Sub-Optimality of a Dyadic Adaptive Control Architecture

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Abstract

The dyadic adaptive control architecture evolved as a solution to the problem of designing control laws for nonlinear systems with unmatched nonlinearities, disturbances and uncertainties. A salient feature of this framework is its ability to work with infinite as well as finite dimensional systems, and with a wide range of control and adaptive laws. In this paper, we consider the case where a control law based on the linear quadratic regulator theory is employed for designing the control law. We benchmark the closed-loop system against standard linear quadratic control laws as well as those based on the state-dependent Riccati equation. We pose the problem of designing a part of the control law as a Nehari problem. We obtain analytical expressions for the bounds on the sub-optimality of the control law.

1 Introduction

In this paper, we are concerned with the control of semilinear systems of the form $\dot{v}(t) = \mathcal{A}v(t) + \mathcal{B}u(t) + f(t,v),\ y(t) = \mathcal{C}v(t)$, where $v(t)$ denotes the system state, $u(t)$ is the control input, and $y(t)$ is the output. The underlying state space may be finite or infinite dimensional, so that the system in question could consist of partial and/or ordinary differential equations (PDEs and/or ODEs). The operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are the drift, control, and output operators, respectively. The forcing term $f(t,v)$ may or may not be known to the control designer a priori.

In this paper, we are interested in the case wherein $f(t,v)$ is potentially nonlinear, not entirely known, and $f(t,v) \notin \mathrm{range}(\mathcal{B})$. In our earlier papers [19, 18], we introduced a dyadic adaptive control architecture for a class of such systems. An extension was later proposed [17] for incorporating optimality into this dyadic adaptive control (DAC) architecture. The specific objective of this paper is to analyze the optimality of the architecture formally. For brevity, we refer to this architecture as the sub-optimal DAC (SDAC) architecture.

1.1 Background

The DAC architecture evolved primarily for addressing boundary control problems in systems of partial differential equations [19]. In such systems, distributed forcing terms are naturally unmatched. It offered an alternative to Lyapunov-based techniques [6, 10, 13, 21, 22] which are naturally suited mainly to well-characterized systems with a small number of degrees of freedom.

In contrast, DAC is able to work readily with systems with an arbitrarily large number of degrees of freedom, while avoiding the need for a finite-dimensional approximation of the PDE as part of the formulation itself. At the same time, it relies on its linear terms (which could be destabilizing) to enable its dyadic structure.

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The DPO architecture, shown in Fig. 1, uses the linear term $\mathcal{A}w(t)$ as a pivot and decouples the system into two components, or halves. The *particular* half filters and estimates the nonlinearity, and its dynamics are not driven by the control signal. The *homogeneous* half is linear and contains the entire control signal (the term $\mathcal{B}u$) as part of its dynamics. The control law is designed to ensure that the output of the homogeneous half $y_h \to (r - y_p)$, where r denotes the reference signal and y_p is the output of the particular half. This is sufficient for ensuring that y tracks r. The two halves are implemented in the form of observers which use full state feedback (i.e., w) to estimate the states of the two halves. The SDAC uses, in particular, the linear quadratic regulator (LQR) theory to design the control signal for the homogeneous half.

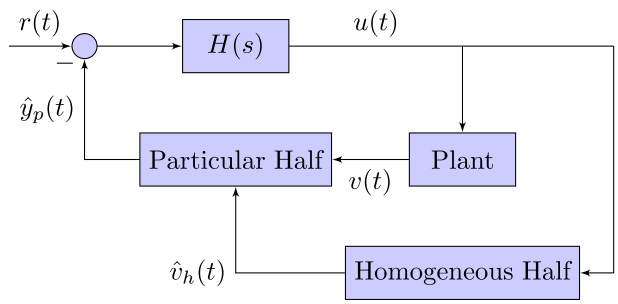

Figure 1: A block diagram of the DAC framework, with the subscripts p and h denoting signals from the particular and homogeneous components. The symbols $v(t)$, $y(t)$, and $r(t)$ denote the system state, output and reference signal, respectively.

In this paper, we investigate the inclusion of optimality in the DAC framework. The theory of optimal control for linear infinite dimensional systems is well-developed (see Chapter 6, [4], for instance). It has been used for solving problems such as determining the actuator schedule in parameter-varying systems [7] and for determining an optimal placement of actuators [15].

Optimal control techniques rely on some knowledge of the future state of the system. This requirement, for fully-known and linear systems, is absorbed fully in the Riccati equation. Since this is generally not the case for uncertain, nonlinear systems, designing optimal controllers for such systems can be challenging. Techniques based on the state-dependent Riccati equation [16, 3] have been developed to accommodate a class of nonlinearities, but proving robustness can be difficult for such systems. Causal approximations have been developed for a class of systems, such as those where the reference input is known for only a part of the time window [2, 1], or where the reference input is the output of a potentially unknown but linear exogenous system [8, 14]. Techniques based on reinforcement learning (RL), or motivated by it, have also been proposed [12, 14, 9]. Techniques based on RL, however, require an adequate amount of "training" in order to ensure, informally, that they stabilize the system and do not inadvertently excite the unstable modes beyond a point. In contrast, the SDAC [17] uses a dynamic system to generate a causal approximation.

1.2 Contribution and Organization

It was explained earlier in this section how the SDAC brings together LQR and a dynamic causal approxia-tion. The purpose of this paper is to investigate the sub-optimality of this architecture systematically.

We start by presenting the preliminaries in Sec. 2 and the problem formulation in Sec. 3, including further details of DAC. We design an optimal control law for SDAC in Sec. 4. In its "pure" form (i.e., without invoking any causal approximations), we benchmark it against LQR (tracking) and SDRE-based control laws. We show, in particular, that the SDAC-based law converges exponentially fast to the LQR-based law, in the sense of Definition 7. In Sec. 5, we consider the problem of designing a causal approximation. We argue that it can be cast into a Nehari problem, and use it to provide guarantees on its sub-optimality when compared to the "pure-form" SDAC law. We summarize the results of stability and robustness analysis in Sec. 6.

2 Preliminaries

2.1 Norms

Definition 1 ($\mathcal{L}_2$ norm) *We define the space $\mathcal{L}_2([0,\,T];\mathbb{R}^n)$, where $T > 0$, as the set of functions $q(t) \in \mathbb{R}^n$ for $t \in [0,\,T]$ satisfying*

$$\|q\|_{\mathcal{L}_{2;T}} \triangleq \int_0^T q(t)^\top q(t)\,dt < \infty$$

This includes the case where $T \to \infty$. The norm of $q \in \mathcal{L}_2([0,\,T];\mathbb{R}^n)$ will be denoted succinctly by $\|q\|_{\mathcal{L}_2}$ (i.e., without T) unless there is room for ambiguity.

Definition 2 *We define the Hilbert space $\mathbb{Z}$ with the inner product $\langle z_1,\, z_2\rangle$ for $z_1,\, z_2 \in \mathbb{Z}$ and the norm $\|z\|_{\mathbb{Z}} = \langle z,\, z\rangle$ for $z \in \mathbb{Z}$. Corresponding to the space $\mathbb{Z}$, we define the Banach space $\mathbb{W} = \mathcal{L}_\infty(\mathbb{R}^+,\,\mathbb{Z})$ with the norm $\|v\|_{\mathbb{W}} = \operatorname{ess\,sup}_{t\geq 0}\|v(t)\|_{\mathbb{Z}}$. We also define the truncated norm $\|v\|_{\mathbb{W},\tau} = \operatorname{ess\,sup}_{0\leq t\leq \tau}\|v(t)\|_{\mathbb{Z}}$.*

2.2 Operators

Definition 3 *The domain of an operator $\mathcal{V}$ is denoted by $\mathcal{D}(\mathcal{V})$. If $\mathcal{V} : X \to Y$ where X and Y are Banach spaces, (obviously, $\mathcal{D}(\mathcal{V}) \subset X$), then we denote the induced norm of $\mathcal{V}$ by $\|\mathcal{V}\|_{(X,Y)}$. If $\mathcal{V} : \mathbb{W} \to \mathbb{W}$, then we use the short-hand notation $\|\mathcal{V}\|_i$ in place of $\|\mathcal{V}\|_{(\mathbb{W},\mathbb{W})}$ for ease of representation.*

Definition 4 ([20], Definition 1.1, Ch. 6) *Let $\mathcal{A}$ be the infinitesimal generator of the C^0 semigroup $\mathcal{T}(t)$. The mild solution of $\dot v = \mathcal{A}v + f(t,v)$, $v(0) = v_0 \in \mathbb{Z}$ is given by*

$$v(t) = \mathcal{T}(t)v_0 + \int_0^t \mathcal{T}(t-\tau)f(\tau, v(\tau))\,d\tau, \tag{1}$$

We succinctly denote $e^{\mathcal{A}t} \triangleq \mathcal{T}(t)$.

Definition 5 (Convolution) *Given a C^0 semigroup $\mathcal{T}(t)$ with the infinitesimal generator $\mathcal{A}$, we define the operator $\mathbf{\Gamma}_{\mathcal{A}}(t) : \mathbf{\Gamma}_{\mathcal{A}}(t)f(t,v(t)) = \int_0^t \mathcal{T}(t-\tau)f(\tau,v(\tau))\,d\tau$. We further define the induced norm $\|\mathbf{\Gamma}_{\mathcal{A}}\|_i \triangleq$* ess $\sup_{(t\geq 0)}\|\mathbf{\Gamma}_A(t)\|_i$

Definition 6 (Inverse) *Let $\mathcal{A}$ be the infinitesimal generator of a C^0 semigroup $\mathcal{T}(t)$ which we also denote as $e^{\mathcal{A}t}$. The inverse of $\mathcal{A}$, denoted $\mathcal{A}^{-1}$, is defined as follows:*

$$\mathcal{A}^{-1}z \triangleq -\int_0^\infty e^{\mathcal{A}\tau}z\,d\tau, \quad z = \text{constant}$$

when the right hand side exists.

The proof of the next lemma is straight-forward, and omitted for brevity.

Lemma 1 *Suppose the operator $\mathbf{\Gamma}_{\mathcal{A}} \in \mathcal{H}_\infty$; i.e., $\mathbf{\Gamma}_{\mathcal{A}}$ maps $\mathcal{L}_2([0,\,T];\mathbb{Z}) \to \mathcal{L}_2([0,\,T];\mathbb{Z})$ for some $T \in (0,\infty]$. Then, the adjoint of the operator $\mathbf{\Gamma}_{\mathcal{A}}$ in Definition 5 is given by*

$$\mathbf{\Gamma}^*_{\mathcal{A}}f(t) = \int_t^T e^{-\mathcal{A}^*(t-\tau)}f(\tau)\,d\tau,\;\; f(t) \in \mathbb{Z}\,\forall\,t \tag{2}$$

The dependence of the adjoint on T (via the definition of the inner product) is omitted in this sequel, unless it is absolutely necessary.

Definition 7 (Control laws) *A control law is a map $u : \mathbb{Z} \to \mathbb{U}$, and we say that two control laws u_1 and u_2 are equal if and only if $u_1(z) = u_2(z)$ for all $z \in \mathbb{Z}$.*

We distinguish a control law from a control signal $u(t)$. The latter is a map from $\mathbb{R} \to \mathbb{U}$, obtained by running (or simulating) the system from a given initial condition. Thus, when starting from different initial conditions, the resulting control signals $u_1(t)$ and $u_2(t)$ need not be identical.

Definition 8 (Further notation on operators) *Given a semilinear system in the standard (infinite or finite dimensional) form $\dot{v} = \mathcal{A}v + \mathcal{B}u + f(t,v), \quad y = \mathcal{C}w$, we define the input-output operator*

$$\mathcal{G}(\mathcal{A},\mathcal{B},\mathcal{C})u = \mathcal{C}\mathbf{\Gamma}_{\mathcal{A}}\mathcal{B}u$$

Furthermore, we define $\mathcal{G}^*(\mathcal{A},\mathcal{B},\mathcal{C}) = \mathcal{B}^*\mathbf{\Gamma}^*_{\mathcal{A}}\mathcal{C}^*$.

3 Problem Formulation

3.1 Plant Model

This paper is concerned with semilinear systems of the form

$$\begin{aligned} &\dot{v} = \mathcal{A}v + \mathcal{B}u + f(v),\ v(0) = v_0 \\ &y(t) = \mathcal{C}v \end{aligned} \tag{3}$$

where $u \in \mathbb{R}^{n_u}$, $y \in \mathbb{R}^{n_y}$ and $v_0 \in \mathbb{Z}$, a suitably chosen Hilbert space consisting of $\mathbb{R}^n$-valued functions. The operators $\mathcal{B}$ and $\mathcal{C}$ are bounded on their respective domains. The operator $\mathcal{A}$ is the infinitesimal generator of an exponential semi-group. The control objective is to design $u(t)$ so that (i) the quadratic penalty function $J = \int_0^T ((y-r)^\top(y-r) + u^\top Ru)\,dt$, where $R > 0$, is minimized, and (ii) the resulting closed-loop system is stable and robust (in a sense which will be made precise later).

The minimization of the cost function does not guarantee asymptotic tracking by itself. One way to get around this problem is to add $\int_0^t (y-r)\,dt$ as a state in the spirit of the internal model principle (see [3], for instance). We will use a different approach in the paper.

Assumption 1 *The system $(\mathcal{A}, \mathcal{B})$ is exponentially stabilizable. Moreover, the initial conditions are restricted to $\|v_0\|_{\mathbb{Z}} < \rho_0$ and $v_0 \in \mathcal{D}(\mathcal{A})$.*

Assumption 2 *The nonlinearity can be expressed as*

$$f(v) = \alpha\phi(v)$$

where $\phi(\cdot)$ is a C^1 function of v and $\alpha \in \mathbb{R}^n$ is constant, but unknown with a known bound $\|\alpha\|_\infty < \nu_\alpha$. We also assume that its rate of change is also bounded.

Moreover, for every $\rho > 0$, we assume that there exist $\nu_{\phi,1}(\rho)$, $\nu_{\phi,2}(\rho) \in \mathcal{K}(\rho)$ such that if $\|v(t)\|_{\mathbb{Z}} < \rho$ for some $t > 0$, then $\|\phi(v)\|_{\mathbb{Z}} \le \nu_{\phi,1}(\rho)\|v(t)\|_{\mathbb{Z}} + \nu_{\phi,2}(\rho)$. It follows that if $\|v\|_{\mathbb{W},\tau} < \rho$ for some $\tau > 0$, then $\|f(v)\|_{\mathbb{W},\tau} \le \nu_1(\rho)\|v\|_{\mathbb{W}^e,\tau} + \nu_2(\rho)$, for some constants $\nu_1(\rho)$ and $\nu_2(\rho)$.

We note that our control design technique is applicable readily [18] to more general linear combination of known basis functions: $f(v) = \sum_{i=1}^N \alpha_i(t)\phi_i(v)$. Since this "extension" would makes our presentation cumbersome without adding to our primary objective, we adhere to Assumption 2.

3.2 Dyadic Adaptive Control

Suppose that we design a control signal of the form $u(t) = -\mathcal{K}v(t) + u_R(t)$ for (3), where the term $u_R(t)$ is added for tracking purposes. Let us write $\mathcal{A}_m = \mathcal{A} - \mathcal{B}\mathcal{K}$ (see Definition 2 for further details), where $\mathcal{A}_m$ generates an exponentially decaying semi-group. The resulting closed-loop system can be viewed as the sum of two sub-systems creating using $\mathcal{A}_m$ as the pivot:

$$\dot{v}_p = \mathcal{A}_m v_p + f(v),\ y_p = C^e v_p \tag{4}$$

$$\dot{v}_h = \mathcal{A}_m v_h + \mathcal{B}u_R(t),\ y_h = C^e v_h \tag{5}$$

The two systems (4) and (5) are referred to as the *particular* and *homogeneous* halves, respectively. The states of these sub-systems can be estimated readily using observers, which rely on knowing the value of the actual state, v. The dynamics of the observers for the two halves, with the observed states denoted by $\hat{v}_p$ and $\hat{v}_h$ respectively, are given by

$$\dot{\hat{v}}_p = \mathcal{A}_m \hat{v}_p + \hat{\alpha}(t)\phi(v),\ \ \hat{y}_p = \mathcal{C}\hat{v}_p \tag{6}$$

$$\dot{\hat{v}}_h = \mathcal{A}_m \hat{v}_h + \mathcal{B}u_R(t),\ \ \hat{y}_h = \mathcal{C}^e\hat{v}_h \tag{7}$$

with the initial conditions at $t = 0$ set to suitable values.

The next assumption asserts the existence of a Lyapunov function corresponding to the generator $\mathcal{A}_m$, which we need for constructing the adaptation law.

Assumption 3 *There exists a* self-adjoint coercive *operator* $\mathcal{P} > 0$ *and a constant* $\lambda_P > 0$ *such that* $\forall t$*,*

$$\begin{aligned}&\langle \mathcal{A}_m z(t), \mathcal{P}z(t)\rangle_{\mathbb{Z}} + \langle \mathcal{P}z(t), \mathcal{A}_m z(t)\rangle_{\mathbb{Z}} \leq -\lambda_P \langle z(t), \mathcal{P}z(t)\rangle_{\mathbb{Z}},\\ &\forall\ z(t) \in \mathcal{D}(\mathcal{A}_m)\end{aligned} \tag{8}$$

The predicted values $\hat{\alpha}(t)$ is found using the projection operator [11]:

$$\begin{aligned}&\dot{\hat{\alpha}}_j(t) = \gamma\,\mathrm{Proj}\left(\hat{\alpha}_j,\ -\langle \mathcal{P}\tilde{v}(t), \phi_j(v)e_j\rangle_{\mathbb{Z}}\right),\\ &|\hat{\alpha}_j(t)| < \nu_\alpha(1+\epsilon)\end{aligned} \tag{9}$$

where $\epsilon \in \mathbb{R}^+$ is arbitrarily small; $\tilde{v} = \hat{v}_p + \hat{v}_h - v$; $\hat{\alpha}_j \in \mathbb{R}$ is the j^{th} component of $\hat{\alpha}$, e_j denotes the j^{th} column of the $n \times n$ identity matrix, and $\gamma > 0$ is the adaptation gain.

The control design described in this section constitutes the dyadic adaptive control (DAC) architecture. It remains to determine $\mathcal{K}$ and $u_R(t)$, for which we turn to tools from optimal control.

3.3 Optimal Control Problem Formulation

We show in Sec. 6 that the observer states $\hat{v}_p$ and $\hat{v}_h$ converge to v_p and v_h, respectively. In order to design and analyse our optimal control law, we confine our analysis to the following system inspired by the converged observer:

$$\begin{aligned}\dot{v}_p &= \mathcal{A}_m v_p + f(v)\\ \dot{v}_h &= \mathcal{A}v_h + \mathcal{B}u(t)\end{aligned} \tag{10}$$

We note the use of $\mathcal{A}$ (rather than $\mathcal{A}_m$) in the dynamics of v_h. We formulate the control design problem as follows: design $u(t)$ to ensure that y_h tracks $\sigma = r - y_p$ while minimizing

$$J = \int_0^T \left((y_h(t)-\sigma(t))^\top (y_h(t)-\sigma(t)) + u(t)^\top R u(t)\right) dt$$

where $R > 0$ and $T \gg 1$. In Sec. 6, we argue that the resulting closed-loop system is stable and robust. We refer to two classes of problems as an aid to our analysis:

1. Non-adaptive regulation: the reference signal $r(t) \equiv r$ (a constant) and $f(v)$ is known.

2. Complete problem: $r(t)$ as well as $f(v)$ are not identically zero.

4 Optimal Control Design and Analysis

To facilitate the design of an optimal control law, we define an extended state space $\mathbb{Z}^f = \mathbb{Z}^e \oplus \mathbb{R}$, and define

$$\dot{w}(t) = \begin{bmatrix}\mathcal{A} & 0\\ 0 & 0\end{bmatrix} w(t) + \begin{bmatrix}\mathcal{B}\\ 0\end{bmatrix} u(t),\ \ w(0) = \begin{bmatrix}v_{h,0}\\ 1\end{bmatrix} \tag{11}$$

The control problem is equivalent to designing $u(t)$ to minimize

$$\begin{aligned}&\min_u \int_0^T \left(\langle w(t),\, Q(t)w(t)\rangle + \langle u(t),\, Ru(t)\rangle\right)\, dt\\ &Q(t) = [\mathcal{C} \quad -\sigma(t)]^*[\mathcal{C} \quad -\sigma(t)];\end{aligned} \tag{12}$$

The control design mirrors the approach in ([4], Chapter 6). Ideally, we would like $T \to \infty$ in (12). However, since the reference signal $\sigma(t)$ is arbitrary, the optimal cost may be infinite as $T \to \infty$. We avoid introducing a discount $e^{-\mu t}$ ($\mu > 0$) in the cost function.

The solution to (12) is given by

$$u(t) = -R^{-1}\mathcal{B}^*\left(\Pi(t)v(t) + q(t)\right) \tag{13}$$

where $\Pi(t)$ is the solution of the Riccati equation

$$\begin{aligned}&\frac{d}{dt}\left(z_2,\, \Pi(t)z_1\right) = -\langle z_2,\, \Pi(t)\mathcal{A}z_1\rangle - \langle \mathcal{A}z_2,\, \Pi(t)z_1\rangle\\ &-\langle \mathcal{C}^e z_1,\, \mathcal{C}^e z_2\rangle + \langle \Pi(t)\mathcal{B}R^{-1}\mathcal{B}^*\Pi(t)z_1,\, z_2\rangle\\ &\Pi(T) = 0, \quad z_1,\, z_2 \in \mathcal{D}(\mathcal{A})\end{aligned} \tag{14}$$

and $q(t)$ is the mild solution of

$$\dot{q}(t) = -\left(\mathcal{A} - \mathcal{B}R^{-1}\mathcal{B}^*\Pi(t)\right)^* q(t) + C^{e*}\sigma(t),\ q(T) = 0 \tag{15}$$

The first part of (13) is identical to the regulation problem with $\sigma \equiv 0$. Therefore, we set $\Pi(t) \equiv \Pi$, the (steady state) solution to the *algebraic* Riccati equation, under the assumption that T is large, and write the control signal as

$$u(t) = -R^{-1}\mathcal{B}^*\Pi v_h(t) - R^{-1}\mathcal{B}^*q(t) \tag{16}$$

$$\dot{q}(t) = -\left(\mathcal{A} - \mathcal{B}R^{-1}\mathcal{B}^*\Pi\right)^* q(t) + C^*\sigma(t),\ q(T) = 0 \tag{17}$$

Lemma 2 (Theorem 5.1.5, Theorem 6.2.7, [4]) *The operator $\mathcal{A}_m \triangleq \mathcal{A} - \mathcal{B}R^{-1}\mathcal{B}^*\Pi$ generates an exponentially stable semigroup, denoted by $e^{\mathcal{A}_m t}$; i.e., there exist constants $M,\, \beta > 0$ such that $\|e^{\mathcal{A}_m t}\|_i \le Me^{-\beta t}$. Moreover, $\|\mathbf{\Gamma}_{\mathcal{A}_m}\|_\infty$ is bounded, and $(sI - \mathcal{A}_m)^{-1} \in \mathcal{H}_\infty$.*

Definition 9 *We denote $\mathcal{G}_m \triangleq \mathcal{G}(\mathcal{A}_m,\, \mathcal{B},\, \mathcal{C})$, and $\mathcal{G}_m(0) = -\mathcal{B}\mathcal{A}_m^{-1}\mathcal{C}$. Note that $\mathcal{G}_m \in \mathcal{H}_\infty$.*

Since $q(T) = 0$, we get

$$q(t) = \int_T^t e^{-\mathcal{A}_m^*(t-s)}\mathcal{C}^*\sigma(s)\, ds = -\mathbf{\Gamma}_{\mathcal{A}_m}^*\mathcal{C}^*\sigma \tag{18}$$

Let us define $u_R = -R^{-1}\mathcal{B}^*q(t)$, which serves the purpose of tracking (as against stabilization). It follows that

$$u_R = R^{-1}\mathcal{G}_m^*\sigma, \quad y_h = \mathcal{G}_m R^{-1}\mathcal{G}_m^*\sigma + \mathcal{C}e^{\mathcal{A}_m t}v_h(0) \tag{19}$$

where $\mathcal{G}_m^*$ is defined (with respect to $\mathcal{G}_m$) through Definition 8. The optimal cost can be written, with the inner product defined over $\mathcal{L}_2([0,T])$ as

$$J_1 = \langle y_h - \sigma,\ y_h - \sigma\rangle + \langle u,\ Ru\rangle \tag{20}$$

where we have used the symbol J_1 to facilitate a comparison later in the paper. It is straight-forward to show that

$$\begin{aligned}&J_1 = \langle \sigma,\, (\mathcal{P}^*\mathcal{P} + I - \mathcal{P})\sigma\rangle + v_h(0)^*W_o v_h(0)\\ &+\langle \mathcal{C}e^{\mathcal{A}_m t}v_h(0), (I - \mathcal{P})\sigma\rangle + \langle (I - \mathcal{P})\sigma, \mathcal{C}e^{\mathcal{A}_m t}v_h(0)\rangle\\ &\mathcal{P} = \mathcal{G}_m R^{-1}\mathcal{G}_m^*\end{aligned} \tag{21}$$

Notice that σ depends on v_h when $f(v) \neq 0$ for some v.

Lemma 3 *Let $T < \infty$ denote the terminal time. Then, the optimal cost J_1 is bounded if and only if $\sigma \in \mathcal{L}_2([0,T])$.*

Proof: We have that $\|\mathcal{G}_m\|_\infty < \infty$, from Lemma 2. Moreover, $\|\mathcal{G}_m^*\|_\infty = \|\mathcal{G}_m\|_\infty$ (Proposition 3.15, [5]), as a result of which $\|\mathcal{P}\|_\infty < \infty$. ■

Next, we investigate the optimality of the closed-loop system with the control law (16) and (17) when compared to traditional linear quadratic regulators. This entails a static causal approximation for $q(t)$, and provides the first glimpse into the optimality of the DAC architecture. In particular, we compare it directly with standard LQR, LQT and SDRE-based control laws.

Remark 1 *We note three important points in connection with the comparisons below. First, although the comparisons are carried for relatively simple systems compared to* (10)*, their value lies in benchmarking the SDAC. Second, we are more interested in benchmarking the control law rather than the actual cost. This is because the cost in most optimal control problems is, informally speaking, a means to an end (stability and robustness) rather than an end in itself. Third, the difference between SDAC and other architectures arises, primarily, due to the feedback of y_p rather than x_p.*

4.1 Comparison with LQR/LQT

Consider the regulation problem, with $f(v) \equiv 0$ and $r \equiv 0$. In this special case, the equation for $q(t)$ in (15) can be simplified further. We recall Eq. (18):

$$q(t) = \int_T^t e^{-\mathcal{A}_m^*(t-s)} \mathcal{C}^* \sigma(s)\, ds$$

We now let $T \to \infty$ and make the coordinate transform $\tau = s - t$. We also note that $v(s) = e^{\mathcal{A}_m(s-t)} v_p(t)$. This gives

$$q(t) = \left(\int_0^\infty e^{\mathcal{A}_m^* \tau} \mathcal{C}^* \mathcal{C} e^{\mathcal{A}_m \tau}\, d\tau \right) v_p(t) = W_o v_p(t)$$

where W_o is the observability Grammian for the closed-loop system. Thus, it follows that the optimal regulator is given by

$$\begin{aligned} u_{\text{reg}}(t) &= -\mathcal{K}\, v_h(t) - R^{-1} \mathcal{B}^* W_o v_p(t) \\ &= -\mathcal{K}\, v(t) + R^{-1} \mathcal{B}^* (\Pi - W_o) v_p(t) \end{aligned} \tag{22}$$

The classic LQR control law for the LTI system $\dot{v} = \mathcal{A}v + \mathcal{B}u$ is given by $u_{\text{lqr}} = -R^{-1}\mathcal{B}^* \Pi\, v$, as in (16). This gives us the first result for the sub-optimality of the DAC.

Lemma 4 *Consider the system* (10) *with $f(\cdot) \equiv 0$ and $r \equiv 0$. The optimal control law, given by* (16) *and* (22)*, converges exponentially fast to the classic LQR-based law. Moreover, the error between them is bounded if $\mathcal{B}^*$ is bounded.*

Proof: Let $v^\star$ denote the trajectory generated by the classic LQR controller. We note that $u_{\text{reg}}(t) - u_{\text{lqr}}(t) = -\mathcal{K}(v(t) - v^\star(t)) + R^{-1}\mathcal{B}^*(\Pi - W_o) v_p(t)$.

It follows from the dynamics of v_p in (10), together with Definition 2, that $\|v_p(t)\|_{\mathbb{Z}}$ is bounded for all t and $\|v_p\|_{\mathbb{Z}} \to 0$ exponentially fast. Thus, u_{reg} converges exponentially fast to u_{lqr}, in the sense of Definition 7.

It follows that the dynamics of $e = v - v^\star$ are given by $\dot{e} = \mathcal{A}_m e + \mathcal{B}R^{-1}\mathcal{B}^*(\Pi - W_o) v_p$. Since $\mathcal{A}_m$ is the generator of an exponentially stable semi-group, it follows that $\|e(t)\|_{\mathbb{Z}}$ is bounded for all t. Hence, $u_{\text{reg}}(t) - u_{\text{lqr}}(t)$ is bounded for all t. This completes the proof. ■

When $r(t) = r \neq 0$ for all t, following a similar approach as above, (18) can be solved to obtain

$$q(t) = W_o v_p(t) + (A_m^*)^{-1} \mathcal{C}^* r, \tag{23}$$

where the operator $(A_m^*)^{-1}$ is defined as per Definition 6. Akin to (22), we get

$$\begin{aligned} u_{\rm trk}(t) &= -\mathcal{K}\, v_h(t) - R^{-1}\mathcal{B}^*\big(W_o v_p(t) + (A_m^*)^{-1}\mathcal{C}^* r\big) \\ &= -\mathcal{K}\, v(t) + R^{-1}\mathcal{B}^*(\Pi - W_o)\, v_p - R^{-1}\mathcal{B}^*(A_m^*)^{-1}\mathcal{C}^* r \end{aligned} \tag{24}$$

The classic linear quadratic tracking control law, following [14], is given by

$$u_{lqt} = -\mathcal{K}\, v(t) - R^{-1}\mathcal{B}^*(A_m^*)^{-1}\mathcal{C}^* r \tag{25}$$

This gives us the following lemma, which generalizes Lemma 4. The proof is identical to that of Lemma 4.

Lemma 5 *Consider the system* (10) *with* $f(\cdot) \equiv 0$ *and* $r(t) = r$ *(a constant) for all* t*. The optimal control law, given by* (16) *and* (22)*, converges exponentially fast to the classic LQT-based law* (25)*. Moreover, the error between them is bounded in the sense of* $\mathcal{L}_\infty$ *if* $\mathcal{B}^*$ *is bounded.*

4.2 Comparison with SDRE

Next, we consider the problem where $f(v)$ is known and $r(t) = r$, a constant for all t. An SDRE-based control law can be obtained, following the usual procedure as in [16]. One important modification that we make to the process of deriving the SDRE is that we do not merge $f(v)$ into $\mathcal{A}v$. Rather, we introduce the state $w \equiv 1$, with the dynamics $\dot{w} = 0$ and $w(0) = 1$.

In preparation for applying SDRE, we rewrite the dynamics of (10), with $v = v_p + v_h$ and with the output denoted as y_{vw}, as

$$\begin{bmatrix} \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & f(v) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} + \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix} u, \; y_{vw} = [C \;\; -r] \begin{bmatrix} v \\ w \end{bmatrix} \tag{26}$$

The control objective is essentially that of regulating y_{vw} while minimizing a cost function of the form (12). Since the dynamics of w are not stabilizable, we obtain the SDRE by introducing a small discount $e^{-\epsilon t}$ into the cost function. Subsequently, we allow $\epsilon \to 0$. We skip the steps, but note that the procedure is similar to [16] and Sec. 4. The Riccati operator can be written as

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$$

of which Π_{11} and Π_{12} are of relevance to the present discussion. Using the shorthand notation introduced earlier, we write the expressions for Π_{11} and Π_{12} as

$$\begin{aligned} &\mathcal{A}^*\Pi_{11} + \Pi_{11}\mathcal{A} + C^*C - \Pi_{11}\mathcal{B}R^{-1}\mathcal{B}^*\Pi_{11} = 0 \\ &\mathcal{A}_m^*\Pi_{12}(v) + \Pi_{11}f(v) - C^*r = 0 \end{aligned} \tag{27}$$

We get that $\Pi_{12} = (\mathcal{A}_m^*)^{-1}(C^*r - \Pi_{11}f(v))$, which yields

$$\begin{aligned} u_{\rm sdre} \;\; = \;\; & -\mathcal{K}v - R^{-1}B^*(\mathcal{A}_m^*)^{-1}C^*r \\ & + R^{-1}\mathcal{B}^*(\mathcal{A}_m^*)^{-1}\Pi_{11}f(v) \end{aligned} \tag{28}$$

Notice that the first two terms are identical to $u_{\rm lqt}$ from (25), and the gain $\mathcal{K}$ is also identical to the optimal controller in (16). To facilitate a comparison, as before, we define

$$u_{R,sdre} = -R^{-1}B^*(\mathcal{A}_m^*)^{-1}C^*r + R^{-1}\mathcal{B}^*(\mathcal{A}_m^*)^{-1}\Pi_{11}f(v)$$

We recall the corresponding expression for u_R from (19):

$$u_R = R^{-1}\mathcal{B}^*\mathbf{\Gamma}_{\mathcal{A}_m}^*\mathcal{C}^*r - R^{-1}\mathcal{B}^*\mathbf{\Gamma}_{\mathcal{A}_m}^*\mathcal{C}^*\mathcal{C}\mathbf{\Gamma}_{\mathcal{A}_m}f(v) \tag{29}$$

where we have assumed that $v_p(0) = 0$. Thus, we get

$$u_R - u_{R,sdre} = R^{-1}\mathcal{B}^*(\mathbf{\Gamma}^*_{\mathcal{A}_m} + (\mathcal{A}^*_m)^{-1})\mathcal{C}^* r$$
$$-R^{-1}\mathcal{B}^*(\mathbf{\Gamma}_{\mathcal{A}^*_m}\mathcal{C}^*\mathcal{C}\mathbf{\Gamma}_{\mathcal{A}_m} f(v) + (\mathcal{A}^*_m)^{-1}\Pi_{11} f(v)) \quad (30)$$

Since we have assumed that r is constant, it can be shown that

$$\begin{aligned}(u_R - u_{R,sdre})_{t\to\infty} \quad = \quad & -R^{-1}\mathcal{B}^*(\mathbf{\Gamma}_{\mathcal{A}^*_m}\mathcal{C}^*\mathcal{C}\mathbf{\Gamma}_{\mathcal{A}_m} f(v) \\ & +(\mathcal{A}^*_m)^{-1}\Pi_{11} f(v))\end{aligned}$$

Here, we have compared the two control laws rather than the time histories of the two control signals. It is clear that, unlike the LQR/LQT case, the two control laws need not converge to each other.

The static approximation works well when the dynamics of the particular half and the reference signal are known *a priori*: this completely mitigates the anti-causal nature of the boundary value problem (17). When these are unknown, the typical course of action (in the context of optimal control) has been to use some form of dynamic approximation [14, 9]. We opt for an alternate approach, which seeks to approximate the backward-in-time equation dynamics of q in (16) by a forward-in-time approximation.

5 Dynamic Causal Approximation for Adaptive Systems

It is a well-known property of the solution to the LQR problem that the adjoint state evolves on the stable manifold of the combined system-adjoint dynamics, and the stable eigenvalues are precisely those of $\mathcal{A}_m$. This motivates us to express the control signal using the following *finite-dimensional* dynamic approximation:

$$\begin{aligned} u(t) &= -R^{-1}\mathcal{B}^*\Pi v_h(t) + R^{-1}H_C p(t), \quad (31)\\ \dot{p}(t) &= H_A p(t) + H_B\sigma(t),\ p(0) = p_0 \in \mathbb{R}^{n_p}\end{aligned}$$

where $H_A \in \mathbb{R}^{n_p\times n_p}$ is Hurwitz, and H_B, $H_C^\top \in \mathbb{R}^{n_p}$. We set $p(0) = 0$, which is in contrast to the approach taken in other causal approximations, such as [2].

We define

$$\mathcal{G}_H = H_C\mathbf{\Gamma}_{H_A}H_B \quad (32)$$

We state the dynamic approximation problem as follows.

Problem 1 *Determine the system (H_A, H_B, H_C; $H_A \in \mathcal{H}_\infty$) which minimizes the cost function and ensures that the closed-loop system in stable.*

From the dynamics of p in (31), with $p(0) = 0$, we get

$$u_R(t) = R^{-1}\mathcal{G}_H\sigma(t), \quad y_h = \mathcal{G}_m R^{-1}\mathcal{G}_H\,\sigma(t) + \mathcal{C}e^{\mathcal{A}_m t}v_h(0) \quad (33)$$

A comparison with the expression for u_R from (19) suggests that we compute a dynamic approximation by solving the Nehari problem:

$$\mathcal{G}^N_H = \arg\min_{X\in\mathcal{H}_\infty}\|\mathcal{G}^*_m - X\|_\infty \quad (34)$$

This allows us to bound the error in the cost function with the dynamic approximation when compared to J_1 in (20). Let J_2 denote the cost function accumulated by employing the controller in (31). Then, we have that

$$J_2 = \|(\mathcal{G}_m R^{-1}\mathcal{G}^N_H - I_{n_y})\sigma\|_{\mathcal{L}_2} + \|R^{1/2}u\|_{\mathcal{L}_2} \quad (35)$$

where

$$\begin{aligned} u &= u_R - \mathcal{K}v_h \\ &= (I_{n_u} - \mathcal{K}\mathbf{\Gamma}_{\mathcal{A}_m}\mathcal{B})R^{-1}\mathcal{G}^N_H\sigma - \mathcal{K}e^{\mathcal{A}_m t}v_h(0) \quad (36)\end{aligned}$$

Note that σ depends on v_p and hence on v_h (through the nonlinear function $f(v)$.). This can complicate the comparison of J_2 and J_1 substantially. A simple comparison can be obtained for the case where $f(v)$ is actually independent of v; i.e., where it is a pure exogenous disturbance.

Theorem 1 *Let $\sigma(t) \in \mathcal{L}_2([0,T])$ be an exogenous signal and independent of the system state v_h, and $T > 0$. Consider the cost functions J_1 and J_2, evaluated in (20) and (35) with $v_h(0)$ being identical in both cases. We have that $|J_2 - J_1| \leq k\|\sigma\|_{\mathcal{L}_2}$, where $k < \infty$.*

Proof: We recall the triangle inequality: $\|z_1\| - \|z_2\| \leq \|z_1 - z_2\|$. We apply it to each of the two terms in the cost functions J_1 and J_2. This gives

$$\begin{aligned} |J_2 - J_1| &\leq \|(\mathcal{G}_m R^{-1}\mathcal{G}_H^N - \mathcal{G}_m R^{-1}\mathcal{G}_m^*)\sigma\| \\ &\quad + \|R^{-1/2}(\mathcal{G}_m^* - \mathcal{G}_H^N)\sigma\|, \end{aligned} \tag{37}$$

from which it follows that

$$\begin{aligned} |J_2 - J_1| &\leq S\|\sigma\|_{\mathcal{L}_2} \\ S &= (\|\mathcal{G}_m\|_\infty + \|R^{-1/2}\|_\infty)\|\boldsymbol{\theta}_{\mathcal{G}_m^*}\| \end{aligned} \tag{38}$$

where $\boldsymbol{\theta}_{\mathcal{G}_m^*}$ is the Hankel operator for $\mathcal{G}_m$. This completes the proof. ■

Theorem 1 calculates an explicit formula for an upper bound on the sub-optimality induced by the dynamic causal approximation, in comparison to the "pure form" control law of (19).

Remark 2 *In Theorem 1, the effect of v on σ is implicitly ignored. In order to factor in the effect of v, it is essential to prove that the closed-loop system is stable. This is the subject of the next section. We use stability in the sense of $\mathcal{L}_\infty$ rather than $\mathcal{L}_2$.*

Remark 3 (Asymptotic tracking) *Although we minimize the cost of tracking, it does not guarantee asymptotic tracking even when the reference input is a constant. In order to ensure asymptotic tracking, one can either extend the state space to include the tracking error $y_h - \sigma$ as a state (as in [3]), or one can solve the constrained Nehari problem*

$$\mathcal{G}_H^N = \arg\min_{X \in \mathcal{H}_\infty; \mathcal{G}_m(0)R^{-1}\hat{X}(0) = I_{n_y}} \|\mathcal{G}_m^* - X\|_\infty \tag{39}$$

where $\hat{X}$ denotes the Laplace transform of X. This option, of course, worsens the bound in Theorem 1.

6 Analysis of the Complete Closed-Loop System

6.1 Summary of the Closed-Loop System

The closed-loop system consists of the original system (3) and the controller. If dynamic causal approximation is employed, the controller consists of the primary control law

$$u(t) = -R^{-1}\mathcal{B}^*\Pi v(t) + R^{-1}H_C p(t) \tag{40}$$

$$\dot{p}(t) = H_A p(t) + H_B(r(t) - \hat{y}_p(t)), \ p(0) = p_0 \tag{41}$$

and the observers equipped with the projection operator [11]:

$$\begin{aligned} &\dot{\hat{v}}_p = \mathcal{A}_m \hat{v}_p + \hat{\alpha}(t)\phi(v), \ \hat{y}_p = \mathcal{C}\hat{v}_p \\ &\dot{\hat{v}}_h = \mathcal{A}_m \hat{v}_h - \mathcal{B}R^{-1}H_c p(t), \ \hat{y}_h = \mathcal{C}\hat{v}_h \\ &\dot{\hat{\alpha}}_j(t) = \gamma \, \mathrm{Proj}\,(\hat{\alpha}_j, \, -\langle \mathcal{P}\tilde{v}(t), \phi_j(v)e_j\rangle_{\mathbb{Z}}), \\ &|\hat{\alpha}_j(t)| < \nu_\alpha(1+\epsilon) \end{aligned}$$

6.2 Stability Analysis

We recall the following results from [17] and [18] for completeness. These results prove the stability of the closed-loop system subject to a small gain condition.

Lemma 6 *Suppose that* $\|v\|_{\mathbb{W},t} < \rho_w$ *for some constant* $\rho_w > 0$. *Then, the observation error* $\|\tilde{v}(t)\|_{\mathbb{Z}}$ *is uniformly bounded and the bound can be made arbitrarily small by increasing* γ. *Furthermore, the observation errors* $\|\tilde{v}_p(t)\|_{\mathbb{Z}}$ *and* $\|\tilde{v}_h\|_{\mathbb{Z}}$ *are uniformly bounded, and can be made arbitrarily small by increasing* γ.

Next, we assert that the control input $u(t)$ is bounded. Recall that the second term in the control signal (40) by $u_R(t) = H_C p(t)$. Let $H(s) = H_C(sI - H_A)H_B$ denote the transfer function between $U_R r(s)$ and $(R(s) - \hat{Y}_p(s))$.

Lemma 7 *Let* $\|v\|_{\mathbb{W},t} < \rho_w$ *for some* t *and* $\rho_w > 0$. *Then, the control input* $u(t)$ *is bounded and a* C^1 *function of time. Moreover, there exist constants* $\delta_{iw} \equiv \delta_{iw}(H(s), \rho)$, $\delta_{ir} \equiv \delta_{ir}(H(s), \rho)$ *and* $\delta_{iu} \equiv \delta_{iu}(H(s), \rho)$ *for* $i = 0, 1$ *such that* $\|u_R\|_{\mathcal{L}_\infty,\tau} \leq \delta_{0w}\|v\|_{\mathbb{W},\tau} + \delta_{0r}\|r\|_{\mathcal{L}_\infty,\tau} + \delta_{0u}$.

Next, we define a small gain condition.

Assumption 4 (Small-gain condition) *We assume that there exists a constant* ρ_w, *an arbitrarily small* $\epsilon_s > 0$, *and a stable strictly proper* $H(s)$ *such that the following inequality is satisfied:*

$$\frac{M\rho_0 + \|\mathbf{\Gamma}_{\mathcal{A}_m}\|_i(\nu_2(\rho_w) + \delta_{0r}\|r\|_{\mathcal{L}_\infty} + \delta_{0u})}{1 - \|\mathbf{\Gamma}_{\mathcal{A}_m}\|_i(\nu_1(\rho_w) + \|\mathcal{B}\|_i(\delta_{0w})} \leq \rho - \epsilon_s$$

where the constants have been defined in Assumption 2 and Lemma 7.

Finally, we state the main result of this section.

Theorem 2 *The closed-loop system summarized in Sec. 6.1 is well-posed and bounded-input-bounded-state stable in the sense of* $\mathcal{L}_\infty$ *if Assumption 4 is satisfied.*

Notice that the satisfaction of the small gain condition also endows the closed-loop system with robustness. The proof of well-posedness relies on Theorems 6.1.4 and 6.1.5 from [20].

6.3 Bounds on the Cost

We would like to derive some bounds on the cost function. In a nonlinear setting, the baseline cost is J_1 from the pure form solution of Sec. 4. Lemma 6 leads us to conclude that there exists a constant ρ_{obs} which can be made arbitrarily small, such that

$$\|\hat{y} - y\|_{\mathcal{L}_\infty,T} \leq \rho_{\text{obs}}$$

Thus, the cost calculated using Theorem 1 and (20) is a reasonable estimate of the cost incurred by the closed-loop system. We note that an exact expression can be found when $\phi(v) = 1$. When $f(v) = \mathcal{S}v$ for some linear $\mathcal{S}$, expressions similar to (19) and (20) would feature a semigroup generated by the time-varying operator $\mathcal{A}_m + \hat{\alpha}(t)\mathcal{S}$. Even in that case, since the same semigroup would feature in the computation of J_2, the expression for the difference would be similar to (38).

7 Conclusion

We analyzed the optimality of LQR-based tracking control laws designed for semilinear systems in the dyadic adaptive control (DAC) framework. We showed that a "pure form" law (i.e., without any dedicated causal approximation) converges exponentially to standard LQR laws for LTI systems. We determined analytical expressions for the cost function in the presence of a dynamic compensator for causal approximation for a class of perturbed linear systems. The compensator design problem was posed, in particular, in the framework of model-matching problems. While an exact expression for the cost function in the presence of nonlinear forcing remains elusive, the present paper has amply demonstrated the utility of the dyadic adaptive architecture for accommodating optimality, robustness and adaptation systematically in a single framework.

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