

Average-case Complexity of Teaching Convex Polytopes via Halfspace Queries

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Abstract

We examine the task of locating a target region among those induced by intersections of n halfspaces in \mathbb{R}^d . This generic task connects to fundamental machine learning problems, such as training a perceptron and learning a ϕ -separable dichotomy. We investigate the *average teaching* complexity of the task, i.e., the minimal number of samples (halfspace queries) required by a *teacher* to help a version-space learner in locating a *randomly* selected target. As our main result, we show that the average-case teaching complexity is $\Theta(d)$, which is in sharp contrast to the worst-case teaching complexity of $\Theta(n)$. If instead, we consider the average-case learning complexity, the bounds have a dependency on n as $\Theta(n)$ for *i.i.d.* queries and $\Theta(d \log(n))$ for actively chosen queries by the learner. Our proof techniques are based on novel insights from computational geometry, which allow us to count the number of convex polytopes and faces in a Euclidean space depending on the arrangement of halfspaces. Our insights allow us to establish a tight bound on the average-case complexity for ϕ -separable dichotomies, which generalizes the known $\mathcal{O}(d)$ bound on the average number of “extreme patterns” in the classical computational geometry literature (Cover, 1965).

Keywords: Teaching dimension, homogeneous halfspaces, average-case complexity

1. Introduction

We consider the problem of locating a target region among those induced by intersections of n halfspaces in d -dimension (Fig. 1(a)). In the basic setting, the learner receives a sequence of instructions, which we refer to as *halfspace queries* (same as membership queries in Angluin, 1987, 1988), each specifying a halfspace the target region is in. Based on the evidence it receives, the learner then determines the location of the target region. This generic task connects to several fundamental problems in machine learning. Consider learning a linear prediction function in \mathbb{R}^d (aka perceptron, see Fig. 1(b)) over n linearly separable data points. Here, every data point specifies a halfspace, and the target hypothesis corresponds to a region in the hypothesis space. The learning task reduces to identifying the convex polytope induced by the n halfspace constraints in the hypothesis spaces Bishop (2006). Similarly, when the set of data points are not linearly separable, but are separable by a ϕ -surface (aka ϕ -separable dichotomy, see Fig. 1(c)), the problem of finding the ϕ -separable dichotomy could be viewed as training a perceptron in the ϕ -induced space (Cover, 1965).

While these fundamental problems have been extensively studied in the *passive learning* setting (Vapnik and Chervonenkis, 1971; Natarajan, 1987; Blumer et al., 1989; Goldman et al., 1993), the underlying *i.i.d.* sampling strategy often requires more data than necessary to learn the target concept (when one is able to control the sampling strategy). Moreover, the majority of existing work focuses on the worst-case complexity measures, which are often too pessimistic and do not reflect the learning complexity in the real-world scenarios (Haussler et al., 1994; Wan, 2010; Nachum and Yehudayoff, 2019). As shown in Table 1, the label complexity of passive learning for the above generic task is $\Theta(n)$. Recently, there has been increasing interest in understanding the complexity of *interactive learning*, which aims to learn under more optimistic, realistic scenarios, in which “representative” examples are selected, and the number of examples needed for successful learning may shrink significantly. For example, under the *active learning* setting, the learner only query data points that are helpful for the learning task, which could lead to exponential savings in the sample complexity as compared with the passive learning setting (Guillory and Bilmes, 2009; Jamieson and Nowak, 2011; Hanneke and Yang, 2015; Kane et al., 2017).

Type	Average-case	Worst-case	Condition on hyperplane arrangement
Passive learning	$\Theta(n)$	$\Theta(n)$	-
Active learning	$\Theta(d' \log n)$	$\Theta(n)$	d' -relaxed general position
Teaching	$\Theta(d')$	$\Theta(n)$	d' -relaxed general position

Table 1: Sample complexity for various types of data selection algorithms for learning intersection of halfspaces. We assume $d' \leq d$ for the d' -relaxed general position arrangement.

An alternative interactive learning scenario is the setting where the learning happens in the presence of a helpful teacher, which identifies useful examples for the learning task. This setting is known as *machine teaching* (Zhu et al., 2018). Importantly, the label complexity of teaching provides a lower bound on the number of samples needed by active learning (Zilles et al., 2011), and therefore can provide useful insights for designing interactive learning algorithms (Brown and Niekum, 2019). Machine teaching has been extensively studied in terms of the worst-case label complexity (Goldman and Kearns, 1995; Anthony et al., 1995; Zilles et al., 2008; Doliwa et al., 2014; Chen et al., 2018; Mansouri et al., 2019). However, to the best of our knowledge, the *average complexity* of machine teaching, even for the fundamental tasks described above, remains significantly underexplored.

In this paper, we investigate the *average teaching* complexity, i.e., the minimal number of examples required by a *teacher* to help a learner in locating a *randomly* selected target. We highlight our key results below.

- We show that under the common assumption that the n hyperplanes are in general position in \mathbb{R}^d , the average-case complexity for teaching such a target is $\Theta(d)$. This is in sharp contrast to the worst-case teaching complexity of $\Theta(n)$ (cf §4).
- We provide a natural extension of the general-position hyperplane arrangement condition, and show that if the n hyperplanes in \mathbb{R}^d are in “ d' -relaxed general position arrangement” where $d' \leq d$, then one can further obtain improved complexity results of $\Theta(d')$ for average-case teaching. Our proof techniques are based on novel insights from computational geometry, which allow us to count the number of convex polytopes and faces in a Euclidean space

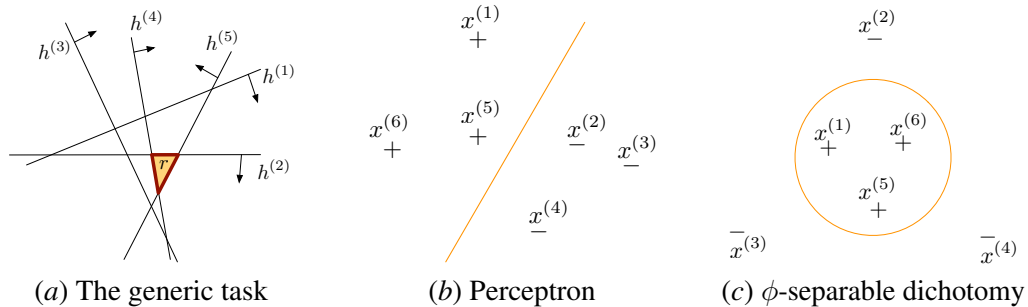


Figure 1: Different tasks as teaching convex polytopes via halfspace queries.

depending on the hyperplane arrangement. Our result improves upon the existing $\mathcal{O}(d)$ result for arbitrary hyperplane arrangement (Fukuda et al., 1991) (cf §4).

- To draw a connection with the learning complexity, we show that without the presence of a teacher, a learning algorithm requires $\Theta(n)$ for *i.i.d.* queries and $\Theta(d \log(n))$ for actively chosen queries. Table 1 summarizes our main complexity results (cf §5).
- Based on our proof framework in §4, we provide complexity results for teaching ϕ -separable dichotomies, which recovers and extends the known $\mathcal{O}(d)$ bound on the average number of “extreme patterns” in the classical computational geometry literature (Cover, 1965) (cf §6).

2. Related Work

Average-case complexity of learning While the majority of complexity measures for concept classes and data selection algorithms focus on the worst-case scenarios, there have been a few works concerning the average-case complexity for various types of learning algorithms. Here we provide a survey on related work concerning average-case complexity under the learning setting. Haussler et al. (1994) studied how the sample complexity depends on properties of a prior distribution on the concept class and over the sequence of examples the algorithm receives. Specifically, they studied the probability of an incorrect prediction for an optimal learning algorithm using the Shannon information gain. Wan (2010) considered the problem of learning DNF-formulas sampled from the uniform distribution. Nachum and Yehudayoff (2019) considered the average information complexity of learning (defined as the average mutual information between the input and the output of the learning algorithm). They show that for a concept class of VC dimension d , there exists a proper learning algorithm that reveals $\mathcal{O}(d)$ bits of information for most concepts. Intuitively, this result aligns with our observation that average complexities of various data selection algorithms are significantly lower than that in the worst-case scenario. Spielman and Teng (2004, 2009) introduce the paradigm of smoothed analysis which differs from our average-case analysis as we don’t allow perturbations to input spaces. Perhaps most similar to our approach, in terms of technical insights, is the work of Jamieson and Nowak (2011), who studied the problem of active ranking via pairwise comparisons, and have used the geometrical properties of hyperplanes in \mathbb{R}^d to achieve an average complexity of $\Theta(d \log n)$ for active ranking over n points. In our work, we extend their results to the general problem of active learning of halfspaces, and also consider the teaching variant of the ranking via pairwise comparison problem.

Connection with the PAC learning framework Intersection of halfspaces have been studied in PAC learning framework (Kwek and Pitt, 1996; Blum and Kannan, 1997; Klivans et al., 2004; Klivans and Sherstov, 2006; Vempala, 2010; Khot and Saket, 2011; Gottlieb et al., 2018). Although we focus on exact teaching of intersections of halfspaces induced by n hyperplanes, our results could be readily extended to analyze the average sample complexity for teaching a PAC learner under the realizable case. It is well known that a single halfspace can be PAC-learnt efficiently by sampling a polynomial number of data points and finding a separating hyperplane via linear programming (Blumer et al., 1989). Relating this to the worst-case sample complexity results in Table 1, we know that the worst-case sample complexity for teaching a halfspace to a PAC learner is also polynomial in the VC dimension, i.e., $n = \mathcal{O}(\text{poly}(d))$ for halfspaces. One can then extend the average-case complexity results in Table 1, based on an argument similar with *pool-based active learning* (McCallumzy and Nigamy, 1998). The idea is for the teacher to draw n unlabeled examples *i.i.d.* from the underlying data distribution in \mathbb{R}^d . Instead of providing all labels, the teacher provides labels to an optimal teaching set such that all unlabeled examples are implied by the given labels. Thus the learner has obtained n labeled examples drawn *i.i.d.*, and classical PAC bounds still apply.

Relevant work in algorithmic machine teaching As discussed above, teaching problem of various concept classes has been explored before. The classic definition of average teaching dimension (Goldman and Kearns, 1995) which is same as our definition in the uniform setting has been studied in various settings: Anthony et al. (1995) showed the bound of $\mathcal{O}(n^2)$ for the class of linearly separable Boolean functions; Kushilevitz et al. (1996) showed an improved upper bound of $\mathcal{O}(|\mathcal{C}|^{\frac{1}{2}})$ for any concept class \mathcal{C} ; Kuhlmann (1999) proved that all classes of VC dimension 1 have an average teaching dimension of less than 2; Lee et al. (2006) have shown an $\mathcal{O}(ns)$ bound on the class of DNFs with at most $s \leq 2^{\Theta(n)}$ terms. In contrast, our work bypasses any dependence on the size of the concept class, and achieves an average teaching complexity of $\Theta(d')$ (where $d' \leq d$). Some more powerful notions of teaching dimension in sequential setting: recursive and preference-based, have been studied in Doliwa et al. (2014); Gao et al. (2017), which differ from our batched setting. There is increasing interest in connecting the VC dimension to the teaching problem of concept classes (stated in Simon and Zilles, 2015; Hu et al., 2017), we notice the VC dimension of n hyperplanes in general position is $\min\{n, d\}$ (Edelsbrunner, 1987) which is closely related to our average-case $\Theta(d')$ result but away from the worst-case $\Theta(n)$ result.

3. Teaching Convex Polytopes via Halfspace Queries: A General Model

Convex polytopes induced by hyperplanes Let $h = \{z \mid \eta \cdot z = b, z \in \mathbb{R}^d\}$ be a hyperplane in \mathbb{R}^d , where $\eta \in \mathbb{R}^d$ and $b \in \mathbb{R}$. We say a point $z \in \mathbb{R}^d$ *satisfies* or *lies* in h if $z \in h$. We define a *halfspace* induced by a hyperplane h to be one of the two connected components of $(\mathbb{R}^d - h)$ i.e. sets corresponding to $\text{sgn}(\eta \cdot z - b)$. We define $\mathcal{H} \triangleq \{h^{(1)}, h^{(2)}, \dots, h^{(n)}\}$ as a set of n hyperplanes in \mathbb{R}^d . The arrangement of the hyperplanes in \mathbb{R}^d , denoted as $\mathcal{A}(\mathcal{H})$, induces *intersections of halfspaces* which create connected components. Any connected component of $\mathbb{R}^d - \cup_{h \in \mathcal{H}} h$ is defined as a *region* or *convex polytope* in \mathbb{R}^d . Equivalently, any region r can be exactly specified by the intersections of halfspaces induced by hyperplanes in \mathcal{H} . We call the smallest subset $B_r \subseteq \mathcal{H}$ that exactly specifies r the *bounding set of hyperplanes* for r . We define connected components induced on hyperplanes (e.g. $h^{(i)} - \cup_{h \in \mathcal{H} \setminus h^{(i)}} h$ for any $h^{(i)} \in \mathcal{H}$) by $\mathcal{A}(\mathcal{H})$ as *faces*. Thus, bounding set B_r forms the faces to the polytope r .

Example 1 (Convex polytopes induced by hyperplanes) Fig. 1(a) provides an example of the arrangement of 5 hyperplanes in \mathbb{R}^2 , where arrows on the hyperplanes specify halfspaces. The bounding set for the highlighted region r , namely $\{h^{(2)}, h^{(4)}, h^{(5)}\}$, forms 3 faces to r .

We use $\mathfrak{R}(\mathcal{A}(\mathcal{H}))$ to denote the regions induced by the arrangement $\mathcal{A}(\mathcal{H})$ and the number of regions $\mathfrak{r}(\mathcal{A}(\mathcal{H})) \triangleq |\mathfrak{R}(\mathcal{A}(\mathcal{H}))|$. We define a labeling function $\ell_r : \mathcal{H} \rightarrow \{-1, +1\}$ for an arbitrary region $r \in \mathfrak{R}(\mathcal{A}(\mathcal{H}))$. Note that r uniquely identifies its labeling function ℓ_r .

The teaching framework We study the problem of teaching *target regions* (convex polytopes) induced by hyperplane arrangement $\mathcal{A}(\mathcal{H})$ in \mathbb{R}^d . Our teaching model is formally stated below. Consider the set of instances \mathcal{H} , with label set $\mathcal{Y} = \{1, -1\}$ corresponding to two halfspaces induced by a hyperplane. Our hypothesis class, denoted as $\mathfrak{R}(\mathcal{A}(\mathcal{H}))$, is the set of regions induced by $\mathcal{A}(\mathcal{H})$. Consider a target region $r^* \in \mathfrak{R}(\mathcal{A}(\mathcal{H}))$. Let $\mathcal{Q} \subseteq \mathcal{H} \times \{1, -1\}$ be the ground set of examples (i.e. labeled instances). We define a labeled subset $Q \subseteq \mathcal{Q}$ as *halfspace queries*. We assume that for any halfspace queries Q wrt r^* , the labels are consistent, i.e., $\forall (h, l) \in Q, \ell_{r^*}(h) = l$. The *version space* induced by Q is the subset of regions $\text{VS}(Q) \subseteq \mathfrak{R}(\mathcal{A}(\mathcal{H}))$ that are consistent with the labels of all the halfspace queries i.e.,

$$\text{VS}(Q) = \{r \in \mathfrak{R}(\mathcal{A}(\mathcal{H})) \mid \forall (h, l) \in Q, \ell_r(h) = l\},$$

or equivalently, set of convex polytopes which satisfy the halfspace queries Q . We define our version space learner as one which upon seeing a set of halfspace queries, maintains a version space containing all the regions that are consistent with all the observed queries. Corresponding to a version space learner and a target region r^* , we define a *teaching set* $\mathcal{TS}(\mathcal{H}, r^*)$ as a minimal set of halfspace queries such that the resulting version space exactly contains $\{r^*\}$. Formally,

$$\mathcal{TS}(\mathcal{H}, r^*) \in \arg \min_{Q \subseteq \mathcal{Q}} |Q|, \text{ s.t. } \text{VS}(Q) = \{r^*\}.$$

Consequently, we want to teach a target hypothesis (regions), say r^* via specifying halfspace queries in the teaching set $\mathcal{TS}(\mathcal{H}, r^*)$ to a learner. Given a target region r^* , the teaching *complexity* (Goldman and Kearns, 1995) is defined as the sample size of the teaching set i.e. $|\mathcal{TS}(\mathcal{H}, r^*)|$.

In section §4, we analyze the teaching complexity of convex polytopes both in the framework of *average-case* and *worst-case*. We define *average teaching complexity* of convex polytopes via halfspace queries as the expected size of the teaching set i.e. $\mathbb{E}_{r \sim \mathcal{U}}[|\mathcal{TS}(\mathcal{H}, r)|]$, when the target region r is sampled uniformly at random. We define *worst-case teaching complexity* as the worst-case sample size of a teaching set corresponding to target regions from the set of hypotheses.

Hyperplanes in general position We adopt a common assumption in computational geometry Feldman and Rojas (2013); Miller et al. (2007) that the hyperplane arrangement is in *general position*, and further provide a relaxed notion of general position hyperplane arrangement, as defined below.

Definition 1 (General position of hyperplanes Miller et al. (2007)) For a set of n hyperplanes \mathcal{H} in \mathbb{R}^d , the arrangement $\mathcal{A}(\mathcal{H})$ is in *general position* if any subset $\mathcal{S} \subseteq \mathcal{H}$ of k hyperplanes where $1 \leq k \leq d$, intersects in a $(d - k)$ -dimensional plane, otherwise has null intersection.

Definition 2 (Relaxed general position of hyperplanes) For a set of n hyperplanes \mathcal{H} in \mathbb{R}^d and $d' \in [d]$, the arrangement $\mathcal{A}(\mathcal{H})$ is in d' -relaxed general position if any subset $\mathcal{S} \subseteq \mathcal{H}$ of k hyperplanes where $1 \leq k \leq d'$, intersects in a $(d - k)$ -dimensional plane, otherwise has null intersection.

As illustrated in Fig. 2, Definition 2 accounts for arrangements beyond general position (Fig. 2(a)) e.g parallel hyperplanes in Fig. 2(c). Definition 1 is a *special case* of Definition 2 which we discuss in details in Appendix F.

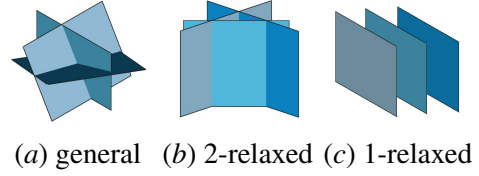


Figure 2: relaxed general position

4. Average-case Teaching Complexity

In this section, we study the generic problem of teaching convex polytopes via halfspace queries as illustrated in Fig. 1(a). Before establishing our main result, we first introduce two important results inherently connected to the average teaching complexity: the number of regions (which corresponds to the target hypotheses) induced by the intersections of n halfspaces, and the number of faces (which corresponds to the teaching sets) induced by the hyperplane arrangement. Our proofs are inspired by ideas from combinatorial geometry and affine geometry, as detailed below.

4.1. Regions and Faces Induced by Intersections of Halfspaces

Consider a set of n hyperplanes \mathcal{H} in \mathbb{R}^d . Generally, it is non-trivial to count the number of regions induced by an arbitrary hyperplane arrangement $\mathcal{A}(\mathcal{H})$. When the hyperplane arrangement is in general position (Definition 1, Fig. 2(a)), Miller et al. (2007) established an exact result for counting the induced regions. However, it remains a challenging problem to identify the number of regions for more general hyperplane arrangements. However, we show that under the relaxed condition of Definition 2, which accounts for various non-trivial arrangements as shown in Fig. 2(a)-2(c), one can exactly count the number of regions.

Theorem 3 (Regions induced by d' -relaxed general position arrangement) *Consider a set \mathcal{H} of n hyperplanes in \mathbb{R}^d . If the hyperplane arrangement $\mathcal{A}(\mathcal{H})$ is in d' -relaxed general position for some $d' \in [d]$, then the following holds: $\mathfrak{r}(\mathcal{A}(\mathcal{H})) = \sum_{i=0}^{d'} \binom{n}{i}$.*

In the following we sketch the proof of Theorem 3. The key insight for the proof is in reducing it to the special case of general position in some d' subspace where $d' \leq d$. We show the reduction by constructing a subspace \mathbb{N} defined as:

$$\mathbb{N} \triangleq \text{span} \left\langle \left\{ \eta_h \mid h \in \mathcal{H}, h := \eta_h \cdot z + b_h = 0, z \in \mathbb{R}^d \right\} \right\rangle.$$

As a key observation, note that \mathbb{N} is d' -dimensional. Let $\hat{\mathcal{H}}_{n,d'}$ be the induced set of hyperplanes in the subspace \mathbb{N} formed by the intersections of \mathcal{H} with \mathbb{N} . Therefore, the number of regions induced¹ by the arrangement of $\hat{\mathcal{H}}_{n,d'}$, denoted as $\mathfrak{r}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$, is exactly $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$. Thus, informatively, it is sufficient to rely on $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ in \mathbb{N} to understand the intersection of halfspaces induced by $\mathcal{A}(\mathcal{H})$ in \mathbb{R}^d . We observe that every region $\hat{r} \in \mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ is contained in exactly one region in $\mathcal{A}(\mathcal{H})$. With this observation, we construct the following map \mathcal{B} from the regions induced by the hyperplane arrangement $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$, to those induced by $\mathcal{A}(\mathcal{H})$:

$$\mathcal{B} : \mathfrak{R}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'})) \longrightarrow \mathfrak{R}(\mathcal{A}(\mathcal{H})) \quad : \quad \hat{r} \longmapsto \text{region}_{\mathcal{A}(\mathcal{H})}(\hat{r}),$$

1. This idea is more formally studied in the hyperplane arrangement literature as *essentialization* (see Miller et al., 2007, chap: An introduction to hyperplane arrangement). See Appendix C.3 for further discussion.

where $\text{region}_{\mathcal{A}(\mathcal{H})}(\hat{r}) := r$ for some $r \in \mathfrak{R}(\mathcal{A}(\mathcal{H}))$ such that $\hat{r} \subseteq r$. The following proposition shows that \mathcal{B} is bijective, thereby providing an alternate way to count $\mathfrak{R}(\mathcal{A}(\mathcal{H}))$.

Proposition 4 *The map \mathcal{B} (as defined above) is a bijection. Thus, $\mathfrak{r}(\mathcal{A}(\mathcal{H})) = \mathfrak{r}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$.*

Note that, if we can resolve $\mathfrak{r}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$ induced by the hyperplane arrangement $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$, then $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$ can be ascertained too. The following key lemma, proved in Appendix C.4, shows that $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ is in d' -relaxed general position.

Lemma 5 *The induced hyperplane arrangement $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ is in d' -relaxed general position.*

This implies that $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ is structurally the same as d' -general position arrangement of n hyperplanes (i.e. Definition 1) in $\mathbb{R}^{d'}$ because any d' -dimensional subspace of $\mathbb{R}^{d'}$ is isomorphic to $\mathbb{R}^{d'}$. Thus, from the relaxed definition of general position, we reduce the problem of counting $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$ to counting $\mathfrak{r}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$ which has the special arrangement of general position. By Miller et al. (2007) we therefore conclude that $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$ can be ascertained in an exact form as in Theorem 3. We defer the full proof of Theorem 3 to Appendix C.

Faces Induced by $\mathcal{A}(\mathcal{H})$ We denote by $\mathfrak{F}(\mathcal{A}(\mathcal{H}))$ the number of faces (i.e. regions induced on the hyperplanes) induced by $\mathcal{A}(\mathcal{H})$ in \mathbb{R}^d . Consider an arbitrary $h^* \in \mathfrak{R}(\mathcal{A}(\mathcal{H}))$. Note if $\mathcal{A}(\mathcal{H})$ is in d' -relaxed general position for $d' > 1$ then $\forall h \in \mathcal{H} \setminus \{h^*\}$, intersection of h and h^* forms a $(d-2)$ -dimensional *flat* on h^* by definition (see Appendix C.1 for formal definitions of the relevant affine geometry concepts). To count the regions induced on h^* is to analyze, wrt $\mathcal{H} \setminus \{h^*\}$, the $n-1$ *flats* of dimension $(d-2)$; thereby reducing the problem to the case of $n-1$ hyperplanes in \mathbb{R}^{d-1} . We would show that these *newly* induced hyperplanes (i.e. *flats*) are in relaxed general position, and thus one can invoke Theorem 3 to count the faces. Proposition 6, as proved in Appendix D, provides the exact count of faces induced by $\mathcal{A}(\mathcal{H})$.

Proposition 6 (Faces induced by hyperplane arrangement) *Consider a set \mathcal{H} of n hyperplanes in \mathbb{R}^d . If the hyperplane arrangement $\mathcal{A}(\mathcal{H})$ is in d' -relaxed general position for some $d' \in [d]$, the number of faces induced by the arrangement satisfies the recursion: $\mathfrak{F}(\mathcal{A}(\mathcal{H})) = n \cdot \sum_{i=0}^{d'-1} \binom{n-1}{i}$.*

4.2. Bound for Average Teaching Complexity: $\Theta(d')$

We are now ready to provide our main result on the average-case teaching complexity, when considering teaching convex polytopes induced by hyperplanes in d' -relaxed general position. We show that using results in §4.1, we achieve an *average-case* teaching complexity of $\Theta(d')$ by Algorithm 1.

Teaching algorithm Let $r \sim \mathcal{U}$ be a region sampled uniformly at random from $\mathfrak{R}(\mathcal{A}(\mathcal{H}))$. To teach r , a teacher has to provide the halfspace queries in $\mathcal{TS}(\mathcal{H}, r)$. Note that these labels is sufficient to teach r since the version space $\text{VS}(\mathcal{TS}(\mathcal{H}, r)) = \{r\}$. In Algorithm 1, the teacher first collects $\mathcal{TS}(\mathcal{H}, r)$ via subroutine **FindTS**(\cdot), and then provides labels to the learner. In particular, the subroutine **FindTS**(\cdot) identifies $\mathcal{TS}(\mathcal{H}, r)$ via linear programming: It checks if each hyperplane intersects the convex body defined by all the $n-1$ constraints (one linear constraint for each hyperplane); each iteration takes polynomial time as it requires solving a linear equation system. In total, it takes n iterations to decide whether any hyperplane is in the teaching set. Thus, the overall computational complexity of this algorithm is $\mathcal{O}(\text{poly}(d) \cdot \text{poly}(n))$ (assuming d is smaller than n).

Average-case analysis Recall that in section §3, we defined B_{r^*} to be the bounding set of hyperplanes for the polytope that contains r^* . To teach r^* , the teacher has to identify the exact subset of hyperplanes in B_{r^*} (i.e. the faces of the polytope), and provides the halfspace labels corresponding to the hyperplanes in B_{r^*} . Thus, teaching a target region corresponds to providing labels for the faces of the bounding set. One can ask if there are pathological arrangements, where teacher has to provide all the n labels? It turns out that, one can construct arrangements of the hyperplane set \mathcal{H} in \mathbb{R}^d where the worst-case teaching complexity is $\Omega(n)$ as shown in Theorem 8. This calls for analyzing the teaching problem under the average-case.

Intuitively, the average teaching complexity of convex polytopes reduces to the average number of faces per region, i.e. the ratio of number of faces induced on \mathcal{H} to number of regions induced in \mathbb{R}^d by $\mathcal{A}(\mathcal{H})$. In arbitrary arrangement of hyperplanes, it is challenging to bound the ratio A.1, as one needs to provide upper bound and lower bound for both terms, and it is unclear how $\mathfrak{F}(\mathcal{A}(\mathcal{H}))$ and $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$ are correlated. However, by imposing the d' -relaxed general position condition (for any $d' \in [d]$) on the hyperplane arrangement, we can leverage our exact results on counting the regions and faces using Theorem 3 and Proposition 6:

$$\mathbb{E}_{r \sim \mathcal{U}}[|\mathcal{TS}(\mathcal{H}, r)|] = \underbrace{\frac{2 \cdot \mathfrak{F}(\mathcal{A}(\mathcal{H}))}{\mathfrak{r}(\mathcal{A}(\mathcal{H}))}}_{d'\text{-relaxed general position}} = \frac{\text{Proposition 6}}{\text{Theorem 3}}. \quad (\text{A.1})$$

Ideally, to bound A.1, $\mathfrak{F}(\cdot)$ and $\mathfrak{r}(\cdot)$ need to be appropriately bounded. We further show (in the Appendix E) that for a relaxed general position of hyperplane arrangement, $\mathfrak{F}(\mathcal{A}(\mathcal{H}))$ can be rewritten in terms of $\mathfrak{r}(\cdot)$ in lower dimensional space. Thus, to bound the ratio in A.1, it suffices to bound $\mathfrak{r}(\cdot)$. Corollary 7, as proved in Appendix E, provides tight bounds on $\mathfrak{r}(\cdot)$.

Corollary 7 *If $\mathcal{A}(\mathcal{H})$ is in d' -relaxed general position, then $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$ satisfies the following for $n > 2d'$: $\binom{n-1}{d'} \leq \mathfrak{r}(\mathcal{A}(\mathcal{H})) \leq \binom{n}{d'} \cdot \frac{n-d'+1}{n-2d'+1}$*

Let M_n denote the sample size of $\mathcal{TS}(\mathcal{H}, r)$ from Algorithm 1 to teach $r \sim \mathcal{U}$, then $\mathbb{E}_{\mathcal{U}}[M_n] = \mathbb{E}_{r \sim \mathcal{U}}[|\mathcal{TS}(\mathcal{H}, r)|]$. Combining (A.1) and Corollary 7, we obtain our main result below.

Theorem 8 (Main theorem) *Assume \mathcal{H} is in d' -relaxed general position. Assume $r \sim \mathcal{U}$. Let the random variable M_n denote the number of halfspace queries that are requested in the teaching Algorithm 1, then, $\mathbb{E}_{\mathcal{U}}[M_n] = \Theta(d')$, i.e. the average teaching complexity of convex polytopes is $\Theta(d')$. Furthermore, the worst-case teaching complexity of convex polytopes is $\Theta(n)$.*

Arbitrary position arrangements of hyperplanes For general position arrangement, exact forms have been established (Miller et al., 2007; Zaslavsky, 1975; Buck, 1943) for $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$. But it is mentioned in Fukuda et al. (1991) that for any arbitrary arrangement one cannot explicitly give a simple formula for $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$ since Vergnas (1980) and Zaslavsky (1975) showed that $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$ depends on the underlying matroid structure. Interestingly, via Theorem 3 we establish an exact form for a non-trivial (d' -relaxed general position) setting. Apparently, Theorem 1.2 of Fukuda et al. (1991)

Algorithm 1 Teaching algorithm

Input: \mathcal{H} ; random target region $r \in \mathfrak{R}(\mathcal{A}(\mathcal{H}))$
begin
 // identifies $\mathcal{TS}(\mathcal{H}, r)$ via linear programming
 $\mathcal{TS}(\mathcal{H}, r) \leftarrow \mathbf{FindTS}(r)$
 for $(h, l) \in \mathcal{TS}(\mathcal{H}, r)$ **do**
 | teacher provides halfspace queries (h, l)
 end
end

establishes that for *any* hyperplane arrangement, average teaching complexity of convex polytopes is $\mathcal{O}(d)$. In contrast, Theorem 8 provides a stronger bound of $\Theta(d')$ in the d' -relaxed general position setting since $d' \leq d$. In addition, as further discussed in the Appendix C.5 the geometrical insights in the proof of Theorem 3 can be leveraged for extending to more general teaching complexity results.

5. Connections to Learning Complexity

In this section, we consider the problem of learning a convex polytope via halfspace queries, *without* the presence of a helpful teacher. We consider both the passive learning setting where learner makes *i.i.d.* queries and the active learning setting with actively chosen queries, and provide sample complexity results accordingly.

Learning convex polytopes via halfspace queries Consider the hyperplane set \mathcal{H} in \mathbb{R}^d and a target region $r^* \in \mathfrak{R}(\mathcal{A}(\mathcal{H}))$. For any hyperplane $h \in \mathcal{H}$ where $h = \{z \mid \eta_h \cdot z = b_h, z \in \mathbb{R}^d\}$, the labeling function ℓ_{r^*} , as defined in §3, specifies its label (halfspace) as $\ell_{r^*}(h) = \text{sgn}(\eta_h \cdot r^* - b_h)$. The problem of learning a region r^* therefore reduces to identifying the corresponding labeling function ℓ_{r^*} . The objective here is to learn the region by querying the reference of the form $q_h := \mathbf{1}\{\ell_{r^*}(h) = 1\}$, where $h \in \mathcal{H}$ and $\mathbf{1}\{\cdot\}$ is the indicator function. Similar to the teaching setting, we assume that the target r^* is sampled uniformly at random. In the following, we establish sample complexity results, i.e., on the minimal number of halfspace queries required to determine a target region, under the settings of *active* and *passive* learning.

5.1. Active Learning of Convex Polytopes

In §4.2, we showed that worst-case teaching complexity for convex polytopes is $\Omega(n)$, this directly implies the lower bound of $\Omega(n)$ on the worst-case for active learning. We now show that when the underlying hyperplane arrangement is in *d' -relaxed general position*, the average-case complexity of active learning has only a $\log n$ dependency on the number of hyperplanes. We achieve this by actively selecting informative queries—a similar characterization of the ambiguous queries as considered by Jamieson and Nowak (2011) for the pairwise ranking problem. Concretely, we consider the following querying strategy: For an (unknown) target region $r \in \mathfrak{R}(\mathcal{A}(\mathcal{H}))$ and a uniformly random ordering of hyperplanes \mathcal{H} , the learner checks in each iteration if a query q_h is ambiguous for randomly selected $h \in \mathcal{H}$ (i.e. intersects the convex body defined by hyperplanes sampled previously); then asks or imputes the labels depending on their ambiguity.

In any iteration k of the above query selection procedure², denote the event of requesting the query for a sampled hyperplane $h^{(k)}$ by B_k . That is, $B_k = \mathbf{1}\{q_{h^{(k)}} \text{ is requested}\}$. Note that each B_k is a Bernoulli distribution with *unknown* parameter (\star) to be ascertained. If we can bound (\star) then we bound the expected number of queries as well. We define by $\mathcal{S} \subseteq \mathcal{H}$ of size k as the set of hyperplanes sampled by the procedure. We notice that the $(k+1)^{\text{th}}$ sampled hyperplane is ambiguous if it intersects the convex body defined by hyperplanes in \mathcal{S} . Thus we want to bound the probability of the event that the query $q_{h^{(k+1)}}$ is ambiguous. Denote the probability of such an event as $P_A(k, d, \mathcal{U})$. Notice $P_A(k, d, \mathcal{U})$ is our (\star) here. In Lemma 9, we show that $P_A(k, d, \mathcal{U})$ is upper bounded by a factor of $1/k$ for relatively small sample size k .

Lemma 9 (Probability of ambiguity) *Assume $r \sim \mathcal{U}$. Let $P_A(k, d, \mathcal{U})$ denote the probability of the event that the query $q_{h^{(k+1)}}$ is ambiguous where $h^{(k+1)}$ is the $(k+1)^{\text{th}}$ sampled hyperplane. If*

2. Full algorithm is detailed in Appendix F.3.

\mathcal{H} is in d' -relaxed general position, then there exists a positive, real number constant a independent of k such that for $k > 2d'$, $P_A(k, d, \mathcal{U}) \leq a \cdot \frac{d'}{k}$.

Lemma 9 allows us to bound the expected value of $\sum_{j=1}^n \mathbf{1}\{q_{h(j)} \text{ is requested}\}$. As detailed in Appendix F, we show that for hyperplane arrangement in d' -relaxed general configuration, the expected value for $\sum_{j=1}^n \mathbf{1}\{q_{h(j)} \text{ is requested}\}$ is $\Theta(d' \log n)$. This leads to the following complexity results for active learning.

Theorem 10 *Assume $r \sim \mathcal{U}$ and that the underlying hyperplane arrangement of \mathcal{H} is in d' -relaxed general position. Let M_n denote a random variable for the number of queries that are requested in the query selection procedure in §5.1, then $\mathbb{E}_{\mathcal{U}}[M_n] = \Theta(d' \log n)$, i.e. the average-case query complexity of convex polytopes is $\Theta(d' \log n)$. Moreover, the worst-case query complexity is $\Theta(n)$.*

5.2. Passive learning of convex polytopes

In the case of passive learning of a target region, the *average* sample complexity is trivially lower bounded by $\Omega(n)$ since the learner gets a label uniform at random. Since there are n hyperplanes n samplings are sufficient to get all the labels which trivially give a $\mathcal{O}(n)$ solution. Thus, it is not very difficult to see that in the case of passive learning the average sample complexity is $\Theta(n)$.

6. Teaching ϕ -separable Dichotomy as Teaching Convex Polytopes

In §4, we discussed the generic problem of teaching convex polytopes induced by intersections of halfspaces via halfspace queries. We now consider the problem of ϕ -separability of points (also see Fig. 1(b)-1(c)) which could be viewed as a variant of teaching convex polytopes. We achieve similar average-case teaching complexity results for the problem. In the seminal work Cover (1965), Cover studied the problem of ϕ -separability of points in which the task is to classify points using various types of classifiers (linear or non-linear).

We first provide useful definitions for the domain of discussion. We define a set of n points in \mathbb{R}^d as $\mathcal{X} \triangleq \{x^{(1)}, x^{(2)}, \dots, x^{(n)}\}$ (referred to as *data space*), and use $x_{[d-1]}$ to represent the first $d-1$ coordinates of a point $x \in \mathbb{R}^d$. A map $\phi : \mathcal{X} \rightarrow \mathbb{R}^{d_\phi}$, is called ϕ -map, and the subset $\phi(\mathcal{X}) \subset \mathbb{R}^{d_\phi}$ is called ϕ -induced space. A dichotomy (i.e., a disjoint partition of a set) $\{\mathcal{X}^+, \mathcal{X}^-\}$ of \mathcal{X} is ϕ -separable if there exists a vector (aka *separator* of the dichotomy) $w \in \mathbb{R}^{d_\phi}$ such that: if $x \in \mathcal{X}^+$ then $w \cdot \phi(x) > 0$ and if $x \in \mathcal{X}^-$ then $w \cdot \phi(x) < 0$.

Definition 11 (Relaxed general position of points) *For a set of n data points in \mathbb{R}^d , say \mathcal{X} , is in d' -general position³ for a fixed $d' \in [d]$ if every d' subset of \mathcal{X} is linearly independent.*

Definition 12 (Relaxed ϕ -general position) *Consider a set of n data points \mathcal{X} in \mathbb{R}^d . For a ϕ -map in \mathbb{R}^{d_ϕ} , \mathcal{X} is said to be in d'_ϕ -relaxed ϕ -general position for a fixed $d'_\phi \in [d_\phi]$ if every d'_ϕ subset of ϕ -induced points $\phi(\mathcal{X})$ is linearly independent.*

We consider the problem of teaching ϕ -separable dichotomy as providing labels to subset $E \subset \mathcal{X}$ such that a separator w_ϕ can be taught which separates the entire dichotomy. In the remaining of this section, we show that the teaching problem of ϕ -separability of dichotomies (Fig. 1(b)-1(c)) can be studied as a special case of teaching convex polytopes. We connect the two problems via duality.

3. See Cover (1965) for the definition of general position of points.

Notice that showing the duality for homogeneous linear separability of dichotomies *i.e.* $\phi = \text{Id}$ (identity function) suffices for general ϕ -separability since it reduces to the homogeneous case.

Naturally, we define *teaching set* for a ϕ -separable dichotomy as the teaching set for the dual convex polytopes of the ϕ -induced space. Following the standard practice, we call the hypothesis space (where each hypothesis/region corresponds to a w) as the *dual space*, and data space as the *primal space*. We discuss the construction and relevant properties of duality below.

WLOG we assume that $x^{(n)} = \mathbf{e}_d$ (standard basis vector in \mathbb{R}^d with coordinate d being 1 and others being 0). Denote the set of all homogeneously linear separable dichotomies of \mathcal{X} by $\mathfrak{D}_{\mathcal{X}}$. We observe that if w is a linear separator of $\{\mathcal{X}^+, \mathcal{X}^-\}$, then $-w$ forms a linear separator for $\{\mathcal{X}^-, \mathcal{X}^+\}$. Based on this observation, we define a *relation* \sim on elements of $\mathfrak{D}_{\mathcal{X}}$ as follows: $\mathbf{u}, \mathbf{v} \in \mathfrak{D}_{\mathcal{X}}$ then $\mathbf{u} \sim \mathbf{v} \iff$ if w separates \mathbf{u} , then w or $-w$ separates \mathbf{v} . Notice that \sim is *reflexive, symmetric, and transitive*. Thus, \sim is an *equivalence relation*. Denote by $\mathfrak{E}(\mathcal{X})$ the set of equivalence classes *i.e.* the quotient set (see Rossen, 2003) $\mathfrak{D}_{\mathcal{X}} / \sim$. It is easy to see that $\# [\mathbf{v}] = 2$, where $[\mathbf{v}]$ denotes an equivalence class for any $\mathbf{v} \in \mathfrak{D}_{\mathcal{X}}$. Before we construct the dual map, wlog, we state a key assumption used in construction as follows:

Assumption 1 *We represent each equivalence class by the dichotomy which labels $x^{(n)}$ as positive.*

This implies that if $w = (w_1, \dots, w_d) \in \mathbb{R}^d$ is a homogeneous linear separator of the *representative* dichotomy of a class then $w_d > 0$ as $w \cdot x^{(n)} > 0$. Thus, dual map exploits this property of each equivalence class *i.e.*

$$\begin{aligned} w \cdot x &= x \cdot w = (x_{[d-1]}, x_d) \cdot (w_{[d-1]}/w_d, 1) && \leq 0 \\ &\Rightarrow x_{[d-1]} \cdot (w_{[d-1]}/w_d) + x_d \triangleq h_{[d-1]} \cdot z_w + x_d && \leq 0 \end{aligned} \quad (\text{I})$$

Hence, points $x \in \mathbb{R}^d$ maps to hyperplane $h_x \triangleq h_{[d-1]} \cdot z + x_d = 0$, $z \in \mathbb{R}^{d-1}$ in \mathbb{R}^{d-1} in the dual space and homogeneous linear hyperplane $w \cdot x = 0$ maps to point $z_w = w_{[d-1]}/w_d$ in \mathbb{R}^{d-1} . Notice that, $x^{(n)}$ maps to a hyperplane which exists in infinity *i.e.* $h_{[d-1]}^{(n)} = \mathbf{0}^{d-1}$. Denote the set of dual hyperplanes by $\mathcal{H}_{n-1, d-1}$ ($=: \bar{\mathcal{H}}$)⁴. Formally, we define our dual map $[\Upsilon_{\text{dual}}, \varphi_{\text{dual}}]$ as follows:

$$\begin{aligned} \Upsilon_{\text{dual}} : \mathcal{X} &\rightarrow \bar{\mathcal{H}} & \varphi_{\text{dual}} : \mathfrak{E}_{\mathcal{X}} &\rightarrow \mathfrak{R}(\mathcal{A}(\bar{\mathcal{H}})) \\ x &\mapsto h_x & [\mathbf{v}] : w_{[\mathbf{v}]} &\mapsto r_{z_{[\mathbf{v}]}} \end{aligned} \quad (\text{D.M})$$

where $z_{[\mathbf{v}]} \in r_{z_{[\mathbf{v}]}} \in \mathfrak{R}(\mathcal{A}(\bar{\mathcal{H}}))$ and $z_{[\mathbf{v}]}$ is dual point of the separator $w_{[\mathbf{v}]}$ to $[\mathbf{v}]$. We state the main result on dual map in Theorem 13 below with detailed proofs in Appendix G.

Theorem 13 (Dual map) *Consider a set of n points \mathcal{X} in \mathbb{R}^d in d' -relaxed general position. The hyperplane arrangement induced by $\mathcal{H}_{n-1, d-1} = \Upsilon_{\text{dual}}(\mathcal{X})$ is in $(d' - 1)$ -relaxed general position. Moreover, φ_{dual} is a bijection.*

Theorem 13 claims that $\Upsilon_{\text{dual}}(\mathcal{X})$ is in $(d' - 1)$ -relaxed general position. Combining the above result with Theorem 8, and the observation that any ϕ -separability reduces to the homogeneous case, we obtain the average teaching complexity of $\mathcal{O}(d'_\phi)$ for ϕ -separable dichotomy.

4. We use this notation to signify that $x^{(n)}$ exists in infinity.

Corollary 14 (Teaching ϕ -separable dichotomies) Consider a ϕ -map in \mathbb{R}^{d_ϕ} . Assume that \mathcal{X} are in d'_ϕ -relaxed ϕ -general position for a fixed $d'_\phi \in [d_\phi]$. If $\mathfrak{E}_{\mathcal{X}}^\phi$ denotes the set of ϕ -separable dichotomies of \mathcal{X} , then the average teaching complexity of dichotomies from $\mathfrak{E}_{\mathcal{X}}^\phi$ is $\mathcal{O}(d'_\phi)$ i.e. $\mathbb{E}_{r_{[u]} \sim \mathcal{U}}[M_n] = \mathcal{O}(d'_\phi)$, where M_n denotes the number of teaching labels for a class $r_{[u]} \in \mathfrak{E}_{\mathcal{X}}^\phi$.

Remark: In §4.2, we discussed that for *any* hyperplane arrangement, Fukuda et al. (1991) established $\mathcal{O}(d)$ result for the average teaching complexity of convex polytopes. We can obtain similar result for *any* arrangement of points for separable dichotomies via duality. The average teaching complexity of linear-separable dichotomies using duality can be established to $\mathcal{O}(d)$ (similarly $\mathcal{O}(d_\phi)$ for ϕ -separable dichotomies).

Connection to the notion of extreme points of Cover (1965) We now establish the connection between teaching set in the dual space and the *extreme points* in the primal space. This implies that our result on the average teaching complexity in Corollary 14 recovers the $\mathcal{O}(d_\phi)$ result on the average number of extreme points, which was proved via a different framework in Cover (1965).

Definition 15 (Extreme points) Consider an arbitrary ϕ -separable dichotomy $\{\mathcal{X}^+, \mathcal{X}^-\}$ of a set of points \mathcal{X} in \mathbb{R}^d . We say a subset $E \subset \mathcal{X}$ to be *extremal points* wrt $\{\mathcal{X}^+, \mathcal{X}^-\}$ if it is minimal and $\{\mathcal{X}^+, \mathcal{X}^-\}$ is ϕ -separable by w_ϕ iff $\{\mathcal{X}^+ \cap E, \mathcal{X}^- \cap E\}$ is ϕ -separable by w_ϕ .

According to Lemma 1 (Cover, 1965), a point y is in the minimal set E of extreme points for a dichotomy $\{X^+, X^-\}$ if it is *ambiguous* wrt the dichotomy i.e. both $\{X^+ \cup \{y\}, X^-\}$ and $\{X^+, X^- \cup \{y\}\}$ are homogeneously linearly separable. We show that this characterization of ambiguous points is *equivalent* to a characterization of hyperplanes in the dual space:

Definition 16 (Ambiguous hyperplanes in the dual space) Let \mathcal{H} be a set of hyperplanes in \mathbb{R}^d , and let r^* be a region induced by the hyperplane arrangement $\mathcal{A}(\mathcal{H})$. Then, an arbitrary hyperplane h' is *informative or ambiguous* with respect to r^* iff \exists a point z in h' such that a normed ball $\mathbb{B}_2(z, \epsilon) \subset r^*$ for some $\epsilon > 0$.

Note that only an ambiguous hyperplane can be contained in the teaching set for r^* . To achieve the equivalence of the two characterizations provided in Definition 15 and Definition 16, our key insight is in noting that Eq. (1) preserves signs of dot products in both the primal and dual spaces. Using this, we realize that (i) every ambiguous data point to dichotomy $\{X^+, X^-\}$ passes through the dual region corresponding to it, and (ii) similarly, every ambiguous hyperplane can be shown to form a data point which intersects a separator of $\{X^+, X^-\}$. Formally, we establish the connection via the following theorem below with detailed discussions and proofs deferred to Appendix H.

Theorem 17 Consider a set of n points \mathcal{X} in \mathbb{R}^d and a ϕ -map where $\phi : \mathcal{X} \rightarrow \mathbb{R}^{d_\phi}$. Assume that \mathcal{X} are in d_ϕ -relaxed ϕ -general position (Definition 12). Let $\{\mathcal{X}^+, \mathcal{X}^-\}$ be a ϕ -separable dichotomy. Now, for a subset $E \subseteq \mathcal{X}$, E is a set of extremal points iff $\Upsilon_{\text{dual}}(E)$ with the appropriate labels forms a teaching set for $\varphi_{\text{dual}}([\{\mathcal{X}^+, \mathcal{X}^-\}])$.

7. Discussion and Conclusion

We have studied the average-case complexity of teaching convex polytopes with halfspace queries, and showed that if the hyperplane arrangement is in d' -relaxed general position, then the average teaching complexity is $\Theta(d')$. In contrast, the average-case sample complexity is $\Theta(d' \log n)$ for

active learning and $\Theta(n)$ for passive learning. We showed that our insights could be applied to teaching ϕ -separable dichotomies. Moreover, as discussed in details in the Appendix I, we further show that our insights in §4 could be further generalized to the problem of teaching rankings over n points $\{x_1, \dots, x_n\} \subseteq \mathbb{R}^d$ (encoded by their distances to an *unknown* reference point $r \in \mathbb{R}^d$) via pairwise comparisons (e.g., “is x_i closer to r than x_j ?”). One interesting line of future work is to understand whether our result could be extended to more general hyperplane arrangement settings. We believe our results provide useful geometrical insights for analyzing the average-case complexity for more complex hypothesis classes.

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Appendix A. List of Appendices

In the appendices, we first provide a table summarizing the notations defined in the main paper. We then provide the proofs of our theoretical results in full detail in the subsequent sections.

The remainder of the appendices are summarized as follows:

- Appendix B provides a list of notations defined in the main paper
- Appendix C provides the proof of Theorem 3 (Number of Regions Induced by Intersections of Halfspaces)
- Appendix D provides the proof of Proposition 6 (Number of Faces Induced by Intersections of Halfspaces)
- Appendix E provides the proof of Theorem 8 (Teaching Complexity of Convex Polytopes)
- Appendix F provides the proof of Theorem 10 (Learning Complexity of Convex Polytopes)
- Appendix G provides the proof of Theorem 13 and Corollary 14 (Teaching Complexity of ϕ -Separable Dichotomy)
- Appendix H provides the proof of Theorem 17 (Equivalence of Teaching Set and Extreme Points)
- Appendix I provides an additional use-case of the problem of teaching convex polytopes via halfspace queries. In particular, we introduce the problem of teaching linear rankings via halfspaces queries, and establish a $\Theta(d)$ bound on the average teaching complexity.

Appendix B. Table of Notations Defined in the Main Paper

For readers' convenience, we summarize the notations used in the main paper in Table 2.

Table 2: Table of Notations

Notations	Use
$h, h^{(i)}$	a hyperplane
$x, x^{(i)}$	a point
r, r^*	target/sampled region/hypothesis/concept
$[u], [v]$	dichotomies equivalence classes
η, η_h	normal vectors of a hyperplane
b, b_h	bias of a hyperpane
\mathcal{X}	data points in \mathbb{R}^d or <i>data space</i>
\mathcal{H}	n hyperplanes set in \mathbb{R}^d or <i>hypothesis space</i>
$\mathcal{A}(\mathcal{H})$	hyperplanes arrangement of set \mathcal{H}
$\mathfrak{R}(\mathcal{A}(\mathcal{H}))$	set of regions induced by hyperplane arrangement \mathcal{H}
$\mathfrak{r}(\mathcal{A}(\mathcal{H}))$	#regions induced by hyperplane arrangement $\mathcal{A}(\mathcal{H})$
$\mathcal{D}_{\mathcal{X}}$	set of dichotomies of \mathcal{X}
$\mathfrak{E}(\mathcal{X})$	the set of equivalence classes of homogeneously linear separable dichotomies
$\mathfrak{E}_{\mathcal{X}}^{\phi}$	the set of equivalence classes of ϕ -separable dichotomies
$r_{[u]}$	random dichotomy (equivalence) class in $\mathfrak{E}_{\mathcal{X}}^{\phi}$
$\mathcal{B}, \phi, \Upsilon_{\text{dual}}, \varphi_{\text{dual}}$	maps
$\mathfrak{F}(\mathcal{A}(\mathcal{H}))$	number of faces
\mathcal{U}	uniform distribution
Θ	set of embedded points
Λ	a matrix
$\mathbb{I}_{[k]}$	set of k indices of naturals

Appendix C. Regions Induced by Intersections of Halfspaces: Proof of Theorem 3

In this section, we would provide the relevant results, with proofs to complete the claim of Theorem 3. The struture of the appendix is: we first introduce basic affine geometry, then construct a subspace in which the underlying hyperplane arrangement is structurally similar to the hyperplane arrangement of discussion *i.e.* $\mathcal{A}(\mathcal{H})$, and establish useful properties in relevant lemmas and proposition to complete the proof of Theorem 3.

Before we proceed to the technical part of the appendix, we provide elementary discussion on affine geometry (Roman, 2007) below.

C.1. Elementary Affine Geometry

Definition 18 (Flats Roman (2007)) *Let S be a subspace of a vector space V . The coset*

$$v + S = \{v + s \mid s \in S\}$$

is called a flat in V with base S and flat representative v . We also refer to $v + S$ as a translate of S . The set $\mathfrak{A}(V)$ of all flats in V is called the affine geometry of V . The dimension $\dim(\mathfrak{A}(V))$ of $\mathfrak{A}(V)$ is defined to be $\dim(V)$.

While a flat may have many flat representatives, it only has one base since $x + S = y + T$ implies that $x \in y + T$ and so $x + S = y + T = x + T$ whence $S = T$.

Definition 19 (Dimension of flats) *The dimension of a flat $v + S$ is $\dim(S)$. A flat of dimension k is called a k -flat. A 0-flat is a point, a 1-flat is a line, and a 2-flat is a plane. A flat of dimension $\dim(\mathfrak{A}(V)) - 1$ is called a hyperplane.*

In the discussion ahead, we would interchangeably use the notation \dim for a *flat* and a *subspace*. With the discussion above, we realize every hyperplane in \mathbb{R}^k has a dual representation as a flat, and a set defined by a *normal* vector and a *bias* (see §3). We would use these representations to our advantage in defining and constructing mathematical objects in the coming discussion.

C.2. Construction of \mathbb{N} and Relevant Lemmas

For any hyperplane $h \in \mathcal{H}$ in \mathbb{R}^d , it can be written as $h \triangleq \eta_h \cdot z + b_h = 0$ where η_h and b_h are a fixed non-zero normal vector and a scalar bias respectively. Consider the subspace \mathbb{N} spanned by the normal vectors of hyperplanes in \mathcal{H} .

$$\mathbb{N} = \mathbf{span} \left\langle \left\{ \eta_h \mid h \in \mathcal{H}, h := \eta_h \cdot z + b_h = 0, z \in \mathbb{R}^d \right\} \right\rangle$$

This construction is interesting pertaining to the arrangement of the hyperplanes which is *d'-relaxed general position*. First, we would show some useful properties of the subspace \mathbb{N} and the manner in which \mathcal{H} intersects \mathbb{N} in Lemma 20 and Lemma 21.

Lemma 20 *Consider a set \mathcal{H} of n hyperplanes in \mathbb{R}^d . If the hyperplane arrangement $\mathcal{A}(\mathcal{H})$ is in d' -relaxed general position, then $\dim(\mathbb{N}) = d'$.*

Proof Let us define an *ordered* subset $\mathbb{N}_{[d']} \triangleq \{\eta_{i_1}, \eta_{i_2}, \dots, \eta_{i_{d'}}\}$ of normal vectors of any d' hyperplanes in \mathcal{H} . Consider the subset $\mathcal{H}_{[d']} \subset \mathcal{H}$ of hyperplanes corresponding to the normal vectors in $\mathbb{N}_{[d']}$. Ideally, if we can show that $\mathbb{N}_{[d]}$ is linearly independent then we have a lower bound on the dimension of \mathbb{N} *i.e.* $\dim(\mathbb{N}) \geq d'$.

We construct the matrix $\mathbf{\Lambda}_{\mathbb{N}_{[d]}}$ such that $\mathbf{\Lambda}_{\mathbb{N}_{[d]}}[k :] = \eta_{i_k}$. Define $\mathbf{b} \triangleq (b_{i_1}, b_{i_1}, \dots, b_{i_{d'}})$. Consider the matrix equation for variable $z \in \mathbb{R}^d$:

$$\mathbf{\Lambda}_{\mathbb{N}_{[d]}} z = -\mathbf{b}^\top \quad (2)$$

But we note that if z is a solution of Eq. (2) *iff* z exists in $(\bigcap_{h \in \mathcal{H}_{[d']}} h)$. Notice that by the definition of d' -relaxed general position, $(\bigcap_{h \in \mathcal{H}_{[d']}} h)$ is a $(d - d')$ -dimensional flat which also forms a solution for Eq. (2). Consider a solution $z_0 \in (\bigcap_{h \in \mathcal{H}_{[d']}} h)$ such that $\mathbf{\Lambda}_{\mathbb{N}_{[d]}} z_0 = -\mathbf{b}^\top$. Thus,

$$\begin{aligned} \mathbf{\Lambda}_{\mathbb{N}_{[d]}} z = \mathbf{\Lambda}_{\mathbb{N}_{[d]}} z_0 &\implies \mathbf{\Lambda}_{\mathbb{N}_{[d]}} (z - z_0) = 0 \\ &\implies \dim(\mathbf{Ker}(\mathbf{\Lambda}_{\mathbb{N}_{[d]}})) = d - d' \end{aligned} \quad (3)$$

But using Theorem 31 (rank-nullity, Appendix G), $\mathbf{rank}(\mathbf{\Lambda}_{\mathbb{N}_{[d]}}) = d'$. It implies $\mathbb{N}_{[d]}$ is a set of d' linearly independent vectors. Thus, $\dim(\mathbb{N}) \geq d'$.

Note, that $\dim(\mathbb{N}) \not\geq d'$ otherwise \exists an *ordered* subset $\mathbb{N}_{[d'+1]} \triangleq \{\eta_{i_1}, \eta_{i_2}, \dots, \eta_{i_{d'+1}}\}$ of $d' + 1$ normal vectors corresponding to a subset $\mathcal{H}_{[d'+1]} \subset \mathcal{H}$, which are linearly independent. Then, the equation $\mathbf{\Lambda}_{\mathbb{N}_{[d'+1]}} z = -(b_{i_1}, b_{i_1}, \dots, b_{i_{d'+1}})^\top$ has a solution because $\mathbf{rank}(\mathbf{\Lambda}_{\mathbb{N}_{[d'+1]}}) = d' + 1$. This implies that $(\bigcap_{h \in \mathcal{H}_{[d'+1]}} h) \neq \emptyset$, which contradicts the d' -relaxed general position arrangement of \mathcal{H} . Thus, $\dim(\mathbb{N}) = d'$. \blacksquare

Any hyperplane $h \in \mathcal{H}$ is a $(d - 1)$ -dimensional *flat* which can be written equivalently as $h \equiv h_{flat} \triangleq v_h + S_h$ for some vector $v_h \in \mathbb{R}^d$ and $(d - 1)$ -dimensional subspace S_h . Notice that \mathbb{N} is a d' -dimensional *flat* which can be written as $(\mathbf{0} + \mathbb{N})$. Using Theorem 16.5 (see Roman, 2007, page 451), the intersection *flat* $X_h = (h_{flat} \cap (\mathbf{0} + \mathbb{N}))$ can be written as $X_h \triangleq y_h + (S_h \cap \mathbb{N})$ for some $y_h \in (h_{flat} \cap (\mathbf{0} + \mathbb{N}))$. Now, we show a straightforward result that X_h has dimension $d' - 1$ which would be useful when we consider the regions induced by the arrangement of *intersection flats* in \mathbb{N} .

Lemma 21 *For the flat X_h constructed as above, $\dim(X_h) = d' - 1$.*

Proof By Theorem 16.6 of Roman (2007), we know that the dimension of the intersection of two subspaces is

$$\dim(S_h \cap \mathbb{N}) = \dim(S_h) + \dim(\mathbb{N}) - \dim(S_h + \mathbb{N})$$

Since S_h is $(d - 1)$ -dimensional and the orthogonal vector (*i.e.* the normal vector) of h (or h_{flat}) exists in \mathbb{N} by definition, the dimension of $(S_h + \mathbb{N}) = d$. This implies that

$$\dim(S_h \cap \mathbb{N}) = (d - 1) + d' - d = d' - 1$$

Since $\dim(X_h) = \dim(S_h \cap \mathbb{N})$, thus the lemma follows. \blacksquare

C.3. Construction of Map \mathcal{B} and Proof of Proposition 4

Now, consider the induced set of hyperplanes in the d' -dimensional subspace \mathbb{N} :

$$\hat{\mathcal{H}}_{n,d'} = \{X_h \mid h \in \mathcal{H}\}$$

With the construction of the induced set of hyperplanes, we can talk about the regions $\mathfrak{R}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$ induced by the arrangement of $\hat{\mathcal{H}}_{n,d'}$ in the d' dimensional subspace \mathbb{N} . We would show that every region induced by the arrangement $\mathcal{A}(\mathcal{H})$ in \mathbb{R}^d contains a point (vector) from a region induced by $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ in the subspace \mathbb{N} . Before we develop ideas, to show that, we provide the following definition which characterizes points contained in different regions:

Definition 22 (Path-connectivity of points) Consider a set of hyperplanes \mathcal{H} in \mathbb{R}^d . For any two points $u, v \in \mathbb{R}^d$, we say u and v are path-connected wrt the regions induced by $\mathcal{A}(\mathcal{H})$ if the following equivalent conditions hold:

- if the line segment $\lambda u + (1 - \lambda)v$ where $\lambda \in (0, 1)$ is not intersected by any hyperplane in \mathcal{H}
- u and v belong to the same region induced by $\mathcal{A}(\mathcal{H})$

Notations Denote the orthogonal projection of a point $u \in \mathbb{R}^d$ onto \mathbb{N} by $\mathbf{proj}_{\mathbb{N}}(u)$. Denote a region (polytope) in $\mathfrak{R}(\mathcal{A}(\mathcal{H}))$ by r . Consider a point $z_r \in r \setminus \mathbb{N}$. Since r contains an open convex polyhedron, for some $\epsilon > 0 \exists$ a normed ball $\mathbb{B}_2(z_r, \epsilon)$ not intersected by any hyperplane.

To prove our intuition developed earlier, we would show that z_r (if it exists) and $\mathbf{proj}_{\mathbb{N}}(z_r)$ are path-connected.

Lemma 23 Following the notations as above, z_r and $\mathbf{proj}_{\mathbb{N}}(z_r)$ are path-connected and, thus every region $r \in \mathfrak{R}(\mathcal{A}(\mathcal{H}))$ has points contained in \mathbb{N} .

Proof For the sake of contraposition, assume that z_r and $\mathbf{proj}_{\mathbb{N}}(z_r)$ are not path-connected. Let $h \triangleq \eta_h \cdot z + b_h = 0 \in \mathcal{H}$ be the intersecting hyperplane. Assume that h intersects the line segment $\lambda z_r + (1 - \lambda) \cdot \mathbf{proj}_{\mathbb{N}}(z_r)$ at the point $z_{h,\cap}$ i.e. $z_{h,\cap} = \lambda' z_r + (1 - \lambda') \cdot \mathbf{proj}_{\mathbb{N}}(z_r)$ for some $\lambda' \in (0, 1)$. By the property of z_r , we realize $z_{h,\cap} \notin \mathbb{B}_2(z_r, \epsilon)$. Since $\mathbf{proj}_{\mathbb{N}}(\cdot)$ is an orthogonal projection, we have

$$\eta_h \perp (\mathbf{proj}_{\mathbb{N}}(z_r) - z_r) \implies \eta_h \cdot \mathbf{proj}_{\mathbb{N}}(z_r) = \eta_h \cdot z_r \quad (4)$$

Using Eq. (4) and noting that $z_{h,\cap}$ lies on h , we have:

$$\eta_h \cdot z_{h,\cap} + b_h = 0 \implies \eta_h \cdot (\lambda' z_r + (1 - \lambda') \cdot \mathbf{proj}_{\mathbb{N}}(z_r)) + b_h = 0 \implies \eta_h \cdot z_r + b_h = 0$$

But this is a contradiction because $\mathbb{B}_2(z_r, \epsilon)$, by definition, is not intersected by any hyperplane in \mathcal{H} . Thus, the lemma follows and this asserts that the subspace \mathbb{N} has at least one point contained in any region induced by $\mathcal{A}(\mathcal{H})$. \blacksquare

This gives us the insight that information theoretically, the regions induced on \mathbb{N} by $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ has similar structure to the regions induced on \mathbb{R}^d by $\mathcal{A}(\mathcal{H})$. We would ascertain this promisingly by showing a bijective map from $\mathfrak{R}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$ to $\mathfrak{R}(\mathcal{A}(\mathcal{H}))$. Before we construct the map, we have certain inferences to make based on the previous discussion.

We observe that every region $\hat{r} \in \mathfrak{R}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$ is contained in *exactly one* region in $\mathcal{A}(\mathcal{H})$ i.e. $\hat{r} \subseteq r$ for some $r \in \mathfrak{R}(\mathcal{A}(\mathcal{H}))$. If it is not so then we have two points $a_r^{ind}, b_r^{ind} \in \hat{r}$ which are *not* path-connected (in \mathbb{R}^d). Thus, there is some hyperplane $h \in \mathcal{H}$ which cuts the line segment at some point z . But then $z \in \mathbb{N}$ because $\forall \lambda \in (0, 1)$ the combination $\lambda a_r^{ind} + (1 - \lambda)b_r^{ind} \in \mathbb{N}$, implying $z \in X_h$. Contradiction because a_r^{ind} and b_r^{ind} are path-connected in \mathbb{N} .

Let us define the map \mathcal{B} as follows:

$$\begin{aligned} \mathcal{B} : \mathfrak{R}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'})) &\longrightarrow \mathfrak{R}(\mathcal{A}(\mathcal{H})) \\ \hat{r} &\longmapsto \mathbf{region}_{\mathcal{A}(\mathcal{H})}(\hat{r}) \end{aligned}$$

where $\mathbf{region}_{\mathcal{A}(\mathcal{H})}(\hat{r})$ is the region (polytope) of $\mathcal{A}(\mathcal{H})$ in which the polytope \hat{r} is contained. Using the observation above, the map is well-defined. Using the observation and Lemma 23, we claim in Proposition 4 that \mathcal{B} is a *bijection*, and thus $\mathfrak{r}(\mathcal{A}(\mathcal{H})) = \mathfrak{r}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$.

Proof [Proof of Proposition 4] Denote by r_i^{ind} and r_j^{ind} two regions in $\mathfrak{R}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$. First, we show that the map \mathcal{B} is an *injection*. For the sake of contraposition, assume it is not injective. Assume that $\mathcal{B}(r_i^{ind}) = \mathcal{B}(r_j^{ind}) = r$ (a region in \mathbb{R}^d). Note that r_i^{ind} and r_j^{ind} are *not* path connected⁵ in the subspace \mathbb{N} . Thus, \exists a *flat* X_h (intersection of flats h and $\mathbf{0} + \mathbb{N}$) which separates r_i^{ind} and r_j^{ind} in \mathbb{N} . Since, $r_i^{ind}, r_j^{ind} \subseteq r$, thus h separates r_i^{ind} and r_j^{ind} in r , which implies r_i^{ind} and r_j^{ind} are *not* path-connected in \mathbb{R}^d . Contradiction! Thus, \mathcal{B} is an injection.

Using Lemma 23, we know any region $r \in \mathfrak{R}(\mathcal{A}(\mathcal{H}))$ has points contained in \mathbb{N} . The observation above implies that \exists a *unique* $\hat{r} \in \mathfrak{R}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$ such that $\hat{r} \subseteq r$. Thus, \mathcal{B} is a surjection. We have shown that \mathcal{B} is both an injection and a surjection, implying it is a bijection. This also implies that:

$$\mathfrak{r}(\mathcal{A}(\mathcal{H})) = \mathfrak{r}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$$

■

Essentialization and Boolean lattices The technique to reduce the counting problem of regions to the normal space, as used above, is studied more formally as *essentialization* as discussed in Miller et al. (2007, chap: An introduction to hyperplane arrangement). One could potentially devise an alternate proof for Proposition 4 using the technique but it would require introducing several other development on characteristics polynomials, inclusion lattices, möbius functions, and inversions among others. On the other hand, our proof technique uses simpler geometric ideas to prove the result from the first principles. Similarly, we could also use *boolean algebra* as discussed in Cohen et al. (2013); Miller et al. (2007, chap: An introduction to hyperplane arrangement) to show the result. Inclusion lattice for a d' -relaxed general position hyperplane arrangement could be shown to be isomorphic to a d' *truncated boolean algebra*, and thus one could arrive at a result similar to Proposition 4.

C.4. Proof of Lemma 5

Using Proposition 4 we have a constructively alternate way to ascertain $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$. The previous discussion and results are useful in the sense that we can indeed find $\mathfrak{r}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$. As it turns

5. Notion of path-connectivity can be extended for two regions (subsets of points) where no two points in the open convex polyhedrons of the regions are path-connected.

out, $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ is in d' -relaxed general position arrangement. Since, counting the regions induced by $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ on the d' -dimensional subspace \mathbb{N} arranged in d' -relaxed general position is same as counting the number of regions induced on $\mathbb{R}^{d'}$ by a size n subset of d' -general position⁶ arranged hyperplanes, thus we can directly count $\mathfrak{R}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$ using Lemma 26 and subsequent Corollary 27. We show in the key Lemma 5 that $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ is in d' -relaxed general position arrangement.

Proof [Proof of Key Lemma 5] Let $1 \leq k \leq d'$. Consider an arbitrary size k subset $\mathcal{S}_k^{ind} \subseteq \hat{\mathcal{H}}_{n,d'}$ of hyperplanes ($d' - 1$ -dimensional flats in \mathbb{N}). We denote the size k subset of corresponding hyperplanes in \mathbb{R}^d by $\mathcal{S}_k \subseteq \mathcal{H}$ ($(d - 1)$ -dimensional flats). Since $\mathcal{A}(\mathcal{H})$ is in d' -relaxed general position we notice that $\dim\left(\bigcap_{h \in \mathcal{S}_k} h\right) = d - k$. Define the orthogonal subspace (complement) of \mathbb{N}

$$\mathbb{N}^\perp = \left\{ z \in \mathbb{R}^d \mid z \cdot v = 0 \forall v \in \mathbb{N} \right\}$$

Using Theorem 16.5 (as shown in Roman, 2007) and noting that for any $h' \in \mathcal{S}_k^{ind}$ we can write $h' \equiv h \cap (\mathbf{0} + \mathbb{N})$ for some $h \in \mathcal{S}_k$, we have:

$$\left(\bigcap_{h' \in \mathcal{S}_k^{ind}} h' \right) = \left(\bigcap_{h \in \mathcal{S}_k} h \right) \cap (\mathbf{0} + \mathbb{N}) \quad (5)$$

Using the representation of *flats*, we can write

$$\bigcap_{h \in \mathcal{S}_k} h = \nu + W_\cap \text{ where } \nu \in \bigcap_{h \in \mathcal{S}_k} h \text{ and } W_\cap \triangleq \bigcap_{z+W \in \mathcal{S}_k} W \quad (6)$$

WLOG we enumerate the hyperplanes in \mathcal{S}_k as $\{h^{(1)}, h^{(2)}, \dots, h^{(k)}\}$. Now, we construct the matrix $\mathbf{\Lambda}_k$ using the normal vectors of the hyperplanes in \mathcal{S}_k i.e. $\mathbf{\Lambda}_k[i :] = \eta^{(i)}$ where $h^{(i)} \triangleq \eta^{(i)} \cdot z + b^{(i)} = 0 \forall i \in [k]$; to solve the system of equations for the intersection of \mathcal{S}_k as follows:

$$\mathbf{\Lambda}_k z = -(b^{(1)}, b^{(2)}, \dots, b^{(k)})^\top \quad (7)$$

Since $\bigcap_{h \in \mathcal{S}_k} h \neq \emptyset$, $\exists z_0 \in \mathbb{R}^d$ such that $\mathbf{\Lambda}_k z_0 = -(b^{(1)}, b^{(2)}, \dots, b^{(k)})^\top$. But then any solution of $\mathbf{\Lambda}_k z = 0$ implies $z - z_0$ is a solution of Eq. (7). We can succinctly write this as follows:

$$\mathbf{\Lambda}_k z = -(b^{(1)}, b^{(2)}, \dots, b^{(k)})^\top \iff \mathbf{\Lambda}_k z = \mathbf{\Lambda}_k z_0 \iff \mathbf{\Lambda}_k(z - z_0) = 0 \quad (8)$$

This implies that solving $\mathbf{\Lambda}_k z = 0$ sufficiently solves Eq. (7). We notice, by definition of \mathbb{N}^\perp and construction of $\mathbf{\Lambda}_k$, $\mathbf{\Lambda}_k \perp \mathbb{N}^\perp$. Thus, \mathbb{N}^\perp is a solution of $\mathbf{\Lambda}_k z = 0$. But then, using Eq. (8)

$$-z_0 + \mathbb{N}^\perp \subseteq \bigcap_{h \in \mathcal{S}_k} h \quad (9)$$

At this point, we observe a small inclusion which would be helpful in claiming the dimension of \mathcal{S}_k^{ind} . We notice that $(-z_0 + \mathbb{N}^\perp)$ and $\bigcap_{h \in \mathcal{S}_k} h$ are flats in \mathbb{R}^d by definition and Eq. (6) respectively. Now, combining Eq. (9) and Theorem 16.1 (as shown in Roman, 2007), we get that $\mathbb{N}^\perp \subseteq W_\cap$.

6. We, interchangeably, use the term d' -general position or general position for d' -relaxed general position arrangement in $\mathbb{R}^{d'}$

Finally, we would argue on the dimension of $\left(\bigcap_{h' \in \mathcal{S}_k^{ind}} h'\right)$ as follows:

$$\dim \left(\bigcap_{h' \in \mathcal{S}_k^{ind}} h' \right) = \dim \left(\left(\bigcap_{h \in \mathcal{S}_k} h \right) \cap (\mathbf{0} + \mathbb{N}) \right) \quad (10)$$

$$= \dim \left(\bigcap_{h \in \mathcal{S}_k} h \right) + \dim(\mathbf{0} + \mathbb{N}) - \dim(W_\cap + \mathbb{N}) \quad (11)$$

$$= (d - k) + d' - d \quad (12)$$

$$= d' - k$$

Eq. (10) is the direct consequence of Eq. (5). Eq. (11) follows from Theorem 16.6 in Roman (2007). Since $\mathbb{N}^\perp \subseteq W_\cap$ and \mathbb{N}^\perp is *orthogonal* to \mathbb{N} , thus $\dim(W_\cap + \mathbb{N}) = d$ (dimension of the space). Since, \mathcal{H} is in d' -relaxed general position and $k \leq d'$, $\dim\left(\bigcap_{h \in \mathcal{S}_k} h\right) = d - k$. These observations yield Eq. (12). Thus, for any arbitrary subset \mathcal{S}_k^{ind} of size $1 \leq k \leq d'$, we have shown that $\dim\left(\bigcap_{h' \in \mathcal{S}_k^{ind}} h'\right) = d' - k$.

Notice that if we select a subset of $\hat{\mathcal{H}}_{n,d'}$ of size more than d' , then they don't intersect at any point since the corresponding subset of hyperplanes in \mathcal{H} has *empty* intersection.

Thus, following Definition 2, we show that $\hat{\mathcal{H}}_{n,d'}$ is in d' -relaxed general position. Hence, the lemma follows. \blacksquare

C.5. Proof of Theorem 3

We note that a subspace of dimension k of \mathbb{R}^d is *isomorphic* to \mathbb{R}^k . Thus, d' -relaxed general position hyperplane arrangement $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ in \mathbb{N} can be *uniquely* mapped to a d' -relaxed general position hyperplane arrangement of n hyperplanes in $\mathbb{R}^{d'}$. It implies that we can use Lemma 26 (discussed and proved in Appendix F.1, provides an exact form for the number of regions induced in \mathbb{R}^d when the hyperplane arrangement is in general position) to ascertain $\mathfrak{r}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'}))$ since $\mathcal{A}(\hat{\mathcal{H}}_{n,d'})$ satisfies all the required premises *i.e.* d' -general position in d' dimensional Euclidean space. Thus, we have

$$\mathfrak{r}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'})) = Q(n, d') = \sum_{i=0}^{d'} \binom{n}{i}$$

Using Proposition 4, we finally show that:

$$\mathfrak{r}(\mathcal{A}(\mathcal{H})) = \mathfrak{r}(\mathcal{A}(\hat{\mathcal{H}}_{n,d'})) = \sum_{i=0}^{d'} \binom{n}{i}$$

This completes the proof of Theorem 3.

Remark One can study the arrangement of hyperplanes $\mathcal{A}(\mathcal{H})$ using the characteristic polynomials as discussed in (An introduction to hyperplane arrangements), Miller et al. (2007). Zaslavsky (1975) connected the computation of the number of regions in an arrangement to the corresponding characteristic polynomials. But it can be extremely tricky to find exact (simple) forms for those polynomials even for rather straight-forward arrangements. Fukuda et al. (1991) explicitly mentioned via citing the work of Vergnas (1980) and Zaslavsky (1975) that computing the number of regions for arbitrary hyperplane arrangement is non-trivial as it depends on the underlying matroid structure. In our work, we are able to establish an exact form for a non-simple setting. The geometric ideas to understand the subspaces spanned by the normals (aka essentialization) corresponding to the hyperplanes can be further leveraged to establish exact forms or average teaching results for more general arrangements than relaxed general position. One possible study could be to understand the induced regions in terms of faces for which intersection of hyperplanes on a given hyperplane could be studied. Our idea of path-connectivity could be a potential direction to find out simple forms for the characteristic polynomials corresponding to more relaxed arrangements.

Appendix D. Faces Induced by Intersections of Halfspaces: Proof of Proposition 6

In this section, we provide the proof of Proposition 6 for the number of faces induced by the hyperplane arrangement $\mathcal{A}(\mathcal{H})$.

Proof [Proof of Proposition 6] To count the number of faces induced by the arrangement $\mathcal{A}(\mathcal{H})$ on the hyperplanes, one way it can be ascertained is by counting the number of regions/faces induced on any hyperplane. If we fix any hyperplane $h^* \in \mathcal{H}$ and look at the intersections of h^* with $\mathcal{H} \setminus \{h^*\}$, we can count the number of regions formed on h^* .

If $d' = 1$, then $\mathfrak{F}(\mathcal{A}(\mathcal{H})) = n$ since all the hyperplanes are parallel to each other. Thus, we assume that $d' > 1$ for further discussion.

Since h^* can be interpreted as a *flat*, we can write $h^* \equiv v^* + W^*$ for some vector $v^* \in \mathbb{R}^d$ and $(d-1)$ -dimensional subspace W^* of \mathbb{R}^d . By Definition 2, $(h^* \cap h)$ is a $(d-2)$ -dimensional flat $\forall h \in \mathcal{H} \setminus \{h^*\}$. Thus, we define by $\mathcal{H}'_{n-1,d-1} \triangleq \{(h^* \cap \ell) \mid \ell \in \mathcal{H} \setminus \{h^*\}\}$ the induced set of $n-1$ flats (intersections) on h^* (which is a $(d-1)$ -dimensional flat). We note that for any $1 \leq k \leq d' - 1$, if $T_k \subset \mathcal{H}'_{n-1,d-1}$ then

$$\dim \left(\bigcap_{\ell \in T_k} \ell \right) = (d-1) - k$$

It holds because if $\dim \left(\bigcap_{\ell \in T_k} \ell \right) \neq (d-1) - k$ then $\dim \left(\left(\bigcap_{\ell \in T_k} \ell \right) \cap h^* \right) \neq d - (k+1)$ since $\left(\bigcap_{\ell \in T_k} \ell \right) \subset h^*$. This violates d' -relaxed general position arrangement of \mathcal{H} . Thus, $\mathcal{H}'_{n-1,d-1}$ is in $(d'-1)$ -relaxed general position arrangement. Since counting the number of regions induced on h^* by $\mathcal{H}'_{n-1,d-1}$ is the same as ascertaining $\mathfrak{r}(\mathcal{A}(\mathcal{H}'_{n-1,d-1}))$ i.e. $(n-1)$ hyperplanes in \mathbb{R}^{d-1} in $(d'-1)$ -relaxed general position, using Theorem 3 we get:

$$\mathfrak{r}(\mathcal{A}(\mathcal{H}'_{n-1,d-1})) = \mathfrak{r}(\mathcal{A}(\mathcal{H}_{n-1,d-1})) = \sum_{i=0}^{d'-1} \binom{n-1}{i}$$

Since, there are n hyperplanes thus the proposition follows,

$$\mathfrak{F}(\mathcal{A}(\mathcal{H})) = n \cdot \sum_{i=0}^{d'-1} \binom{n-1}{i}$$

which completes the proof. ■

Appendix E. Teaching Complexity of Convex Polytopes: Proof of Theorem 8

In this section, we provide the proof of the main Theorem 8. It is divided in three subsections: (i) worst-case of teaching complexity of convex polytopes of $\Theta(n)$ as part of Theorem 8 in §E.1, (ii) bounds on $\tau(\mathcal{A}(\mathcal{H}))$ via proof of Corollary 7 in §E.2 and (iii) proof of average-teaching complexity of Main Theorem 8 in §E.3.

E.1. Worst-case Complexity for Teaching: $\Theta(n)$

We would show the lower bound on the *worst-case* of $\Omega(n)$ and notice that upper bound is trivial. Consider n -dimensional hypersphere \mathbb{S} in \mathbb{R}^d and \mathbb{S}_{pos} the restriction in the positive quadrant *i.e.* all coordinates are positive.

To give an intuition of the worst-case scenario, we start with \mathbb{R}^2 . Consider the unit circle $x^2 + y^2 = 1$ restricted in the positive quadrant. We randomly drop n points on the arc and draw tangents to them. Notice that *no* three tangents can intersect at a point. Moreover, since all the tangents lie in a single quadrant, they can't be parallel. Thus, any two have a *non-empty* intersection. It implies the n hyperplanes thus constructed are in *2-relaxed general position*. Notice that the arc forms a convex connected set with all the hyperplanes sharing a point. Thus, arrangement of the tangents induces a region which has n many sides or faces.

We use the similar idea to construct n hyperplanes in \mathbb{R}^d . Let us consider \mathbb{S}_{pos} the restriction of unit hypersphere in \mathbb{R}^d . Now, drop n points on the restriction in such a way that any d are *linearly independent*. Denote the n points as $\{x^{(1)}, x^{(2)}, \dots, x^{(n)}\}$. Now, consider the matrix Λ defined by $x^{(i)}$ as row for each $i \in [n]$. Thus, for $k \in [d]$, any k rows are *linearly independent*. Consider the hyperplanes defined by the n points. Notice that the *bias* is same for all the hyperplanes. Denote the hyperplanes by $\{h^{(1)}, h^{(2)}, \dots, h^{(n)}\}$. It is easy to see that we can equivalently write $h^{(i)} \equiv x^{(i)} \cdot y + 1$ for variable $y \in \mathbb{R}^d \forall i \in [n]$. Let us define for $k \in [d]$ $\mathbb{I}_{[k]} \triangleq \{i_1, i_2, \dots, i_k\}$ as k indices for rows. Denote by $\Lambda_{\mathbb{I}_{[k]}} = \Lambda_{[\mathbb{I}_{[k]}] \times [d]}$ (rows of Λ corresponding to $\mathbb{I}_{[k]}$) If we consider the linear system equation

$$\Lambda_{\mathbb{I}_{[k]}} \cdot y = \mathbf{1}^k \quad (13)$$

Notice that $\mathbf{rank}(\Lambda_{\mathbb{I}_{[k]}}) = k$ because row rank is k . Thus, Eq. (13) has a solution, call it y_0 .

Using rank-nullity (Theorem 31), we realize that $\dim(\{y \mid \Lambda_{\mathbb{I}_{[k]}} \cdot (y - y_0) = \mathbf{0}\})$ is k . Define a matrix Λ_h with each row as $(x^{(i)}, 1) \forall i \in [n]$. Now, if rewrite Eq. (13) as :

$$\Lambda_{\mathbb{I}_{[k]}} \cdot y = \mathbf{1}^k \Leftrightarrow (\Lambda_h)_{\mathbb{I}_{[k]}} \cdot \begin{pmatrix} y \\ 1 \end{pmatrix} = 0 \quad (14)$$

Eq. (14) implies that $\dim(\{y \mid \Lambda_{\mathbb{I}_{[k]}} \cdot (y - y_0) = \mathbf{0}\}) = \dim(\{y \mid (\Lambda_h)_{\mathbb{I}_{[k]}} \cdot \begin{pmatrix} y \\ 1 \end{pmatrix} = 0\}) = d - k$. But solving Eq. (14) is same as finding an intersection point of the hyperplane corresponding to rows $\mathbb{I}_{[k]}$ in Λ_h . Thus, we show that for any $k \in [n]$ subset of hyperplanes in $\{h^{(1)}, h^{(2)}, \dots, h^{(n)}\}$, they intersect in a $(d - k)$ -dimensional plane. Thus, these hyperplanes are in *d-relaxed general position*. Since, \mathbb{S}_{pos} is contained in exactly one halfspace of every hyperplane touching it implies it is contained in one region induced by the hyperplanes arrangement. Since all the hyperplanes share

one point in that region, thus we show that there is one region with n faces for arbitrary d -dimensional Euclidean space. This implies, the *worst-case* of teaching complexity of convex polytopes is $\Theta(n)$.

This completes the second part of Theorem 8.

E.2. Upper and Lower Bound on number of regions

In this subsection, we establish bounds on $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$ as Corollary 7.

Proof [Proof of Corollary 7]

We'll prove the corollary in two parts – by establishing the upper and lower bounds on $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$.

The proof for the upper bound is based on a geometric series argument and uses the definition of a binomial term. First note that, using Theorem 3, we have:

$$\mathfrak{r}(\mathcal{A}(\mathcal{H})) = \sum_{i=0}^{d'} \binom{n}{i}$$

Now, we observe the following computation:

$$\begin{aligned} \frac{\sum_{i=0}^{d'} \binom{n}{i}}{\binom{n}{d'}} &= \sum_{i=d'}^0 \frac{\binom{n}{i}}{\binom{n}{d'}} \\ &= 1 + \frac{d'}{(n-d'+1)} + \frac{d'(d'-1)}{(n-d'+1)(n-d'+2)} + \cdots + \frac{d!}{(n-d'+1) \cdots (n-d'+d')} \\ &\leq 1 + \frac{d'}{(n-d'+1)} + \frac{(d')^2}{(n-d'+1)^2} + \cdots + \frac{(d')^{d'}}{(n-d'+1)^{d'}} \\ &\leq \sum_{i=0}^{\infty} \left(\frac{d'}{(n-d'+1)} \right)^i \\ &= \frac{1}{1 - \frac{d'}{n-d'+1}} = \frac{n-d'+1}{n-2d'+1} \end{aligned}$$

The last inequality establishes the upper bound in the corollary.

For the lower bound we note that:

$$\mathfrak{r}(\mathcal{A}(\mathcal{H})) = \sum_{i=0}^{d'} \binom{n}{i} \geq \binom{n-1}{d'}$$

Hence, the corollary is proven. ■

E.3. Proof of Theorem 8

In the subsection E.2, we proved the key corollary to show tight bounds on $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$. We use Corollary 7 to show the stated bounds on A.1– upper bound in Lemma 24 and lower bound in Lemma 25. We combine Lemma 24 and Lemma 25 to prove the Main Theorem 8.

To simplify the notations, we use $Q(n, d)$ (discussed in details in Appendix F.1) to denote the number of regions induced by n hyperplanes in \mathbb{R}^d arranged in general position (*cf* Definition 1). We note that, in the case of d' -relaxed general position arrangement, $\mathfrak{r}(\mathcal{A}(\mathcal{H})) = Q(n, d')$ and $\mathfrak{F}(\mathcal{A}(\mathcal{H})) = n \cdot Q(n-1, d'-1)$. This follows from the recursion on $Q(\cdot, \cdot)$ *i.e.* $Q(n, d) = Q(n-1, d) + Q(n-1, d-1)$ (for $n > d$), as discussed in Lemma 26 and the subsequent exact form in Corollary 27 (in Appendix F.1). We rewrite $\mathfrak{r}(\mathcal{A}(\mathcal{H}))$ and $\mathfrak{F}(\mathcal{A}(\mathcal{H}))$ in terms of $Q(\cdot, \cdot)$ so that any bound on $Q(\cdot, \cdot)$ would help us in bounding $\mathfrak{F}(\mathcal{A}(\mathcal{H}))/\mathfrak{r}(\mathcal{A}(\mathcal{H}))$. We leverage tight bounds (upper and lower) on the ratio $Q(n-1, d')/Q(n-1, d'-1)$ to achieve the results in the main theorem. We would formally state the two lemmas and provide their proofs before we complete the proof of the main theorem of the section.

Lemma 24 (Upper bound) *Assume \mathcal{H} is in d' -relaxed general position. Assume $r \sim \mathcal{U}$. Let the random variable M_n denote the number of halfspace queries that are requested in the teaching Algorithm 1, then*

$$\mathbb{E}_{\mathcal{U}}[M_n] = \mathcal{O}(d')$$

i.e. the average teaching complexity of convex polytopes is upper bounded by $\mathcal{O}(d')$.

Proof Since the *target hypotheses* are sampled uniformly at random, each hypothesis is enclosed by $\mathfrak{F}(\mathcal{A}(\mathcal{H}))/\mathfrak{r}(\mathcal{A}(\mathcal{H}))$ hyperplanes on average.

We first provide an upper bound on the average teaching complexity and using similar technique show a lower bound.

Combining Theorem 3, Lemma 26, upper bound in Corollary 7, and Proposition 6, we prove the lemma in two cases:

Case 1: $n > 2d'$ (n is sufficiently large)

$$\frac{\mathfrak{F}(\mathcal{A}(\mathcal{H}))}{\mathfrak{r}(\mathcal{A}(\mathcal{H}))} = \frac{n \cdot Q(n-1, d'-1)}{Q(n, d')} \quad (15)$$

$$= \frac{n \cdot Q(n-1, d'-1)}{Q(n-1, d') + Q(n-1, d'-1)} \quad (16)$$

$$= n \cdot \left(1 / \left(\frac{Q(n-1, d')}{Q(n-1, d'-1)} + 1 \right) \right) \\ \leq n \cdot \left(1 / \left(\frac{n-1}{2d'} + 1 \right) \right) \quad (17)$$

$$= 2d' \cdot \left(1 / \left(1 + \frac{2d'-1}{n} \right) \right) \\ \leq 2d'. \quad (18)$$

Eq. (15) follows using Theorem 3 and Proposition 6, Eq. (16) is based on the recursion mentioned in Lemma 26, Eq. (17) is bounded using Lemma 30 and in Eq. (18), we observe that $0 < \frac{2d'-1}{n}$.

Case 2: $n \leq 2d' \implies n = \mathcal{O}(d')$. This trivially gives $\mathcal{O}(d')$ as each *target hypothesis* is enclosed by at the most n hyperplanes.

Thus, in the two cases we have shown that the average teaching complexity of the algorithm is upper bounded by $\mathcal{O}(d')$. \blacksquare

Lemma 25 (Lower bound) *Assume \mathcal{H} is in d' -relaxed general position, and $r \sim \mathcal{U}$. Let the random variable M_n denote the number of halfspace queries that are requested in the teaching Algorithm 1, then*

$$\mathbb{E}_{\mathcal{U}}[M_n] = \Omega(d')$$

i.e. the average teaching complexity of convex polytopes is lower bounded by $\Omega(d')$.

Proof Following similar steps as Lemma 24; for sufficiently large $n > d$ we get:

$$\frac{\mathfrak{F}(\mathcal{A}(\mathcal{H}))}{\mathfrak{r}(\mathcal{A}(\mathcal{H}))} = \frac{n \cdot Q(n-1, d'-1)}{Q(n, d')} = \frac{n \cdot Q(n-1, d'-1)}{Q(n-1, d') + Q(n-1, d'-1)} \quad (19)$$

$$= n \cdot \left(1 / \left(\frac{Q(n-1, d')}{Q(n-1, d'-1)} + 1 \right) \right)$$

$$= n \cdot \left(1 / \left(\frac{\binom{n-1}{d'}}{Q(n-1, d'-1)} + 2 \right) \right) \quad (20)$$

$$\geq n \cdot \left(1 / \left(\frac{n-1}{d'} + 2 \right) \right) \quad (21)$$

$$= d' \cdot \left(1 / \left(\frac{n-1}{n} + \frac{2d'}{n} \right) \right)$$

$$\geq \frac{d'}{1+2} \quad (22)$$

Eq. (19) follows using Theorem 3, Lemma 26, and Proposition 6. Eq. (20) is a direct consequence of Corollary 27. By carefully noting the lower bound in Corollary 7, we get the bound in Eq. (21). We observe that $\frac{n-1}{n} + \frac{2d'}{n} < 1 + 2$. Thus for sufficiently large $n > d$, we show that the average teaching complexity of intersection of halfspaces is lower bounded by $\Omega(d')$. \blacksquare

Proof [Proof of Theorem 8] Using Lemma 24 and Lemma 25, it is straightforward that $\mathbb{E}_{\mathcal{U}}[M_n] = \Theta(d')$. \blacksquare

Appendix F. Learning Complexity of Convex Polytopes: Proof of Theorem 10

In this section, we would discuss the problem of active learning of convex polytopes induced by the hyperplanes arrangement in \mathbb{R}^d . We would provide some relevant results on the counting of the number of regions induced by the arrangement of n hyperplanes in \mathbb{R}^d in *general position* (Definition 1). We would provide a procedure (shown in Algorithm 2) which actively and sequentially learns a uniformly randomly sampled region. We show that the average query(sample) complexity for the algorithm is $\Theta(d' \log n)$. We would provide the proof of Theorem 10 when the hyperplane arrangement is in *general position* (Definition 1) and then show the extension to the case of d' -relaxed *general position* arrangement.

First we would start with some illustration of the Definition 1 and see how it is an special case of Definition 2. To illustrate and understand the definition, we can take a look at euclidean spaces \mathbb{R}^2 and \mathbb{R}^3 . For \mathbb{R}^2 , consider three lines denoted by l_1, l_2 and l_3 (hyperplanes). Note, k can take two values. For $k = 1$, the given line l_i intersects in a line which is vacuously true. For any two lines, they need to intersect in a point. For the three lines, they have an empty intersection. For \mathbb{R}^3 , consider four planes denoted by P_1, P_2, P_3 and P_4 . We can understand the definition from Table 3.

Table 3: General position of planes in \mathbb{R}^3

k	Intersection
1	A plane, \mathbb{R}^2
2	A line, \mathbb{R}
3	A point
4	Null

We notice that Definition 1 is a *special* case of Definition 2. If we fix, say $k = 2$ and assume that for intersections of planes upto k follow Table 3 but if any subset of hyperplanes of size more than k , they intersect only in null *i.e.* if we pick three planes then they don't intersect in a common point. This would rightly give an example of an arrangement in d' -relaxed *general position* for $d' = 2$. We illustrate this arrangement in Fig. 2(b). If $k = 1$, then that would give 1-relaxed *general position* as illustrated in Fig. 2(c) which accounts for case when hyperplanes are parallel to each other. In the case of $k = 3$, we get 3-relaxed *general position* (Fig. 2(a)) which is also the case of *general position* (Definition 1) arrangement. Relaxed general position is a natural extension to general position. It takes into account arrangements which can't be structurally explained by general position setting in higher dimension as discussed above. From a learning point of view, data is usually embedded sparsely in spaces with much higher dimension than the information they contain. There has been extended study on learning the sparse representation using component analyses. Interestingly, relaxed notion of general position captures the essence of arrangements where hyperplanes could be sparsely embedded in high dimensional space but are in general position in a much smaller subspace. We interchangeably use d' -general position or *general position* when $d' = d$ if the hyperplane arrangement is in d' -relaxed *general position*.

We are interested in the notion of *general position* of hyperplanes for a variety of reasons. First, we show an existing duality (see §6) between a problem instance of finding the number of ϕ -separable dichotomies (primal space) (Cover, 1965) to a problem instance of teaching intersection of halfspaces (dual space). This duality would be achieved when the points in primal space and hyperplanes in

dual space are in general position of points (see Definition 1, Definition 2) and general position of hyperplanes (see Definition 11, Definition 12) respectively. Second, Miller et al. (2007, chap: An Introduction to Hyperplane Arrangements) mentions an exact form for the number of regions induced by the *general position* arrangement of hyperplanes \mathcal{H} . This key result would be used in our significant contributions (see §4): Theorem 3 and Proposition 6, where we would try to reduce from the case of *d'-relaxed general position* to a case of *general position*.

To prove Theorem 10, we would show some relevant results in the following subsection:

F.1. Bounds on Number of Regions Induced by General Position Arrangement

Consider a set of n hyperplanes in \mathbb{R}^d , denoted by \mathcal{H} , and the underlying arrangement $\mathcal{A}(\mathcal{H})$ is in general position (Definition 1). Denote by $Q(n, d)$ the number of regions induced by $\mathcal{A}(\mathcal{H})$. Although Miller et al. (2007) provides an exact form for $Q(n, d)$, we would provide a recursion similar to Jamieson and Nowak (2011) with a proof for continuity and flow of ideas.

Lemma 26 (Regions induced by general-position hyperplane arrangement) *Let $Q(n, d)$ denote the number of d -cells or regions induced by the general position hyperplane arrangement. $Q(n, d)$ satisfies the recursion:*

$$Q(n, d) = Q(n - 1, d) + Q(n - 1, d - 1) \quad (23)$$

where $Q(1, d) = 2$ and $Q(n, 0) = 1$.

Proof The proof is based on a recursive argument on how hyperplanes are added to the d -dimensional space. Consider an arbitrary ordering on the hyperplanes. Denote the last hyperplane added by $h^{(n)}$. We observe that the number of new regions induced by $h^{(n)}$ to $\mathcal{A}(\mathcal{H} \setminus \{h^{(n)}\})$ is equal to the number of regions/faces induced on $h^{(n)}$ by the intersections of $(\mathcal{H} \setminus \{h^{(n)}\})$ on it. Since, the hyperplanes are in general position, thus all the other $(n - 1)$ hyperplanes intersect $h^{(n)}$ on $(d - 2)$ -plane. Thus, we have $(n - 1)$ of $(d - 2)$ - dimensional hyperplanes⁷ arranged on a $(d - 1)$ -plane. Denote this induced set of hyperplanes by $\hat{\mathcal{H}}_{n-1, d-1}$, which can be defined as $\hat{\mathcal{H}}_{n-1, d-1} \triangleq \{(h^{(n)} \cap \ell) \mid \ell \in (\mathcal{H} \setminus \{h^{(n)}\})\}$ the induced set of $n - 1$ flats (intersections) on $h^{(n)}$. We note that for any $1 \leq k \leq d - 1$, if $T_k \subset \hat{\mathcal{H}}_{n-1, d-1}$ then

$$\dim \left(\bigcap_{\ell \in T_k} \ell \right) = (d - 1) - k$$

It holds because if $\dim \left(\bigcap_{\ell \in T_k} \ell \right) \neq (d - 1) - k$ then $\dim \left(\left(\bigcap_{\ell \in T_k} \ell \right) \cap h^{(n)} \right) \neq d - (k + 1)$ since $\left(\bigcap_{\ell \in T_k} \ell \right) \subset h^{(n)}$. This violates the general position arrangement of \mathcal{H} . Thus, $\hat{\mathcal{H}}_{n-1, d-1}$ is in general position arrangement. But by definition, number of faces induced on $h^{(n)}$ by $\hat{\mathcal{H}}_{n-1, d-1}$ is $Q(n - 1, d - 1)$.

Hence, the total number of regions in the d -dimensional space is $Q(n - 1, d) + Q(n - 1, d - 1)$. Thus, the lemma follows. \blacksquare

$Q(\cdot)$ as defined above has the following exact form:

⁷ Proof follows similar steps as in Proposition 6.

Corollary 27 (An introduction to hyperplane arrangement Miller et al. (2007)) *The recursion in Lemma 26 has the form:*

$$Q(n, d) = \sum_{i=0}^d \binom{n}{i}$$

for $n > d$. If $n \leq d$, then $Q(n, d) = 2^n$.

We prove a simple corollary which claims an asymptotic bound on $Q(\cdot)$ that would be used in a number of results:

Corollary 28 *For sufficiently large $n > d$, there exist positive real number k_1 such that:*

$$k_1 \frac{n^d}{d!} < Q(n, d)$$

Proof Using Corollary 27, we can write:

$$\begin{aligned} Q(n, d) &= \sum_{i=0}^d \binom{n}{i} \\ &= \sum_{i=0}^d \Theta \left(\frac{n^i}{i!} \right) \text{ (for sufficiently large } n \text{ each term is bounded by above and below)} \\ &> k_1 \frac{n^d}{d!} \text{ (by definition, } \exists k_1 > 0, N_0 \text{ such that } \forall n > N_0 \text{ condition holds)} \end{aligned}$$

Specifically, we can show that for $n \geq d^2$, the condition holds. This is true because there exists a constant c such that $c \cdot \prod_{i=0}^{d-1} (n - i) > n^d$. ■

F.2. Average-case Analysis of Active Learning Complexity

In subsection §4, we introduced the problem of teaching convex polytopes via halfspace queries for a set of hyperplanes \mathcal{H} in \mathbb{R}^d arranged in d' -relaxed general position. In Theorem 8, we showed that the teaching complexity for the arrangement is $\Theta(d')$. Now, we would discuss the problem of active learning of convex polytopes induced by $\mathcal{A}(\mathcal{H})$, via halfspace queries. Using motivations from Jamieson and Nowak (2011) in which they explore the problem of ranking, we provide Algorithm 2 to actively learn the enclosing region for a randomly sampled *target region* via adaptive and sequential selection of halfspaces queries for a hyperplane. We analyze the problem in the framework of the average-case analysis as motivated in Traub (2003) and section 1.1 of Jamieson and Nowak (2011). We achieve $\Theta(d' \cdot \log n)$ average label complexity for active learning through our Algorithm 2. The lower bound is straight forward using Corollary 28. We need at least $|\log_2(\mathfrak{R}(\mathcal{A}(\mathcal{H})))|$ bits of information to specify (enumerate) all the possible target concepts *i.e.* $\log_2(Q(n, d')) = \Omega(d' \cdot \log n)$ many for sufficiently large n . As discussed in Jamieson and Nowak (2011), we note that the overall computational complexity of the algorithm is

$\mathcal{O}(n \mathbf{poly}(d) \mathbf{poly}(\log n))$ because in total the number of queries requested are at $\max^8 \mathcal{O}(d \log n)$ and the complexity of each test is polynomial in the number of queries requested because each one is a linear constraint. In fact, we could also show that our Algorithm 2 is attribute efficient (Klivans and Servedio, 2006). As defined, we could think of finding the exact labelling function (cf. §3) as learning the boolean function (Klivans and Servedio, 2006). If $d = \mathbf{poly}(n)$ (or d is small compared to n) then algorithm runs in $\mathcal{O}(\mathbf{poly}(n) \cdot \mathbf{poly}(\log n)) = \mathbf{poly}(n)$, and hence is attribute efficient.

Our key observation is that the sequential algorithm doesn't ask for labels for *non-trivial* number of hyperplanes since they are *unambiguous* or *uninformative* wrt to the target region. Our adaptive algorithm filters out such queries irrespective of the ordering in which the hyperplanes are queried for the enclosing region. In the following subsection, we formally provide the characterization of *ambiguous* hyperplane queries which is based on our Definition 16.

F.3. Characterization of an Ambiguous Query of a Hyperplane

In Definition 16, we gave the characterization for an ambiguous hyperplane wrt to a subset $\mathcal{S} \subset \mathcal{H}$. Jamieson and Nowak (2011) gave similar characterization but for bisecting hyperplanes corresponding to pairwise queries of embedded objects. With our characterization we are able to show similar results which we use to give a bound on the query complexity.

Algorithm 2 Query Selection Algorithm

```

1 Input:  $n$  hyperplanes in  $\mathbb{R}^d$ 
  begin
2   Initialize: hyperplanes  $\mathcal{H} = \{h^{(1)}, h^{(2)}, \dots, h^{(n)}\}$  in uniformly random order
   for  $i \in [n]$  do
3     if  $h^{(i)}$  is ambiguous then
       | request  $h^{(i)}$ 's label from reference
     end
4     else
       | impute  $h^{(i)}$ 's label from previously labeled queries.
     end
   end
  end
5 Output: target region(region)
  
```

As mentioned in Jamieson and Nowak (2011), we call the arrangement of the set of n hyperplanes in \mathbb{R}^d as an *n-partition* and a region induced by the arrangement as a *d-cell*. Now consider the basic sequential procedure of Algorithm 2. WLOG, assume that the algorithm samples the k hyperplanes in the order $\{h^{(1)}, \dots, h^{(k)}\}$. It is not very difficult to see that the target region r is contained within a *d-cell*, C_k (defined by the labels of the queried hyperplanes from $h^{(1)}$ through $h^{(k)}$). Assume that $h^{(k+1)}$ is sampled in the next iteration. Querying $h^{(k+1)}$ for labels is informative (*i.e.*, ambiguous) *iff* it intersects this *d-cell* C_k . We realize that this observation is significant because if k is sufficiently larger than d , then the probability that the next sampled hyperplane intersects C_k is very small; in fact the probability is on the order of $1/k$ (proved in Lemma 9). In the next subsection, we provide

8. In the case of d' -relaxed general position, the number of queries requested is $\mathcal{O}(d' \log n)$.

the proof of Lemma 9 which ascertains a bound on the probability that a sampled hyperplane is ambiguous for query.

F.4. Probability of Ambiguity: Proof of Lemma 9

In this subsection, we would show that on a random ordering of hyperplanes, it is highly likely that a hyperplane query is unambiguous. This is the essential component of the query selection algorithm. We would start by stating an important result which would allow us to argue the probability with which a randomly sampled hyperplane is ambiguous. We denote a target hypothesis(region) by r .

Lemma 29 *Assume $r \sim \mathcal{U}$. Consider the subset $\mathcal{S} \subset \mathcal{H}$ with $|\mathcal{S}| = k$ that is randomly selected from \mathcal{H} such that all $\binom{n}{k}$ subsets are equally probable. If $\mathfrak{R}(\mathcal{A}(\mathcal{S}))$ denotes the set of regions induced by the arrangement of \mathcal{S} , then every $r \in \mathfrak{R}(\mathcal{A}(\mathcal{S}))$ is equally probable (where $Q(k, d) = |\mathfrak{R}(\mathcal{A}(\mathcal{S}))|$).*

Proof This lemma follows immediately using Jamieson and Nowak (2011, Lemma 3). Any uniformly random selection of k -tuple of hyperplanes induces k -partition of the d -dimensional space. Each k -partition contains some d -cells of n -partition induced by the arrangement of all the hyperplanes. Since the k -tuple has been uniform randomly selected and each d -cell of the n -partition is equally probable, thus there are $Q(n, d)/Q(k, d)$ d -cells of the n -partition in any d -cell of the k -partition. As each d -cell of the n -partition is equally probable which implies, probability mass in each d -cell of k -partition is $Q(n, d)/Q(k, d) \times 1/Q(n, d) = 1/Q(k, d)$. Hence, the lemma follows. \blacksquare

We would state an easy inequality that we would use in the subsequent lemmas.

Lemma 30 *For $k > 2d$, the following inequality holds:*

$$\frac{Q(k, d-1)}{Q(k, d)} \leq \frac{d}{k/2}$$

Proof First note that,

$$d + \frac{(k-d+1)(k-2d+3)}{(k-d+2)} \geq \frac{k}{2} \quad (24)$$

Using the following simplification, Eq. (24) holds.

$$\begin{aligned} & 2d(k-d+2) + 2(k-d+1)(k-2d+3) - k(k-d+2) \\ &= (2d-k)(k-d+2) + 2(k-d+1)(k-2d+3) \\ &= (2d-k)(k-d+1) + (2d-k) + 2(k-d+1)(k-2d+3) \\ &= (k-d+1)[2(k-2d+3) + (2d-k)] + (2d-k) \\ &= (k-d+1)(k-2d+6) - (k-2d) \geq 0 \end{aligned}$$

Now, we would the result in the following computation:

$$\frac{Q(k, d-1)}{Q(k, d)} = 1 / \left(1 + \frac{\binom{k}{d}}{Q(k, d-1)} \right) \quad (25)$$

$$\leq 1 / \left(1 + \frac{\binom{k}{d}}{\binom{k}{d-1} \frac{k-d+2}{k-2d+3}} \right) \quad (26)$$

$$\begin{aligned}
 &= 1 / \left(1 + \frac{(k-d+1)(k-2d+3)}{d(k-d+2)} \right) \\
 &= d / \left(d + \frac{(k-d+1)(k-2d+3)}{(k-d+2)} \right) \\
 &\leq \frac{d}{k/2}
 \end{aligned} \tag{27}$$

Using Lemma 26 and Corollary 27, we have $Q(k, d) = Q(k, d-1) + \binom{k}{d}$, which gives Eq. (25). Eq. (26) is the straight forward consequence of Corollary 7 *i.e.* $Q(k, d-1) \leq \binom{k}{d-1} \frac{k-d+2}{k-2d+3}$. Finally, we use Eq. (24) to get Eq. (27). ■

Now, we would talk about the probability of ambiguity of any randomly selected hyperplane. If we assume that k hyperplanes have been selected uniformly at random, they induce a k -partition. We can ascertain the probability of the event of $(k+1)$ th sampled hyperplane to be ambiguous conditioned on the labels queried/imputed of the first k hyperplanes. We state the result in the Lemma 9.

Proof [Proof of Lemma 9] The first k sampled hyperplanes induce a k -partition. The target region r belongs to one of the d -cells, say C_k in the k -partition. According to the characterization, hyperplane query for $h^{(k+1)}$ is ambiguous if it intersects C_k . Let $P(k, d)$ denote the number of d -cells in the k -partition that are intersected by the hyperplane $h^{(k+1)}$. Using Lemma 29, we know that each of the d -cell in the k -partition is equally probable. Thus, probability of $q_{h^{(k+1)}}$ being ambiguous is same as the probability of each d -cell that $h^{(k+1)}$ intersects times the number of d -cells it intersects in the k -partition. Thus we have:

$$P_A(k, d, \mathcal{U}) = \frac{P(k, d)}{Q(k, d)} = \frac{Q(k, d-1)}{Q(k, d)} \stackrel{\text{Lemma 30}}{\leq} \frac{d}{k/2}$$

Thus, for $a = 2$, we have achieved a bound on the probability of the event of a hyperplane query being ambiguous. ■

E.5. Proof of Theorem 10

We denote by M_n the number of queries asked for by the algorithm. But this is same as the number of queries being requested by the Query Selection Algorithm. Thus, we have $M_n = \sum_{i=1}^n \mathbf{1}\{q_{h^{(i)}} \text{ is requested}\}$.

We would provide the proof of the bound for the average-case complexity for active learning of convex polytopes in the main theorem of the section Theorem 10.

Proof [Proof of Theorem 10] Let us denote the event of requesting the query for hyperplane $h^{(k)}$ for each k by B_k . Note that each $B_k = \mathbf{1}\{q_{h^{(k)}} \text{ is requested}\}$ is a bernoulli distribution with parameter $P_A(k, d, \mathcal{U})$. Since, the bounds of $P_A(k, d, \mathcal{U})$ makes sense when $k > 2d$ so we assume that for $k \leq 2d$, all the queries are ambiguous.

$$\mathbb{E}_{\mathcal{U}}[M_n] = \sum_{i=1}^n \mathbb{E}_{\mathcal{U}}[B_i]$$

$$\begin{aligned}
 &\leq \sum_{i=1}^{2d} \mathbb{E}_{\mathcal{U}}[B_i] + \sum_{i=2d+1}^n \mathbb{E}_{\mathcal{U}}[B_i] \\
 &\leq 2d + \sum_{i=2d+1}^n \frac{2d}{i} \\
 &\leq 2d + 2d \log_2 \left(\frac{n}{2d+1} \right) \\
 &= 2d \log_2 \left(\frac{2n}{2d+1} \right) \leq 2d \log_2(n)
 \end{aligned}$$

This completes the proof. ■

Thus, for a set of hyperplanes \mathcal{H} arranged in general position, we provide an algorithm with $\mathcal{O}(d \cdot \log n)$ average query complexity for active learning of an enclosing region for target region.

Generalization to d' -relaxed general position We note that with similar arguments we can achieve the bound of $\mathcal{O}(d' \cdot \log n)$ if the set of hyperplanes are arranged in d' -relaxed general position. It is not very difficult to see that Theorem 3 and Proposition 6 would yield similar results as Lemma 29 and Lemma 9 and then a result similar to Theorem 10 follows. We note that in the case of d' -relaxed general position arrangement, the number of regions induced in \mathbb{R}^d by n hyperplanes is $Q(n, d')$. Similarly, the number of faces induced on a hyperplane turns out to be $Q(n-1, d'-1)$ (intersection of n hyperplanes). Lemma 29 and Lemma 9 can be extended for the relaxed case by straight-forward replacement of $Q(\cdot, d')$ and $Q(\cdot, d'-1)$ for number of regions and faces accordingly.

Earlier we argued on the lower bound which turns out to be $\Omega(d' \log n)$ (see Appendix F.2). With the upper bound of $\mathcal{O}(d' \cdot \log n)$ on the label complexity, thus we achieve the strong bound of $\Theta(d' \cdot \log n)$ for active learning of convex polytopes as shown in Table 1.

For the *worst-case* complexity of active learning of convex polytopes, we notice that it has to be $\Theta(n)$ since the lower bound holds because of the lower bound of $\Omega(n)$ for worst-case teaching complexity as shown in Appendix E.1. It implies that there exists a worst-case construction of a target regions such that no matter how the ordering of the hyperplanes are initialized, every sampled hyperplane in any iteration of Algorithm 2 would be ambiguous requiring all the halfspace queries to be made to determine the target region. Since n queries are sufficient thus the worst-case sample complexity of active learning of convex polytopes is $\Theta(n)$.

This completes the proof of the main theorem of the section.

Appendix G. Dual Map for ϕ -Separable Dichotomy: Proof of Theorem 13

In this appendix, we provide the proof of our main result for the construction of dual map *i.e.* Theorem 13. Using the properties of the dual map and bounds on the average teaching complexity for convex polytopes *i.e.* Theorem 8, we provide the proof of Corollary 14 which establishes similar bound on the average teaching complexity of ϕ -separable dichotomies. We first state and prove the necessary lemmas and results in order to prove Theorem 13. Before that, we mention a fundamental result from linear algebra (also mentioned in Roman, 2007, Theorem 2.8) which would be used in a number of lemmas across appendices.

Theorem 31 (Rank-Nullity Theorem) *Let V and W be vector spaces over a field F , and let $T: V \rightarrow W$ be a linear transformation. Assuming the dimension of V is finite, then*

$$\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T)) \quad (28)$$

where $\dim(\text{Ker}(T))$ is nullity of T and $\dim(\text{Im}(T))$ is the rank of T .

G.1. Relevant Lemmas for Proof of Theorem 13

First, we would prove a straight-forward result for homogeneous linear separability which forms the basis for the equivalence relation we obtained in §6.

Lemma 32 *If w is the normal vector for the homogeneous linear separator of $\{\mathcal{X}^+, \mathcal{X}^-\}$ then, $-w$ is the normal vector for the homogeneous linear separator of $\{\mathcal{X}^-, \mathcal{X}^+\}$.*

Proof If w is the normal vector for a homogeneous linear separator of $\{\mathcal{X}^+, \mathcal{X}^-\}$, then,

$$\begin{aligned} w \cdot x > 0 &\Leftrightarrow (-w) \cdot x < 0 \text{ if } x \in \mathcal{X}^+ \\ w \cdot x < 0 &\Leftrightarrow (-w) \cdot x > 0 \text{ if } x \in \mathcal{X}^- \end{aligned}$$

Thus, $-w$ is the the normal vector for a homogeneous linear separator of $\{\mathcal{X}^-, \mathcal{X}^+\}$ ■

To study the arrangement of dual hyperplanes, we define the matrices $\Lambda_{[(n-1) \times d]}$ and $[\Lambda_h]_{[(n-1) \times (d-1)]}$ such that $\forall i \in [n-1] \Lambda[i, :] = x^{(i)}$ and $\Lambda_h[i, :] = x_{[d-1]}^{(i)}$ where $x_{[d-1]}$ is first $d-1$ components of x . Using the d' -relaxed general position arrangement of \mathcal{X} and nullity of $x^{(n)}$ as a dimension, in Lemma 33 we show that $\text{rank}(\Lambda_h) = d' - 1$ and any $(d' - 1)$ rows of Λ_h are linearly independent.

Lemma 33 *For the matrices constructed above, $\text{rank}(\Lambda_h) = d' - 1$, and any $(d' - 1)$ rows of Λ_h are linearly independent.*

Proof First part of the lemma is straight-forward since, by definition any d' vectors in \mathcal{X} are linearly independent which means d' columns of Λ are linearly independent, implying $(d' - 1)$ columns of Λ_h are linearly independent.

For the second part, for an indexed set $\mathbb{I}_{[d'-1]} \triangleq \{i_1, i_2, \dots, i_{d'-1}\}$ consider the $(d' - 1)$ rows $\left\{ x_{[d-1]}^{(i_1)}, x_{[d-1]}^{(i_2)}, \dots, x_{[d-1]}^{(i_{d'-1})} \right\}$ of Λ_h which are linearly dependent. Thus, \exists scalars α_j 's (not all zeros) such that:

$$\sum_{j=1}^{d'-1} \alpha_j \cdot x_{[d-1]}^{(i_j)} = 0 \implies \sum_{j=1}^{d'-1} \alpha_j \cdot \left(x_{[d-1]}^{(i_j)}, x_d^{(i_j)} \right) - \left(\sum_{j=1}^{d'-1} \alpha_j \cdot x_d^{(i_j)} \right) \cdot x^{(n)} = 0 \quad (29)$$

$$\implies \sum_{j=1}^{d'-1} \alpha_j \cdot x^{(i_j)} - \left(\sum_{j=1}^{d'-1} \alpha_j \cdot x_d^{(i_j)} \right) \cdot x^{(n)} = 0 \quad (30)$$

In Eq. (29) we use that $x^{(n)} = \mathbf{e}_d$. Eq. (30) implies that we have d' vectors of \mathcal{X} linearly dependent. Contradiction! Thus, for any indexed set $\mathbb{I}_{[d'-1]}$, the corresponding submatrix of dimension $[d' - 1 \times d' - 1]$ of $\mathbf{\Lambda}_h$, is full rank. Hence, the second part of the lemma is proven. \blacksquare

Now, we would give the proof of the key lemma of duality which shows that the mapped hyperplanes follow the criterion of $(d' - 1)$ -relaxed general position. For the sake of clarity and flow, we would restart with the construction of sets. Let us define $\mathbb{I}_{[k]} \triangleq \{i_1, i_2, \dots, i_k\}$ as k indices for rows. Denote by $\mathbf{\Lambda}_{\mathbb{I}_{[k]}} = \mathbf{\Lambda}_{[\mathbb{I}_{[k]} \times d]}$ (rows of $\mathbf{\Lambda}$ corresponding to $\mathbb{I}_{[k]}$) and by $(\mathbf{\Lambda}_h)_{\mathbb{I}_{[k]}} = (\mathbf{\Lambda}_h)_{[\mathbb{I}_{[k]} \times d-1]}$ (rows of $\mathbf{\Lambda}_h$ corresponding to $\mathbb{I}_{[k]}$). As in §6, we redefine $\mathcal{X} \triangleq \{x^{(1)}, x^{(2)}, \dots, x^{(n)}\}$.

Lemma 34 (Key lemma of duality) *If $\mathcal{S}_{\cap}^k = \left\{ x \in \mathbb{R}^{d-1} \mid \mathbf{\Lambda}_{\mathbb{I}_{[k]}} \begin{pmatrix} x \\ 1 \end{pmatrix} = \mathbf{0}^k \right\}$ for $1 \leq k \leq (d' - 1)$, then $\dim(\mathcal{S}_{\cap}^k) = (d - 1) - k$. Moreover, no d' rows of $\mathbf{\Lambda}$ intersects in dual space i.e. $\left\{ x \in \mathbb{R}^{d-1} \mid \mathbf{\Lambda}_{\mathbb{I}_{[d']}} \begin{pmatrix} x \\ 1 \end{pmatrix} = \mathbf{0}^{d'} \right\} = \emptyset$.*

Proof [Proof of Lemma 34 of Duality] Define by $\mathbf{b} \triangleq (x_d^{(i_1)}, x_d^{(i_2)}, \dots, x_d^{(i_k)})$. Notice that,

$$\mathbf{\Lambda}_{\mathbb{I}_{[k]}} \begin{pmatrix} x \\ 1 \end{pmatrix} = 0 \iff (\mathbf{\Lambda}_h)_{\mathbb{I}_{[k]}} x = -\mathbf{b}^{\top} \quad (31)$$

If $k = d' - 1$ then $(\mathbf{\Lambda}_h)_{\mathbb{I}_{[d'-1]}}$ is $d' - 1$ rank invertible matrix implying Eq. (31) has a *unique* solution. Note that using Lemma 33, $(\mathbf{\Lambda}_h)_{\mathbb{I}_{[k]}}$ has rank k for $k < d'$. This implies that there is some $x_0 \in \mathbb{R}^{d-1}$ such that $(\mathbf{\Lambda}_h)_{\mathbb{I}_{[k]}} x_0 = -\mathbf{b}^{\top}$. Thus, we rewrite Eq. (31) as

$$\mathbf{\Lambda}_{\mathbb{I}_{[k]}} \begin{pmatrix} x \\ 1 \end{pmatrix} = 0 \iff (\mathbf{\Lambda}_h)_{\mathbb{I}_{[k]}} x = (\mathbf{\Lambda}_h)_{\mathbb{I}_{[k]}} x_0 \iff (\mathbf{\Lambda}_h)_{\mathbb{I}_{[k]}} (x - x_0) = 0$$

But using Theorem 31, $\mathbf{Ker}((\mathbf{\Lambda}_h)_{\mathbb{I}_{[k]}}) = (d-1) - k$. This implies that $\dim \left(\left\{ x \in \mathbb{R}^{d-1} \mid (\mathbf{\Lambda}_h)_{\mathbb{I}_{[k]}} (x - x_0) = 0 \right\} \right) = (d - 1) - k$. Thus, $\dim(\mathcal{S}_{\cap}^k) = (d - 1) - k$.

Notice that if $\mathbf{\Lambda}_{\mathbb{I}_{[d']}} \begin{pmatrix} x \\ 1 \end{pmatrix} = \mathbf{0}^{d'}$ has a solution then we can define $\begin{pmatrix} x \\ 1 \end{pmatrix}$ as a homogeneous linear separator and the points of \mathcal{X} corresponding to $\mathbf{\Lambda}_{\mathbb{I}_{[d'']}}$ lie on a $(d - 1)$ -dimensional halfspace (subspace) defined by $\begin{pmatrix} x \\ 1 \end{pmatrix}$. Note, $x^{(n)}$ doesn't lie on that subspace. On the other hand, because of d' -relaxed general position arrangement of \mathcal{X} , rows of $\mathbf{\Lambda}_{\mathbb{I}_{[d'']}}$ are linearly independent and lie on the subspace. It implies $\mathbf{rows}(\mathbf{\Lambda}_{\mathbb{I}_{[d']}}) \cup \{x^{(n)}\}$ are linearly independent. Contradiction. Thus, $\left\{ x \in \mathbb{R}^{d-1} \mid \mathbf{\Lambda}_{\mathbb{I}_{[d']}} \begin{pmatrix} x \\ 1 \end{pmatrix} = \mathbf{0}^{d'} \right\} = \emptyset$. Hence, the lemma follows. \blacksquare

With Lemma 34 and Eq. (1), we can formally prove our main theorem of the section on the dual map which says the dual set of hyperplanes are in $(d' - 1)$ -relaxed general position and each equivalence class of dichotomies $\mathfrak{E}(\mathcal{X})$ maps *uniquely* to all the concepts (hypotheses) in $\mathfrak{R}(\mathcal{A}(\mathcal{H}_{n-1, d-1}))$.

G.2. Proof of Theorem 13 and Corollary 14

In this subsection, we provide the proof of the results of interest. Following the notations in §6, we use slightly different notations in the proofs for the sake of clarity. For a dichotomy class $[v] \in \mathfrak{E}(\mathcal{X})$, we denote the dual point to a separator $w_{[v]}$ of the representative dichotomy by $z_{w_{[v]}}$ and region corresponding to $z_{w_{[v]}}$ as⁹ $r_{z_{w_{[v]}}} \in \mathfrak{R}(\mathcal{A}(\overline{\mathcal{H}}))$ such that $z_{w_{[v]}} \in r_{z_{w_{[v]}}}$ i.e. $r_{z_{w_{[v]}}} = \varphi_{\text{dual}}([v])$.

Proof [Proof of Theorem 13] By the definition of D.M, we get $\mathcal{H}_{n-1,d-1} = \Upsilon_{\text{dual}}(\mathcal{X})$. We constructed the matrices $\Lambda_{[n-1 \times d]}$ and $[\Lambda_h]_{[n-1 \times d-1]}$ to study the arrangement of dual hyperplanes. In the Key Lemma 34 of Duality, we proved that $\forall 1 \leq k \leq d' - 1$, any size k subset of $\mathcal{H}_{n-1,d-1}$ intersects in a flat of dimension $(d - 1 - k)$ and no d' dual hyperplanes intersect at a point. Thus, we show that $\mathcal{H}_{n-1,d-1}$ is in $(d' - 1)$ -relaxed general position arrangement which proves the first part of the theorem.

First, we notice that φ_{dual} is well-defined since Eq. (1) is a sign preserving construction. To prove the bijection of φ_{dual} , we first show that it is an *injection*. We assume that $\#\mathfrak{E}(\mathcal{X}) > 1$ since the other case can be handled trivially. Denote by $[u], [v]$ two different equivalence classes of $\mathfrak{E}(\mathcal{X})$. Let $w_{[u]}$ and $w_{[v]}$ be two corresponding linear separators respectively. Since $[u] \neq [v]$, \exists at least one point $x' \neq x^{(n)} \in \mathcal{X}$ which is classified/labeled *differently*. Consider the dual hyperplane $h_{x'} = \Upsilon_{\text{dual}}(x')$, and the dual points $z_{w_{[u]}}$ and $z_{w_{[v]}}$ of $w_{[u]}$ and $w_{[v]}$ respectively using the construction shown in Eq. (1). Since $w_{[u]}$ and $w_{[v]}$ classify x' differently, $z_{w_{[u]}}$ and $z_{w_{[v]}}$ belongs to two different regions of $h_{x'}$, implying $r_{z_{w_{[u]}}} \neq r_{z_{w_{[v]}}}$ where $r_{z_{w_{[u]}}} = \varphi_{\text{dual}}([u])$ and $r_{z_{w_{[v]}}} = \varphi_{\text{dual}}([v])$. Thus, φ_{dual} is an injection. Consider a region $r \in \mathfrak{R}(\mathcal{A}(\overline{\mathcal{H}}))$. Pick a point $z_0 \in r$. Now, define $w_z \triangleq (z_0^\top, 1)$. Since $z_0 \in r$, w_z is a homogeneous linear separator of a dichotomy in the primal space corresponding to r where dichotomy is defined by signs using Eq. (1). Note that it is a valid dichotomy since $0 \cdot z_0 + 1 > 0$ implying $(z_0^\top, 1)$ labels $x^{(n)}$ positively. We represent the dichotomy using the class $[u]$. Since, z_0 is arbitrary, thus $\varphi_{\text{dual}}^{-1}(r) = [u]$ implying surjection of φ_{dual} . Hence, we show φ_{dual} is a bijection. \blacksquare

The properties of the dual map is key in showing the bound on the teaching complexity of ϕ -separable dichotomies. We note that the dual map retains the arrangement of the general position of points (Definition 11) to relaxed general position of hyperplanes in the dual space (Definition 2). Thus, our bound on the average teaching complexity of convex polytopes in Theorem 8 applies in the case of average teaching complexity of ϕ -separable dichotomies which we show in Corollary 14. We present the proof of the corollary here.

Proof [Proof of Corollary 14] For the set \mathcal{X} , we consider the set of ϕ -induced points $\phi(\mathcal{X}) = \{\phi(x^{(1)}), \phi(x^{(2)}), \dots, \phi(x^{(n)})\}$ in the ϕ induced primal space \mathbb{R}^{d_ϕ} . For the ϕ -separable dichotomies of \mathcal{X} , we denote the quotient set of equivalence classes of dichotomies as $\mathfrak{E}_{\mathcal{X}}^\phi$. Since \mathcal{X} are in d'_ϕ -relaxed ϕ -general position for a fixed $d'_\phi \in [d_\phi]$, we can apply the dual map $[\Upsilon_{\text{dual}}, \varphi_{\text{dual}}]$ on the pair $[\phi(\mathcal{X}), \mathfrak{E}_{\mathcal{X}}^\phi]$. We denote the set of $d'_\phi - 1$ -relaxed general position dual hyperplanes by $\mathcal{H}_{n-1,d'_\phi-1} \triangleq \Upsilon_{\text{dual}}(\phi(\mathcal{X}))$, and the set of dual regions as $\mathfrak{R}(\mathcal{A}(\mathcal{H}_{n-1,d'_\phi-1})) \triangleq \varphi_{\text{dual}}(\mathfrak{E}_{\mathcal{X}}^\phi)$. Using the definition of the teaching set for ϕ -separable dichotomies and bijection of $\varphi_{\text{dual}}(\cdot)$ (using

9. In section §6, we denote the dual point of the separator $w_{[v]}$ to $[v]$ as $z_{[v]}$ and region containing $z_{[v]}$ as $r_{z_{[v]}}$.

Theorem 13), we can write:

$$\mathbb{E}_{r_{[u]} \sim \mathcal{U}}[M_n] = \mathbb{E}_{r \sim \mathcal{U}} \left[|\mathcal{TS}(\mathcal{H}_{n-1, d'_\phi-1}, r)| \right] \quad (32)$$

where $r_{[u]}$ is a random class in $\mathfrak{E}_{\mathcal{X}}^\phi$ and r is a uniformly random region in $\mathfrak{R}(\mathcal{A}(\mathcal{H}_{n-1, d'_\phi-1}))$. But, using Theorem 8, we know that rhs in Eq. (32) is bounded by $\mathcal{O}(d'_\phi)$. Thus, we show that the average teaching complexity of ϕ -separable dichotomies is $\mathcal{O}(d'_\phi)$. This proves the corollary. ■

Appendix H. Equivalence of Teaching Set and Extreme Points: Proof of Theorem 17

In this section, we would talk about the connection of teaching set in the dual space and extreme points in primal space as mentioned in Cover (1965). In order to complete the proof of the main result Theorem 17 we would prove two lemmas: Lemma 36 and Lemma 37.

In §6, we discussed the characterization of *ambiguous* points in the primal space. Formally, we state the lemma mentioned in Cover (1965) to characterize ambiguous points.

Lemma 35 (Lemma 1, Cover (1965)) *Let X^+ and X^- be subsets of \mathbb{R}^d , and let y be a point other than the origin in \mathbb{R}^d . Then the dichotomies $\{X^+ \cup \{y\}, X^-\}$ and $\{X^+, X^- \cup \{y\}\}$ are both homogeneously linearly separable if and only if $\{X^+, X^-\}$ is homogeneously linearly separable by a $(d - 1)$ -dimensional subspace containing y .*

Using this lemma we can argue on the equivalence of the ambiguous points in the primal space and ambiguous hyperplanes in the dual space. Let P^+ and P^- be subsets of X^+ and X^- respectively, whose classes/labels are ascertained (known). Denote by H^+ and H^- (for P^+ and P^-) the corresponding subsets of dual hyperplanes in the dual space. Assume that y is a new point in the primal space. Due to the nature of the dual map which uses the information of the vector $x^{(n)}$, we assume that the label for $x^{(n)}$ is known and $x^{(n)} \in P^+$. In the asymptotic analysis of our algorithms, this much information can be trivially included. We state this as a key assumption as mentioned in Assumption 1.

In section §6, we constructed a teaching set for a dichotomy via dual map. With the virtue of the D.M, we show the equivalence of extreme points in the primal space and teaching set in the dual space. In other words, extreme points are exactly the inverse of the teaching set in the dual space under $[\Upsilon_{\text{dual}}, \varphi_{\text{dual}}]$. In the next two lemmas we show that for the points $\mathcal{P}^+ \cup \mathcal{P}^-$ mapped to $\Upsilon_{\text{dual}}(\mathcal{P}^+ \cup \mathcal{P}^-)$, y is ambiguous wrt $\mathcal{P}^+ \cup \mathcal{P}^-$ iff $\Upsilon_{\text{dual}}(y)$ is ambiguous wrt to the region $\varphi_{\text{dual}}(\{\mathcal{P}^+, \mathcal{P}^-\})$. The key insights in establishing the connection is in using Eq. (1) and noting how Lemma 35 is essentially same as the characterization in Definition 16.

Lemma 36 *If y is ambiguous with respect to the partial dichotomy $\{\mathcal{P}^+, \mathcal{P}^-\}$, then $h_y := \Upsilon_{\text{dual}}(y)$ (dual hyperplane) is ambiguous with respect to $\varphi_{\text{dual}}(\{\mathcal{P}^+, \mathcal{P}^-\})$ i.e. the region induced by the hyperplane arrangement of $\mathcal{H}_{\mathcal{P}^+ \cup \mathcal{P}^-}$.*

Proof Denote the region representing the partial dichotomy in the dual space by r_{partial} . To show that, h_y is ambiguous, we need to show that h_y intersects r_{partial} . Using Lemma 35, we know that y is ambiguous with respect to $\{P^+, P^-\}$ iff there exists homogeneous linear separator w_y for $\{P^+, P^-\}$ passing through y . Notice that w_y has a dual image (as a point) since $(w_y)_d > 0$ as $w_y \cdot x^{(n)} > 0$. Say z_{w_y} is the dual point then using Eq. (1) $z_{w_y} \in r_{\text{partial}}$ and since $w_y \cdot y = 0$, it implies that hyperplane h_y contains z_{w_y} . Hence, h_y intersects r_{partial} . Thus, lemma follows. ■

Now, we would show that the pre-image (of dual map) of an ambiguous hyperplane with respect to a region in a hyperplane arrangement is an *extreme point* for the corresponding dichotomy. Assume that the dual hyperplane of the point y (in primal) is h_y and it is ambiguous i.e. it intersects the region corresponding to the partial dichotomy $\{P^+, P^-\}$ in the dual space.

Lemma 37 *If a hyperplane h_y is ambiguous in the dual space, then $y := \Upsilon_{\text{dual}}^{-1}(h_y)$ is ambiguous in the primal space, where inverse of Υ_{dual} is taken over the restriction $H^+ \cup H^-$.*

Proof To show that y is ambiguous, we need to show that there is a homogeneous linear separator, say w_y which separates the partial dichotomy $\{P^+, P^-\}$ and passes through y . Similar to Lemma 36, define the region representing the partial dichotomy in the dual space by r_{partial} . Since, h_y intersects r_{partial} , we know that there exists a point $z_0 \in r_{\text{partial}}$ which lies on the hyperplane h_y . As shown in the construction in Eq. (1), $h_y \equiv y_{[d-1]} \cdot z + y_d = 0$ for $z \in \mathbb{R}^{d-1}$. Now, define $w_y \triangleq (z_0^\top, 1)$. Note that, $y_{[d-1]} \cdot z_0 + y_d = 0$, thus implies $w_y \cdot y = 0$. Also, w_y is a homogeneous linear separator of the partial dichotomy in the primal space since $z \in r_{\text{partial}}$. Hence, we have shown that there exists a homogeneous linear hyperplane passing through y and separating the partial dichotomy. Thus, y is ambiguous. Hence, the lemma follows. \blacksquare

Given that we have established the equivalence of *ambiguous* points in the primal space and *ambiguous* hyperplanes in the dual space, we can show the equivalence of extreme points and teaching set. We provide the proof of Theorem 17 here.

Proof [Proof of Theorem 17] WLOG we assume that $x^{(n)} \in \mathcal{X}^+$ as stated in Assumption 1. We denote the ϕ -separable dichotomy class $[\{\mathcal{X}^+, \mathcal{X}^-\}]$ by $[\mathbf{u}]$. First, we show (\Rightarrow) i.e. if condition. Consider the mapped concept (dual region) $r_{z_{[\mathbf{u}]}} = \varphi([\mathbf{u}])$. Using Eq. (1) it is easy to see, if T_s is the teaching set for $r_{z_{[\mathbf{u}]}}$, then using Lemma 37, $\Upsilon_{\text{dual}}^{-1}(T_s)$ is *ambiguous* wrt $[\mathbf{u}]$ following the characterization mentioned in Lemma 35. This implies that $\Upsilon_{\text{dual}}^{-1}(T_s) \subseteq E$. Now, using Lemma 36, since E is ambiguous wrt $[\mathbf{u}]$ in the primal space, $\varphi_{\text{dual}}(E)$ is ambiguous wrt $r_{z_{[\mathbf{u}]}}$ in the dual space. This implies $\Upsilon_{\text{dual}}(E) \subseteq T_s$. Using the two sides of the containment, we have $\Upsilon_{\text{dual}}(E) \equiv T_s$. This implies that $\Upsilon_{\text{dual}}(E)$ is the teaching set for $\varphi_{\text{dual}}([\{\mathcal{X}^+, \mathcal{X}^-\}])$.

Now, we show (\Leftarrow) i.e. only if condition. Since $\Upsilon_{\text{dual}}(E)$ is the teaching set for $\varphi_{\text{dual}}([\{\mathcal{X}^+, \mathcal{X}^-\}])$, this implies E is ambiguous in the primal space using Lemma 37, implying a subset of extremal points. We need to ascertain that E is sufficiently a set of extremal points. Now, if $y' \notin E$ is ambiguous in the primal space, then $\Upsilon_{\text{dual}}(y')$ is *ambiguous* in the dual space using Lemma 36. Thus, $\Upsilon_{\text{dual}}(y') \in \Upsilon_{\text{dual}}(E)$ using the characterization of teaching set as stated in Definition 16. Hence, E is sufficient. Thus, E is a minimal set of extremal points.

Thus, we have proven the theorem. We show that the teaching set in the dual space is optimally recoverable as extreme points in the primal space. \blacksquare

Appendix I. Additional Use-case: Teaching Linear Ranking via Pairwise Comparisons

In this section, we would talk about the problem of teaching a randomly selected ranking of n objects embedded in a d -dimensional space. Consider a set Θ of n objects embedded in \mathbb{R}^d (in general position). We define a ranking on the objects as an ordering $\sigma : [n] \rightarrow [n]$ of the form:

$$\sigma(\Theta) := \theta_{\sigma(1)} \prec \theta_{\sigma(2)} \cdots \prec \theta_{\sigma(n-1)} \prec \theta_{\sigma(n)}$$

where $\theta_i \prec \theta_j$ implies θ_i precedes θ_j in ranking. The problem of interest is to construct a random ranking using *pairwise comparisons* of the form:

$$q_{i,j} := \{\theta_i \prec \theta_j\}$$

The response or label of $q_{i,j}$ is binary and denoted as $y_{i,j} := \mathbf{1}\{q_{i,j}\}$ where $\mathbf{1}$ is the indicator function; ties are not allowed. This is a well-studied problem in the literature and in the general setting it requires $\Theta(n \log n)$ bits of information to specify a ranking. But by imposing certain constraints on the embedding of the objects into the d -dimensional Euclidean space, Jamieson and Nowak (2011) shows we can get rid of the n factor in the active query complexity.

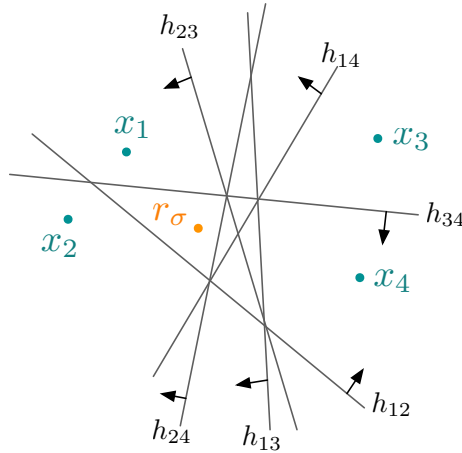


Figure 3: Teaching ranking via pairwise comparisons.

We assume that for any ranking σ , there is a *reference point* r_σ such that if σ ranks $\theta_i \prec \theta_j$, then $\|\theta_i - r_\sigma\| < \|\theta_j - r_\sigma\|$. We refer to such assumption as **E1**—This leads to an interpretation of a query “is θ_i closer to r_σ than θ_j ”, as identifying which side of the bisecting hyperplane (as shown in Definition 38) of θ_i and θ_j does r_σ lie in (as shown in Fig. 3). Before we discuss our teaching results and connections to the prior work of Jamieson and Nowak (2011), we mention our key assumption (also mentioned in Jamieson and Nowak, 2011) over the space of rankings as follows:

Assumption 2 (E1 embedding) *The set of n objects are embedded in \mathbb{R}^d (in general position) and we will also use $\theta_1, \theta_2, \dots, \theta_n$ to refer to their (known) locations in \mathbb{R}^d . Every ranking σ can be specified by a reference point $r_\sigma \in \mathbb{R}^d$, as follows. The Euclidean distances between the reference and objects are consistent with the ranking in the following sense: if the σ ranks $\theta_i \prec \theta_j$, then $\|\theta_i - r_\sigma\| < \|\theta_j - r_\sigma\|$. Let $\Sigma_{n,d}$ denote the set of all possible rankings of the n objects that satisfy this embedding condition.*

We assume that every pairwise comparison is consistent with the ranking to be learned. That is, if the reference ranks $\theta_i \prec \theta_j$, then θ_i must precede θ_j in the (full) ranking. We define the notion of bisecting hyperplane corresponding to objects θ_i and θ_j as follows:

Definition 38 (Bisecting hyperplane) *A hyperplane $h_{i,j}$ in \mathbb{R}^d is a bisecting hyperplane to objects θ_i and θ_j if both are equidistant from $h_{i,j}$ and $h_{i,j} \cdot (\theta_i - \theta_j) = 0$.*

Thus, n objects lead to $\binom{n}{2}$ hyperplanes (one query for each pair of objects) in \mathbb{R}^d :

$$\underbrace{n \text{ embedded objects in } \mathbb{R}^d}_{\text{Rankings of } \Theta} \xrightarrow{\mathbf{E1}} \underbrace{\binom{n}{2} \text{ hyperplanes in } \mathbb{R}^d}_{\text{Convex polytopes : reference points}}$$

Each convex polytope corresponds to a reference point, thereby to a ranking of objects.

Geometric interpretation of E1 We summarize the geometric interpretation of the key assumption which follows similar motivations as given in Jamieson and Nowak (2011, section 3). If we consider two objects θ_i and θ_j in \mathbb{R}^d , querying for $y_{i,j}$ corresponding to $q_{i,j}$ is equivalent to ascertaining to which halfspace of the orthogonal bisecting hyperplane of θ_i and θ_j , r_σ belongs to. The set of all possible pairwise comparison queries can be represented as $\binom{n}{2}$ distinct halfspaces in \mathbb{R}^d . The intersections of these halfspaces partition \mathbb{R}^d into a number of cells termed as d -cells, and each one corresponds to a unique ranking of Θ . Arbitrary rankings are not possible due to the embedding assumption **E1**. Similar to Jamieson and Nowak (2011), we represent the set of rankings possible under **E1** by $\Sigma_{n,d}$. The cardinality of $\Sigma_{n,d}$ is equal to the number of cells in the partition.

Now, we formulate the teaching problem of linear rankings under the mentioned assumptions here.

Teaching rankings as teaching convex polytopes Denote the $\binom{n}{2}$ hyperplanes induced by pairwise comparison of n embedded objects by $\mathcal{H}_{(n),d}$. Following our teaching framework in §3, we know that $\mathfrak{R}(\mathcal{A}(\mathcal{H}_{(n),d}))$ induced by $\mathcal{A}(\mathcal{H}_{(n),d})$ forms the underlying hypothesis class; with instances $\mathcal{H}_{(n),d}$ and corresponding labeling set $\{1, -1\}$. Thus, teaching a ranking r_σ corresponds to providing the teaching set $\mathcal{TS}(\mathcal{H}_{(n),d}, r_\sigma)$ to a learner.

Interestingly, we note that the hyperplanes induced by pairwise comparison of objects are no longer in general position. For example, in Fig. 3, the three bisecting hyperplanes induced by any three points (in \mathbb{R}^2) intersect at an 1-d subspace. When the embedded objects follow the assumption **E1**(embedding) (Jamieson and Nowak, 2011)¹⁰ show that the average query complexity for active ranking is $\mathcal{O}(d \log n)$. In contrast, we would show that the average teaching complexity of ranking via pairwise comparisons is $\mathcal{O}(d)$ via our Algorithm 3.

1.1. Algorithm for Teaching Rankings

We present our basic algorithm for teaching a ranking via pairwise comparisons. We assume we are given a set of n objects Θ embedded in \mathbb{R}^d in general position and a uniformly random ranking $r_\sigma \in \Sigma_{n,d}$ over it.

Note that to teach the ranking r_σ teacher has to provide the labels in $\mathcal{TS}(\mathcal{H}_{(n),d}, r_\sigma)$. Since, $\mathcal{TS}(\mathcal{H}_{(n),d}, r_\sigma)$ corresponds to the labels of the query hyperplanes which form the bounding set for

10. We work in noise-free setting thus consistency is assumed similar to Jamieson and Nowak (2011)

Algorithm 3 Teaching Ranking via Pairwise Comparisons

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1 Input:  $n$  objects in  $\mathbb{R}^d$ , random ranking  $r_\sigma \in \Sigma_{n,d}$ 
   begin
2    $\mathcal{TS}(\mathcal{H}_{(n),d}, r_\sigma) \leftarrow \mathbf{FindLabels}(r_\sigma)$  /* identifies  $\mathcal{TS}(\mathcal{H}_{(n),d}, r_\sigma)$  via linear
   programming */
3   for  $(h, l) \in \mathcal{TS}(\mathcal{H}_{(n),d}, r_\sigma)$  do
   |   teacher provides halfspace queries  $(h, l)$ 
   end
   end

```

r_σ , thus the entire ranking can be *inferred*. Algorithm 3 is straight forward in which for the set of objects Θ and a random ranking r_σ teacher identifies the pair of comparisons using the subroutine **FindLabels**(\cdot) and iteratively provides the labels (or halfspace queries) wrt the reference r_σ . As discussed for Algorithm 1, the subroutine **FindLabels**(\cdot) can obtain the enclosing region in $\mathcal{O}(n^4)$ iteration by solving linear equations system corresponding to $\mathcal{O}(n^2)$ constraints.

I.2. Average Complexity of Teaching Linear Ranking Functions

Before we delve into the relevant results of the subsection, we would motivate the notations.

Notations Consider the set of n objects $\Theta = (\theta_1, \theta_2, \dots, \theta_n)$ embedded in \mathbb{R}^d in general position. We denote by $h_{i,j}$ the bisecting hyperplane for the pairwise comparison $q_{i,j}$ for objects θ_i and θ_j . We use $C(n, d)$ to denote the number of regions or equivalently d -cells induced by query hyperplanes corresponding to pairwise comparisons of the embedded objects. $F(n, d)$ denotes the number of faces induced on all the query hyperplanes by their intersections.

The ideas behind the bound share similar motivations as for Theorem 8. Since the rankings are selected uniform at random, if we ascertain the number of faces for any region on average we get the bound. Thus, first we mention a recursion on $C(n, d)$ stated in Jamieson and Nowak (2011). Then, we provide the result for the total number of faces induced on all the bisecting hyperplanes.

Lemma 39 (Lemma 1, Jamieson and Nowak (2011)) *Assume E1. Let $C(n, d)$ denote the number of d -cells (regions) defined by the hyperplane arrangement of pairwise comparisons between these objects (i.e. $C(n, d) = |\Sigma_{n,d}|$). $C(n, d)$ satisfies the recursion:*

$$C(n, d) = C(n-1, d) + (n-1)C(n-1, d-1)$$

Lemma 40 *Assume E1. Let $F(n, d)$ denote the number of faces induced by the hyperplane arrangement of pairwise comparisons between these objects. $F(n, d)$ satisfies the recursion:*

$$F(n, d) = \binom{n}{2} \cdot C(n-1, d-1)$$

Proof If we consider any object say θ_k , then the pairwise comparison induced hyperplane $h_{k,i}$ for a fixed $i \neq k$ is *uniquely* intersected by query hyperplanes induced by pairwise comparison of other objects since they are in general position. Thus, on the $(d-1)$ -dimensional hyperplane $h_{k,i}$ there

are $\binom{n-1}{2}$ intersections (flats of dimension $d - 2$). Following the discussion for Lemma 1, Jamieson and Nowak (2011) we note that the number of regions or $(d - 1)$ -cells induced on the bisecting hyperplane $h_{k,i}$ for a query is *exactly* $C(n - 1, d - 1)$. Since there are $\binom{n}{2}$ hyperplanes for all the pairwise queries, thus the lemma follows. \blacksquare

Corollary 41 (Corollary 1, Jamieson and Nowak (2011)) *There exist positive real numbers k_1 and k_2 such that*

$$k_1 \frac{n^{2d}}{2^d d!} < C(n, d) < k_2 \frac{n^{2d}}{2^d d!}$$

for $n > d + 1$. If $n \leq d + 1$, then $C(n, d) = n!$.

The following result shows that even under this special arrangement of hyperplanes, the average complexity for teaching such a ranking is $\Theta(d)$.

Theorem 42 *Assume E1 and $r_\sigma \sim \mathcal{U}$. There exists a teaching algorithm which requests $\Theta(d)$ pairwise comparisons on average for ranking i.e. $\mathbb{E}_{\mathcal{U}}[M_n] = \Theta(d)$ where M_n denotes a random variable for the number of pairwise comparisons requested by an algorithm. In other words, the average teaching complexity of ranking via pairwise comparisons is $\Theta(d)$.*

I.3. Proof of Theorem 42

We would prove the main result in two parts: (i) Lemma 44 claims the upper bound on the average teaching complexity and (ii) Lemma 43 claims the average teaching complexity. Thus, we show the proof of the main result by combining (i) and (ii). Similar to §4.2, we analyze the following ratio to achieve the bounds:

$$\underbrace{\mathbb{E}_{r_\sigma \sim \mathcal{U}}[|\mathcal{TS}(\mathcal{H}_{(n),d}, r_\sigma)|]}_{\text{Average teaching complexity of ranking}} = \frac{2 \cdot \mathfrak{F}(\mathcal{A}(\mathcal{H}_{(n),d}))}{\mathfrak{r}(\mathcal{A}(\mathcal{H}_{(n),d}))} = \frac{\text{Lemma 40}}{\text{Lemma 39}} \quad (\text{A.7})$$

Key idea of the proofs is to control the rate in A.7. Let us denote by M_n a random variable for the number of labels provided by the teacher for a uniformly random sampled ranking $r_\sigma \in \Sigma_{n,d}$. We say $\sigma \sim \mathcal{U}$ for ease of notation. We would show that Algorithm 3, runs for at most $\mathcal{O}(d)$ in the following lemma ¹¹.

Lemma 43 *Assume E1 and $\sigma \sim \mathcal{U}$. Let the random variable M_n denote the number of pairwise comparisons that are requested in the teaching Algorithm 3, then*

$$\mathbb{E}_{\mathcal{U}}[M_n] \leq c \cdot d$$

for some positive constant c .

11. Note that Fukuda et al. (1991) established an $\mathcal{O}(d)$ average complexity for teaching convex polytopes under any hyperplane arrangement. Therefore one can apply Fukuda et al. (1991) to achieve the upper bound in Theorem 42. Here, we provide an alternative proof of the upper bound, which could be of separate interest.

Proof For teaching, the labels of enclosing query hyperplanes of the reference point r_σ induced by the objects, should be specified. Since the rankings are sampled uniformly at random, each ranking is enclosed by $F(n, d)/C(n, d)$ hyperplanes on average. We prove the theorem in two cases using the Corollary 41 and Lemma 40.

Case 1: $n > d + 1$ (n is sufficiently large)

$$\frac{F(n, d)}{C(n, d)} = \frac{\binom{n}{2} \cdot C(n-1, d-1)}{C(n, d)} \leq \binom{n}{2} \cdot \left(k_2 \frac{(n-1)^{2(d-1)}}{2^{d-1}(d-1)!} \right) \cdot \frac{1}{k_1 \frac{n^{2d}}{2^{d!}}} = \left(1 - \frac{1}{n}\right)^{2d-1} \frac{k_2}{k_1} d \leq c \cdot d$$

The second inequality follows from Corollary 41.

Case 2: $n \leq d + 1$

$$\frac{F(n, d)}{C(n, d)} = \frac{\binom{n}{2} \cdot (n-1)!}{n!} = \frac{n-1}{2} \leq \frac{d}{2}$$

Thus, in the two cases we have shown that $\frac{F(n, d)}{C(n, d)} = \mathcal{O}(d)$. This proves the lemma. \blacksquare

We would show that Algorithm 3, runs for at least $\Omega(d)$ in the following lemma for sufficiently large n .

Lemma 44 *Assume E1 and $\sigma \sim \mathcal{U}$. Let the random variable M_n denote the number of pairwise comparisons that are requested in the teaching Algorithm 3, then for sufficiently large $n > d$:*

$$\mathbb{E}_{\mathcal{U}}[M_n] \geq c \cdot d$$

for some positive constant c .

Proof Following similar steps in upper bound provided in Lemma 43, but instead using opposite side of bounds in Corollary 41, we get:

For $n > d + 1$ (n is sufficiently large)

$$\frac{F(n, d)}{C(n, d)} = \frac{\binom{n}{2} \cdot C(n-1, d-1)}{C(n, d)} \geq \binom{n}{2} \cdot \left(k_1 \frac{(n-1)^{2(d-1)}}{2^{d-1}(d-1)!} \right) \cdot \frac{1}{k_2 \frac{n^{2d}}{2^{d!}}} = \left(1 - \frac{1}{n}\right)^{2d-1} \frac{k_1}{k_2} d \geq c \cdot d$$

The second inequality follows from Corollary 41. In the last inequality we note that $\left(1 - \frac{1}{n}\right)^{2d-1}$ is bounded since $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$ and is increasing for large enough n .

Thus, we have shown that $\frac{F(n, d)}{C(n, d)} = \Omega(d)$. This proves the lemma. \blacksquare

Proof [Proof of Theorem 42] In Lemma 43 and Lemma 44, we showed the required bounds of $\mathcal{O}(d)$ and $\Omega(d)$, and thus $\mathbb{E}_{\mathcal{U}}[M_n] = \Theta(d)$, which completes the proof. \blacksquare