Blow-Up Analysis for Focusing Many-Body Quantum Systems

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Contents

Zusammenfassung

In dieser Dissertation wird die Bildung von Singularitäten in fokussierenden quantenmechanischen Vielteilchensystemen untersucht. Insbesondere wird das Verhalten der Grundzustandsenergie und der zugehörigen Grundzustände solcher Systeme analysiert, wenn diese sich dabei befinden, zu kollabieren.

Artikel A und B sind der Untersuchung von Neutronensternen gewidmet, welche klassische Beispiele für fokussierende, fermionische Vielteilchensysteme sind. Bekanntlich kollabieren Neutronensterne, wenn ihre Masse die sogenannte kritische Chandrasekhar-Masse übersteigt. Mathematisch besteht der Kollaps des Sterns darin, dass die Grundzustandsenergie gleich minus Unendlich ist. Hier wird der Vorgang des Kollapses im Rahmen zweier Näherungsmodelle genauer untersucht, nämlich der Chandrasekhar-Theorie und der Hartree–Fock–Bogoliubov-Theorie. Wir analysieren das asymptotische Verhalten der Energie im massenkritischen Limes und zeigen, dass ein universelles Singularitätenprofil auftritt, welches eine Lösung der Lane–Emden-Gleichung ist.

Artikel C und D behandeln Bosonensterne. Obwohl die Existenz solcher Sterne derzeit nicht durch astronomische Beobachtungen belegt ist, stellen sie für die Kosmologie und die mathematische Physik ein interessantes Studienmodell dar. Ähnlich wie Neutronensterne kollabieren Bosonensterne bei genügend großer Masse. Wir untersuchen das Kollaps-Phänomen im Rahmen zweier Modelle, nämlich der Hartree-Theorie und der vollständigen Vielteilchentheorie. Bei letzterer ist es nötig, ein externes Potential einzuführen, um die Existenz eines Grundzustandes zu garantieren. Im massenkritischen Limes zeigen wir, dass Kondensation der Grundzustände auf die Menge der optimierenden Funktionen einer nicht-lokalen Interpolationsungleichung auftritt.

Artikel E befasst sich mit Kondensatgemischen aus Bosegasen im Rahmen der vollständigen quantenmechanischen Vielteilchentheorie. Wir betrachten die Grundzustandsenergie eines nicht-relativistischen, bosonischen fokussierenden Vielteilchensystems, welches aus zwei verschiedenen Teilchenspezies besteht, die sich jeweils in einem lokalisierenden Potential befinden. Die Wechselwirkung der Teilchen innerhalb derselben Spezies ist dabei attraktiv, wohingegen die Wechselwirkung zwischen den verschiedenen Spezies attraktiv oder repulsiv sein kann. Im Grenzwert, der den Kollaps beschreibt, zeigen wir, dass die Grundzustände Bose–Einstein-Kondensation aufweisen und, bis auf Reskalierung, zu dem Optimierer der Gagliardo–Nirenberg-Interpolationsungleichung konvergieren.

Abstract

This thesis is focused on the blow-up analysis for focusing many-body quantum systems. The central object of study is the behavior of the ground state energies and ground states in the collapse regime.

Papers A and B are concerned with neutron stars which are classic examples of focusing fermionic many-body systems. It is a fundamental fact that neutron stars collapse as soon as their masses exceed the so-called Chandrasekhar limit mass. Mathematically, the collapse corresponds to the unboundedness from below of the ground state energy. Here we study the details of the collapse in two approximate models: the Chandrasekhar theory and the Hartree–Fock–Bogoliubov theory. We investigate the asymptotic behavior of the energy in the mass critical limit and prove that the ground states develop a universal blow-up profile which solves the Lane–Emden equation.

Papers C and D treat boson stars. Until now, there is no observational evidence that such stars exist. Nevertheless, they are interesting objects in astronomy and mathematics. Similarly to neutron stars, boson stars collapse when their masses are too big. We will study the collapse phenomenon in two models: the Hartree theory and the full many-body theory. For the latter, we have to include an external potential to guarantee the existence of ground states. In the mass critical limit, we show that the ground states condensate on the optimizers of a non-local interpolation inequality.

Paper E deals with the mixture condensate of Bose gases in the full many-body quantum theory. We consider the ground state energy of a confined, non-relativistic bosonic many-body system consisting of two species in the focusing regime and assume attractive intra-species and either attractive or repulsive inter-species interactions between the particles. In the collapse regime, we show that the ground states exhibit the Bose–Einstein condensation and, up to rescaling, converge to the optimizer of the Gagliardo–Nirenberg interpolation inequality.

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Preface

This thesis is concerned with some focusing many-body quantum systems. In the collapse regime, we provide the asymptotic behavior of the ground state energies and the convergence of ground states in terms of the single (or double, for mixture) reduced density (matrices). This interpolates material from five papers by the author which are listed in the following by their alpha numerals.

List of Publications

[A] *Blow-Up Profile of Neutron Stars in the Chandrasekhar Theory*,
Journal of Mathematical Physics, 60 (2019), p. 071508.
DOI 10.1063/1.5085277.

[B] *Blow-Up Profile of Neutron Stars in the Hartree–Fock–Bogoliubov Theory*,
Calculus of Variations and Partial Differential Equations, 58 (2019), p. 202.
DOI 10.1007/s00526-019-1641-x.

[C] *On Blow-Up Profile of Ground States of Boson Stars with External Potential*,
Journal of Statistical Physics, 169 (2017), pp. 395–422.
DOI 10.1007/s10955-017-1872-1.

[D] *Many-Body Blow-Up Profile of Boson Stars with External Potentials*,
Review in Mathematical Physics, 31 (2019), p. 1950034.
DOI 10.1142/S0129055X1950034X.

[E] *Blow-Up Profile of 2D Focusing Mixture Bose Gases*,
Zeitschrift für Angewandte Mathematik und Physik, 71 (2020), p. 81.
DOI 10.1007/s00033-020-01302-y.

CHAPTER 1

Introduction

In this chapter, we introduce the mathematical models for relativistic gravitational fermion and boson stars and for non-relativistic mixture of Bose gases. In the end of this introduction we shall give a short summary of the papers included in this thesis.

1.1. Relativistic Gravitational Fermions and Bosons

In this section, let us consider N relativistic quantum particles of mass $m > 0$, which are either all fermions or all bosons in $\mathbb{R}^3$. The N-particle gravitating system is described by the Hamiltonian

$$H_N = \sum_{i=1}^{N} \sqrt{-\Delta_{x_i} + m^2} - \kappa \sum_{1 \le i < j \le N} |x_i - x_j|^{-1}, \tag{1.1.1}$$

where $\kappa = gm^2 > 0$ and g is the Newton's gravitational constant. The Hamiltonian H_N in (1.1.1) consists of the pseudo-relativistic kinetic operator $\sqrt{-\Delta + m^2}$ and an attractive Newtonian interaction potential. Here, we use units such that Planck's constant $\hbar$ and the speed of light c satisfy $\hbar = c = 1$.

An N-fermion wave function describing the system (1.1.1) is a normalized function in the Hilbert space $\mathcal{H}_N := \bigwedge^N L^2(\mathbb{R}^3, \mathbb{C}^q)$ which is anti-symmetric with respect to exchange of particles. Here $\bigwedge$ stands for the anti-symmetric tensor product, $L^2(\mathbb{R}^3)$ is the space of square-integrable one-particle wave functions on physical space $\mathbb{R}^3$ and $\mathbb{C}^q$ is the space of states of the spin degrees of freedom $q \ge 1$. For simplicity, we will assume that particles are spinless ($q = 1$) because the spin number does not play any important role in our analysis. On the other hand, an N-boson wave function is a normalized function in the Hilbert space $\mathcal{H}_N := \bigotimes_{\text{sym}}^N L^2(\mathbb{R}^3)$ which is symmetric with respect to exchange of particles. Here $\bigotimes_{\text{sym}}$ stands for the symmetric tensor product.

The ground state energy per particle of the N particles is the bottom of the spectrum of $\frac{H_N}{N}$, defined by

$$E_N^{\mathrm{Q}} := \frac{1}{N} \inf \operatorname{spec} H_N = \frac{1}{N} \inf \left\{ \langle \Psi_N, H_N \Psi_N \rangle : \Psi_N \in \mathcal{H}_N, \int_{\mathbb{R}^{3N}} |\Psi_N(x)|^2 \mathrm{d}x = 1 \right\}. \tag{1.1.2}$$

Since (1.1.1) is an attractive many-body quantum system, it may collapse, in the sense that $E_N^{\mathrm{Q}} = -\infty$. In fact, after a mean-field approximation, kinetic and gravitational energies behave the same under scaling, so that there is a critical number of particles above which the system is unstable. The maximum number of particles of a stable star is the famous **Chandrasekhar limit mass**. It is named after the physicist Chandrasekhar who computed this number in 1930 and earned the 1983 Nobel Prize in Physics for it. We remark that the number of particles needed for collapse does not depend on the mass-factor $m > 0$ in the kinetic energy. This follows from a standard scaling argument together with the operator inequality

$$\sqrt{-\Delta} \leq \sqrt{-\Delta + m^2} \leq \sqrt{-\Delta} + m^2. \tag{1.1.3}$$

In this thesis, we are interested in the behavior of the ground state energy per particles E_N^{Q} and its ground states in the collapse regime. Although the many-body problem is linear, it is very complicated to analyze because there are too many variables. Therefore, it is useful to introduce approximate one-body models which are non-linear but easier to deal with.

1.1.1. Chandrasekhar Theory of Neutron Stars. The Chandrasekhar theory is the relativistic analogue of the famous Thomas–Fermi theory of non-relativistic electrons in atomic physics. It involves two semi-classical approximations. First, for the kinetic energy,

$$\left\langle \Psi, \sum_{i=1}^{N} \sqrt{-\Delta_{x_i} + m^2}\Psi \right\rangle \approx \int_{\mathbb{R}^3} j_m(\rho_\Psi(x)) \mathrm{d}x.$$

Here the one-particle density ρ_Ψ associated to the anti-symmetric N-particle wave function Ψ is given by

$$\rho_\Psi(x) := N \int_{\mathbb{R}^{3(N-1)}} |\Psi(x, x_2, \ldots, x_N)|^2 \mathrm{d}x_2 \ldots \mathrm{d}x_N.$$

The function $j_m(\rho)$ is obtained by integrating $\sqrt{-\Delta + m^2}$ over the subset of phase-space in which $\sqrt{-\Delta} \leq (6\pi^2\rho)^{\frac{1}{3}} =: \eta$. It is given by

$$j_m(\rho) := \frac{1}{16\pi^2}\left[\eta(2\eta^2 + m^2)\sqrt{\eta^2 + m^2} - m^4 \ln\left(\frac{\eta + \sqrt{\eta^2 + m^2}}{m}\right)\right]. \tag{1.1.4}$$

The second approximation we need in the Chandrasekhar theory is the approximation between particles self-interaction

$$\left\langle \Psi, \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1}\Psi \right\rangle \approx \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho_\Psi(x)\rho_\Psi(y)}{|x - y|} \mathrm{d}x\mathrm{d}y.$$

This is commonly used in the study of quantum Coulomb systems of charged particles [**34**]. Putting together the above approximations, we obtain the Chandrasekhar functional

$$\mathcal{E}_\kappa^{\mathrm{Ch}}(\rho) = \int_{\mathbb{R}^3} j_m(\rho(x)) \mathrm{d}x - \frac{\kappa}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x - y|} \mathrm{d}x\mathrm{d}y. \tag{1.1.5}$$

The Chandrasekhar energy is defined by

$$E_\kappa^{\mathrm{Ch}}(N) := \inf\left\{\mathcal{E}_\kappa^{\mathrm{Ch}}(\rho) : 0 \leq \rho \in L^1 \cap L^{\frac{4}{3}}(\mathbb{R}^3), \int_{\mathbb{R}^3} \rho(x)\mathrm{d}x = N\right\}. \tag{1.1.6}$$

By rescaling $\tilde{\rho}(x) = \rho(N^{\frac{1}{3}}x)$ we have

$$\mathcal{E}_\kappa^{\mathrm{Ch}}(\rho) = N\mathcal{E}_\tau^{\mathrm{Ch}}(\tilde{\rho}) \quad \text{and} \quad E_\kappa^{\mathrm{Ch}}(N) = NE_\tau^{\mathrm{Ch}}(1),$$

where

$$\tau = \kappa N^{\frac{2}{3}}.$$

We note that, by (1.1.3),

$$E_\tau^{\mathrm{Ch}}(1)_{|m=0} \leq E_\tau^{\mathrm{Ch}}(1) \leq E_\tau^{\mathrm{Ch}}(1)_{|m=0} + m.$$

In case $m = 0$, since the kinetic and potential energies behave the same under scaling, there exists a critical value τ_c such that $E_\tau^{\mathrm{Ch}}(1)_{|m=0} = -\infty$ as soon as $\tau > \tau_c$. Therefore, the

Chandrasekhar limit mass for neutron stars is $\left(\frac{\tau_c}{\kappa}\right)^{\frac{3}{2}}$. In fact, τ_c is the optimal constant in the Hardy–Littlewood–Sobolev inequality

$$\frac{\tau_c}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho(x)\rho(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \le K_{\mathrm{cl}}\|\rho\|_{L^{\frac{4}{3}}}^{\frac{4}{3}}\|\rho\|_{L^1}^{\frac{2}{3}}, \quad \forall 0\le\rho\in L^1\cap L^{\frac{4}{3}}(\mathbb{R}^3), \tag{1.1.7}$$

where $K_{\mathrm{cl}} = \frac{3}{4}(6\pi^2)^{\frac{1}{3}}$.

By standard methods in the calculus of variations, one can prove that there is a (unique, up to translations) minimizer for $E_\tau^{\mathrm{Ch}}(1)$ as long as $\tau < \tau_c$. In this subcritical regime, Lieb–Yau [**36**] proved that the Chandrasekhar theory is a correct description of the full many-body theory in the limit $\kappa \to 0$ and $N \to \infty$ with $\tau = \kappa N^{\frac{2}{3}}$ fixed.

In Paper A, we consider the Chandrasekhar variational problem. We analyze in detail the blow-up behavior of the Chandrasekhar energy $E_\tau^{\mathrm{Ch}}(1)$ and its minimizer when $\tau \nearrow \tau_c$. This is the first step to understand the stellar collapse.

1.1.2. Hartree–Fock–Bogoliubov (HFB) Theory of Neutron Stars. While the Chandrasekhar theory involves only the density functionals, the Hartree–Fock–Bogoliubov (HFB) theory involves the one-body density matrices associated to the quasi-free states in Fock space. See Bach–Lieb–Solovej [**3**] for the general rigorous discussion on the HFB theory. The HFB functional of neutron stars associated to (1.1.1) is given by

$$\mathcal{E}_\tau^{\mathrm{HFB}}(\gamma,\alpha) = \mathrm{Tr}\left(\sqrt{-\Delta+m^2}\gamma\right) - \frac{\kappa}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_\gamma(x)\rho_\gamma(y) - |\gamma(x,y)|^2 + |\alpha(x,y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{1.1.8}$$

Here we use the subscript $\tau = \kappa N^{\frac{2}{3}}$, which will play an important role in our analysis. The terms in the integral are the so-called *direct term*, *exchange term* and *pairing term*, respectively. The density matrix γ is a non-negative self-adjoint operator on $L^2(\mathbb{R}^3,\mathbb{C})$ and $\rho_\gamma(x) = \gamma(x,x)$. The pairing density matrix α is a Hilbert–Schmidt operator on $L^2(\mathbb{R}^3,\mathbb{C})$, i.e., $\mathrm{Tr}\,\alpha^*\alpha < \infty$, and its kernel is a (2×2)-matrix which is assumed to be anti-symmetric in the sense $\alpha^T = -\alpha$. The HFB minimization problem associated to (1.1.8) reads

$$E_\tau^{\mathrm{HFB}}(N) = \inf\left\{\mathcal{E}_\tau^{\mathrm{HFB}}(\gamma,\alpha) : (\gamma,\alpha)\in\mathcal{K}_{\mathrm{HFB}}, \mathrm{Tr}\,\gamma = N\right\}, \tag{1.1.9}$$

where the set of HFB states is given by

$$\mathcal{K}_{\mathrm{HFB}} = \left\{(\gamma,\alpha) = (\gamma^*, -\alpha^T)\in\mathcal{X}_{\mathrm{HFB}} : \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix} \le \begin{pmatrix}\gamma & \alpha\\ \alpha^* & 1-\overline{\gamma}\end{pmatrix} \le \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\right\} \tag{1.1.10}$$

with the Sobolev-type space $\mathcal{X}_{\mathrm{HFB}}$ being defined as

$$\mathcal{X}_{\mathrm{HFB}} := \left\{(\gamma,\alpha)\in\mathfrak{S}_1\times\mathfrak{S}_2 : \|(1-\Delta)^{\frac{1}{4}}\gamma(1-\Delta)^{\frac{1}{4}}\|_{\mathfrak{S}_1} + \|(1-\Delta)^{\frac{1}{4}}\alpha\|_{\mathfrak{S}_2} < \infty\right\}.$$

Here $\mathfrak{S}_p$, with $1\le p<\infty$, denotes the Schatten class of operators acting on $L^2(\mathbb{R}^3,\mathbb{C})$.

Without the pairing term, (1.1.8) becomes the Hartree–Fock (HF) functional. It is the expectation value of H_N in (1.1.1) in a determinantal wave function $\Psi = \psi_1\wedge\ldots\wedge\psi_N$ made of an orthonormal family $\{\psi_i\}_{i=1}^N$ in $L^2(\mathbb{R}^3,\mathbb{C})$. Such a wave function is also called Slater determinant and can be rewritten as

$$\Psi(z_1,\ldots,z_N) := (N!)^{-\frac{1}{2}}\det\{\psi_i(z_j)\}_{i,j=1}^N. \tag{1.1.11}$$

The density matrix γ_Ψ associated to Ψ in (1.1.11) is a finite rank orthogonal projection, i.e.,[1]

$$\gamma_\Psi(x,y) := N\int_{\mathbb{R}^{3(N-1)}} \overline{\Psi(x,x_2,\dots,x_N)}\Psi(y,x_2,\dots,x_N)\mathrm{d}x_2\dots\mathrm{d}x_N = \sum_{1\le i\le N}\overline{\psi_i(x)}\psi_i(y).$$

It is straightforward from the above that the HF energy is an upper bound to the ground state energy. On the other hand, it can be seen from Bessel's inequality that $0\le\gamma_\Psi\le 1$. This is the condition to take the Pauli exclusion principle into account. In practice, one could ignore this feature and apply (1.1.8) to any mixed state γ satisfying $0\le\gamma\le 1$. In fact, it was confirmed by Lenzmann–Lewin [**24**] that if a non-trivial pairing term $\alpha\not\equiv 0$ is taken into account, then a HFB minimizer (if it exists) has infinite rank. Furthermore, the appearance of an attractive pairing term will decrease the HF energy and one hopes that the HFB theory approximates the full many-body Schrödinger theory better than the Chandrasekhar theory.

The existence of HFB minimizers has been proved by Lenzmann–Lewin [**24**]. It is obtained for $0<N<N^{\mathrm{HFB}}(\kappa)$, $0<\kappa<4/\pi$ and $m>0$. The finite number $N^{\mathrm{HFB}}(\kappa)$ is asymptotically equivalent, as $\kappa\to 0$, to the Chandrasekhar limit mass $\left(\frac{\tau_c}{\kappa}\right)^{\frac{3}{2}}$.

In Paper B, we analyzed the behavior of the HFB energy and its minimizers when $N\to\infty$ simultaneously $\tau:=\tau_N\nearrow\tau_c$. We remark that the following operator inequality for $(\gamma,\alpha)\in\mathcal{K}_{\mathrm{HFB}}$ can be derived from (1.1.10) (see [**3**])

$$\gamma^2+\alpha\alpha^*\le\gamma. \tag{1.1.12}$$

In the mean field limit of large N with $\kappa N^{\frac{2}{3}}$ kept fixed, the exchange and pairing terms are of smaller order compared with the direct term, due to their coupling with the small parameter $\kappa=\mathcal{O}(N^{-\frac{2}{3}})$. In fact, by (1.1.12) and the Hardy–Kato inequality $|x-y|^{-1}\le\frac{\pi}{2}\sqrt{-\Delta_x}$ (see, e.g., [**35**, Lemma 8.2]), we have

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\gamma(x,y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y\le\frac{\pi}{2}\operatorname{Tr}(\sqrt{-\Delta}\gamma^2)\le\frac{\pi}{2}\operatorname{Tr}(\sqrt{-\Delta}\gamma) \tag{1.1.13}$$

and

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\alpha(x,y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y\le\frac{\pi}{2}\operatorname{Tr}(\sqrt{-\Delta}\alpha\alpha^*)\le\frac{\pi}{2}\operatorname{Tr}(\sqrt{-\Delta}\gamma). \tag{1.1.14}$$

Therefore, they do not show up in the leading order of the blow-up profile.

1.1.3. Hartree Theory of Boson Stars. For bosons, the Hartree theory is the simplest semi-classical theory which can be obtained by assuming that all particles are independent and identically distributed. This leads to the celebrated non-linear model introduced by Hartree [**17**]. Mathematically, it corresponds to the choice of the wave function

$$\Psi_N(x_1,\dots,x_N)=\prod_{i=1}^N u(x_i), \tag{1.1.15}$$

for some normalized $u\in H^{\frac{1}{2}}(\mathbb{R}^3)$. Computing the ground state energy gives $\langle\Psi_N,H_N\Psi_N\rangle = N\mathcal{E}^{\mathrm{H}}_\omega(u)$ where $\omega=\kappa N$ and[2]

$$\mathcal{E}^{\mathrm{H}}_\omega(u):=\|(-\Delta+m^2)^{\frac{1}{4}}u\|^2_{L^2}-\frac{\omega}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{1.1.16}$$

[1] For fermions, the one-particle density matrix is not normalized to have trace equal to one.

[2] In fact, the coupling constant in (1.1.16) is $\frac{\omega}{2}\left(1-\frac{1}{N}\right)$ but we discarded $\frac{1}{N}$ because we are interested in the limit $N\to\infty$.

The Hartree energy is defined by

$$E^{\mathrm{H}}_{\omega}(1) := \inf\left\{\mathcal{E}^{\mathrm{H}}_{\omega}(u) : u \in H^{\frac{1}{2}}(\mathbb{R}^3), \int_{\mathbb{R}^3} |u(x)|^2 \mathrm{d}x = 1\right\}. \tag{1.1.17}$$

We note that, by (1.1.3),

$$E^{\mathrm{H}}_{\omega}(1)_{|m=0} \le E^{\mathrm{H}}_{\omega}(1) \le E^{\mathrm{H}}_{\omega}(1)_{|m=0} + m.$$

In case $m = 0$, since the kinetic and potential energies behave the same under scaling, there exists a critical value ω_c such that $E^{\mathrm{H}}_{\omega}(1)_{|m=0} = -\infty$ as soon as $\omega > \omega_c$. Therefore, the Chandrasekhar limit mass for boson stars is $\frac{\omega_c}{\kappa}$. In fact, ω_c is the optimal constant in the Gagliardo–Nirenberg-type inequality

$$\frac{\omega_c}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} \mathrm{d}x\mathrm{d}y \le \|(-\Delta)^{\frac{1}{4}}u\|^2_{L^2}\|u\|^2_{L^2}, \quad \forall u \in H^{\frac{1}{2}}(\mathbb{R}^3). \tag{1.1.18}$$

The value ω_c is not known explicitly but we have that $4/\pi < \omega_c < 2.7$ where the lower bound follows from the Hardy–Kato inequality $|x-y|^{-1} \le \frac{\pi}{2}\sqrt{-\Delta_x}$.

By standard methods in the calculus of variations, one can prove that there exists a ground state for $E^{\mathrm{H}}_{\omega}(1)$ in (1.1.17) as long as $\omega < \omega_c$. In this subcritical regime, Lieb–Yau [**36**] proved that the Hartree theory is a correct to the leading order of the full many-body theory in the limit $\kappa \to 0$ and $N \to \infty$ with $\omega = \kappa N$ fixed. Obviously, if $u \in H^{\frac{1}{2}}(\mathbb{R}^3)$ in (1.1.15) minimizes (1.1.16), then the Hartree energy $E^{\mathrm{H}}_{a}(1)$ is an upper bound to the ground state energy per particle E^{Q}_{N}. The lower bound is the most difficult part of the proof in [**36**]. This result was later obtained again by Lewin–Nam–Rougerie [**29**] with a different method that exploits the quantum de Finetti theorem.

In Paper C, where we consider the Hartree variational problem (1.1.17) with or without external potentials, we analyze in detail the blow-up behavior of the Hartree energy $E^{\mathrm{H}}_{\omega}(1)$ and its ground states when $\omega \nearrow \omega_c$. This is the first step to understand the gravitational collapse of boson stars.

1.1.4. Many-Body Boson Stars with External Potentials. Our goal is to study the gravitational collapse of boson stars from the view point of quantum mechanics. In that case, Bose–Einstein condensates are formulated by the k-particle reduced density matrices associated to the ground state Ψ_N. It is defined by taking the partial trace of the orthogonal projection $\gamma_{\Psi_N} := |\Psi_N\rangle\langle\Psi_N|$ over the last $N-k$ particles[3]

$$\gamma^{(k)}_{\Psi_N} := \mathrm{Tr}_{k+1\to N}\,\gamma_{\Psi_N}. \tag{1.1.19}$$

Equivalently, $\gamma^{(k)}_{\Psi_N}$ is defined as a non-negative trace class operator on $L^2_{\mathrm{sym}}(\mathbb{R}^{3k})$ with kernel

$$\gamma^{(k)}_{\Psi_N}(x_1,\dots,x_k;y_1,\dots,y_k) = \int_{\mathbb{R}^{3(N-k)}} \overline{\Psi_N(x_1,\dots,x_k;Z)}\Psi_N(y_1,\dots,y_k;Z)\mathrm{d}Z.$$

We remark that (1.1.1) has no ground states, due to its translation invariance. A way out of this difficulty is to include an external potential, as we do here. A similar technique was

[3]For bosons, we use the convention that the k-particle reduce density matrices are normalized to have trace equal to one, for all $k \in \mathbb{N}$.

used by Lieb–Yau [**36**] for neutron stars. We consider the following modified Hamiltonian[4]

$$H_N = \sum_{i=1}^{N} \left(\sqrt{-\Delta_{x_i} + m^2} + V(x_i) \right) - \frac{\omega}{N-1} \sum_{1\le i<j\le N} |x_i - x_j|^{-1}, \tag{1.1.20}$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is an external potential. For the sake of simplicity, we consider the case where V is trapping. Then the existence of many-body ground states is easily obtained, whenever $0 < \omega < \omega_c$, by a standard compactness argument.

In Paper D, we study the asymptotic behaviors of the ground state energy per particle E_N^{Q} of the system (1.1.20) and its ground states in the limit $\omega_N = \kappa N \nearrow \omega_c$ at the same time as the mean-field limit $N \to \infty$ is taken. Our main tool is the quantum de Finetti theorem and its quantitative version developed recently by Lewin–Nam–Rougerie [**29, 30**].

1.2. Mixture Condensates of Bose Gases

In this section, let us consider $N = N_1 + N_2$ non-relativistic quantum bosons in $\mathbb{R}^2$ consisting of two different families of N_1 and N_2 identical particles. We consider the N-particle bosonic system described by the Hamiltonian

$$\begin{aligned} H_N = H_{N_1,N_2} = & \sum_{i=1}^{N_1} \big(-\Delta_{x_i} + V_1(x_i) \big) - \frac{1}{N_1 - 1} \sum_{1\le i<j\le N_1} w_N^{(1)}(x_i - x_j) \\ & + \sum_{r=1}^{N_2} \big(-\Delta_{y_r} + V_2(y_r) \big) - \frac{1}{N_2 - 1} \sum_{1\le r<s\le N_2} w_N^{(2)}(y_r - y_s) \\ & - \frac{1}{N} \sum_{i=1}^{N_1} \sum_{r=1}^{N_2} w_N^{(12)}(x_i - y_r), \end{aligned} \tag{1.2.1}$$

on the Hilbert space

$$\mathcal{H}_N = \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2} := L^2_{\mathrm{sym}}(\mathbb{R}^{2N_1}, \mathrm{d}x_1, \ldots, \mathrm{d}x_{N_1}) \otimes L^2_{\mathrm{sym}}(\mathbb{R}^{2N_1}, \mathrm{d}y_1, \ldots, \mathrm{d}y_{N_2}).$$

Here $\mathcal{H}_N$ consists of two Hilbert spaces $\mathcal{H}_{N_\sigma} = L^2_{\mathrm{sym}}(\mathbb{R}^{2N_\sigma})$ of square-integrable functions in $(\mathbb{R}^2)^{N_\sigma}$ which are symmetric under permutations of the N_σ variables, for $\sigma \in \{1, 2\}$. The exchange symmetry is not present among variables of different type.

The Hamiltonian (1.2.1) consists of two attractive one-component systems among each species and interactions between the two species. The potentials V_1 and V_2 are trapping for each species and can be chosen to be different. The inter-species interactions can be either attractive or repulsive. The choice of coupling constants proportional to $\frac{1}{N_\sigma - 1}$ and $\frac{1}{N}$ ensures that the kinetic and the potential energy are comparable in the limit $N \to \infty$. Furthermore, all of the interactions terms are chosen of the form

$$w_N^{(\sigma)}(x) = N^{2\beta} w^{(\sigma)}(N^\beta x) \in L^1(\mathbb{R}^2), \quad \sigma \in \{1, 2, 12\}, \tag{1.2.2}$$

for a fixed parameter $0 \le \beta \le 1$, and fixed functions $w^{(\sigma)}$ satisfying

$$w^{(\sigma)}(x) = w^{(\sigma)}(-x) \quad \text{and} \quad (1 + |x|) w^{(\sigma)}, \widehat{w^{(\sigma)}} \in L^1(\mathbb{R}^2). \tag{1.2.3}$$

[4] $\kappa = \frac{\omega}{N}$ in (1.1.1) but we choose here $\kappa = \frac{\omega}{N-1}$ to ensure that the kinetic and interaction energy are comparable. This change has of course no effect in the limit $N \to \infty$ with κN kept fixed.

Here $\beta = \frac{1}{2}$ is the dividing line between two different physical pictures. The case $0 < \beta < \frac{1}{2}$ corresponds to a high density regime where the particles meet frequently but interact weakly since the typical interaction length is larger than the average distance between the particles. The case $\frac{1}{2} < \beta < 1$ is more subtle and corresponds to a low density regime where the particles meet rarely but interact strongly. Finally, we assume that there exist $0 < c_1, c_2 < 1$ such that

$$c_1 = \lim_{N\to\infty} \frac{N_1}{N} \quad \text{and} \quad c_2 = \lim_{N\to\infty} \frac{N_2}{N}. \tag{1.2.4}$$

This realistic requirement guarantees that the two populations are comparable. It is not restrictive to assume that the ratios $\frac{N_1}{N}$ and $\frac{N_2}{N}$ are fixed and equal to c_1 and c_2, respectively, and so shall we henceforth.

As usual, the ground state energy per particle of H_N is denoted by $E_N^{\mathrm{Q}} = N^{-1} \inf \operatorname{spec} H_N$. We may have $E_N^{\mathrm{Q}} = -\infty$ because (1.2.1) consists two attractive many-body quantum systems. Since these systems are confined, a (normalized) mixture ground state exists under certain assumptions on the potentials. It is an N-boson wave function in $\mathcal{H}_N$ with two distinct sets of variables. The most natural example of state which models a two-component condensate is a state $\Psi_N \in \mathcal{H}_N$ of the form

$$\Psi_N(x_1, \ldots, x_{N_1}; y_1, \ldots, y_{N_2}) = \prod_{i=1}^{N_1} u_1(x_i) \otimes \prod_{r=1}^{N_2} u_2(y_r) \tag{1.2.5}$$

for some normalized functions $u_1, u_2 \in H^1(\mathbb{R}^2)$. Computing the ground state energy gives $\langle \Psi_N, H_N \Psi_N \rangle = N \mathcal{E}_N^{\mathrm{H}}(u_1, u_2)$ where

$$\begin{aligned}
\mathcal{E}_N^{\mathrm{H}}(u_1, u_2) = {} & c_1 \int_{\mathbb{R}^2} \left[|\nabla u_1(x)|^2 + V_1(x)|u_1(x)|^2 - \frac{1}{2}|u_1(x)|^2 (w_N^{(1)} \star |u_1|^2)(x) \right] \mathrm{d}x \\
& + c_2 \int_{\mathbb{R}^2} \left[|\nabla u_2(x)|^2 + V_2(x)|u_2(x)|^2 - \frac{1}{2}|u_2(x)|^2 (w_N^{(2)} \star |u_2|^2)(x) \right] \mathrm{d}x \\
& - c_1 c_2 \int_{\mathbb{R}^2} |u_1(x)|^2 (w_N^{(12)} \star |u_2|^2)(x) \mathrm{d}x,
\end{aligned} \tag{1.2.6}$$

with c_1 and c_2 given by (1.2.4). In the above, the symbol $\star$ stands for the convolution. The N-dependent Hartree energy is defined by

$$E_N^{\mathrm{H}} := \inf \left\{ \mathcal{E}_N^{\mathrm{H}}(u_1, u_2) : u_1, u_2 \in H^1(\mathbb{R}^2), \int_{\mathbb{R}^2} |u_1(x)|^2 \mathrm{d}x = 1 = \int_{\mathbb{R}^2} |u_2(x)|^2 \mathrm{d}x \right\}.$$

Obviously, if the couple $(u_1, u_2) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ in (1.2.5) minimizes (1.2.6), then the Hartree energy E_N^{H} is an upper bound to the ground state energy per particle E_N^{Q}.

When $\beta > 0$ and $N \to \infty$, the scaled interactions potentials (1.2.2) converge to a delta function at the origin in the sense of measures, i.e.,

$$w_N^{(\sigma)} \rightharpoonup \left(\int_{\mathbb{R}^2} w^{(\sigma)}(x) \mathrm{d}x \right) \delta_0 =: a_\sigma \delta_0, \quad \sigma \in \{1, 2, 12\}. \tag{1.2.7}$$

Here a_1, a_2 and a_{12} measure the strengths of the intra-species and inter-species interactions. The non-linear Schrödinger (NLS) functional which is obtained from (1.2.6) is

$$\mathcal{E}^{\mathrm{NLS}}(u_1, u_2) = c_1 \int_{\mathbb{R}^2} \left[|\nabla u_1(x)|^2 + V_1(x)|u_1(x)|^2 - \frac{a_1}{2}|u_1(x)|^4 \right] \mathrm{d}x$$

$$+ c_2 \int_{\mathbb{R}^2} \left[|\nabla u_2(x)|^2 + V_2(x)|u_2(x)|^2 - \frac{a_2}{2}|u_2(x)|^4 \right] \mathrm{d}x$$
$$- c_1 c_2 a_{12} \int_{\mathbb{R}^2} |u_1(x)|^2 |u_2(x)|^2 \mathrm{d}x. \quad (1.2.8)$$

The NLS energy is defined by

$$E^{\mathrm{NLS}} := \inf \left\{ \mathcal{E}^{\mathrm{NLS}}(u_1, u_2) : u_1, u_2 \in H^1(\mathbb{R}^2), \int_{\mathbb{R}^2} |u_1(x)|^2 \mathrm{d}x = 1 = \int_{\mathbb{R}^2} |u_2(x)|^2 \mathrm{d}x \right\}. \quad (1.2.9)$$

For the defocusing case, i.e., $a_1, a_2, a_{12} < 0$ in (1.2.7), the derivation of the Hartree and NLS theories from the many-body theory has been achieved by Michelangeli–Nam–Olgiati [**38**]. However, for the focusing case, things are different because the system is unstable. In that case, the NLS theory is the main tool to understand the collapse of the many-body system (1.2.1). The asymptotic behaviors of the ground state energy per particle E_N^{Q} and its ground states are always of our main interests. The mixture condensates are formulated by the *double* (k, ℓ)-reduced density matrices associated to the ground state Ψ_N. It is defined by taking the partial trace of $\gamma_{\Psi_N} := |\Psi_N\rangle\langle\Psi_N|$ over the last $N_1 - k$ and $N_2 - \ell$ particles

$$\gamma_{\Psi_N}^{(k,\ell)} := \mathrm{Tr}_{k+1 \to N_1} \otimes \mathrm{Tr}_{\ell+1 \to N_2} \gamma_{\Psi_N}, \quad \forall k, \ell \in \mathbb{N}. \quad (1.2.10)$$

Equivalently, $\gamma_{\Psi_N}^{(k,\ell)}$ is defined as a non-negative trace class operator on $\mathcal{H}_{k+\ell} = \mathcal{H}_k \otimes \mathcal{H}_\ell$ with kernel

$$\gamma_{\Psi_N}^{(k,\ell)}(X, Y; X', Y') = \int_{\mathbb{R}^{2(N_1-k)}} \int_{\mathbb{R}^{2(N_2-\ell)}} \Psi_N(X, Z; Y, T) \overline{\Psi_N(X', Z; Y', T)} \mathrm{d}Z \mathrm{d}T$$

where $X, X' \in (\mathbb{R}^2)^k$ and $Y, Y' \in (\mathbb{R}^2)^\ell$.

Recently, the collapse of the one-component focusing many-body system has been studied [**13, 31**]. See also [**28, 29, 30, 32**] and references therein for results of the mean field approximation and the validity of the NLS theory. In the two-component setting, the mixture condensates of Bose gases in the NLS theory have been studied by Guo–Zeng–Zhou[5] [**15, 16**]. Most of the results in [**15, 16**] (also in [**13, 31**]) are related to the critical strength for the stability of both one- and two-component NLS functionals. Such a critical number, denoted by a_*, is the optimal constant of the Gagliardo–Nirenberg inequality

$$\frac{a_*}{2} \|u\|_{L^4}^4 \le \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^2, \quad \forall u \in H^1(\mathbb{R}^2). \quad (1.2.11)$$

Equivalently, $a_* = \|Q\|_{L^2}^2$ where Q is the unique (up to translation) symmetric radial positive solution of the following equation in $\mathbb{R}^2$

$$-\Delta Q + Q - Q^3 = 0. \quad (1.2.12)$$

In Paper E, we study the collapse of the two-component system (1.2.1). We consider the case where $a_1 > 0$, $a_2 > 0$ and either $a_{12} > 0$ or $a_{12} < 0$. We establish quantitative bounds on the difference between the ground state energy per particle E_N^{Q} and the N-dependent Hartree energy E_N^{H}. Then by passing to the limit $N \to \infty$ in the latter and using (1.2.3), we obtain the NLS energy E^{NLS}. Furthermore, we analyze in detail the blow-up behavior of E_N^{Q} and its ground states in the collapse regimes. We show that, up to rescaling, the ground states fully condensate on the unique solution of (1.2.12).

[5]In [**15, 16**], the authors consider only the effective NLS theory and c_1, c_2 can be chosen such that $c_1 = 1 = c_2$ for simplicity. In this thesis, those constants are taken into account to fit our general picture.

CHAPTER 2

Overview of Results

2.1. Overview of Papers A and B. Blow-Up of Neutron Stars

In this section, we summarize the results in the papers A and B. In these papers, we study the gravitational collapse of neutron stars which are described by the Hamiltonian (1.1.1) on the anti-symmetric space $\bigwedge^N L^2(\mathbb{R}^3;\mathbb{C})$[1]. The neutron mass $m>0$ is assumed to be strictly positive and $\kappa = gm^2$ with g the gravitational constant. The *collapse* of neutron stars refers to the fundamental fact that the ground state energy per particle of H_N, which is defined by $E_N^{\mathrm{Q}} := N^{-1}\inf \mathrm{spec} H_N$, is not bounded from below if $\tau := \kappa N^{\frac{2}{3}} > \tau_c$. Here the critical value τ_c is given by (1.1.7).

For the reasons explained in Chapter 1, we will focus on effective models in order to study the stellar collapse.

2.1.1. Blow-Up of Neutron Stars in the Chandrasekhar Theory. As a starting point, we first study the gravitational collapse of neutron stars in the Chandrasekhar theory since this is the simplest approximate theory from the full many-body Schrödinger theory. We will consider the following variational problem

$$E_\tau^{\mathrm{Ch}}(1) := \inf\left\{\mathcal{E}_\tau^{\mathrm{Ch}}(\rho) : 0 \le \rho \in L^1\cap L^{\frac{4}{3}}(\mathbb{R}^3), \int_{\mathbb{R}^3}\rho(x)\mathrm{d}x = 1\right\}, \tag{2.1.1}$$

where the Chandrasekhar energy functional is given by

$$\mathcal{E}_\tau^{\mathrm{Ch}}(\rho) := \int_{\mathbb{R}^3} j_m(\rho(x))\mathrm{d}x - \frac{\tau}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho(x)\rho(y)}{|x-y|}\mathrm{d}x\mathrm{d}y. \tag{2.1.2}$$

Here $j_m(\rho)$ which was introduced in (1.1.4) is the relativistic kinetic energy at the density ρ.

It is well-known that there exists a (unique) minimizer for (2.1.1) whenever $0 < \tau < \tau_c$ (see [**36**]). We focus on the analysis of the blow-up behavior of the Chandrasekhar energy $E_\tau^{\mathrm{Ch}}(1)$ and its minimizer when $\tau \nearrow \tau_c$. This is based on a detailed analysis of the associate Euler–Lagrange equation. We will show that the Chandrasekhar minimizer develops a universal blow-up profile given explicitly by the optimizer of (1.1.7). Let us briefly recall some results regarding the properties of (1.1.7). From [**34**, Appendix A] we have that (1.1.7) has a unique optimizer, up to dilation and translation. Such an optimizer, called $Q \in L^1\cap L^{\frac{4}{3}}(\mathbb{R}^3)$, has compact support and can be chosen uniquely to be non-negative symmetric decreasing, by rearrangement inequality (see [**33**, Chapter 3]). The dilation can be fixed by setting

$$\sigma_f\int_{\mathbb{R}^3} Q(x)^{\frac{4}{3}}\mathrm{d}x = \int_{\mathbb{R}^3} Q(x)\mathrm{d}x = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{Q(x)Q(y)}{|x-y|}\mathrm{d}x\mathrm{d}y = 1, \tag{2.1.3}$$

[1]More generally, we consider $\bigwedge^N L^2(\mathbb{R}^3;\mathbb{C}^q)$ where $q\ge 1$ denotes the internal spin degree of freedom.

where $\sigma_f := K_{\rm cl}\tau_c^{-1} \approx 1.092$, numerically. Moreover, Q solves the Lane–Emden equation [**21**]

$$\frac{4}{3}\sigma_f Q(x)^{\frac{1}{3}} - (|\cdot|^{-1}\star Q)(x) + \frac{2}{3}\begin{cases} = 0 & \text{if } Q(x) > 0, \\ \geq 0 & \text{if } Q(x) = 0. \end{cases} \tag{2.1.4}$$

An equivalent way to write (2.1.4) is, with $[f(x)]_+ := \max\{f(x), 0\}$,

$$\frac{4}{3}\sigma_f Q(x)^{\frac{1}{3}} = \left[(|\cdot|^{-1}\star Q)(x) - \frac{2}{3}\right]_+.$$

Furthermore, it can be seen from (2.1.4) that Q has compact support.

Now we are able to describe our first result.

THEOREM 1 ([**44**]). *Fix $m > 0$. For each $0 < \tau < \tau_c$, let ρ_τ be a minimizer for $E_\tau^{\rm Ch}(1)$ in* (2.1.1)*. Then for every sequence $\{\tau_n\}$ with $\tau_n \nearrow \tau_c$ as $n \to \infty$, there exist a sequence $\{x_n\} \subset \mathbb{R}^3$ and a subsequence of $\{\tau_n\}$ (still denoted by $\{\tau_n\}$) such that*

$$\lim_{n\to\infty}(\tau_c - \tau_n)^{\frac{3}{2}}\rho_{\tau_n}((\tau_c - \tau_n)^{\frac{1}{2}}x + x_n) = \Lambda^3 Q(\Lambda x) \tag{2.1.5}$$

strongly in $L^1 \cap L^{\frac{4}{3}}(\mathbb{R}^3)$. Here Q is the (unique) non-negative symmetric function satisfying (2.1.3)–(2.1.4) *and*

$$\Lambda = \frac{3}{4}m\left(\frac{1}{K_{\rm cl}}\int_{\mathbb{R}^3} Q(x)^{\frac{2}{3}}{\rm d}x\right)^{\frac{1}{2}}. \tag{2.1.6}$$

Let us discuss briefly the strategy of the proof of Theorem 1. Heuristically, assume that the minimizer ρ_τ for (2.1.1) collapses at a length $\ell \to 0$, namely $\ell^3\rho_\tau(\ell x) \approx Q(x)$. By using the formal approximation of the function $j_m(\rho_\tau) \approx K_{\rm cl}\rho_\tau^{\frac{4}{3}} + \frac{9}{16}m^2 K_{\rm cl}\rho_\tau^{\frac{2}{3}}$, which follows from that of the operator

$$\sqrt{-\Delta + m^2} \approx \sqrt{-\Delta} + \frac{m^2}{2\sqrt{-\Delta}}, \tag{2.1.7}$$

we obtain

$$E_\tau^{\rm Ch}(1) = \mathcal{E}_\tau^{\rm Ch}(\rho_\tau) \approx \frac{1}{\ell}(\tau_c - \tau) + \ell\frac{9}{16}m^2 K_{\rm cl}\int_{\mathbb{R}^3} Q(x)^{\frac{2}{3}}. \tag{2.1.8}$$

Then the result in Theorem 1 essentially follows by optimizing over $\ell > 0$ on the right hand side of (2.1.8). As a by-product, we also obtain the asymptotic behavior of the Chandrasekhar energy. It is given by

$$\lim_{\tau\nearrow\tau_c}\frac{E_\tau^{\rm Ch}(1)}{(\tau_c - \tau)^{\frac{1}{2}}} = \frac{3}{2}m\left(\frac{1}{K_{\rm cl}}\int_{\mathbb{R}^3} Q(x)^{\frac{2}{3}}{\rm d}x\right)^{\frac{1}{2}}. \tag{2.1.9}$$

Roughly speaking, for each $0 < \tau_n < \tau_c$, the corresponding minimizer $\rho_{\tau_n} =: \rho_n$ for $E_{\tau_n}^{\rm Ch}(1)$ solves the Euler–Lagrange equation

$$\sqrt{\eta_n(x)^2 + m^2} = \left[\tau_n(|\cdot|^{-1}\star\rho_n)(x) + \mu_n\right]_+ \tag{2.1.10}$$

where $\eta_n = (6\pi^2\rho_n)^{\frac{1}{3}}$ and the Lagrange multiplier $\mu_n \in \mathbb{R}$ can be calculated as follows

$$\mu_n = \int_{\mathbb{R}^3}\sqrt{\eta_n(x)^2 + m^2}\rho_n(x){\rm d}x - \tau_n\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\rho_n(x)\rho_n(y)}{|x-y|}{\rm d}x{\rm d}y. \tag{2.1.11}$$

Next, delicate estimates on kinetic and potential energies show that $(\tau_c - \tau_n)^{\frac{1}{2}}\mu_n$ stay away from 0 as $\tau_n \nearrow \tau_c$. This implies the compactness of the rescaling of Chandrasekhar minimizer.

In the literature, the Chandrasekhar theory without external potential is the most physical relevant. Nevertheless, there are also motivations from physics and mathematics (as in [**2, 1, 9, 37**]) to include an external potential $V : \mathbb{R}^3 \to \mathbb{R}$. In Paper A, we considered

$$V(x) = -\sum_{i=1}^{M} \frac{z_i}{|x - x_i|^{s_i}}, \tag{2.1.12}$$

where $0 < z_i$, $0 < s_i < \frac{3}{4}$, $x_i \in \mathbb{R}^3$ and $x_i \neq x_j$, for $1 \leq i \neq j \leq M$. Based on the concentration-compactness method [**37**], one can prove the existence of minimizers below criticality τ_c and absence thereof above and exactly at the critical coupling. The behavior of the Chandrasekhar minimizers depends on that of V near its minima. In [**44**], we prove that they concentrate at the most singular points of $V(x)$ in (2.1.12).

2.1.2. Blow-Up of Neutron Stars in the Hartree–Fock–Bogoliubov Theory. The Hartree–Fock–Bogoliubov (HFB) theory is believed to be much more precise than the Chandrasekhar theory. To see this, let us assume for the moment that the exchange and pairing terms are trivial. Let ρ^{Ch} be the unique (up to translation) minimizer for $E_\kappa^{\mathrm{Ch}}(N)$ in (1.1.6). By the non-negativity of the direct term, we have

$$-\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho_\gamma(x)\rho_\gamma(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \leq \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho^{\mathrm{Ch}}(x)\rho^{\mathrm{Ch}}(y)}{|x-y|}\mathrm{d}x\mathrm{d}y - 2\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho^{\mathrm{Ch}}(x)\rho_\gamma(y)}{|x-y|}\mathrm{d}x\mathrm{d}y$$

for the density functional ρ_γ of any trace-class self-adjoint operator γ. This implies that

$$\begin{aligned} E_\tau^{\mathrm{HFB}}(N) \leq &\inf\left\{\mathrm{Tr}\left[\left(\sqrt{-\Delta+m^2} - \kappa(|\cdot|^{-1}\star\rho^{\mathrm{Ch}})\right)\gamma\right] : 0 \leq \gamma = \gamma^* \leq 1, \mathrm{Tr}\,\gamma = N\right\} \\ &+ \frac{\kappa}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho^{\mathrm{Ch}}(x)\rho^{\mathrm{Ch}}(y)}{|x-y|}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{2.1.13}$$

Now, if we consider the trial state $\gamma := \mathbb{1}(\sqrt{-\Delta} \leq \eta^{\mathrm{Ch}})$ with $\eta^{\mathrm{Ch}} = (6\pi^2\rho^{\mathrm{Ch}})^{\frac{1}{3}}$, then the right hand side of (2.1.13) is $E_\kappa^{\mathrm{Ch}}(N) = NE_\tau^{\mathrm{Ch}}(1)$, where $\tau = \kappa N^{\frac{2}{3}}$. Furthermore, we have that $NE_N^{\mathrm{Q}} \leq E_\tau^{\mathrm{HFB}}(N)$, by the variational principle. Therefore, the HFB theory interpolates between the Schrödinger many-body and the Chandrasekhar theories.

Having the knowledge of the stellar collapse in the Chandrasekhar theory, we now study it in the HFB theory (1.1.8). In this case, the limit $N \to \infty$ is taken into account besides the original limit $\tau := \tau_N \nearrow \tau_c$.

We have the following result.

THEOREM 2 ([**45**]). *Fix $m > 0$. Assume that $0 < \tau_N = \tau_c - N^{-\beta}$ with $0 < \beta < \frac{1}{9}$. Then we have*

$$\frac{1}{N}E_{\tau_N}^{\mathrm{HFB}}(N) = (\tau_c - \tau_N)^{\frac{1}{2}}(2\Lambda + o(1)_{N\to\infty}), \tag{2.1.14}$$

where Λ is given by (2.1.6).

Furthermore, assume that (γ_N, α_N) is a minimizer for $E_{\tau_N}^{\mathrm{HFB}}(N)$ and $\rho_{\gamma_N}(x) = \gamma_N(x,x)$. Then there exist a sequence $\{x_N\} \subset \mathbb{R}^3$ and a subsequence of $\{\rho_{\gamma_N}\}$ (still denoted by $\{\rho_{\gamma_N}\}$) such that

$$\lim_{N\to\infty} (\tau_c - \tau_N)^{\frac{3}{2}}\rho_{\gamma_N}((\tau_c - \tau_N)^{\frac{1}{2}}N^{\frac{1}{3}}x + x_N) = \Lambda^3 Q(\Lambda x) \tag{2.1.15}$$

strongly in $L^r(\mathbb{R}^3)$ for $1 \leq r < \frac{4}{3}$ and weakly in $L^{\frac{4}{3}}(\mathbb{R}^3)$. Here Q is the (unique) non-negative symmetric function satisfying (2.1.3)–(2.1.4).

REMARK 3.
- *We only obtain the $L^{\frac{4}{3}}(\mathbb{R}^3)$-weak convergence in* (2.1.15)*. To get the strong convergence, our proof needs the Lieb–Thirring-type inequality with optimal constant. This is a long-standing open question. See* [**6, 35**] *for thorough discussions.*
- *The pairing term in* (1.1.8) *does not show up in the leading order of the blow-up profile since its contribution is too small.*

Let us discuss briefly the strategy of the proof of Theorem 2. The asymptotic behavior of the HFB energy follows from that of the Chandrasekhar energy. See Lenzmann–Lewin [**24**] for the energy estimate in the subcritical regime. In fact, it essentially follows from the analysis of Lieb–Yau in [**36**] where they compared the ground state energy and the Chandrasekhar energy. They proved that the error term in the energy estimate is of order $N^{-\frac{1}{9}}$. Therefore, it is necessary to assume that $\tau_N = \tau_c - N^{-\beta}$ with $0 < \beta < \frac{1}{9}$ in order to obtain (2.1.14).

On the other hand, we note that a HFB minimizer (γ_N, α_N) exists, whenever N is less than the Chandrasekhar limit mass, and there is an Euler–Lagrange equation associated to this minimizer. Unlike the Chandrasekhar equation (2.1.10), such a HFB equation is more complicated and therefore more difficult to analyze. What we do instead is to apply the concentration-compactness principle [**37**]. In such a method, the relative compactness of the minimizing sequence is usually consequence of the strict binding inequality. In the HFB case, the HFB energy formally reduces to the Chandrasekhar energy in the limit $N \to \infty$ and $\tau_N \nearrow \tau_c$. This means we would need a binding inequality like

$$E^{\rm Ch}_{\tau_c}(1)_{|m=0} < E^{\rm Ch}_{\tau_c}(\nu)_{|m=0} + E^{\rm Ch}_{\tau_c}(1-\nu)_{|m=0}, \quad \forall 0 < \nu < 1.$$

However, the above is not true since $E^{\rm Ch}_{\tau}(\nu)_{|m=0} = 0$ for any $0 \le \nu \le 1$ and $0 \le \tau \le \tau_c$. Such a binding inequality holds only if $m > 0$ and $\tau < \tau_c$, which yields the existence of the Chandrasekhar minimizer for $E^{\rm Ch}_{\tau}(1)$. Another difficulty arising in the dichotomy argument is the lack of the sharp Lieb–Thirring-type inequality. For our purpose, we only need that such an inequality holds in the weak sense. More precisely, for any sequence of density matrices $\{\gamma_N\}$ such that $0 \le \gamma_N \le 1$ and the density $\rho_{\gamma_N}(N^{\frac{1}{3}}x) = \gamma_N(N^{\frac{1}{3}}x, N^{\frac{1}{3}}x)$ converges to ρ weakly in $L^{\frac{4}{3}}(\mathbb{R}^3)$, we have

$$\liminf_{N\to\infty} \frac{1}{N}\,{\rm Tr}(\sqrt{-\Delta}\gamma_N) \ge K_{\rm cl}\int_{\mathbb{R}^3} \rho(x)^{\frac{4}{3}}{\rm d}x. \tag{2.1.16}$$

See also [**11**] for the non-relativistic analogue version.

Heuristically, suppose that (γ_N, α_N) is a minimizer for $E^{\rm HFB}_{\tau_N}$ and set the collapse length $\ell_N := (\tau_c - \tau_N)^{\frac{1}{2}} \to 0$ as $N \to \infty$. In this limit, the exchange and pairing terms give no contribution to the leading order, by (1.1.13) and (1.1.14). Let $\tilde{\gamma}_N(x,y) = \ell_N^3\gamma_N(\ell_N x, \ell_N y)$ be the scaling of the HFB minimizing sequence. Then delicate estimates between the HFB and the Chandrasekhar energies together with (2.1.9) show that $\tilde{\gamma}_N(N^{\frac{1}{3}}x, N^{\frac{1}{3}}y)$ does not vanish for sufficient large N. Next, we assume that such a sequence is not relatively compact. After splitting the energy we obtain

$$\frac{\ell_N}{N}E^{\rm HFB}_{\tau_N} = \frac{1}{N}\mathcal{E}^{\rm HFB}_{\tau_N}(\tilde{\gamma}_N) \ge \frac{1}{N}\mathcal{E}^{\rm HFB}_{\tau_N}(\tilde{\gamma}_N^{(\bullet)}) + \frac{1}{N}\mathcal{E}^{\rm HFB}_{\tau_N}(\tilde{\gamma}_N^{(\circ)}) + o(1)_{N\to\infty}, \tag{2.1.17}$$

where $\tilde{\gamma}_N^{(\bullet)}$ is the localized state of $\tilde{\gamma}_N$ in a bounded domain and $\tilde{\gamma}_N^{(\circ)}$ is the state at infinity. It is obvious that $E^{\rm HFB}_{\tau_N} \le NE^{\rm Ch}_{\tau_N}(1)$, by variational principle, and $E^{\rm Ch}_{\tau_N}(1)$ behaves like $\ell_N \to 0$

as $N \to \infty$, by (2.1.9). On the other hand, (2.1.16) allows us to conclude that

$$\liminf_{N\to\infty} \frac{1}{N}\mathcal{E}^{\rm HFB}_{\tau_N}(\tilde\gamma_N^{(\bullet)}) \geq E^{\rm Ch}_{\tau_c}(\nu)_{|m=0} = 0$$

for some $0 < \nu < 1$ and that $E^{\rm Ch}_{\tau_c}(\nu)_{|m=0}$ admits a minimizer. However, this will never happen, due to the positivity of the direct term.

On the other hand, when $\operatorname{Tr}\tilde\gamma_N^{(\circ)}$ is close to $\operatorname{Tr}\tilde\gamma_N$, the non-sharp Lieb–Thirring-type inequality (see [**6, 35**]) is not enough to control the energy of mass at infinity. In the following, we present another argument differently from [**45**] that allows us to obtain the conclusion quickly. By the same reason as above, the sequence $\{\tilde\gamma_N^{(\circ)}\}$ cannot vanish. Furthermore, if it is relatively compact then we obtain again from (2.1.16) that

$$\liminf_{N\to\infty} \frac{1}{N}\mathcal{E}^{\rm HFB}_{\tau_N}(\tilde\gamma_N^{(\circ)}) \geq E^{\rm Ch}_{\tau_c}(1-\nu)_{|m=0} = 0.$$

Otherwise, we may split the energy one more time the second term on the right hand side of (2.1.17). In fact, we can repeat this process as many times as we please. By doing this, we decrease the mass at infinity. When this is small enough, we can therefore apply (1.1.7) and the non-sharp Lieb–Thirring-type inequality to conclude that the energy of mass at infinity is non-negative and we are done.

2.2. Overview of Papers C and D. Blow-Up of Boson Stars

In this section, we summarize the results in the papers C and D. In these papers, we study the gravitational collapse of boson stars, which are a class of models from relativistic many-body quantum mechanics inspired by stellar collapse (see Section 2.1). Although rather unphysical, this model has very interesting mathematical features and has been extensively studied in the literature [**36, 7, 10, 22, 25, 40, 29**].

Mathematically, boson stars are also described by the same Hamiltonian in (1.1.1), which now acts on the symmetric space $\bigotimes_{\rm sym}^N L^2(\mathbb{R}^3)$. Note that all the force carrier particles now are bosons since we neglect the Pauli exclusion principle. Furthermore, the boson mass $m > 0$ is again assumed to be strictly positive. The *collapse* of boson stars refers to the fundamental fact that the ground state energy per particle of H_N, which is defined by $E_N^{\rm Q} := N^{-1}\inf{\rm spec}H_N$, is not bounded from below if $\omega := \kappa N > \omega_c$. Here the critical value ω_c is given by (1.1.18).

2.2.1. Blow-Up of Boson Stars in the Hartree Theory.

As a starting point, we first study the gravitational collapse of boson stars in the Hartree theory since this is the simplest approximate theory from the full many-body Schrödinger theory (1.1.20). We consider the following variational problem

$$E^{\rm H}_\omega(1) := \inf\left\{\mathcal{E}^{\rm H}_\omega(u) : u \in H^{\frac12}(\mathbb{R}^3), \int_{\mathbb{R}^2} |u(x)|^2 {\rm d}x = 1\right\}, \tag{2.2.1}$$

where the general Hartree energy functional is given by

$$\mathcal{E}^{\rm H}_\omega(u) = \|(-\Delta+m^2)^{\frac14}u\|^2_{L^2} + \int_{\mathbb{R}^3} V(x)|u(x)|^2{\rm d}x - \frac{\omega}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}{\rm d}x{\rm d}y. \tag{2.2.2}$$

The case $V = 0$ is allowed and is the most physically relevant. But it is also mathematically interesting to include a general external potential. Here, we consider two cases that either V

is trapping, i.e.,

$$0 \le V \in L^\infty_{\rm loc}(\mathbb{R}^3) \quad \text{and} \quad \lim_{|x|\to\infty} V(x) \to \infty \tag{2.2.3}$$

or V is periodic, i.e.,

$$0 \le V \in C(\mathbb{R}^3) \quad \text{and} \quad V(x+z) = V(x) \quad \text{for all } z \in \mathbb{Z}^3.$$

Under the above assumptions, we first prove the existence of ground states for (2.2.1) below criticality ω_c and absence thereof above and exactly at the critical coupling. In the trapping case, the existence result follows from standard methods in the calculus of variations. On the other hand, the proof in the periodic case is an application of the concentration-compactness method [**37**]. Note that we can restrict the minimization problem (2.2.1) to non-negative functions since $\mathcal{E}^{\rm H}_\omega(u) \ge \mathcal{E}^{\rm H}_\omega(|u|)$, for any $u \in H^{\frac12}(\mathbb{R}^3)$. This follows from the fact that $\|(-\Delta+m^2)^{\frac14}u\|_{L^2} \ge \|(-\Delta+m^2)^{\frac14}|u|\|_{L^2}$ (see [**33**, Theorem 7.13]). In particular, a ground state for $E^{\rm H}_\omega(1)$ (if it exists) can be chosen to be non-negative. Furthermore, if V is radial increasing, then one can actually restrict the minimization problem $E^{\rm H}_\omega(1)$ to radial decreasing functions, by rearrangement inequalities (see [**33**, Chapter 3 and Lemma 7.17]).

Having the existence of Hartree ground states, we next give an explicit blow-up profile for the Hartree energy and its ground states. Here we only describe our results in the case of a trapping potential. We refer the reader to [**43**] for the other interesting cases such as periodic and ring-shaped potentials (which have infinitely many minimizers). Let us consider the case where V is trapping and has finitely many minimizers, i.e., $V \ge 0$ and $V^{-1}(0) = \{x_i\}_{i=1}^n \subset \mathbb{R}^3$. Furthermore, assume that there exist constants $p_i > 0$ and $\nu_i > 0$ such that

$$\lim_{x\to x_i} \frac{V(x)}{|x-x_i|^{p_i}} = \nu_i, \quad \forall i = 1,\dots,n. \tag{2.2.4}$$

Moreover, we denote by $\mathcal{Z}$ the set of positions of the flattest global minima of $V(x)$, given by

$$\mathcal{Z} := \{x_i : p_i = p,\, \nu_i = \nu\}, \quad \text{where} \quad p = \max\{p_i : 1 \le i \le n\} \quad \text{and} \quad \nu = \min\{\nu_i : p_i = p\}.$$

One easily sees that V in (2.2.4) satisfies (2.2.3). Hence, the existence of ground states for (2.2.1) is obtained whenever $\omega < \omega_c$. Next, we analyze the blow-up behavior of the Hartree ground states when $\omega \nearrow \omega_c$. We will see that this depends crucially on the local behavior of V close to its minimizers only in the case $0 < p \le 1$. The analysis will be based on a detailed analysis of the Euler–Lagrange equation associated to them. We will show that the Hartree ground states develop a universal blow-up profile given explicitly by the optimizers of (1.1.18). Let us briefly recall some results regarding the properties of (1.1.18). From [**36**] we have that (1.1.18) has an optimizer, called $Q \in H^{\frac12}(\mathbb{R}^3)$. It can be chosen to be non-negative symmetric decreasing, by rearrangement inequality, and satisfies

$$\|(-\Delta)^{\frac14}Q\|^2_{L^2} = \|Q\|^2_{L^2} = \frac{\omega_c}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|Q(x)|^2|Q(y)|^2}{|x-y|}\mathrm{d}x\mathrm{d}y = 1. \tag{2.2.5}$$

Moreover, Q solves the *massless boson star equation*

$$\sqrt{-\Delta}Q + Q - \omega_c(|\cdot|^{-1} \star |Q|^2)Q = 0. \tag{2.2.6}$$

The uniqueness (up to translation and dilation) of the optimizers of (1.1.18), as well as the uniqueness (up to translation) of the positive solutions of (2.2.6), are still major *open problems*. Let us define the following

$$\mathcal{GN} = \{\text{positive symmetric decreasing functions satisfying (2.2.5)–(2.2.6)}\}. \tag{2.2.7}$$

We have the following result.

THEOREM 4 ([**43**]). *Let V satisfy (2.2.3) and the assumption (2.2.4). For each $0<\omega<\omega_c$, let u_ω be a non-negative ground state for $E^{\rm H}_\omega(1)$ in (2.2.1). Then for every sequence $\{\omega_n\}$ with $\omega_n\nearrow\omega_c$ as $n\to\infty$, there exist a $Q\in\mathcal{GN}$ in (2.2.7) and a subsequence of $\{\omega_n\}$ (still denoted by $\{\omega_n\}$) such that*

- *If $p\leq 1$, then there exists an $x_0\in\mathcal{Z}$ such that*
$$\lim_{n\to\infty}(\omega_c-\omega_n)^{\frac{3}{2(p+1)}}u_{\omega_n}\left((\omega_c-\omega_n)^{\frac{1}{p+1}}x+x_0\right)=\Lambda^{\frac{3}{2}}Q\left(\Lambda x\right) \tag{2.2.8}$$
strongly in $H^{\frac12}(\mathbb{R}^3)$, where
$$\Lambda=\begin{cases}\inf\limits_{W\in\mathcal{GN}}\left(\omega_c p\nu\displaystyle\int_{\mathbb{R}^3}|x|^p|W(x)|^2\mathrm{d}x\right)^{\frac{1}{p+1}} & \text{if } 0<p<1,\\ \inf\limits_{W\in\mathcal{GN}}\left(\dfrac{m^2\omega_c}{2}\|(-\Delta)^{-\frac14}W\|^2_{L^2}+\omega_c\nu\displaystyle\int_{\mathbb{R}^3}|x||W(x)|^2\mathrm{d}x\right)^{\frac12} & \text{if } p=1.\end{cases} \tag{2.2.9}$$
- *If $p>1$, then there exists a sequence $\{x_n\}\subset\mathbb{R}^3$ such that*
$$\lim_{n\to\infty}(\omega_c-\omega_n)^{\frac34}u_{\omega_n}\left((\omega_c-\omega_n)^{\frac12}x+x_n\right)=\Lambda^{\frac32}Q\left(\Lambda x\right) \tag{2.2.10}$$
strongly in $H^{\frac12}(\mathbb{R}^3)$, where
$$\Lambda=m\sqrt{\frac{\omega_c}{2}}\inf_{W\in\mathcal{GN}}\|(-\Delta)^{-\frac14}W\|_{L^2}. \tag{2.2.11}$$

REMARK 5.
- *The infima in (2.2.9) and (2.2.11) are attained at Q in (2.2.8) and (2.2.10), respectively.*
- *It can be seen from (2.2.9) and (2.2.11) that V has contribution to the leading order of $E^{\rm H}_\omega(1)$ only in the case $0<p\leq 1$. In the reverse case $p>1$ as well as in the case $V\equiv 0$, therefore, we lose information about the sequence $\{x_n\}$ in (2.2.10). However, if a non-trivial strictly radial increasing V is included then we can choose $x_n=0$.*

A similar minimization problem of (2.2.1) was considered independently in [**14, 50**]. The authors in [**14**] studied a problem with non-local non-linear terms, while the authors in [**50**] considered a case of a trapping potential as in (2.2.4) with $0<p<1$.

The collapse scales in (2.2.8) and (2.2.10) are set by the subleading contribution of the kinetic energy in a large momentum expansion. More precisely, assume that the ground state u_ω collapse at a length $\ell\to 0$ around x_0, namely
$$\ell^{\frac32}u_\omega(\ell x+x_0)\approx Q(x),$$
where $Q\in\mathcal{GN}$ in (2.2.7). By using the formal approximation (2.1.7) and the assumption that $V(x)\approx\nu|x-x_0|^p$ around x_0 we obtain
$$E^{\rm H}_\omega(1)=\mathcal{E}^{\rm H}_\omega(u_\omega)\approx\frac{1}{\ell}\left(1-\frac{\omega}{\omega_c}\right)+\ell\frac{m^2}{2}\|(-\Delta)^{-\frac14}Q\|^2_{L^2}+\ell^p\nu\int_{\mathbb{R}^3}|x|^p|Q(x)|^2\mathrm{d}x. \tag{2.2.12}$$
Then the result in Theorem 4 essentially follows by optimizing over $\ell>0$ on the right hand side of (2.2.12). As a by-product, we also obtain the asymptotic behavior of the Hartree energy
$$\lim_{\omega\nearrow\omega_c}\frac{E^{\rm H}_\omega(1)}{(\omega_c-\omega)^{\frac{q}{q+1}}}=\frac{q+1}{q}\cdot\frac{\Lambda}{\omega_c}\quad\text{where}\quad q=\min\{p,1\}. \tag{2.2.13}$$

Note that if $V \equiv 0$, then we also obtain (2.2.13) with $q = 1$ and Λ is given by (2.2.11).

Roughly speaking, for each $0 < \omega_n < \omega_c$, the corresponding ground state $u_{\omega_n} =: u_n$ for $E^{\rm H}_{\omega_n}(1)$ solves the Euler–Lagrange equation

$$\sqrt{-\Delta + m^2} u_n(x) + V(x)u_n(x) - \omega_n(|\cdot|^{-1} \star |u_n|^2)(x)u_n(x) = \mu_n u_n(x),$$

where the Lagrange multiplier $\mu_n \in \mathbb{R}$ can be calculated as follows

$$\mu_n = E^{\rm H}_{\omega_n}(1) - \frac{\omega_n}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u_n(x)|^2|u_n(y)|^2}{|x-y|} \mathrm{d}x\mathrm{d}y. \tag{2.2.14}$$

Next, delicate estimates on kinetic and potential energies show that $(\omega_c - \omega_n)^{\frac{1}{q+1}} \mu_n$ stay away from 0 as $\omega_n \nearrow \omega_c$. This implies the compactness of the rescaling of the Hartree ground state. We summarize this in the following lemma which is also used in the proofs for periodic and ring-shaped potentials.

LEMMA 6 ([**43**]). *For any sequence $\{z_n\} \subset \mathbb{R}^3$ and $\epsilon_n > 0$ such that $\epsilon_n \to 0$ as $n \to \infty$, let $v_n(x) := \epsilon_n^{\frac{3}{2}} u_n(\epsilon_n x + z_n)$ be L^2-normalized of u_n. Assume that v_n is bounded in $H^{\frac{1}{2}}(\mathbb{R}^3)$ and $\epsilon_n\mu_n \to -\lambda < 0$ as $n \to \infty$. Then there exists a non-negative $v \in H^{\frac{1}{2}}(\mathbb{R}^3)$ such that $v_n \to v$ strongly in $H^{\frac{1}{2}}(\mathbb{R}^3)$. Moreover if $v > 0$ then, up to translation, we have*

$$v(x) = \lambda^{\frac{3}{2}} Q(\lambda x),$$

where $Q \in \mathcal{GN}$ in (2.2.7).

2.2.2. Blow-Up of Boson Stars in the Many-Body Theory. To study the collapse of the full many-body system (1.1.1) of boson stars, the limit $N \to \infty$ is taken into account besides the original limit $\omega := \omega_N \nearrow \omega_c$. The asymptotic behavior of the ground state energy per particle can be derived from that of the Hartree energy, which was given by (2.2.13) with $q = 1$. We have

$$E^{\rm Q}_N = (\omega_c - \omega_N)^{\frac{1}{2}} \Big(2\frac{\Lambda}{\omega_c} + o(1)_{N\to\infty} \Big), \tag{2.2.15}$$

where Λ is given by (2.2.11). It is obvious that the Hartree energy is an upper bound for the ground state energy per particle. On the other hand, the lower bound was analyzed in great mathematical detail by Lieb–Yau [**36**]. They proved that the error term in the energy estimate is of order $N^{-\frac{1}{3}}$. Therefore, it is necessary to assume that $\omega_N = \omega_c - N^{-\beta}$ with $0 < \beta < \frac{1}{3}$ in order to obtain (2.2.15).

However, it is difficult to study the blow-up behavior of the many-body ground states. In fact, a ground state does not exist since (1.1.1) is translation invariant. In the following, we consider the modified Hamiltonian (1.1.20) which included an external potential V. For precise analysis and for simplicity, we assume that V is trapping and has only one minimum[2], i.e.,

$$V(x) = \nu |x|^p \tag{2.2.16}$$

for fixed parameters $p > 0$ and $\nu > 0$. Since V is trapping, the existence of many-body ground states is easily obtained, whenever $0 < \omega < \omega_c$, by a standard compactness argument. Having the existence of many-body ground states, we next give an explicit blow-up profile for the many-body system in the collapse regime $N \to \infty$ simultaneously $\omega_N = \kappa N \nearrow \omega_c$. The asymptotic behavior of the ground state energy per particle essentially follows from the

[2] $\nu = 1$ in [**46**] but we include here $\nu > 0$, as in [**43**], in order to fit in the overall picture of this thesis.

analysis of Lieb–Yau [**36**]. Our main interest is the behavior of the many-body ground states. It is formulated using the k-particle reduced density matrices associated to them which was introduced in (1.1.19).

We have the following result.

THEOREM 7 ([**46**]). *Assume that $m > 0$ and V is given by (2.2.16). Let $\omega_N = \omega_c - N^{-\beta}$ with $0 < \beta < \frac{1}{3}$. Then we have*

$$E_N^{\mathrm{Q}} = (\omega_c - \omega_N)^{\frac{q}{q+1}} \left(\frac{q+1}{q} \cdot \frac{\Lambda}{\omega_c} + o(1)_{N\to\infty} \right)$$

where $q = \min\{p, 1\}$ and Λ is given in Theorem 4.

In addition, assume that $0 < p \leq 1$ and $0 < \beta < \frac{p}{17p+15}$. Let Ψ_N be a ground state for H_N in (1.1.20). Then there exists a Borel probability measure $\mathrm{d}\mu$ supported on $\mathcal{GN}$ defined in (2.2.7) such that, along a subsequence of the rescaled states $\Phi_N = \ell_N^{-\frac{3N}{2}} \Psi_N(\ell_N^{-1} \cdot)$, where $\ell_N = \Lambda(\omega_c - \omega_N)^{-\frac{1}{q+1}}$, we have

$$\lim_{N\to\infty} \mathrm{Tr} \left| \gamma_{\Phi_N}^{(k)} - \int |u^{\otimes k}\rangle\langle u^{\otimes k}| \mathrm{d}\mu(u) \right| = 0, \quad \forall k \in \mathbb{N}. \tag{2.2.17}$$

REMARK 8.

- *If $\mathcal{GN} = \{Q_0\}$, as conjectured in [**36**], then for $p > 0$ and $0 < \beta < \frac{1}{3}$ we have*

$$\lim_{N\to\infty} \mathrm{Tr} \left| \gamma_{\Phi_N}^{(k)} - |Q_0^{\otimes k}\rangle\langle Q_0^{\otimes k}| \right| = 0, \quad \forall k \in \mathbb{N},$$

without the constraints $p \leq 1$ and $\beta < \frac{p}{17p+15}$. Moreover, the convergence holds for the whole sequence as $N \to \infty$.

- *Although the convergence of the Hartree ground states in Theorem 4 is obtained for any $p > 0$, we are not able to prove the Bose–Einstein condensation (2.2.17) for $p > 1$. In this case, V has no impact to the leading order of E_N^{Q}. A loss of some compactness of the many-body ground states arises similarly to the translation-invariant case when $V \equiv 0$.*

Let us discuss briefly the strategy of the proof of Theorem 7. The asymptotic behavior of the many-body ground state energy follows from that of the Hartree energy which was pointed out in Theorem 4. Note that the energy estimate in [**36**] still holds when an external potential is included. On the other hand, it is more complicated to obtain the convergence of the ground states. The lack of uniqueness of the limiting profile does not allow us to use a Feynman–Hellman-type argument, which was used by Lieb–Yau [**36**] in the study of neutron stars. In the proof of (2.2.17), the crucial ingredient is the quantum de Finetti theorem and its quantitative (finite dimensional) version [**49, 19, 29, 30**]. Those theorems allow us to fix the structure of limits of reduced density matrices. The asymptotic behavior of the ground state energy per particle ensures the strong compactness of the density matrices of the rescaled state Φ_N of the ground state Ψ_N and we have

$$\lim_{N\to\infty} \mathrm{Tr} \left| \gamma_{\Phi_N}^{(k)} - \int_{SL^2(\mathbb{R}^3)} |u^{\otimes k}\rangle\langle u^{\otimes k}| \mathrm{d}\mu(u) \right| = 0, \quad \forall k \in \mathbb{N}.$$

Here $\mathrm{d}\mu$ is a Borel probability measure supported on the unit sphere $SL^2(\mathbb{R}^3)$ which is given by the quantum de Finetti theorem. The main difficulty is to prove that $\mathrm{d}\mu$ is supported on the set $\mathcal{GN}$, defined in (2.2.7), of positive solutions of the equation (2.2.6).

The idea is to estimate the ground state energy per particle by the Hartree functional in terms of the quantum de Finetti measure and second moment of the one-body Hamiltonian. Together with the asymptotic behavior of the ground state energy per particle, this allows us to reduce the problem of convergence of the reduced density matrix to that of the Hartree *approximate* ground states, in the sense of energy. The energy upper bound is trivial by taking a factorized ansatz. To obtain a lower bound, we localize the two-body Hamiltonian

$$H_2 = h_x + h_y - \frac{\omega}{|x-y|}, \quad \text{where } h = \sqrt{-\Delta + m^2} + V \geq m > 0$$

using the spectral projector $P = \mathbb{1}(h \leq L)$ associated to the one-body operator h. Here $L \geq 0$ is an energy cut-off which will be optimized. Since h has compact resolvent, the dimension of the low-lying subspace (or equivalently, the number of eigenvalues of h below the energy cut-off L) is finite and we have

$$N_L := \dim(PL^2(\mathbb{R}^3)) \leq CL^{3+\frac{3}{p}}.$$

This is the relativistic version of the Cwikel–Lieb–Rosenblum bound and is a particular case of Lieb–Thirring-type inequality [**35**, Chapter 4]. After applying the quantitative de Finetti theorem, the error in the energy estimate is given by N_L and the second moment which will be controlled by delicate new inequalities and a priori estimates. We want this error term to be of order $E^{\mathrm{H}}_{\omega_N}(1)$, which is achieved by optimizing over L and taking the restriction $0 < \beta < \frac{p}{17p+15}$. Consequently, this shows that the final de Finetti measure must be concentrated on the Gagliardo–Nirenberg-type optimizers, which gives the convergence of density matrices.

2.3. Overview of Paper E. Blow-Up of 2D Focusing Mixture Bose Gases

In this section, we summarize the results in the paper E. We study the collapse of the many-body system (1.2.1) which is used to model two-component Bose–Einstein condensates with attractive intra-species interactions and either attractive or repulsive inter-species interactions. Let us rewrite H_N in (1.2.1) as follows

$$H_N = H_{N_1} + H_{N_2} - \mathcal{V}_{N_1,N_2} \tag{2.3.1}$$

where H_{N_σ}, with $\sigma \in \{1,2\}$, denotes the single-component Hamiltonian

$$H_{N_\sigma} = \sum_{i=1}^{N_\sigma} \big(-\Delta_{x_i} + V_\sigma(x_i) \big) - \frac{1}{N_\sigma - 1} \sum_{1 \leq i < j \leq N_\sigma} w_N^{(\sigma)}(x_i - x_j) \tag{2.3.2}$$

and $\mathcal{V}_{N_1,N_2}$ denotes the inter-species interaction between two species

$$\mathcal{V}_{N_1,N_2} = \frac{1}{N} \sum_{i=1}^{N_1} \sum_{r=1}^{N_2} w_N^{(12)}(x_i - y_r). \tag{2.3.3}$$

The natural Hilbert space associated to (2.3.2) is $\mathcal{H}_{N_\sigma} := L^2_{\mathrm{sym}}(\mathbb{R}^{2N_\sigma})$ and the Hilbert space associated to (2.3.1) is $\mathcal{H}_N = \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}$. The symmetry by exchanging particles from the two species is not allowed. This is the main difficulty in the mixture condensates to compare with the single condensates.

We are intersted in the large-N behavior of the ground state energy per particle of H_N in (2.3.1), given by $E^{\mathrm{Q}}_N := N^{-1} \inf \mathrm{spec} H_N$, and the corresponding (mixture) ground states in the collapse regimes. As a starting point, we first consider the collapse phenomenon in the effective non-linear Schrödinger (NLS) theory, given by (1.2.9). Let us rewrite the NLS functional associated to (2.3.1) as follows

$$\mathcal{E}^{\rm NLS}(u_1,u_2)=c_1\mathcal{E}_1^{\rm NLS}(u_1)+c_2\mathcal{E}_2^{\rm NLS}(u_2)-c_1c_2a_{12}\int_{\mathbb{R}^2}|u_1(x)|^2|u_2(x)|^2{\rm d}x, \tag{2.3.4}$$

where $c_1=\frac{N_1}{N}$ and $c_2=\frac{N_2}{N}$ are fixed. Here $\mathcal{E}_1^{\rm NLS}$ and $\mathcal{E}_2^{\rm NLS}$ are one-component NLS functionals associated to (2.3.2), given by

$$\mathcal{E}_\sigma^{\rm NLS}(u_\sigma)=\int_{\mathbb{R}^2}\left[|\nabla u_\sigma(x)|^2+V_\sigma(x)|u_\sigma(x)|^2-\frac{a_\sigma}{2}|u_\sigma(x)|^4\right]{\rm d}x, \tag{2.3.5}$$

for $\sigma\in\{1,2\}$. The one-component NLS energy is defined analogously to (1.2.9), i.e.,

$$E_\sigma^{\rm NLS}:=\inf\left\{\mathcal{E}_\sigma^{\rm NLS}(u_\sigma):u_\sigma\in H^1(\mathbb{R}^2),\int_{\mathbb{R}^2}|u_\sigma(x)|^2{\rm d}x=1\right\}. \tag{2.3.6}$$

We recall that the parameters $a_1>0$ and $a_2>0$ in (2.3.5) correspond to the scattering length of attractive intra-species interactions for each one-component system. Moreover, the parameter a_{12} in (2.3.4) corresponds to the scattering length of inter-species interactions between two components of the system. It can be either attractive or repulsive.

Note that we can restrict the minimization problem (2.3.4) to non-negative functions since $\mathcal{E}^{\rm NLS}(u_1,u_2)\ge\mathcal{E}^{\rm NLS}(|u_1|,|u_2|)$, for any $u_1,u_2\in H^1(\mathbb{R}^3)$. The same situation applies for (2.3.6). This follows from the fact that $\|\nabla u\|_{L^2}\ge\|\nabla|u|\|_{L^2}$, for any $u\in H^1(\mathbb{R}^3)$ (see [**33**, Theorem 7.8]). In particular, a ground state for $E^{\rm NLS}$ (if it exists) can be chosen to be non-negative. The existence and non-existence of two-component NLS ground states and its blow-up profile have been proved by Guo–Zeng–Zhou [**15, 16**]. See also Guo–Seiringer [**13**] for (2.3.6). Their results were established for potentials having finitely many minima. Let us assume here, for precise analysis and for simplicity, that the external potentials V_1 and V_2 have only one minima and are chosen of the typical forms

$$V_\sigma(x)=|x-z_\sigma|^{p_\sigma},\quad \sigma\in\{1,2\}, \tag{2.3.7}$$

where $z_\sigma\in\mathbb{R}^2$ and $p_\sigma>0$. Those are generalizations of the harmonic trapping potentials commonly used in laboratory experiments.

In the one-component setting, Guo–Seiringer [**13**] showed that if $V_\sigma\ge 0$ is a trapping potential, i.e., $\lim_{|x|\to\infty}V_\sigma(x)=\infty$, then $E_\sigma^{\rm NLS}$ in (2.3.6) has a ground state u_{a_σ}, for $0<a_\sigma<a_*$. Moreover, they also proved that if V_σ is of the form (2.3.7) then, up to a subsequence of u_{a_σ}, we have[3]

$$\lim_{a_\sigma\nearrow a_*}\Lambda_\sigma^{-1}(a_*-a_\sigma)^{\frac{1}{p_\sigma+2}}u_{a_\sigma}(\Lambda_\sigma^{-1}(a_*-a_\sigma)^{\frac{1}{p_\sigma+2}}x+z_\sigma)=(a_*)^{-\frac12}Q(x)=:Q_0(x) \tag{2.3.8}$$

strongly in $L^q(\mathbb{R}^2)$ for all $2\le q<\infty$. Here z_σ are minimum points of V_σ,

$$\Lambda_\sigma=\left(\frac{p_\sigma}{2}\int_{\mathbb{R}^2}|x|^{p_\sigma}|Q(x)|^2{\rm d}x\right)^{\frac{1}{p_\sigma+2}}, \tag{2.3.9}$$

[3] In fact, Guo–Seiringer [**13**] considered the more general situation where V_σ has finitely many minima similarly to (2.2.4).

and Q is the unique solution of (1.2.12). In fact, the convergence (2.3.8) holds in $H^1(\mathbb{R}^2)$. In addition, we also have the asymptotic behavior of the one-component NLS energy

$$\lim_{a_\sigma \nearrow a_*} \frac{E_\sigma^{\mathrm{NLS}}}{(a_* - a_\sigma)^{\frac{p_\sigma}{p_\sigma+2}}} = \frac{p_\sigma + 2}{p_\sigma} \cdot \frac{\Lambda_\sigma^2}{a_*}.$$

Recently, Lewin–Nam–Rougerie [**31**] extended the above results to the rotating Bose gases and obtained the blow-up profile of NLS *approximate* ground states. In addition, together with a Feynman–Hellmann-type argument, they also studied the collapse of the one-component many-body system (2.3.2) in the collapse regime $N_\sigma \to \infty$ simultaneously $a_\sigma := a_{N_\sigma} \nearrow a_*$. More precisely, the asymptotic behavior of the ground state energy per particle, which is defined by $E_{N_\sigma}^{\mathrm{Q}} = N^{-1} \inf \operatorname{spec} H_{N_\sigma}$, follows from that of the one-component NLS energy E_σ^{NLS}. It is given by

$$E_{N_\sigma}^{\mathrm{Q}} = E_\sigma^{\mathrm{NLS}} + o(E_\sigma^{\mathrm{NLS}})_{N_\sigma \to \infty} = (a_* - a_{N_\sigma})^{\frac{p_\sigma}{p_\sigma+2}} \left(\frac{p_\sigma + 2}{p_\sigma} \cdot \frac{\Lambda_\sigma^2}{a_*} + o(1)_{N_\sigma \to \infty} \right).$$

In fact, the following conditions arise from the energy estimate,

$$0 < \beta < \frac{1}{2} \quad \text{and} \quad a_{N_\sigma} = a_* - N_\sigma^{-\gamma_\sigma} \quad \text{with} \quad 0 < \gamma_\sigma < \min\left\{ \frac{p_\sigma + 2}{p_\sigma + 3}\beta, \frac{p_\sigma + 2}{p_\sigma}(1 - 2\beta) \right\}.$$

Furthermore, the many-body ground states for (2.3.2) exhibit condensation on the unique (normalized) solution Q_0 of (1.2.12). It is formulated by the k-particle reduced density matrices which was introduced in (1.2.10) and is given by

$$\lim_{N_\sigma \to \infty} \operatorname{Tr} \left| \gamma_{\Phi_{N_\sigma}}^{(k)} - |Q_0^{\otimes k}\rangle\langle Q_0^{\otimes k}| \right| = 0, \quad \forall k \in \mathbb{N}. \tag{2.3.10}$$

Here $\Phi_{N_\sigma} = \ell_{N_\sigma}^{-N_\sigma} \Psi_{N_\sigma}(\ell_{N_\sigma}^{-1} \cdot)$, with $\ell_{N_\sigma} = \Lambda_\sigma (a_* - a_{N_\sigma})^{-\frac{1}{p_\sigma+2}}$, is the rescaling of a many-body ground state Ψ_{N_σ} for (2.3.2).

In the two-component setting, the situation is more complicated because of the presence of the inter-species interactions. So far, in the literature, the mixture condensates of Bose gases have been carried out only in the level of the NLS theory [**15, 16**]. In the repulsive case, i.e., $a_{12} < 0$, it is obvious that $E^{\mathrm{NLS}} \geq E_1^{\mathrm{NLS}} + E_2^{\mathrm{NLS}}$. Moreover, we have that the ground states u_σ for E_σ^{NLS} in (2.3.6) decays exponentially at the collapse length $\ell_\sigma := (a_* - a_\sigma)^{\frac{1}{p_\sigma+1}} \to 0$ as $a_\sigma \nearrow a_*$ (see, e.g., [**16**, Proposition A]) and their behaviors depend crucially on the minima z_σ of V_σ. If $z_1 \neq z_2$, then we do not see the contribution of the cross term $-a_{12} \int_{\mathbb{R}^2} |u_1(x)|^2 |u_2(x)|^2 \mathrm{d}x$ in the collapse regime $(a_1, a_2) \nearrow (a_*, a_*)$. Hence, we have formally that $E^{\mathrm{NLS}} \approx E_1^{\mathrm{NLS}} + E_2^{\mathrm{NLS}}$ and the existence of ground states for E^{NLS} is obtained under the conditions that $a_{12} < 0$ is fixed and $0 < a_1, a_2 < a_*$. Furthermore, the collapse phenomenon of the system (2.3.1), in the limit regime $(a_1, a_2) := (a_{1,N}, a_{2,N}) \nearrow (a_*, a_*)$ as $N \to \infty$, is somehow similar to the one of (2.3.2). The latter case was analyzed in great mathematical detail by Lewin–Nam–Rougerie [**31**].

In our two-component setting, we have the following result.

THEOREM 9 ([**47**]). *Assume that $z_1 \neq z_2$ in (2.3.7), $0 < \beta < \frac{1}{2}$, $a_{12} < 0$ is fixed and $a_\sigma := a_{\sigma,N} = a_* - N^{-\gamma_\sigma}$, for $\sigma \in \{1, 2\}$, with*

$$\frac{\gamma_1}{\gamma_2} = \frac{p_1 + 2}{p_2 + 2} \quad \text{and} \quad 0 < \gamma_\sigma < \min\left\{ \frac{p_\sigma + 2}{p_\sigma + 3}\beta, \frac{p_\sigma + 2}{p_\sigma}(1 - 2\beta) \right\}.$$

Let Ψ_N *be a (mixture) ground state for* H_N *in* (2.3.1). *Let*

$$\Phi_N(x_1,\dots,x_{N_1};y_1,\dots,y_{N_2}) = \frac{\Psi_N\left(\frac{x_1}{\ell_{1,N}}+z_1,\dots,\frac{x_{N_1}}{\ell_{1,N}}+z_1;\frac{y_1}{\ell_{2,N}}+z_2,\dots,\frac{y_{N_2}}{\ell_{2,N}}+z_2\right)}{\ell_{1,N}^{N_1}\ell_{2,N}^{N_2}}$$

where $\ell_{\sigma,N} = \Lambda_\sigma(a_* - a_{\sigma,N})^{-\frac{1}{p_\sigma+2}}$ *with* Λ_σ *are given by* (2.3.9). *Then, up to extraction of a subsequence, we have*

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_{\Phi_N}^{(k,\ell)} - \left|Q_0^{\otimes k}\otimes Q_0^{\otimes \ell}\right\rangle\left\langle Q_0^{\otimes k}\otimes Q_0^{\otimes \ell}\right|\right| = 0, \quad \forall k,\ell\in\mathbb{N}, \tag{2.3.11}$$

where Q_0 *is the unique (normalized) solution of* (1.2.12). *In addition, we have*

$$E_N^{\mathrm{Q}} = \sum_{\sigma=1}^{2} c_\sigma E_\sigma^{\mathrm{NLS}} + o(E_\sigma^{\mathrm{NLS}}) = \sum_{\sigma=1}^{2} c_\sigma (a_* - a_{\sigma,N})^{\frac{p_\sigma}{p_\sigma+2}}\left(\frac{p_\sigma+2}{p_\sigma}\cdot\frac{\Lambda_\sigma^2}{a_*} + o(1)_{N\to\infty}\right).$$

REMARK 10. *The condition* $\frac{\gamma_1}{\gamma_2} = \frac{p_1+2}{p_2+2}$ *is a technical assumption which yields that* $\ell_{1,N}$ *and* $\ell_{2,N}$ *have the same asymptotic behavior when* $N\to\infty$. *This will only be used to prove the convergence of ground states in* (2.3.11)*, but not the asymptotic behavior of the ground state energy per particle.*

The case of a totally attractive system presents a more interesting problem. The behavior of the NLS energy and its ground states depend crucially on the scattering length of attractive inter-species interactions. By standard methods in the calculus of variations, the existence of ground states for E^{NLS} is obtained under the conditions $0 < a_1, a_2 < a_*$ and $0 < a_{12} < \sqrt{c_1^{-1}c_2^{-1}(a_*-a_1)(a_*-a_2)}$. Furthermore, $E^{\mathrm{NLS}} = -\infty$ if either $a_1 > a_*$ or $a_2 > a_*$ or $a_{12} > 2^{-1}c_1^{-1}c_2^{-1}(a_* - c_1a_1 - c_2a_2)$. The case

$$\sqrt{c_1^{-1}c_2^{-1}(a_*-a_1)(a_*-a_2)} \le a_{12} \le 2^{-1}c_1^{-1}c_2^{-1}(a_* - c_1a_1 - c_2a_2) \tag{2.3.12}$$

is left open. In this case, the existence and non-existence of the NLS ground states depend crucially on V_1, V_2 and all the parameters in (2.3.12) (see, e.g., [**15**, Theorems 1.2 and 1.3]). Of course there is no more discussion about this issue when $c_1(a_*-a_1) = c_2(a_*-a_2)$. In that case, it was proven that a NLS ground state exists at the threshold point $(a_1,a_2,a_{12}) = (a_*-c_2a_{12}, a_*-c_1a_{12}, a_{12})$ if potentials have different minimum point (see, e.g., [**15**, Theorem 1.3 and Example 1.1]). Therefore, in order to study the behavior of the NLS ground states, we assume that V_1 and V_2 in (2.3.7) have a common minimum $z_1 = z_2$. Those points can be assumed to be at the origin without loss of generality.

We will consider two collapse regimes: either we fix $0 < a_{12} < a_*\min\{c_1^{-1}, c_2^{-1}\}$ and we take $(a_1,a_2) \nearrow (a_*-c_2a_{12}, a_*-c_1a_{12})$, or we fix $0 < a_1, a_2 < a_*$ and we take $a_{12} \nearrow \alpha_*$ for some critical constant α_* depending on a_1, a_2 and a_*. In the first case, the blow-up profile of E^{NLS} and its ground states was analyzed in detail by Guo–Zeng–Zhou [**15**]. They showed that, up to extraction of a subsequence, the NLS ground state (u_{a_1}, u_{a_2}) converges to the unique (normalized) solution Q_0 of (1.2.12), i.e.,

$$\lim_{(a_1,a_2)\nearrow(a_*-c_2a_{12},a_*-c_1a_{12})} \Lambda^{-1}(a_*-a)^{\frac{1}{p_0+2}} u_{a_\sigma}(\Lambda^{-1}(a_*-a)^{\frac{1}{p_0+2}}x) = Q_0(x) \tag{2.3.13}$$

strongly in $H^1(\mathbb{R}^2)$, for $\sigma \in \{1,2\}$, where $a = c_1a_1 + c_2a_2 + 2c_1c_2a_{12}$, $p_0 = \min\{p_1,p_2\}$ and

$$\Lambda = \left(\frac{p_0\nu}{2}\int_{\mathbb{R}^2}|x|^{p_0}|Q(x)|^2\mathrm{d}x\right)^{\frac{1}{p_0+2}} \quad \text{with} \quad \nu = \lim_{x\to 0}\frac{c_1V_1(x)+c_2V_2(x)}{|x|^{p_0}}. \tag{2.3.14}$$

In addition, we also have the asymptotic behavior of the NLS energy

$$\lim_{(a_1,a_2)\nearrow(a_*-c_2a_{12},a_*-c_1a_{12})}\frac{E^{\mathrm{NLS}}}{(a_*-a)^{\frac{p_0}{p_0+2}}} = \frac{p_0+2}{p_0}\cdot\frac{\Lambda^2}{a_*}.$$

The following is the many-body version of the above results.

Theorem 11 ([**47**]). *Assume that* $z_1 = 0 = z_2$ *and* $p_0 = \min\{p_1,p_2\}$ *in* (2.3.7), $0<\beta<\frac{1}{2}$, $0 < a_{12} < a_*\min\{c_1^{-1},c_2^{-1}\}$ *is fixed and* $(a_1,a_2) := (a_{1,N},a_{2,N}) \nearrow (a_*-c_2a_{12}, a_*-c_1a_{12})$ *such that* $a_N := c_1a_{1,N}+c_2a_{2,N}+2c_1c_2a_{12} = a_*-N^{-\gamma}$ *with*

$$0<\gamma<\min\left\{\frac{p_0+2}{p_0+3}\beta, \frac{p_0+2}{p_0}(1-2\beta)\right\}.$$

Let Ψ_N *be a (mixture) ground state for* H_N *in* (2.3.1). *Let* $\Phi_N = \ell_N^{-N}\Psi_N(\ell_N^{-1}\cdot)$ *where* $\ell_N = \Lambda(a_*-a_N)^{-\frac{1}{p_0+2}}$ *with* Λ *given by* (2.3.14). *Then, up to extraction of a subsequence, we have*

$$\lim_{N\to\infty}\operatorname{Tr}\left|\gamma_{\Phi_N}^{(k,\ell)} - \left|Q_0^{\otimes k}\otimes Q_0^{\otimes \ell}\right\rangle\left\langle Q_0^{\otimes k}\otimes Q_0^{\otimes \ell}\right|\right| = 0, \quad \forall k,\ell\in\mathbb{N}, \tag{2.3.15}$$

where Q_0 *is the unique (normalized) solution of* (1.2.12). *In addition, we have*

$$E_N^{\mathrm{Q}} = E^{\mathrm{NLS}} + o(E^{\mathrm{NLS}})_{N\to\infty} = (a_*-a_N)^{\frac{p_0}{p_0+2}}\left(\frac{p_0+2}{p_0}\cdot\frac{\Lambda^2}{a_*}+o(1)_{N\to\infty}\right).$$

Remark 12. *The condition* $0<\gamma<\frac{p_0+2}{p_0}(1-2\beta)$ *implies that we consider mean-field interactions. This corresponds to a high density regime where the particles meet frequently but interact weakly since the typical interaction length is larger than the average distance between the particles. On the other hand, the condition* $\gamma < \frac{p_0+2}{p_0+3}\beta$ *ensures that the Hartree and NLS energies are close in the limit* $N\to\infty$.

There exists another setting for which it is reasonable to study the blow-up behavior of ground states in the case of attractive inter-species interactions: fix $0 < a_1, a_2 < a_*$ and take $a_{12} := \alpha_N \nearrow \alpha_*$ as $N\to\infty$, for some critical value $0<\alpha_*<2^{-1}c_1^{-1}c_2^{-1}(a_*-c_1a_1-c_2a_2)$. Since there is a gap in the existence theory for NLS ground states, we will consider only the case $c_1(a_*-a_1) = c_2(a_*-a_2)$. In this special case, we note that there will be no more discussion on (2.3.12). Furthermore, Theorem 11 covers the case $c_1(a_*-a_1) = c_2(a_*-a_2)$. We have the following.

Theorem 13 ([**47**]). *Assume that* $z_1 = 0 = z_2$ *and* $p_0 = \min\{p_1,p_2\}$ *in* (2.3.7), $0<\beta<\frac{1}{2}$, $0 < a_1, a_2 < a_*$ *are fixed such that* $c_1(a_*-a_1) = c_1c_2\alpha_* = c_2(a_*-a_2)$ *and* $0 < a_{12} := \alpha_N = \alpha_* - N^{-\gamma}$ *with*

$$0<\gamma<\min\left\{\frac{p_0+2}{p_0+3}\beta, \frac{p_0+2}{p_0}(1-2\beta)\right\}.$$

Let Ψ_N *be a (mixture) ground state for* H_N *in* (2.3.1). *Let* $\Phi_N = \ell_N^{-N}\Psi_N(\ell_N^{-1}\cdot)$ *where* $\ell_N = \Theta(\alpha_*-\alpha_N)^{-\frac{1}{p_0+2}}$ *and*

$$\Theta = \left(\frac{p_0\nu}{4c_1c_2}\int_{\mathbb{R}^2}|x|^{p_0}|Q(x)|^2\mathrm{d}x\right)^{\frac{1}{p_0+2}} \quad \textit{with} \quad \nu = \lim_{x\to 0}\frac{c_1V_1(x)+c_2V_2(x)}{|x|^{p_0}}. \tag{2.3.16}$$

Then, up to extraction of a subsequence, we have

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_{\Phi_N}^{(k,\ell)} - \left|Q_0^{\otimes k}\otimes Q_0^{\otimes \ell}\right\rangle\left\langle Q_0^{\otimes k}\otimes Q_0^{\otimes \ell}\right|\right| = 0, \quad \forall k,\ell\in\mathbb{N}, \tag{2.3.17}$$

where Q_0 is the unique (normalized) solution of (1.2.12)*. In addition, we have*

$$E_N^{\mathrm{Q}} = E^{\mathrm{NLS}} + o(E^{\mathrm{NLS}})_{N\to\infty} = (\alpha_* - \alpha_N)^{\frac{p_0}{p_0+2}}\left(2c_1c_2\frac{p_0+2}{p_0}\cdot\frac{\Theta^2}{a_*} + o(1)_{N\to\infty}\right).$$

Remark 14. *To avoid the assumption $c_1(a_*-a_1) = c_2(a_*-a_2)$, one might consider a more evolved NLS model, where the constraint condition in* (1.2.9) *is replaced by $\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2 = 1$ (see, e.g.,* [**12**]*). However, it is not clear to us the many-body theory behind this.*

Let us discuss briefly the strategy of the proofs of Theorems 9, 11 and 13. As first step, we study the collapse of the ground state energy per particle by giving explicitly error terms compared to the NLS energy. This will be done via the Hartree energy. In fact, by adapting Lewin's arguments [**28**, Section 3] for the one-component system we obtain that

$$E^{\mathrm{H}} \geq E_N^{\mathrm{Q}} \geq E^{\mathrm{H}} - CN^{2\beta-1}. \tag{2.3.18}$$

Here we focus on the lower bound since the upper bound is trivial, by the variational principle. The proof contains two ingredients. One is to apply the Hoffmann-Ostenhof inequality [**18**] to bound the full kinetic energy from below by that of the square roots of $(1,0)$- and $(0,1)$-particle density. This is done separately for each species. The other ingredient is to get lower bounds on the interactions. This follows by a variant of Onsager's lemma [**48**] and a trick due to Lévy-Leblond [**26**] and works for arbitrary (regular) interaction. The price to pay in this complicated process is $N^{2\beta-1}$. This comes from the assumption in (1.2.3) that $\widehat{w^{(\sigma)}} \in L^1(\mathbb{R}^2)$, for $\sigma\in\{1,2,12\}$. In the next step, we compare the Hartree and NLS energies. We first note that the intra-species interactions can be estimated by the Cauchy–Schwarz inequality as follows

$$\begin{aligned}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |u_\sigma(x)|^2 w_N^{(\sigma)}(x-y)|u_\sigma(y)|^2\mathrm{d}x\mathrm{d}y &\leq \iint_{\mathbb{R}^2\times\mathbb{R}^2} w_N^{(\sigma)}(x-y)\frac{|u_\sigma(x)|^4+|u_\sigma(y)|^4}{2}\mathrm{d}x\mathrm{d}y\\ &= \int_{\mathbb{R}^2}|u_\sigma(x)|^4\mathrm{d}x \end{aligned}\tag{2.3.19}$$

The reverse inequality, however, is more subtle and we will need the technical assumption (1.2.3) that $(1+|x|)w^{(\sigma)}\in L^1(\mathbb{R}^2)$, for $\sigma\in\{1,2,12\}$. This will also be used to estimate the inter-species interaction since it cannot be done by (2.3.19). The final estimate gives an error term depending on N and β, i.e.,

$$E^{\mathrm{NLS}} - CN^{-\beta} \geq E_N^{\mathrm{Q}} \geq E^{\mathrm{NLS}} - CN^{-\beta} - CN^{2\beta-1}. \tag{2.3.20}$$

Thus, it is natural to require $0<\beta<\frac{1}{2}$ in order to obtain the convergence of the ground state energy per particle to the NLS energy. Furthermore, the asymptotic behavior of the ground state energy per particle follows from that of the latter which was given in [**15, 16**].

Having the blow-up profile of the ground state energy par particle, we are able to give an explicit blow-up profile for its many-body ground states. This will be done via a Feynman–Hellmann-type argument (see, e.g., [**36**]). Such an argument was recently used by Lewin–Nam–Rougerie [**31**] to study the collapse of the many-body system arising in a one-component BEC with an attractive interaction (2.3.2). We emphasize that this argument relies on the energy estimate (2.3.20) and the uniqueness of the limiting profile, i.e., the positive solution of (1.2.12). In the one-component setting, this allows us to reduce the problem of convergence of the single-reduced density matrix to that of the NLS *approximate* ground states (in the sense of energy) whose physical properties are now easier to obtain. In the two-component setting, such an argument can be reused when the inter-species interactions are repulsive since we can ignore them in the energy estimate. On the other hand, the situation for the totally attractive system is more complicated and we will need to study the blow-up behavior in the following variational problem

$$E_N^{\mathrm{mH}} := \inf\left\{\mathcal{E}_N^{\mathrm{mH}}(u_1,u_2) : u_1,u_2 \in H^1(\mathbb{R}^2), \int_{\mathbb{R}^2}|u_1(x)|^2\mathrm{d}x = 1 = \int_{\mathbb{R}^2}|u_2(x)|^2\mathrm{d}x\right\}. \quad (2.3.21)$$

Here the (modified) Hartree functional is given by

$$\mathcal{E}_N^{\mathrm{mH}}(u_1,u_2) = c_1\mathcal{E}_1^{\mathrm{NLS}}(u_1) + c_2\mathcal{E}_2^{\mathrm{NLS}}(u_2) - c_1c_2a_{12}\int_{\mathbb{R}^2}|u_1(x)|^2(w_N^{(12)}\star|u_2|^2)(x)\mathrm{d}x,$$

which interpolates between Hartree (1.2.6) and NLS (1.2.9).

Considering the perturbed Hamiltonian in the group of N_σ particles and using a Feynman–Hellmann-type argument twice, we obtain the convergence of the marginals $\gamma_{\Psi_N}^{(1,0)}$ and $\gamma_{\Psi_N}^{(0,1)}$ of the ground state Ψ_N to a rank-one projector. It turns out that this implies the convergence of the generic $\gamma_{\Psi_N}^{(k,\ell)}$, for all $k,\ell \in \mathbb{N}$. This was discussed in great mathematical detail by Michelangeli–Olgiati [**39**, Section 3]. More precisely, we estimate the ground state energy per particle of a perturbed Hamiltonian from below by the Hartree energy. However, the proof of (2.3.18) uses the Hoffmann-Ostenhof inequality, which is not available here, to bound the one-body energy from below. Fortunately, the Onsager's lemma is general and we can still use it to estimate the ground state energy per particle from below by the ground state energy of a Hartree-type functional defined on trace-class self-adjoint operators on $L^2(\mathbb{R}^2)$ (note that the $(1,0)$- and $(0,1)$-reduced density matrices of a mixture many-body ground state are). Using (2.3.19) and the attractiveness of intra-species interactions, the latter is bounded from below by the (modified) Hartree energy (2.3.21). Finally, the convergence of the many-body ground states follows from that of the (modified) Hartree *approximate* ground states.

CHAPTER 3

Conclusions and Perspectives

In this chapter, we describe some problems for future research, which are related to the subject presented in the present thesis.

3.1. The Uniqueness of Hartree and HFB Minimizers

In [**45**] we showed that the HFB minimizers converge strongly to the (unique) Lane–Emden solution. The uniqueness of HFB minimizers is therefore expected. This might be a very difficult problem even if we consider the reduce HF minimizers. This is mainly due to the non-locality of the pseudo-differential operator $\sqrt{-\Delta+m^2}$ and the convolution-type non-linearity as well. The first question could be to address the case where the exchange and pairing terms are trivial and the minimizer is a pure state $|u\rangle\langle u|$. This corresponds to the problem of the uniqueness for the boson star equation

$$\sqrt{-\Delta+m^2}u-(|\cdot|^{-1}\star|u|^2)u=\mu u$$

with $m\geq 0$ and some constants $\mu<0$ that depends on u. However, this is still a major open problem. For small values of $\|u\|_{L^2}$, this was proved by Lenzmann [**23**]. See also [**8**] for the uniqueness of non-linear ground states for fractional Laplacian.

3.2. Stability of 2D Focusing Mixture Bose Gases

In [**47**] we showed that the quantum energy per particles of a two-component system converges to the NLS energy for any $0<\beta<\frac{1}{2}$, i.e.,

$$\lim_{N\to\infty}E_N^{\mathrm{Q}}=E^{\mathrm{NLS}}. \tag{3.2.1}$$

The proof of (3.2.1) was done via the Hartree energy E_N^{H}. While the convergence $\lim_{N\to\infty}E_N^{\mathrm{H}}=E^{\mathrm{NLS}}$ follows from standard analysis, it is complicated to estimate E_N^{Q} by E_N^{H}. In fact, we obtained the latter for a more general model where we made only the positivity preserving of kinetic energies. What makes this possible is the Hoffmann-Ostenhof inequality applying to the kinetic energy in each species. Furthermore, the intra-species and inter-species interactions can be either attractive or repulsive.

The next step is to find the maximum value $\beta_{\max}>0$ such that (3.2.1) holds for any $0<\beta\leq\beta_{\max}$. In the one-component setting, the convergences of energy and of (approximate) ground states were first proved by Lewin [**28**] for $0<\beta<\frac{1}{2}$ and were later extended to the range $0<\beta<1$ by Nam–Rougerie [**42**] (see also Lewin–Nam–Rougerie [**32**] for an earlier result). The major ingredient in [**32, 42**] is a quantitative version of the quantum de Finetti theorem and a localization method in the Fock space by Lewin [**27**]. In the two-component setting, an analogous version of the first ingredient was given by Michelangeli–Nam–Olgiati

[**38**]. However, it is not clear how to use the second ingredient, due to the presence of the inter-species interactions.

3.3. Blow-Up of Mixture Bose Gases with Repulsive Intra-Species Interaction(s)

An interesting open problem in the mixture BEC is to study the collapse of the two-component many-body system in the cases if either $a_1 < 0$ or $a_2 < 0$ or $a_1, a_2 < 0$ and $a_{12} > 0$. Note that there might be blow-up in those cases since there is at least an attractive interaction. By (3.2.1), we obtain the behavior of the quantum energy per particle as soon as we know that of the NLS energy. Unfortunately, the arguments in [**47**] do not allow us to obtain the behavior of the many-body ground states since we need the attractive intra-species interactions in the energy estimate in order to use a Feynman–Hellman-type argument.

When there is at least one repulsive intra-species interaction, one possible approach to obtain the convergence of many-body ground states is to use the quantitative version of quantum de Finetti theorem and a localization method. When these will be understood, one can adapt the arguments in [**46**] which were given for the blow-up of boson stars.

3.4. Focusing (Mixture) Quantum Dynamics

From the time-dependent viewpoint, it would be nice if one can prove the validity of the effective dynamics for focusing mixtures condensates which is governed by a system of two coupled non-linear Schrödinger equations

$$\begin{aligned} i\partial u_t &= (-\Delta + V_1(x) - a_1|u_t|^2 - c_2 a_{12}|v_t|^2)u_t, \\ i\partial v_t &= (-\Delta + V_2(x) - a_2|v_t|^2 - c_1 a_{12}|u_t|^2)v_t, \end{aligned} \tag{3.4.1}$$

with initial condition $u_{|t=0} = u_0$ and $v_{|t=0} = v_0$. In the subcritical regime, one want to prove that if the initial state $\Psi_{N,0}$ is asymptotically factorized, in the sense that,

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma^{(1,1)}_{\Psi_{N,0}} - \big|u_0 \otimes v_0\big\rangle\big\langle u_0 \otimes v_0\big|\right| = 0,$$

then for every time $t > 0$, the evolved state $\Psi_{N,t} = e^{-itH_N}\Psi_{N,0}$ with H_N given by (1.2.1) condensates on the solution of the system of time-dependent NLS equations (3.4.1), i.e.,

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma^{(k,\ell)}_{\Psi_{N,t}} - \big|u_t^{\otimes k} \otimes v_t^{\otimes \ell}\big\rangle\big\langle u_t^{\otimes k} \otimes v_t^{\otimes \ell}\big|\right| = 0, \quad \forall k, \ell \in \mathbb{N}.$$

In the one-component setting, the derivation of the time-dependent 2D focusing NLS in a harmonic trap from many-body quantum dynamics has been achieved by Chen–Holmer [**5**] (see also Jeblick–Pickl [**20**]). The results in [**5**] are obtained for any $0 < \beta < \frac{3}{4}$ as soon as the stability of the second kind, i.e., $H_N \geq -CN$, is verified for that range of β, which was confirmed by Lewin–Nam–Rougerie [**32**]. Recently, such a result was obtained again by Nam–Napiórkowski [**41**] with a different method that emploits the Bogoliubov approximation [**4**]. Their results hold for the range $0 < \beta < 1$ and without using the stability condition.

Another very interesting open problem is to study the dynamical collapse for the focusing mixture system. In the supercritical regime, one expects that if the solution of the system of NLS equations (3.4.1) blows up at finite time T, in the sense that $\|u_t\|^2_{H^1} + \|v_t\|^2_{H^1} \to \infty$ as $t \nearrow T$, then also the solution of the evolved Schrödinger equation collapses, in the sense that $\operatorname{Tr}\left(-\Delta\gamma^{(1,0)}_{\Psi_{N,t}}\right) + \operatorname{Tr}\left(-\Delta\gamma^{(0,1)}_{\Psi_{N,t}}\right) \to \infty$ as $t \to T^-$ simultaneously $N \to \infty$. The first question should be to address the one-component case. In that case, the dynamical collapse of boson stars (with a regularized Newton potential) has been studied by Michelangeli–Schlein [**40**].

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