THEORETICAL AND COMPUTATIONAL ANALYSIS OF SPECTRALLY HYPERVISCOUS MODELS OF TURBULENT FLOW

by

Chang Xiao

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Approved by:

____
Dr. Joel Avrin

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Dr. Shaozhong Deng

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Dr. Vincent(Tobi) Ogunro

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Dr. You-Lan Zhu

ABSTRACT

CHANG XIAO. Theoretical and computational analysis of spectrally hyperviscous models of turbulent flow. (Under the direction of Dr. JOEL AVRIN)

Computing turbulent flow is very difficult but forms the basis for computational experiments in Meteorology and Oceanography. To overcome the difficulty and complexity in turbulence computation, a spectrally hyperviscous version of Navier-Stokes equations (SHNSE) has been suggested(see [4] and the references contained therein).

My PhD research has been focusing on the theoretical and computational analysis for the SHNSE. This dissertation bases on my research under the advise of Dr. Avrin and Dr. Deng [5,7,8,77]. The theoretical results that we obtained are the convergence of Galerkin solutions and the continuous dependence on data for the SHNSE [5], the estimates for the number of determining nodes and determining modes [8], and the inviscid limit $\nu \to 0$ in the case of unforced turbulence [7].

Let u_N denote the Galerkin solutions which approximate the solution u, and let $w_N = u - u_N$ then by spectral decomposition, we have $w_N = P_m w_N + Q_m w_N$ where P_m is the projection onto the first m eigenspaces of $A = -\Delta$ and $Q_m = I - P_m$. For assumptions on λ_m that compare well with those in previous results, the convergence of $\|Q_m w_N(t)\|_{H^\beta}$ depends linearly on key parameters (and on negative powers of λ_m), which is reflective of Kolmogorov-theory predictions that in high wavenumber modes viscous (i.e. linear) effects dominate. Meanwhile $\|P_m w_N(t)\|_{H^\beta}$ satisfies a more standard exponential estimate, but with fractional dependence on λ_m. Similar results demonstrate continuous dependence on data.

The estimates for the number of determining nodes of the three dimensional SHNSE are proportional to G^3, where G denotes the Grashof number, comparing well with the exponential determining-node results for the two dimensional no-slip NSE. The estimates for the number of determining modes also compare well with previous results, and in particular as long as $\alpha > 9/4$ these estimates are less than those for the two dimensional space-periodic NSE. If $\alpha \geq 5/2$, explicit non-exponential estimates for the dependence of

the high-wavenumber modes on the determining modes can be obtained with improved computationally utility. We discuss these results in the context of physical and computational experiments, and in terms of the potential of the SHNSE to reduce the number of degrees of freedom required for physical and computational experiments.

We can use the pairing of the 3-D Euler system with spectrally applied hyperviscous terms and the SHNSE as a platform to study the inviscid limit $\nu \to 0$ in the case of unforced turbulence. Let u_ν be the solution of the SHNSE, let u be the solution of the spectrally-hyperviscous Euler equations, and let $w_\nu = u - u_\nu$, then we will show that $w_\nu \to 0$ strongly. The characteristic feature of our convergence methodology is its multiscale approach, which seems to optimize the results and yields behavior in each regime suggested by experiments and physical theory.

The computational analysis implements the SHNSE for a periodic box by using pseudo-spectral methods, and numerical results obtained from large eddy simulations for the decaying turbulence are compared with those obtained by direct numerical simulation [77]. Numerical experiments are conducted to validate some of the theoretical properties of the SHNSE and to investigate optimal parameter choices. Numerical results indicate that the SHNSE model has strong potential to be a highly robust platform for studying turbulence which can retain spectral accuracy while significantly reducing the number of degrees of freedom needed for accurate simulation.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1: INTRODUCTION TO PROGRESS OF RESEARCH ON SHNSE

1.1 Introduction to the SHNSE

The 3-D incompressible homogeneous spectrally-hyperviscous Navier-Stokes equations(SHNSE) are

$$u_t + \nu Au + \mu A_\varphi u + (u \cdot \nabla) u + \nabla p = g, \tag{1.1a}$$

$$\nabla \cdot u = 0. \tag{1.1b}$$

where $u = (u_1, u_2, u_3)$ is the velocity field, p is the pressure, and $g = (g_1, g_2, g_3)$ is the external force. We have that $u_i = u_i(x,t)$, $g_i = g_i(x,t)$, and $p = p(x,t)$ where $x \in \Omega$, a domain in $\mathbb{R}^3$, and $t \geq 0$. Here we assume Ω is a periodic box, and for simplicity assume that $\Omega = (0,l) \times (0,l) \times (0,l)$. If we "mode out" the constant vectors as in standard practice the operator A has eigenvalues $0 < \lambda_1 < \lambda_2 < ...$ with corresponding eigenspaces $E_1, E_2, ...$; let P_m be the projection onto $E_1 \oplus E_2 \oplus ... \oplus E_m$, let $Q_m = I - P_m$, and let P_{E_j} be the projection onto E_j, then we consider the general class of operators $A_\varphi = \sum a(\lambda_j) P_{E_j}$ such that $A_\varphi \geq Q_m A^\alpha$ in the sense of quadratic forms, i.e. $(A_\varphi v, v) \geq (Q_m A^\alpha v, v)$.

The equations (1.1) describe a number of models of fluid turbulence for $\alpha \geq 1$. In [48], Kraichnan argued for the spectral dependence of eddy-viscosity models; further details of spectral eddy viscosity (SEV) models were discussed in [19] and [20], and further motivation for the necessity of spectral dependence was discussed in [11]. Karamanos and Karniadakis ([43]) applied the spectral vanishing viscosity (SVV) method to a three dimensional turbulence model as an approximation to SEV. The SVV methodology corresponds to a particular distinguished class of A_φ in which $a(\lambda_j) = 0$ for $j \leq m_0 \leq m$, $0 \leq a(\lambda_j) \leq \lambda_j^\alpha$ for $m_0 \leq j \leq m$, and $a_j(\lambda_j) \geq \lambda_j^\alpha$ for $j \geq m+1$, where $m_0 \to \infty$ as $m \to \infty$. In [12] $\alpha = 1$ but it was noted therein that SVV could be implemented with hyperviscosity using e.g. discontinuous Galerkin techniques. Typically in such a hyperviscous version of SVV we would have $\alpha \geq 2$ and such models were studied theoretically in [35],[34] and [4], and referred to in the latter as the spectrally hyperviscous Navier-Stokes equations (SHNSE). In [34], J.-L.Guermond and S. Prudhomme proved global regularity for a subclass of such models and established subsequence convergence to a weak solution of the Navier-Stokes equations. In [4] global regularity was established for a general class of the SHNSE as well as estimates of attractor dimension and the existence of inertial manifolds. The attractor-dimension estimates in particular obtain uniquely robust agreement with the Landau-Lifschitz estmates for the degrees of freedom in turbulent flow.

In the next section, we will introduce the strong convergence of Galerkin solutions for the SHNSE model.

1.2 Convergence of Galerkin Solutions and Continuous Dependence on Data

In [21] the NSE with spectral hyperviscosity, i.e. (1.1), was first studied theoretically and therein it was argued that the model (1.1) improves spectral accuracy and regularizing properties compared with SEV. In [4] we obtained estimates on the dimension of the attractor for trajectories of (1.1) by adapting elements of the "CFT" framework ([21], [23], [24], [74], [75]), and in particular techniques using the Lieb-Thirring inequalities as developed in [74], [75]. For typical A_φ in the distinguished-class case we also adapted the concepts and techniques developed in [30], [31] to establish the existence of an inertial manifold of dimension m_0. The attractor results compare favorably with those obtained for the NS-α model ([27]; see also the references contained therein and the origins of NS-α in [37], [38]). In fact the properties of the attractor and the results on existence of intertial manifolds for (1.1) established in [4] are arguably unique among realistic closure models for the NSE. In particular, let ϵ be Kolmogorov's mean rate of dissipation of energy and let

$$l_\epsilon = \left(\frac{\nu}{\epsilon}\right)^{4/3}. \tag{1.2}$$

Then for $l_0 = \lambda_1^{-1/2}$ we found in [4] that the Hausdorff and fractal dimensions of the attractor $\mathcal{A}$ are bounded by

$$K_\alpha \left(\nu/\mu\right)^{9/(10\alpha)} \left[\lambda_m/\lambda_1\right]^{9(\alpha-1)/(10\alpha)} \left[l_0/l_\epsilon\right]^{(6\alpha+9)/(5\alpha)} \tag{1.3}$$

where K_α depends only on α and is on the order of magnitude of unity, $c_\alpha = (\nu/\mu)^{9/(10\alpha)}$ is (as discussed in [4]) within an order of magnitude of unity for typical choices of ν and μ, $b_\alpha = (6\alpha+9)/(5\alpha) < 3$, and $d_\alpha = 9\,(\alpha-1)\,/(10\alpha)$ gives an overall growth in m generally significantly less than $m^{3/5}$. Also, K_α, c_α, and λ_m/λ_1 are dimensionless and scale invariant. In particular, as long as

$$\lambda_m \leq \left(1/l_\epsilon\right)^2 \tag{1.4}$$

the estimate (1.2) becomes

$$K_\alpha c_\alpha \left[l_0/l_\epsilon\right]^3 \tag{1.5}$$

which is virtually straight-up agreement with the Landau-Lifschitz predictions for the number of degrees of freedom in turbulent flow ([53]), with m so large that one would expect machine-indistinguishable agreement with NSE solutions; indeed, using the fact that $\lambda_m \sim c\lambda_1 m^{2/3}$ for a dimensionless constant c that depends on the shape, but not the size, of Ω, we can write (1.4) as $m \leq (c\lambda_1)^{-3/2}\left(1/l_\epsilon\right)^3$. This implies that the NSE only needs to be regularized in the highest modes in order to conform to the Landau-Lifschitz predictions, and thus (1.3) - (1.5) contain both information on the NSE itself and the smoothing and approximating power of (1.1), as well as implying for lower values of λ_m the ability of (1.1) to reduce the number of degrees of freedom in turbulence simulation.

With simulation in mind as well as further exploration of the finite-dimensional character of (1.1) we study here convergence results of Galerkin approximations to (1.1); as corollaries of our estimates we will also obtain results on the continuous dependence on data. We begin with some basic results on the convergence of Galerkin solutions. Let P_N be the projection onto $E_1 \oplus \cdots \oplus E_N$ for some $N > m$, then the Galerkin approximation

u_N to u satisfies

$$(u_N)_t + \nu A u_N + \mu A_\varphi u_N + P_N (u_N \cdot \nabla) u_N + \nabla p_N = g_N, \tag{1.6a}$$

$$\nabla \cdot u_N = 0 \tag{1.6b}$$

where $g_N = P_N g$. Let $w_N = u - u_N$, let $G_N = g - g_N$, and let L_g and U_g be defined as in (2.8) and (2.12) below, then we have

Theorem 1 *Let $T > 0$ then for $\gamma = (\alpha - \beta)/2$, for any $\beta \geq 0$, and for some $\theta \geq 1/2$ we have for all $0 \leq t \leq T$ that if $\alpha \geq 2$*

$$\left\| A^{\beta/2} w_N(t) \right\|_2^2 \leq W_N(t) e^{\frac{4}{\mu}\lambda_m^{\alpha-1} C_0 C_{0,1} T} \tag{1.7a}$$

where C_0 is a generic constant and $C_{0,1}$ is a polynomial in U_g and T of degree depending on θ. If $\alpha - 3 \leq \beta \leq \alpha$ then we obtain

$$\left\| A^{\beta/2} w_N(t) \right\|_2^2 \leq W_N(t) \exp\left(\frac{8}{\mu} \lambda_m^{\alpha-1} \frac{C_0}{(\nu\lambda_1^2)} \int_0^t \|g\|_2^2 \, ds \right). \tag{1.7b}$$

Here

$$W_N(t) \equiv \left\| A^{\beta/2} w_N(0) \right\|_2^2 + \frac{4\lambda_m^{\alpha-1}}{\mu} \int_0^t \left(\frac{1}{\lambda_1^\gamma} \|G_N\|_2^2 + \left\| A^{-\gamma} Q_N (u \cdot \nabla) u \right\|_2^2 \right) ds. \tag{1.7c}$$

We now show how the Dominated Convergence Theorem applies to (1.7c) to obtain uniform convergence of $\left\| A^{\beta/2} w_N(t) \right\|_2$ to zero on each $[0, T]$. In particular for the term $\|A^{-\gamma} Q_N (u \cdot \nabla) u\|_2^2$, we note that for $\alpha \geq 2$ we will show below (see Lemmas 13 and 14) that there is a generic constant M_0 such that

$$\begin{aligned} \left\| A^{-\gamma} Q_N (u \cdot \nabla) u \right\|_2^2 &= \left\| Q_N A^{-\gamma} (u \cdot \nabla) u \right\|_2^2 \\ &\leq \left\| A^{-\gamma} (u \cdot \nabla) u \right\|_2^2 \leq M_0 \|\nabla u\|_2^2 \left\| A^{\beta/2} u \right\|_2^2. \end{aligned} \tag{1.8}$$

Now $\left\| A^{\beta/2} u \right\|_2^2$ is uniformly bounded for any β by the regularity results in [4, section 2], and

$$\nu \int_0^t \|\nabla u\|_2^2 \, ds \leq \|u_0\|_2^2 + \frac{1}{\nu\lambda_1} \int_0^t \|g\|_2^2 \, ds \tag{1.9}$$

by the standard energy inequality (see e.g. (2.6) below). Meanwhile $\|G_N\|_2 \leq \|g_N\|_2 + \|g\|_2 \leq 2\|g\|_2$, thus by the Dominated Convergence Theorem, since both the left-hand side of (1.10) and $\|G_N\|_2^2$ go to zero as $N \to \infty$ for each t, we have that (1.7c) goes to zero as $N \to \infty$.

We now explore a sufficient condition for the bound in (1.7b), (1.7c) to give uniform convergence on $[0, \infty)$. We define *decaying turbulence* (DT) to be the case

$$g \in L^2\left([0,\infty]\,; L^2(\Omega)\right). \tag{1.10}$$

Before specializing (1.7b), (1.7c) to the DT case, we note the following property of DT which generalizes the exponential decay property of the $g \equiv 0$ case:

Theorem 2 *Let (1.1) be such that g is in the DT case, then*

$$\left\|A^{\beta/2}u(t)\right\|_2 \to 0 \text{ as } t \to \infty \tag{1.11}$$

for all $\beta > 0$.

Now suppose g in (1.7b), (1.7c) is in the DT case, i.e. (1.10) holds, and let

$$G_\infty = \int_0^\infty \|g\|_2^2\, ds,$$

then we have the following result:

Theorem 3 *Suppose g is in the DT case, and suppose that $\alpha \geq 2$ and $\beta \leq \alpha$. Then for $W_N(t)$ as in (1.7c) we have for each $t \geq 0$ that*

$$\left\|A^{\beta/2}w_N(t)\right\|_2^2 \leq W_N(t) e^{\frac{8}{\mu}\lambda_m^{\alpha-1}C_0(\nu\lambda_1)^2 G_\infty} \tag{1.12}$$

and so by the remarks above the convergence $u_N \to u$ in H^β is uniform on $[0,\infty)$ as $N \to \infty$ by the Dominated Convergence Theorem.

The results in Theorems 1 and 3 overlap with those in [79], in which C^1-convergence of Galerkin solutions was established on each $[0,T]$ for a general class of semilinear parabolic

PDE which include the HNSE, i.e. (1.1) with $m = 0$. In addition to considering arbitrary m here, we have convergence in C^n for all n by the Sobolev embedding theorems since in Theorem 1 $\beta > 0$ is arbitrary. Also, since a generalization of the basic energy inequality applies to (1.1), we can obtain uniform H^β-convergence on all of $[0, \infty)$ in the case of decaying turbulence. Moreover, here we can allow for arbitrary initial data while in [79] $P_N u_0$ needs to be in a compact trapping region. Finally, the error bounds given by (1.7b) and (1.12) depend explicitly on generic constants and the data.

While these are reasonably satisfying theoretical results, the exponential dependence on $\lambda_m^{\alpha-1}$ detracts somewhat from its practical implications, especially for $\alpha > 2$. Our studies indicate that it seems impossible to completely eliminate the exponential dependence on m, but is worthwhile to try to achieve exponential dependence like m^b where $b \leq 1/2$; this would mean that for even reasonably large m the contribution of m in the exponential term is within an order of magnitude of the other constants present.

We can achieve such results for additional conditions imposed on β and for sufficiently large λ_m by using a spectral decomposition technique. Specifically, let P_m be the projection onto $E_1 \oplus \cdots \oplus E_m$ and let $Q_m = I - P_m$ as above; without loss of generality we can assume that $N > m$. We will look at the convergence of $P_m w_N$ and $Q_m w_N$ separately in the next theorems. Our first result in this direction in fact borrows some techniques directly from the determining-modes theory ([22], [29], [40]; also see [28, Chapter III]), which like inertial-manifold theory represents a measure for how lower frequencies dominate the high frequencies. The Kolmogorov theory of turbulence ([46]) predicts such dominance, in that the highest wavenumber components receive dynamic input from the lower-wavenumber components as a result of the energy cascade and then decay so rapidly as to no longer be of dynamical consequence.

Let $\eta \geq 0$ be such that

$$\|u_0\|_2^2 \leq (1 + \eta) \left(\frac{L_g}{\nu \lambda_1} \right)^2 \tag{1.13}$$

and set

$$T_1 = (1 + \eta) \tag{1.14}$$

then for

$$\gamma_m \equiv \lambda_{m+1}^{\alpha} - \lambda_m^{\alpha}, \tag{1.15}$$

for a constant Γ depending on the size of the data (see (4.12) below and the preceeding discussion), and for

$$\Gamma' \equiv e^{(\gamma_m + \Gamma)T_1} \tag{1.16}$$

we have the following:

Theorem 4 *Assume that $\alpha \geq 2$ and $\beta \leq 3/2$, let $T_1, \gamma_m, \Gamma,$ and Γ' be as above, let M be an integer such that $t/M \leq 1$, let $\sigma = \alpha - 3/4$, and suppose that m is large enough so that*

$$\lambda_{m+1}^{\alpha} - \frac{8C_0\lambda_{m+1}^{-2\sigma}}{\mu^2}\left(1 + \frac{1}{T_1} + \nu\lambda_1\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 > \lambda_{m+1}^{\alpha} - \lambda_m^{\alpha} = \gamma_m, \tag{1.17}$$

then for a generic constant C_1 we have that

$$\begin{aligned}\left\|A^{\beta/2}Q_m w(t)\right\|_2^2 \ &\leq \ \left\|A^{\beta/2}Q_m w\left(0\right)\right\|_2^2 \Gamma' e^{-\gamma_m t} + \Gamma' \int_0^t e^{-\gamma_m (t-s)} F_{Q,N}(s) ds \\ &\quad + \frac{8C_1\lambda_{m+1}^{-2\sigma}\Gamma'}{\nu\mu}\mathcal{E}_{m,M}(t)(2 + \eta + \nu\lambda_1)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \rho_P\left(t\right) \end{aligned} \tag{1.18}$$

where

$$\rho_P\left(t\right) = \sup_{0 \leq s \leq t} \left\|A^{\beta/2} P_m w_N\left(s\right)\right\|_2^2, \tag{1.19a}$$

$$F_{Q,N}\left(t\right) = \frac{4}{\mu}\left\|Q_m G_N\right\|_2^2 + \frac{4}{\lambda_1^{\gamma}}\left\|A^{-\gamma} Q_N\left(u \cdot \nabla\right) u\right\|_2^2, \tag{1.19b}$$

and

$$\mathcal{E}_{m,M}(t) = \frac{1 - e^{-\gamma_m(1-(1/M))t}}{e^{(\gamma t)/M} - 1} + 1. \tag{1.19c}$$

Thus in particular, by what we know about $F_{Q,N}\left(t\right)$ from before, $\left\|A^{\beta/2}Q_m w_N\left(t\right)\right\|_2^2$ converges uniformly to zero as $N \to \infty$ on any interval on which $\rho_P\left(t\right)$ converges uniformly; no additional restrictions are needed on U_g except as $\rho_P\left(t\right)$ may require. The exponential factor Γ' is time-independent without requiring a DT condition. Also note that the right-hand side of (1.18) is nonincreasing in m, and in fact this result will lead to better results for $P_m w_N$ as well. Just as significantly, we see that in some sense the convergence of $Q_m w_N$ to zero is controlled in large part by the convergence of $P_m w_N$, in analogy with results on determining modes for the NSE and with the inertial-manifold results noted above.

The condition (1.17) basically requires that $\lambda_{m+1}^{\alpha-3/4} > c_1(\nu/\mu)G$ where $G = L_g/(\nu^2\lambda_1^{3/4})$ is the Grashoff number and c_1 is a generic constant. Our best estimates on the dimension of the inertial manifolds constructed in [4] were of the form $\lambda_m > c_2 G^2$ when $\alpha \geq 5/2$; such a lower bound is improved here for any $\alpha \geq 2$ and further improves as α grows. Using $\lambda_m \sim c\lambda_1 m^{2/3}$ the lower-bound condition is satisfied if $m^{(2/3)\alpha-1/2} > c_3(\nu/\mu)G$. This gives $m > c_3(\nu/\mu)^{5/6}G^{5/6}$ when $\alpha \geq 2$, comparing well with the 2-d no-slip estimate $m > c_4 G^2$ in [29], and for $\alpha \geq 9/4$ we match the condition $m > c_4 G$ derived in [40] for the 2-d periodic case. The value $\alpha = 3 > 9/4$ was used in [20] for $m = 0$. The Grashoff number is an upper bound for $[l_0/l_\epsilon]^2$; assuming that this is a sharp upper bound makes $G^{1/2}$ the wavenumber boundary of the inertial range. We cross this boundary as α approaches 4, obtaining the estimate $m > c_5(\nu/\mu)^{6/13}G^{6/13}$ when $\alpha \geq 4$; the values $\alpha = 4, 8$ were used in [11], [12] in applications of (1.1) when $m = 0$.

While Theorem 4 improves on the previous estimates in terms of the power on λ_m, we still have exponential dependence on the data in the coefficient Γ as we will see below. In the next results we employ new techniques to achieve linear dependence on the data and on $\rho_P(t)$, as well as on $A^{\beta/2}Q_m w(0)$, $Q_m G_N$, and $Q_N(u \cdot \nabla)u$. The tradeoff is a somewhat larger lower estimate on m; the simplest case of these results additionally assumes larger α and resultingly adjusted assumptions on β.

Theorem 5 *Let L_g, η, and $F_{Q,N}$ be as above, assume that $\alpha \geq 5/2$, that $\alpha - 3/2 \leq \beta \leq \alpha - 1$, and that*

$$\lambda_{m+1}^{2\alpha-5/2} \geq \frac{16C_0(2+\eta)}{\mu^2}\left(\frac{L_g}{\nu\lambda_1}\right)^2 \tag{1.20}$$

then for all $t \geq 0$, for $d = \mu/2$, and for a generic constant C_2 we have that

$$\left\|A^{\beta/2}Q_m w_N(t)\right\|_2^2 \leq \left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 e^{-dt\lambda_{m+1}^{\alpha}} + \int_0^t e^{-d(t-s)\lambda_{m+1}^{\alpha}} F_{Q,N}(s)ds + \frac{16C_2}{\mu^2\lambda_{m+1}^{2\alpha-5/2}}(2+\eta)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \rho_P(t). \quad (1.21)$$

From the above discussion we see that (1.20) compares well with our inertial manifold conditions on λ_m in [4], as well as the 2-d no-slip results in [29], in that for $\alpha \geq 2$ we obtain estimates of the form $m > c_6(\nu/\mu)^2 G^2$. When $\alpha = 3$, condition (1.20) is

$$\lambda_{m+1} \geq \frac{2\sqrt[3]{3C_2(2+\eta)}(\nu/\mu)^{2/3}}{\lambda_1^{1/6}} G^{2/3} \quad (1.22)$$

which is satisfied if

$$m \geq \frac{2^{3/2}\sqrt{3C_2(2+\eta)}(\nu/\mu)}{\lambda_1^{1/4}} G \quad (1.22)$$

which matches the 2-d periodic result in [40] as noted above. When $\alpha > 17/4$ the lower bound enters into the inertial range. Meanwhile, condition (1.21) is the kind of linear dependence on the data and on $\rho_P(t)$ that we seek, and the estimate improves as m grows; moreover again the convergence is uniform on any interval for which uniform convergence holds for $\rho_P(t)$.

Except for the term involving $Q_N(u\cdot\nabla)u$, the estimate (1.21) is exactly the same as would be obtained if the nonlinear term were not present. In fact the Kolmogorov theory predicts that viscous (i.e. linear) effects dominate the dynamics in these modes. The reverse is true for the low wavenumber modes, and indeed we will see this reflected in our new estimates for $P_m w_N$ (see Theorem 7 below), where we obtain improved but still exponential estimates involving the data.

Meanwhile, we can obtain similar linear dependence as in (1.21) for $2 \leq \alpha \leq 5/2$, but for more involved calculations.

Theorem 6 *Assume that $\alpha \geq 2$ and that $\beta \leq 3/2$. Let L_g and η be as above, let*

$$\rho_Q(t) = \sup_{0\leq s\leq t}\left\|A^{\beta/2}Q_m w_N(s)\right\|_2^2, \quad (1.23a)$$

and let

$$\mathcal{F}_{Q,N}\left(t\right)=\sup_{0\leq s\leq t}\int_0^t e^{-\mu(s-\tau)\lambda_{m+1}^{\alpha}}F_{Q,N}\left(\tau\right)d\tau, \tag{1.23b}$$

then if we assume that

$$\lambda_{m+1}^{2\alpha-5/2}\geq\frac{16C_1}{\mu^2}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \tag{1.24}$$

we have for a generic constant C_1 as above that

$$\begin{aligned}\rho_Q\left(t\right) &\leq 2\left\|A^{\beta/2}Q_m w_N\left(0\right)\right\|_2^2+2\mathcal{F}_{Q,N}\left(t\right)\\ &\quad+\frac{16C_1}{\mu^2\lambda_{m+1}^{2\alpha-5/2}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2\rho_P\left(t\right).\end{aligned} \tag{1.25}$$

The condition (1.24) is similar to and essentially represents the same lower bound as (1.20), and thus compares in essentially the same way to the inertial-manifold and determining-mode conditions mentionded above. Both Theorems 5 and 6 can be adapted to produce results on the continuous dependence on data that reflect the same spectral characteristics; we will discuss these in section 3.1.3.

Note that under the minimum conditions on m in both Theorems 5 and 6, and replacing μ by $b=\mu/2$ in the definition of $\mathcal{F}_{Q,N}$ in (1.23) in the case of Theorem 6, we have the combined condition that

$$\rho_Q\left(t\right)\leq 2\left\|A^{\beta/2}Q_m w_N\left(0\right)\right\|_2^2+2\mathcal{F}_{Q,N}\left(t\right)+\rho_P\left(t\right) \tag{1.26}$$

This simple expression can be used to obtain an estimate on $\rho_P\left(t\right)$ that improves the estimate in Theorem 4 and depends only on the data on the right-hand side of (1.26) and on U_g.

Theorem 7 *Let C_0, L_g, η, and $F_{Q,N}$ be as above and let*

$$\mathcal{L}_{G_N}\equiv\sup_{0\leq s\leq t}\frac{3}{\nu\lambda_1}\int_0^s e^{-\nu\lambda_1(s-\tau)}\left\|G_N\left(\tau\right)\right\|_2^2d\tau \tag{1.27}$$

then we have on each interval $[0,T]$ *that for* $w_{N,0} = w_N(0)$*,* $G_0 = L_g/(\nu\lambda_1)$ *and for*

$$W_{N,0}(t) \equiv \left\|A^{\beta/2}P_m w_{N,0}\right\|_2^2 + \frac{3\mathcal{L}_{G_N}}{(\nu\lambda_1)^2} + \frac{3\lambda_m^{1/2}C_0\left(3+2\eta+\nu\right)}{\nu^2}G_0^2(\left\|A^{\beta/2}Q_m w_{N,0}\right\|_2^2 + \mathcal{F}_{Q,N}\left(t\right)) \qquad (1.28)$$

we have that

$$\rho_P\left(t\right) \leq W_{N,0}(t)\exp\left(\frac{12\lambda_m^{1/2}C_0}{\nu^2}\left[(1+\eta)G_0^2 + \frac{1}{\nu\lambda_1}\int_0^t \|g\|_2^2\, ds\right]\right) \qquad (1.29)$$

and in the case of decaying turbulence we can replace $\int_0^t \|g\|_2^2\, ds$ *by* $G_\infty \equiv \int_0^\infty \|g\|_2^2\, ds$ *and have uniform convergence for all* $t \geq 0$.

Note the fractional dependence on m in both (1.28)and (1.29); in fact, since $\lambda_m \sim c\lambda_1 m^{2/3}$, this dependence is like $m^{1/3}$. We will discuss further the significance of Theorems 4 - 7 in the conclusion. After making some preliminary observations and calculations in the next section, as well as proving Theorem 2, we will prove Theorems 1 and 3 in section 3.1.1. In section 3.1.2 we will prove Theorems 4 - 7, and in section 3.1.3 we will discuss our results for continuous dependence on data in the context of spectral decomposition.

1.3 Determining Nodes and Determining Modes Result

When dynamical systems are implemented in physical and computational experiments, questions arise concerning how many and which elements or modes of the solution rule the dynamic behavior. Attractor estimates and inertial manifold results represent two methods for addressing these questions, but the former cannot identify which modes are dominant and the estimates for the dimension of the latter are generally quite large. Estimates for the number of determining modes, however, provide a sharper estimate for a number N such that in a reasonable way the first N modes control the dynamics, and estimates for the number of determining nodes give a reasonable estimate of the number of nodal points needed to capture a good approximation of the dynamics. We estimate the number of determining nodes and modes for the 3-D incompressible spectrally-hyperviscous Navier-Stokes equations (SHNSE).

In practice, experimentalists take their measurements at a relatively small number of nodes in the physical domain. The number of nodes used is directly related to the accuracy of the experiment results; determining-node results, as developed in [28, Chapter III], attempt to establish sufficient conditions for the nodal measurements to capture the solution behavior. In particular they obtain lower bounds on the number of nodes such that asymptotic continuous dependence on data at these nodes implies asymtotic continuous dependence on data throughout the domain. Such a relationship is detailed and established in the following result:

Theorem 8 *For the three dimensional SHNSE model with spectral hyperviscosity* $\mu A_\varphi \geq \mu Q_m A^\alpha$ *and* $\alpha \geq 2$*, let the domain* Ω *be covered by* N *identical cubic boxes. Consider a set* $\varepsilon = \left\{X^1, X^2, X^3..., X^N\right\}$ *of points in* Ω*, distributed one in each cubic box. Let* f *and* g *be two forcing terms in* $L^\infty(0,\infty;H)$ *that satisfy* $\int_\Omega \mid f(x,t) - g(x,t) \mid^2 dx \to 0$ *as* $t \to \infty$*. Let* $F = \limsup_{t\to\infty} \mid f(t) \mid = \limsup_{t\to\infty} \mid g(t) \mid$*, let* M_1 *be as in (2.1), and let* C_1 *be the same as in (3.67). If*

$$N^{\frac{2}{3}} > (\frac{1}{\mu}\lambda_m^{-(\alpha-\frac{5}{2})} + \frac{1}{\nu}\lambda_m^{3/2})\frac{4M_1^4}{\nu^2\mu\lambda_1^3 C_1}F^2 \tag{1.30a}$$

then ε *is a set of determining nodes in the sense that* $max_j \mid u(X^j,t) - v(X^j,t) \mid \to 0$ *as* $t \to \infty$ *implies* $\parallel A^{\frac{1}{2}}(u(x,t) - v(x,t)) \parallel_2^2 \to 0$ *as* $t \to \infty$*. Set* $\mu_1 = min(\mu,\nu)$ *then since it is safe to assume that* $\lambda_m \geq 1$*, we can replace (1.30a) with the simpler condition*

$$N^{\frac{2}{3}} > \frac{8}{\nu^2\mu\mu_1}C_1^{-1}\lambda_1^{-3}M_1^4\lambda_m^{3/2}F^2. \tag{1.30b}$$

Let $L_g = sup_{t\geq 0} \parallel f \parallel_2^2$ and let $G \equiv L_g/(\nu^2\lambda_1^{3/4})$ be the Grashoff number in 3-D (see e.g. [28]). Then the estimate (1.2b) is satisfied in particular if

$$N > \left(\frac{\nu}{\mu}\right)^3 \left(\frac{\mu}{\mu_1}\right)^{3/2} C\lambda_1^{-9/4}\lambda_m^{9/4}G^3. \tag{1.31}$$

This is larger than the 2-D determining-node estimate for the periodic case, but significantly smaller than the corresponding 2-D estimate in the no-slip case, which is exponential in the data.

As noted above, attractor dimension results suggest the number of modes determining the dynamics of the turbulent flow without identifying their location in the mode hierarchy. In contrast determining modes show that the first N modes control the dynamics of the turbulent flow in the sense defined in [28, Chapter III] and as stated below in the following results for solutions of (1.1).

Theorem 9 *For $\alpha \geq 3/2$ and $1 < \beta < \frac{3}{2}$, suppose that $N \geq m$ such that whenever $n \geq N$ we have that*

$$2\lambda_n \nu + \mu\lambda_n^\alpha > \frac{4}{\mu\nu^2\lambda_1}\frac{1}{\lambda_n^{(\alpha-\frac{3}{2})}}M_3F^2 \tag{1.32}$$

where M_3 is as in (3.6) below and F is as in Theorem 8. Let P_n be the projection onto $E_1 \oplus E_2 \oplus ... \oplus E_n$, and let $Q_m = I - P_m$. Then the first N modes are determining modes for (1.1) in the sense that $\| A^{\frac{\beta}{2}} P_n(u-v) \|_2 \to 0$ as $t \to \infty$ implies that $\| Q_n A^{\frac{\beta}{2}}(u-v) \|_2 \to 0$ as $t \to \infty$.

In particular it is reasonable to expect that $\mu\lambda_n^\alpha \gg 2\lambda_n\nu$, especially given the size of the parameter corresponding to μ in typical SVV computations. Under this assumption the condition (1.4) becomes

$$\lambda_n \geq (\frac{4M_3}{\lambda_1})^{\frac{2}{4\alpha-3}}(\frac{F}{\nu\mu})^{\frac{4}{4\alpha-3}} \tag{1.33a}$$

and in terms of the Grashoff number we have the condition

$$\lambda_n \geq (\frac{4M_3}{\lambda_1})^{\frac{2}{4\alpha-3}}(\frac{\nu}{\mu})^{\frac{4}{4\alpha-3}}(G)^{\frac{4}{4\alpha-3}}. \tag{1.33b}$$

Noting that $\lambda_n \sim c\lambda_1 n^{2/3}$, we can express (1.33b) as

$$n \geq [1/(c\lambda_1)]^{3/2}(\frac{4M_3}{\lambda_1})^{\frac{3}{4\alpha-3}}(\frac{\nu}{\mu})^{\frac{6}{4\alpha-3}}(G)^{\frac{6}{4\alpha-3}}. \tag{1.33c}$$

The power on G is $6/5$ when $\alpha = 2$, comparing well with the estimate $n \geq CG^2$ in the 2-D no-slip case ([2], [41]), and almost matching the estimate $n \geq CG$ in the 2-D periodic case; note that we improve on the latter estimate when $\alpha > 9/4$. Outside of new estimates of the nonlinear term using appropriate Sobolev inequalities, the proof of Theorem 9 is a straightforward adaptation of the techniques used in [4], [5]. As in those works we use Lemma 17(also see e.g. [28, Chapter III, Lemma 1.1]) which uses the assumptions (1.32), (1.33) to obtain an estimate of the form

$$\limsup_{t\to\infty} \| A^{\beta/2}Q_n(u-v)(t) \|_2 \leq KTe^{\gamma T} \sup_{t\geq t_0} \frac{1}{T}\int_t^{t+T} \| A^{\beta/2}P_n(u-v)(\tau) \|_2^2 \, d\tau \quad (1.34)$$

for appropriately chosen T and t_0. From (1.33c) the conclusion of this lemma is reached, and from it the conclusion of Theorem 9 is reached. In particular the constant $KTe^{\gamma T}$ is multiply exponential in the data. We can improve on this estimate and obtain one that is more computationally relevant if we assume an extra condition on α:

Theorem 10 *For three dimensional SHNSE with spectral hyperviscosity $\mu A_\varphi \geq \mu Q_m A^\alpha$, $\alpha \geq \frac{5}{2}$ and $\beta < \frac{3}{2}$, suppose that $n \in N$ and $n \geq m$, is such that*

$$2\lambda_n \nu + \mu\lambda_n^\alpha > \frac{8}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} U_g^2 + d \quad (1.35)$$

suppose $\rho_P(\tau,t) = \limsup_{\tau<s<t} \| A^{\frac{\beta}{2}}P_n(u-v) \|_2^2$ and $\rho_G(\tau,t) = \limsup_{\tau<s<t} \| A^{-\frac{\beta}{2}}G \|_2^2$, the estimate for $\xi(t) = \| A^{\frac{\beta}{2}}Q_n(u-v) \|_2^2$ is

$$\overline{\lim}_{t\to\infty}\xi(t) \leq [\rho_P(\tau,\infty)U_g^2 \frac{8}{\mu} M_4 \frac{1}{\lambda_n^{\alpha-\frac{5}{2}}} + \frac{1}{d}\rho_G(\tau,\infty)] \quad (1.36)$$

where M_4 is the same as in (4.3), $U_g^2 = \| u_0 \|_2^2 + (\frac{L_g}{\nu\lambda_1})^2$ and $L_g = sup_{t\geq 0} \| f \|_2$. Then the first n modes are determining modes in the sense that $\| A^{\frac{\beta}{2}}P_n(u-v) \|_2 \to 0$ as $t\to\infty$ implies $\| A^{\frac{\beta}{2}}(u-v) \|_2 \to 0$ as $t\to\infty$.

Note that the dependence now on the key parameters on the right-hand side of (1.36) is at most quadratic. In section 3.2.1, we will develop some preliminary results and prove Theorem 8. Theorem 9 will be proven in section 3.2.2, with Theorem 10 proven in section 3.2.3. We will then close with some concluding remarks.

1.4 Convergence to the Inviscid Limit in the SHNSE Model

For a nonnegative constant μ, viscosity coefficient ν, and certain regularizing operators A_φ, we consider 3-D turbulence models in a general class

$$(u_\nu)_t + \mu A_\varphi u_\nu + \nu A u_\nu + (u_\nu \cdot \nabla) u_\nu + \nabla p_\nu = g, \tag{1.37a}$$

$$div\ u_\nu = 0 \tag{1.37b}$$

where we use the subscript to regard (1.37) as having a family of solutions parametrized by ν. We will study the limit $\nu \to 0$, for which the target equations are

$$u_t + \mu A_\varphi u + (u \cdot \nabla) u + \nabla p = g, \tag{1.38a}$$

$$div\ u = 0. \tag{1.38b}$$

The system (1.38) represents a class of approximations to the Euler system. The method of viscosity solutions (see e.g. [54]) sets $A_\varphi = \epsilon \Delta u$, i.e. $m = m_0 = 0$, $\alpha = 1$, and $\mu = \epsilon$, and it has been applied extensively to conservation laws and symmetric hyperbolic systems as well to suppress Gibbs oscillations, stabilize the system, enforce entropy dissipation, and obtain a unique entropy solution. However, it is an "uncontrollable process that may compromise the solution accuracy ([43])." To accomplish the same goals while preserving spectral accuracy, Tadmor ([72]), in application to the 1-d inviscid Burger's equation, introduced the spectral vanishing viscosity (SVV) method, applying extra viscosity only to the high wavenumber modes. Assuming L^∞ stability Tadmor proved the convergence of the SVV approximations to the unique entropy solution of Burger's equation. These results were generalized to multidimensional conservation laws in [16], and in [2] and [41] SVV methodology was applied in 2-d versions of (1.38) with $\alpha = 1$ and $\mu, m_0 > 0$, retaining spectral accuracy while successfully stabilizing the computations and ensuring convergence to the entropic solution of the Euler system.

This form of SVV methodology was first applied to the SVV-modified incompressible 3-D Navier-Stokes equations (i.e. (1.37) with $\nu > 0$), in [43] for Reynolds numbers up to Re

= 395 in the simulation of turbulent channel flows. In this context SVV is comparable to the spectral eddy viscosity (SEV) introduced by Kraichnan ([48]; also see the discussion in [19], [20], [12], [15]), but SVV does not affect the low modes and its spectral profile rises more sharply in the higher wavenumber ranges. In [43] excellent agreement with benchmark DNS was obtained, whereas standard LES models were overdissipative in the near-wall region. In [45] Re = 1250 in a study of a triangular duct and in [63] Re = 768000 in a treatment of the Ahmed car-body problem. In these works μ is on the order of $1/N$ where N is the polynomial order; since $N = 21, 16$ in [43], [45] and $N \sim 150$ in [63], μ is thus much larger than the viscosity coefficient ν.

Though SVV is implemented in [43], [45], [63] using $\alpha = 1$, it was mentioned in [43] that the case $\alpha > 1$ could be realized using e.g. discontinuous Galerkin techniques. Indeed, "SVV can be thought of as using hyperviscous dissipation that will affect only the high Fourier modes" ([43]). Global regularity results for (1.1) were obtained in [4], [35] for $\alpha \geq 5/4$, generalizing the classical results in [58] for the case $m = 0$. In [4] for $\alpha \geq 3/2$ we demonstrated the existence of a compact global attractor $\mathcal{A}$ with Hausdorff and fractal dimensions bounded by $Km^a\kappa_d^b$ where κ_d is the Kolmogorov wavenumber, K is generally within an order of magnitude of unity, a is a fractional power, and $b < 3$. In particular $m^a\kappa_d^b \leq \kappa_d^3$ for any $m \leq \kappa_d^3$, i.e. even for m so large as to suggest machine-indistinguishability from NSE solutions. The property that we only need to regularize the highest wavenumber modes to achieve robust conformance with the Landau-Lifschitz estimates ([53]) appears to be unique among NSE closure models, and as such the results compare well with the groundbreaking attractor results for the 3-D NS-α system in [27]. For computationally realistic choices of m we have that b is significantly lower, implying the potential for the system (1.38) to supply spectrally-accurate simulation with reduced degrees of freedom. Also in [4] we obtained for $\alpha \geq 3/2$ the existence of inertial manifolds which in particular imply that for m large enough eigenmodes free of hyperviscosity control the essential dynamics.

Meanwhile, a form of SVV as in (1.38) with $\alpha > 1$ has been proposed for 1-D conservation laws in [73]; therein H^1-stability is established, as well as convergence to enropic solutions of the conservation laws assuming L^∞-stability. Given these results, and that the results in [72] led to the 2-D applications of SVV to the Euler system as described above, it is thus natural to assume that (1.38) for $\alpha > 1$ can serve as a viable model of inviscid turbulence in 3-D. Since both (1.37) and (1.38) share a spectrally-applied hyperviscous term $\mu A_\varphi u$, and since in the SVV applications to 3-D viscous flow in [43], [45], [63] μ is significantly larger than ν with a significant respective increase in the ratio μ/ν, it would also seem natural to fix the term $\mu A_\varphi u$ in both equations in an appropriate scaling while studying the limit $\nu \to 0$.

We will assume as in [3] - [5] that Ω is a periodic box in 3-D and establish, for $m_0 > 0$, $\alpha \geq 3/2$, and (for simplicity) $g = 0$, strong convergence of solutions of (1.37) to solutions of (1.38) as $\nu \to 0$ on compact time intervals. First we establish global regularity; here P is the Leray projection.

Theorem 11 *Let $u_0 \in PH^s(\Omega)$ for some $s > 0$, then for $\alpha \geq 5/4$ there exists a unique global regular solution u of (1.38) such that $u \in C(0,\infty); PH^s(\Omega) \cap C((0,\infty); PH^s(\Omega))$.*

Details of the proof of Theorem 11 will be discussed in section 3.3. Now let P_n project onto the first n eigenspaces of $A = -\Delta$, and let $Q_n = I - P_n$ for $n \geq m$; let $w_\nu = u - u_\nu$ with u as in (1.38) and $u_\nu$as in (1.37) with $\nu > 0$, and for simplicity let $\Omega = [0, L]^3$, then we have the following result:

Theorem 12 *For constants M_4 and M_5 arising from the Sobolev inequalities, suppose that $\alpha \geq 3/2$ and that $n \geq m$ is large enough so that*

$$\lambda_{n+1}^{2\alpha-5/2} \geq \frac{6M_4 \|u_0\|_2^2}{\mu^2} \tag{1.39}$$

then for $d = \mu/2$, for $\rho_P(t) = \sup_{0\leq s\leq t} \|P_n w_\nu(s)\|_2^2$, and for $U_{\nu,0} = \|u_0\|_2^2 + \|u_{\nu,0}\|_2^2$ we have for all $\alpha \geq 3/2$ and for all $t \geq 0$ that

$$\|Q_n w_\nu(t)\|_2^2 \leq \|Q_n w_\nu(0)\|_2^2 e^{-d\lambda_{n+1}^\alpha t} + \frac{3\nu}{\mu^2 \lambda_1^{(\alpha-2)/2} \lambda_{n+1}^\alpha} \|u_{\nu,0}\|_2^2 + \frac{6C_n M_4}{\mu^2 \lambda_{n+1}^\alpha} U_{\nu,0} \rho_P(t) \tag{1.40}$$

where $C_n = \lambda_n^{5/2-\alpha}$ *for* $2 \leq \alpha \leq 5/2$ *and* $C_n = \lambda_{n+1}^{-(\alpha-5/2)}$ *for* $\alpha \geq 5/2$. *Thus if* $\|Q_n w_\nu(0)\|_2^2 \to 0$ *then* $\|Q_n w_\nu(t)\|_2 \to 0$ *uniformly as* $\nu \to 0$ *on each interval on which* $\|Q_n w_\nu(0)\|_2^2 \to 0$ *and* $\rho_P(t) \to 0$ *uniformly. Setting*

$$V_{\nu,0} = 3\nu\lambda_n\|u_{\nu,0}\|_2^2 + 3\lambda_n^{5/4} M_5 U_{\nu,0}\left[\|Q_n w_\nu(0)\|_2^2 + \frac{3\nu}{\mu^2\lambda_1^{(\alpha-2)/2}\lambda_{n+1}^{\alpha}}\|u_{\nu,0}\|_2^2\right] \tag{1.41a}$$

and

$$W_{\nu,0} = \frac{6C_n M_4}{\mu^2\lambda_{n+1}^{\alpha}} U_{\nu,0} + 3(\lambda_n^{5/4} M_5 U_{\nu,0} + \lambda_n) \tag{1.41b}$$

we have that

$$\rho_P(t) \leq V_{\nu,0} T \exp W_{\nu,0} t \tag{1.42}$$

for all $t \in [0, T]$, *and hence* $\rho_P(t) \to 0$ *uniformly as* $\nu \to 0$ *on each such interval.*

Noting that $\lambda_n \sim c\lambda_1 n^{2/3}$ we see that the growth of all constants in (1.41a), (1.41b) with n is at most $n^{5/6}$. The estimate (1.40) depends on negative powers of n, is linear in the initial data, and sees nonlinear (at most quadratic) input only from the term involving $\|P_n w\|_2$ acting as a forcing function. The Galerkin convergence estimates in [5] for (1.37) are similar to (1.40) for the high-wavenumber modes, reflecting the Kolmogorov theory ([46]) predictions that in these modes dissipative effects dominate; in particular the high-wavenumber modes have the better behavior of a nearly linear parabolic equation. The estimate (1.40) suggests that in the inviscid case similar behavior is produced by enforced entropy dissipation.

The estimate (1.40) is not possible without using the new spectral decomposition techniques developed below; otherwise we would have at best a single Gronwall-type estimate for $\|w_\nu\|_2^2$ similar to (1.39), and even in that case we would have to assume that $\alpha \geq 5/2$, as will be seen from our estimates (see (3.117) - (3.119) below). Theorem 12 will be proven in section 3 below; next we develop some preliminary observations and estimates and prove Theorem 11.

1.5 Large Eddy Simulation Results for Flows with High Reynolds Number

It was demonstrated that the SHNSE model has unique and robust theoretical properties including: (1) global regularity and enforced energy dissipation in the microscale viscous range, (2) strong potential to simultaneously retain spectral accuracy in modeling the inertial range, (3) possession of a compact finite-dimensional global attractor whose Hausdorff and fractal dimensions are in virtual straight-up agreement with the Landau-Lifschitz degrees-of-freedom estimates even with closure parameter values so extreme as to imply machine-indistinguishability from NSE solutions, (4) possession of an inertial manifold for SHNSE subclasses that represent viable and practical models of turbulence, (5) convergence of Galerkin approximate solutions with unprecedented significant improvements in the convergence for high wave numbers, (6) strong convergence to the inviscid limit for fixed spectral hyperviscous terms (with convergence optimized for high wavenumbers), suggesting that the SHNSE is a viable platform for studying this limit computationally; numerical results described below support this strategy with evidence that the coefficient of the spectral hyperviscous term (μ in (2.1) below) is independent of the viscosity coefficient, and (7) determining mode and determining node results that are sharper than those for the two-dimensional (2D) NSE with either periodic or no-slip boundary conditions.

All these properties imply that the SHNSE model has the potential to be a highly robust and accurate platform for studying and modeling turbulence while simultaneously reducing the number of degrees of freedom required for accurate simulation. In the computational research part, as our first numerical investigation of the SHNSE model, we shall evaluate the performance of the SHNSE model and determine optimal choices of key parameters of the model through modeling homogeneous isotropic turbulent flows at high Reynolds numbers. In the future, we will further investigate its capabilities in handling complex geometries and general boundary conditions by modeling the benchmark wall-bounded turbulent channel flow [10,43,44,65] or even the more challenging flow over the "Ahmed body" car model [1,62,63], where the Reynolds number could be as high as

$Re = 768000$. One motivation for us to focus on LES for very high Reynolds number flows come from the fact that the Reynolds numbers in real atmosphere and ocean are about 1×10^5 and 5×10^5. For example, the default values are for water at $60C^\circ$ with a kinematic viscosity of 1.13×10^{-6} m^2/s in a schedule 40 steel pipe and the characteristic length (hydraulic diameter) of the pipe is 0.102 m. If the velocity of the water is 5 m/s, then the Reynolds number is 451,327, and the corresponding Kolmogorov wave number is several thousand, definitely not easily accessible by DNS on current computers.

At the beginning of our computational research, we tried to find a model more accurate than NSE in direct numerical simulation. In fact, we find somewhat of a self-similarity phenomena and the model may be used for large eddy simulation for very high Reynolds number flows. The definition for the spectrally hyperviscous term in terms of parameter choices has been figured out and tested by dimensional analysis and numerical experiments to make the SHNSE meet the goals of the LES for almost all Reynolds numbers. In fact, the SHNSE can be considered as a viable subgrid scale model. That the spectrally hyperviscous term consumes energy as much as the $Q_m \triangle$ from SVV LES has been carried out for a broad range of high Reynolds numbers, from about 100 to 7×10^6.

The computational analysis part is organized as follows. In Sec.2.2, we discuss how to discretize the incompressible SHNSE on a cubic spatial domain with periodic boundary conditions (PBCs). Detailed numerical results and analysis are then given in Sec.4.1 to demonstrate the SHNSE' capabilities in modeling homogeneous isotropic turbulence at high Reynolds numbers and to show optimal choices of key parameters of the SHNSE model. Finally, some concluding remarks are given in Chapter 5.

CHAPTER 2: PRELIMINARIES

2.1 Preliminaries for Theoretical Analysis

We express the Sobolev inequalities on Ω in terms of the operator $A = -\triangle$:

$$\|\upsilon\|_q \leq M_1 \left\|A^\theta \upsilon\right\|_p \tag{2.1}$$

where $q \leq 3p/\left(3 - 2\theta p\right)$ and $M_1 = M_1\left(\theta, p, q, \Omega\right)$. For the semigroup $\exp\left(-tA\right)$ we have the decay estimate

$$\left\|e^{-tA}\upsilon\right\|_2 \leq \|\upsilon\|_2 e^{-\lambda_1 t} \tag{2.2}$$

and, since A is analytic there is a constant c_2 such that

$$\left\|A^\beta e^{-tA}\upsilon\right\|_2 \leq c_2 t^{-\beta}\|\upsilon\|_2 \tag{2.3}$$

for any $\beta > 0$ where A^β is defined by $A^\beta = \sum_{j=1}^{\infty} \lambda_n^\beta P_{E_j}$ where as above P_{E_j} is the projection onto the jth eigenspace. Like the standard NSE, (1.1) satisfies an energy inequality, which we derive as follows: taking the inner product of both sides of (1.1) with u we have that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \nu\left\|A^{1/2}u\right\|_2^2 + \mu\left\|A_\varphi^{1/2}u\right\|_2^2 = (g, u) \tag{2.4}$$

noting that since $div\ u = 0$ we have that $(\nabla p, u) = 0$ and $\left(\left(u \cdot \nabla\right)u, u\right) = -\left(\left(div\ u\right)u, u\right) = 0$.

Now

$$(g, u) = \left(A^{-1/2}g, A^{1/2}u\right) \leq \frac{\nu}{2}\left\|A^{1/2}u\right\|_2^2 + \frac{1}{2\nu}\left\|A^{-1/2}g\right\|_2^2; \tag{2.5}$$

combining (2.5) with (2.4) using $A_\varphi \geq Q_m A^\alpha$ and multiplying by 2 we have our basic energy inequality

$$\frac{d}{dt}\|u\|_2^2 + \nu\left\|A^{1/2}u\right\|_2^2 + 2\mu\left\|Q_m A^\alpha u\right\|_2^2 \leq \frac{1}{\nu\lambda_1}\|g\|_2^2 \tag{2.6}$$

where we note that by Poincaré's inequality $\left\|A^{-1/2}g\right\|_2 \leq \lambda_1^{-1/2}\left\|g\right\|_2$; note that (2.6) reduces to the standard NSE energy inequality when $\mu = 0$. We will use 2 consequences of (2.6), the first obtained by discarding the term $u\left\|A_{\varphi}^{1/2}u\right\|_2^2$ and again using Poincaré to obtain

$$\frac{d}{dt}\left\|u\right\|_2^2 + \nu\lambda_1\left\|u\right\|_2^2 \leq \frac{1}{\nu\lambda_1}\left\|g\right\|_2^2 \tag{2.7}$$

so that, setting

$$L_g = \sup_{t\geq 0}\left\|g\right\|_2^2 \tag{2.8}$$

we have that

$$\frac{d}{dt}\left\|u\right\|_2^2 + \nu\lambda_1\left\|u\right\|_2^2 \leq \frac{L_g^2}{\nu\lambda_1}. \tag{2.9}$$

Solving the differential inequality (2.9) we have that for $u_0 = u(x,0)$

$$\left\|u(t)\right\|_2^2 \leq \left\|u_0\right\|_2^2 e^{-\nu\lambda_1 t} + \int_0^t \left(\frac{L_g^2}{\nu\lambda_1}\right) e^{-\nu\lambda_1(t-s)}ds \tag{2.10}$$

or, since $L_g^2/(\nu\lambda_1)$ is a constant,

$$\left\|u(t)\right\|_2^2 \leq \left\|u_0\right\|_2^2 e^{-\nu\lambda_1 t} + \left(\frac{L_g}{\nu\lambda_1}\right)^2. \tag{2.11}$$

Thus, we have a priori estimate

$$\left\|u(t)\right\|_2^2 \leq \left\|u_0\right\|_2^2 + \left(\frac{L_g}{\nu\lambda_1}\right)^2 \equiv U_g^2. \tag{2.12}$$

Next we prove the following technical lemma which will be used several times in section 3.1.

Lemma 13 *Let $v \in H^1(\Omega)$, let $w \in H^{\beta/2}(\Omega)$, and suppose that $\alpha \geq 2$ and $\alpha \geq \beta$. Then for $\gamma = (\alpha - \beta)/2$ there exists constants M_0 and M_0' such that*

$$\left\|A^{-\gamma}(v\cdot\nabla)w\right\|_2 \leq M_0\left\|\nabla v\right\|_2\left\|A^{\beta/2}w\right\|_2 \tag{2.13a}$$

and

$$\left\|A^{-\gamma}(w\cdot\nabla)v\right\|_2 \leq M_0'\left\|A^{\beta/2}w\right\|_2\left\|\nabla v\right\|_2. \tag{2.13b}$$

For $\sigma = \alpha/2 - 3/4$ and $\beta \leq 3/2$ there exists constants K_0 and K_0' such that

$$\left\|A^{-\gamma} Q_m (v \cdot \nabla) w\right\|_2 \leq \lambda_{m+1}^{-\sigma} K_0 \|\nabla v\|_2 \left\|A^{\beta/2} w\right\|_2 \tag{2.13c}$$

and

$$\left\|A^{-\gamma} Q_m (w \cdot \nabla) v\right\|_2 \leq \lambda_{m+1}^{-\sigma} K_0' \left\|A^{\beta/2} w\right\|_2 \|\nabla v\|_2 . \tag{2.13d}$$

To prove this, we first treat the case $0 \leq \beta \leq 1$. Write $(v \cdot \nabla) w = div\,(v \otimes w)$ where $\otimes$ denotes the appropriate tensor product. We note the standard fact that $A^{-1/2} div$ extends to a bounded operator on $L^2(\Omega)$ of norm 1, which we also denote by $A^{-1/2} div$. We also note that $A^{-1/2} div$ commutes with A under the periodic boundary conditions. Then

$$\begin{aligned}
\left\|A^{-\gamma} (v \cdot \nabla) w\right\|_2 &= \left\|A^{-\gamma} div\,(v \otimes w)\right\|_2 \\
&= \left\|A^{-(\gamma-1/2)} \left(A^{-1/2} div\right) (v \otimes w)\right\|_2 \\
&= \left\|\left(A^{-1/2} div\right) A^{-(\gamma-1/2)} (v \otimes w)\right\|_2 \\
&\leq \left\|A^{-(\gamma-1/2)} (v \otimes w)\right\|_2 .
\end{aligned} \tag{2.14}$$

Applying (2.1) to the right-hand side of (2.14) with $M_2 = M_1(\gamma - 1/2, p, 2, \Omega)$ we have that

$$\begin{aligned}
\left\|A^{-\gamma} (v \cdot \nabla) w\right\|_2 &\leq \left\|A^{-(\gamma-1/2)} v \otimes w\right\|_2 \\
&\leq M_2 \|v \otimes w\|_p \leq M_2 \|v\|_{rp} \|w\|_{sp}
\end{aligned} \tag{2.15}$$

where $p = \max\{1, 6/(4\gamma+1)\}$ and $1/r + 1/s = 1$. Since we need $\gamma \geq 1/2$, we have $\alpha - \beta \geq 1$ or $\alpha - 1 \geq \beta$ which holds with $\alpha \geq 2$ and $0 \leq \beta \leq 1$. If $p = 1$ then we take $r = s = 2$ and then (2.13a) holds by Poincaré. Otherwise by (2.1) there exists M_3 such that $\|v\|_{rp} \leq M_3 \|\nabla v\|_2$ provided that $rp = 6$ which holds if $r = 4\gamma + 1$. Then $s = r/(r-1) = (4\gamma+1)/(4\gamma)$ so that $sp = 3/(2\gamma) = 3/(\alpha - \beta)$. Again by (2.1) there exists an M_4 such that $\|w\|_{sp} \leq M_4 \left\|A^{\beta/2} w\right\|_2$ provided that $sp = 3/(\alpha-\beta) \leq 6/(3-2\beta)$ or $\alpha - \beta \geq 3/2 - \beta$ or $\alpha \geq 3/2$. The last is already satisfied since $\alpha \geq 2$. Combining the above remarks we have (2.13a) with $M_0 = M_2 M_3 M_4$ in the case $0 \leq \beta \leq 1$. By interchanging the roles of v and w we obtain (2.13b) in this case.

Next we assume that $\beta \geq 1$. Now by (2.1) there is an M_5 such that

$$\begin{aligned} \left\| A^{-\gamma} (v \cdot \nabla) w \right\|_2 &\leq M_5 \| v \cdot \nabla w \|_p \\ &\leq M_5 \| v \|_{rp} \| \nabla w \|_{sp} \end{aligned} \tag{2.16a}$$

where now $p = 6/(4\gamma + 3)$. We again want $rp = 6$ so that by (2.1) $\|v\|_{rp} \leq M_6 \|\nabla v\|_2$ for $M_6 = M_1 (1/2, 2, rp, \Omega)$, which means that $r = 4\gamma + 3$. Then $s = (4\gamma + 3) / (4\gamma + 2)$ so that $sp = 3/(2\gamma + 1)$. From (2.1) there exists an M_7 such that $\|\nabla w\|_{sp} \leq M_7 \left\| A^{(\beta-1)/2} (\nabla w) \right\|_2 = M_7 \left\| A^{(\beta-1)/2} A^{1/2} w \right\|_2 = M_7 \left\| A^{\beta/2} w \right\|_2$ provided that $sp = 3/(2\gamma + 1) \leq 6/(3 - 2(\beta - 1))$ which holds provided that $4\gamma + 2 \geq 3 - 2\beta + 2$ or $2\alpha - 2\beta + 2 \geq 3 - 2\beta + 2$ which again requires that $\alpha \geq 3/2$. Thus we obtain (2.13a) with $M_0 = M_5 M_6 M_7$ for $\beta \geq 1$.

For (2.13c) with $1 \leq \beta \leq 3/2$ we modify (2.16a) by looking for ω such that

$$\begin{aligned} \left\| A^{-\omega} (v \cdot \nabla) w \right\|_2 &\leq M_5' \| v \cdot \nabla w \|_p \\ &\leq M_5' \| v \|_{rp} \| \nabla w \|_{sp} \end{aligned} \tag{2.16b}$$

where now $p = 6/(4\omega + 3)$. We again want $rp = 6$ so now $r = 4\omega + 3$, so that $sp = 3/(2\omega + 1)$; by (2.1) we have that $\|\nabla w\|_{sp} \leq M_8 \left\| A^{(\beta-1)/2} (\nabla w) \right\|_2 = M_8 \left\| A^{(\beta-1)/2} A^{1/2} w \right\|_2 = M_8 \left\| A^{\theta/2} w \right\|_2$ provided that $\beta \geq 1$ and that $sp = 3/(2\gamma + 1) \leq 6/(3 - 2(\beta - 1))$ which leads to the condition $3/4 - \beta/2 \leq \omega$; since in this proof we have assumed that $\omega \geq 0$ we need $\beta \leq 3/2$ as well; with this we obtain (2.13c) by Poincaré with $K_0 = M_5' M_6 M_8$ since $\gamma - \omega = \sigma$.

For (2.13b) with $\beta \geq 1$ we have that

$$\begin{aligned} \left\| A^{-\gamma} (w \cdot \nabla) v \right\|_2 &\leq M_9 \| w \cdot \nabla v \|_p \\ &\leq M_9 \| w \|_{rp} \| \nabla v \|_{sp} \end{aligned} \tag{2.16c}$$

where $p = \max\{1, 6/(4\gamma + 3)\}$ and if $p = 1$ then we again take $r = s = 2$ so that (2.13b) and 2.13d) hold by Poincaré. Otherwise we now want $sp = 2$ which says that $s = (4\gamma + 3)/3$ so that $r = (4\gamma + 3)/(4\gamma)$ and thus $rp = 3/(2\gamma)$. We want for some M_{10}

that $\|w\|_{rp} \leq M_{10} \left\|A^{\beta/2}w\right\|_2$ or $3/\left(2\gamma\right) \leq 6/\left(3-2\beta\right)$ or $3-2\beta \leq 4\gamma$ which again leads to the condition $\alpha \geq 3/2$. Thus we obtain (2.13b) with $M_0' = M_9 M_{10}$.

For (2.13d) with $1 \leq \beta \leq 3/2$ we modify (2.16c) by looking for ω such that

$$\begin{aligned}\left\|A^{-\omega}\left(w\cdot\nabla\right)v\right\|_2 &\leq M_{11}\left\|w\cdot\nabla v\right\|_p \\ &\leq M_{11}\left\|w\right\|_{rp}\left\|\nabla v\right\|_{sp} \qquad (2.16d)\end{aligned}$$

where as in (2.16b) $p = 6/\left(4\omega+3\right)$. We now want $sp = 2$ which says that $s = \left(4\omega+3\right)/3$ so that $r = \left(4\omega+3\right)/\left(4\omega\right)$ and thus $rp = 3/\left(2\omega\right)$. We want for some M_{12} that $\|w\|_{rp} \leq M_{12}\left\|A^{\beta/2}w\right\|_2$ or $3/\left(2\omega\right) \leq 6/\left(3-2\beta\right)$ or $3-2\beta \leq 4\omega$; this again for $1 \leq \beta \leq 3/2$ leads to the condition $3/4-\beta/2 \leq \omega$ and thus we have (2.13d) by Poincaré with $K_0' = M_{11}M_{12}$.

For (2.13c) and (2.13d) with $0 \leq \beta \leq 1/2$ we use techniques similar to those used to obtain (2.13a); for $1/2 \leq \beta \leq 1$ we use techniques similar to those used to obtain the proof of Lemma 14 below. This finishes the proof of Lemma 13.

Next we consider the case $\beta \geq \alpha$, for which we will prove the following:

Lemma 14 *Let $\beta \geq \alpha \geq 2$ and let $\gamma = \left(\beta-\alpha\right)/2$. Then for $v \in H^{2\gamma+1}\left(\Omega\right)$ and $w \in H^{\beta/2}\left(\Omega\right)$ we have for a constant M_0'' that*

$$\left\|A^{\gamma}\left(w\cdot\nabla\right)v\right\|_2 \leq M_0''\left\|A^{\gamma+1/2}v\right\|_2\left\|A^{\beta/2}w\right\|_2^2 \qquad (2.17a)$$

and that

$$\left\|A^{\gamma}\left(v\cdot\nabla\right)w\right\|_2 \leq M_0''\left\|A^{\gamma+1/2}v\right\|_2\left\|A^{\beta/2}w\right\|_2^2 \qquad (2.17b)$$

To prove this, we have for the appropriate tensor product $\otimes$ that $v\cdot\nabla w = div\left(v\otimes w\right)$ so that for integer values of $\gamma+1/2$ we have that

$$\begin{aligned}\left\|A^{\gamma}\left(v\cdot\nabla\right)w\right\|_2 &= \left\|A^{\gamma+1/2}\left(A^{-1/2}div\right)\left(v\otimes w\right)\right\|_2 \\ &= \left\|\left(A^{-1/2}div\right)A^{\gamma+1/2}\left(v\otimes w\right)\right\|_2 \\ &\leq \left\|A^{\gamma+1/2}\left(v\otimes w\right)\right\|_2. \qquad (2.17c)\end{aligned}$$

The leading terms of $A^{\gamma+1/2}\left(v\otimes w\right)$ are by Leibniz $\left(A^{\gamma+1/2}\right)\otimes w$ and $v\otimes A^{\gamma+1/2}$. We have that $\left\|v \otimes A^{\gamma+1/2} w\right\|_2 \leq \|v\|_{2r} \left\|A^{\gamma+1/2} w\right\|_{2s}$. Note that $\beta/2 - \gamma + 1/2 = (\alpha - 1)/2$ so that $\left\|A^{\gamma+1/2} w\right\|_{2s} \leq M_{13} \left\|A^{(\alpha-1)/2}\left(A^{\gamma+1/2} w\right)\right\|_2 = M_{13} \left\|A^{\beta/2} w\right\|_2^2$ for $2s = 6/\left[3 - 2(\alpha - 1)\right]$ or $s = 3/\left[3 - 2(\alpha-1)\right]$, therefore $r = s/(s-1) = 3/\left[2(\alpha - 1)\right]$ so that $2r = 3/(\alpha - 1)$. We have that $\|v\|_{3/(\alpha-1)} \leq M_{14} \|\nabla v\|_2$ if $3/(\alpha - 1) \leq 6/(3-2) = 6$ which holds provided that $1/(\alpha - 1) \leq 2$ or $\alpha \geq 3/2$. Thus for some M_{15} we have that $\|v\|_{2r} \leq M_{15} \left\|A^{\gamma+1/2} v\right\|_2$ by Poincaré. Since $\alpha \geq 2$ we thus have that

$$\left\|v \otimes A^{\gamma+1/2} w\right\|_2 \leq M_{13} M_{15} \|\nabla v\|_2 \left\|A^{\beta/2} w\right\|_2.$$

We also have that $\left\|\left(A^{\gamma+1/2} v\right) \otimes w\right\|_2 \leq \left\|\left(A^{\gamma+1/2} v\right)\right\|_{2a} \|w\|_{2b}$; we need $\|w\|_{2b} \leq M_{16} \left\|A^{\beta/2} w\right\|_2$, but since $\beta \geq \alpha \geq 2$ we have that $\beta/2 \geq 1 \geq 3/4$ so we can take $b = 2b = \infty$. Then $a = 1$ so that $2a = 2$ and so $\left\|\left(A^{\gamma+1/2} v\right) \otimes w\right\|_2 \leq M_{17} \left\|A^{\gamma+1/2} v\right\|_2 \left\|A^{\beta/2} w\right\|_2^2$. For the j middle terms in the Leibniz expansion we use Hölder to get the same bounds as above for constants M_{18}^j. By interpolation for noninteger $\gamma + 1/2$ we get (2.17a), and by switching the roles of v and w we get (2.17b). This completes the proof of Lemma 14.

Next we prove Theorem 2. From (2.7) we obtain that

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 e^{-\nu\lambda_1 t} + \frac{1}{\nu\lambda_1} \int_0^t \|g(s)\|_2^2 e^{-\nu\lambda_1 (t-s)} ds. \tag{2.18}$$

Let $f_t(s) = \|g(s)\|_2^2 e^{-\nu\lambda_1(t-s)}$ then note that $|f_t| \leq \|g(s)\|_2^2 \in L^1(0,\infty)$ and that $f_t(s) \to 0$ as $t \to \infty$ for each s; given $\epsilon > 0$ choose N large enough such that $\int_N^\infty \|g(s)\|_2^2 ds < \epsilon/2$ then

$$\begin{aligned} 0 &\leq \int_0^t \|g(s)\|_2^2 e^{-\nu\lambda_1(t-s)} ds \\ &\leq \int_0^N \|g(s)\|_2^2 e^{-\nu\lambda_1(t-s)} ds + \int_N^\infty \|g(s)\|_2^2 s < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned} \tag{2.19}$$

for large enough t by the Dominated Convergence Theorem. From (2.19) and (2.18) we see that $\|u(t)\|_2^2 \to 0$ as $t \to \infty$. By a standard interpolation inequality we have that

$$\|u(t)\|_{\beta,2} \leq \|u(t)\|_{\theta\beta,2}^{1/\theta} \|u(t)\|_2^{1-1/\theta} \tag{2.20}$$

where $\|u\|_{\gamma,2}$ denotes the norm in the Sobolev space $W^{\gamma,2}(\Omega)$. Since $\left\|A^{\beta/2} u\right\|_2$ is a norm equivalent to $\|u\|_{\beta,2}$, and since each of the norms $\|u(t)\|_{\theta\gamma,2}$ was shown to be uniformly bounded for all $\theta\gamma \geq 0$ in [4, section 2], we thus have Theorem 2 from (2.18), (2.19), and (2.20).

The following result will be integral in proving Theorem 4.

Lemma 15 *Let $h \in L^{\infty}(0,\infty)$, then for each $\lambda > 0$ and each $M > t$ we have that*

$$\int_0^t e^{-\lambda(t-s)} h(s)\, ds \leq \left[\frac{1-e^{-\lambda t(1-1/M)}}{e^{\frac{t}{N}\lambda}-1}+1\right] \sup_{0\leq k\leq N} \int_{[(k-1)t]/M}^{(kt)/M} h(s)\, ds. \tag{2.21}$$

To prove this, partition the interval $[0,t]$ into N equal parts of length t/N, so that

$$\begin{aligned}
\int_0^t e^{-\lambda(t-s)} h(s)\, ds &= e^{-\lambda t}\int_0^t e^{\lambda s} h(s)\, ds \\
&= e^{-\lambda t}\sum_{k=1}^{M}\int_{[(k-1)t]/M}^{(kt)/M} e^{\lambda s} h(s)\, ds \\
&\leq e^{-\lambda t}\sum_{k=1}^{M} e^{(\lambda k t)/M}\int_{[(k-1)t]/M}^{(kt)/M} h(s)\, ds \\
&= e^{-\lambda t}\sum_{k=1}^{M}\left(e^{(\lambda t)/M}\right)^k \sup_{0\leq k\leq N}\int_{[(k-1)t]/M}^{(kt)/M} h(s)\, ds \\
&= \left[e^{-\lambda t}\left(\sum_{k=1}^{M-1}\left(e^{(\lambda t)/M}\right)^k\right)+1\right] \sup_{0\leq k\leq N}\int_{[(k-1)t]/M}^{(kt)/M} h(s)\, ds \\
&= \left[e^{-\lambda t}\left(\frac{e^{(\lambda t)/M}-e^{\lambda t}}{1-e^{\lambda t/M}}\right)+1\right] \sup_{0\leq k\leq N}\int_{[(k-1)t]/M}^{(kt)/M} h(s)\, ds \\
&= \left[\frac{1-e^{-\lambda(1-(1/M))t}}{e^{(\lambda t)/M}-1}+1\right] \sup_{0\leq k\leq N}\int_{[(k-1)t]/M}^{(kt)/M} h(s)\, ds
\end{aligned} \tag{2.22}$$

where in the fourth line of (2.22) we applied the geometric summation formula. This proves Lemma 10; it is similar in flavor to lemmas that appeared in [29], [40] (see also [28, Lemma 1.1, Chapter III]), but needs to serve non-asymptotic purposes.

The following result, of some interest in its own right, will be key to proving Theorems 6 and 7.

Lemma 16 *For any $\lambda > 0$ we have that the solution u of (1.1) satisfies*

$$\nu \int_0^t e^{-\lambda(t-s)} \|\nabla u\|_2^2 \, ds + 2\mu \int_0^t e^{-\lambda(t-s)} \left\|Q_m A^{\alpha/2} u\right\|_2^2 ds$$
$$\leq \|u_0\|_2^2 e^{-\lambda t} + U_g^2 + \frac{L_g^2}{\nu\lambda_1\lambda}. \tag{2.23}$$

To prove this, we multiply both sides of (2.6) by $e^{\lambda s}$ and add $\lambda \|u\|_2^2 e^{\lambda s}$ to both sides to obtain

$$\frac{d}{ds}\left(\|u\|_2^2 e^{\lambda s}\right) + \nu \|\nabla v\|_2^2 e^{\lambda s} + 2\mu \left\|Q_m A^{\alpha/2} u\right\|_2^2 e^{\lambda s}$$
$$\leq \lambda \|u\|_2^2 e^{\lambda s} + \frac{1}{\nu\lambda_1} \|g\|_2^2 e^{\lambda s}. \tag{2.24}$$

Now integrate both sides of (2.24) from 0 to t to obtain

$$\|u\|_2^2 e^{\lambda t} + \nu \int_0^t e^{\lambda s} \|\nabla u\|_2^2 + 2\mu \int_0^t e^{\lambda s} \left\|Q_m A^{\alpha/2} u\right\|_2^2 ds$$
$$\leq \|u_0\|_2^2 + \lambda \int_0^t \|u\|_2^2 e^{\lambda s} ds + \frac{1}{\nu\lambda_1} \int_0^t e^{\lambda s} \|g\|_2^2 \, ds. \tag{2.25}$$

Multiplying both sides of (2.25) by $e^{-\lambda t}$ we obtain

$$\|u\|_2^2 + \nu \int_0^t e^{-\lambda(t-s)} \|\nabla u\|_2^2 \, ds + 2\mu \int_0^t e^{-\lambda(t-s)} \left\|Q_m A^{\alpha/2} u\right\|_2^2 ds$$
$$\leq \|u_0\|_2^2 e^{-\lambda t} + \lambda \int_0^t \|u\|_2^2 e^{-\lambda(t-s)} ds + \frac{1}{\nu\lambda_1} \int_0^t e^{-\lambda(t-s)} \|g\|_2^2 \, ds. \tag{2.26}$$

Using (2.8) and (2.12), together with

$$\int_0^t e^{-\lambda(t-s)} ds = \int_0^t e^{-\lambda s} ds \leq \int_0^\infty e^{-\lambda s} ds \leq \frac{1}{\lambda} \tag{2.27}$$

we obtain Lemma 16.

The below is the generalized Gronwall lemma,

Lemma 17 *Let $\alpha = \alpha(t)$ and $\beta = \beta(t)$ be locally integrable real-valued functions on $[0, +\infty)$ that satisfy the following conditions for some $T > 0$:*

$\liminf_{t\to\infty} \frac{1}{T} \int_t^{t+T} \alpha(\tau) d\tau > 0$

$\limsup_{t\to\infty} \frac{1}{T} \int_t^{t+T} \alpha^-(\tau) d\tau < \infty$

$\lim_{t\to\infty} \frac{1}{T} \int_t^{t+T} \beta^+(\tau) d\tau = 0$

where $\alpha^-(t) = max(-\alpha, 0), \beta^+(t) = max(\beta, 0)$. *Suppose that* $\xi = \xi(t)$ *is an absolutely continuous nonnegative function on* $[0, \infty)$ *that satisfies the following inequality almost everywhere on* $[0, \infty)$:

$\frac{d\xi}{dt} + \alpha\xi \leq \beta.$ *Then* $\xi(t) \to 0$ *as* $t \to \infty$"

The proof of Lemma 17 is in [28, p.156].

In [28, p.156], there is a "Lemma 2.1" in support of the result of Determining Nodes for two dimensional case. Following the idea in the proof of two dimensional case, a three dimensional version is given below:

Lemma 18 *Let the domain* Ω *be covered by* N *identical cubic boxes. Consider the set* $\varepsilon = \left\{x^1, x^2, ..., x^N\right\}$ *of points in* Ω*, distributed one in each box. Assume that* $A = -\triangle$ *and that* $\eta^2(w) = max_j(\mid w(x^j) \mid^2)$*, then for each vector* w *in* $D\left\{A\right\}$*, the following inequality holds:*

$$\parallel w \parallel_2^2 \leq a_0 \frac{1}{\lambda_1^2 N^{\frac{4}{3}}} \parallel Aw \parallel_2^2 + a_1 \frac{1}{\lambda_1^3 N^2} \parallel A^{\frac{3}{2}} w \parallel_2^2 + 8a_2 \frac{1}{\lambda_1^{\frac{3}{2}}} \eta^2(w) \tag{2.28}$$

where the constant c, c_3 *depends only on the shape of the domain* Ω.

Proof. Consider a small cubic box $Q = (0, l) \times (0, l) \times (0, l)$, with $l > 0$. Fix $X^0 = (x_0, y_0, z_0)$ in Q, and let $u = u(X)$ be a smooth function defined on Q. For any $X = (x, y, z)$, in Q, $w(x, y, z) - w(x_0, y_0, z_0) = \int_{x_0}^{x} w_x(\xi, y, z)d\xi + \int_{y_0}^{y} w_y(x_0, \eta, z)d\eta + \int_{z_0}^{z} w_z(x_0, y_0, \theta)d\theta$. Hence, after using the inequality
$(a + b + c)^2 \leq 2(a^2 + b^2 + c^2)$, and Cauchy Schwarz inequality
$(\int_0^l wdx)^2 \leq (\int_0^l 1^2 dx) \int_0^l w^2 dx = l \int_0^l w^2 dx$, we have

$$\begin{aligned}
&\mid w(x, y, z) - w(x_0, y_0, z_0) \mid^2 \\
&\leq 2(\int_{x_0}^{x} w_x(\xi, y, z)d\xi)^2 + 2(\int_{y_0}^{y} w_y(x_0, \eta, z)d\eta)^2 + 2(\int_{z_0}^{z} w_z(x_0, y_0, \theta)d\theta)^2 \\
&\leq 2l \int_{x_0}^{x} w_x^2(\xi, y, z)d\xi + 2l \int_{y_0}^{y} w_y^2(x_0, \eta, z)d\eta + 2l \int_{z_0}^{z} w_z^2(x_0, y_0, \theta)d\theta \\
&\leq 2l \int_0^l w_x^2(\xi, y, z)d\xi + 2l \int_0^l w_y^2(x_0, \eta, z)d\eta + 2l \int_0^l w_z^2(x_0, y_0, \theta)d\theta
\end{aligned}$$

after integration over Q on the dummy variables (x, y, z), we find

$$\int_0^l \int_0^l \int_0^l \mid w(x,y,z) - w(x_0,y_0,z_0) \mid^2 dxdydz$$
$$\leq 2l^2 \int_0^l \int_0^l \int_0^l w_x^2(\xi,y,z)d\xi dydz + 2l^3 \int_0^l \int_0^l w_y^2(x_0,\eta,z)d\eta dz$$
$$+ 2l^4 \int_{z_0}^{z} w_z^2(x_0,y_0,\theta)d\theta \tag{2.29}$$

We need to estimate the second term and the third term on the right hand side of (2.29). It will be an estimate of the type of trace theorem(traces in the sense of Sobolev space). To simplify the deduction, we need to insert the relation between integral and norms as below,

$$\int_0^l \int_0^l \int_0^l w_z^2(x,y,\theta)dxdyd\theta = \parallel w_z \parallel^2_{L^2(Q)}$$

$$2l \int_0^l \int_0^l \int_0^l \mid w_z(\xi,y,\theta)w_{zx}(\xi,y,\theta) \mid d\xi dyd\theta \leq \int_0^l \int_0^l \int_0^l w_z^2(\xi,y,\theta) + l^2 w_{zx}^2(\xi,y,\theta)d\xi dyd\theta$$
$$= \parallel w_z \parallel^2_{L^2(Q)} + l^2 \parallel w_{zx} \parallel^2_{L^2(Q)}$$

$$2l^2\int_{z_0}^{z}\int_0^l\{[| w_z(x,\eta,\theta) | +\int_0^l | w_{zx}(\xi,\eta,\theta) | d\xi] * [| w_{zy}(x,\eta,\theta) | +\int_0^l | w_{zyx}(\xi,\eta,\theta) | d\xi]\}d\eta d\theta$$

$$= 2l^2\int_{z_0}^{z}\int_0^l\{| w_z(x,\eta,\theta) || w_{zy}(x,\eta,\theta) | + | w_z(x,\eta,\theta) | \int_0^l | w_{zyx}(\xi,\eta,\theta) | d\xi$$

$$+ | w_{zy}(x,\eta,\theta) | \int_0^l | w_{zx}(\xi,\eta,\theta) | d\xi + \int_0^l | w_{zx}(\xi,\eta,\theta) | d\xi \int_0^l | w_{zyx}(\xi,\eta,\theta) | d\xi\}d\eta d\theta$$

$$\leq [\int_0^l\int_0^l\int_0^l | w_z(x,\eta,\theta) |^2 + l^2 | w_{zy}(x,\eta,\theta) |^2 \, d\eta d\theta dx] + \int_0^l\int_0^l l \, | w_z(x,\eta,\theta) |^2$$

$$+ l^3(\int_0^l | w_{zyx}(\xi,\eta,\theta) | d\xi)^2 d\eta d\theta \int_0^l\int_0^l l^3 | w_{zy}(x,\eta,\theta) |^2 + l(\int_0^l | w_{zx}(\xi,\eta,\theta) | d\xi)^2 d\eta d\theta$$

$$+ \int_0^l\int_0^l l(\int_0^l | w_{zx}(\xi,\eta,\theta) | d\xi)^2 + l^3(\int_0^l | w_{zyx}(\xi,\eta,\theta) | d\xi)^2 d\eta d\theta$$

$$\leq \int_0^l\int_0^l\int_0^l | w_z(x,\eta,\theta) |^2 \, d\eta d\theta dx + l^2\int_0^l\int_0^l\int_0^l | w_{zy}(x,\eta,\theta) |^2 \, d\eta d\theta dx$$

$$+ \int_0^l\int_0^l\int_0^l | w_z(x,\eta,\theta) |^2 \, dx d\eta d\theta + l^4\int_0^l\int_0^l\int_0^l w_{zyx}^2(\xi,\eta,\theta) d\xi d\eta d\theta \int_0^l\int_0^l\int_0^l l^2 w_{zy}^2(x,\eta,\theta) d\eta d\theta$$

$$+ l^2\int_0^l\int_0^l\int_0^l w_{zx}^2(\xi,\eta,\theta) d\xi d\eta d\theta + \int_0^l\int_0^l\int_0^l l^2 w_{zx}^2(\xi,\eta,\theta) d\xi + l^4\int_0^l\int_0^l\int_0^l w_{zyx}^2(\xi,\eta,\theta) d\xi d\eta d\theta$$

$$\leq \| w_z \|_{L^2(Q)}^2 + l^2 \| w_{zy} \|_{L^2(Q)}^2 + \| w_z \|_{L^2(Q)}^2 + l^4 \| w_{zxy} \|_{L^2(Q)}^2 + l^2 \| w_{zy} \|_{L^2(Q)}^2 + l^2 \| w_{zx} \|_{L^2(Q)}^2 + l^2 \| w$$

$$\leq 2 \| w_z \|_{L^2(Q)}^2 + 2l^2(\| w_{zy} \|_{L^2(Q)}^2 + \| w_{zx} \|_{L^2(Q)}^2) + 2l^4 \| w_{zxy} \|_{L^2(Q)}^2$$

For the second term on the right hand side of (2.29),

$$\begin{aligned} w_y^2(x_0,\eta,z) &= w_y^2(x,\eta,z) + 2\int_x^{x_0} w_y(\xi,\eta,z) w_{yx}(\xi,\eta,z) d\xi \\ &\leq w_y^2(x,\eta,z) + 2\int_0^l | w_y(\xi,\eta,z) || w_{yx}(\xi,\eta,z) | d\xi \end{aligned}$$

therefore

$$l\int_0^l\int_0^l w_y^2(x_0,\eta,z)d\eta dz = \int_0^l\int_0^l\int_0^l w_y^2(x_0,\eta,z)d\eta dz dx$$
$$\leq \int_0^l\int_0^l\int_0^l w_y^2(x,\eta,z)+[2\int_0^l \mid w_y(\xi,\eta,z)\mid\mid w_{yx}(\xi,\eta,z)\mid d\xi]d\eta dz dx$$
$$= \int_0^l\int_0^l\int_0^l w_y^2(x,\eta,z)d\eta dz dx + 2l\int_0^l\int_0^l\int_0^l \mid w_y(\xi,\eta,z)\mid\mid w_{yx}(\xi,\eta,z)\mid d\xi d\eta d$$
$$\leq \parallel w_y \parallel_{L^2(Q)}^2 + \int_0^l\int_0^l\int_0^l (\mid w_y(\xi,\eta,z)\mid^2 + l^2 \mid w_{yx}(\xi,\eta,z)\mid^2)d\xi d\eta dz$$
$$\leq \parallel w_y \parallel_{L^2(Q)}^2 + \parallel w_y \parallel_{L^2(Q)}^2 + l^2 \parallel w_{yx} \parallel_{L^2(Q)}^2 = 2\parallel w_y \parallel_{L^2(Q)}^2 + l^2 \parallel w_{yx} \parallel_{L^2(Q)}^2 \tag{2.30}$$

For the third term on the right hand side,

$$w_z^2(x_0,y_0,\theta) = w_z^2(x,y,\theta) + 2\int_x^{x_0} w_z(\xi,y,\theta)w_{zx}(\xi,y,\theta)d\xi + 2\int_y^{y_0} w_z(x_0,\eta,\theta)w_{zy}(x_0,\eta,\theta)d\eta$$
$$= w_z^2(x,y,\theta) + 2\int_x^{x_0} w_z(\xi,y,\theta)w_{zx}(\xi,y,\theta)d\xi$$
$$+ 2\int_y^{y_0}[w_z(x,\eta,\theta) + \int_x^{x_0} w_{zx}(\xi,\eta,\theta)d\xi]w_{zy}(x_0,\eta,\theta)d\eta$$
$$= w_z^2(x,y,\theta) + 2\int_x^{x_0} w_z(\xi,y,\theta)w_{zx}(\xi,y,\theta)d\xi$$
$$+ 2\int_y^{y_0}\{[w_z(x,\eta,\theta) + \int_x^{x_0} w_{zx}(\xi,\eta,\theta)d\xi] * [w_{zy}(x,\eta,\theta) + \int_x^{x_0} w_{zyx}(\xi,\eta,\theta)d\xi]\}d\eta$$
$$\leq w_z^2(x,y,\theta) + 2\int_0^l w_z(\xi,y,\theta)w_{zx}(\xi,y,\theta)d\xi + 2\int_0^l\{[\mid w_z(x,\eta,\theta)\mid$$
$$+ \int_0^l \mid w_{zx}(\xi,\eta,\theta)\mid d\xi] * [\mid w_{zy}(x,\eta,\theta)\mid + \int_0^l \mid w_{zyx}(\xi,\eta,\theta)\mid d\xi]\}d\eta$$

therefore

$$l^2 \int_{z_0}^{z} w_z^2(x_0, y_0, \theta) d\theta$$
$$= \int_0^l \int_0^l \int_{z_0}^{z} w_z^2(x_0, y_0, \theta) d\theta dx dy$$
$$\leq \int_0^l \int_0^l \int_0^l w_z^2(x, y, \theta) dx dy d\theta + 2l \int_0^l \int_0^l \int_0^l \mid w_z(\xi, y, \theta) w_{zx}(\xi, y, \theta) \mid d\xi dy d\theta$$
$$+ 2l^2 \int_{z_0}^{z} \int_0^l \{[\mid w_z(x, \eta, \theta) \mid + \int_0^l \mid w_{zx}(\xi, \eta, \theta) \mid d\xi] * [\mid w_{zy}(x, \eta, \theta) \mid + \int_0^l \mid w_{zyx}(\xi, \eta, \theta) \mid d\xi]\} d\eta d\theta$$
$$\leq \| w_z \|_{L^2(Q)}^2 + [\| w_z \|_{L^2(Q)}^2 + l^2 \| w_{zx} \|_{L^2(Q)}^2] + [2 \| w_z \|_{L^2(Q)}^2 + 2l^2(\| w_{zy} \|_{L^2(Q)}^2 + \| w_{zx} \|_{L^2(Q)}^2) + 2l^4 \|$$
$$\leq 4 \| w_z \|_{L^2(Q)}^2 + l^2(\| w_{zx} \|_{L^2(Q)}^2 + 2 \| w_{zy} \|_{L^2(Q)}^2 + 2 \| w_{zx} \|_{L^2(Q)}^2) + 2l^4 \| w_{zxy} \|_{L^2(Q)}^2$$
$$(2.31)$$

Inserting (2.30) and (2.31) into (2.29),

$$\int_0^l \int_0^l \int_0^l \mid w(x, y, z) - w(x_0, y_0, z_0) \mid^2 dx dy dz$$
$$\leq 2l^2 \int_0^l \int_0^l \int_0^l w_x^2(\xi, y, z) d\xi dy dz + 2l^3 \int_0^l \int_0^l w_y^2(x_0, \eta, z) d\eta dz + 2l^4 \int_{z_0}^{z} w_z^2(x_0, y_0, \theta) d\theta$$
$$\leq 2l^2 \| w_x \|_{L^2(Q)}^2 + 2l^2(2 \| w_y \|_{L^2(Q)}^2 + l^2 \| w_{yx} \|_{L^2(Q)}^2) + 2l^2[4 \| w_z \|_{L^2(Q)}^2 + l^2(\| w_{zx} \|_{L^2(Q)}^2 + 2 \| w_{zy} \|_{L^2}^2$$
$$+ 2l^4 \| w_{zxy} \|_{L^2(Q)}^2]$$
$$= 2l^2(\| w_x \|_{L^2(Q)}^2 + 2 \| w_y \|_{L^2(Q)}^2 + 4 \| w_z \|_{L^2(Q)}^2) + 2l^4(\| w_{yx} \|_{L^2(Q)}^2 + 2 \| w_{zy} \|_{L^2(Q)}^2 + 3 \| w_{zx} \|_{L^2(Q)}^2)$$
$$+ 4l^6 \| w_{zxy} \|_{L^2(Q)}^2$$

therefore

$$\int_0^l \int_0^l \int_0^l w^2(x, y, z) dx dy dz$$
$$\leq 2 \int_0^l \int_0^l \int_0^l \mid w(x, y, z) - w(x_0, y_0, z_0) \mid^2 dx dy dz + 2 \int_0^l \int_0^l \int_0^l w^2(x_0, y_0, z_0) dx dy dz$$
$$\leq 4l^2(\| w_x \|_{L^2(Q)}^2 + 2 \| w_y \|_{L^2(Q)}^2 + 4 \| w_z \|_{L^2(Q)}^2) + 4l^4(\| w_{yx} \|_{L^2(Q)}^2 + 2 \| w_{zy} \|_{L^2(Q)}^2 + 3 \| w_{zx} \|_{L^2(Q)}^2)$$
$$+ 8l^6 \| w_{zxy} \|_{L^2(Q)}^2 + 2l^3 w^2(x_0, y_0, z_0)$$

which means the estimate for $\| w \|_{L^2(Q)}^2$ in every small cubic box $Q = (0, l) \times (0, l) \times (0, l)$

is

$$\| w \|_{L^2(Q)}^2 \leq 4l^2(\| w_x \|_{L^2(Q)}^2 + 2 \| w_y \|_{L^2(Q)}^2 + 4 \| w_z \|_{L^2(Q)}^2) + 4l^4(\| w_{yx} \|_{L^2(Q)}^2 + 2 \| w_{zy} \|_{L^2(Q)}^2 + 3 \| w_{zx} \|_{L^2(Q)}^2) + 8l^6 \| w_{zxy} \|_{L^2(Q)}^2 + 2l^3 w^2(x_0, y_0, z_0) \tag{2.32}$$

The estimate over the whole domain $\Omega = (0, L) \times (0, L) \times (0, L)$ can be obtained by summing up the inequality (2.32) for all N small cubic boxes. To simplify the expression, the norm $\| \|_2$ represents $\| \|_{L^2(\Omega)}$ below.

$$\begin{aligned} \| w \|_{L^2(\Omega)}^2 \leq 4l^2(\| w_x \|_{L^2(\Omega)}^2 + 2 \| w_y \|_{L^2(\Omega)}^2 + 4 \| w_z \|_{L^2(\Omega)}^2) + 4l^4(\| w_{yx} \|_{L^2(\Omega)}^2 + 2 \| w_{zy} \|_{L^2(\Omega)}^2 \\ + 3 \| w_{zx} \|_{L^2(\Omega)}^2) + 8l^6 \| w_{zxy} \|_{L^2(\Omega)}^2 + \sum_{i=0}^{N} 2l^3 w^2(x_{i0}, y_{i0}, z_{i0}) \\ \leq 16l^2 \| \nabla w \|_2^2 + c_1 l^4 \| Aw \|_2^2 + c_2 l^6 \| A^{3/2} w \|_2^2 + 2Nl^3 \eta^2(w) \end{aligned} \tag{2.33}$$

Using interpolation$\| \nabla w \|_2^2 \leq c_0 \| w \|_2 \| Aw \|_2$, therefore

$$16l^2 \| \nabla w \|_2^2 \leq 16l^2 c_0 \| w \|_2 \| Aw \|_2 \leq \frac{1}{2} \| w \|_2^2 + 128 c_0^2 l^4 \| Aw \|_2^2$$

Inserting into (2.33),

$$\| w \|_2^2 \leq \frac{1}{2} \| w \|_2^2 + (256 c_0^2 + 2c_1) l^4 \| Aw \|_2^2 + 2c_2 l^6 \| A^{3/2} w \|_2^2 + 4Nl^3 \eta^2(w)$$

therefore

$$\| w \|_2^2 \leq c l^4 \| Aw \|_2^2 + c_3 l^6 \| A^{3/2} w \|_2^2 + 8Nl^3 \eta^2(w) \tag{2.34}$$

The volume of Ω is $L^3 = Nl^3$. The Grashof number is defined in terms of λ_1, the first eigenvalue of the stokes operator, so we would like to relate l and N to λ_1. By the relation $\lambda_1 \sim (\frac{\pi}{L})^2$ and $l^3 = \frac{L^3}{N}$, we have $l = \frac{C_l}{N^{\frac{1}{3}} \lambda_1^{\frac{1}{2}}}$, where C_l is a constant that depends only on the shape of Ω (i.e., the constant does not change if we rescale the domain). Then (2.34) becomes

$$\begin{aligned}\| w \|_2^2 &\le cl^4 \| Aw \|_2^2 + c_3 l^6 \| A^{\frac{3}{2}} w \|_2^2 + 8Nl^3\eta^2(w)\\ &= C_l^4 \frac{c}{\lambda_1^2 N^{\frac{4}{3}}} \| Aw \|_2^2 + c_3 C_l^6 \frac{1}{\lambda_1^3 N^2} \| A^{\frac{3}{2}} w \|_2^2 + 8C_l^3 \frac{1}{\lambda_1^{\frac{3}{2}}}\eta^2(w)\\ &= a_0 \frac{1}{\lambda_1^2 N^{\frac{4}{3}}} \| Aw \|_2^2 + a_1 \frac{1}{\lambda_1^3 N^2} \| A^{\frac{3}{2}} w \|_2^2 + 8a_2 \frac{1}{\lambda_1^{\frac{3}{2}}}\eta^2(w) \qquad (2.35)\end{aligned}$$

with $a_0 = C_l^4 c, a_1 = c_3 C_l^6, a_2 = C_l^3$. which is the same as (2.1) now, so we finished the proof of Lemma 17. ■

To prove Theorem 8,9,10, we need to use bounds of $\| u \|_2^2$ and $\limsup_{t\to\infty} \int_t^{t+T} \| A^{\frac{1}{2}} u \|_2^2 ds$ as in Lemma 19.

Lemma 19 *For SHNSE model as* (1.1)*, suppose* F*,* λ_1 *and* L_g *are the same as in Theorem 8, then* $\limsup_{t\to\infty} \int_t^{t+T} \| A^{\frac{1}{2}} u \|_2^2 \, ds \le \frac{1}{2\nu^2} F^2 T + \frac{F^2}{\lambda_1^2 \nu^2}$*, and* $\| u \|_2^2 \le \| u_0 \|_2^2 + \frac{L_g^2}{\lambda_1^2 \nu^2} = U_g^2$

Proof. by $((1.1a), u)$

$\frac{1}{2}\frac{d}{dt} \| u \|_2^2 + \nu \| A^{\frac{1}{2}} u \|_2^2 + \mu (A_\varphi u, u) + ((u\cdot\nabla) u, u) = (g, u),$

by $(1.1b)$

$((u\cdot\nabla) u, u) = (-(\nabla\cdot u)u, u) = 0,$

for the general hyperviscous term $(A_\varphi u, u) \ge (Q_m A^\alpha, u) = \| Q_m A^{\frac{\alpha}{2}} u \|_2^2 \ge 0$

therefore $\frac{1}{2}\frac{d}{dt} \| u \|_2^2 + \nu \| A^{\frac{1}{2}} u \|_2^2 \le \frac{1}{2b} \| A^{-1/2} g \|_2^2 + \frac{b}{2} \| A^{\frac{1}{2}} u \|_2^2,$

set $b = \nu$, then

$\frac{d}{dt} \| u \|_2^2 + \nu \| A^{\frac{1}{2}} u \|_2^2 \le \frac{1}{\nu} \| A^{-1/2} g \|_2^2$

Inserting $\| A^{\frac{1}{2}} u \|_2^2 \ge \lambda_1 \| A^{\frac{1}{2}} u \|_2^2$ and $\| A^{-\frac{1}{2}} g \|_2^2 \le \lambda_1^{-1} \| g \|_2^2,$

$\frac{d}{dt} \| u \|_2^2 + \nu\lambda_1 \| u \|_2^2 \le \frac{1}{\nu\lambda_1} \| g \|_2^2$, therefore

$$\begin{aligned}\| u \|_2^2 &\le e^{-\nu\lambda_1 t} \| u_0 \|_2^2 + \int_0^t \frac{1}{\nu\lambda_1} e^{-\nu\lambda_1 (t-s)} \| g \|_2^2 \, ds\\ &\le \| u_0 \|_2^2 + \frac{L_g^2}{\lambda_1^2 \nu^2} = U_g^2 \qquad (2.36a)\end{aligned}$$

for $\frac{d}{dt} \| u \|_2^2 + \nu \| A^{\frac{1}{2}} u \|_2^2 \le \frac{1}{\nu} \| A^{-1/2} g \|_2^2 \le \frac{1}{\nu\lambda_1} \| g \|_2^2$, integral with the time from t to

$t+T$,

$\| u(t+T) \|_2^2 - \| u(t) \|_2^2 + \nu \int_t^{t+T} \| A^{\frac{1}{2}} u(s) \|_2^2 \, ds \leq \frac{1}{\nu\lambda_1} \int_t^{t+T} \| g(s) \|_2^2 \, ds$

then $\int_t^{t+T} \| A^{\frac{1}{2}} u(s) \|_2^2 \, ds \leq \frac{1}{\nu^2\lambda_1} \int_t^{t+T} \| g(s) \|_2^2 \, ds + \frac{1}{\nu} \| u(t) \|_2^2$.

Inserting $\| u \|_2^2 \leq e^{-\nu\lambda_1 t} \| u_0 \|_2^2 + \int_0^t \frac{1}{\nu\lambda_1} e^{-\nu\lambda_1(t-s)} \| g \|_2^2 \, ds$,

$\int_t^{t+T} \| A^{\frac{1}{2}} u \|_2^2 \, ds \leq \frac{1}{\nu^2\lambda_1} \int_t^{t+T} \| g(s) \|_2^2 \, ds + \frac{1}{\nu} (e^{-\nu\lambda_1 t} \| u_0 \|_2^2 + \int_0^t \frac{1}{\nu\lambda_1} e^{-\nu\lambda_1(t-s)} \| g \|_2^2 \, ds)$

Because $\frac{1}{\nu} e^{-\nu\lambda_1 t} \| u_0 \|_2^2 \to 0$ as $t \to +\infty$, and

$$\begin{aligned}
\int_0^t \frac{1}{\lambda_1\nu^2} e^{-\nu\lambda_1(t-s)} \| g \|_2^2 \, ds &= \int_0^t \frac{1}{\lambda_1\nu^2} e^{-\nu\lambda_1\tau} \| g \|_2^2 \, d\tau \\
&\leq (limsup \| g \|_2^2) \frac{1}{\lambda_1^2\nu^3} (-e^{-\nu\lambda_1 t} + 1) \\
&\leq (limsup \| g \|_2^2) \frac{1}{\lambda_1^2\nu^3}
\end{aligned}$$

by $F = \limsup_{t\to\infty} \| g \|_2^2$,

$$\begin{aligned}
\limsup_{t\to\infty} \int_t^{t+T} \| A^{\frac{1}{2}} u \|_2^2 \, ds &\leq \frac{1}{\nu^2\lambda_1} \int_t^{t+T} \| g(s) \|_2^2 \, ds + \frac{1}{\nu} (e^{-\nu\lambda_1 t} \| u_0 \|_2^2 \\
&\quad + \int_0^t \frac{1}{\nu\lambda_1} e^{-\nu\lambda_1(t-s)} \| g \|_2^2 \, ds) \\
&\leq \frac{1}{\nu^2\lambda_1} F^2 T + 0 + \frac{F^2}{\lambda_1^2\nu^3} \qquad (2.36b)
\end{aligned}$$

■

The bounds of nonlinear term $\| A^{-\gamma}(w \cdot \nabla)u \|_2$ and $\| A^{-\gamma}(v \cdot \nabla)w \|_2$ are given in the following Lemma 20, Lemma 21, Lemma 22 and all the coefficients M_1 in this section have the same definition as in (2.1) . The next lemma is for the case $\gamma \geq \frac{3}{4}$.

Lemma 20 *For $\gamma \geq \frac{3}{4}$, there are estimates for the L^2 norm of the nonlinear term: $\| A^{-\gamma}(w \cdot \nabla)u \|_2 \leq M_1^2 \| A^{\frac{1}{2}} w \|_2 \| u \|_2$, and $\| A^{-\gamma}(v \cdot \nabla)w \|_2 \leq M_1^2 \| A^{\frac{1}{2}} w \|_2 \| v \|_2$, for $\gamma > \frac{1}{2}$, $\| A^{-\gamma}(w \cdot \nabla)u \|_2 \leq M_1^2 \| A^{\frac{5}{4}-\gamma} w \|_2 \| u \|_2$, and $\| A^{-\gamma}(v \cdot \nabla)w \|_2 \leq M_1^2 \| A^{\frac{5}{4}-\gamma} w \|_2 \| v \|_2$*

Proof. Because $A^{-\frac{1}{2}} div \leq 1$, $\| A^{-\gamma}(w \cdot \nabla)u \|_2 = \| A^{-\gamma+\frac{1}{2}}(A^{-\frac{1}{2}} div)(w \otimes u) \|_2 \leq \|$

$A^{-\gamma+\frac{1}{2}}(w \otimes u) \parallel_2$. By Sobolev inequality and Holder inequality, if $q = 6/(4\gamma+1), sq = 3/(2\gamma-1), \sigma = \frac{5}{4} - \gamma$, and by $\gamma > \frac{1}{2}$, $sq > 0$,then

$$\parallel A^{-\gamma}(w \cdot \nabla)u \parallel_2 \leq \parallel A^{-\gamma+\frac{1}{2}}(w \otimes u) \parallel_2 \leq M_1 \parallel (w \otimes u) \parallel_q$$
$$\leq M_1 \parallel w \parallel_{sq} \parallel u \parallel_2 \leq M_1^2 \parallel A^{\sigma} w \parallel_2 \parallel u \parallel_2^2$$

so, if $\gamma \geq \frac{3}{4}$, then $\parallel A^{-\gamma}(w \cdot \nabla)u \parallel_2 \leq M_1^2 \parallel A^{\sigma} w \parallel_2 \parallel u \parallel_2^2 \leq M_1^2 \parallel A^{\frac{1}{2}} w \parallel_2 \parallel u \parallel_2^2$ and $\sigma \leq \frac{1}{2}$.

For the other way, $\parallel A^{-\gamma}(v \cdot \nabla)w \parallel_2 = \parallel A^{-\gamma+\frac{1}{2}}(A^{-\frac{1}{2}} div)(v \otimes w) \parallel_2 \leq \parallel A^{-\gamma+\frac{1}{2}}(v \otimes w) \parallel_2$, following the same way as above, there is a similar estimate, if $\gamma \geq \frac{3}{4}$, then $\parallel A^{-\gamma}(v \cdot \nabla)w \parallel_2 \leq M_1^2 \parallel A^{\sigma} w \parallel_2 \parallel v \parallel_2^2 \leq M_1^2 \parallel A^{\frac{1}{2}} w \parallel_2 \parallel v \parallel_2^2$ and $\sigma \leq \frac{1}{2}$. ■

The next lemma is the bounds of the nonlinear terms $\parallel A^{-\gamma}(w \cdot \nabla)u \parallel_2$ and $\parallel A^{-\gamma}(v \cdot \nabla)w \parallel_2$ under the condition $\gamma \geq \frac{1}{4}$.

Lemma 21 *There are estimates for the L^2 norm of the nonlinear term:*
When $\gamma \geq \frac{1}{4}$, $\parallel A^{-\gamma}(w \cdot \nabla)u \parallel_2 \leq M_1^2 \parallel A^{-\gamma+\frac{3}{4}} w \parallel_2 \parallel \nabla u \parallel_2 \leq M_1^2 \parallel A^{\frac{1}{2}} w \parallel_2 \parallel \nabla u \parallel_2$, and
$\parallel A^{-\gamma}(v \cdot \nabla)w \parallel_2 \leq M_1^2 \parallel A^{\frac{1}{2}} w \parallel_2 \parallel \nabla v \parallel_2$
especially as $\gamma = \frac{1}{4}$, $\parallel A^{-\gamma}(v \cdot \nabla)w \parallel_2 \leq M_1^2 \parallel \nabla v \parallel_2 \parallel A^{\frac{1}{2}} w \parallel_2$, $\parallel A^{-\gamma}(w \cdot \nabla)u \parallel_2 \leq M_1^2 \parallel A^{\frac{1}{2}} w \parallel_2 \parallel \nabla u \parallel_2$.

Proof. By sobolev inequality in (2.1) and holder inequality, if $q = 6/(4\gamma+3), sq = 3/(2\gamma)$, and $\gamma \geq \frac{1}{4}$ makes $-\gamma + \frac{3}{4} \leq \frac{1}{2}$ then $\parallel A^{-\gamma}(w \cdot \nabla)u \parallel_2 \leq M_1 \parallel (w \cdot \nabla)u \parallel_q \leq M_1 \parallel w \parallel_{sq} \parallel \nabla u \parallel_2 \leq M_1^2 \parallel A^{-\gamma+\frac{3}{4}} w \parallel_2 \parallel \nabla u \parallel_2 \leq M_1^2 \parallel A^{\frac{1}{2}} w \parallel_2 \parallel \nabla u \parallel_2$ if $q = 6/(4\gamma+3), sq = 3/(2\gamma+1)$, then $sq \leq 2$, and $\parallel A^{-\gamma}(v \cdot \nabla)w \parallel_2 \leq M_1 \parallel (v \cdot \nabla)w \parallel_q \leq M_1 \parallel v \parallel_6 \parallel \nabla w \parallel_{sq} \leq M_1^2 \parallel A^{\frac{1}{2}} w \parallel_2 \parallel \nabla v \parallel_2$ ■

Next lemma is for the case γ between 0 and $\frac{1}{4}$.

Lemma 22 *Assume $0 < \gamma < \frac{1}{4}$, then we have estimates for the L^2 norm of nonlinear terms: $\parallel A^{-\gamma}(w \cdot \nabla)u \parallel_2 \leq M_1^2 \parallel A^{-\gamma+\frac{3}{4}} w \parallel_2 \parallel \nabla u \parallel_2$, and $\parallel A^{-\gamma}(v \cdot \nabla)w \parallel_2 \leq M_1^3 \parallel A^{-\gamma+\frac{3}{4}} w \parallel_2 \parallel \nabla v \parallel_2$.*

Proof. Suppose $q = \frac{6}{3+4\gamma}, sq = \frac{3}{2\gamma}$, and $0 < \gamma < \frac{1}{4}$ makes $\frac{1}{2} < -\gamma + \frac{3}{4} < \frac{3}{4}$, then by Sobolev inequality and Hölder inequality, $\| A^{-\gamma}(w \cdot \nabla)u \|_2 \leq M_1 \| (w \cdot \nabla)u \|_q \leq M_1 \| w \|_{sq} \| \nabla u \|_2 \leq M_1^2 \| A^{-\gamma+\frac{3}{4}} w \|_2 \| \nabla u \|_2$. Suppose $q = \frac{6}{3+4\gamma}, sq = \frac{3}{2\gamma+1}$, and $0 < \gamma < \frac{1}{4}$ makes $\frac{1}{2} < -\gamma + \frac{3}{4} < \frac{3}{4}$, then by Sobolev inequality and Hölder inequality, $\| A^{-\gamma}(v \cdot \nabla)w \|_2 \leq M_1 \| (v \cdot \nabla)w \|_q \leq M_1 \| A^{\frac{1}{2}} w \|_{qs} \| v \|_6 \leq M_1^3 \| A^{-\gamma+\frac{3}{4}} w \|_2 \| \nabla v \|_2$ ■

Like the standard NSE, (1.38) satisfies an energy inequality, which we derive as follows: taking the inner product of both sides of (1.38) with u we have that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \mu \left\|A_\varphi^{1/2}u\right\|_2^2 = 0 \tag{2.37}$$

noting that since $div\ u = 0$ we have that $(\nabla p, u) = 0$ and $((u \cdot \nabla)\, u, u) = -\,((div\ u)\, u, u) = 0$. Integrating both sides of (2.37) for $u_0 = u\,(x, 0)$ we obtain that

$$\|u\,(t)\|_2^2 + \mu \int_0^t \left\|A_\varphi^{1/2}u(s)\right\|_2^2 ds \leq \|u_0\|_2^2 \tag{2.38}$$

which is the basic energy inequality in its most useful form here. Using the same techniques with (1.37) we obtain

$$\|u_\nu\,(t)\|_2^2 + \nu \int_0^t \left\|A^{1/2}u_\nu(s)\right\|_2^2 ds + \mu \int_0^t \left\|A_\varphi^{1/2}u_\nu(s)\right\|_2^2 ds \leq \|u_{\nu,0}\|_2^2 \tag{2.39}$$

so that in particular

$$\|u_\nu\,(t)\|_2^2 + \mu \int_0^t \left\|A_\varphi^{1/2}u_\nu(s)\right\|_2^2 ds \leq \|u_{\nu,0}\|_2^2 \tag{2.40}$$

and so u, u_ν satisfy the same estimate independent of ν whenever (as here) $\|u_0\|_2$ and $\|u_{\nu,0}\|_2$ share a common bound.

We now discuss the proof of Theorem 11. The local existence theory in [4] can be straightforwardly adapted once we observe that $A_\varphi = A^\alpha - B_m$ where $B_m = P_m B_m$ is a bounded operator with bound satisfying $\|B_m\|_2 \leq \lambda_m^\alpha$; the main observation in this regard is that with this decomposition the semigroup e^{-tA_φ}, which would be substituted for e^{-tA^α}, satisfies the estimate $\|e^{-tA_\varphi}\|_2 \leq e^{\lambda_m^\alpha t}$ given that the operators in the decomposition commute. This would mean, for example, that on a potential local existence interval $[0,T]$ the factor $T^{(1-5/(4\alpha))}$ would be replaced by $e^{\lambda_m^\alpha T}T^{(1-5/(4\alpha))}$ in the estimates at the beginning of [3, section 4], and it is still the case that the latter factor can still be made arbitrarily small by making T small enough. Somewhat more complex, but still relatively similar and staightforward modifications, would result in a generalization of the local existence result [3, Theorem 1], as well as the regularity arguments in [3, section 4].

To obtain a priori estimates for global existence and regularity, we first observe that the a priori estimates for $\|A^\beta Q_m u\|_2$ established in [4, section 2] for $\beta < \alpha - 5/4$ hold independently of ν, and in particular hold for (1.38). Specifically, since $A_\varphi Q_m u = A^\alpha Q_m u$ we have that $Q_m u$ satisfies the integral equation

$$Q_m u\left(t\right) = e^{-\mu t A^\alpha} Q_m u_0 + \int_0^t e^{-\mu(t-s)A^\alpha} Q_m P((u\left(s\right)\cdot\nabla)\,u\left(s\right) - f(s))ds \tag{2.41}$$

where P is the Leray projection. We then have that

$$\begin{aligned}\left\|A^\beta Q_m u\left(t\right)\right\|_2 \le \mu^{-\beta/\alpha} &\left\|\left(\mu A^\alpha\right)^{\beta/\alpha}\left[e^{-\frac{t}{2}(\mu A^\alpha)}\right]^2 u_0\right\|_2 \\ &+ \mu^{-\beta/\alpha}\int_0^t \left\|\left(\mu A^\alpha\right)^{\beta/\alpha}\left[e^{-\frac{(t-s)}{2}(\mu A^\alpha)}\right]^2 f\left(s\right)\right\|_2 ds \\ &+ \mu^{-\beta/\alpha}\int_0^t \left\|\left(\mu A^\alpha\right)\left[e^{-\frac{(t-s)}{2}(\mu A^\alpha)}\right]^2 \left(u\left(s\right)\cdot\nabla\right)u\left(s\right)\right\|_2 ds\end{aligned} \tag{2.42}$$

from which it can be seen from a straightforward adaptation of [3, (2.19) - (2.22)] where we set $f = 0$ that for all $\beta < \alpha - 5/4$ we have that

$$\begin{aligned}\left\|A^\beta Q_m u\left(t\right)\right\|_2 \le\; & c_2\left(2/\mu\right)^{\beta/\alpha}\left\|A^\beta Q_m u_0\right\|_2 e^{-\lambda_m^\alpha(\mu/2)t} \\ &+ c_2 M_2 \sup_{s\ge\tau}\left\|u\left(s\right)\right\|_2^2\left(1-\gamma\right)^{-1} e^{-\gamma}\frac{2}{\mu}\left[\frac{1}{\lambda_m}\right]^{\alpha-(\beta+5/4)}\end{aligned} \tag{2.43}$$

where $\gamma = \left(\beta + 5/4\right)/\alpha$, c_2 is as in (2.36), and M_2 is an appropriate Sobolev constant from (2.1). Meanwhile

$$\|A^{\beta}P_n u(t)\|_2 \leq \lambda_n \|u(t)\|_2 \tag{2.44}$$

and then we can apply (2.38) above to (2.43) and (2.44). Since it is clear from e.g. [3, section 4] that obtaining such a bound for any $\beta \in (0, \alpha - 5/4)$ is enough to obtain global regularity, we thus establish Theorem 11.

2.2 Discretization of the SHNSE

Numerical simulation of turbulent flows is one of the greatest challenges in computational fluid dynamics (CFD). Generally speaking, DNS is not likely to become feasible in the foreseeable future since it is computationally very expensive, although high-resolution DNS simulation of incompressible turbulence with the number of grid points up to $4,096^3$ has been performed [42] (also see [39] for a latest survey of studies of high-Reynolds number isotropic turbulence by DNS). Therefore, the use of a turbulence model such as LES and variational multiscale (VMS) models becomes necessary. In the computational research part, we focus on numerical simulation of isotropic homogeneous decaying turbulent flows using the SHNSE model to explore its utility in providing a physically accurate and computationally efficient model for simulating turbulent phenomena in practical applications.

As in [18, 64, 52], in the present numerical research we will test the SHNSE model mainly by considering the incompressible SHNSE on a cubic spatial domain, say $\Omega = [0, 2\pi]^3 \subset \mathbb{R}^3$, with periodic boundary conditions (PBCs) in all three spatial directions. For such a partial differential equation (PDE) system, as is widely-known, it is natural to use a spectral method in which the solution is written as a sum of Fourier modes with time-dependent coefficients (mode amplitudes). Let us consider $A_\phi = Q_m A^\alpha$ with $\alpha = 2$. In this case, the SHNSE model takes on the form

$$\mathbf{u}_t + \nu A\mathbf{u} + \mu Q_m A^2 \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{g}, \tag{2.45a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.45b}$$

subject to $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ where $\mathbf{u}_0$ is the given initial velocity, assumed divergence free.

The Fourier series representations of the solutions are

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{\mathbf{u}}_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{2.46a}$$

$$p(\mathbf{x}, t) = \sum_{\mathbf{k}} \hat{p}_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{2.46b}$$

where $\mathbf{k}{=}(k_1, k_2, k_3)$ is the wavenumber vector, and $\hat{\mathbf{u}}_{\mathbf{k}}$ and $\hat{p}_{\mathbf{k}}$ are the Fourier coefficients of $\mathbf{u}$ and p, respectively. The Fourier versions of (2.45a) and (2.45b) are, respectively,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu |\mathbf{k}|^2 + \mu |\mathbf{k}|^4 Q_m \right) \hat{\mathbf{u}}_{\mathbf{k}} = -\mathrm{i}\mathbf{k}\hat{p}_{\mathbf{k}} - \mathrm{i}\mathbf{k} \cdot \widehat{(\mathbf{u} \otimes \mathbf{u})}_{\mathbf{k}} + \hat{\mathbf{g}}_{\mathbf{k}}, \tag{2.47a}$$

$$\mathrm{i}\mathbf{k} \cdot \hat{\mathbf{u}}_{\mathbf{k}} = 0, \tag{2.47b}$$

where $\hat{\mathbf{g}}_{\mathbf{k}}$ are the Fourier coefficients of the external force $\mathbf{g}$. Note here that using the divergence free constraint (2.45b), the nonlinear term has been written in the divergence form

$$\mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \nabla \cdot \mathbf{u} \equiv \nabla \cdot (\mathbf{u} \otimes \mathbf{u}),$$

where $\mathbf{u} \otimes \mathbf{v} := u_i v_j, i, j{=}1, 2, 3$. The Fourier-Galerkin approximation truncates the sums (2.47a) and (2.47b), such that $-N/2 \leq k_j \leq N/2 - 1, j{=}1, 2, 3$. The modes for which $k_j{=}-N/2, j = 1, 2, 3$, are omitted for reasons concerning commonly used FFT's [14]. All terms may have to be dealiased with the 3/2 rule. In practice, one does not use the pressure. The velocity is simply projected on its divergence-free part employing (2.47b).

(2.47a)-(2.47b) form a system of ordinary differential equations (ODEs) for the Fourier mode amplitudes $\hat{\mathbf{u}}_{\mathbf{k}}(t)$, which can be solved by certain time-advancement methods. For example, in case of $\mathbf{g}{=}0$, if one chooses to integrate analytically the viscous terms in time (by the principle of integrating factor methods [14, 13, 33, 67]), and on the other hand, to discretize the other terms using a two-step second-order Adams-Bashforth scheme, and to calculate the nonlinear terms using pseudo-spectral methods [17], the time evolution of (2.47a) using time step Δt can be written as

$$\frac{\hat{\mathbf{u}}_{\mathbf{k}}^{n+1} - \hat{\mathbf{u}}_{\mathbf{k}}^{n} \mathrm{e}^{\left((-\nu|\mathbf{k}|^2-\mu|\mathbf{k}|^4 Q_m)\Delta t\right)}}{\Delta t}$$

$$= \hat{P}(\mathbf{k}) \left(\frac{3}{2} \widehat{(\mathbf{u}\otimes\mathbf{u})}_{\mathbf{k}}^{n} \mathrm{e}^{\left((-\nu|\mathbf{k}|^2-\mu|\mathbf{k}|^4 Q_m)\Delta t\right)} - \frac{1}{2} \widehat{(\mathbf{u}\otimes\mathbf{u})}_{\mathbf{k}}^{n-1} \mathrm{e}^{\left(2(-\nu|\mathbf{k}|^2-\mu|\mathbf{k}|^4 Q_m)\Delta t\right)} \right), \quad (2.48)$$

where $\hat{P}(\mathbf{k})$, defined through $\hat{P}_{ij}{=}\delta_{ij}{-}k_i k_j/|\mathbf{k}|^2$, is the incompressible projection operator.

In Fourier space, the incompressible SHNSE under periodic boundary condition can be written as follows:

$$\frac{\partial u_k}{\partial t} = -(\widehat{u \cdot \nabla u})_k - \nu k^2 u_k - \mu Q_m k^{2\alpha} - (\widehat{\nabla P})_k + f_k \quad (2.49)$$

$$k \cdot u_k = 0 \quad (2.50)$$

The explicit expression for the pressure term comes from inner product between (1.1) and ∇ by inserting (2.49)

$$P_k = ((\nabla \cdot \widehat{(u \cdot \nabla u)})_k - (\widehat{\nabla \cdot f})_k)/k^2 \quad (2.51)$$

The viscous and spectrally hyperviscous term can be integrated analytically in time. The nonlinear terms are calculated using pseudo-spectral methods[68]. The other terms are discretized using a second-order Adams-Bashforth scheme. Using B_k^n represents $-(\widehat{u \cdot \nabla u})_k^n - (\widehat{\nabla P})_k^n + f_k^n$ and $a(k)$ represents $-\nu k^2 - \mu Q_m k^{2\alpha}$, the time evolution of (2.49) can be written:

$$\frac{u_k^{n+1} - u_k^n exp(a(k)\triangle t)}{\triangle t} = \frac{3}{2} B_k^n exp(a(k)\triangle t) - \frac{1}{2} B_k^{n-1} exp(2a(k)\triangle t) \quad (2.52)$$

The pressure term in B_k^n can be calculated by (2.52). We may use $u_k^0 = u_k^1$ to initialize the flow.

CHAPTER 3: THEORETICAL ANALYSIS

3.1 Convergence of Galerkin Solutions and Continuous Dependence on Data

3.1.1 Proofs of Theorems 1 and 3

Let $w_N = u - u_N$ then subtracting (1.6a) from (1.1a) we obtain the following equations for w_N:

$$\begin{aligned} & (w_N)_t + \nu A w_N + \mu A_\varphi w_N + P_N (u_N \cdot \nabla) w_N + P_N (w_N \cdot \nabla) u \\ = \; & G_N + \nabla \mathcal{P}_N + Q_N (u \cdot \nabla) u \end{aligned} \tag{3.1}$$

where $G_N = g - g_N$ and $\mathcal{P}_N = \mathcal{P} - P_N$.

Taking the inner product of both sides of (3.1) with $A^\beta w_N$ we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|A^{\beta/2} w_N\right\|_2^2 + \left(\nu A w_N + \mu A_\varphi w_N, A^\beta w_N\right) = \left(V, A^\beta w_N\right) \tag{3.2}$$

where $V =$ the right-hand side of (3.1) minus the nonlinear terms of the left-hand side. We assume for simplicity that $\mu \le \nu$ (otherwise replace μ by $\mu_0 \equiv \min\{\mu, \nu\}$) then since the operators A and A_φ are positive we have that

$$\begin{aligned} \left(\nu A + \mu A_\varphi w_N, A^\beta w_N\right) & \ge \mu \left(\left(A + A_\varphi\right) w_N, A^\beta w_N\right) \\ & \ge \mu \left(\left(P_m A + Q_m A^\alpha\right) w_N, A^\beta w_N\right). \end{aligned} \tag{3.3}$$

Now $P_m A + Q_m A^\alpha = N^{\alpha-1} A^\alpha$ where $N = P_m A^{-1} + Q_m I$.

The smallest eigenvalue of $N^{\alpha-1}$ is $1/\lambda_m^{\alpha-1}$, thus from (3.3)

$$
\begin{aligned}
\left(vA+\mu A_{\varphi}w_N, A^{\beta}w_N\right) &\geq \mu\left(N^{\alpha-1}A^{\alpha}w_N, A^{\beta}w_N\right) \\
&= \mu\left(N^{\alpha-1}A^{(\alpha+\beta)/2}w_N, A^{(\alpha+\beta)/2}w_N\right) \\
&\geq \frac{\mu}{\lambda_m^{\alpha-1}}\left\|A^{(\alpha+\beta)/2}w_N\right\|_2^2. \qquad (3.4)
\end{aligned}
$$

Meanwhile $\left(\nabla P_N, A^{\beta}w_N\right)=0$ since $\nabla\cdot A^{\beta}w_N = A^{\beta}\left(\nabla\cdot w_N\right)=0$ on a periodic box, while for $V_1 = V-\nabla P_N$ we have $\left(V_1, A^{\beta}w_N\right)=\left(A^{-\gamma}V_1, A^{(\alpha+\beta)/2}w_N\right)$ for $\gamma=(\alpha-\beta)/2$. By Young's inequality

$$
\begin{aligned}
&\left(A^{-\gamma}V_1, A^{(\alpha+\beta)/2}w_N\right) \\
\leq\ & \frac{2\lambda_m^{\alpha-1}}{\mu}\left\|A^{-\gamma}V_1\right\|_2^2+\frac{\mu}{8\lambda_m^{\alpha-1}}\left\|A^{(\alpha+\beta)/2}w_N\right\|_2^2 \qquad (3.5)
\end{aligned}
$$

where $v_i, i=1,\cdots,4$ is one of the terms of V_1. Combining (3.2), (3.4), and (3.5), collecting the terms of $\left\|A^{(\alpha+\beta)/2}w_N\right\|_2^2$, and multiplying by 2 we have

$$
\begin{aligned}
&\frac{d}{dt}\left\|A^{\beta/2}w_N\right\|_2^2+\frac{\mu}{\lambda_m^{\alpha-1}}\left\|A^{(\alpha+\beta)/2}w_N\right\|_2^2 \\
\leq\ & \frac{4\lambda_m^{\alpha-1}}{\mu}(\left\|A^{-\gamma}P_N\left(u_N\cdot\nabla\right)w_N\right\|_2^2+\left\|A^{-\gamma}P_N\left(w_N\cdot\nabla\right)u\right\|_2^2) \\
&+\left\|A^{-\gamma}G_N\right\|_2^2+\left\|A^{-\gamma}Q_N\left(u\cdot\nabla\right)u\right\|_2^2). \qquad (3.6)
\end{aligned}
$$

By Lemmas 13 and 14, with $v=u_N$ or u and $w=w_N$, we have for $C_0=\max\left\{M_0, M_0'\right\}$ that

$$
\begin{aligned}
\frac{d}{dt}\left\|A^{\beta/2}w_N\right\|_2^2 &\leq \frac{4\lambda_m^{\alpha-1}C_0}{\mu}\left[\left\|A^{\theta}u_N\right\|_2^2+\left\|A^{\theta}u\right\|_2^2\right]\left\|A^{\beta/2}w_N\right\|_2^2 \\
&\quad+\frac{4\lambda_m^{\alpha-1}}{\mu}\left[\frac{1}{\lambda_1^{\gamma}}\left\|G_N\right\|_2^2+\left\|A^{-\gamma}Q_N\left(u\cdot\nabla\right)u\right\|_2^2\right] \qquad (3.7)
\end{aligned}
$$

where we have discarded the term $\mu\lambda_m^{1-\alpha}\left\|A^{(\alpha+\beta)/2}w_N\right\|_2^2$, used Poincaré on $\left\|A^{-\gamma}G_N\right\|_2^2$, and used the fact that P_N is an orthogonal projection; here $\theta=1/2$ if $\alpha\geq\beta$ and $\theta=\gamma+1/2$ if $\alpha<\beta$.

Integrating on $(0,t)$ and using Gronwall's inequality we have for $W_N(t)$ as in (1.7c) that

$$\left\|A^{\beta/2}w_N(t)\right\|_2^2 \le W_N(t)\exp\left(\frac{4\lambda_m^{\alpha-1}C_0}{\mu}\int_0^t\left[\left\|A^\theta u_N\right\|_2^2+\left\|A^\theta u\right\|_2^2\right]ds\right). \tag{3.8}$$

For $\beta > \alpha$ we obtain from (3.8) and the regularity results in [4, Section 2] (which show that $\int_0^t \left\|A^\theta u\right\|_2 ds \le C_1(U_g, T, \theta)$ where C_1 is a polynomial of degree depending on θ) that

$$\left\|A^{\beta/2}w_N(t)\right\|_2^2 \le W_N(t)\exp\left(\frac{4\lambda_m^{\alpha-1}C_0C_1T}{\mu}\right) \tag{3.9}$$

where again $W_N(t)$ is as in (1.7c).

Let $C_2(u_0,\beta)$ be a bound from [4, Section 2] on $\left\|A^{\beta(\gamma)}u\right\|_2^2$, then Theorem 1 for $\beta > \alpha$ follows from (3.9), the integrability from the energy inequality of $\|A^{-\gamma}(u\cdot\nabla)u\|_2^2 \le M_0\left\|A^{\beta/2}u\right\|_2^2\|\nabla u\|_2^2 \le M_0C_2\|\nabla u\|_2^2$, the integrability of $\|G_N\|_2^2 \le 2\|g\|_2^2$, and the Dominated Convergence Theorem.

For $\beta \le \alpha$ so that $\gamma = 1/2$, we use (2.6) to obtain in standard fashion

$$\int_0^t \left\|A^{1/2}u\right\|_2^2 ds \le \frac{1}{\nu^2\lambda_1}\int_0^t \|g\|_2^2\, ds \tag{3.10}$$

and

$$\int_0^t \left\|A^{1/2}u_N\right\|_2^2 ds \le \frac{1}{\nu^2\lambda_1}\int_0^t \|g_N\|_2^2\, ds \le \frac{1}{\nu^2\lambda_1}\int_0^t \|g\|_2^2\, ds \tag{3.11}$$

so that from (3.8)

$$\left\|A^{\beta/2}w_N(t)\right\|_2^2 \le W_N(t)\exp(8\mu^{-1}\lambda_m^{\alpha-1}C_0(\lambda_1\nu)^{-2}\int_0^t \|g\|_2^2\, ds). \tag{3.12}$$

Using the discussion involving the Dominated Convergence Theorem and C_2 from above we see that the convergence as $N \to \infty$ in (3.12) is uniform on $[0,T]$, proving Theorem 1 in the case $\beta \le \alpha$. The convergence is uniform on $[0,\infty)$ if (1.8) is satisfied, again using the discussion involving C_2; this proves Theorem 3, and concludes this section.

3.1.2 Proofs of Theorems 4, 5, 6, and 7.

We apply Q_m to both sides of (3.1) and take the inner product with $A^\beta Q_m w_N$, noting that $\left(A^{-\gamma}V_1, A^{(\alpha+\beta)/2}Q_m w_N\right) = \left(A^{-\gamma}V_1, A^{(\alpha+\beta)/2}Q_m w_N\right) = \left(A^{-\gamma}Q_m V_1, A^{(\alpha+\beta)/2}Q_m w_N\right)$.

We then proceed as in (3.6), (3.7), only now we use (2.13c), (2.13d) of Lemma 13, and Young's inequality in a similar way to obtain for $\beta \leq 3/2$ and for $C_1 = \max\{K_0, K_0'\}$ that

$$\begin{aligned} & \frac{d}{dt}\left\|A^{\beta/2}Q_m w_N\right\|_2^2 + \mu\left\|A^{(\alpha+\beta)/2}Q_m w_N\right\|_2^2 \\ \leq \;& \frac{4C_1\lambda_{m+1}^{-2\sigma}}{\mu}\left[\left\|A^{1/2}u_N\right\|_2^2 + \left\|A^{1/2}u\right\|_2^2\right]\left\|A^{\beta/2}w_N\right\|_2^2 \\ & + \frac{4}{\mu}\left\|A^{-\gamma}Q_N\left(u\cdot\nabla\right)u\right\|_2^2 + \frac{4}{\mu\lambda_1^{\gamma}}\left\|Q_m G_N\right\|_2^2 \end{aligned} \tag{3.13}$$

where since $\alpha \geq \beta$ we can take $\theta = 1/2$ and where we note that we can assume that $N > m$; note also we have not included the term $\lambda_m^{\alpha-1}$ when using Young's inequality since we will not use the operator $N^{\alpha-1}$ in this section. Set

$$U_N = \left\|A^{1/2}u_N\right\|_2^2 + \left\|A^{1/2}u\right\|_2^2 \tag{3.14}$$

and

$$F_{Q,N}(t) = \frac{4}{\mu}\left\|Q_m G_N\right\|_2^2 + \frac{4}{\lambda_1^{\gamma}}\left\|A^{-\gamma}Q_N\left(u\cdot\nabla\right)u\right\|_2^2 \tag{3.15}$$

then applying Poincaré to (3.13) we have that

$$\begin{aligned} & \frac{d}{dt}\left\|A^{\beta/2}Q_m w_N\right\|_2^2 + \mu\lambda_{m+1}^{\alpha}\left\|A^{\beta/2}Q_m w_N\right\|_2^2 \\ \leq \;& \frac{4C_1\lambda_{m+1}^{-2\sigma}}{\mu}U_N\left\|A^{\beta/2}w_N\right\|_2^2 + F_{Q,N}(t) \\ \leq \;& \frac{4C_1}{\mu}U_N\left(\lambda_{m+1}^{-2\sigma}\left\|A^{\beta/2}P_m w_N\right\|_2^2 + \lambda_{m+1}^{-2\sigma}\left\|A^{\beta/2}Q_m w_N\right\|_2^2\right) + F_{Q,N}(t) \end{aligned} \tag{3.16}$$

which we write as

$$\begin{aligned} & \frac{d}{dt}\left\|A^{\beta/2}Q_m w_N\right\|_2^2 + \left(\mu\lambda_{m+1}^{\alpha} - \frac{4C_1\lambda_{m+1}^{-2\sigma}}{\mu}U_N\right)\left\|A^{\beta/2}Q_m w_N\right\|_2^2 \\ \leq \;& \frac{4C_1\lambda_{m+1}^{-2\sigma}}{\mu}U_N\left\|A^{\beta/2}P_m w_N\right\|_2^2 + F_{Q,N}(t) \end{aligned} \tag{3.17}$$

or

$$\frac{d}{dt}\left\|A^{\beta/2}Q_m w_N\right\|_2^2 + a(t)\left\|A^{\beta/2}Q_m w_N(t)\right\|_2^2 \leq b(t) \tag{3.18}$$

where

$$a = \mu \lambda_{m+1}^{\alpha} - \frac{4C_1 \lambda_{m+1}^{-2\sigma}}{\mu} U_N \tag{3.19a}$$

and

$$b = \frac{4C_1 \lambda_{m+1}^{-2\sigma}}{\mu} U_N \left\| A^{\beta/2} P_m w_N \right\|_2^2 + F_{Q,N}(t). \tag{3.19b}$$

For the proof of Theorem 4 we apply Gronwall's inequality to (3.18) so that as in e.g. [28, III. A1] we have that

$$\begin{aligned} \left\| A^{\beta/2} Q_m w_N \right\|_2^2 &\le \left\| A^{\beta/2} Q_m w_N(0) \right\|_2^2 e^{-\int_0^t a(s) ds} \\ &\quad + \int_0^t e^{-\int_s^t a(\tau) d\tau} b(s)\, ds. \end{aligned} \tag{3.20}$$

This part of the proof of Theorem 4 will be similar to that in [28, III.A1], with a difference in the end of the argument for our purpose here.

If $\|u_0\|_2^2 \le [L_g/(\nu\lambda_1)]^2$ then set $T_1 = 1$, otherwise if $\|u_0\|_2^2 \le (1+\eta)[L_g/(\nu\lambda_1)]^2$ for some $\eta > 0$, set $T_1 = 1+\eta$; it is reasonable to expect that η is not that large, otherwise we are essentially dealing with a small forcing-data situation. We then have for any $t_0 \ge 0$ that

$$\begin{aligned} \frac{1}{T_1}\int_{t_0}^{t_0+T_1} U_N d\tau &= \frac{1}{\nu}\left(\frac{1}{T_1}\|u_N(t_0)\|_2^2 + \frac{1}{T_1}\int_{t_0}^{t_0+T_1}\frac{\|g_N\|_2^2}{\nu\lambda_1}d\tau\right) \\ &\quad + \frac{1}{\nu}\left(\frac{1}{T_1}\|u(t_0)\|_2^2 + \frac{1}{T_1}\int_{t_0}^{t_0+T_1}\frac{\|g\|_2^2}{\nu\lambda_1}d\tau\right) \\ &\le \frac{1}{\nu}\left(\frac{1}{T_1}\left[\|u_N(0)\|_2^2 + \left(\frac{L_g}{\nu\lambda_1}\right)^2\right] + \frac{L_g^2}{\nu\lambda_1}\right) \\ &\le \frac{2}{\nu}\left(\frac{T_1+1}{T_1}\left(\frac{L_g}{\nu\lambda_1}\right)^2 + \frac{L_g^2}{\nu\lambda_1}\right) \\ &\le \frac{2}{\nu}\left(1 + \frac{1}{T_1} + \nu\lambda_1\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2. \end{aligned} \tag{3.21}$$

Let k be an integer such that $s+kT_1 \leq t \leq s+(k+1)\,T_1$, then by (3.21)

$$\begin{aligned}
\int_s^t a(\tau)\,d\tau &= -\int_s^{s+kT_1} a(\tau)\,d\tau - \int_{s+kT_1}^t a(\tau)\,d\tau \\
&= -T_1 \sum_{j=1}^{k} \frac{1}{T_1} \int_{s+(j-1)T_1}^{s+jT_1} \frac{4C_1\lambda_{m+1}^{-2\sigma}}{\mu} U_N - \mu\lambda_{m+1}^{\alpha} d\tau + \int_{s+kT_1}^t a(\tau)\,d\tau \\
&= -T_1 k \left[\frac{8C_1\lambda_{m+1}^{-2\sigma}}{\mu}\left(1+\frac{1}{T_1}+\nu\lambda_1\right)\left(\frac{L_g}{\upsilon\lambda_1}\right)^2 - \mu\lambda_{m+1}^{\alpha}\right] \\
&\quad +T_1\left(\frac{1}{T_1}\int_{s+kT_1}^{s+(k+1)T_1} a^-(\tau)\,d\tau\right) \qquad (3.22)
\end{aligned}$$

where $a^- = -\min\{a,0\}$.

Choose m large enough so that

$$\lambda_{m+1}^{\alpha} - \frac{8C_1\lambda_{m+1}^{-2\sigma}}{\mu^2}\left(1+\frac{1}{T_1}+\nu\lambda_1\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 > \lambda_{m+1}^{\alpha} - \lambda_m^{\alpha} \equiv \gamma_m \qquad (3.23)$$

and set

$$\Gamma \equiv \sup_{k\geq 0}\frac{1}{T_1}\int_{s+kT_1}^{s+(k+1)T_1} a^-(\tau)\,d\tau \qquad (3.24)$$

then from (3.22) - (3.24) we have that

$$\begin{aligned}
e^{-\int_s^t \alpha(\tau)d\tau} &\leq e^{-T_1 k\gamma_m} e^{\Gamma T_1} = e^{-[(s+kT_1)-s]\gamma_m} e^{\Gamma T_1} \\
&= e^{-[(s+(k+1)T_1-s)]\gamma_m} e^{T_1\gamma_m} e^{\Gamma T_1} \\
&\leq e^{-(t-s)\gamma_m} e^{(\gamma_m+\Gamma)T_1} \\
&\equiv \Gamma' e^{-\gamma_m(t-s)}. \qquad (3.25)
\end{aligned}$$

Applying (3.25) to (3.20) we have that

$$\begin{aligned}
\left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 &\leq \left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 \Gamma' e^{-\gamma_m t} \\
&\quad +\Gamma'\int_0^t e^{-\gamma_m(t-s)} b(s)\,ds \\
&\leq \left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 \Gamma' e^{-\gamma_m t} + \Gamma'\int_0^t e^{-\gamma_m(t-s)} F_{Q,N}(s)\,ds \\
&\quad +\frac{4C_1\lambda_{m+1}^{-2\sigma}}{\mu}\Gamma'\left(\int_0^t e^{-\gamma_m(t-s)} U_N(s)\,ds\right)\rho_P(t) \qquad (3.26)
\end{aligned}$$

where

$$\rho_P(t) = \sup_{0 \le s \le t} \left\| A^{\beta/2} P_m w_N(s) \right\|_2^2 . \tag{3.27}$$

Now for a given positive integer M such that $(t/M) \le 1$ we have from (2.6), (2.12), and $\|g_N\|_2^2 \le \|g\|_2^2$ that

$$\begin{aligned}
\sup_{0 \le s \le t} \int_{[(k-1)t]/M}^{(kt)/M} U_N(s)\, ds &= \sup_{0 \le l \le t} \int_{[(k-1)t]/M}^{(kt)/M} \left(\|\nabla u_N\|_2^2 + \|\nabla u\|_2^2 \right) ds \\
&\le \frac{1}{\nu} \left(\|u_N([(k-1)t]/N)\|_2^2 + \frac{(t/M)\, L_g^2}{\nu \lambda_1} \right) \\
&\quad + \frac{1}{\nu} \left(\|u([(k-1)t]/M)\|_2^2 + \frac{(t/M)\, L_g^2}{\nu \lambda_1} \right) \\
&\le \frac{1}{\nu} \left[\left(\|u_N(0)\|_2^2 + \left(\frac{L_g}{\nu \lambda_1} \right)^2 \right) + \frac{L_g^2}{\nu \lambda_1} \right] \\
&\quad + \frac{1}{\nu} \left[\left(\|u_0\|_2^2 + \left(\frac{L_g}{\nu \lambda_1} \right)^2 \right) + \frac{L_g^2}{\nu \lambda_1} \right] \\
&\le \frac{2}{\nu} \left[(2+\eta) \left(\frac{L_g}{\nu \lambda_1} \right)^2 + \frac{L_g^2}{\nu \lambda_1} \right] \\
&= \frac{2}{\nu} \left[(2 + \eta + \nu \lambda_1) \left(\frac{L_g}{\nu \lambda_1} \right)^2 \right].
\end{aligned} \tag{3.28}$$

Thus from Lemma 15 with $\lambda = \gamma_m$ and $h = U_N$ we have using (3.26) - (3.28) that

$$\begin{aligned}
\left\| A^{\beta/2} Q_m w_N \right\|_2^2 &\le \left\| A^{\beta/2} Q_m w_N(0) \right\|_2^2 \Gamma' e^{-\gamma_m t} + \Gamma' \int_0^t e^{-\gamma_m (t-s)} F_{Q,N}(s)\, ds \\
&\quad + \frac{8 C_0 \lambda_{m+1}^{-2\sigma} \Gamma'}{\nu \mu} \mathcal{E}_{m,M}(t) (2 + \eta + \nu \lambda_1) \left(\frac{L_g}{\nu \lambda_1} \right)^2 \rho_P(t)
\end{aligned} \tag{3.29}$$

where from Lemma 15

$$\mathcal{E}_{m,M}(t) = \frac{1 - e^{-\gamma_m (1-(1/M))t}}{e^{(\gamma t)/M} - 1} + 1. \tag{3.30}$$

For $C_1' \equiv 8 C_1 (1 + (1/T) + \nu \lambda_1)$ we have e.g. for $\alpha = 3$ that the requirement on m given by (3.23) becomes

$$\lambda_{m+1} > \frac{(\nu/\mu)^{2/9} (C_1')^{2/9}}{\lambda_1^{1/9}} G^{4/9} \tag{3.31}$$

and the other lower-bound estimates in the introduction for this case follow similarly. Meanwhile, clearly the first term on the right-hand side of (3.29) goes to zero on $N \to$

∞, and so does the second term by the remarks of section 3.1.1.Thus the rest of the convergence of $\left\|A^{\beta/2}Q_m w_N\right\|_2^2$ to zero is entirely dependent on $\rho_P(t) \to \infty$. This finishes the proof of Theorem 4.

Next, in proving Theorem 5 we obtain better estimates while obtaining somewhat larger conditions on λ_{m+1}^{α} as compared to those in (3.23) by assuming that $\alpha \geq 5/2$; as in the case with inertial manifolds in [4], the analysis is simpler in this case as well. Here for $\alpha \geq 5/2$, for $\alpha - 3/2 \leq \beta \leq \alpha - 1$, and now setting $\omega = \alpha/2 - 5/4$ we obtain, by retracing the steps of Lemma 13, that there is a generic constant C_2 such that

$$\left\|A^{-\gamma}Q_m\left(u_N \cdot \nabla\right) w_N\right\|_2 \leq \lambda_{m+1}^{-2\omega} C_2 \left\|u_N\right\|_2 \left\|A^{\beta/2} w_N\right\|_2 \tag{3.32a}$$

and

$$\left\|A^{-\gamma}Q_m\left(w_N \cdot \nabla\right) u\right\|_2 \leq \lambda_{m+1}^{-2\omega} C_2 \left\|u\right\|_2 \left\|A^{\beta/2} w_N\right\|_2 . \tag{3.32b}$$

The corresponding version of (4.6), for $U_N^0 \equiv \left\|u_N\right\|_2 + \left\|u\right\|_2$, becomes

$$\frac{d}{dt}\left\|A^{\beta/2}Q_m w_N\right\|_2^2 + a_0(t)\left\|A^{\beta/2}Q_m(t)\right\|_2^2 \leq b_0(t) \tag{3.33}$$

where a_0 and b_0 are as in (3.19) but with U_N replaced by U_N^0 and $\lambda_1^{-2\sigma}$, $\lambda_{m+1}^{-2\sigma}$ replaced by $\lambda_1^{-2\omega}$, $\lambda_{m+1}^{-2\omega}$. Now by (2.12) $U_N^0 \leq 2U_g^2$ since $\left\|u_N(0)\right\|_2 \leq \left\|u_0\right\|_2$ and $\left\|g_N\right\|_2 \leq \left\|g\right\|_2$, so choose m large enough so that

$$a_0 = \mu\lambda_{m+1}^{\alpha} - \frac{4C_2\lambda_{m+1}^{-2\omega}}{\mu}U_N^0 \geq \mu\lambda_{m+1}^{\alpha} - \frac{8C_2\lambda_{m+1}^{-2\omega}}{\mu}U_g^2 \geq \frac{\mu}{2}\lambda_{m+1}^{\alpha} \tag{3.34}$$

which requires that

$$\lambda_{m+1}^{2\alpha-5/2} \geq \frac{16C_2}{\mu^2}U_g^2 \tag{3.35}$$

or

$$\lambda_{m+1}^{\alpha-5/4} \geq \frac{4\sqrt{C_2}}{\mu}U_g. \tag{3.36}$$

Using $\left\|u_0\right\|_2^2 = (1+\eta)\left[L_g/(\nu\lambda_1)\right]^2$, this becomes

$$\lambda_{m+1}^{\alpha-5/4} \geq \frac{4\sqrt{C_2(2+\eta)}}{\mu}\left(\frac{L_g}{\nu\lambda_1}\right) = \frac{4(\nu/\mu)\sqrt{C_2(2+\eta)}}{\lambda_1^{1/4}}G. \tag{3.37a}$$

For example, if $\alpha = 3$ then from (3.37a) we obtain

$$\lambda_{m+1} \geq \frac{[4(\nu/\mu)]^{4/7}[C_2(2+\eta)]^{2/7}}{\lambda_1^{1/7}} G^{4/7} \tag{3.37b}$$

and the other lower bounds for other values of α mentioned in the introduction follow similarly. Now let $d = \mu/2$, then by replacing a and b by a_0 and b_0 in (3.20), and using $U_N^0 \leq 2U_g^2$ and $\|u_0\|_2^2 = (1+\eta)\left[L_g/\left(v\lambda_1\right)\right]^2$, we have that

$$\begin{aligned}
\left\|A^{\beta/2}Q_m w_N\right\|_2^2 &\leq \left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 e^{-d\lambda_{m+1}^{\alpha}t} \\
&\quad + \int_0^t e^{-d(t-s)\lambda_{m+1}^{\alpha}} b_0(s)\,ds \\
&\leq \left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 e^{-d\lambda_{m+1}^{\alpha}t} \\
&\quad + \frac{4C_2\lambda_{m+1}^{-2\omega}}{\mu} \int_0^t e^{-d(t-s)\lambda_{m+1}^{\alpha}} U_N^0 \left\|A^{\beta/2}P_m w_N\right\|_2^2 ds \\
&\quad + \int_0^t e^{-d(t-s)\lambda_{m+1}^{\alpha}} F_{Q,N}(s)\,ds \\
&\leq \left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 e^{-d\lambda_{m+1}t} + \int_0^t e^{-d(t-s)\lambda_{m+1}^{\alpha}} F_{Q,N}(s)\,ds \\
&\quad + \frac{8C_2\lambda_{m+1}^{-2\omega}}{\mu} U_g^2 \left(\int_0^t e^{-d\lambda_{m+1}^{\alpha}(t-s)} ds\right) \rho_P(t) \\
&\leq \left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 e^{-d\lambda_{m+1}t} + \int_0^t e^{-d(t-s)\lambda_{m+1}^{\alpha}} F_{Q,N}(s)\,ds \\
&\quad + \frac{16C_2}{\mu^2\lambda_{m+1}^{2\alpha-5/2}}(2+\eta)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \rho_P(t).
\end{aligned} \tag{3.38}$$

Thus we have similar dependence on $\rho_p(t)$ as in (3.29), but with linear dependence on the right-hand side, although the conditions on λ_{m+1} are not quite as good as in (3.30), (3.31). This finishes the proof of Theorem 5.

Next for Theorem 6 we consider $2 \leq \alpha \leq 5/2$ and $\beta \leq 3/2$, and obtain similar results by using different and more involved techniques. By Poincaré we have

$$\lambda_{m+1}^{\alpha-1}\left\|Q_m A^{1/2}u\right\|_2^2 \leq \left\|Q_m A^{(\alpha-1)/2}A^{1/2}u\right\|_2^2 = \left\|Q_m A^{\alpha/2}u\right\|_2^2 \tag{3.39}$$

so that, from Lemma 16 with $\lambda = \mu\lambda_{m+1}^{\alpha}$,

$$\begin{aligned}
\int_0^t e^{-\mu(t-s)\lambda_{m+1}^{\alpha}} \left\|Q_m A^{1/2} u\right\|_2^2 ds &\leq \frac{1}{2\mu\lambda_{m+1}^{\alpha-1}}\left[\|u_0\|_2^2 + U_g^2 + \frac{L_g^2}{\mu\nu\lambda_1\lambda_{m+1}^{\alpha}}\right] \\
&\leq \frac{1}{2\mu\lambda_{m+1}^{\alpha-1}}\left[2\|u_0\|_2^2 + \left(1+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2\right] \\
&\leq \frac{1}{\mu\lambda_{m+1}^{\alpha-1}}\left(3+2\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \qquad (3.40)
\end{aligned}$$

Meanwhile $\left\|P_m A^{1/2} u\right\|_2^2 \leq \lambda_m \|u\|_2^2 \leq \lambda_m U_g$ so that

$$\begin{aligned}
&\int_0^t e^{-\mu(t-s)\lambda_{m+1}^{\alpha}} \left\|P_m A^{1/2} u\right\|_2^2 ds \\
\leq\ & \lambda_m U_g \int_0^t e^{-\mu(t-s)\lambda_{m+1}^{\alpha}} dt \\
\leq\ & \lambda_m U_g \lambda_{m+1}^{-\alpha} \leq (\mu\lambda_{m+1})^{-(\alpha-1)}(2+\eta)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \qquad (3.41)
\end{aligned}$$

Combining (3.40) and (3.41) we have that

$$\int_0^t e^{-\mu(t-s)\lambda_{m+1}^{\alpha}} \|\nabla u\|_2^2 ds \leq \frac{1}{\mu\lambda_{m+1}^{\alpha-1}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \qquad (3.42)$$

We note that (3.42) holds with u replaced by u_N, since Lemma 16 holds with u_N replacing u, and since $\|u_N(0)\|_2 \leq \|u_0\|_2$ and $\|g_N\|_2 \leq \|g\|_2$. Hence

$$\int_0^t e^{-\mu(t-s)\lambda_{m+1}^{\alpha}} U_N ds \leq \frac{2}{\mu\lambda_{m+1}^{\alpha-1}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2. \qquad (3.43)$$

Integrating the differential inequality (3.16) and using (2.13c), (2.13d) with $C_1 = \max\{K_0, K_0'\}$ as above we obtain

$$\begin{aligned}
\left\|A^{\beta/2} Q_m w_N\right\|_2^2 \leq\ & \left\|A^{\beta/2} Q_m w_N(0)\right\|_2^2 e^{-\mu\lambda_{m+1}^{\alpha} t} \\
&+ \frac{4C_1\lambda_{m+1}^{-2\sigma}}{\mu}\int_0^t e^{-\mu\lambda_{m+1}^{\alpha}(t-s)} U_N \left\|A^{\beta/2} Q_m w_N\right\|_2^2 ds \\
&+ \frac{4C_1\lambda_{m+1}^{-2\sigma}}{\mu}\int_0^t e^{-\mu\lambda_{m+1}^{\alpha}(t-s)} U_N \left\|A^{\beta/2} P_m w_N\right\|_2^2 ds \\
&+ \int_0^t e^{-\mu\lambda_{m+1}^{\alpha}(t-s)} F_{Q,N}(s)\, ds. \qquad (3.44)
\end{aligned}$$

Setting $\rho_P(t) = \sup_{0\leq s\leq t} \left\|A^{\beta/2} P_m w_N\right\|_2^2$ as before and setting $\rho_Q(t) = \sup_{0\leq s\leq t} \left\|A^{\beta/2} Q_m w_N(s)\right\|_2^2$, we have from (3.43) and (3.44) that

$$\begin{aligned}
\left\|A^{\beta/2}Q_m w_N\right\|_2^2 &\leq \left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 \\
&+\frac{4C_1\lambda_{m+1}^{-2\sigma}}{\mu}\left(\int_0^t e^{-\mu(t-s)\lambda_{m+1}^{\alpha}}U_N ds\right)\rho_Q(t) \\
&+\frac{4C_1\lambda_{m+1}^{-2\sigma}}{\mu}\left(\int_0^t e^{-\mu(t-s)\lambda_{m+1}^{\alpha}}U_N ds\right)\rho_P(t) \\
&+\left(\int_0^t e^{-\mu(t-s)\lambda_{m+1}^{\alpha}}F_{Q,N}(s)\,ds\right) \\
&\leq \left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2+\int_0^t e^{-\mu(t-s)\lambda_{m+1}^{\alpha}}F_{Q,N}(s)\,ds \\
&+\frac{8C_1}{\mu^2\lambda_{m+1}^{2\alpha-5/2}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2\rho_Q(t) \\
&+\frac{8C_1}{\mu^2\lambda_{m+1}^{2\alpha-5/2}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2\rho_P(t) \qquad (3.45)
\end{aligned}$$

Set $\mathcal{F}_{Q,N}(t) = \sup\limits_{0\leq s\leq t}\int_0^s e^{-\mu(s-\tau)\lambda_{m+1}^{\alpha}}F_{Q,N}(\tau)\,d\tau$ then (4.33) holds with $\mathcal{F}_{Q,N}$ replacing $\int_0^t e^{-\mu(t-s)\lambda_{m+1}^{\alpha}}F_{Q,N}(\tau)\,ds$ on the right-hand side so that

$$\begin{aligned}
\rho_Q(t) &\leq \left\|A^{\beta/2}Q_m w_N\right\|_2^2+\mathcal{F}_{Q,N}(t) \\
&+\frac{8C_1}{\mu^2\lambda_{m+1}^{2\alpha-5/2}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2\rho_Q(t) \\
&+\frac{8C_1}{\mu^2\lambda_{m+1}^{2\alpha-5/2}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2\rho_P(t) \qquad (3.46)
\end{aligned}$$

so that if

$$\frac{8C_1}{\mu^2\lambda_{m+1}^{2\alpha-5/2}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2\leq\frac{1}{2} \qquad (3.47)$$

which requires that

$$\lambda_{m+1}^{2\alpha-5/2}\geq\frac{16C_1}{\mu^2}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \qquad (3.48)$$

then

$$\begin{aligned}
\rho_Q(t) &\leq 2\left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2+2\bar{\mathcal{F}}_{Q,N}(t) \\
&+\frac{16C_1}{\mu^2\lambda_{m+1}^{2\alpha-5/2}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2\rho_P(t)\,. \qquad (3.49)
\end{aligned}$$

Thus with larger m then before we have linear dependence of ρ_Q on all convergence factors, and as is the case with Theorem 5, there are no exponential terms except as may appear in estimates of $\rho_P(t)$. This finishes the proof of Theorem 6.

Now we use the above results to obtain a bound on $\left\|A^{\beta/2}P_m w_N\right\|_2^2$, and thus establish Theorem 7. First we apply P_m to both sides of (3.2) and take the inner product with $A^{\beta/2}P_m w_N$ to obtain in similar fashion to the calculations (3.2) - (3.6) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|A^{\beta/2}P_m w_N\right\|_2^2 + \nu\left\|A^{\frac{1+\beta}{2}}P_m w_N\right\|_2^2 \\ \leq\ & C_0(\|\nabla u_N\|_2 + \|\nabla u\|_2)\left\|A^{\frac{\alpha+\beta}{2}}P_m w_N\right\|_2 \\ &+\frac{1}{\lambda_1^{1/2}}\|G_N\|_2\left\|A^{\frac{1+\beta}{2}}w_N\right\|_2 \end{aligned} \tag{3.50}$$

where we note that $P_m Q_N = 0$.

Set $\alpha = 3/2$ then $\left\|A^{\frac{\alpha+\beta}{2}}P_m w_N\right\|_2 = \left\|A^{\frac{(3/2)+\beta}{2}}P_m w_N\right\|_2 \leq \lambda_m^{1/4}\left\|A^{\frac{1+\beta}{2}}P_m w_N\right\|_2$; applying Young's inequality we have

$$\begin{aligned} \frac{d}{dt}\left\|A^{\beta/2}P_m w_N\right\|_2^2 + \nu\left\|A^{\frac{1+\beta}{2}}P_m w_N\right\|_2^2 \ \leq\ & \frac{3}{\nu\lambda_1}\|G_N\|_2^2 \\ &+\frac{3\lambda_m^{1/2}C_0}{\nu}U_N\left\|A^{\beta/2}Q_m w_N\right\|_2^2 \\ &+\frac{3\lambda_m^{1/2}C_0}{\nu}U_N\left\|A^{\beta/2}P_m w_N\right\|_2^2. \end{aligned} \tag{3.51}$$

Noting that $\left\|A^{\frac{1+\beta}{2}}P_m w_N\right\|_2^2 \leq \lambda_1\left\|A^{\beta/2}P_m w_N\right\|_2^2$ we integrate the differential inequality (3.51) to obtain ν

$$\begin{aligned} \left\|A^{\beta/2}P_m w_N\right\|_2^2 \ \leq\ & \left\|A^{\beta/2}P_m w_N\right\|_2^2 e^{-\nu\lambda_1 t} + \frac{3}{\nu\lambda_1}\int_0^t e^{-\nu\lambda_1(t-s)}\|G_N\|_2^2\,ds \\ &+\frac{3\lambda_m^{1/2}C_0}{\nu}\int_0^t e^{-\nu\lambda_1(t-s)}U_N(s)\,\rho_Q(s)\,ds \\ &+\frac{3\lambda_m^{1/2}C_0}{\nu}\int_0^t e^{-\nu\lambda_1(t-s)}U_N(s)\,\rho_P(s)\,ds \end{aligned} \tag{3.52}$$

Combining (3.37) and (3.38), or by combining (3.38) and (3.39), and noting that $\mathcal{F}_{Q,N}(s) \leq \mathcal{F}_{Q,N}(t)$ for $s \leq t$ we have that (replacing μ by ν in the definition of $\mathcal{F}_{Q,N}$ in (1.23b))

$$\rho_Q(s) \leq 2\left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 + 2\mathcal{F}_{Q,N}(t) + \rho_P(s). \tag{3.53}$$

In similar fashion to (3.40)

$$\int_0^t e^{-\nu\lambda_1(t-s)} U_N(s)\, ds \leq \frac{1}{\nu}(3+2\eta+\nu)\left(\frac{L_g}{\nu\lambda_1}\right)^2 \tag{3.54}$$

so combing (3.52) - (3.54) we have for

$$\mathcal{L}_{G_N} \equiv \sup_{0\leq s\leq t} \frac{3}{\nu\lambda_1}\int_0^t e^{-\nu\lambda_1(s-\tau)} \left\|G_N(\tau)\right\|_2^2 d\tau \tag{3.55}$$

$$\begin{aligned} \left\|A^{\beta/2}P_m w_N\right\|_2^2 &\leq \left\|A^{\beta/2}P_m w_N(0)\right\|_2^2 + \frac{3}{(\nu\lambda_1)^2}\mathcal{L}_{G_N} \\ &+\frac{3\lambda_m^{1/2}C_0(3+2\eta+\nu)}{\nu^2}\left(\frac{L_g}{\nu\lambda_1}\right)^2\left[\left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 + \mathcal{F}_{Q,N}(t)\right] \\ &+\frac{6\lambda_m^{1/2}C_0}{\nu}\int_0^t e^{-\upsilon\lambda_1(t-s)} U_N(s)\,\rho_P(s)\, ds. \end{aligned} \tag{3.56}$$

Neglecting the exponential term in the last term of (3.56) and taking the sup over $[0,t]$ on the left-hand side we obtain

$$\begin{aligned} \rho_P(t) &\leq \left\|A^{\beta/2}P_m w_N\right\|_2^2 + \frac{3}{(\nu\lambda_1)^2}\mathcal{L}_{G_N} \\ &+\frac{3\lambda_m^{1/2}C_0(3+2\eta+\nu)}{\nu^2}\left(\frac{L_g}{\nu\lambda_1}\right)^2\left[\left\|A^{\beta/2}Q_m w_N(0)\right\|_2^2 + \mathcal{F}_{Q,N}(t)\right] \\ &+\frac{6\lambda_m^{1/2}C_0}{\nu}\int_0^t U_N(s)\,\rho_P(s)\, ds \end{aligned} \tag{3.57}$$

from which by Gronwall we obtain, for $w_{N,0} \equiv w_N(0)$, for $G_0 = L_g/(\nu\lambda_1)$, and

$$\begin{aligned} W_{N,0}(t) &\equiv \left\|A^{\beta/2}P_m w_{N,0}\right\|_2^2 + \frac{3\mathcal{L}_{G_N}}{(\nu\lambda_1)^2} \\ &+\frac{3\lambda_m^{1/2}C_0(3+2\eta+\nu)}{\nu^2}G_0^2\left(\left\|A^{\beta/2}Q_m w_{N,0}\right\|_2^2 + \mathcal{F}_{Q,N}(t)\right) \end{aligned} \tag{3.58}$$

that

$$\begin{aligned} \rho_P(t) &\leq W_{N,0}(t)\exp\left(\frac{6\lambda_m^{1/2}C_0}{\nu}\int_0^t U_N(s)\, ds\right) \\ &\leq W_{N,0}(t)\exp\left(\frac{12\lambda_m^{1/2}C_0}{\nu^2}\left[(1+\eta)G_0^2 + \frac{1}{\nu\lambda_1}\int_0^t \|g\|_2^2\, ds\right]\right) \end{aligned} \tag{3.59}$$

and in the case of decaying turbulence we can replace $\int_0^t \|g\|_2^2\, ds$ by $\int_0^\infty \|g\|_2^2\, ds$. Since

$\lambda_m \sim c\lambda_1 m^{2/3}$, we have now fractional dependence like $m^{1/3}$ in the exponential in (3.59). This completes the proof of Theorem 7.

3.1.3 Continuous Dependence on Data

Suppose we have two solutions $v(t)$ and $u(t)$ to (1.1) with forcing data $f(t)$ and $g(t)$ and pressure terms $p_1$and p_2 respectively. Their difference $w(t) \equiv v(t) - u(t)$ satisfies an equation simlar to (3.1), namely:

$$(w)_t + \nu Aw + \mu A_\varphi w + (v \cdot \nabla) w + (w \cdot \nabla) u = G + \nabla P \tag{3.60}$$

where $G = f - g$ and $P = p_1 - p_2$. Using the same techniques that led to (3.18) and (3.19) with just a slightly different use of Young's inequality (there is no term like $A^{-\gamma} Q_N (u \cdot \nabla) u$) we obtain that $Q_m w$ satisfies

$$\frac{d}{dt} \left\| A^{\beta/2} Q_m w \right\|_2^2 + a(t) \left\| A^{\beta/2} Q_m w(t) \right\|_2^2 \leq b(t) \tag{3.61}$$

where for $U \equiv \left\| A^{1/2} u \right\|_2^2 + \left\| A^{1/2} v \right\|_2^2$

$$a = \mu \lambda_{m+1}^{\alpha} - \frac{3C_1 \lambda_{m+1}^{-2\sigma}}{\mu} U \tag{3.62a}$$

and

$$b = \frac{3C_1 \lambda_{m+1}^{-2\sigma}}{\mu} U \left\| A^{\beta/2} P_m w \right\|_2^2 + \frac{3}{\mu} \left\| Q_m G \right\|_2^2 . \tag{3.62b}$$

Based on these estimates it is straightforward to adapt the techniques used to prove Theorem 6 to obtain the following result relating the continuous dependence of $Q_m w$ on its corresponding data and on $P_m w$:

Corollary 23 *Let L_g and η be as above, suppose $\alpha \geq 2$, and assume that*

$$\lambda_{m+1}^{2\alpha - 5/2} \geq \frac{16C_1}{\mu^2} \left(5 + 3\eta + \frac{\nu\lambda_1}{\mu \lambda_{m+1}^{\alpha}} \right) \left(\frac{L_g}{\nu \lambda_1} \right)^2 \tag{3.63}$$

then if I is any interval over which $\left\| A^{\beta/2} P_m w(t) \right\|_2^2$ and $\left\| Q_m G(t) \right\|_2^2$ have suprema, then *we have for a generic constant C_1 as above that*

$$\sup_{t\in I}\left\|A^{\beta/2}Q_m w\left(t\right)\right\|_2^2 \leq 2\left\|A^{\beta/2}Q_m w(0)\right\|_2^2+\frac{6}{\mu}\sup_{t\in I}\left\|Q_m G(t)\right\|_2^2 + \frac{16C_1}{\mu^2\lambda_{m+1}^{2\alpha-5/2}}\left(5+3\eta+\frac{\nu\lambda_1}{\mu\lambda_{m+1}^{\alpha}}\right)\left(\frac{L_g}{\nu\lambda_1}\right)^2\sup_{t\in I}\left\|A^{\beta/2}P_m w\left(t\right)\right\|_2^2 \quad (3.64)$$

It is also straightforward to adapt the techniques of Theorem 5 to obtain a correspondingly slightly simpler result with $\alpha \geq 5/2$. To in turn establish estimates on $\left\|A^{\beta/2}P_m w\left(t\right)\right\|_2^2$, we can straightforwardly adapt the techniques used to prove Theorem 7. We note again the connection to the Kolmogorov theory (deeper here because there is no $Q_N\left(u\cdot\nabla\right)u$ term) in terms of the linear dependence on the data in (3.64).

3.2 The Estimates for Determining Nodes and Determining Modes

3.2.1 The Estimate for Determining Nodes (Theorem 8)

Now Theorem 8 could be proved by using Lemma 17 Lemma 18, Lemma 19, Lemma 21 and Lemma 22.

Proof. First, by Lemma 18 we have

$$\| w \|_{L^2(\Omega)}^2 \leq a_0\frac{1}{\lambda_1^2 N^{\frac{4}{3}}} \| Aw \|_2^2 + a_1\frac{1}{\lambda_1^3 N^2} \| A^{\frac{3}{2}} w \|_2^2 + 8a_2\frac{1}{\lambda_1^{\frac{3}{2}}}\eta^2(w) \quad (3.65)$$

begin at $\| A^{1/2}w \|_2^2 \leq c_2 \| w \|_2 \| Aw \|_2 \leq l^{-2} \| w \|_2^2 + \frac{1}{4}c_2 l^2 \| Aw \|_2^2$,

after inserting (3.65), the relation between l and λ_1 used in proof of (3.65): $l = \frac{C_l}{N^{\frac{1}{3}}\lambda_1^{\frac{1}{2}}}$, and $\| A^{\frac{3}{2}}w \|_2^2 \geq \lambda_1 \| Aw \|_2^2$, it becomes

$$\begin{aligned}
\| A^{1/2}w \|_2^2 &\leq l^{-2}(a_0\frac{1}{\lambda_1^2 N^{\frac{4}{3}}} \| Aw \|_2^2 + a_1\frac{1}{\lambda_1^3 N^2} \| A^{\frac{3}{2}}w \|_2^2 + 8a_2\frac{1}{\lambda_1^{\frac{3}{2}}}\eta^2(w)) + \frac{1}{4}c_2 l^2 \| Aw \|_2^2 \\
&\leq N^{\frac{2}{3}}\lambda_1 C_l^{-2}(a_0\frac{1}{\lambda_1^3 N^{\frac{4}{3}}} \| A^{\frac{3}{2}}w \|_2^2 + a_1\frac{1}{\lambda_1^3 N^2} \| A^{\frac{3}{2}}w \|_2^2 + 8a_2\frac{1}{\lambda_1^{\frac{3}{2}}}\eta^2(w)) \\
&+ \frac{1}{4}c_2 N^{-\frac{2}{3}}\lambda_1^{-1}C_l^{-2}\lambda_1^{-1} \| A^{\frac{3}{2}}w \|_2^2 \\
&\leq N^{-\frac{2}{3}}\lambda_1^{-2} \| A^{\frac{3}{2}}w \|_2^2 \,(C_l^2 a_0 + a_1\lambda_1^{-1}N^{-\frac{3}{4}} + \frac{1}{4}c_2 C_l^{-2}) + 8a_2 N^{\frac{2}{3}}\lambda_1^{-\frac{1}{2}}C_l^{-2}\eta^2(w) \\
&\leq N^{-\frac{2}{3}}\lambda_1^{-2} \| A^{\frac{3}{2}}w \|_2^2 \,(C_l^2 a_0 + a_1\lambda_1^{-1} + \frac{1}{4}c_2 C_l^{-2}) + 8a_2 N^{\frac{2}{3}}\lambda_1^{-\frac{1}{2}}C_l^{-2}\eta^2(w)
\end{aligned} \quad (3.66)$$

If we assume $C_1 = (C_l^2 a_0 + a_1\lambda_1^{-1} + \frac{1}{4}c_2 C_l^{-2})^{-1}$ and $C_2 = 8a_2 C_l^{-2} C_1$, then

$$\| A^{3/2}w \|_2^2 \geq C_1\lambda_1^2 N^{\frac{2}{3}} \| A^{1/2}w \|_2^2 - C_2\lambda_1^{\frac{3}{2}} N^{\frac{4}{3}}\eta^2(w) \tag{3.67}$$

Let's consider two velocity fields $u = u(X,t)$, and $v = v(X,t)$, satisfying the three dimensional Spectrally-Hyperviscous Navier-Stokes equations(SHNSE) under incompressible condition

$$u_t + \nu Au + \mu A_\varphi u + (u\cdot\nabla)u = f_1 - \nabla p_1 = g_1 \tag{3.68a}$$

$$\nabla \cdot u = 0 \tag{3.68b}$$

and

$$v_t + \nu Av + \mu A_\varphi v + (v\cdot\nabla)v = f_2 - \nabla p_2 = g_2 \tag{3.69a}$$

$$\nabla \cdot v = 0 \tag{3.69b}$$

corresponding to two different forcing terms $f_1 = f_1(X,t)$ and $f_2 = f_2(X,t)$; the corresponding pressure terms are $p_1 = p_1(X,t)$ and $p_2 = p_2(X,t)$. The boundary conditions for both problems are either no-slip on a bounded smooth domain or periodic with vanishing space average. We assume, as in the case of determining nodes, that f_1 and f_2 have the same asymptotic behavior, that is, $\int_\Omega \mid f_1(X,t) - f_2(X,t) \mid^2 dx \to 0$, as $t \to \infty$. let $w = u - v$, and we consider a set of N cubic box covering the domain Ω. We pick up one node from each cubic box,and we call them determining nodes, denoted by $\varepsilon = \left\{X^1, X^2, ...X^N\right\}$.

$$w_t + \nu Aw + \mu A_\varphi w + (u\cdot\nabla)\, u - (v\cdot\nabla)v = g_1 - g_2 = G \tag{3.70}$$

doing inner product between Aw and (3.70)

$$\frac{1}{2}\frac{d}{dt} \| A^{\frac{1}{2}}w \|_2^2 + (\nu Aw + \mu A_\varphi w, Aw) + ((u\cdot\nabla)\, u - (v\cdot\nabla)v, Aw) = (G, Aw) \tag{3.71}$$

the nonlinear term could be divided into two terms,

$$\mid ((u\cdot\nabla)\, u - (v\cdot\nabla)v, Aw) \mid \leq \mid ((w\cdot\nabla)u, Aw) \mid + \mid ((v\cdot\nabla)w, Aw) \mid \tag{3.72}$$

For all $\alpha \geq 2$, we have $-(\frac{\alpha-1}{2}) + \frac{1}{4} \leq 0$, and by the help of Lemma 21, the estimate for the high frequency part is

$$
\begin{aligned}
((w\cdot\nabla)u, Q_mAw) &= \mid Q_m(A^{(1-\alpha)/2}(w\cdot\nabla)u, Q_mA^{(\alpha+1)/2}w)\mid \\
&\leq \| Q_mA^{-(\alpha-1)/2+1/4}A^{-1/4}(w\cdot\nabla)u\|_2 \| Q_mA^{(\alpha+1)/2}w\|_2 \\
&\leq \lambda_m^{-(\alpha-1)/2+1/4} \| A^{-1/4}(w\cdot\nabla)u\|_2 \| Q_mA^{(\alpha+1)/2}w\|_2 \\
&\leq M_1^4\lambda_m^{-(\alpha-1-\frac{1}{2})}\frac{1}{2a} \| A^{\frac{1}{2}}w\|_2^2 \|\nabla u\|_2^2 + \frac{a}{2} \| Q_mA^{(\alpha+1)/2}w\|_2^2
\end{aligned}
\tag{3.73}
$$

For the low frequency part, by Lemma 21

$$
\begin{aligned}
\mid ((w\cdot\nabla)u, P_mAw)\mid &= \mid (P_m(w\cdot\nabla)u, P_mAw)\mid \\
&\leq \| P_m(w\cdot\nabla)u\|_2 \| P_mAw\|_2 \\
&= \| A^{\frac{1}{4}}A^{-\frac{1}{4}}P_m(w\cdot\nabla)u\|_2 \| P_mAw\|_2 \\
&\leq \lambda_m^{\frac{1}{4}} \| A^{-\frac{1}{4}}P_m(w\cdot\nabla)u\|_2 \| P_mAw\|_2 \\
&\leq M_1^2\lambda_m^{\frac{1}{4}} \| A^{\frac{1}{2}}w\|_2 \|\nabla u\|_2 \| Aw\|_2 \\
&\leq M_1^4\frac{1}{2b}\lambda_m^{\frac{1}{2}} \| A^{\frac{1}{2}}w\|_2^2 \|\nabla u\|_2^2 + \frac{b}{2} \| Aw\|_2^2
\end{aligned}
\tag{3.74}
$$

follow the same way in proving (3.73), (3.74). The estimate of the other two inner products is:

$$
\mid ((v\cdot\nabla)w, AQ_mw)\mid \leq M_1^4\lambda_m^{-(\alpha-\frac{3}{2})}\frac{1}{2a} \| A^{\frac{1}{2}}w\|_2^2 \|\nabla u\|_2^2 + \frac{a}{2} \| Q_mA^{(\alpha+1)/2}w\|_2^2 \tag{3.75}
$$

and

$$
\mid ((v\cdot\nabla)w, AP_mw)\mid \leq M_1^4\frac{1}{2b}\lambda_m^{\frac{1}{2}} \| A^{\frac{1}{2}}w\|_2^2 \|\nabla u\|_2^2 + \frac{b}{2} \| Aw\|_2^2 \tag{3.76}
$$

plug $(3.73),(3.74)$ and $(3.75),(3.76)$ into (3.72), we have:

$$\begin{aligned}|((u\cdot\nabla)\,u-(v\cdot\nabla)v,Aw)| \leq& |((w\cdot\nabla)u,Aw)|+|((v\cdot\nabla)w,Aw)|\\ &\leq M_1^4\lambda_m^{-(\alpha-\frac{3}{2})}\frac{1}{2a}\parallel A^{\frac{1}{2}}w\parallel_2^2(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)+a\parallel Q_mA^{(\alpha+1)/2}w\parallel_2^2\\ &+M_1^4\frac{1}{2b}\lambda_m^{\frac{1}{2}}\parallel A^{\frac{1}{2}}w\parallel_2^2(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)+b\parallel Aw\parallel_2^2\end{aligned} \tag{3.77}$$

for the linear term on the right of (3.71),

$(\nu Aw+\mu A_\varphi w,Aw)\geq\frac{\nu}{2}\parallel Aw\parallel_2^2+\frac{\mu}{2}\parallel Q_mA^{(\alpha+1)/2}w\parallel_2^2+\frac{1}{2}(\nu Aw+\mu A_\varphi w,Aw)$

Because $\nu>\mu,\alpha\geq 2$, $(\nu Aw+\mu A_\varphi w,Aw)\geq\mu(P_mAw+Q_mA^\alpha w,Aw)=\mu(A^2(P_mA^{-1}w+Q_mA^{\alpha-2}w),Aw)\geq\mu\lambda_m^{-1}\parallel A^{\frac{3}{2}}w\parallel_2^2$. Therefore

$$(\nu Aw+\mu A_\varphi w,Aw)\geq\frac{\nu}{2}\parallel Aw\parallel_2^2+\frac{\mu}{2}\parallel Q_mA^{(\alpha+1)/2}w\parallel_2^2+\frac{\mu}{2}\lambda_m^{-1}\parallel A^{\frac{3}{2}}w\parallel_2^2 \tag{3.78}$$

Insert $(3.77),(3.78)$ into (3.71), we have that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\parallel A^{\frac{1}{2}}w\parallel_2^2+\frac{\nu}{2}\parallel Aw\parallel_2^2+\frac{\mu}{2}\parallel Q_mA^{(\alpha+1)/2}w\parallel_2^2+\frac{\mu}{2}\lambda_m^{-1}\parallel A^{\frac{3}{2}}w\parallel_2^2\\ &\leq M_1^4\lambda_m^{-(\alpha-\frac{3}{2})}\frac{1}{2a}\parallel A^{\frac{1}{2}}w\parallel_2^2(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)+a\parallel Q_mA^{(\alpha+1)/2}w\parallel_2^2\\ &+M_1^4\frac{1}{2b}\lambda_m^{\frac{1}{2}}\parallel A^{\frac{1}{2}}w\parallel_2^2(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)+b\parallel Aw\parallel_2^2+\frac{1}{2d}\parallel A^{-\frac{1}{2}}G\parallel_2^2+\frac{d}{2}\parallel A^{\frac{1}{2}}w\parallel_2^2\end{aligned}$$

if we set $a=\frac{\mu}{2}$, $b=\frac{\nu}{2}$, then

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\parallel A^{\frac{1}{2}}w\parallel_2^2+\frac{\nu}{2}\parallel Aw\parallel_2^2+\frac{\mu}{2}\parallel Q_mA^{(\alpha+1)/2}w\parallel_2^2+\frac{\mu}{2}\lambda_m^{-1}\parallel A^{\frac{3}{2}}w\parallel_2^2\\ &\leq M_1^4\lambda_m^{-(\alpha-\frac{3}{2})}\frac{1}{\mu}\parallel A^{\frac{1}{2}}w\parallel_2^2(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)+\frac{\mu}{2}\parallel Q_mA^{(\alpha+1)/2}w\parallel_2^2\\ &+M_1^4\frac{1}{\nu}\lambda_m^{\frac{1}{2}}\parallel A^{\frac{1}{2}}w\parallel_2^2(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)+\frac{\nu}{2}\parallel Aw\parallel_2^2+\frac{1}{2d}\parallel A^{-\frac{1}{2}}G\parallel_2^2+\frac{d}{2}\parallel A^{\frac{1}{2}}w\parallel_2^2\end{aligned}$$

therefore

$$\frac{1}{2}\frac{d}{dt}\parallel A^{\frac{1}{2}}w\parallel_2^2+\frac{\mu}{2}\lambda_m^{-1}\parallel A^{\frac{3}{2}}w\parallel_2^2$$
$$\leq M_1^4\lambda_m^{-(\alpha-\frac{3}{2})}\frac{1}{\mu}\parallel A^{\frac{1}{2}}w\parallel_2^2(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)+M_1^4\frac{1}{\nu}\lambda_m^{\frac{1}{2}}\parallel A^{\frac{1}{2}}w\parallel_2^2(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)$$
$$+\frac{1}{2d}\parallel A^{-\frac{1}{2}}G\parallel_2^2+\frac{d}{2}\parallel A^{\frac{1}{2}}w\parallel_2^2$$

rewrite and only keep the force term on the right side

$$\frac{1}{2}\frac{d}{dt}\parallel A^{\frac{1}{2}}w\parallel_2^2+\frac{\mu}{2}\lambda_m^{-1}\parallel A^{\frac{3}{2}}w\parallel_2^2-M_1^4\lambda_m^{-(\alpha-\frac{3}{2})}\frac{1}{\mu}\parallel A^{\frac{1}{2}}w\parallel_2^2(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)$$
$$-M_1^4\frac{1}{\nu}\lambda_m^{\frac{1}{2}}\parallel A^{\frac{1}{2}}w\parallel_2^2(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)-\frac{d}{2}\parallel A^{\frac{1}{2}}w\parallel_2^2$$
$$\leq\frac{1}{2d}\parallel A^{-\frac{1}{2}}G\parallel_2^2 \tag{3.79}$$

After inserting (3.67) to (3.79) , we have that

$\frac{1}{2}\frac{d}{dt}\parallel A^{\frac{1}{2}}w\parallel_2^2+\{\frac{\mu}{2}\lambda_m^{-1}C_1\lambda_1^2N^{\frac{2}{3}}-M_1^4\lambda_m^{-(\alpha-\frac{3}{2})}\frac{1}{\mu}(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)-M_1^4\frac{1}{\nu}\lambda_m^{\frac{1}{2}}(\parallel\nabla v\parallel_2^2$
$+\parallel\nabla u\parallel_2^2)-\frac{d}{2}\}\parallel A^{\frac{1}{2}}w\parallel_2^2\leq\frac{1}{2d}\parallel A^{-\frac{1}{2}}G\parallel_2^2+2\nu\lambda_m^{-1}cN^{4/3}\lambda_1^{4/3}\eta^2(w)$

the coefficient of $\parallel A^{\frac{1}{2}}w\parallel_2^2$ is:

$\xi_1(t)=\{\frac{\mu}{2}\lambda_m^{-1}C_1\lambda_1^2N^{\frac{2}{3}}-M_1^4\lambda_m^{-(\alpha-\frac{3}{2})}\frac{1}{\mu}(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)-M_1^4\frac{1}{\nu}\lambda_m^{\frac{1}{2}}(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2$
$)-\frac{d}{2}\}$

then

$$\frac{1}{2}\frac{d}{dt}\parallel A^{\frac{1}{2}}w\parallel_2^2+\xi_1(t)\parallel A^{\frac{1}{2}}w\parallel_2^2\leq\frac{1}{2d}\parallel A^{-\frac{1}{2}}G\parallel_2^2+2\nu\lambda_m^{-1}cN^{4/3}\lambda_1^{4/3}\eta^2(w) \tag{3.80}$$

For applying Lemma 17, we need $\liminf_{t\to\infty}\frac{1}{T}\int_t^{t+T}\xi_1(\tau)d\tau>0$. Therefore

$$N^{\frac{2}{3}}>(\frac{2}{\mu}C_1^{-1}\lambda_1^{-2}M_1^4\lambda_m^{-(\alpha-\frac{5}{2})}\frac{1}{\mu}+\frac{2}{\mu}C_1^{-1}\lambda_1^{-2}M_1^4\frac{1}{\nu}\lambda_m^{\frac{3}{2}})\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}(\parallel\nabla v\parallel_2^2+\parallel\nabla u\parallel_2^2)d\tau$$
$$+\frac{d}{2}\frac{2}{\mu}\lambda_mC_1^{-1}\lambda_1^{-2}$$

Inserting $(2.36b)$ of Lemma 19, and because d can be arbitrarily small positive number and T can be arbitrarily large,

$$N^{\frac{2}{3}}>(\frac{1}{\mu}\lambda_m^{-(\alpha-\frac{5}{2})}+\frac{1}{\nu}\lambda_m^{\frac{3}{2}})\frac{4M_1^4}{\nu^2\mu\lambda_1^3C_1}F^2 \tag{3.81}$$

If (3.81) is satisfied, then $\xi_1(t) > 0$ and we may apply the Lemma 17 to (3.80): $\eta(t) \to 0$ and $\| A^{-\frac{1}{2}} G \|_2^2 \to 0$ as $t \to \infty$ implies $\| A^{\frac{1}{2}} w \|_2^2 \to 0$ as $t \to 0$.

so the number of determining nodes for the SHNSE is N, which has been bounded by (3.81) for $\alpha \geq 2$. This finishes the proof of Theorem 8. ■

3.2.2 The First Estimate for Determining Modes (Theorem 9)

Proof. The proof of determining modes begins from (3.76):

$$w_t + \nu A w + \mu A_\varphi w + (u \cdot \nabla) u - (v \cdot \nabla) v = g - g_1 = G \tag{3.82}$$

Let n be the number of determining modes, $n \geq m$.

So $A_\varphi \geq Q_m A^\alpha \geq Q_n A^\alpha$, with $\alpha \geq \frac{3}{2}$.

First taking the inner product between $Q_n A^\beta w$ and (3.82)

$$\frac{1}{2}\frac{d}{dt} \| Q_n A^{\frac{\beta}{2}} w \|_2^2 + (\nu A w + \mu A_\varphi w, Q_n A^\beta w) + ((u \cdot \nabla) u - (v \cdot \nabla) v, Q_n A^\beta w) = (G, Q_n A^\beta w) \tag{3.83}$$

For the nonlinear term

$$\begin{aligned} | ((u \cdot \nabla) u - (v \cdot \nabla) v, Q_n A^\beta w) | &\leq | ((w \cdot \nabla) u, Q_n A^\beta w) | + | ((v \cdot \nabla) w, Q_n A^\beta w) | \\ &\leq | ((P_n w \cdot \nabla) u, Q_n A^\beta w) | + | ((Q_n w \cdot \nabla) u, Q_n A^\beta w) | \\ &+ | ((v \cdot \nabla) P_n w, Q_n A^\beta w) | + | ((v \cdot \nabla) Q_n w, Q_n A^\beta w) | \end{aligned} \tag{3.84}$$

With $\alpha \geq \frac{3}{2}$, $1 < \beta < \frac{3}{2}$, and using Lemma 22 with $0 < -(\frac{\beta}{2} - \frac{3}{4}) < \frac{1}{4}$, the estimate for the first term is

$$
\begin{aligned}
&| ((P_n w \cdot \nabla)u, Q_n A^{\beta} w) | \\
=&| (Q_n A^{-\frac{\alpha-\beta}{2}}(P_n w \cdot \nabla)u, A^{\frac{\beta+\alpha}{2}} Q_n w) | \\
\leq&\| Q_n A^{-\frac{\alpha-\beta}{2}}(P_n w \cdot \nabla)u \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
=&\| Q_n A^{-(\frac{\alpha}{2}-\frac{3}{4})+(\frac{\beta}{2}-\frac{3}{4})}(P_n w \cdot \nabla)u \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
\leq& \lambda_n^{-(\frac{\alpha}{2}-\frac{3}{4})} \| Q_n A^{(\frac{\beta}{2}-\frac{3}{4})}(P_n w \cdot \nabla)u \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
\leq& M_1^2 \lambda_n^{-(\frac{\alpha}{2}-\frac{3}{4})} \| A^{\frac{\beta}{2}} P_n w \|_2 \| \nabla u \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
\leq& \frac{1}{2a} M_1^4 \lambda_n^{-(\alpha-\frac{3}{2})} \| A^{\frac{\beta}{2}} P_n w \|_2^2 \| \nabla u \|_2^2 + \frac{a}{2} \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2^2 \qquad (3.85)
\end{aligned}
$$

by Lemma 22, with $0 < -(\frac{\beta}{2} - \frac{3}{4}) < \frac{1}{4}$, the estimate for the forth term is

$$
\begin{aligned}
&| ((v \cdot \nabla) Q_n w, Q_n A^{\beta} w) | \\
=&| (A^{\frac{\beta-\alpha}{2}}(v \cdot \nabla) Q_n w, A^{\frac{\beta+\alpha}{2}} Q_n w) | \\
\leq&\| A^{\frac{\beta-\alpha}{2}}(v \cdot \nabla) Q_n w \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
=&\| A^{-(\frac{\alpha}{2}-\frac{3}{4})+(\frac{\beta}{2}-\frac{3}{4})}(v \cdot \nabla) Q_n w \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
\leq& \lambda_n^{-(\frac{\alpha}{2}-\frac{3}{4})} \| A^{(\frac{\beta}{2}-\frac{3}{4})}(v \cdot \nabla) Q_n w \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
\leq& M_1^3 \lambda_n^{-(\frac{\alpha}{2}-\frac{3}{4})} \| A^{\frac{\beta}{2}} Q_n w \|_2 \| \nabla v \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
\leq& \frac{1}{2a} M_1^6 \lambda_n^{-(\alpha-\frac{3}{2})} \| A^{\frac{\beta}{2}} Q_n w \|_2^2 \| \nabla v \|_2^2 + \frac{a}{2} \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2^2 \qquad (3.86)
\end{aligned}
$$

suppose $M_3 = max\,(M_1^6, M_1^4)$, then following the same way in (3.85) and (3.86), we can get the estimate of (3.84),

$$
\begin{aligned}
&| ((u \cdot \nabla)\, u - (v \cdot \nabla) v, Q_n A^{\beta} w) | \\
&\leq 2a \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2^2 + \frac{1}{2a} M_3 \lambda_n^{-(\alpha-\frac{3}{2})} \| A^{\frac{\beta}{2}} P_n w \|_2^2 \, (\| \nabla u \|_2^2 + \| \nabla v \|_2^2) \\
&+ \frac{1}{2a} M_3 \lambda_n^{-(\alpha-\frac{3}{2})} \| Q_n A^{\frac{\beta}{2}} w \|_2^2 \, (\| \nabla u \|_2^2 + \| \nabla v \|_2^2) \qquad (3.87)
\end{aligned}
$$

by $A_\varphi \geq Q_m A^\alpha$, and by $n \geq m$, $Q_m \geq Q_n$, the estimate for the linear term in (3.83) is

$$((\nu Aw + \mu A_\varphi w, Q_n A^\beta w)) \geq \nu \parallel A^{\frac{1+\beta}{2}} Q_n w \parallel_2^2 + \mu \parallel A^{\frac{\alpha+\beta}{2}} Q_n w \parallel_2^2 \tag{3.88}$$

Using young's inequality, the forcing term $(G, Q_n w)$ may be bounded as ,

$$(G, Q_n w) \leq \parallel A^{-\frac{\beta}{2}} G \parallel_2 \parallel Q_n A^{\frac{\beta}{2}} w \parallel_2 \leq \frac{1}{2d} \parallel A^{-\frac{\beta}{2}} G \parallel_2^2 + \frac{d}{2} \parallel Q_n A^{\frac{\beta}{2}} w \parallel_2^2 \tag{3.89}$$

Inserting (3.87), (3.88) and (3.89) into (3.83),it becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 + \nu \parallel A^{\frac{1+\beta}{2}} Q_n w \parallel_2^2 + \mu \parallel A^{\frac{\alpha+\beta}{2}} Q_n w \parallel_2^2 \\
&\leq 2a \parallel A^{\frac{\beta+\alpha}{2}} Q_n w \parallel_2^2 + \frac{1}{2a} M_3 \frac{1}{\lambda_n^{(\alpha-\frac{3}{2})}} \parallel A^{\frac{\beta}{2}} P_n w \parallel_2^2 (\parallel \nabla u \parallel_2^2 + \parallel \nabla v \parallel_2^2) \\
&+ \frac{1}{2a} M_3 \frac{1}{\lambda_n^{(\alpha-\frac{3}{2})}} \parallel Q_n A^{\frac{\beta}{2}} w \parallel_2^2 (\parallel \nabla u \parallel_2^2 + \parallel \nabla v \parallel_2^2) + \frac{1}{2d} \parallel A^{-\frac{\beta}{2}} G \parallel_2^2 + \frac{d}{2} \parallel Q_n A^{\frac{\beta}{2}} w \parallel_2^2
\end{aligned} \tag{3.90}$$

if we set $a = \mu/4$, then (3.90) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 + \nu \parallel A^{\frac{1+\beta}{2}} Q_n w \parallel_2^2 + \mu \parallel A^{\frac{\alpha+\beta}{2}} Q_n w \parallel_2^2 \\
&\leq \frac{\mu}{2} \parallel A^{\frac{\beta+\alpha}{2}} Q_n w \parallel_2^2 + \frac{2}{\mu} \frac{1}{\lambda_n^{(\alpha-\frac{3}{2})}} M_3 \parallel A^{\frac{\beta}{2}} P_n w \parallel_2^2 (\parallel \nabla u \parallel_2^2 + \parallel \nabla v \parallel_2^2) \\
&+ \frac{2}{\mu} \frac{1}{\lambda_n^{(\alpha-\frac{3}{2})}} M_3 \parallel Q_n A^{\frac{\beta}{2}} w \parallel_2^2 (\parallel \nabla u \parallel_2^2 + \parallel \nabla v \parallel_2^2) + \frac{1}{2d} \parallel A^{-\frac{\beta}{2}} G \parallel_2^2 + \frac{d}{2} \parallel Q_n A^{\frac{\beta}{2}} w \parallel_2^2
\end{aligned}$$

Inserting $\parallel A^{\frac{1+\beta}{2}} Q_n w \parallel_2^2 \geq \lambda_n \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2$ and $\parallel A^{\frac{\alpha+\beta}{2}} Q_n w \parallel_2^2 \geq \lambda_n^\alpha \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2$,

$$\begin{aligned}
&\frac{d}{dt} \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 + 2\nu\lambda_n \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 + \mu\lambda_n^\alpha \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 \\
&\leq \frac{4}{\mu} \frac{1}{\lambda_n^{(\alpha-\frac{3}{2})}} M_3 \parallel A^{\frac{\beta}{2}} P_n w \parallel_2^2 (\parallel \nabla u \parallel_2^2 + \parallel \nabla v \parallel_2^2) \\
&+ \frac{4}{\mu} \frac{1}{\lambda_n^{(\alpha-\frac{3}{2})}} M_3 \parallel Q_n A^{\frac{\beta}{2}} w \parallel_2^2 (\parallel \nabla u \parallel_2^2 + \parallel \nabla v \parallel_2^2) + \frac{1}{d} \parallel A^{-\frac{\beta}{2}} G \parallel_2^2 + d \parallel Q_n A^{\frac{\beta}{2}} w \parallel_2^2
\end{aligned} \tag{3.91}$$

assume $\alpha(t) = 2\lambda_n \nu + \mu\lambda_n^\alpha - \frac{4}{\mu} \frac{1}{\lambda_n^{(\alpha-\frac{3}{2})}} M_3 (\parallel \nabla u \parallel_2^2 + \parallel \nabla v \parallel_2^2) - d$,
and $\beta(t) = \frac{4}{\mu} \frac{1}{\lambda_n^{(\alpha-\frac{3}{2})}} M_3 \parallel A^{\frac{\beta}{2}} P_n w \parallel_2^2 (\parallel \nabla u \parallel_2^2 + \parallel \nabla v \parallel_2^2) + \frac{1}{d} \parallel A^{-\frac{\beta}{2}} G \parallel_2^2$

therefore (3.91) becomes

$$\frac{d}{dt} \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 + \alpha(t) \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 \leq \beta(t) \tag{3.92}$$

For applying Lemma 17 to (3.92), we need:

$$\liminf_{t \to +\infty} \frac{1}{T} \int_t^{t+T} \alpha(s) ds > 0$$

and considering d can be an arbitrarily small positive number,

$2\lambda_n \nu + \mu \lambda_n^{\alpha} > \frac{4}{\mu} \frac{1}{\lambda_n^{(\alpha - \frac{3}{2})}} M_3 \limsup_{t \to +\infty} \frac{1}{T} \int_t^{t+T} (\parallel \nabla u \parallel_2^2 + \parallel \nabla v \parallel_2^2) ds > 0$

Inserting (2.36b) of Lemma 19

$2\lambda_n \nu + \mu \lambda_n^{\alpha} > \frac{4}{\mu} \frac{1}{\lambda_n^{(\alpha - \frac{3}{2})}} M_3 (\frac{1}{\nu^2 \lambda_1} F^2 + \frac{F^2}{\lambda_1^2 \nu^3 T})$

Considering T can be arbitrarily large,

$$2\lambda_n \nu + \mu \lambda_n^{\alpha} > \frac{4}{\mu \nu^2 \lambda_1} \frac{1}{\lambda_n^{(\alpha - \frac{3}{2})}} M_3 F^2 \tag{3.93}$$

which finished the proof of Theorem 9. ■

If $\frac{\mu}{\nu}$ is not extremely small, $\mu \lambda_n^{\alpha} \gg 2\lambda_n \nu$. It is reasonable to generate a simpler and sufficient result from (3.83):

Corollary 24 *If* $\lambda_n \geq \sqrt[2\alpha - \frac{3}{2}]{\frac{4M_3}{\lambda_1}} * \sqrt[2\alpha - \frac{3}{2}]{\frac{F^2}{\mu^2 \nu^2}} = C(\frac{F}{\nu \mu})^{\frac{4}{4\alpha - 3}}$*, the estimate for the number of determining modes, (3.93) of Theorem 9, can be satisfied .*

Proof. $2\lambda_n \nu + \mu \lambda_n^{\alpha} > \mu \lambda_n^{\alpha}$, so if

$\mu \lambda_n^{\alpha} > \frac{4}{\mu \nu^2 \lambda_1} \frac{1}{\lambda_n^{(\alpha - \frac{3}{2})}} M_3 F^2$, (3.93) can also be satisfied.

$\mu \lambda_n^{\alpha} > \frac{4}{\mu \nu^2 \lambda_1} \frac{1}{\lambda_n^{(\alpha - \frac{3}{2})}} M_3 F^2$

$\lambda_n \geq \sqrt[2\alpha - \frac{3}{2}]{\frac{4M_3}{\lambda_1}} * \sqrt[2\alpha - \frac{3}{2}]{\frac{F^2}{\mu^2 \nu^2}} = C(\frac{F}{\nu \mu})^{\frac{4}{4\alpha - 3}}$ ■

3.2.3 The Second Estimate of Determining Modes (Theorem 10)

Proof. The estimate for the number of determining modes with $\alpha \geq \frac{5}{2}$, $\beta < \frac{3}{2}$ and $n \geq m$ does not need Lemma 17. After getting the same result as in (3.83),(3.84), we use Lemma 20 to estimate the nonlinear terms: by applying Lemma 20 with $\frac{5}{4} - \frac{\beta}{2} > \frac{1}{2}$, the estimate for the first term in (3.84) is

$$\begin{aligned}
&| ((P_n w\cdot\nabla)u, Q_n A^{\beta} w) | \\
&=| (Q_n A^{-\frac{\alpha-\beta}{2}}(P_n w\cdot\nabla)u, A^{\frac{\beta+\alpha}{2}} Q_n w) | \\
&\leq \| Q_n A^{-\frac{\alpha-\beta}{2}}(P_n w\cdot\nabla)u \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
&=\| Q_n A^{-(\frac{\alpha}{2}-\frac{5}{4})-(\frac{5}{4}-\frac{\beta}{2})}(P_n w\cdot\nabla)u \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
&\leq \lambda_n^{-(\frac{\alpha}{2}-\frac{5}{4})} \| Q_n A^{-(\frac{5}{4}-\frac{\beta}{2})}(P_n w\cdot\nabla)u \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
&\leq M_1^2 \lambda_n^{-(\frac{\alpha}{2}-\frac{5}{4})} \| A^{\frac{\beta}{2}} P_n w \|_2 \| u \|_2 \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2 \\
&\leq \frac{1}{2a} M_1^4 \lambda_n^{-(\alpha-\frac{5}{2})} \| A^{\frac{\beta}{2}} P_n w \|_2^2 \| u \|_2^2 + \frac{a}{2} \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2^2
\end{aligned} \tag{3.94}$$

similarly, the estimate to the fourth term is

$$\begin{aligned}
&| ((v\cdot\nabla)Q_n w, Q_n A^{\beta} w) | \\
&\leq \frac{1}{2a} M_1^4 \lambda_n^{-(\alpha-\frac{5}{2})} \| A^{\frac{\beta}{2}} Q_n w \|_2^2 \| v \|_2^2 + \frac{a}{2} \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2^2
\end{aligned} \tag{3.95}$$

Following the same choice of a in (3.94) and (3.95), and assuming $M_4 = M_1^4$, where M_1 is the same as in (2.1), we can get the estimate of (3.84),

$$\begin{aligned}
&| ((u\cdot\nabla)\, u - (v\cdot\nabla)v, Q_n A^{\beta} w) | \\
&\leq 2a \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2^2 + \frac{1}{2a} M_4 \lambda_n^{-(\alpha-\frac{5}{2})} \| A^{\frac{\beta}{2}} P_n w \|_2^2 (\| u \|_2^2 + \| v \|_2^2) \\
&+ \frac{1}{2a} M_4 \lambda_n^{-(\alpha-\frac{5}{2})} \| Q_n A^{\frac{\beta}{2}} w \|_2^2 (\| u \|_2^2 + \| v \|_2^2)
\end{aligned} \tag{3.96}$$

Inserting (3.96),(3.88) and (3.89) into (3.83), then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \| A^{\frac{\beta}{2}} Q_n w \|_2^2 + \nu \| A^{\frac{1+\beta}{2}} Q_n w \|_2^2 + \mu \| A^{\frac{\alpha+\beta}{2}} Q_n w \|_2^2 \\
&\leq \frac{\mu}{2} \| A^{\frac{\beta+\alpha}{2}} Q_n w \|_2^2 + \frac{2}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} \| A^{\frac{\beta}{2}} P_n w \|_2^2 (\| u \|_2^2 + \| v \|_2^2) \\
&+ \frac{2}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} \| Q_n A^{\frac{\beta}{2}} w \|_2^2 (\| u \|_2^2 + \| v \|_2^2) + \frac{1}{2d} \| A^{-\frac{\beta}{2}} G \|_2^2 + \frac{d}{2} \| Q_n A^{\frac{\beta}{2}} w \|_2^2
\end{aligned}$$

Therefore

$$\frac{d}{dt} \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 + 2\nu\lambda_n \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 + \mu\lambda_n^\alpha \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2$$
$$\leq \frac{4}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} \parallel A^{\frac{\beta}{2}} P_n w \parallel_2^2 (\parallel u \parallel_2^2 + \parallel v \parallel_2^2) + \frac{4}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} \parallel Q_n A^{\frac{\beta}{2}} w \parallel_2^2 (\parallel u \parallel_2^2 + \parallel v \parallel_2^2)$$
$$+ \frac{1}{d} \parallel A^{-\frac{\beta}{2}} G \parallel_2^2 + d \parallel Q_n A^{\frac{\beta}{2}} w \parallel_2^2$$

This can be rewrite as

$$\frac{d}{dt}\xi + a(t)\xi \leq b(t) \tag{3.97}$$

where $\xi = \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2$, $a(t) = 2\lambda_n \nu + \mu\lambda_n^\alpha - \frac{4}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} (\parallel u \parallel_2^2 + \parallel v \parallel_2^2) - d$,
and $b(t) = \frac{4}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} \parallel A^{\frac{\beta}{2}} P_n w \parallel_2^2 (\parallel u \parallel_2^2 + \parallel v \parallel_2^2) + \frac{1}{d} \parallel A^{-\frac{\beta}{2}} G \parallel_2^2$.
In $a(t)$ and $b(t)$ there are no ∇u as in Theorem 9. We just need to bound the L^2 norm of u. Inserting (2.36a) of Lemma 19, then

$$a(t) \geq 2\lambda_n \nu + \mu\lambda_n^\alpha - \frac{4}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} (2U_g^2) - d = a_2 \tag{3.98}$$

$$b(t) \leq \frac{4}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} \parallel A^{\frac{\beta}{2}} P_n w \parallel_2^2 (2U_g^2) + \frac{1}{d} \parallel A^{-\frac{\beta}{2}} G \parallel_2^2 = b_2(t) \tag{3.99}$$

by (3.97), (3.98), (3.99) and $\xi(t) = \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 \geq 0$, we have
$\frac{d}{dt}\xi + a_2\xi \leq \frac{d}{dt}\xi + a(t)\xi \leq b(t) \leq b_2(t)$
Therefore

$$\frac{d}{dt}\xi + a_2\xi \leq b_2(t) \tag{3.100}$$

Multiplying factor $exp(a_2 t)$ on both sides
$exp(a_2 t)\frac{d}{dt}\xi + a_2 exp(a_2 t)\xi \leq b_2(t) exp(a_2 t)$
therefore
$\frac{d}{dt}(\xi exp(a_2 t)) \leq b_2(t) exp(a_2 t)$
After doing integral on both sides from time τ to time t, it becomes
$\xi(t) exp(a_2 t) - \xi(\tau) exp(a_2 \tau) \leq \int_\tau^t b_2(s) exp(a_2 s) ds$
then
$\xi(t) \leq \xi(\tau) exp(-a_2(t-\tau)) + exp(-a_2 t) \int_\tau^t b_2(s) exp(a_2 s) ds$

if we suppose

$\rho_P(\tau,t) = \limsup_{\tau<s<t} \parallel A^{\frac{\beta}{2}} P_n w \parallel_2^2$

$\rho_G(\tau,t) = \limsup_{\tau<s<t} \parallel A^{-\frac{\beta}{2}} G \parallel_2^2$

and insert $b_2(t) = \frac{4}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} \parallel A^{\frac{\beta}{2}} P_n w \parallel_2^2 (2U_g^2) + \frac{1}{d} \parallel A^{-\frac{\beta}{2}} G \parallel_2^2$, then

$$\begin{aligned}
\xi(t) &\le \xi(\tau)\exp(-a_2(t-\tau)) + \exp(-a_2 t)\int_\tau^t \exp(a_2 s)(\frac{8}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} \parallel A^{\frac{\beta}{2}} P_n w \parallel_2^2 U_g^2) ds \\
&+ \exp(-a_2 t)\int_\tau^t \exp(a_2 s)(\frac{1}{d} \parallel A^{-\frac{\beta}{2}} G \parallel_2^2) ds \\
&\le \xi(\tau)\exp(-a_2(t-\tau)) + \exp(-a_2 t)\rho_P(\tau,t) U_g^2 \frac{8}{\mu} M_4 \frac{1}{\lambda_n^{\alpha-\frac{5}{2}}} \int_\tau^t \exp(a_2 s) ds \\
&+ \frac{1}{d}\exp(-a_2 t)\rho_G(\tau,t)\int_\tau^t \exp(a_2 s) ds \\
&= \xi(\tau)\exp(-a_2(t-\tau)) + [\rho_P(\tau,t) U_g^2 \frac{8}{\mu} M_4 \frac{1}{\lambda_n^{\alpha-\frac{5}{2}}} + \frac{1}{d}\rho_G(\tau,t)](1-\exp(-a_2(t-\tau)))
\end{aligned} \tag{3.101}$$

first let $t \to \infty$. If $a_2 > 0$,then $\exp(-a_2(t-\tau)) \to 0$, so (3.101) becomes

$$\overline{\lim}_{t\to\infty}\xi(t) \le [\rho_P(\tau,\infty) U_g^2 \frac{8}{\mu} M_4 \frac{1}{\lambda_n^{\alpha-\frac{5}{2}}} + \frac{1}{d}\rho_G(\tau,\infty)] \tag{3.102}$$

second, let $\tau \to \infty$. Because $\parallel A^{\frac{\beta}{2}} P_n w \parallel_2^2 \to 0$ and $\parallel A^{-\frac{\beta}{2}} G \parallel_2^2 \to 0$ as $t \to \infty$, therefore$\rho_P(\tau,\infty) \to 0$ and $\rho_G(\tau,\infty) \to 0$ as $\tau \to 0$. Then (3.102) becomes:

$\overline{\lim}_{t\to\infty}\xi(t) \le [\rho_P(\tau,\infty) U_g^2 \frac{8}{\mu} M_4 \frac{1}{\lambda_n^{\alpha-\frac{5}{2}}} + \frac{1}{d}\rho_G(\tau,\infty)] \to 0$ as $\tau \to 0$

and by $\xi \ge 0$, so $\xi \to 0$.

We have supposed that $a_2 > 0$ to obtain (3.102), so

$a_2 = 2\lambda_n \nu + \mu\lambda_n^\alpha - \frac{4}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} (2U_g^2) - d > 0$, therefore

$$2\lambda_n \nu + \mu\lambda_n^\alpha > \frac{8}{\mu} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} U_g^2 + d \tag{3.103}$$

which finished the proof of Theorem 10 and (3.103) is the estimate for the number of determining modes for $\alpha \ge \frac{5}{2}$. ∎

If $d = 2\lambda_n \nu$, (3.103) can be simplified as in Corollary 25.

Corollary 25 *Under the same condition in Theorem 10, the estimate for the number of determining modes is:*
$\lambda_n^{\alpha} > (\frac{8}{\mu^2} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} U_g^2)^{\frac{2}{4\alpha-5}}$ *and the estimate for* $\overline{\lim}_{t\to\infty} \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2$ *is*
$\overline{\lim}_{t\to\infty} \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 \leq [\rho_P(\tau,\infty) U_g^2 \frac{8}{\mu} M_4 \frac{1}{\lambda_n^{\alpha-\frac{5}{2}}} + \frac{1}{2\lambda_n \nu} \rho_G(\tau,\infty)]$

Proof. Suppose $d = 2\lambda_n \nu$, then we rewrite (3.102) and (3.103):

$$\lambda_n^{\alpha} > (\frac{8}{\mu^2} M_4 \frac{1}{\lambda_n^{(\alpha-\frac{5}{2})}} U_g^2)^{\frac{2}{4\alpha-5}} \tag{3.104}$$

$$\overline{\lim}_{t\to\infty} \parallel A^{\frac{\beta}{2}} Q_n w \parallel_2^2 \leq [\rho_P(\tau,\infty) U_g^2 \frac{8}{\mu} M_4 \frac{1}{\lambda_n^{\alpha-\frac{5}{2}}} + \frac{1}{2\lambda_n \nu} \rho_G(\tau,\infty)] \tag{3.105}$$

■

3.3 The Inviscid Limit Results (Proof of Theorem 12)

Let $w_\nu = u - u_\nu$; for simplicity we denote w_ν by w. Then subtracting (1.37a) from (1.38a) we obtain the following equations for w:

$$w_t + \nu A u_\nu + \mu A_\varphi w + (w\cdot\nabla)\, u + (u_\nu \cdot \nabla)\, w + \nabla P_\nu = 0 \tag{3.106}$$

where $P_\nu = p - p_\nu$. First we obtain an estimate for $Q_n w$. Taking the inner product of both sides of (3.106) with $Q_n w$ and noting that $(\nabla P_\nu, Q_n w) = 0$ we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|Q_n w\right\|_2^2 + (\nu A u_\nu, Q_n w) + \mu \left\|A^{\alpha/2} Q_n w\right\|_2^2 + (Q_n (w\cdot\nabla)\, u, Q_n w) + (Q_n (u_\nu\cdot\nabla)\, P_n w, Q_n w) = 0 \tag{3.107}$$

where we have noted that since $n \geq m$ and thus $Q_n A_\varphi w = Q_n A^\alpha Q_n$ we have that $(A_\varphi w, Q_n w) = (Q_n A_\varphi w, Q_n w) = (A_\varphi Q_n w, Q_n w) \geq (Q_n A^\alpha Q_n w, Q_n w) = (A^\alpha Q_n w, Q_n w)$ and that $((u_\nu\cdot\nabla)\, w, Q_n w) = ((u_\nu\cdot\nabla)\, P_n w, Q_n w) + ((u_\nu\cdot\nabla)\, Q_n w, Q_n w) = ((u_\nu\cdot\nabla)\, P_n w, Q_n w)$ since $((u_\nu\cdot\nabla)\, Q_n w, Q_n w) = 0$. Using that $((u_\nu\cdot\nabla)\, P_n w, Q_n w) = (A^{-\alpha/2}(u_\nu\cdot\nabla)\, P_n w, A^{\alpha/2} Q_n w)$, similarly for $((w\cdot\nabla)\, u, Q_n w)$, and using Young's inequality we have that

$$\begin{aligned} &\frac{d}{dt}\left\|Q_n w\right\|_2^2 + \mu\left\|A^{\alpha/2} Q_n w\right\|_2^2 \\ &\leq \frac{3\nu}{\mu}\left\|A^{(\alpha-2)/2} u_\nu\right\|_2^2 + \frac{3}{\mu}\left\|Q_n A^{-\alpha/2}(w\cdot\nabla)\, u\right\|_2^2 + \frac{3}{\mu}\left\|Q_n A^{-\alpha/2}(u_\nu\cdot\nabla)\, P_n w\right\|_2^2. \end{aligned} \tag{3.108}$$

Using (2.1) and the appropriate tensor product we have for $2 \leq \alpha \leq 5/2$ that

$$\begin{aligned}
\left\|Q_n A^{-\alpha/2}(w\cdot\nabla)u\right\|_2 &\leq \left\|A^{-\alpha/2}(w\cdot\nabla)u\right\|_2 \\
&= \left\|A^{-\alpha/2} div\,(w\otimes u)\right\|_2 \\
&= \left\|A^{-(\alpha-1)/2}\left(A^{-1/2} div\right)(w\otimes u)\right\|_2 \\
&= \left\|\left(A^{-1/2} div\right)A^{-(\alpha-1)/2}(w\otimes u)\right\|_2 \\
&\leq \left\|A^{-(\alpha-1)/2}(w\otimes u)\right\|_2 \\
&\leq M_1\left\|w\otimes u\right\|_p \leq M_1\left\|w\right\|_{rp}\left\|u\right\|_{sp} \qquad (3.109)
\end{aligned}$$

where $M_1 = M_1(\alpha/2, p, 2, L)$ and p satisfies $2 = 3p/(3-(\alpha-1)p)$ or $p = 6/(1+2\alpha)$. We want $sp = 2$ so that $s = (1+2\alpha)/3$ which using $1/r + 1/s = 1$ implies that $r = (2\alpha+1)/(2\alpha-2)$ which in turn means that $rp = 3/(\alpha-1)$. Using (2.1) again we have that

$$\left\|w\right\|_{rp} \leq M_2\left\|A^{\beta/2}w\right\|_2 \qquad (3.110)$$

provided that $3/(\alpha-1) = 6/(3-2\beta)$ or $5/2-\alpha = \beta$. Thus for $M_3 = M_1M_2$ and for β so chosen we have from (3.109) and (3.110) that

$$\left\|Q_n A^{-\alpha/2}(w\cdot\nabla)u\right\|_2 \leq M_3\left\|A^{\beta/2}w\right\|_2\left\|u\right\|_2. \qquad (3.111)$$

Setting $M_4 = (M_3)^2$ and using Poincaré and (2.38) we have that

$$\begin{aligned}
\left\|Q_n A^{-\alpha/2}(w\cdot\nabla)u\right\|_2^2 &\leq M_4\left\|A^{\beta/2}w\right\|_2^2\left\|u\right\|_2^2 \\
&= M_4\left\|A^{\beta/2}P_n w\right\|_2^2\left\|u\right\|_2^2 + M_4\left\|A^{\beta/2}Q_n w\right\|_2^2\left\|u\right\|_2^2 \\
&\leq \lambda_n^{5/2-\alpha}M_4\left\|P_n w\right\|_2^2\left\|u_0\right\|_2^2 + \frac{M_4}{\lambda_{n+1}^{2\alpha-5/2}}\left\|u_0\right\|_2^2\left\|A^{\alpha/2}Q_n w\right\|_2^2. \qquad (3.112)
\end{aligned}$$

Using similar arguments to those involved with (3.109)-(3.111), using (2.40), and the operator bound of A on P_nH_σ we have that

$$\begin{aligned}
\left\|Q_n A^{-\alpha/2}(u_\nu\cdot\nabla)P_n w\right\|_2^2 &\leq M_4\left\|A^{\beta/2}P_n w\right\|_2^2\left\|u_\nu\right\|_2^2 \\
&\leq \lambda_n^{5/2-\alpha}M_4\left\|P_n w\right\|_2^2\left\|u_{\nu,0}\right\|_2^2. \qquad (3.113)
\end{aligned}$$

If $\alpha = 5/2$ we have (3.111) with $\beta = 0$ so for the case $\alpha \geq 5/2$ we set $\gamma = \alpha/2 - 5/4$ in which case, using (3.111) and Poincaré we have that

$$\begin{aligned} \left\| Q_n A^{-\alpha/2} (w \cdot \nabla) u \right\|_2 &= \left\| A^{-\gamma} Q_n A^{-5/4} (w \cdot \nabla) u \right\|_2 \\ &\leq \lambda_{n+1}^{-\gamma} \left\| Q_n A^{-5/4} (w \cdot \nabla) u \right\|_2 \\ &\leq \lambda_{n+1}^{-\gamma} M_3 \|w\|_2 \|u\|_2 \end{aligned} \tag{3.114}$$

so that the versions of (3.112) and (3.113) for the case $\alpha \geq 5/2$ respectively become

$$\left\| Q_n A^{-\alpha/2} (w \cdot \nabla) u \right\|_2^2 \leq \lambda_{n+1}^{-(\alpha-5/2)} M_4 \|P_n w\|_2^2 \|u_0\|_2^2 + \frac{M_4}{\lambda_{n+1}^{\alpha-5/2}} \|u_0\|_2^2 \|Q_n w\|_2^2 \tag{3.115a}$$

$$\leq \lambda_{n+1}^{-(\alpha-5/2)} M_4 \|P_n w\|_2^2 \|u_0\|_2^2 + \frac{M_4}{\lambda_{n+1}^{2\alpha-5/2}} \|u_0\|_2^2 \left\| A^{\alpha/2} Q_n w \right\|_2^2 \tag{3.115b}$$

and

$$\left\| Q_n A^{-\alpha/2} (u_\nu \cdot \nabla) P_n w \right\|_2^2 \leq \lambda_{n+1}^{-(\alpha-5/2)} M_4 \|P_n w\|_2^2 \|u_{\nu,0}\|_2^2 . \tag{3.116}$$

Combining (3.108), (3.115a), and (3.116), we have for $\alpha \geq 5/2$ that

$$\begin{aligned} &\frac{d}{dt} \|Q_n w\|_2^2 + \mu \left\| A^{\alpha/2} Q_n w \right\|_2^2 \\ &\leq \frac{3\nu}{2\mu\lambda_1^{(\alpha-2)/2}} \|u_{\nu,0}\|_2^2 + \frac{3M_4}{\mu\lambda_{n+1}^{\alpha-5/2}} U_{\nu,0} \|P_m w\|_2^2 + \frac{3M_4}{\mu\lambda_{n+1}^{\alpha-5/2}} \|u_0\|_2^2 \|Q_m w\|_2^2 \end{aligned} \tag{3.117}$$

where we set

$$U_{\nu,0} = \|u_0\|_2^2 + \|u_{\nu,0}\|_2^2 . \tag{3.118}$$

For the case $2 \leq \alpha \leq 5/2$ we combine (3.108), (3.112), and (3.113) to obtain that

$$\begin{aligned} &\frac{d}{dt} \|Q_n w\|_2^2 + \mu \left\| A^{\alpha/2} Q_n w \right\|_2^2 \\ &\leq \frac{3\nu}{2\mu\lambda_1^{(\alpha-2)/2}} \|u_{\nu,0}\|_2^2 + \frac{3\lambda_n^{5/2-\alpha} M_4}{\mu} U_{\nu,0} \|P_n w\|_2^2 + \frac{3M_4}{\lambda_{n+1}^{2\alpha-5/2}} \|u_0\|_2^2 \left\| A^{\alpha/2} Q_n w \right\|_2^2 . \end{aligned} \tag{3.119}$$

Now choosing n large enough so that $(3/\mu)(M_4/\lambda_{n+1}^{2\alpha-5/2}) \|u_0\|_2^2 \leq \mu/2$ or

$$\frac{6M_4 \|u_0\|_2^2}{\mu^2} \leq \lambda_{n+1}^{2\alpha-5/2}, \tag{3.120}$$

we have from (3.117), (3.119), (3.120), and Poincaré that

$$\frac{d}{dt}\left\|Q_n w\right\|_2^2+\frac{\mu}{2}\lambda_{n+1}^{\alpha}\left\|Q_n w\right\|_2^2 \leq \frac{3\nu}{2\mu\lambda_1^{(\alpha-2)/2}}\left\|u_{\nu,0}\right\|_2^2+\frac{3C_n M_4}{\mu}U_{\nu,0}\left\|P_n w\right\|_2^2 \tag{3.121}$$

where $C_n=\lambda_n^{5/2-\alpha}$ for $2\leq\alpha\leq 5/2$ and $C_n=\lambda_{n+1}^{-(\alpha-5/2)}$ for $\alpha\geq 5/2$. Setting $d\equiv\mu/2$ and

$$\rho_P(t)\equiv\sup_{0\leq s\leq t}\left\|P_n w_N(s)\right\|_2^2 \tag{3.122}$$

we integrate (3.121) to obtain

$$\begin{aligned}
&\|Q_n w(t)\|_2^2 \leq \left\|Q_n w(0)\right\|_2^2 e^{-d\lambda_{n+1}^{\alpha}t}+\\
&\int_0^t e^{-d(t-s)\lambda_{n+1}^{\alpha}}\left[\frac{3\nu}{2\mu\lambda_1^{(\alpha-2)/2}}\left\|u_{\nu,0}\right\|_2^2+\frac{3C_n M_4}{\mu}\left\|P_n w\right\|_2^2\right]ds\\
&\leq \left\|Q_n w(0)\right\|_2^2 e^{-d\lambda_{m+1}^{\alpha}t}+\left[\frac{3\nu}{2\mu\lambda_1^{(\alpha-2)/2}}\left\|u_{\nu,0}\right\|_2^2+\frac{3C_n M_4}{\mu}U_{\nu,0}\rho_P(t)\right]\int_0^t e^{-d(t-s)\lambda_{n+1}^{\alpha}}ds\\
&\leq \left\|Q_n w(0)\right\|_2^2 e^{-d\lambda_{n+1}^{\alpha}}+\frac{3\nu}{\mu^2\lambda_1^{(\alpha-2)/2}\lambda_{n+1}^{\alpha}}\left\|u_{\nu,0}\right\|_2^2+\frac{6C_n M_4}{\mu^2\lambda_{n+1}^{\alpha}}U_{\nu,0}\rho_P(t).
\end{aligned} \tag{3.123}$$

which is (1.40).

To estimate $\|P_n w\|_2$ we take the inner product of both sides of (3.106) with $P_n w$ and in similar fashion to (3.107) obtain

$$\frac{1}{2}\frac{d}{dt}\left\|P_n w\right\|_2^2+(\nu A u_\nu,P_n w)+\mu\left\|A^{\alpha/2}P_n w\right\|_2^2+(P_n(w\cdot\nabla)u,P_n w)+(P_n(u_\nu\cdot\nabla)Q_n w,P_n w)=0. \tag{3.124}$$

Now $(P_n(w\cdot\nabla)u,P_n w)=(A^{-5/8}P_n(w\cdot\nabla)u,A^{5/8}P_n w)$ while with calculations similar to (3.110) we have for a constant M_5, which is the same as M_1 from (2.1), that

$$\begin{aligned}
\|A^{-5/8}P_n(w\cdot\nabla)u\|_2 &= \|A^{-5/8}A^{5/4}P_n A^{-5/4}div(w\otimes u)\|_2=\|A^{5/8}P_n A^{-3/4}A^{-1/2}div(w\otimes u)\|_2\\
&\leq \lambda_n^{5/8}\|A^{-1/2}divA^{-3/4}(w\otimes u)\|_2\leq\lambda_n^{5/8}\|A^{-3/4}(w\otimes u)\|_2\\
&\leq \lambda_n^{5/8}M_5\|w\otimes u\|_1\leq\lambda_n^{5/8}M_5\|w\|_2\|u\|_2\leq\lambda_n^{5/8}M_5\|u_0\|_2\|w\|_2
\end{aligned} \tag{3.125}$$

and similarly $(P_n(u_\nu\cdot\nabla)Q_nw, P_nw) = (A^{-5/8}P_n(u_\nu\cdot\nabla)Q_nw, A^{5/8}P_nw)$ with

$$\|A^{-5/8}P_n(u_\nu\cdot\nabla)Q_nw\|_2 \le \lambda_n^{5/8}M_5\|u_{\nu,0}\|_2\|Q_nw\|_2 \le \lambda_n^{5/8}M_5\|u_{\nu,0}\|_2\|w\|_2. \tag{3.126}$$

Combining the above inner-product arguments with (3.125) and (3.126), along with $(\nu Au_\nu, P_nw) = (\nu P_nA^{1/2}u_\nu, A^{1/2}P_nw)$, we have from (2.38), (3.124), using Young's inequality, and neglecting the term $\mu\left\|A^{\alpha/2}P_nw\right\|_2$, that

$$\begin{aligned}\frac{d}{dt}\|P_nw\|_2^2 &\le 3\nu\lambda_n\|u_{\nu,0}\|_2^2 + 3\lambda_n^{5/4}M_5U_{\nu,0}\|w\|_2^2 + 3\|A^{1/2}P_nw\|_2^2\\ &\le 3\nu\lambda_n\|u_{\nu,0}\|_2^2 + 3\lambda_n^{5/4}M_5U_{\nu,0}\|Q_nw\|_2^2 + 3(\lambda_n^{5/4}M_5U_{\nu,0}+\lambda_n)\|P_nw\|_2^2. \end{aligned}\tag{3.127}$$

We integrate (3.127) on $(0,t)$ for $0\le t\le T$ to obtain

$$\begin{aligned}\|P_nw\|_2^2 &\le 3\nu\lambda_n\|u_{\nu,0}\|_2^2T\\ &+ 3\lambda_n^{5/4}M_5U_{\nu,0}\int_0^t\|Q_nw\|_2^2\,ds + 3(\lambda_n^{5/4}M_5U_{\nu,0}+\lambda_n)\int_0^t\|P_nw\|_2^2\,ds\end{aligned}\tag{3.128}$$

which we combine with (3.122) and (3.123) to obtain

$$\begin{aligned}&\|P_nw\|_2^2 \le 3\nu\lambda_n\|u_{\nu,0}\|_2^2T\\ &+ 3\lambda_n^{5/4}M_5U_{\nu,0}T\left[\|Q_nw(0)\|_2^2 + \frac{3\nu}{\mu^2\lambda_1^{(\alpha-2)/2}\lambda_{n+1}^\alpha}\|u_{\nu,0}\|_2^2\right] + \frac{6C_nM_4}{\mu^2\lambda_{n+1}^\alpha}U_{\nu,0}\int_0^t\rho_P(t)\,ds+\\ &3(\lambda_n^{5/4}M_5U_{\nu,0}+\lambda_n)\int_0^t\|P_nw\|_2^2\,ds.\end{aligned}\tag{3.129}$$

This in turn after combining terms and taking appropriate suprema in the right order becomes

$$\begin{aligned}\rho_P(t) &\le \left(3\nu\lambda_n\|u_{\nu,0}\|_2^2 + 3\lambda_n^{5/4}M_5U_{\nu,0}\left[\|Q_nw(0)\|_2^2e^{-d\lambda_{m+1}^\alpha} + \frac{3\nu}{\mu^2\lambda_1^{(\alpha-2)/2}\lambda_{n+1}^\alpha}\|u_{\nu,0}\|_2^2\right]\right)T\\ &+\left[\frac{6C_nM_4}{\mu^2\lambda_{n+1}^\alpha}U_{\nu,0} + 3(\lambda_n^{5/4}M_5U_{\nu,0}+\lambda_n)\right]\int_0^t\rho_P(s)\,ds.\end{aligned}\tag{3.130}$$

Setting

$$V_{\nu,0} = 3\nu\lambda_n\|u_{\nu,0}\|_2^2 + 3\lambda_n^{5/4}M_5U_{\nu,0}\left[\|Q_nw(0)\|_2^2 + \frac{3\nu}{\mu^2\lambda_1^{(\alpha-2)/2}\lambda_{n+1}^\alpha}\|u_{\nu,0}\|_2^2\right]\tag{3.131}$$

and

$$W_{\nu,0} = \frac{6C_n M_4}{\mu^2 \lambda_{n+1}^{\alpha}} U_{\nu,0} + 3(\lambda_n^{5/4} M_5 U_{\nu,0} + \lambda_m) \tag{3.132}$$

we have from (3.130) - (3.132) that

$$\rho_P(t) \leq V_{\nu,0} T + W_{\nu,0} \int_0^t \rho_P(s)\, ds \tag{3.133}$$

from which we obtain by Gronwall that

$$\rho_P(t) \leq V_{\nu,0} T \exp W_{\nu,0} t \tag{3.134}$$

which is (1.42). This concludes the proof of Theorem 12.

CHAPTER 4: NUMERICAL ANALYSIS RESULT

4.1 Numerical examples

In our computational research we first focus on the simulations of decaying isotropic homogeneous turbulence since it is a more realistic idealization of a turbulent flow than the forced case. The initial condition in the present computational research follows [68] (see page 52 therein). In particular, the initial divergence free velocity field is a Gaussian field with a prescribed energy spectrum given by

$$E(k) = Ak^{\sigma} \exp\left(-\frac{\sigma}{2}\left(\frac{k}{k_p}\right)^2\right) / k_p^{\sigma+1} \,, \tag{4.1}$$

where σ determines the width of the distribution of $E(k)$, k_p is the wave number of the peak of the energy spetrum, and A is a normalizing factor, respectively. In all of the simulations presented in the present study we use $\sigma = 4$ and $k_p = 2$, but similar results can be obtained with other choices of σ and k_p as well.

In this section, we adopt the same definitions as those used in Ref. [18]: the mean velocity fluctuation u' is defined as

$$u' = \left(\frac{2}{3}\int_0^{\infty} E(k)\mathrm{d}k\right)^{1/2},$$

where the energy spectrum $E(k)$ is defined as

$$E(k) = \sum_{k-\frac{1}{2}\leq k' < k+\frac{1}{2}} u_{k'} \cdot u_{k'}.$$

The mean dissipation rate ϵ, and the Taylor microscale λ, are defined, respectively, as follows:

$$\epsilon = 2\nu\int_0^{\infty} k^2 E(k)\mathrm{d}k, \qquad \lambda = \left(\frac{15\nu}{\epsilon}\right)^{1/2} u'.$$

The large eddy turnover time τ, the Kolmogorov dissipation scale η, and the Taylor microscale Reynolds number Re_λ, are defined respectively as follows:

$$\tau = \frac{L_f}{u'}, \qquad \eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}, \qquad \mathrm{Re}_\lambda = \frac{u'\lambda}{\nu},$$

where L_f is the integral length. Note that the corresponding Kolmogorov wave number is $k_d = 1/\eta$. In addition, the total kinetic energy of the velocity field T, is define as $T = \sum_k E(k)$.

In our numerical experiments, the spatial computational domain is fixed as Ω=$[0, 2\pi]^3$. The resolution of the simulations, namely, the dimension of the velocity matrix is chosen as $N \times N \times N$ (denoted by N^3 in the sequel for simplicity); after dealiasing, we have $-N/2+1 \leq k_i \leq N/2$ for $i = 1, 2, 3$, and the corresponding cut-off wave number N_c is less than $\sqrt{3}N/2$. The computational resource available to the authors, however, limits the highest cut-off wave number N_c. And particularly in this paper, unless otherwise specified, the resolution of a simulation is 200^3, and the corresponding cut-off wave number is $N_c = 173$. Recall that the primary goal of this study is to explore the utility of the SHNSE in LES of turbulent flows at very high Reynolds numbers, so the Kolmogorov wave number k_d is assumed to be far beyond the computing ability, namely, $k_d \gg N$. On the other hand, without otherwise specified, plotted in the following figures are energy spectra at the nondimensional time $t = 3.2$, but in order to maintain a statistical steady state, most simulations are carried out for several large eddy turnover times before recording any data. The time step employed in the time evolution Δt, depending on many factors such as the resolution parameter N, the initial total energy T_0 and the hyperviscous coefficient μ, ranges between 2×10^{-4} and 1×10^{-3}. In addition, in our simulations, the default normalizing factor in (4.1) is chosen as $A = 8.53$, with corresponding total kinetic energy of the initial velocity field T_0 about .5.

For the spectrally hyperviscous term $\mu Q_m A^2 \mathbf{u}$, we choose $Q_m = 1$ as $k \geq M$ and $Q_m = 0$ as $k < M$, where M is a prescribed cut-off wave number. To give the hyperviscosity enough "space" to consume energy, we choose

$$M \leq M_0 = 0.8 \times \left(\frac{N}{2}\right), \tag{4.2}$$

and M should satisfy $M \leq k_d$. Unless otherwise specified, in this paper the cut-off wave number M used in a SHNSE simulation is set to M_0 given by (4.2). On the other hand, as to be discussed in detail in section 4.4, the hyperviscosity coefficient μ used in a SHNSE simulation is determined by the total energy using (4.3) below by default.

4.2 Comparison with DNS

As widely noted, for turbulent flows at high Reynolds numbers, it is impossible to perform LES using the NSE with a resolution of only 200^3 because the Kolmogorov wave number can be well over 1000 for these cases. On the contrary, since the spectrally hyperviscous term in the SHNSE can help dissipate energy for high frequency modes, it can keep large eddies running as an expensive DNS with as high as 4096^3 resolution does [39].

For turbulent flows of not very high Reynolds numbers, the results of SHNSE simulations are still similar to those obtained from DNS simulations if we choose the cut-off wave number M in the SHNSE close to the Kolmogorov wave number k_d. For instance, for a flow of viscosity $\nu = .001$, with the corresponding initial Taylor Reynolds number $\mathrm{Re}_\lambda = 187$ and the Kolmogorov wave number $k_d = 83$, LES have been carried out using the NSE and the SHNSE with the same $\mu = 2 \times 10^{-7}$, and M=80. In Fig. 4.1, the straight line represents a slope of $-5/3$, the dot-dashed line is by using NSE, and plus-dashed line is by using SHNSE with $M = 80$ close to k_d. It can be seen that all four simulations have comparable energy spectra, and clearly there is a well-developed $k^{-5/3}$ region over one decade in wave numbers.

4.3 Optimal choices of M

Intuitively, M should be placed in the neighborhood of the Kolmogorov wave number k_d so that the spectral accuracy in the inertial range is maintained. But a lower value

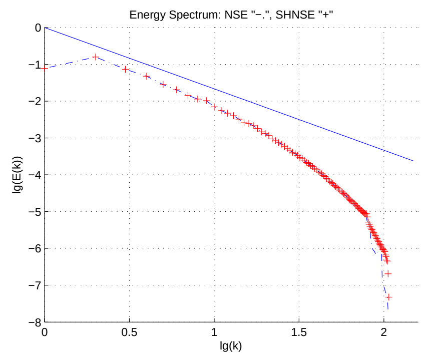
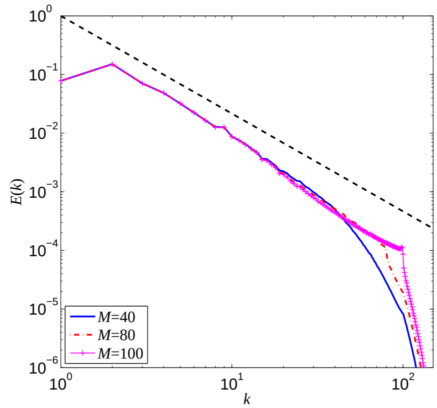


Figure 4.1: Energy Spectrum $E(k)$ at Re= 187.

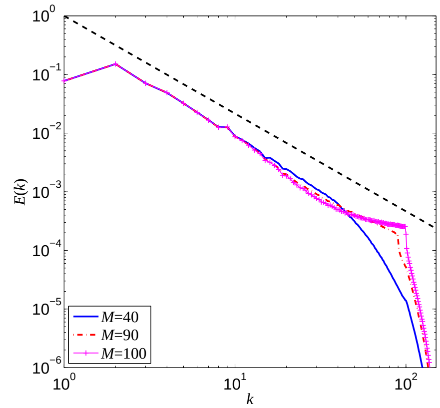
of this parameter means more significant degrees-of-freedom reduction. Therefore, it would be of interest to experiment with various values of this parameter to find the right balance between these considerations. To this end, we have carried out LES for three different flows using the SHNSE with different choices of M, and the energy spectra of these simulations are depicted in Fig. 4.2. More specifically, the Taylor Reynold number and the Kolmogorov wave number are Re=677 and k_d=262 for Fig. 4.2(a), Re=55951 and k_d=2906 for Fig. 4.2(b), and Re=5973270 and k_d=28155 for Fig. 4.2(c), respectively. The hyperviscosity coefficient μ is fixed at 2×10^{-7}. Also, since $N = 200$, the number M_0 as given by (4.2) is $M_0 = 80$.

As can be seen, the LES results by the SHNSE using $M \leq M_0$ are similar for all three cases. In particular, for the large eddy part which corresponds to $k \leq M$ (note that $\log(40) \approx 1.6$ and $\log(80) \approx 1.9$), the energy spectrum is parallel to $E(k) = k^{-5/3}$, while in the high frequency regime of $k \geq M$ where the hyperviscous term works, the energy spectrum decreases sharply. On the other hand, for the SHNSE simulations with $M > M_0$ such as $M = 90, 95$ or 100, although the corresponding energy spectrum curves are still parallel to $E(k) = k^{-5/3}$ for the large eddy part, they clearly have a bump at around $k = M$, which may be just because there is not enough "space" for the hyper-viscosity to absorb energy.

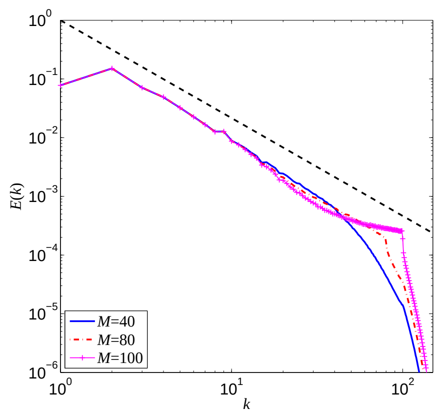
Recall that in our computational research, we adopt a step-like dependence of Q_m on the wave number k, namely, we choose $Q_m = 1$ as $k \geq M$ and $Q_m = 0$ as $k < M$. It should



(a)



(b)



(c)

Figure 4.2: Energy Spectrum $E(k)$ at Re$= 677, 55951, 5973270$.

be pointed out that we can also adopt an Arrhenius-like distribution function of the wave number suggested in [15] or a smoothed-out Heaviside function as an alternative to the Arrhenius-like distribution proposed by practitioners of the SVV method [70], namely, we prescribe two numbers M_1 and M_2 such that $M_1 < M_2 \leq k_d$, and then choose $Q_m = 1$ for $k \geq M_2$, $Q_m = 0$ for $k < M_1$, and Q_m as a monotonically increasing sequence between 0 to 1 for $M_1 < k < M_2$.

4.4 Optimal choices of μ

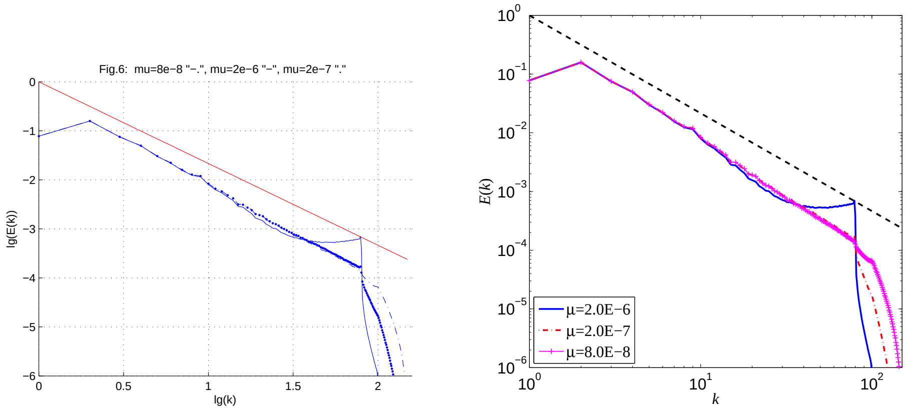
The SEV and SVV methodology suggests higher values of μ than the Chapman-Enskog value $\mu \sim \nu^\alpha$, and in [Figure 15, 15] measurements of what is essentially the ratio μ/ν are plotted against the wavenumber k. For the Kolmogorov constant set at 2.1 this ratio is on the order of 15 to 20 percent as k approaches the viscous range. This would seem to be a good starting point for the selection of μ in numerical implementation, but higher values would theoretically result in greater degrees-of-freedom reduction. In fact the coefficient of the extra viscosity kernels in the SVV implementation [39] is significantly larger than ν itself. This would suggest that larger values of μ may give better computational behavior without sacrificing spectral accuracy, a reasonable premise for numerical comparison.

By extensive numerical experiments, the hyperviscosity coefficient μ is found to depend ONLY on the total energy T. In particular, we obtain an empirical relation between μ and T as

$$\mu = 2 \times \sqrt{2T} \times 10^{-7}. \tag{4.3}$$

For example, in Fig. 4.3, we plot the energy spectra of two sets of SHNSE simulations with using several different μ values. The initial total energy is $T_0 = .5$, so by (4.3), μ should be 2×10^{-7}. The viscosity and the Taylor Reynolds number are $\nu = 1.73 \times 10^{-5}$ and $\mathrm{Re}_\lambda = 6608$ for Fig. 4.3 (a), and $\nu = 1.73 \times 10^{-8}$ and $\mathrm{Re}_\lambda = 6454850$ for Fig. 4.3 (b), respectively. The hyperviscosity coefficients tested are $\mu = 8 \times 10^{-8}$, 2×10^{-7}, and 2×10^{-6}, respectively. Note that Figure 4.3 clearly shows that if the spectrally hyperviscosity coefficient μ is less than that given by (4.3), nice LES results can be obtained (but of course with a higher computational cost), while on the contrary, if it is far larger than that given by (4.3), the LES will simply blow up. The figure a and b are the same, just because the difference of the total energy is tiny since the viscosities are very small and the Kolmogorov wavenumbers are far more than M.The straight line represents a slope of $-5/3$. (a) $\nu = 1.73 \times 10^{-5}$ and (b) $\nu = 1.73 \times 10^{-8}$.

On the other hand, we also carried out LES with the SHNSE using the same initial total energy $T_0 = .5$ but with different viscous coefficients, namely, $\nu = 1.73 \times 10^{-3}$, 1.73×10^{-4}, and 1.73×10^{-7}, respectively, and the resulting energy spectra are displayed in Fig. 4.4. The corresponding Taylor Reynolds number, determined by the initial total energy T_0 and the viscous coefficient ν, are 148, 733 and 645637, respectively. The hyperviscosity coefficient μ, calculated by (4.3), is 2×10^{-7}. As can be seen, the energy spectra are all parallel to $E(k) = k^{-5/3}$, except for the case of $\nu = 1.73 \times 10^{-3}$ whose



(a) (b)

Figure 4.3: Energy Spectrum $E(k)$ with Different Hyperviscous Coefficients.

Kolmogorov wavenumber is less than cutoff wavenumber.

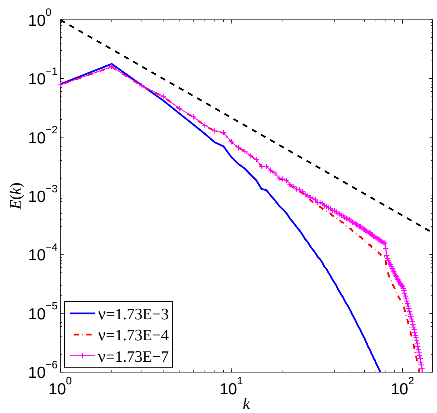


Figure 4.4: Energy Spectrum $E(k)$ with Different Viscous Coefficients.

4.5 LES for different N

To investigate how the choice of N affects the performance of the SHNSE simulations, we carried out several simulations using different N values, including $N = 120, 160, 200$, and the results are displayed in Fig. 4.5. In these simulations, the initial condition , the coefficients ν and μ (determined by (4.3)) are all fixed. The Taylor Reynolds number is $\text{Re}_\lambda = 6552$. Again, we note that the energy spectra of the large eddy part, saying $k \leq M$, are all parallel to $E(k) = k^{-5/3}$. Moreover, the numerical results show (4.2) and (4.3) are fit for different positive N.

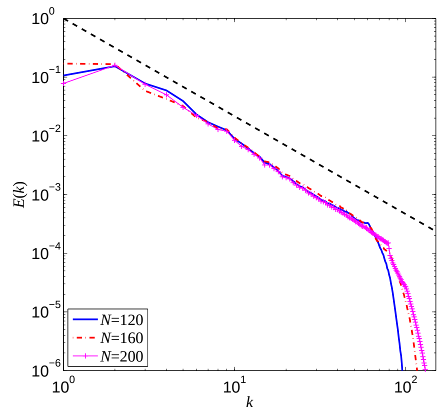


Figure 4.5: Energy Spectrum $E(k)$ with Different N.

4.6 LES for different initial total energy T_0

In modeling a decaying turbulent flow by any turbulence model, it is important to investigate the effects of the initial energy spectrum on the performance of the model. The initial velocity employed in the present paper, namely, that given by (4.1), has a sharply peaked initial energy spectrum with the peak of the energy spectrum at a relatively low wave number. We conducted LES by the SHNSE with using different initial total energy T_0, and the results are displayed in Fig. 4.6, where the viscosity $\nu = 1.73 \times 10^{-5}$. The initial total energies tested include $T_0 = 2, 15, 45$. The hyperviscosity coefficient μ is determined by (4.3), and the Taylor Reynolds numbers are 10852, 30791, and 48457 respectively. As can be observed, all the large scale energy spectra satisfy the Kolmogorov 5/3 rule, regardless of the magnitude of the initial total energy.

4.7 Energy transfer spectrum

To see how the energy cascades from the large scale modes to the small scale modes, we show the energy transfer spectra

$$\Pi(k) \equiv u(k) \cdot [\widehat{P(k)}(u \times \widehat{(\nabla \times u)})_k]. \tag{4.4}$$

as a function of the wave number k in Fig. 4.7 for several different cases, where the viscosity $\nu = 1.73 \times 10^{-5}$ and 1.73×10^{-8}, and the Taylor Reynolds number Re_λ =8383,

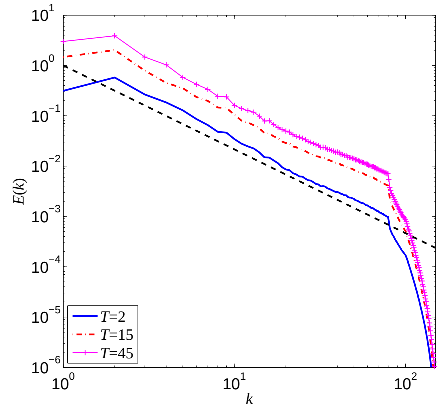


Figure 4.6: Energy Spectrum $E(k)$ with Different Initial Total Energy.

5973270, 38619 and 46260 for the four curves from the above to the bottom and the straight line represents a slope of 3/5. Because here the Reynolds numbers are very high, all the energy spectra are still in the inertial range. We see that slops for all the energy spectra under $k \leq M$ are almost a constant, which indicates that the SHNSE preserves the fundamental properties for Kolmogorov energy cascades in the inertial range very well. For the dissipation range ($k > M$), however, the change of the energy spectrum is significant for all cases.

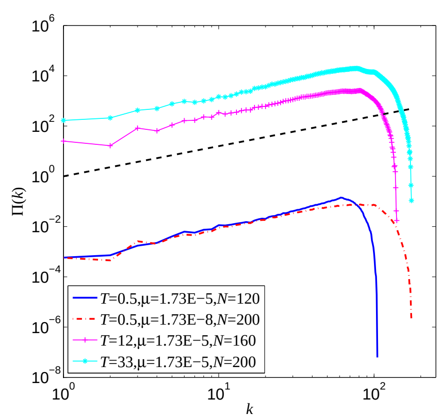


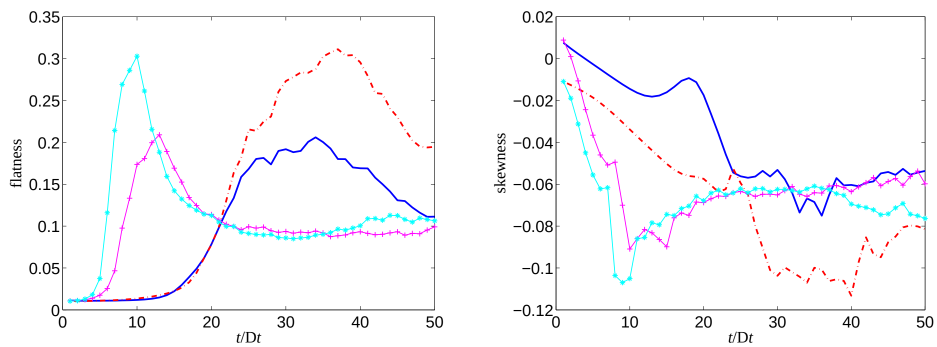
Figure 4.7: Energy Transfer Spectrum $\Pi(k)$.

4.8 Skewness and Flatness

In Fig. 4.8, we show the skewness and flatness. Under the same viscosity and total energy, flatness and skewness curves for different N has similar patterns. For different total energy, the positions of the peak are different. The coefficient using in the four simulations are: Dashed line: $T = 0.5, \mu = 1.73 \times 10^{-5}$ and $N = 120$; Dot-dashed line: $T = 0.5, \mu = 1.73 \times 10^{-8}$ and $N = 200$; Plus-dashed line: $T = 12, \mu = 1.73 \times 10^{-5}$ and $N = 160$; Star-dashed line: $T = 33, \mu = 1.73 \times 10^{-5}$ and $N = 200$.

$$skewness = < u_{1,1}^3 > / (< u_{1,1}^2 >)^{3/2} \tag{4.5}$$

$$flatness = < u_{1,1}^4 > / (< u_{1,1}^2 >)^2 \tag{4.6}$$

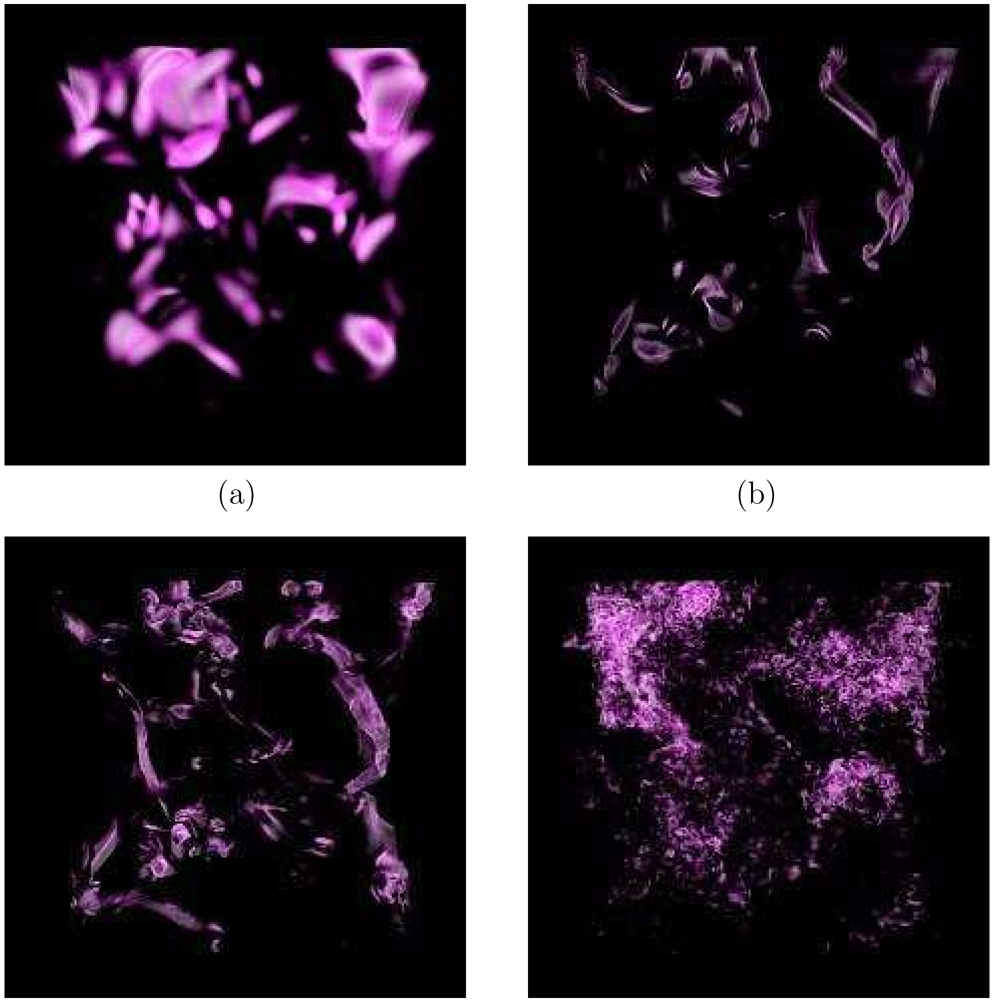


(a) (b)

Figure 4.8: Flatness (a) and Skewness (b).

4.9 3D visualization of turbulence

Finally, in Fig. 4.9 we show several snapshots of simulations of isotropic homogeneous decaying turbulence with $Re = 426$. The data are from 200^3 simulation by the SHNSE with using $\nu = 1.73 \times 10^{-4}$, $\mu = 2 \times 10^{-7}$. As can be seen, the images appear to record in a pretty decent way a transition to full turbulence. In the Fig. 4.9a, the energy concentrate in large eddies, while in Fig. 4.9d, significant energy has been transferred from large eddies to small eddies. The vector norm of the flow velocity data are calculated and rendered in 3D. Then a 2D histogram is constructed from norm values at their gradient magnitude and used to select interesting data regions. In these 4 images, we select high gradient regions, meaning that they have highest pressures in the data.



(a) (b) (c) (d)

Figure 4.9: Snapshots of Turbulence at Time 1.2, 2.4, 3.6, 4.8.

CHAPTER 5: CONCLUSION

Since (2.13c) and (2.13d) hold for all $\beta \leq 3/2$, we can for the moment replace β by $3/2$ in Theorems 5 and 6 and use Poincaré for $\beta < 3/2$ to note that

$$\lambda_{m+1}^{(3/2)-\beta} \left\| A^{\beta/2} Q_m w(t) \right\|_2^2 \leq \left\| A^{3/4} Q_m w(t) \right\|_2^2 . \tag{5.1}$$

Using (5.1) on the left-hand sides of the estimates (1.21) and (1.25) and dividing through by $\lambda_{m+1}^{(3/2)-\beta}$ we obtain stronger results that reduce the size of our estimates. For example, applying this procedure to the estimate (1.25) we obtain that

$$\begin{aligned} \sup_{0\leq t\leq T} \left\| A^{\beta/2} Q_m w(t) \right\|_2^2 \leq & \frac{2}{\lambda_{m+1}^{(3/2)-\beta}} \left\| A^{3/4} Q_m w_N(0) \right\|_2^2 + \frac{2}{\lambda_{m+1}^{(3/2)-\beta}} \mathcal{F}_{Q,N}(t) \\ & + \frac{16 C_1}{\mu^2 \lambda_{m+1}^{2\alpha+3/2-(\beta+5/2)}} \left(5 + 3\eta + \frac{\nu\lambda_1}{\mu \lambda_{m+1}^{\alpha}} \right) \left(\frac{L_g}{\nu\lambda_1} \right)^2 \sup_{0\leq t\leq T} \left\| A^{3/4} P_m w(t) \right\|_2^2 \end{aligned} \tag{5.2}$$

where we note that now we replace β by $3/2$ in the definition of $\mathcal{F}_{Q,N}(t)$.

Note that this paper reinforces the significance of certain "magic numbers" related to the exponent α in spectral hyperviscosity. For $\alpha \geq 5/4$ we have global existence of regular solutions; the assumption $\alpha \geq 3/2$ was essential to our attractor and inertial-manifold results in [4] as well as several of our estimates in section 2.1 above; and the assumption $\alpha \geq 5/2$ gave the best estimates for the size of the inertial manifolds constructed in [4] as well as giving here the easiest case of our spectral-decomposition-dependent estimates.

We have noted that all of our results here that use spectral decomposition have similarities to those involving bounds on the number of determining modes, as well as to those involving the existence of inertial manifolds. Such estimates for the NSE and its closure models, along with estimates for the number of degrees of freedom, are a measure of the complexity of the system. Further results in this regard include lower bounds on the dimension of the attractor for the 2-d NSE; see the discussion and references in [28] and [75]. Another interesting way to obtain a lower-bound estimate on the complexity is to provide upper bounds on the size of the nodal set for the vorticity, as was done in [50, 51] for periodic solutions of the 2-d NSE.

It would have been fairly straightforward to also include results on the convergence of the attractors and inertial manifolds of the Galerkin systems to those corresponding to (1.1), along the lines of the corresponding results in [31], [76]. We hope to discuss these in a separate paper in which we will also seek to establish trajectory-tracking results in which trajectories of Galerkin solutions will be compared with trajectories on the attractor for (1.1). In any future exploration of spectrally-hypervisous models, we expect that the sharpest results will be obtained by employing spectral-decomposition techniques similar to the ones developed in [4] as well as those appearing in these pages. In essence, this approach reflects an underlying philosophy of treating turbulence modeling as a multiscale phenomenon.

In [28, p.136], the estimate of the determining nodes is $N > CG^2$ for two dimensional Navier-Stokes equation. By (3.81) in Theorem 8 , the estimates for determining nodes of three dimensional SHNSE is $N > CG^3$.

In [28, p.128], the estimates for determining modes of two dimensional Navier-Stokes equation with no-slip boundary condition is $\lambda_n > \frac{2c_1 F}{\nu^2 \lambda_1} = CG^2$, with $G = \frac{F}{\lambda_1 \nu^2}$ the Grashof number. In [28, p.130], the estimates for determining modes of two dimensional Navier-Stokes equation with periodic boundary condition is $\lambda_n > \frac{2c_1 F}{\nu^2 \lambda_1} = CG$. Comparing with the results of three dimensional SHNSE in Theorem 9: $\lambda_n > C(\frac{F}{\nu\mu})^{\frac{4}{4\alpha-3}} = CG^{\frac{4}{4\alpha-3}}$, if $\alpha > \frac{7}{4}$, the exponent of Grashof number is less than 1, which means the number of determining modes for three dimensional SHNSE is less than those for two dimensional space-periodic case NSE.

In Theorem 10, we not only obtain the estimate for the number of determining modes in (3.103), but also get an explicit expression for the estimate of $\| A^{\frac{\beta}{2}} Q_n w \|_2^2$ in (3.101) and (3.102). 'd' is a common constant in (3.101), (3.102) and (3.103). The choice of d depends on case by case analysis. In future research, numerical experiments will be held to further explore the properties of turbulence, especially in the context of the hyperviscosity term $\mu Q_m A^\alpha$. To obtain the optimal choice of μ, α and m, lots of numerical experiments are necessary to be executed. Right now, it is one of my research directions.

Our spectral decomposition techniques used to establish Theorem 12 have their roots in the study of determining modes ([22], [29], [40], also see [28, chapter III]), and the theory of inertial manifolds ([30], [31], [76], [4]), which seek in some sense to show that the high-frequency modes are dominated and controlled by the low-frequency modes. The estimate (1.38) reflects this dependence by showing that most of the dynamical input for the high-frequency modes comes from the low frequencies via the source term involving $\rho_P(t)$. Our methods here and in [5] resemble those used in the determining-mode theory, but only at the outset; we have had to develop new techniques to avoid estimates which are exponential in the data (see the key relevant lemmas in e.g. [27]). Our techniques differ from those in [5] in that we need to avoid estimates which depend on ν.

That (1.38) is a good approximation for the Euler system ((1.38) with $\mu = 0$) is strongly supported as noted by the results in [72], [73], [15] and the SVV applications to 2-d versions of (1.38) in [2] and [41]. To connect the Euler system with (1.38) more rigorously, we believe Tadmor's techniques using compensated compactness could be adapted to obtain similar convergence results to those in [72] as $m \to \infty$. In fact a straightforward argument using the overlapping assumption that $\left\| A^\beta u(t) \right\|_2$ is bounded for any $\beta > 0$ results in subsequence convergence to a weak solution of the Euler system on any interval $[0, T]$ since in this case by compacness we have a subsequence converging in $L^2(\Omega)$. On local intervals $[0, T]$ of regularity for solutions of the Euler system, we believe methods similar to those applied in [61] to the 2-d NSE as $\nu \to 0$ could be applied to show strong convergence to solutions the Euler system on $[0, T]$ as $\mu \to 0$ or $m \to \infty$; the arguments would also resemble those used in [60] in a similar situation involving the NS-α and Lagrangian-averaged Euler equations. Since the methods of characteristics and compensated compactness needed are somewhat beyond the scope of this paper, we leave these convergence considerations for future work.

Our primary object for the computational analysis has been to investigate the utility of the recently proposed SHNSE for three-dimensional isotropic homogeneous turbulence in a periodic box with periodic boundary conditions. Our systematic numerical study has demonstrated that the SHNSE model has real potential to be a highly robust and accurate platform for studying and modeling turbulence while simultaneously reducing the number of degrees of freedom required for accurate simulation. In particular, our numerical experiments have shown that: (1) The SHNSE simulations can achieve significantly closer agreement with the Kolmogorov energy power law than that achieved by DNS ([18, 64]), with significantly lower computational cost, and (2) Stable numerical experiments with Reynolds numbers on the order of 10^6 using the SNSE are still with very good agreement with the Kolmogorov energy power law and with manageable computational cost. In addition, the results *appear to reflect self-similarity features and* suggest remarkably that the coefficient of the spectral hyperviscous term is independent of the viscosity coefficient, depending instead on a small fixed parameter times the square root of the initial total energy.

However, before the SHNSE can be used as a practical platform in modeling industrial flows, a number of other issues need to be addressed. Most significantly, since higher powers of the Laplacian are applied to high wave number modes, basic numerical stability considerations require the use of smaller time steps in these ranges. Also, since the model is inherently connected with spectral methods, significant adaptation needs to be made to handle the case of general domains and non-periodic boundary conditions. To address the first issue, we propose to develop spectrally-sensitive techniques based on the principles of exponential time differencing [9, 25] or some multiple time stepping strategies based on a quasi-steady-state assumption for the high wave numbers. On the other hand, to address the second issue, we propose to first develop and implement space-continuous spectral element methods [47, 59, 66, 10, 78] to solve the system (1.1) on a general domain or with non-peroidic boundary conditions, and then model the benchmark wall-bounded turbulent channel flow [10,43,44,65] or even the more challenging flow over the "Ahmed body" car model [1,36,49,55,56,62,63] which has a Reynolds number as high as 768,000. We will report any significant progress in these directions in future publications.

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