ √1
Domain: [-1, 1]
Range: [0, π]
lim x→-1 arccos x = π
lim x→1 arccos x = 0
lim x→∞ arccos x = DNE
lim x→-∞ arccos x = DNE
lim x→1 f(x) = DNE
lim x→-1 f(x) = DNE
Domain: (-1, 1)
Range: (0, π)
lim x→∞ f(x) = DNE
lim x→-∞ f(x) = DNE
Domain includes x = 1, -1 but limit
d.n.e. b/c of the hole. x=1, x=-1 is
not included as a point (solid dot) in domain.
Domain: (-∞, ∞)
Range: (-∞, ∞)
lim x→∞ e^-x cos x = 0
lim x→-∞ e^-x cos x = DNE
oscillates
Domain: (-∞, ∞)
Range: [-1, 1]
lim x→0 cos(1/x) = DNE
oscillates
lim x→∞ cos(1/x) = 1
lim x→-∞ cos(1/x) = 1

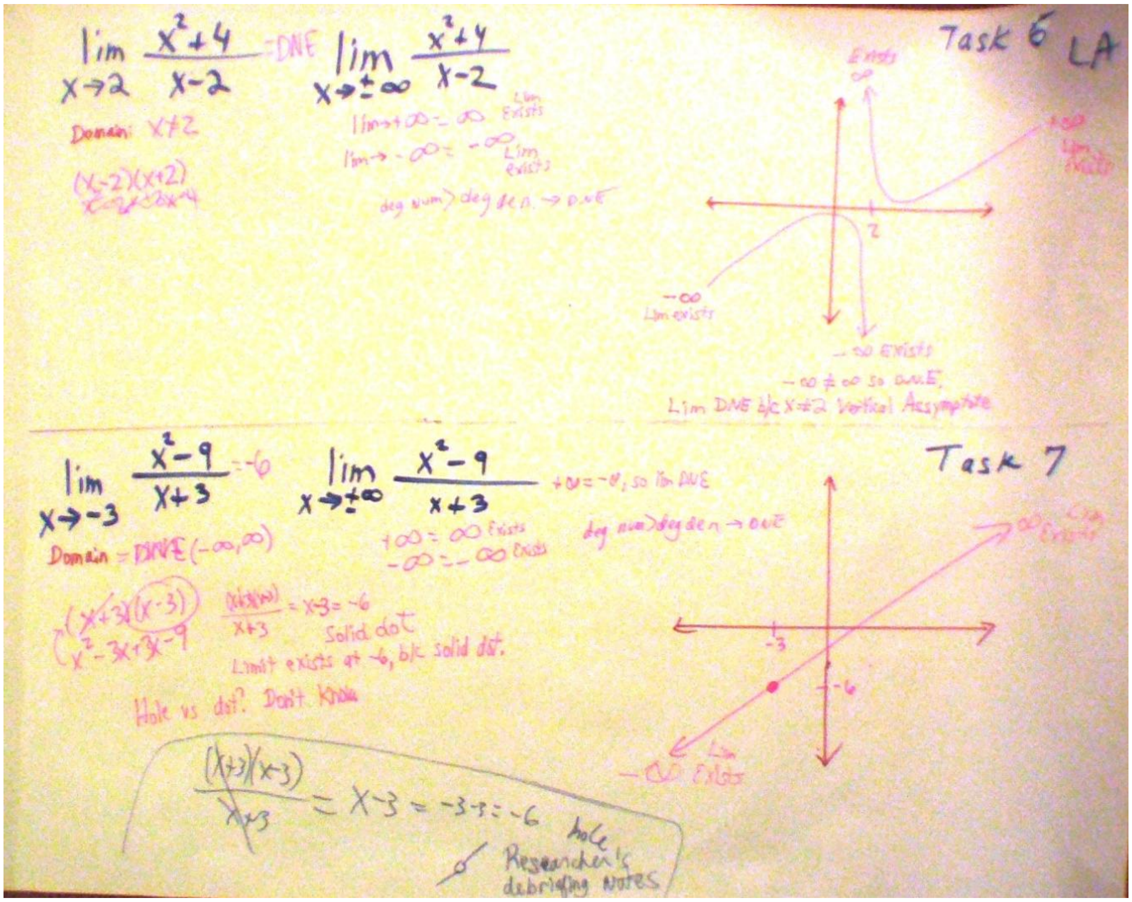

√1−x²
arccos(x)
e^−x cos x
cos(1/x)
Domain: x ≥ √1 ; [−1, 1]
Range: [0, 1]
lim x→−1 √1−x² = 0
lim x→1 √1−x² = 0
lim x→∞ √1−x² = DNE
not in domain
lim x→−∞ √1−x² = DNE
lim x→0 √1−x² = 1
Domain: [−1, 1]
Range: [0, π]
lim x→−1 arccos x = π
lim x→1 arccos x = 0
lim x→∞ arccos x = DNE
lim x→−∞ arccos x = DNE
Domain: [−∞, ∞]
Range: (−∞, ∞)
lim x→∞ e^−x cos x = 0
lim x→−∞ e^−x cos x = DNE
oscillates
Domain: (−∞, ∞)
Range: [−1, 1]
lim x→0 cos(1/x) = DNE
oscillates
lim x→∞ cos(1/x) = 1
lim x→−∞ cos(1/x) = 1
Domain:
x² − 1 ≥ 0
x² ≥ 1
x ≥ √1
lim x→1 f(x) = DNE
lim x→−1 f(x) = DNE
Domain: (−1, 1)
Range: (0, π)
lim x→∞ f(x) = DNE
lim x→−∞ f(x) = DNE
Domain includes x = 1, −1 but limit d.n.e. b/c of the hole. x = 1, x = −1 is not included as a point (solid dot) in domain.

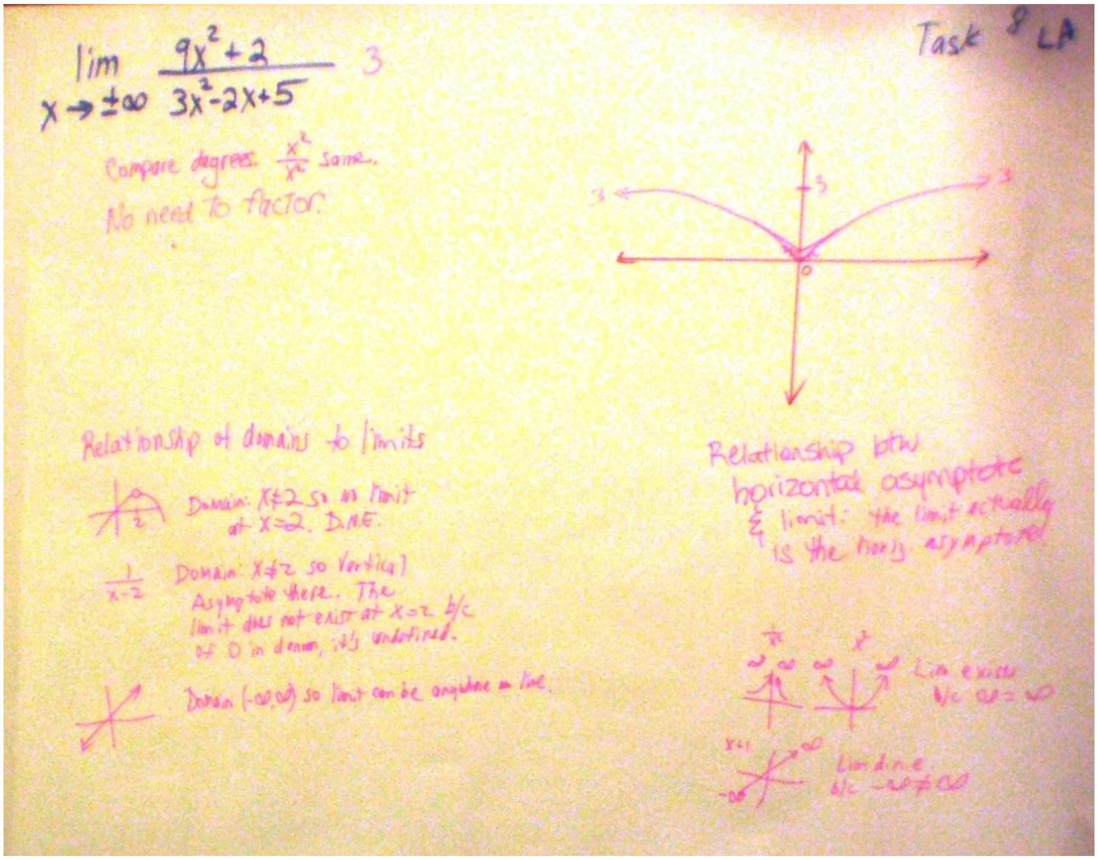

Task 3 LA
(2,6)
Lim dne
−∞
−∞
Function? Quadratic f(x) = (x² + 2)
lim = DNE, not continuos If hole, no limit.
x→2
Is (2,6) on the graph of the function? yes because x=2, y=6
Plug in 2, you get 6. 4+2=6
limit = −∞ Lim exists
x→∞
lim = −∞ Lim exists.
x→−∞
Domain: (−∞,∞)
Range: (−∞,6], x≠3
if x=2,
if x><2, (−∞,3)
Function? linear
lim = DNE, left ≠ right
x→−2
lim = 2
x→∞
lim = −1
x→−∞
Domain: (−∞,∞)
Range: [−1,2]


LA
Task 4
Domain: (-∞,∞)
Range: (-∞,∞)
lim f(x) = 3
x→2
lim f(x) = ∞
x→∞
lim exists
lim f(x) = -∞
x→-∞
lim exists
What's the function?
f(x)= (x-1)+1
Domain: x≠2
lim f(x)= DNE
x→2
(hole)
lim f(x)= ∞
x→∞
lim f(x)= -∞
x→-∞
lim f(x)= DNE b/c
x→±∞
Left ≠ right
-∞ ≠ +∞
What's the function?
f(x)= (x-1)+1 x≠2
Domain: (-∞,∞)
lim f(x)= DNE b/c not continuous
x→2
If there's a hole, then no limit.
Is (2,4) on the graph of the function? no?
Yes, but not on the function. It's a random point that has nothing to do with the function
What's the function?
G(x)= (x-1)+1

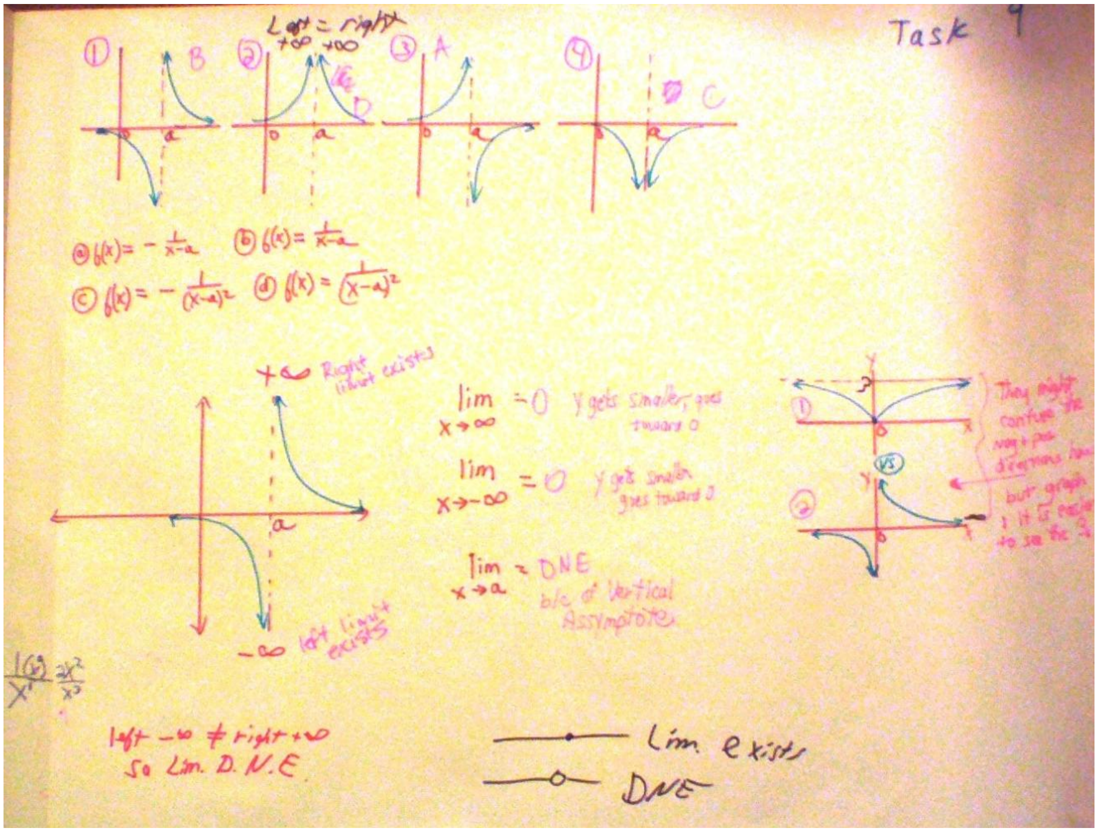


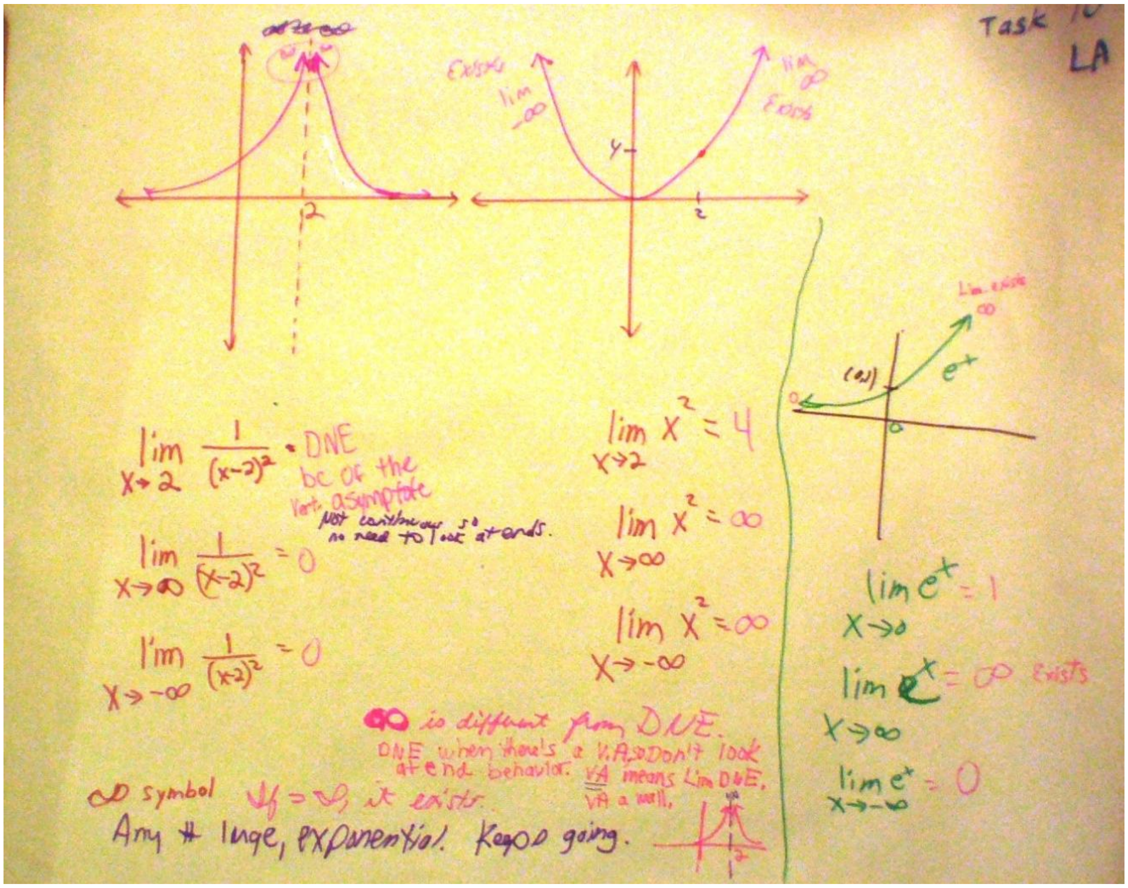
Domain: (-∞,∞)
Range: (-∞,∞)
Lim = 3
X→2
Lim = ∞
X→∞
Lim Exists.
Lim = -∞
X→-∞
Lim exists
Domain: X≠2
lim = DNE
x→2
lim = ∞
x→∞
lim = -∞
x→-∞
lim = d.n.e b/c
x→±∞
left (-∞) ≠ right (∞)
Domain: (-∞,∞)
lim = DNE b/c not continuous
x→2
It hole, no limit there.
Is the point (2,4) on the graph of the function?
Yes
VLT
Only 1 y for x=2
hole at (2,3)
Task 5
G(x)= 2X+1
Will do graph first.
lim 2X+1 = 7
X→3
lim 2X+1 = 5
X→2
lim 2X+1 = ∞
X→∞
lim 2X+1 = -∞
X→-∞
Is (3,8) on the graph of the function?
NO b/c there's 2 y's for x=3, and b/c then it is not a function any more
lim not valid to ask.
x→±∞
but limit would not exist because
left -∞ ≠ right ∞
Draw Graph
Step 1:
2(0)+1=1
Set x=0.
slope is 2, y-int: 1
y = mx+b


Task 6 LA
lim x→2 (x²+4)/(x−2) = DNE
lim x→±∞ (x²+4)/(x−2)
Domain: x≠2
deg num > deg den → DNE
Lim DNE b/c x≠2 Vertical Assymptote
Task 7
lim x→−3 (x²−9)/(x+3) = −6
lim x→±∞ (x²−9)/(x+3)
Domain = DNE (−∞,∞)
Solid dot
Limit exists at −6, b/c solid dot.
Hole vs dot? Don't Know
Researcher's debriefing notes

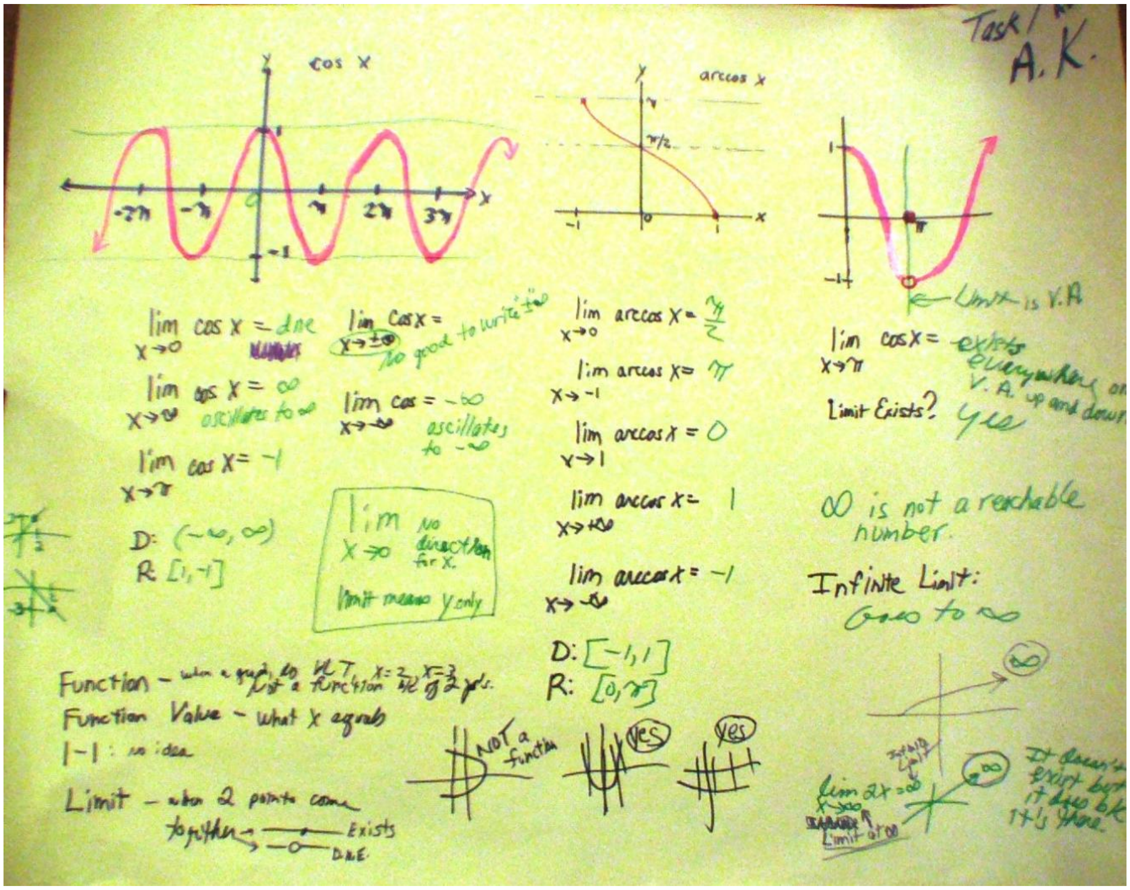


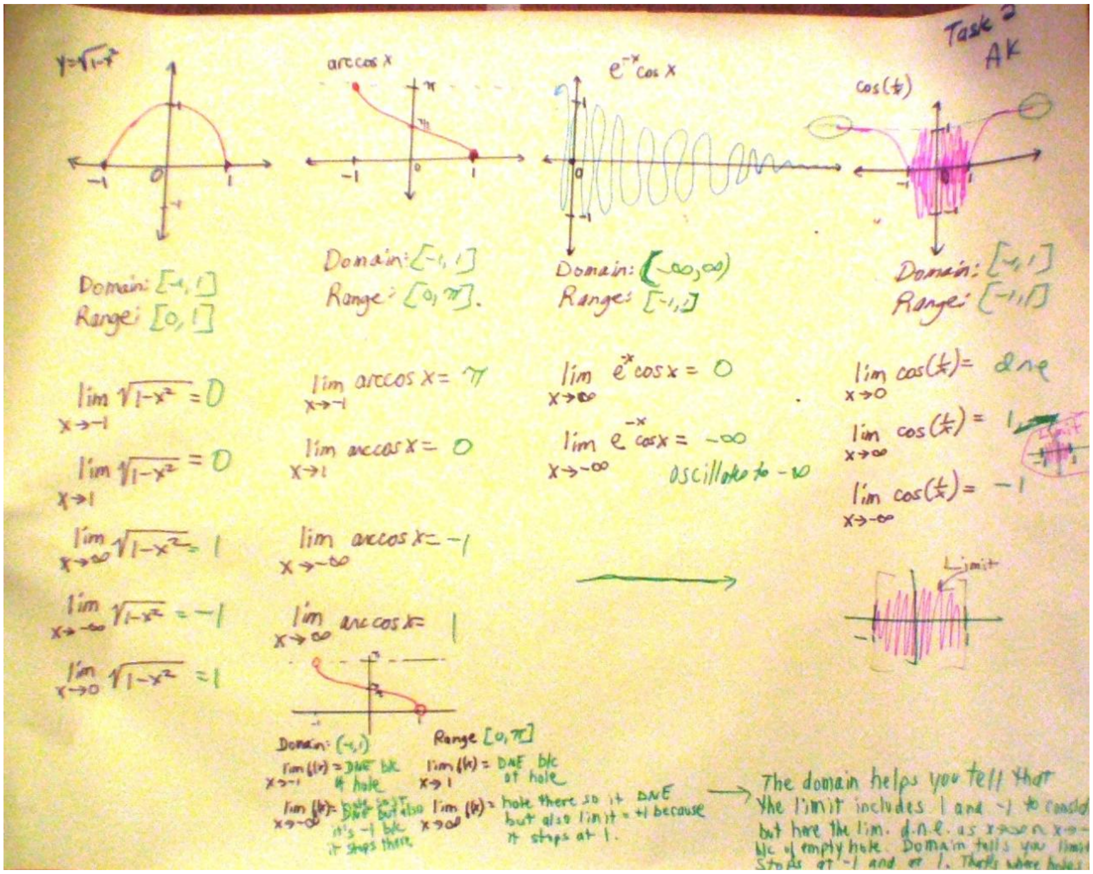
Task 8 LA
$\lim_{x \to \pm\infty} \frac{9x^2+2}{3x^2-2x+5}$ 3
Compare degrees: $\frac{x^2}{x^2}$ same.
No need to factor.
3
3
5
0
Relationship of domains to limits
Domain: $x \neq 2$ so no limit at $x=2$. D.N.E.
$\frac{1}{x-2}$ Domain: $x \neq 2$ so Vertical Asymptote there. The limit does not exist at $x=2$ b/c of 0 in denom, it's undefined.
Domain $(-\infty,\infty)$ so limit can be anywhere on line.
Relationship btw horizontal asymptote & limit: the limit actually is the horiz. asymptote.
x^2
Lim exists b/c $\infty = \infty$
Lim d.n.e b/c $-\infty \neq \infty$

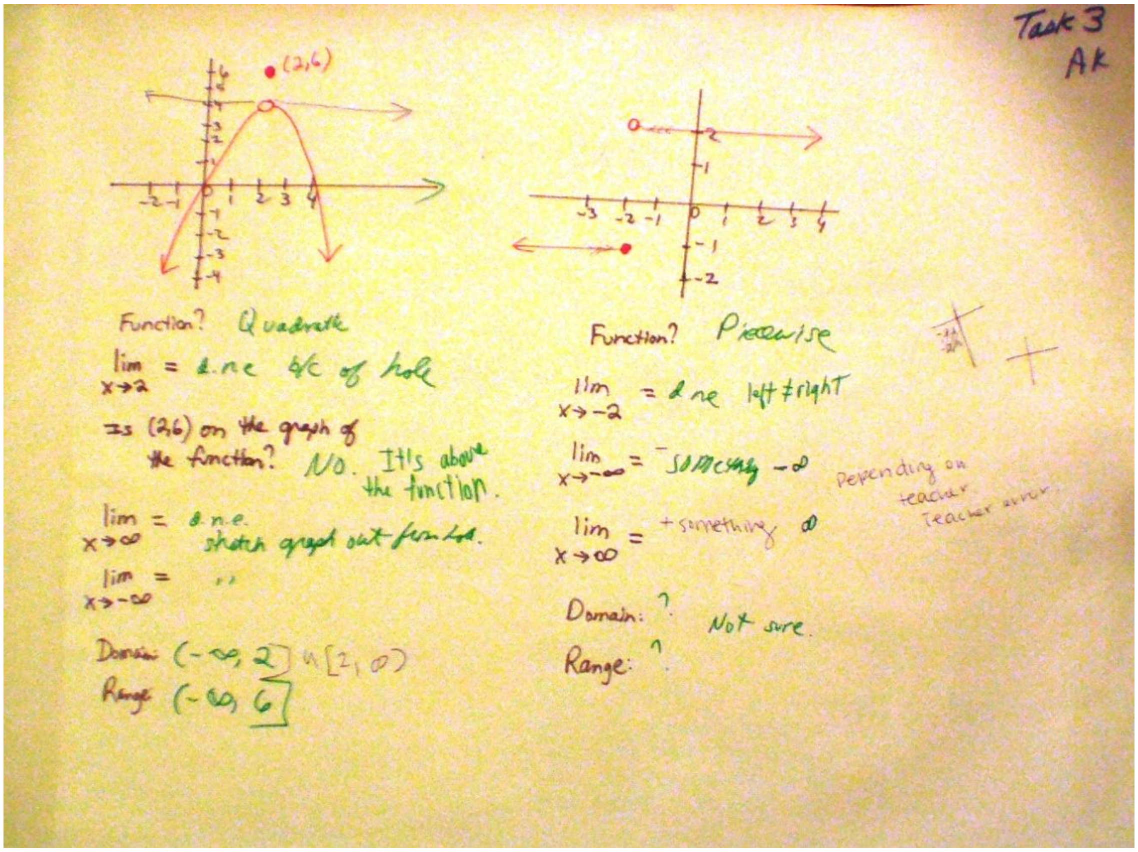

Task 9
Left = right
+∞ +∞
f(x) = $-\frac{1}{x-a}$
f(x) = $\frac{1}{x-a}$
f(x) = $-\frac{1}{(x-a)^2}$
f(x) = $\frac{1}{(x-a)^2}$
+∞ Right limit exists
−∞ left limit exists
lim x→∞ = 0 y gets smaller, goes toward 0
lim x→−∞ = 0 y gets smaller goes toward 0
lim x→a = DNE b/c of Vertical Assymptote
left −∞ ≠ right +∞ so Lim. D.N.E
Lim. exists
DNE
vs

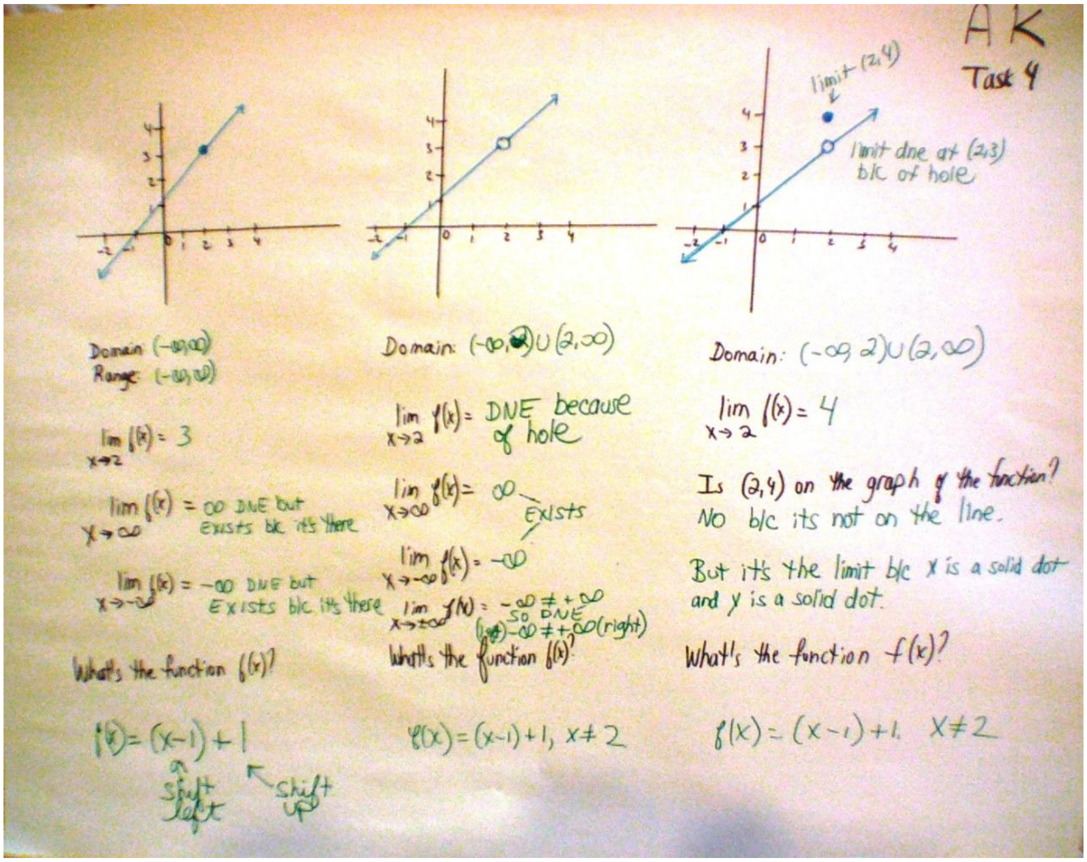

Task
LA
lim ∞ Exists
lim −∞ Exists
lim x→2 1/(x−2)² = DNE bc of the vert. asymptote
not continuous so no need to look at ends.
lim x→∞ 1/(x−2)² = 0
lim x→−∞ 1/(x−2)² = 0
lim x→2 x² = 4
lim x→∞ x² = ∞
lim x→−∞ x² = ∞
lim x→0 eˣ = 1
lim x→∞ eˣ = ∞ Exists
lim x→−∞ eˣ = 0
∞ is different from DNE.
DNE when there's a V.A. so Don't look at end behavior. VA means Lim DNE, VA a wall.
∞ symbol
Any # large, exponential. Keeps going.

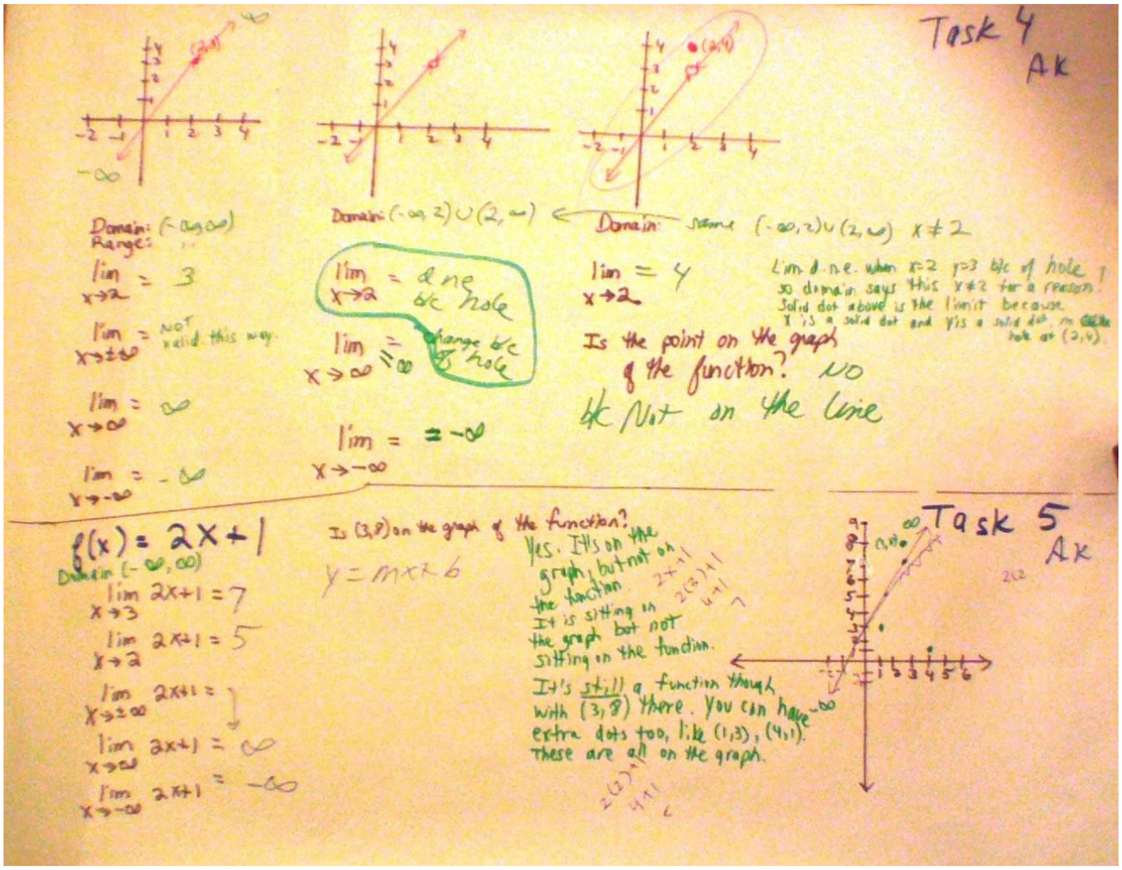
APPENDIX J-4: Images of AK'S Work

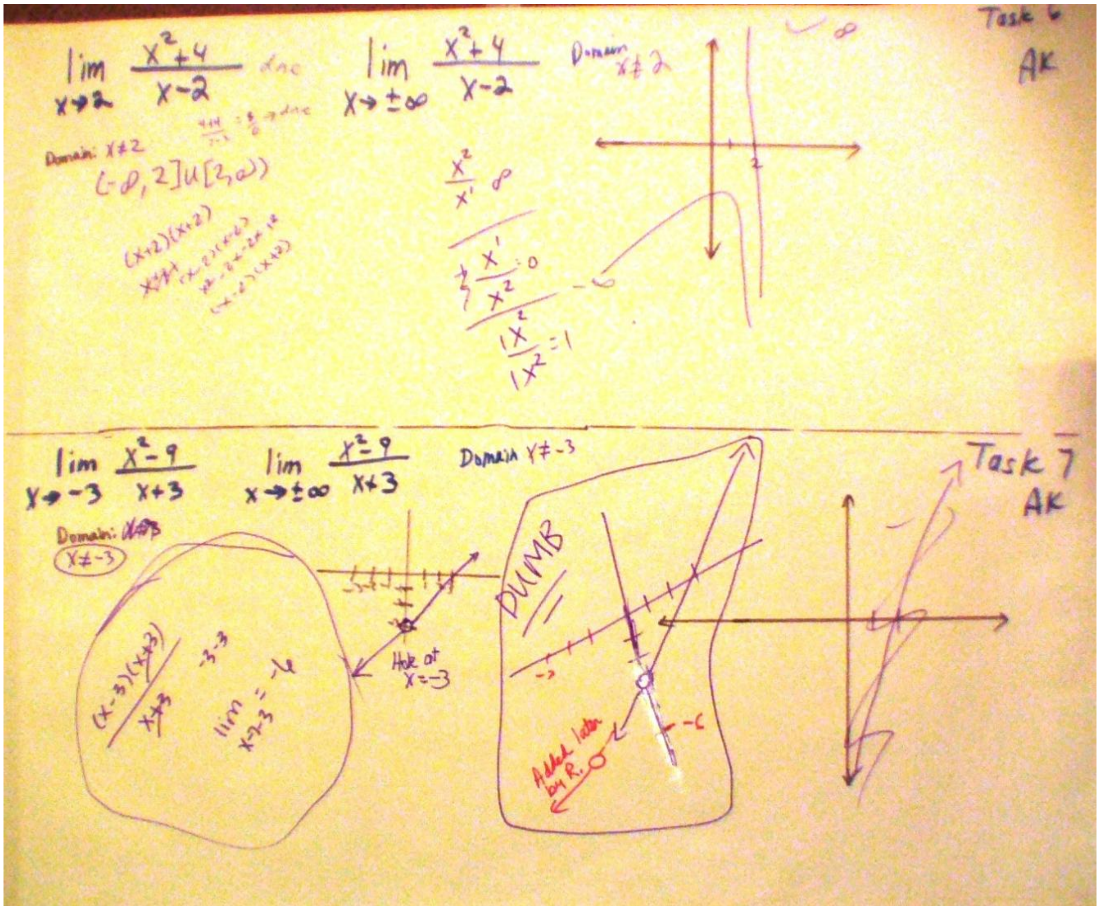


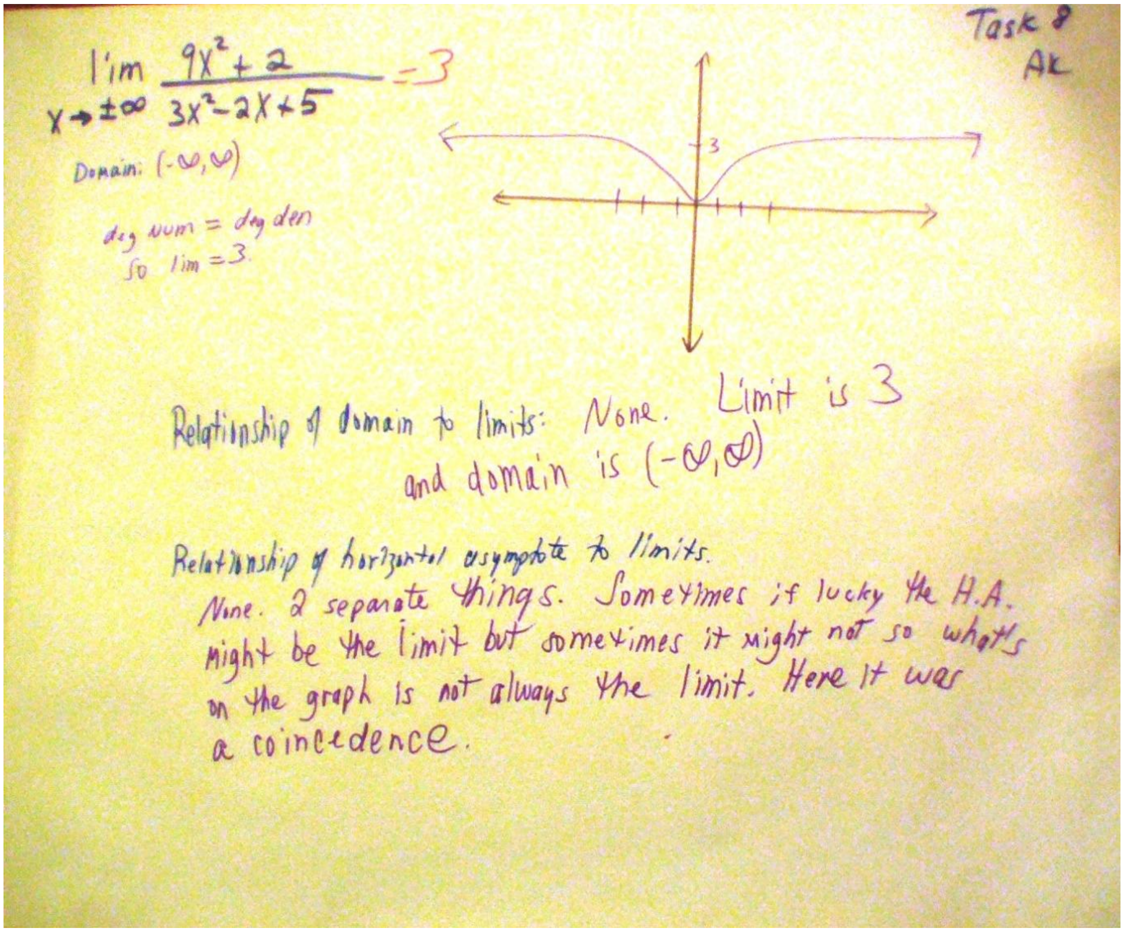

Task 2
AK
y=√(1-x²)
arccos x
e^-x cos x
cos(1/x)
Domain: [-1, 1]
Range: [0, 1]
Domain: [-1, 1]
Range: [0, π].
Domain: (-∞, ∞)
Range: [-1, 1]
Domain: [-1, 1]
Range: [-1, 1]
lim x→-1 √(1-x²) = 0
lim x→1 √(1-x²) = 0
lim x→∞ √(1-x²) = 1
lim x→-∞ √(1-x²) = -1
lim x→0 √(1-x²) = 1
lim x→-1 arccos x = π
lim x→1 arccos x = 0
lim x→-∞ arccos x = -1
lim x→∞ arccos x = 1
lim x→∞ e^-x cos x = 0
lim x→-∞ e^-x cos x = -∞
oscillate to -∞
lim x→0 cos(1/x) = dne
lim x→∞ cos(1/x) = 1
lim x→-∞ cos(1/x) = -1
Limit
Domain: (-1, 1)
Range [0, π]
The domain helps you tell that the limit includes 1 and -1 to consider but here the lim. d.n.e. as x→

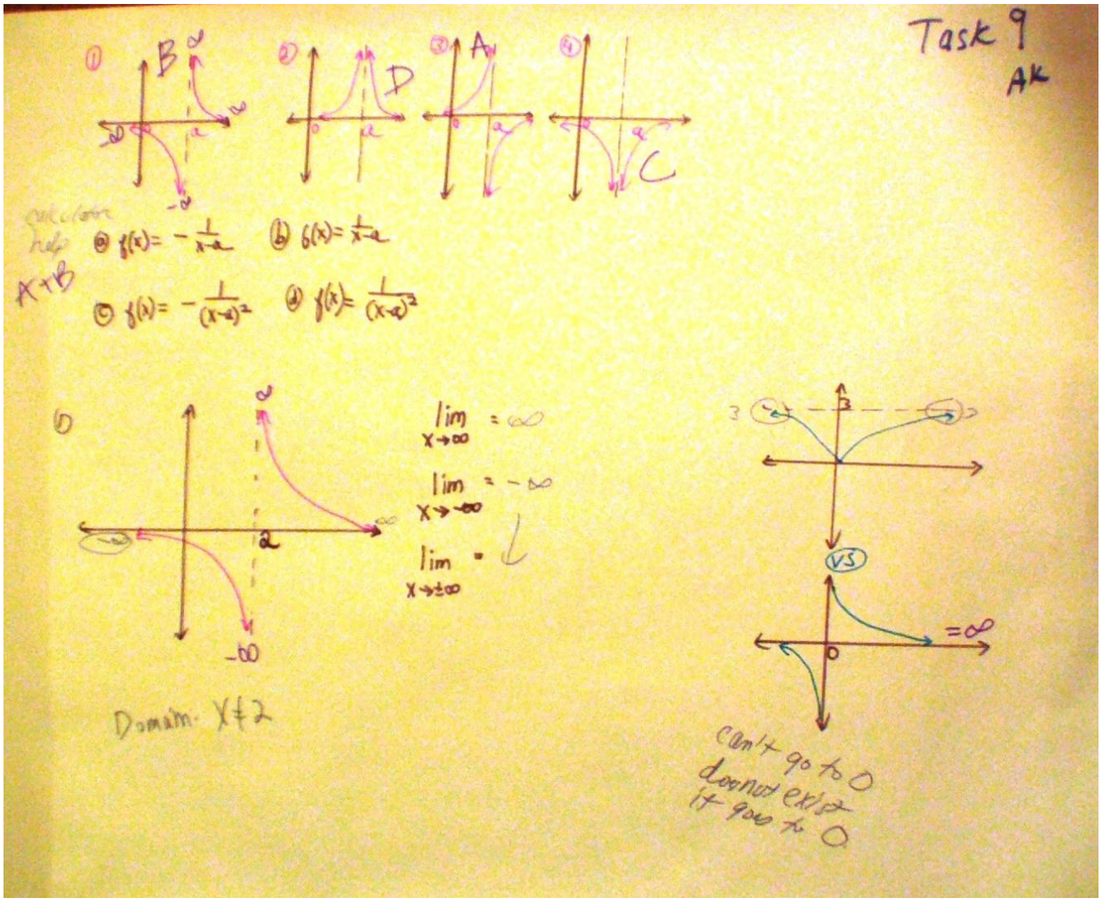

Task 3
AK
(2,6)
Function? Quadratic
lim x→2 = d.n.e b/c of hole
Is (2,6) on the graph of the function? No. It's above the function.
lim x→∞ = d.n.e.
sketch graph out further.
lim x→−∞ = "
Domain (−∞, 2] ∪ [2, ∞)
Range (−∞, 6]
Function? Piecewise
lim x→−2 = d.n.e left ≠ right
lim x→−∞ = −something −∞
lim x→∞ = +something ∞
Domain: ? Not sure.
Range: ?
Depending on teacher
Teacher error.

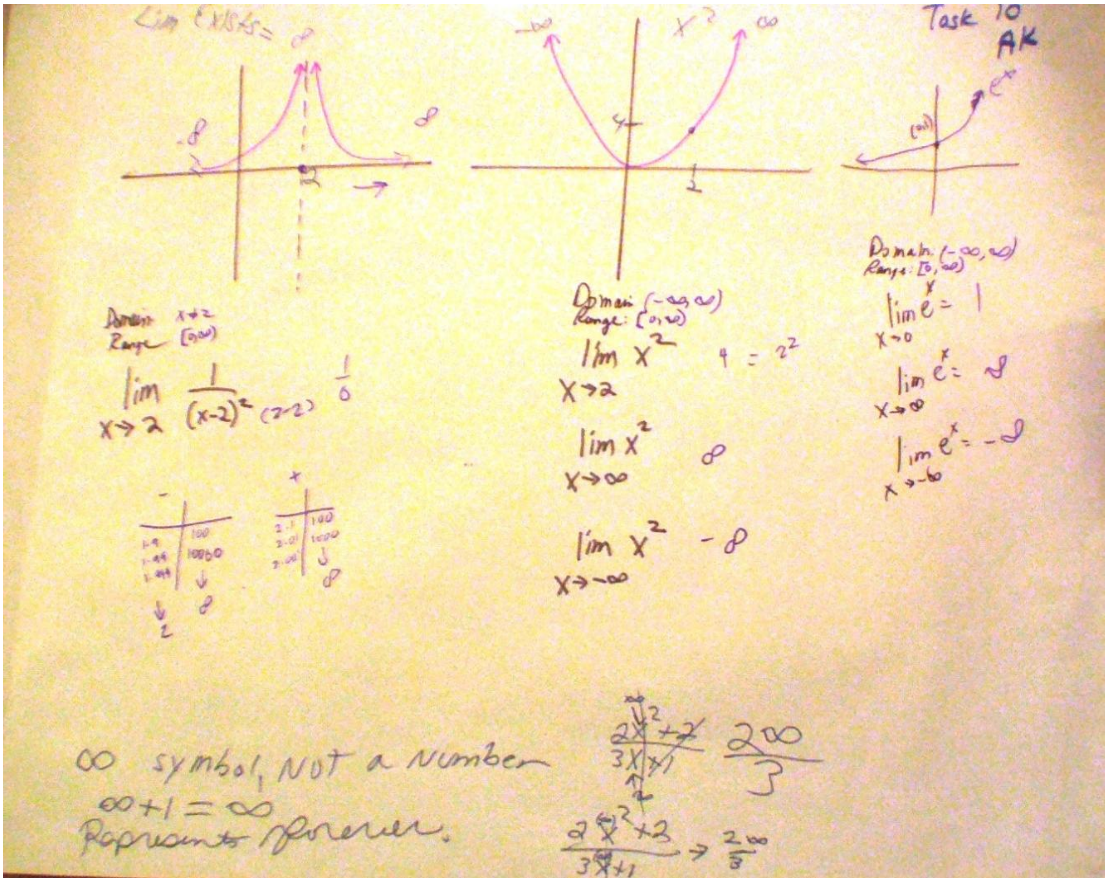

AK
Task 4
Domain (-∞,∞)
Range (-∞,∞)
lim f(x) = 3
x→2
lim f(x) = ∞ DNE but Exists b/c it's there
x→∞
lim f(x) = -∞ DNE but Exists b/c it's there
x→-∞
What's the function f(x)?
f(x) = (x-1) + 1
shift left
shift up
Domain: (-∞,2) ∪ (2,∞)
lim f(x) = DNE because of hole
x→2
lim f(x) = ∞ Exists
x→∞
lim f(x) = -∞
x→-∞
What's the function f(x)?
f(x) = (x-1)+1, x≠2
limit (2,4)
limit dne at (2,3) b/c of hole
Domain: (-∞,2) ∪ (2,∞)
lim f(x) = 4
x→2
Is (2,4) on the graph of the function?
No b/c its not on the line.
But it's the limit b/c x is a solid dot and y is a solid dot.
What's the function f(x)?
f(x) = (x-1)+1, x≠2


Task 4
AK
Domain: (-∞,∞)
Range:
lim x→2 = 3
lim x→±∞ = NOT valid this way.
lim x→∞ = ∞
lim x→-∞ = -∞
Domain: (-∞,2)∪(2,∞)
lim x→2 = dne b/c hole
lim x→∞ = ∞
change b/c of hole
lim x→-∞ = -∞
Domain: same (-∞,2)∪(2,∞) x≠2
lim x→2 = 4
Is the point on the graph of the function? NO
b/c Not on the line
Task 5
AK
f(x) = 2x+1
Domain (-∞,∞)
lim x→3 2x+1 = 7
lim x→2 2x+1 = 5
lim x→±∞ 2x+1 =
lim x→∞ 2x+1 = ∞
lim x→-∞ 2x+1 = -∞
Is (3,8) on the graph of the function?
y = mx+b
Yes. It's on the graph, but not on the function.
It is sitting on the graph but not sitting on the function.
It's still a function though with (3,8) there. You can have extra dots too, like (1,3), (4,1).
These are all on the graph.


Task 6
AK
lim x→2 (x²+4)/(x−2) dne
lim x→±∞ (x²+4)/(x−2)
Domain: x≠2
Domain x≠2
(−∞,2]U[2,∞)
Task 7
AK
lim x→−3 (x²−9)/(x+3)
lim x→±∞ (x²−9)/(x+3)
Domain x≠−3
Domain: x≠−3
(x−3)(x+3)/(x+3)
lim x→−3 = −6
Hole at x=−3
DUMB
Added later by R.


Task 8
AK
$\lim_{x \to \pm\infty} \frac{9x^2+2}{3x^2-2x+5} = 3$
Domain: $(-\infty,\infty)$
deg num = deg den
so lim = 3.
3
Relationship of domain to limits: None. Limit is 3
and domain is $(-\infty,\infty)$
Relationship of horizontal asymptote to limits.
None. 2 separate things. Sometimes if lucky the H.A.
might be the limit but sometimes it might not so what's
on the graph is not always the limit. Here it was
a coincedence.


Task 9
AK
calculator help
A+B
(a) f(x) = − 1/(x−a)
(b) f(x) = 1/(x−a)
(c) f(x) = − 1/(x−a)²
(d) f(x) = 1/(x−a)²
lim x→∞ = ∞
lim x→−∞ = −∞
lim x→±∞ =
Domain: X≠2
VS
can't go to 0
does not exist
it goes to 0


Lim Exists = ∞
Task 10
AK
Domain: x≠2
Range: [0,∞)
lim x→2 1/(x−2)²
Domain: (−∞,∞)
Range: [0,∞)
lim x→2 x²
4 = 2²
lim x→∞ x²
lim x→−∞ x²
Domain: (−∞,∞)
Range: [0,∞)
lim x→0 eˣ = 1
lim x→∞ eˣ = ∞
∞ symbol, NOT a number
∞+1 = ∞
Represents forever.

APPENDIX K: STUDENT QUESTIONNAIRES FOR FUTURE RESEARCH

APPENDIX K-1: Comparing 3 Graphs

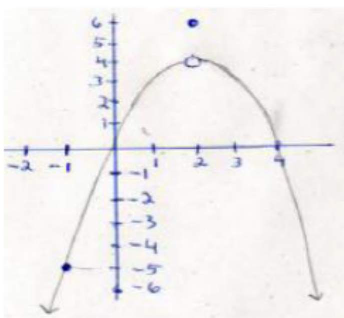
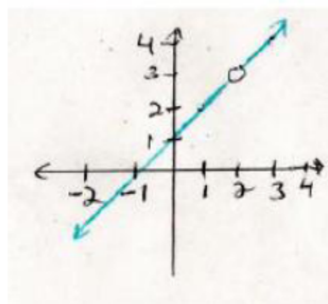
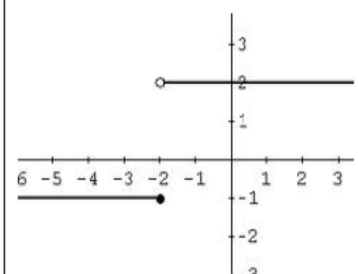
APPENDIX K-2: Comparing 2 Graphs

APPENDIX K1: STUDENT QUESTIONNAIRE FOR FUTURE RESEARCH

COMPARING 3 GRAPHS

Directions:

1. Define (or describe) what a limit is: ______________________________ ____
2. Define what the ∞ symbol represents and/or what infinity means: _____ ____
3. Refer to the graphs in each column and answer the questions below.

 Graph A	 Graph B	 Graph C
Domain is:	**Domain is:**	**Domain is:**
Is this function quadratic? **Yes** **No**	Is this function linear? **Yes** **No**	Is this function piecewise? **Yes** **No**
Is this function piecewise? **Yes** **No**	Is this a rational or piecewise function? **Yes** **No** If yes, which one?	Compute: $\lim_{x\to -1} f(x) =$
Write a formula for this graph: f(x) =	Write a formula for this graph: f(x) =	Write a formula for this graph: f(x) =
$\lim_{x\to 2} f(x) =$	$\lim_{x\to 2} f(x) =$	$\lim_{x\to -2} f(x) =$
Does the limit exist as $x \to 2$? **Yes** **No**	Does the limit exist as $x \to 2$? **Yes** **No**	Does the limit exist as $x \to -2$? **Yes** **No**
Is the point (2,6) on the graph of this function? **Yes** **No**	The limit does not exist as $x \to 2$ because there is a hole. **True** **False**	As $x \to -2$, does the limit = -1? **Yes** **No**
The point (2,6): a. Is a random point that sits above the quadratic function. b. Is part of the function.	$\lim_{x\to \infty} f(x) =$ Does this limit exist? **Yes** **No** In general, when $\lim_{x\to\infty} f(x) = \infty$, I would say which of the following: a. The limit exists b. The limit does not exist.	$\lim_{x\to -2^+} f(x) =$ Does this limit exist? **Yes** **No** I really do not like problems with limits! **True** **False**

Write any comments in this space that you may have including if you'd like to be in a study on limits:

APPENDIX K2: STUDENT QUESTIONNAIRE FOR FUTURE RESEARCH

COMPARING 2 GRAPHS

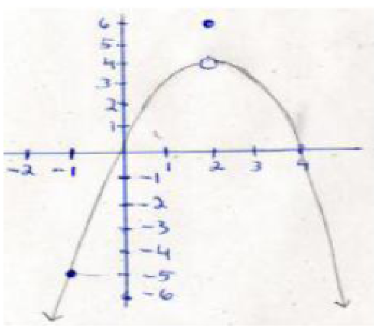
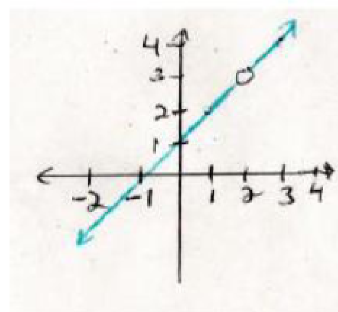
Name________________________ **Course** ______________ **Fall 2012 Limit Study**

Directions: Answer all of the following questions.

A. Explain what a "limit" is in Calculus. ____

B. Explain what the ∞ symbol represents (is it a real large number or not). ____

C. Refer to the graphs in each column and answer the questions below.

 Graph A	 Graph B
1. State the domain:	**1. State the domain:**
2. Is this a piecewise function? **Yes** **No**	2. Is this a rational or piecewise function? **Yes** **No**
3. Write a formula for this graph: f(x) =	3. Write a formula for this graph: f(x) =
4. $\lim_{x \to 2} f(x) =$	4. $\lim_{x \to 2} f(x) =$
5. Does the limit exist as $x \to 2$? **Yes** **No**	5. Does the limit exist as $x \to 2$? **Yes** **No**
6. Is the point (2,6) on the graph of this function? **Yes** **No**	6. The limit does not exist as $x \to 2$ because there is a hole. **True** **False**
7. The point (2,6): a. Is a random point that sits above the quadratic function. b. Is part of the function.	7. $\lim_{x \to \infty} f(x) =$ Does this limit exist? **Yes** **No**
8. $\lim_{x \to \infty} f(x) =$ Does this limit exist? **Yes** **No**	8. As $x \to 2$, the point x=2 must be in the domain for the limit to exist at the point (2,3). **True** **False**

Write any comments in this space that you may have. If you'd like to be in a study on limits, please leave contact information:

APPENDIX L: SAMPLE LESSON PLAN ON PIECEWISE FUNCTIONS

Limits of Piecewise Functions

Subject: AP Calculus
Grade: 11 and 12
Topic: Introduction to Limits
Class Length: 90 minutes

I. Initial Planning
This lesson is planned for a relatively small class of about 20 students. The class has 12 boys and 8 girls and consists of 12 Whites, 3 African American, 3 Asian, and 2 Hispanic students. Two students are special needs. One involves a behavioral-emotional disability; the other is ESL but is fluent with English.

Following the completion of this lesson, students should be able to:

- Define limit and infinity.
- Understand the limit notation.
- Identify piecewise functions in general.
- Identify a piecewise function in which one piece is a function and the other piece is a point (function value) that is not equal to the limit.
- Distinguish between limits and function values with piecewise functions.
- Construct graphs from function formulas.
- Derive the formulas from graphs.

Objective: Students will be able to distinguish limits from function values on graphs of piecewise functions. They will be able to write formulas from graphs of piecewise functions, and be able to construct graphs given the formulas. By looking closely at the intervals, they will be able to tell if a point is in the domain of the function. They will also capitalize on learning that with limits, as $x \to a$, the "a" need not be in the domain of the function for a limit to exist. Finally, they will be able to determine that when there is a discontinuous function with a function value, the limit exists but is not equal to the function value.

NCSCOS:
AP Calculus Mathematics Competency Goal 1:*
The learner will demonstrate an understanding of the behavior of functions.

Objective 1.01 Demonstrate an understanding of limits both local and global.

a. Estimate limits from graphs or tables of data.

Objective 1.03 Identify and demonstrate an understanding of continuity of functions.

a. Develop an intuitive understanding of continuity. (Closed values of the domain lead to closed values of the range).

b. Understand continuity in terms of limits.
c. Develop geometric understanding of graphs of continuous.

* www.dpi.state.nc.us/curriculum/mathematics/scos

Prior Knowledge: There should be familiarity with piecewise functions and their domains. They previously studied limits from a graphical perspective and learned with graphs that when there is a hole, the limit exists, which will mean now more formally considering the domain, as $x \to a$, the "a" need not be in the domain of the function for a limit to exist. They also learned how to interpret the limit notation so they can read it correctly from left to right.

Knowledge of Student Characteristics:

- How well do students work in groups?
- What algebra deficiencies are common to this group?
- What are the writing strengths of this group?
- Have they used K-L-W and graphic organizers before?
- What prior experiences with functions, physical systems and limits do they bring to this class?
- Do students know how to take notes?
- Do students have good study skills?
- Do students correctly transcribe what I say or what they "think" they heard me say?
- Do students know the difference between how to read a math book and how to read a novel?
- Are students able to critique themselves and their peers? How do they accept criticism?
- Are there particular IEP goals that must be met that are best met with certain learning situations?

Resources/Materials: Graphing calculator, paper, pencil and ruler.

- Textbook for practice problems and supplemental information
- Graphic organizers
- K-L-W Noteboo.
- Practice and homework worksheets
- Graphic Organizer for Note Taking
- Think Pair Share sheet
- Assessments

II. Lesson Introduction

- Go over assessment, cooperative group activity and homework from Day 1 (Appendices).
- Remind students of their knowledge of functions and limits at a point.

- Go over meaning of function and limit notation.
- Review definitions of limits and infinity via written and graphical examples.

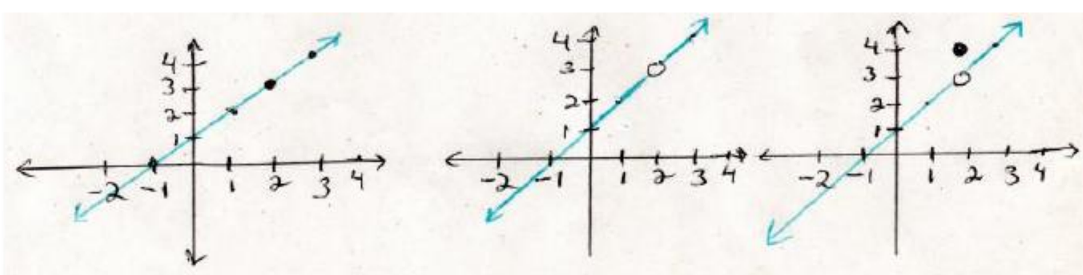
Focus and Review:   In both cases, the limit exists. Now when we look at this graph, we will see that the limit also exists, but is not equal to the function value.  To better understand this graph, we will carefully look at piecewise functions and study their domains, reviewing that the domain is about what the x-values are doing, and that piecewise functions have two components, both of which should be considered when finding limits at a point for piecewise functions.

III. Lesson Development

Instructional Method and Student Involvement**:** This lesson will involve mostly lecturing by direct instruction so students will be listening; watching examples presented, asking questions and taking notes. Graphic organizers and KLW sheets will help students put the main ideas into a meaningful format.

Teacher Input: <u>Part I</u>: Last time we said the limit and function value are equal. This is represented by f(x)=L. In limits, the letter "L" is denoted to represent a number and we will say that this number L stands for Limit. This means that if you have a continuous function, the function value is equal to the limit. When you see this notation $\lim_{x \to a} f(x) = L$, this means that as x is approaching a, the function values are getting close to L. In this instance, you can have either a dot or a hole. In both symbolic representations, since you see "=L" you conclude that the limit exists.

Part II: Lets compare the domains of 3 graphs now in order to study their domains.



Graph A Graph B Graph C

` Notice that graph A and C have the same domains $(-\infty, \infty)$. Graph B, though, has a discontinuity and therefore it's domain is different, $x \neq 2$. Now we consider this graph as piecewise or even rational (if it had common factors that divided out in the numerator and denominator). At this time, we only want to emphasize that 2 is not in the domain.

As $x \to 2$ the limits exist for all 3 graphs—graph A is continuous and graph B is discontinuous but even though x=2 is not in the domain for Graph B, the limit still exists. This is because as $x \to a$, the a does not have to be in the domain for the limit to exist. This is a major point which should help clarify why when you see a hole, the limit still exists. Limits are about what is happening "near" a, not just "at" a. In Graph C, notice there is a function value above the hole. People tend to say the limit exists and equals the function value at (2,4). This is not correct. The limit is still equal to 3 in the problem, for the reason in graph 2, that the graph is tending toward the same point in the plane (2,3). Even though (2,3) is not on the graph of the function, it is still in the plane and therefore since the function values are tending toward 3 from the left and from the right, the limit exists. The limit has nothing to do with the function value in Graph C. The point (2,4) is in the domain, but with limits, the point does not have to be in the domain for the limit to exist. Also, (2,4) is the function value, not the limit, and it cannot be a limit because there are no function values tending toward an isolated point on a graph. So if you see graphs that are similar to Graph C, you should distinguish between the limit and the value of the function, in which case, they are different. The limit is 3 and the function value is 4.

Look again at Graph C. Is the point (2,4) on the graph of the function? Yes. The reason is that (2,4) happens to be part of the function. If you do not know that there are 2 parts of the function that match this graph, then you will not understand why the limit is not equal to the function value. To think about this in more general terms, ask what does it mean for a point to be on the graph of the function, and then answer that with "the second coordinate is f(of) the first coordinate". This essentially constitutes a point in the plane, (x,y).

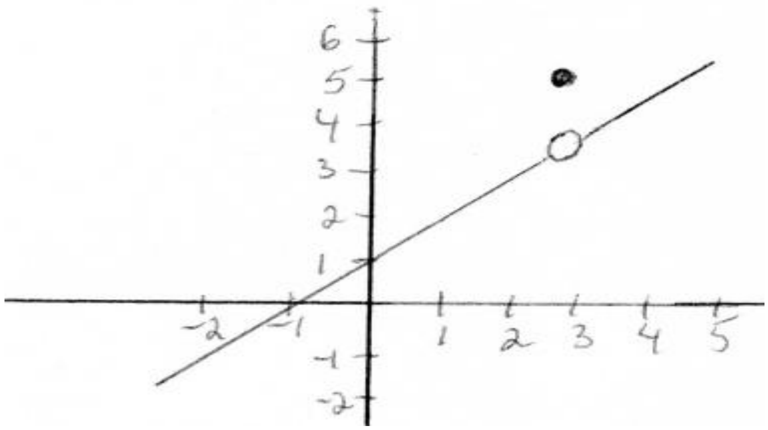
IV. Lesson Implementation

Guided Practice:

Let's look at this function. Write it down the construct its graph. Specify the domain. What you will see is that the limit exists but it is not equal to the function value. This is a piecewise function.

$$f(x)=\begin{cases}5 & \text{if } x=3\\ x+1 & \text{if } x\neq 3\end{cases}$$

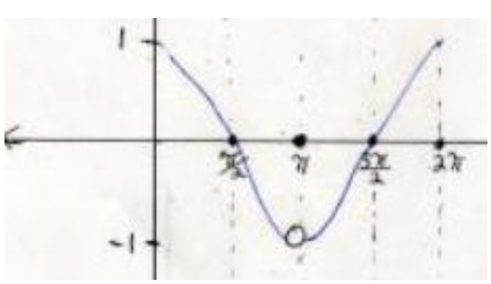
Here is the graph we should construct:



Continued Guided Practice and Supplemental Instruction for Diverse Learners:

This is a good time to review some algebra involving shifts and translations. We have f(x)=x+1. This means the slope is 1, and the y-intercept is 1. We can think of the vertical shift as being (x+0) units or (x-0) units because when we refer the x next to the equal sign, this tells us to go left or right. Since there is no value to translate left or right, we simply pretend there is a 0 in its place, so 0 is a placeholder. Then reading the equation from left to right, we see that we have +1 which is not only the y-intercept from the formula y=mx+b, but it also means to shift up 1 unit. What is attached to the x is where you go left or right, and what comes last in the equation is up or down. Now if we want to check to see where the x-intercept is, we are going to set the whole equation to 0, so we replace y with 0. In that case y=x+1, 0=x+1, x=? subtract 2 from both sides and x=-1, so the x-intercept is 1. Now that we have 2 points, we can sketch the line on the graph but must leave a hole at the point (3,3) since x=3 is not in the domain of the function.

Let's now begin with a graph. From the graph we want to construct the function. Specify the domain of the function. Note: should write [0,2π]



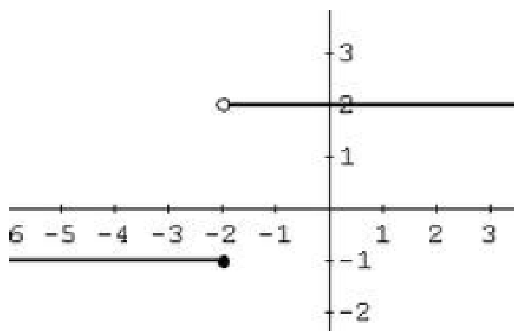
Answer: The formula is: $f(x)=\begin{cases}\cos x, \text{ if } x\neq\pi \\ 0, \text{ if } x=\pi\end{cases}$

Is the point $(\pi,0)$ on the graph of this function? Explain.

Answer: Yes, because when x= π , f(x)=0. This point is part of the piecewise function.

Now, does the limit exist as $x\to\pi$? Yes, the limit is -1 because the left hand limit is equal to the right hand limit if you were to slide both fingers down towards the hole on the graph. The limit is -1 and the function value is π, so we have just shown how the limit can be different from the function value when we have a piecewise function.

Finally, consider the graph of this piecewise function below. As $x\to-2$ note visually how the left hand limit of -1 is not equal to the right hand limit of 2; therefore the limit does not exist as $x\to-2$. Now each limit exists separately as one-sided limits, but not together at x=-2. Note that even though x=-2 is in the domain, the limit in this case does not exist because the left hand limit does not equal the right hand limit.



Independent Practice:

Come up with a piecewise function of your own, one that contains a formula and a point in the plane as we have just done. Then come up with a graph of your choice that has a two components, one of which should be a point, and then write it's formula. In each case, explain if the limit exists.

Day 2 Worksheet is found in the Appendices.

Cooperative Group Learning: (none for this particular lesson)

Closure: We have learned about piecewise functions, domains and limits of piecewise functions. We learned that as $x\to a$ the "a" does not have to be in the domain in order for a limit to exist. For continuous functions, the limit equals the function value. When there is a discontinuity without a function value, the limit exists in spite that the "a" is not in the domain and this is because limits are about how close x gets to "a" not what "a" equals. In cases where there are piecewise functions with two distinct components to the graph, a line with a hole and dot above or below, the limit is different from the function value.

Assessment:

There will be a several questions asking once again what a limit is to reinforce the definition; given a graph state the formula; given a formula, construct the graph; does the limit exist as x approaches the "a" in each case; is the point on the graph of the function?

Plans for Students who Finish Early: Can help those having difficulty.

Plans for Individual Differences:

Students with ADHD can continue doing a hands-on calculator exploration with different parts of piecewise functions. Gifted students can construct piecewise functions of 3 pieces and those that involve $\leq or \geq$ in the domain. Students who finish early and those of above average ability can sit next to ESL students to help write and recite the definition of limit.

Technology Use: TI-84 graphing calculator.

Project for this Topic: (already assigned)

V. Lesson Evaluation

The main short-term evaluation of this lesson will come from the short assessment given at the end of this lesson plan that they will complete in class and hand in. The final project and final test will also be a long-term evaluation. I will also assess students based on the nature of their questions and responses to my questions. For instance, are questions just "what does that say up there" versus "can you explain the difference between the two graphs?" I will evaluate how well they work in their cooperative groups and the accuracy of their independent practice homework assignment in which case all work must be shown and KLW sheet completed and put in their binders.

Homework: Piecewise Function Homework: limits and graphs
Piecewise Function Assignment using Limits

APPENDIX M: SAMPLE ASSESSMENT ON PIECEWISE FUNCTIONS

Name____

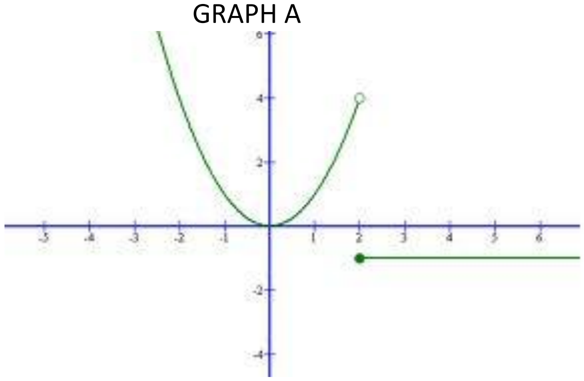
Assessment on Limits of Piecewise Functions

1. What is a limit?

2. What does "L" represent when you see the notation f(x)=L versus $\lim_{x \to a} f(x) = L$

3. Study GRAPH A below. Write a formula for this graph.

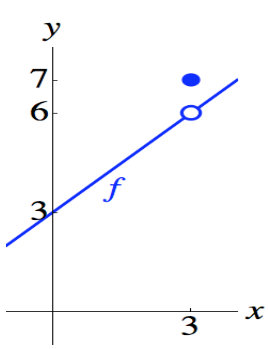
GRAPH A



4. Explain if the limit exist as x approaches 2? Remember to consider the left hand limit versus the right hand limit only by visually inspecting the graph.

5. In GRAPH B below, what is the limit as x approaches 3?
6. Is the point (3,7) on the graph of the function? Explain.
7. Write a formula for Graph B.

GRAPH B:

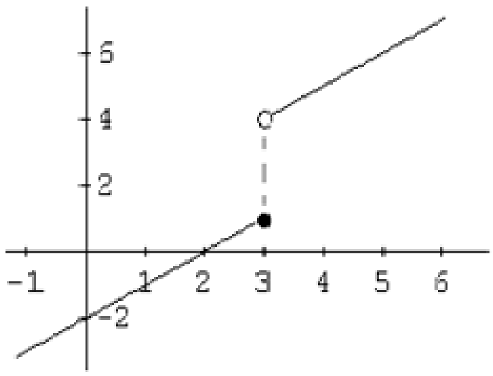


8. In Graph B, what is the limit?

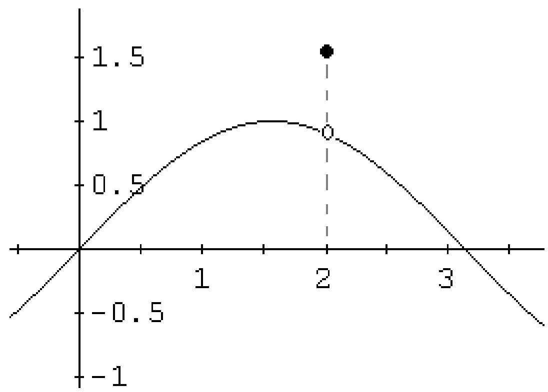
9. In Graph B, what is the function value?

10. Does x=3 have to be in the domain for the limit to exist? Explain your answer.

Graph C



Graph D



11. REFER TO THE ABOVE GRAPHS, C and D.

a. What do the following two graphs have in common? Explain their similarities and/or differences.

b. What's the limit as x approaches 3 in Graph C? Does it exist? Explain.

c. What's the limit as x approaches 2 in graph D? Does it exist? Explain.

d. True or False. In Graph D, the limit is (2, 1.5).

e. Does x=2 have to be in the domain for the limit to exist as x approaches 2? Explain.

12. Is there anything you did not understand from this discussion that you still need to know?

APPENDIX N: SAMPLE LESSON PLAN ON RATIONAL FUNCTIONS

Limits of Rational Functions Part 2

Subject: AP Calculus
Grade: 11 and 12
Topic: Introduction to Limits of Rational Functions
Class Length: 90 minutes

I. Initial Planning
This lesson is planned for a relatively small class of about 20 students. The class has 12 boys and 8 girls and consists of 12 Whites, 3 African American, 3 Asian, and 2 Hispanic students. Two students are special needs. One involves a behavioral-emotional disability; the other is ESL but is fluent with English.

Following the completion of this lesson, students should be able to:

- Define limit and infinity and understand the limit notation for rational functions.
- Compute limits of rational functions, at a point and at infinity; explain what it means.
- Identify whether or not limits exist on graphs of rational functions.
- Show evidence of understanding the definitions provided in a list. Identify the 3 rules for finding asymptotes in addition to factoring.
- Show evidence knowing functions can cross horizontal asymptotes, but not cross vertical asymptotes.
- Show evidence of knowing the limiting behavior of function values can be identified by moving fingers along the graph.

Objective: Students will be able to determine limits of rational functions by stating the domain, performing any necessary polynomial division and transferring the information to graphs. Knowing the meaning of infinity, they will be able to explain that the limit does not exist when function values increase without bound in either the positive or negative directions. Students will know the difference between what procedures to use for limits at a point and limits at infinity for rational functions.

NCSCOS:
AP Calculus Mathematics Competency Goal 1:*
The learner will demonstrate an understanding of the behavior of functions.

Objective 1.01 Demonstrate an understanding of limits both local and global.

a. Calculate limits, including one-sided, using algebra.
b. Estimate limits from graphs or tables of data.

Objective 1.02 Recognize and describe the nature of aberrant behavior caused by asymptotes and unboundedness.

a. Understand asymptotes in terms of graphical behavior.
b. Describe asymptotic behavior in terms of limits involving infinity.

Objective 1.03 Identify and demonstrate an understanding of continuity of functions.

a. Develop an intuitive understanding of continuity. (Closed values of the domain lead to closed values of the range).
b. Understand continuity in terms of limits.
c. Develop geometric understanding of graphs of continuous functions (Intermediate Value Theorem and Extreme Value Theorem).

* www.dpi.state.nc.us/curriculum/mathematics/scos

Prior Knowledge: Students must know the definition of a function and limit, know how to identify the domain of a function, and know that infinity means that numbers keep increasing without bound. They must also know how to factor and perform polynomial division.

Knowledge of Student Characteristics:

- How well do students work in groups?
- What algebra deficiencies are common to this group?
- What are the writing strengths of this group?
- Have they used K-L-W and graphic organizers before?
- What prior experiences with functions, physical systems and limits do they bring to this class?
- Do students know how to take notes?
- Do students have good study skills?
- Do students correctly transcribe what I say or what they "think" they heard me say?
- Do students know the difference between how to read a math book and how to read a novel?
- Are students able to critique themselves and their peers? How do they accept criticism?
- Are there particular IEP goals that must be met that are best met with certain learning situations?

Resources/Materials: Ruler, TI-84 Calculator, paper and pencil.

- Textbook for practice problems and supplemental information
- Graphic organizers
- K-L-W Notebook
- Graphic Organizer for Note Taking
- Think Pair Share sheet
- Practice and homework worksheets
- Assessments

II. Lesson Introduction

- Remind students of their knowledge of fractions, domains and rational functions.
- Go over meaning of limits at a point and limits at infinity.
- Review asymptotic behavior of functions.
- Identify the 3 rules for finding asymptotes in addition to factoring.
- Functions can cross horizontal asymptotes, but not cross vertical asymptotes.
- Limiting behavior of function values can be identified by moving fingers along the graph.

Focus and Review: We will go over Day 3 Assessment and Homework (Appendices).

Last time we looked at the graphs of an elementary rational function and transferred that information we learned to similar graphs of that nature. Now we will look more closely at limits at a point of rational functions by studying domains and doing some factoring or polynomial division. We will see that there are two different procedures for computing limits at a point and limits at infinity. After we compute the limits of rational functions, we will translate that information to graphs and determine whether or not limits exist, being able to explain why.

III. Lesson Development

Teacher Input: Start with the rational function below.

$$\lim_{x \to -3} \frac{x^2 - 9}{x + 3} \qquad \lim_{x \to \pm\infty} \frac{x^2 - 9}{x + 3}$$

Limit 1 **Limit 2**

Describe the domain. Compute the limits if they exist. Explain the behavior of the function values.

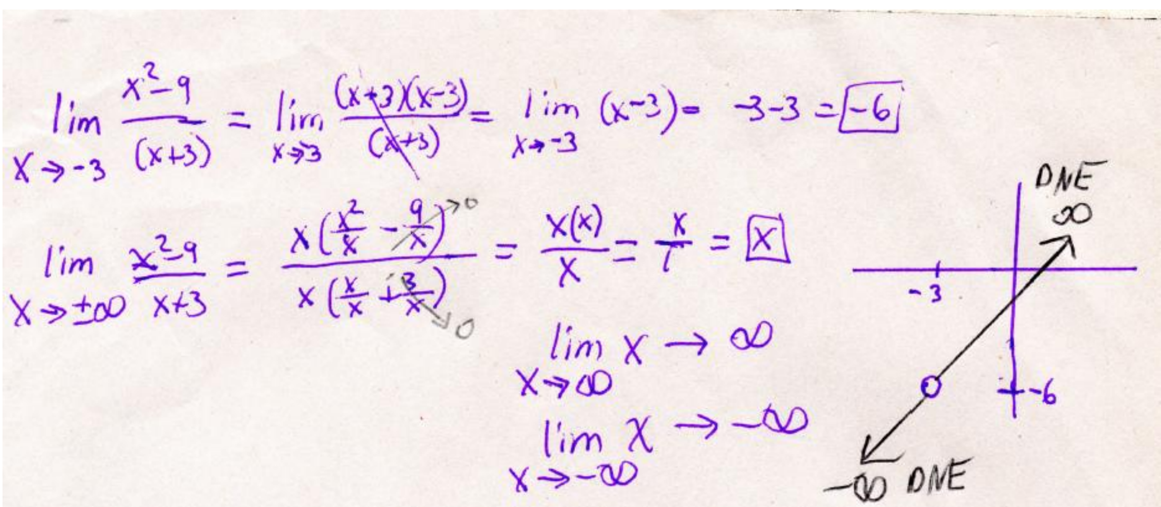
What is the domain? Well we know that x=3 cannot be in the domain or the function is not defined. So we know that the domain is $\{x | x \neq 3\}$.

Look at the denominator first. If we cannot factor that, we look at the numerator. If we can factor the numerator, we will see that we have (x+3)(x-3). The two factors of (x+3) in the numerator divide out and we are left with (x-3) in the numerator. Note here that when we factor, we have literally changed the function to a linear function. The domain of this linear function now is different from the domain of the rational function. Now it is $(-\infty, \infty)$. It is important to note that whenever you change the domain, you have changed the function. If you were to graph these in the calculator, you would get the exact same straight line, but you must acknowledge that x=3 is not in the domain of the rational function and so you will not have a vertical asymptote there, but rather, a hole to represent that point in the domain. You get holes when there are common factors that divide out. You get vertical asymptotes when you cannot factor and if you were to plug x into the denominator, the denominator would be 0.

So for a limit at a point, you first try to plug in. If you get $\frac{0}{0}$ by plugging in, then you must try to factor. If common factors are found in the numerator and denominator, you divide them out and remember to put a hole in the graph to represent the discontinuity. . If you cannot divide anything out, you draw a vertical asymptote to represent the discontinuity.

Once you divide out, plug the x=-3 into the limit as follows. $\lim_{x \to -3} \frac{x^2 - 9}{x + 3}$ = $\lim_{x \to 3}(x - 3)$ =-3-3=-6. So your limit is -6. This is the y-value. (-3,-6) is in the plane but it is not on the graph of the function because it is a hole, so you must represent it as such in the graph.

To compute the limit at infinity, you must factor out the highest power of x separately from the numerator and denominator. $\lim_{x \to \pm\infty} \frac{x^2 - 9}{x + 3}$ The terms to the right go to 0, as 9 and 3 divided by a large x tend to 0. If you divide out the x-squared, you are left with x in the numerator and 1 in the denominator. As x goes to infinity, so do the function values. You can use the calculator to plug in some large positive numbers for x and some large negative numbers for x to see how the function values behave in each direction. Below is the work and complete graph once it is all done.



There are rules for finding asymptotes that you've learned earlier on that will assist you here as well to check your work. It should not be used as a substitute.

Degrees of numerator<degrees of denominator: Horizontal asymptote at y=0 *Degrees of numerator=degrees of denominator: Asymptote is ratio of leading coefficient* *Degrees of numerator>degrees of denominator: No horizontal asymptote. Other types occur such as slant (oblique) if degrees differ by 1, and parabolic asymptote if degrees differ by 2.*

IV. Lesson Implementation

Guided Practice: Now that you have some skills, let's try one together that is a little bit different.

$$\lim_{x \to 2} \frac{x^2 + 4}{x - 2} \qquad \lim_{x \to \pm\infty} \frac{x^2 + 4}{x - 2}$$

Limit 1 **Limit 2**

Describe the domain. Compute the limits if they exist. Explain the behavior of the function values.

If we plug 2 into the denominator, what happens? We get $\frac{2}{0}$ so the limit does not exist. Let's look at the numerator. Can we factor the numerator? Be careful. The answer is no, we cannot factor. Since we won't get any common factors this time, what happens? We have a vertical asymptote at x=2. What does this mean? Well, take our your calculator and plug this function into Y1. Then do VARS and Y-VARS to plug in test values, or you can use your table. Select test values close to 2 from the left and right. Do you see that as x approaches 2 from the left, the function values are negative but from the right the function values are positive? Well lets use the calculator to draw this function. We see 2 curves in the plane and what appears to be asymptotic behavior at x=2, so this makes sense and corresponds to our original result when we plugged 2 into the numerator and denominator and got 2. As x is going to plus or minus infinity, you can now refer to your calculator and write down some of those numerical values as well as look at the graph itself to see that as x approaches minus infinity, the limit does not exist because function values increase in the negative direction without bound and as x approaches positive infinity, the limit does not exist because the function values increase in the positive direction without bound.

Independent Practice: Now I want you to try this function. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ and $\lim_{x \to \pm\infty} \frac{x^2 - 4}{x - 2}$

Your graph should be similar to the example above**.** A x approaches 2, the limit is 4 and there is a hole in the graph at (2,4). The limits do not exist as x approaches plus or minus infinity because the function values increase in both directions without bound. Day 4 Worksheet will be given.

Cooperative Group Learning:

$$\lim_{x \to \pm\infty} \frac{9x^2 + 2}{3x^2 - 2x + 5}$$

Describe the domain and the limiting behaviors. Sketch the graph and explain how the function behaves for large x in the positive and negative directions. Do the limits exist? What's the relationship between horizontal asymptotes and limits, if any. Describe how knowing the domain is associated with knowing how to find the limits.

Let's explore a different function together. Use KLW and Think-Pair-Share. This problem looks different but you have several skill sets you can use to determine what the limit is as x gets very large in either direction. You might be able to look right at it and tell what the limit is, but do not reveal any information until the end, so that there is an opportunity for group members to discuss how to determine the limiting behavior of this function. It is not about getting the answer, but about how to systematically work through a task in an algorithmic, cooperative manner given the skill sets you have.

We will work in groups of 4, each person will be A, B, C and D. Person A will read the task aloud. Person B will see if it can be factored. Person C will agree or disagree. Person D will draw a table of values, perhaps 6 different values altogether, picking some large positive and negative numbers. Person A will put these same numbers into a table on the calculator. Person B will then sketch a rough graph of the function. Person C will plot some points from the hand drawn table and calculator to see what they find. By now, everyone should be seeing that there is asymptotic behavior around y=3. Now Person D asks the rest of the group members if there is any relationship between the table of values they computed, the sketch of the graph, and the limit they computed for this function. The answer should be that there is limiting behavior around y=3, which happens to be a horizontal asymptote. Person A now refers to the table of the rules for asymptotes to see if any of those rules apply here. The other group members do the same. Person A asks if any of the rules apply and each group member will state their opinion.

Finally, someone in the group can volunteer to say if they see any relationship between the graph's limiting behavior of the function values and the limit they identified for this rational function.

The answer should be that the limit is 3, which is also the horizontal asymptote. A way to check is by noting that the degrees of the numerator and denominator are the same, therefore you take the leading term's coefficient as being the limit. So the graph drawn corresponds with the limit identified.

Note: Some people may try to do polynomial division to find the limit, or may try to factor out the highest power of x separately from the numerator and denominator and either way, you will still end up with y=3, or the limit is 3.

Closure: Students learned to determine limits of rational functions by stating the domain, performing any necessary polynomial division and transferring the information to graphs. The limit does not exist when function values increase without bound in either the positive or negative directions. There are different algebraic procedures to use for limits at a point and limits at infinity for rational functions.

Assessment: An open ended type worksheet will be given.

Plans for Students who Finish Early: Using the calculator, they can explore different types of rational functions, such as when the numerator is 0 and the denominator is not 0, and use polynomial division if necessary then graph the final result.

Plans for Individual Differences**:** ESL students will sit together in pairs within a cooperative group and will write their ideas down on paper if they cannot articulate them verbally just yet. Gifted students can create two other rational functions for their peers to consider exploring to reinforce the current concepts.

Technology Use: TI-84 Calculator and Equation Grapher Software on the computer.

V. Lesson Evaluation

The main short-term evaluation of this lesson will come from the short assessment given at the end of this lesson plan that they will complete in class and hand in. The final project and final test will also be a long-term evaluation. I will also assess students based on the nature of their questions and responses to my questions. For instance, are questions just "what does that say up there" versus "can you explain the difference between the two graphs?" I will evaluate how well they work in their cooperative groups and the accuracy of their independent practice homework assignment in which case all work must be shown and KLW sheet completed and put in their binders.

Homework: Day 4 Homework Sheet (Appendices)

Project for this Topic: (already given out on day 1)

APPENDIX O: SAMPLE ASSESSMENT ON RATIONAL FUNCTIONS

Name____

1. What is a limit?

2. If you compute a limit and it equals infinity, explain if the limit exists or not.

3. State the domain. Compute the following limit at a point and at infinity and construct its graph.
   Show all of your work and steps that you use before constructing the graph. Explain what procedural techniques are used for each type, limit at a point and limit at infinity. For example, when do apply rules of asymptotes and what kind of factoring might be done with each case?

   $$\lim_{x \to -5} \frac{x^2 - 25}{x + 5} \text{ and } \lim_{x \to \pm\infty} \frac{x^2 - 25}{x + 5}$$

4. State the domain. Compute the following limit at infinity and construct its graph. Show your algebraic work and explain any rules of for finding asymptotes that you may have used to check your result. Explain if there is a relationship between the limit you computed and the result you are graphing.

   $$\lim_{x \to \infty} \frac{x^2 + 1}{2x^2 - 1}$$

5. State the domain. Compute the following limits at infinity and construct the graph.
   Is there going to be any horizontal asymptote on this graph? Explain why or why not.
   What kind of asymptotic behavior would the graph have?
   $\lim_{x \to \pm\infty} \frac{3x^2 - 1}{x}$

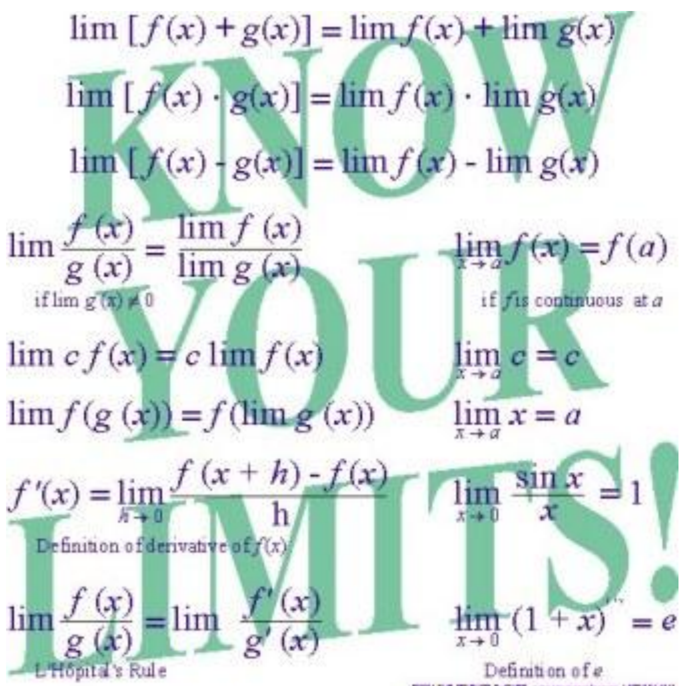
   Break this problem down into two separate limits, as $x \to \infty$ and $x \to -\infty$ then for each one explain if the limit exists.

6. Construct a rational function of your own that you might like to give to the class. It can be a limit at a point, a limit at infinity or a combination of both.

7. Is there anything presented in today's discussion you were unclear about and still need to know more. Write down your question here:

Limits in Calculus

Literacy Toolkit



Margaret Adams

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INTRODUCTION

Limits are the heart of calculus, yet many students are completely lost when they encounter this topic at the beginning of the calculus sequence. Derivatives are limits, integrals are limits, series are limits, and there is limiting behavior all around us. Knowing how to compute limits of functions differs from conceptually understanding what the computations mean. The North Carolina Standard Course of Study attempts includes AP calculus and understanding of functions, and so knowing the basic, most fundamental information is imperative to students learning calculus for the first time.

Given toolkits have become more and more popular in school districts, some curriculum coaches are encouraging teachers to have toolkits. In math, this is very helpful and useful because it gives an opportunity for teachers to incorporate reading and writing in math, and have resources for specific topics in math. The toolkit is also important to teachers because if they know ahead of time what kinds of misconceptions students have about any given topic, they can account for this up front and implement the literacy toolkit to help eliminate the misconceptions from occurring.

Students must learn to read mathematics texts. Martinez and Martinez (2001)[1] contend that students learn to use learn to use language to focus on and work through math problems, to communicate ideas coherently, organize ideas and structure arguments and extend their thinking and knowledge to encompass other perspectives and experiences. They also use mathematical language to understand their own problem-solving and thinking processes as well as those of others and develop flexibility in representing and interpreting ideas. Moreover, students begin to see mathematics not as some series of isolated discrete topics but more as part of the world. They ultimately are able to make connections to concepts and knowledge encountered across the curriculum.

Another reason students need to learn how to read mathematics is that reading mathematics requires unique knowledge and skills not taught in other content areas. Math students must be able to read from left to right as in other subject areas, but also from right to left when reading an integer number line, and from top to bottom when reading tables, and even reading diagonally when analyzing graphs.

Mathematics texts contain more concepts per word, per sentence and per paragraph than any other kind of text (Brennan & Dunlap, 1095; Culyer, 1988) [2,3]. In addition, the concepts are often abstract so it is difficult for readers to visualize their meaning. Also, math is written in terse language. Sentences contain a lot of information with little redundancy. Sentences and words have precise meanings and connect logically to surrounding sentences. Students who want to read mathematics texts quickly, like they do a short story in their language arts class, is in trouble because they may miss significant details, explanations, vocabulary and underlying logical principles.

Mathematics also requires students to be proficient at decoding not just words but numeric and nonnumeric symbols (Barton & Heidema, 2002) [4] . Consequently, the reader must shift from "sounding out" words such as plus or minus to instantly recognize their counterparts, + and -. Even the layout of a math textbook can inhibit comprehension. Students often scan a page looking for examples, graphics or problems to be solved, skipping worded passages filled with critical information. Also, many math texts are written above grade level for which they are intended, so the vocabulary and sentence structure in math textbooks are often very difficult for students using these texts. Lastly, publishers of many math texts include longer passages of verbal text (prose). So both students and teachers have to understand how to navigate these passages successfully. Other more traditional texts apply short verbal passages, a few examples and a set of problems to solve.

Most likely very few educators particularly in administration and in other content areas think about the challenges of teaching math, and the reasons why literacy in mathematics is ever so important. In this toolkit, there are many strategies for reading, comprehension, vocabulary and technology. The overall theoretical framework for this toolkit is constructivism, as students assimilate external events into mental structures and construct knowledge accordingly into various schemas. Sometimes these schemas are correct and appropriate, and other times, they are not given they contain misconceptions and deficits due to deficiencies in reading, vocabulary, comprehension and vocabulary. The overall theoretical teaching strategy for this toolkit is Before-During-After as many of the tools can be incorporated at any time and over the course of time. Finally, The N.C. strategic plan for students is to be globally competitive and to promote reading literacy and literacy strategies in every curriculum content area (dpi.state.nc.us).

1. Martinez, J. & Martinez, N. (2001*). Reading and writing to learn mathematics: A guide and resource book*. Boston: Allyn & Bacon.

2. Brennan, A. & Dunlap, W. (1985). What are the prime factors of reading mathematics? *Reading Improvement, 22,* 152-159.

3. Culyer, R.C. (1988). Reading and mathematics go hand in hand. *Reading Improvement, 25,* 189-195.

4. Barton, M.L. & Heidema, C. (2002). *Teaching Reading in Mathematics,* 2nd Ed. CO: Mid-continent Research for Education and Learning.

LITERACY TOOLS

Reflective Memo

AP Calculus is part of the high school curriculum. Calculus is the study of rates of change. Reading is the basic fundamental skill needed to be successful in this course and to understand the content. Being able to read definitions, examples and problems posed requires knowing that math books are read differently from novels and social studies texts. Each part of the text is read slowly and then read over again several times. Next, the examples are read and then read over again before looking at the mathematical steps. Each author has a writing style in mathematics that is portrayed in the textbook, and so the student can become familiar with the author's style and assimilate it into their reading so that it appears as if a real person wrote the textbook, which is the case. In addition to examples, there are also websites, applets or calculator activities that involve the use of technology.

To demonstrate students' comprehension skills, we would use the K-W-L plus Vocabulary Chart so that students could access prior knowledge. K-W-L is a discussion based strategy where the teacher is scaffolding the discussion as the facilitator. The students participating in class discussions support content literacy. Students learn when they are engaged in a collaborative environment. The discussion web is another strategy that could be used in the Calculus classroom. This strategy develops students' ability to exchange point of view with others in a group or pairs after reading over a particular topic of in the textbook. By doing so, misconceptions can be identified and corrected. In the process, it can be determined how many students are having the same misconceptions. a book. It encourages them to perceive the content presented and make interpretations of what they have read before they draw their conclusions.

Starting with Algebra I, concepts are tied together through verbal instruction, proper vocabulary usage in discussions, and guided and independent written practice. Anticipation guides and KWL charts would be useful in preparing students to review trigonometry or learn related rates and sequences. Mnemonics (FOIL, PEMDAS, SOHCOHTOA) are traditionally used in math classrooms. Graphic organizers are excellent comprehension tools for showing interrelationships between mathematical concepts such as between computed limits and how they are graphed, with an emphasis on the behavior of function values. Organizers can take students through the process of understanding directions, recognizing types of problems, and then picking the most efficient way to approach them. (A graphic organizer for working on limit problems would help students recognize and subsequently solve derivatives, integrals and series since each of these are really limits. Students must understand and be able to properly use math vocabulary in class and be able to interpret directions. There are a variety of additional literacy tools to enhance comprehension and vocabulary in math.

COMPREHENSION TOOLS

Comprehension in math is a struggle for many students including those who are considered to be proficient readers. Poor comprehension combined with questionable algebra proficiency negatively affect attitudes towards math as well as self-efficacy. Although there are many useful, powerful tools, we will describe what they are but will only be attaching a few as examples.

Comprehension is a skill that is learned over time with ongoing reading practice and reinforcement. Utilizing strategies such as Anticipation/Prediction Guides, which are a set of carefully selected statements that serve as a pre/post inventory for a reading selection. They are designed to activate and assess students' prior knowledge, to focus reading, and to motivate reluctant readers by stimulating their interest in the topic because the statements are selected to focus on and pay attention to this information. Students should read closely in order to get evidence that supports or answers their predictions. In math, these guides help students focus and pay attention to critical information, and may help them become more actively involved as they search for supporting information and for answers to their own questions. Teachers may find these useful in identifying students' misconceptions with mathematical concepts or procedures and thus, adapt their instruction to reduce the opportunities for misconceptions to occur.

KWL plus vocabulary forms, IPEC (Imagine, Elaborate, Predict, and Confirm) charts, concept maps, graphic organizers, vocabulary cards, and vocabulary self awareness charts are good ways to have students be aware of their understanding before, during and after. By having students review prior knowledge and use brainstorming prior to reading, it will stimulate readers to want to know more. ReQuest is a strategy to help struggling readers think as they read. It encourages students to think and ask questions themselves about the material they are reading. Concept maps for functions and limits will help students improve their comprehension because they will construct vocabulary, the meaning of new words and ultimately assimilate external information being taught into their mental structures, which will foster ongoing development of comprehension. The student can break down the information from large sections of the text or by breaking down word problems. SQ3R is a comprehension method that has been around for a long time. It is a simple mnemonic that helps students develop effective consistent strategies for reading math textbooks. Subsequently, SQ3R can be used more specifically with understanding and solving word problems in calculus, as the word problems can be difficult to conceptualize, visualize and decipher. Graphic organizers are useful aids for students at different levels of proficiency. Such creative strategies can be used across content areas.

Anticipation Guide for Limits

Directions: Put a check under "Likely" if you believe that the statement has any truth to it. Put a Check under "unlikely" if you believe that it is not true. BE READY TO EXPLAIN YOUR CHOICES.

Likely Unlikely

__________ __________ A limit is a number.

__________ __________ A limit is not a reachable number.

__________ __________ Infinity is a very large number.

__________ __________ The limit exists if it equals infinity.

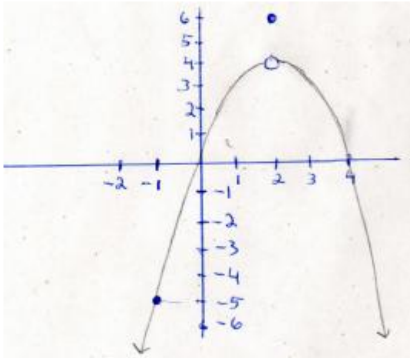
__________ __________ If there is a hole at x=2 in the graph of f(x)=2x+1, the limit does not exist because of the hole at (2,5).

__________ __________ If the same function has a function with the hole also has a function value above it at (2,7), the limit exists and equals the the function value at (2,7).

__________ __________ .For the oscillator function cosine, as x approaches positive Infinity, the limit exists and equals infinity.

_________ _________ Given the oscillating function, sine, as x approaches negative infinity, the limit does not exist because the function values oscillate and do not settle down to any particular number.

_________ _________ The following graph is piecewise:



_________ _________ $\lim_{x \to \pm\infty} \frac{9x^2+2}{3x^2-2x+5}$ The limit has no relationship to the horizontal asymptote.

Imagine, Elaborate, Predict, Confirm Chart

I	E	P	C

I= Imagine

E=Elaborate

P=Predict

C=Confirm

KWL CHART plus Vocabulary

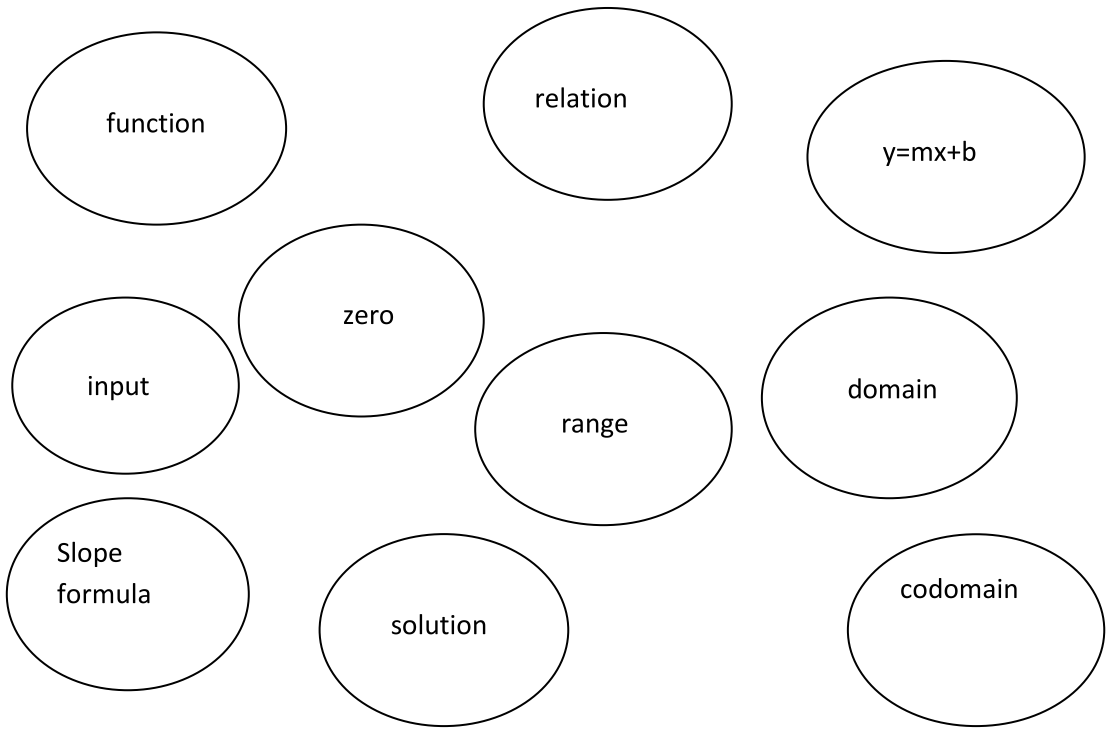
What I *KNOW*	What I *WANT* to Know	What I *LEARNED*

Vocabulary Words (nouns or verbs)		
A.		
B.		
C.		
D.		
E.		
F.		
G.		

Concept Map for Functions

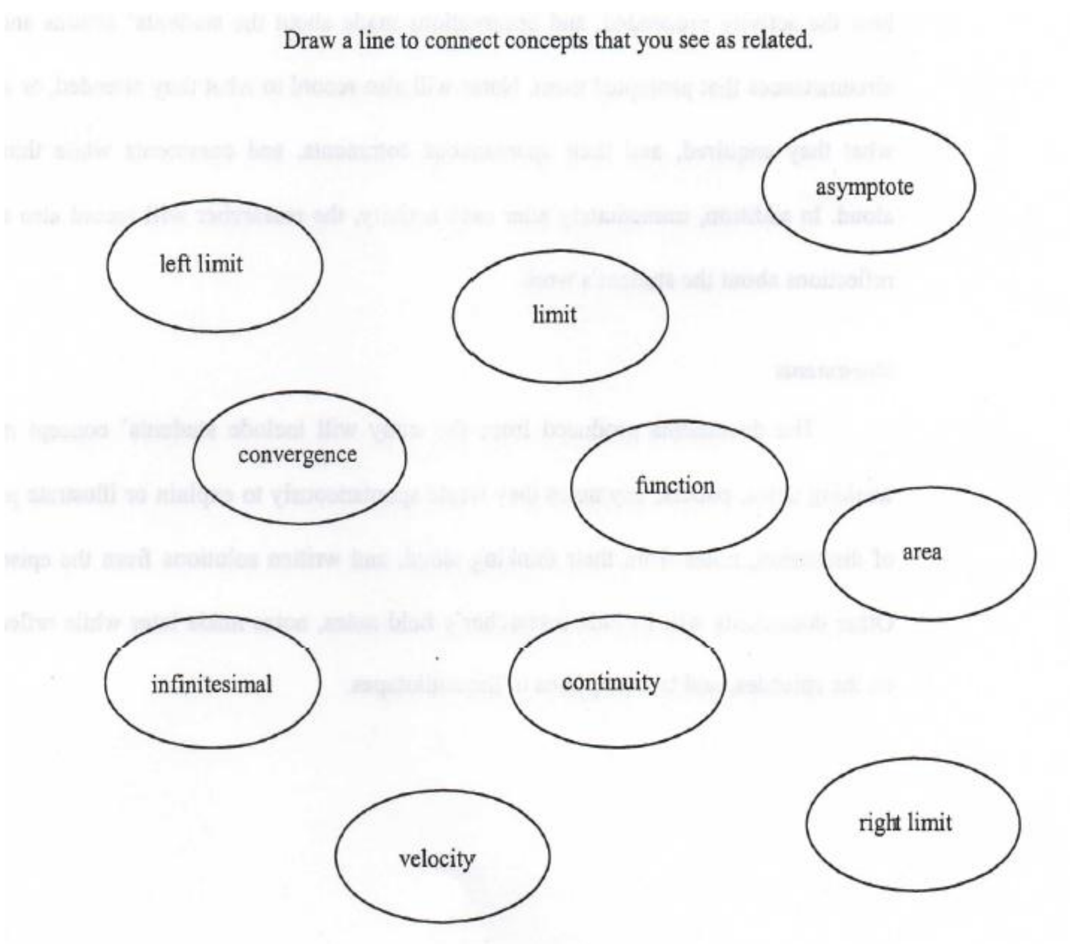
Name:_______________________________________

Directions: Draw a line to connect concepts that you see as related.



Name ______________________________________

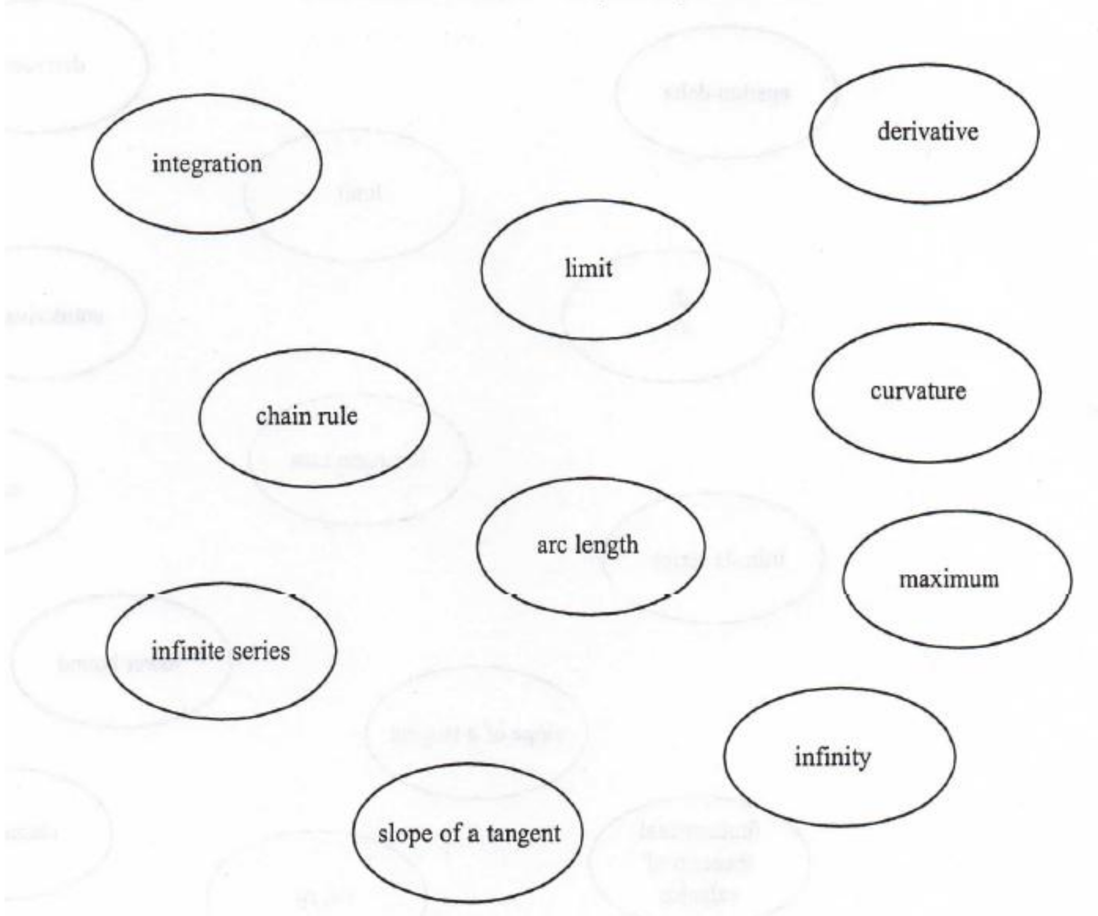
Concept Map Worksheet 1: Limits



Name ____

Concept Map Worksheet 2: Limits

Draw a line to connect concepts that you see as related.

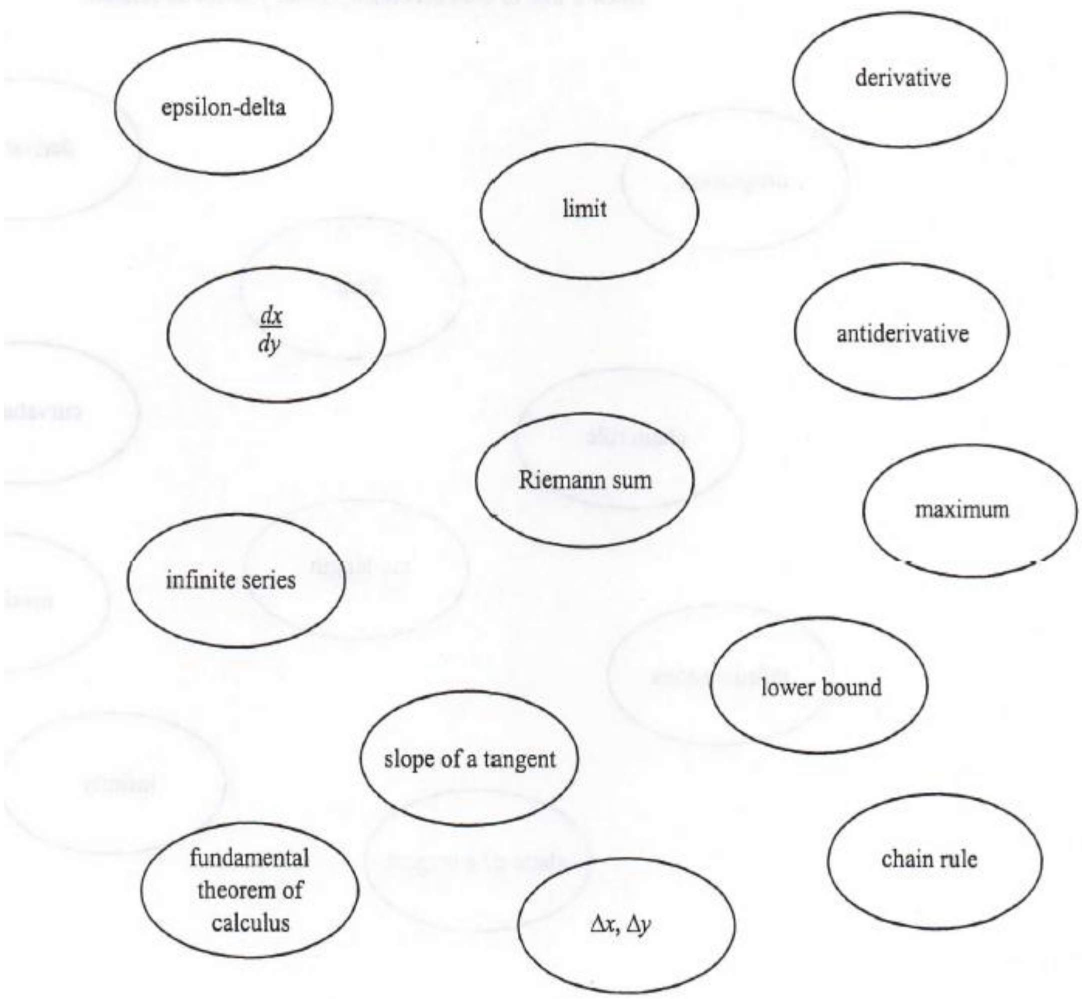


Name ______________________________________

Concept Map Worksheet 3: Limits

(Analytical)

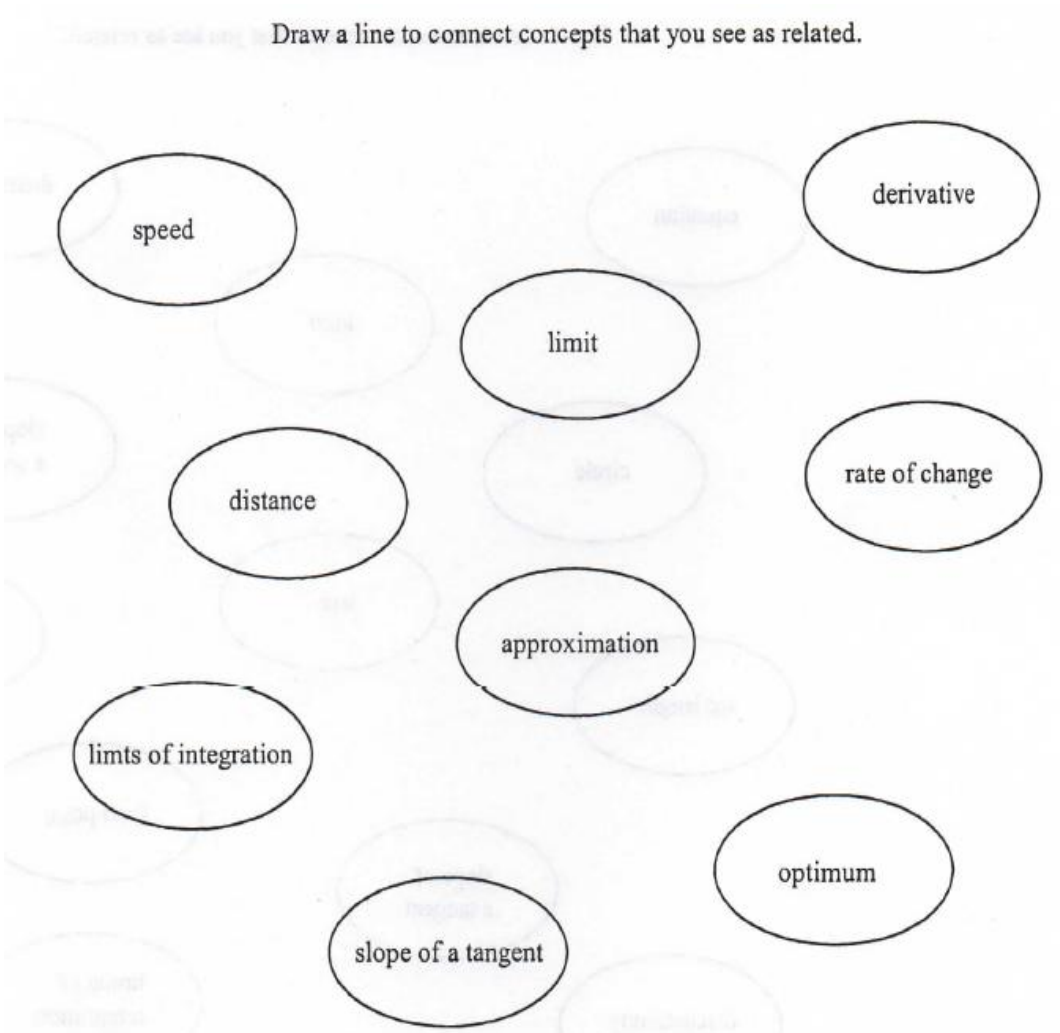
Draw a line to connect concepts that you see as related.



Name ____

Concept Map Worksheet 4: Limits

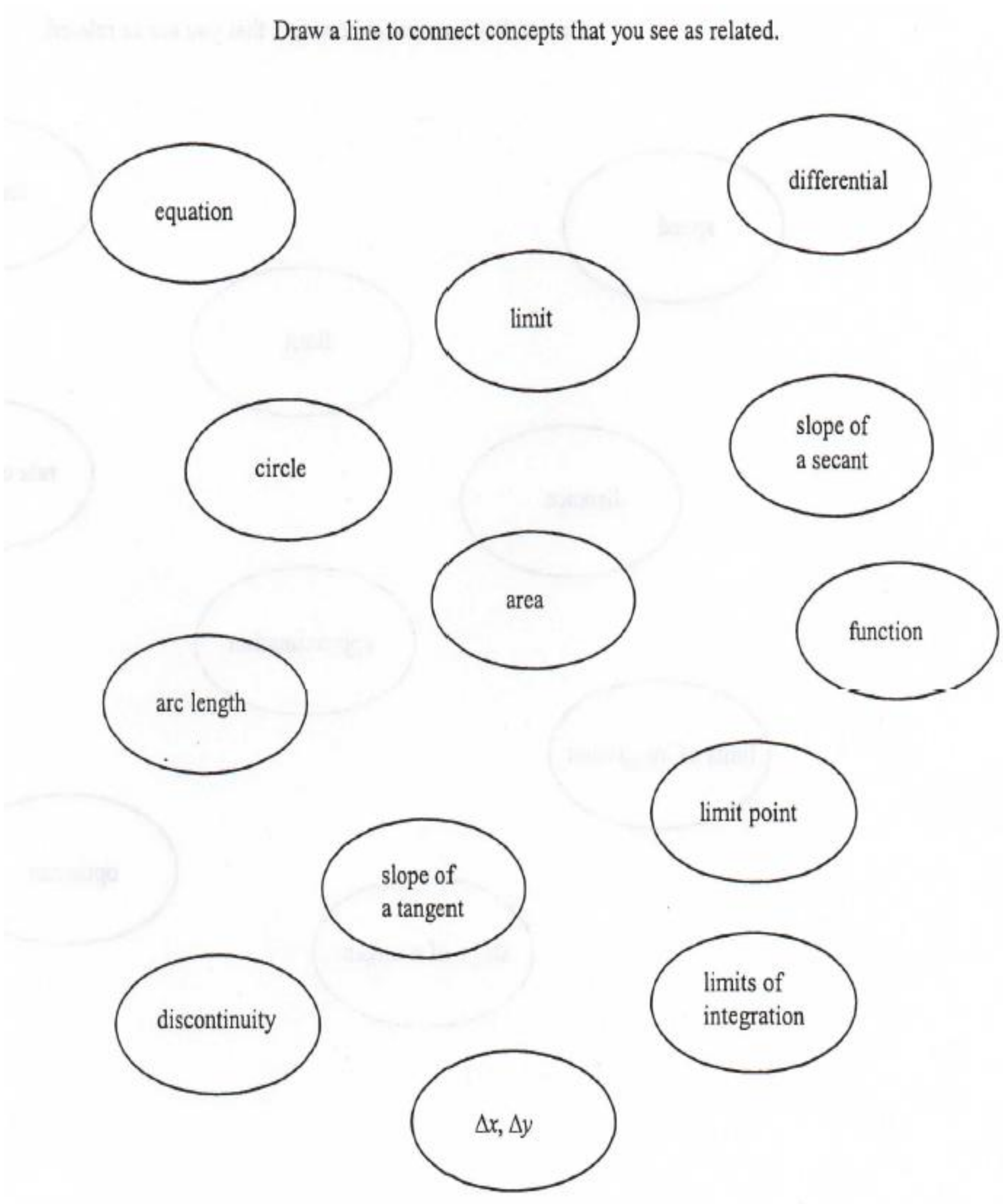
(application)



Name ____

Concept Map Worksheet 5: Limits

(geometrical/topological)



VOCABULARY TOOLS

Students must know vocabulary in calculus in order to understand the meaning of the content. Vocabulary is built over a period of years, but it is never too late to start from the beginning and introduce some older terms to make sure students have an accurate understanding of what things mean. For instance, they often confuse the words "terms" and "factors". Terms are separated by signs of addition and subtraction, whereas factors are separated by multiplication and division. However, many students have vague notions and use these words interchangeably.

Reviewing vocabulary and introducing new words is imperative. A variety of creative tools can be employed to make learning vocabulary more interesting and effective. The vocabulary tools below are discussed in reference to math, but they may easily be adapted to fit the needs of any content area. The section begins with definitions.

Index Cards can be used to reinforce math concepts. They can write the word on the front with a definition, and draw an example such as a graph or a solved problem with steps on the back. This makes a good reference tool when studying for tests on limits. while the student is creating the flip chart as well as providing a quick reference during the unit of limits. These would appeal to the ELL students, people with learning disabilities, kinesthetic learners, as well as students who enjoy adding color and creativity to visual aids.

List-Group-Label is helpful graphic organizer in which the teacher will ask students what do they know about limits and classify terms they know to each category.

Many students in math tend to see each concept and each problem separately and fail to see the interrelationship among them. For example, once they study domains of functions, they go onto solve problems about functions but do not think it's important to identify the domain because it was already covered the week before. Students often keep lists of terms in their minds without recognizing how the terms work together.

Crossword Puzzles, either student or teacher-generated at http://www.readwritethink.org/, or created from scratch by the teacher as per the example below can be fun, creative ways to master vocabulary related to limits. Puzzles would be created by concept and would involve interrelated subtopics. For example the attached crossword puzzle could reinforce the concepts of functions.

Vocabulary Self-Awareness Charts are not only usable to improve reading comprehension, but is a great tool for students to list the vocabulary words from the sections in each chapter, and then check off if they think they know it, know it a little, or don't know it at all. Then they give either an example or a definition, or even both. The writing component helps them to construct knowledge and produce an authentic product.

Definitions of Common Terms

Function – a relation between elements in the domain and range.
A relation is a function when one element in the domain maps to exactly one element in the range. Functions are also considered to be models of physical systems, such that, if you manipulate your independent variable x, you can measure the output dependent variable y. A function is an operation, what you do, such as square x's to get x^2.

Function Value – the output of a function, the y-value only. Function values are what you get.
Infinite Limit – refers to when a limit equals infinity. For example, $\lim_{x \to \infty} 2x^2 = \infty$. More precisely, an infinite limit describes the behavior of function values increasing or decreasing without bound. It is what the y is doing, unlike limits at infinity which are about what x is doing.

Infinity – the behavior of either x or y values increasing or decreasing without bound. Infinity does not represent a large number and should not be used for arithmetical operations.

Limit – a number. A limit is a number, referred to in textbooks as "L", that describes the behavior of a function near a point or increasing/decreasing without bound.

Limit at a Point – refers to $x \to a$. As x is approaching some value "a" on the x-axis, the limiting behavior of the function values are found near that point.

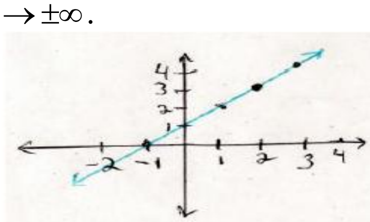
Limit at Infinity – refers to $x \to \infty$. As x is increasing without bound in either direction positive or negative, the limiting behavior of the function values are found for large x.

One Sided Limit – refers to the limiting behavior that occurs at a point from one side at a time. For example, $\lim_{x \to 2} x - 1$ can be separated into one sided limits as follows: From the right: $\lim_{x \to 2^+} x - 1$ and also from the left $\lim_{x \to 2^-} x - 1$.

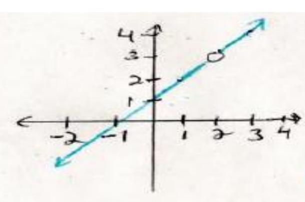
Rational Function – a quotient of 2 polynomials. It is not a fraction, as fraction is the ratio of integers only. Rational functions involve variables. Ex: $f(x) = \frac{x^2 - 16}{(x+4)}$ or $f(x) = \frac{1}{x}$ are rational functions.

Graphical Images of Limiting Behavior to Know

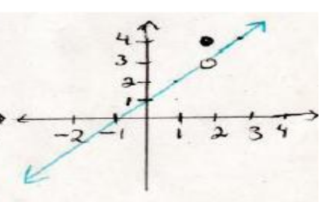
The limit exists at a point as $x \to 2$ and equals 3 in all cases below, but does not exist as $x \to \pm\infty$.



Graph A



Graph B



Graph C

List-Group-Label

Limits at a Point	Limits at Infinity	Both

Teacher will ask students what they know about limits then classify terms they know in each category.

Vocabulary Self-Awareness Chart					
Word	+	√	-	Example	Definition

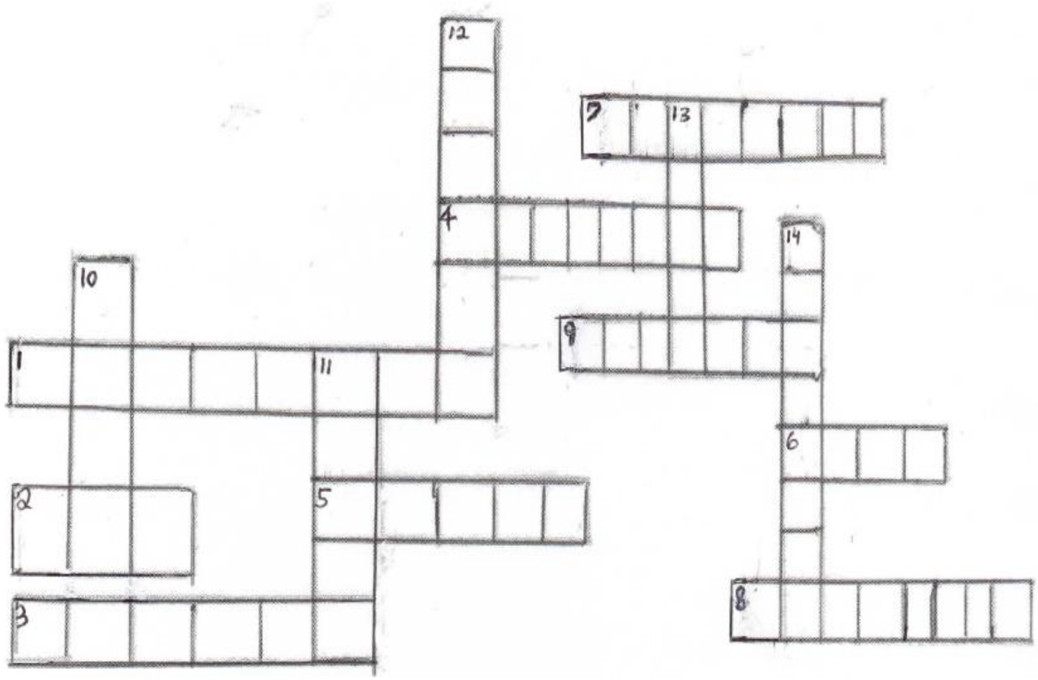
Procedure:
1. Examine the list of words you have written in the first column.
2. Put a "+" next to each word you know well, and give an accurate example and definition of the word. Your
definition and example must relate to what we are studying.
3. Put a "√ " next to any words for which you can write only a definition or example, but not both.
4. Put a "-" next to words that are new to you.

This chart will be used throughout our unit. By the end of the unit you should have the entire chart completed.
Since you will be revising this chart, write in pencil.-

Adapted from Goodman (2001)

Crossword Puzzle on Functions

(Designed by Margaret Adams)



Across	Down
1. A relation or rule that maps elements from the domain to the range. 2. A well defined group of objects 3. The y-value 4. The input 5. An ordered pair 6. If every value from the co-domain comes from an element in the domain or if every element of the co-domain is a function value then the function is ________. 7. The letter that represents a number 8. A subset of ordered pairs 9. Function must be 1-1 and onto.	10. A standard method or procedure 11. Argument 12. Set of all possible values for the argument of a function. 13. Set of all possible values for the output of a function. 14. For each y-value, there is exactly one x-value.

WRITING AND INQUIRY TOOLS

Writing to learn mathematics is a method that can and should be incorporated across the curriculum. This approach helps students personalize learning so they can understand their course work better and retain what they have learned longer. It also encourages high-level thinking skills. Writing activities can be used to help students reflect on and explore ideas and concepts they are reading in class, which helps them to construct meaning. As with learning log entries, writing activities are intended to be brief and can be assigned at any point during the class period. In math, writing activities can engage students in thinking and reasoning about a concept. When students communicate their understanding of a concept in writing, they confront what they know and need to learn about this concept, i.e., how well they understand the concept. Writing can also be used for students to reflect on what they have learned.

The teacher would select a topic to address then assign the writing to learn activity at any time during class depending on the topic and the purpose. They can be done with pre-reading, during reading and after reading activities. When designing a writing topic, the task should not require students to merely restate facts from the text but should ask students to reflect on or apply what they are learning. Once the topic is assigned, students can have 5 minutes of thinking time to consider their responses. Some of the strategies and activities I chose are from the following resource: Barton, M. & Heidema, C. (2002). Teaching Reading in Mathematics, 2nd Ed. Colorado: Mid-continent Research for Education and learning. www.mcrel.org

Prereading: Alphabet Soup can be used to activate students' prior knowledge. On the first day of a unit, students work in groups to complete the pre-reading sections of a K-W-L: worksheet or pre-reading plan. After group collaboration, they spend 5 minutes individually writing about their prior knowledge. If the topic is new to most students, it can be explained to them in brief terms and suggest students to write for five minutes about any impressions they have about the subject. Anticipation guides involving true/false anticipation on the subject is helpful. Before they discuss their answers as a class, students select one or two of the guide's statements and write for five minutes, defending their answer for that statement**.** Problematic Situations: Instead of discussing potential solutions to a problematic situation as a group, students would write up their own solutions to the problem. One-liners: At the beginning of class, students can write one sentence about the importance or relevance of something they learned in the previous lesson.

During Math Class/Reading: Students can use Fast Food for Thought. After explaining a particular concept, process or vocabulary term, students can write a question they still have about the topic. Students can be asked to exchange papers and either answer the writer’s question or suggest resources they could use to locate the answer. Out of This World is a real interesting strategy that can be used at a convenient point during a lesson. Students can write something for 10 minutes about this: they are an alien from another galaxy, their spacecraft just landed outside the school and their first stop is your classroom. The student pretends to be an alien and writes an “outside” observation of the lesson, the teacher and students in the class.

After the Math Lesson/Reading: Students can write The Last Word, spending 10 minutes of class time writing a letter to the teacher about something they do not understand or need help with in the current unit. In addition to revealing to students what they do not know, this writing task can inform the teacher about what needs to be reviewed or clarified during the next lesson. Dear Diary is another strategy. Students can be asked to assume the identity of an historic figure who is/was involved in the lesson topic and to write a diary entry as if they were that person. For instance, students studying Limits might pretend to be Cauchy, Leibnitz, or Weierstrauss and compose a diary entry chronicling one of their thoughts and ideas on the development of limits.

Throughout the lesson, a great way to assess understanding is by means of writing and there are many instructional strategies that link math with language and writing. Using strategies such as KWL and K-W-H-L-S, students can work individually as they do independent practice, work on homework or even work in collaborative groups. By using these, students can construct the information they know and organize their knowledge into categories. Students can form small groups to discuss what they know and what they want to know.

Another really useful strategy for calculus is “How Well Do I Know These Words” because students can separate out the words or mathematical phrases they think they have mastered versus though they think they are pretty sure they know, versus those that they either are not sure about or never heard of before. After reading, students can come back together to discuss what they learned and then share with the class. Writing about what they learned is a great way for students to connect with the lesson and retain the information and improve comprehension.

Group work is often beneficial for low achieving students. These tools would be helpful to use with students reading difficult mathematical information. These various activities will help students gain mastery and proficiency and hopefully even reduce math anxiety. GIST is a great comprehension strategy that teaches students how to summarize what they read, as they read it. Many students, even proficient readers, cannot summarize what they read due to comprehension problems. This strategy breaks down the task by paragraph to make it easier for students to grasp. By writing a summary sentence for each paragraph read, a student may gain the knowledge and the skills to improve their comprehension of the entire reading. RAFT would be a real interesting way for the students to identify with the noun or verb that is a mathematical term. For example, a person could pretend to be a limit at a point and could write about themselves so their classmates can learn about them as if a real person.

A writing assessment for limits is also helpful either as a formative or self-assessment. It helps a student to recall and reconstruct the definition of limit, definition and meaning of the infinity symbol, and it requires them to interpret the formula with the $\lim_{x \to \infty} f(x)$ notation. It has them look at graphs and decide in which cases limits exist; asks them to write down some example of real-world situations where limiting behavior occurs, and also has them reflect on what information they either understand or still do not understand that they want to learn more about, in full sentences.

K-W-H-L-S Chart for Content Areas

Topic:

What I Know (K)	What I Want to Learn (W)	How I Can Learn More About This Topic (H)	What I Learned (L)	How I Can Share This New Knowledge (S)

How Well Do I Know These Words

Don't Know at All	Have seen or heard—don't know what it means	I think I know what it means	I know the meaning

GIST with a Math Word Problem

Name: ______________________________

Topic or problem and page number:

____

Read the word problem and list the 10 to 15 most important words/ideas in the problem:

________	________	________
________	________	________
________	________	________
________	________	________
________	________	________

Write a 15-word summary:

____

____

____

____

RAFT Form

Name:______________________________ Partners:__________________________

1. Tell the following about your chosen RAFT writing assignment:

R: Role of the writer (Who or what are you?) ______________________________

____

A: Audience (To whom is this written? A friend, a police officer, a parent?) ______

____

F: Format (What kind of form will it take? A journal entry, a letter, a memo, a list, a poem, a song, an advertisement?) ____

____

T: Topic (Persuade a company to hire you, demand for fair treatment as a slave, plead for a ride on a rocket ship.) ____

____

2. Use the space below to compose your idea.

____

____

____

____

____

____

____

____

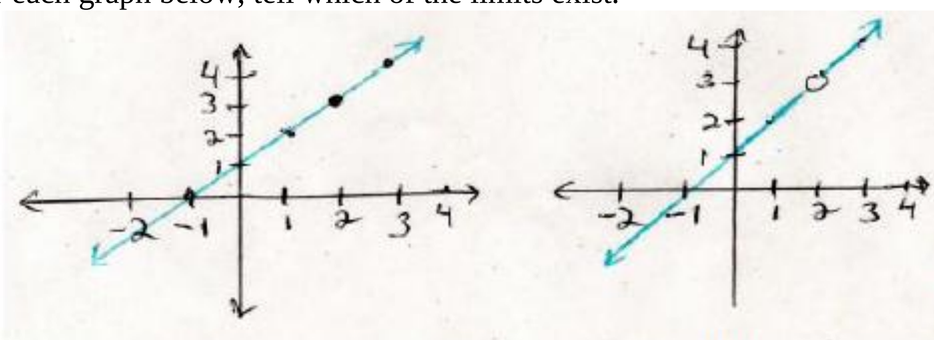
____

____

Name____

Writing Assessment on Limits

1. What is a limit?
2. What is the relationship between limits and function values?
3. What does the infinity symbol mean?
4. Translate in your own words what this means: $\lim_{x \to 2} 9x + 3$.
5. Compute this limit $\lim_{x \to 2} 9x + 3 =$
6. Translate in your own words what this means: $\lim_{x \to \infty} \cos x$.
7. For each graph below, tell which of the limits exist.



Graph A   Graph B

Graph A: $\lim_{x \to 2} f(x) =$ ________

Graph B: $\lim_{x \to 2} f(x) =$ ________

8. As $x \to 2$ does the limit exist only in Graph A?   YES   NO
   Explain your answer:

9. Describe at least 2 real world situations where you think limiting behavior occurs.

10. Write down anything you did not understand about today's discussion and any questions you still may have.

Text Selections and Reading Tools

Reflective Memo

Mathematics standards require students to be knowledgeable of content and conceptual understanding. Due to inconsiderate text, some texts are more difficult to read than others. Using the strategies listed above helps students digest information and text. Due to the many goals set for calculus, trade books and workbooks tend to be used as supplements to or in place of text books. Trade books help make learning more enjoyable and meaningful because the teacher can plan lessons from them and the students can often benefit from the formal language of math being translated into simpler colloquial terms. Applying word problems to a culture or to popular real-world situations that students can relate to will improve understanding. In turn, they will be able to connect with the author and the purpose of the material. There are many trade books available to teachers and students that would be beneficial for enhancing learning and understanding of limits in calculus.

Supplemental texts are valuable classroom resources but in addition there are really good trade books out there. These include workbooks, picture books, activity books, riddles, puzzles, and challenges can introduce and reinforce math concepts and vocabulary in a fun manner. These math books can engage students in a way that a traditional text book cannot and can engage students with diverse needs, including students with giftedness and talent; those with learning disabilities, ELL's, and also students who read at different levels. Whether set out as part of an informal classroom library or used in teacher-directed activity, these books should enhance math instruction and help students to construct new knowledge by visualizing and learning new concepts.

Mathematics/Calculus Bibliographies (annotated list of book references)

1. The book *The Humongous Book of Calculus Problems by W. Kelley (2007)* is phenomenal because it highlights vocabulary, breaks down questions and explaining what they mean on the side inside of captions with scull bones, and it gives step by step solutions with reasons and easy to follow explanations along the way that really makes the subject matter more enjoyable. In addition, other useful books are:
2. *How to Ace Calculus: The Streetwise Guide,* by C. Adams (1998)
   This is a study guide written by teachers and provides humorous and easy to read explanations of key topics of calculus without the technical details and fine print found in a formal text. Capturing the tone of students exchanging ideas among themselves, this gook explains how calculus is taught, how to get the best teachers, what to study and what is likely to be on exams.
3. *Calculus for Dummies (2008)*
   A wonderful supplement that gives basic examples of calculus problems with detailed step by step solutions, and explanations of what things mean. There are practice problems with detailed answers.

4. *The Idiots Guide to Calculus (W. Kelley, 2006).*
   This is another reference type book that gives numerous examples of calculus problems with detailed steps, solutions, explanations and practice problems.
5. *The Universal Encyclopedia of Mathematics (1965) by J. Newman*
6. *Calculus Made Easy* (1988) by S. Thompson.
   This is a very simple, classic introduction to differential and integral calculus. The author explains topics like limits in colloquial English and being such a short paperback book, it appears and reads like an interesting novel.
7. *Mathematics for the Non-mathematicians* (1967) by Kline
   This book provides liberal aerts students with a detailed treatment of math in a cultural and historical context. The book is also a means of self-study for bright high school students focusing on practical, scientific, philosophical and artistic problems that caused men to investigate math. It simplifies complex subjects for the nonspecialist.
8. *How Mathematicians Think* (2010) by W. Byers
   Ambuity, contradiction and paradox is used to create mathematics. The book reveals that mathematics is a profoundly creative activity not just a body of formalized rules and results. It shows that the nature of mathematical thinking can teach us a great deal about the human condition itself.
9. *Historical Topics for the Mathematics Classroom* (NCTM, 1989).
   A complete, compact and fully illustrated mathematical reference book that lists mathematical terms from A to Z, and gives examples of each. It is literally a world encyclopedia of math.
10. *The Historical Development of the Calculus* by Edwards
    This book explains calculus from the beginnings of geometry in antiquity to the nonstandard analysis of the twentieth century. It emphasizes the genesis and evolution of both fundamental concepts and computational techniques. The intended audiences are students interested in the history of math and people who study, teach and use calculus. There are historically motivated exercises and carefully chosen illustration type examples. Many sections of the book are good for use in courses in introductory calculus.
11. *The Calc Handbook (1991) (Conceptual Activities for Learning the Calculus)* by DeTemple & Robertson
    This large workbook type paperback is a supplement to any standard calculus text. The authors collected materials over many years of teaching. It was piloted in 4 high schools. Explanations of topics are followed by exercises and solutions that you can photocopy for students. The pick and choose format gives total flexibility you can use. It has activities and real world applications to aid conceptual understanding. There is a visual framework for each topic and numerous exercises for students to construct or interpret figures.. The topics are introduced in a general, non-technical way so it gives students a big picture about where these topics fit into the large scheme of mathematical study.

12. *How to Solve Word Problems in Algebra* (2000) by M. Johnson and T. Johnson
    This is an easy to use pocket guide that shows how solving word problems in algebra can become easy and fun. It is an anxiety-quelling guide that helps you get students ready for daunting word problems, and does it one step at a time with fully explained examples. It shows how to translate word problems into solvable algebraic formulas and get answers right.
13. *501 Math Word Problems Solved by Learning Express* (2007) (LearnATest.com)
    Anoter great book that contains only word problems the kinds encountered at school and on high stakes tests. Each chapter has questions that move from easy to advanced and gives test takers the chance to gain confidence in many math areas, so it has questions, answers, and detailed explanations to reinforce learning and understanding. It is great for students who have any deficits in prior knowledge in algebra or math that they bring to the calculus class.
14. *Calculus Connections (Mathematics for Middle School Students)* (2006) by John Beem
    This is an actual textbook used in some places. It has a number of classroom connections and classroom discussions designed to deepen the connections between calculus that students are studying and what they might someday have to present in a project or be assessed on. The book accommodats high school teachers who wish to go slow and spend more time on things like classroom connections, and for instructors who wish to move more quickly and cover more topics.
15. *The Calculus Lifesaver (All the Tools you need to Excel at Calculus)* (2007) by Adrian Banner
16. TI-83/84 Graphing Calculator Manual
    A must-have supplement to use with the graphing calculator, to learn how to use special features of the calculator, create programs and more.

Additional Artifacts

Picture/Illustration of Limits Activity and Homework Assignment

Below is an example of an introductory activity that can be used in AP Calculus when beginning the topic of limits. Students may complete as classwork and as homework. They will complete a companion checklist on which they indicate whether they found the questions comfortable, challenging, or confusing. They will bring the checklist as well as their own questions about the material to class for additional in-class small group discussion. Students will work together to help each other on problems they missed as well as narrow the list of more complicated questions which will be reviewed as a class.

Pictures of Phenomenon Representing Limits

Introduction to Limits

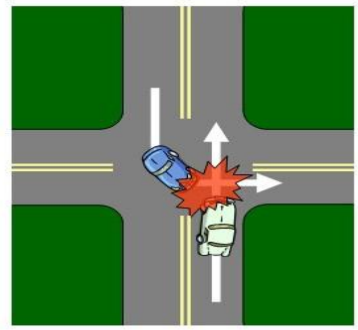
If there is a hole in graph, does the limit exist? Limits are really numbers, but here, limits are represented by holes to illustrate the notion of nearness. The golfer taps the ball along a straight path, which you can describe on a graph with a line that has a hole at the end. The ball will approach but may or may not fall into the hole. Whether it ends up at the edge or falls in, the limit exists because limits are about nearness. If the holes were filled in, the limits would still exist. A straight line on a graph does not have to be continuous for a limit to exist; it can have a hole instead of a function value.

www.istockphoto.com

www.graphicleftovers.com

Speed limits are examples of limits that exist. You can approach or "near" the limit by going over or under. 25 mph can be represented as a straight line with a solid dot.

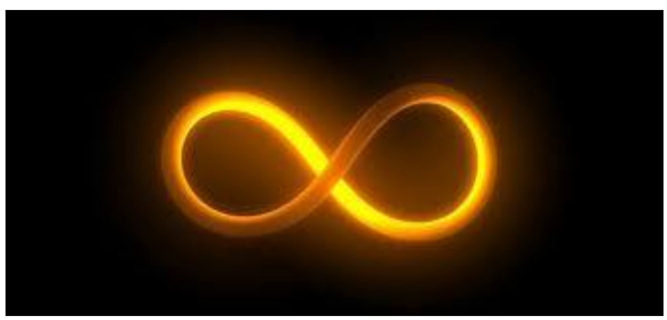
Compare the 3 pictures below. The intersection is the limit because the cars will all end up at the same place, including in a wreck. So the limit exists. The 3rd picture has an object in the middle of the traffic circle but the limit still exists because that is where all of the cars meet and must travel around.



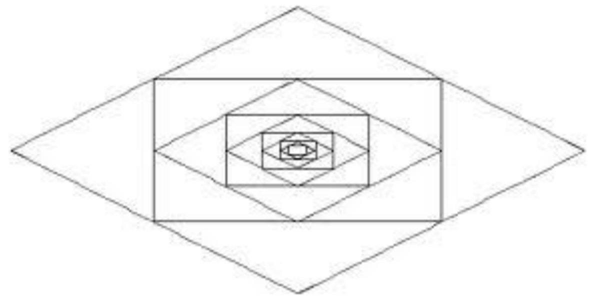
Limits can approach an object (or number) and they can also get there and touch the object (or number in which case the limit would equal the function value). So limits can do both—approach as well as equal, but in any event, if they are heading to the exact same object (or number) then we would say that the limit exists, regardless of an opening as in the first intersection or if there is a solid object there (in the case of the car crash or traffic circle). In all 3 graphs, the limit exists.

A rocket can travel around the earth an infinite number of times—especially if it has a refueling mission going on in space.

The infinity symbol is something to think about. Infinity and infinitesimals are polar opposites. Infinity gets larger without bound, infinitesimals get smaller and smaller without bound.



The geometric figure below represents an infinitesimal of squares and triangles getting infinitely smaller and smaller. On the other hand, you can say the figures get infinitely larger going in the opposite direction.



The telescope image represents exploring infinite distance in space. The man with the magnifying glass is making the fly appear infinitely larger because he can get infinitely closer and closer without bound.

Infinite Distance

This homework assignment will help set the stage for the project you will select to do on this unit. You should feel free to explore different fields and applications where limiting behavior occurs in everyday life. It could be at a machine shop, it could be something that happens in health care, in astronomy, physics, mechanical engineering, psychology, neuroscience, ecology, etc.

1. Compare the 2 images below. Pretend you are riding a bicycle in a straight line with the pothole directly in your path. The first picture has an deep pothole. In the 2^{nd} picture, the hole was filled in with bricks. Explain if the limit exists where the potholes are.

	
Does the limit exist? Why or why not?	Does the limit exist? Why or why not?

2. Search Google-images on topics of your choice. Think about limiting behavior of these topics in every- day life. Pick two interesting types that would represent a limit that meets near a point or limits that go off to infinity. This is where you do a search on something of your interests. (Ex: Maybe you're a civil engineer designing a bridge and you want to know what the maximum weight of cars on top can be (limit) for the bridge to keep standing; a musician tuning a flute or electric guitar; a nurse administering insulin to a diabetic patient; the treads on your tires wearing down; pendulum, springs—all these involve limiting behavior). Come up with a few examples of your own like these. Be sure to copy and paste the image then type up a description or explanation of the type of limiting behavior each has.

Technology Tools

Reflective Memo

Mathematics can come alive with the technology available today. Students can use the internet to find information. There are many websites that teachers can use to pull information to use in the classroom. Many schools have lap tops for students and some have to resort to visits to the computer lab to complete an assignment. To prepare students for the 21st century, school systems must make technology a priority. Teachers should incorporate technology in the classroom as much as possible.

The Web Evaluation Sheet can help students to collect, compare and choose different websites with a nice organized chart. This will help students to remember which websites they visited and what they had to offer, and reminds them which websites to access later on.

CLVG can be used in conjunction with videos, power points, or web-searches to help students organize notes as they add personal and group notes to information they already know. In addition, CLVG will help students with overarching concepts, helping them to anticipate future concepts in derivatives and integration, max and min problems, optimization, sequences and series which all involve limits.

A variety of websites can provide math help for calculus. Additionally, many sites can provide quick reference and further explanation as students seek to solidify understanding of math concepts. Teachers should also use resources that are made available on line.

Below are examples of a few websites that could be used by teachers and students. Teachers can assure meaningful learning by using new ideas and multiple strategies. Items made available by other teachers and educational websites on line will allow teachers to spend more time on how they will teach and what strategies they will use. Quick access to different materials online is fast and gives students opportunities to practice and achieve success while learning limits for the first time.

DVD Resource

Gibson, J. (2005). The Calculus 1 & 2 Tutor: 8 hour video course. MathTutorDVD.com

The Calculus 1 & 2 Tutor is an easy way to improve grades. The instructor in the video works out hundreds of examples with each step fully narrated so no one gets lost. Students can learn by examples and the course covers essential material that is presented in Calculus 1 and 2.

Annotated Website List

www.edhelper.com

This website does require a small annual fee but it has an extensive list of practice worksheets. Students could work on these sheets during free time, such as waiting for others to complete a test. They could also be used for extra credit opportunities.

www.dpi.state.nc.us/accountability/testing/eog

This state website is also a good resource that gives examples of calculus problems pulled from previous tests.

http://www.mymathtest.com/login_mmt.htm

MyMathTest is a testing tool that assesses students' strengths and knowledge gaps. From a teacher's prospective, it has online test banks for teachers to create tests for students' individual needs. Teachers can also create short refresher courses. From a student's prospective, you can take practice tests and see your results instantly so that you can practice the concepts you still need help with. The practice questions come with tutorial help such as videos and step-by-step instruction.

http://college.cengage.com/mathematics/larson/precalculus_limits/1e2/ins_resources/ap.html

This site outlines great student help. For each topic a student can study the concept. The concept is given in a tutorial style video with the option of listening to the instructor or just reading the in the notes on your own. It offers study tips to view on the side when plausible. After studying the concept, the student can try and example and participate in practice exercises. There is an end of the chapter mastery test as well.

http://www.calculus-help.com/tutorials

This portion of the website is all about limits. It covers concepts from what is a limit all the way to the intermediate value theorem.

http://www.education.com/study-help/article/rapid-review1/

Education.com offers loads of information. This particular part of the website is a great source for an AP Calculus review of limits and continuity. It supplies the students with the most needed information. It would make a nice study guide for an AP exam.

_http://quizlet.com/subject/calculus-limits/

Quizlet has excellent study features. The key vocabulary words are given in an animated flash card way. You can choose to view and study the flash cards in several different formats, speller, learn, and test. If that is not enough versatility, a student can also play a few different games with the flash cards, scatter, and space race. If you are a person that likes to manually hold the flash cards in your hand, there are several different print options.

http://www.interactmath.com/

This website offers tutorial exercises accompanied by the end-of-section exercises that follow the student's textbook. Simply select your text and begin.

http://www.learnerstv.com/Free-Maths-Video-lectures-ltv332-Page1.htm

Learnerstv.com is metaphorically an educational YouTube. It offers video Lectures, science animations and more. It's all in one website.

http://www.rootmath.org/calculus/solving-limits

Rootmath.org is an animated website that works through countless examples. It shows the smallest and easily forgotten steps.

http://www.intuitive-calculus.com/limits-at-infinity.html

This website has good notes on limits. The notes include good wording, sample problems, and techniques made easy.

http://www.mymathlab.com/titles-available

MyMathLab offer a range of textbooks for online learning and assessment. They have pre-assigned assignments covering each chapter and section for students.

Other Technology Artifacts

www.youtube.com

You Tube is growing in popularity and items are uploaded daily. The appealing thing with you tube is that there is so much variety on the website to help students to connect with math. For example, some high school students created a video as a funny math project and so they sing funny songs about calculus by changing the words from popular rap music. There are other more informational ones that involve more theatrical actions but focus on definitions.

http://www.pbs.org/teachers/

This website provides additional resources, links and other opportunities.

Additional Technology Artifacts

Web Page Evaluation Sheet

The Three C's of Web Use: Collect, Compare, Choose

	Website:	Website:	Website:
COLLECT			
COMPARE			
CHOOSE			

CLVG will help students to anticipate future concepts in derivatives and integration, max and min problems, optimization, sequences and series which all involve limits.

APPENDIX Q: ALTERNATIVE REPRESENTATION OF THE DOMAIN

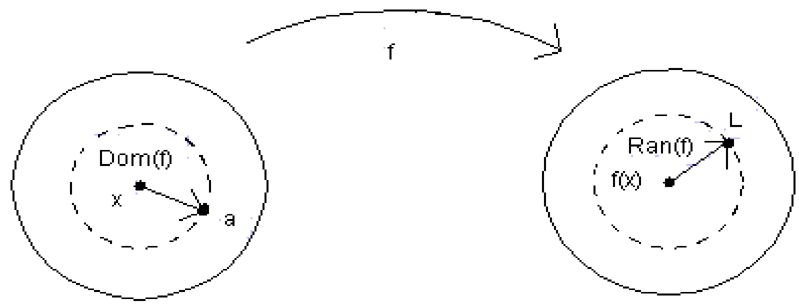


Figure Q.1: Visual Representation of $\lim_{x \to a} f(x) = L$.

The figure shows schematically the relationship between the various objects appearing in the notation used for the limit, $\lim_{x \to a} f(x) = L$. Every function has a domain. The points within the dashed circle on the left constitute the domain of the function, where the $x's$ “live”. The "a" is not necessarily in the domain as depicted, being on the boundary of the dashed circle. The picture suggests that x is approaching a. The points within the dashed circle on the right are in the range of f -i.e. the set of values $f(x)$ for $x's$ in the domain. The limit L is not necessarily a function value so it is on the boundary of the dashed circle. It could be inside the dashed circle, especially in the continuous case. The picture suggests that $f(x)$ is approaching L.