Supporting Material for "Perturbed Sachdev-Ye-Kitaev model: a polaron in the hyperbolic plane."

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I. THE EFFECTIVE ACTION

In this part, we will describe the solution of the problem using a geometrical approach. The logic will be the same as in the main text. We derive the effective action in adiabatic approximation and then the first non-adiabatic correction. Full action is provided in Eq.(22).

A. Adiabatic approximation

The action of the SYK model at the Hyperbolic plane (we use Poincaré disk model) was presented at the main text. After proper regularization it has the form:

$$S = \int_0^{\tilde\beta} \left\{ \frac{1}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu - \gamma \omega_\mu \dot{X}^\mu \right\} d\tau - \frac{g\gamma}{4} \int_0^{\tilde\beta} d\tau_1 d\tau_2 \chi^{1/2}_{z(\tau_1), z(\tau_2)} \tag{1}$$

Here $g_{\mu\nu}$ is a metric tensor and ω_μ is the spin connection. We also introduced the following notations:

$$\tilde\beta = \frac{J\beta}{\gamma}, \qquad g = \frac{b^{2\Delta}}{2} \frac{N\Gamma^2}{J^2} \gamma^{2-4\Delta} = \frac{N\gamma}{4\sqrt{\pi}} \frac{\Gamma^2}{J^2}. \qquad \chi = \frac{(1-|z_1|^2)(1-|z_2|^2)}{|1-z_1^* z_2|^2} \tag{2}$$

Here z is a complex coordinate of the point at the model. We will use coordinates ξ and φ which are defined as $z = \tanh(\xi/2) e^{i\varphi}$ to solve our problem. We also perform Hubbard–Stratonovich transformation, as a result the action of the problem will be:

$$\begin{aligned} S_{SYK} &= \frac{1}{2} \int_0^{\tilde\beta} \left[\frac{\dot\xi^2}{2} + \sinh^2(\xi) \frac{\dot\varphi^2}{2} - \gamma \cosh(\xi) \dot\varphi \right] d\tau \\ S_\Phi &= \frac{1}{4g\gamma} \int d\mu \Phi(x) (-L - \tfrac{1}{4} + \delta^2) \Phi(x) \\ S_{int} &= \int_0^{\tilde\beta} \Phi(x(\tau)) d\tau \end{aligned} \tag{3}$$

Here L is the Laplace operator and $d\mu$ is the invariant measure on the hyperbolic plane and we should take a limit $\delta \to 0$. If we integrate the bosonic field Φ we will obtain the previous action. We employ an adiabatic approximation, assuming that the motion along the phase φ is much slower than along radial coordinate ξ. Then functional integral over trajectories $\xi(\tau)$ can be done at fixed value of φ, which is the way to find an effective action for $\dot\varphi(\tau)$. Since parameter $\gamma \gg 1$, we can use saddle point approximation for $\dot\varphi$, which leads to the relation $\dot\varphi = \frac{\gamma \cosh(\xi)}{\sinh^2(\xi)}$. The effective action is then defined in the following way:

$$S_{eff}[\varphi(\tau)] = \ln \left(\int D\Phi D\xi \delta \left(\dot\varphi - \frac{\gamma \cosh(\xi)}{\sinh^2(\xi)} \right) e^{-S} \right) \tag{4}$$

A Lagrange variable $\lambda(\tau)$ is used to remove the δ-function. Then we need to calculate the functional integral with the action dependent of trajectories $\xi(\tau)$ and $\lambda(\tau)$:

$$S = S_\Phi + S_{int} + \int_0^{\tilde\beta} \left[\frac{\dot\xi^2}{2} - \frac{1}{2}\gamma^2 \frac{\cosh^2(\xi)}{\sinh^2(\xi)} - \lambda(\tau) \left(\dot\varphi - \frac{\gamma \cosh(\xi)}{\sinh^2(\xi)} \right) \right] d\tau \tag{5}$$

$$\simeq S_\Phi + S_{int} + \int_0^{\tilde\beta} \left[\frac{1}{2}\dot\xi^2 - \lambda(\tau) \left(\dot\varphi - 2\gamma e^{-\xi(\tau)} \right) \right] d\tau - \int_0^{\tilde\beta} 2\gamma^2 e^{-2\xi(\tau)} d\tau \tag{6}$$

Representation (6) follows from Eq.(5) since the condition $\gamma \gg 1$ leads also to $\xi \gg 1$; we also omit irrelevant constant $\gamma^2/2$. Now calculation of the functional integral over $\xi(\tau)$ is reduced to the solution of the 1D quantum-mechanical problem with the Hamiltonian

$$H = -\frac{\partial_\xi^2}{2} + 2\gamma\lambda(\tau)e^{-\xi} + \Phi(\xi, \varphi(\tau)) \tag{7}$$

It is the same Hamiltonian as one presented in the main text. Its eigenfunctions and eigenvalues will be presented below. Last term in the action (6) was neglected in the Hamiltonian (7) due to its smallness w.r.t. other terms; however, we will need this term later. The term $\Phi(\xi, \varphi)$ in Eq.(7) came from S_{int} term in Eq.(6). Explicit form of $\Phi(\xi, \varphi)$ is to be obtained variationally. Variation of the full action over Φ leads to the relation

$$\Phi_0(\varphi, \xi) = -\int G_\Phi(\xi, \varphi|\xi', \varphi')\psi_g^2(\xi', \varphi')\frac{d\varphi'}{\varepsilon(\varphi')}d\xi' \tag{8}$$

where G_Φ is the Green function of the operator $-L - \frac{1}{4} + \delta^2$, and the limit $\delta \to 0$ is implied. Full analysis of this Green function is provided in Sec.IV below; here we need its asymptotic expression only (it coincides with Eq.(49) in the end of Sec.IV). $G_\Phi(\xi_1, \varphi_1|\xi_2, \varphi_2) = 2g\gamma\left(\frac{e^{-\xi_1-\xi_2}}{\varphi_{12}^2}\right)^{1/2}$, where $\varphi_{12} = 2\sin(\frac{\varphi_1-\varphi_2}{2})$.

Using Eq.(8) and the result of variation of the full action over $\lambda(\tau)$, we obtain, as explained in the main text:

$$\Phi_0(\xi, \varphi) = -\frac{\kappa\sqrt{\lambda\gamma}}{2}e^{-\xi/2} \quad \text{where } \lambda(\tau) = \frac{\kappa(\kappa-1)}{32\dot{\varphi}} \quad \text{and} \quad \kappa^2 = 32g\ln\left(\frac{\kappa\beta}{16\pi}\right) \tag{9}$$

We start our analysis of Eq.(7)) from the simplest case of $\dot{\varphi} = \varepsilon_0 \equiv 2\pi/\tilde{\beta}$. Then Schrodinger equation (7) with potential (9) allows for exact ground-state ψ_g and excited bound-state solutions ψ_n. We provide these functions below together with corresponding eigenvalues, assuming $\kappa > 1$:

$$\psi_g(\chi) = \frac{e^{-\chi/2}\chi^{\kappa/2-1/2}}{\sqrt{2\Gamma(\kappa-1)}}; \qquad E_g = -\frac{(\kappa-1)^2}{32} \tag{10}$$

$$\psi_n(\chi) = \frac{1}{\sqrt{\frac{2\Gamma(n+1)\Gamma(\kappa-n)}{\kappa-2n-1}}}e^{-\chi/2}\chi^{(-1-2n+\kappa)/2}U(-n, -2n+\kappa, \chi); \qquad E_n = -\frac{(1+2n-\kappa)^2}{32} \tag{11}$$

where $\chi = 8\sqrt{\gamma\lambda}e^{-\xi/2}$ and $U(n, m, \chi)$ is confluent hypergeometric function; line (11) is valid for $1 + 2n < \kappa$.

Now we need to generalize the above result for non-constant but slowly varying $\dot{\varphi} \equiv \varepsilon(\varphi)$. Our goal is to determine effective action $S_{eff}[\varphi(\tau)]$; equivalent representation can be obtained in terms of $S_{eff}[\varepsilon(\varphi)]$, since it is always assumed that $\dot{\varphi} \equiv \varepsilon(\varphi) > 0$. Formally, this functional can be written as

$$S_{eff}[\varphi(\tau)] = \left[S_\Phi + \int_0^\beta E_g(\lambda(\tau), \Phi)d\tau - \int_0^\beta \lambda(\tau)\dot{\varphi}d\tau\right]_{saddle} \tag{12}$$

where "saddle" means that Φ and λ should be determined from the saddle point equations.

To find the energy of the ground state for a general choice of $\varepsilon(\varphi)$ it is convenient to consider three terms in the Hamiltonian (7) separately and notice that the term which contains $\lambda(\varphi)$ is canceled out in the effective action (12). Then we need to calculate the average of the two other terms in the Hamiltonian over the deformed (dependent on $\varepsilon(\varphi)$) ground state:

$$\tilde{E}_g = \frac{\kappa-1}{32} - \int G_\Phi(\xi, \varphi|\xi', \varphi')\psi_g^2(\xi', \varphi')\psi_g^2(\xi, \varphi)\frac{d\varphi'}{\varepsilon(\varphi')}d\xi' d\xi \tag{13}$$

The first term in (13) comes from kinetic term in the Hamiltonian (7), its dependence on $\varepsilon(\varphi)$ is weak and we neglect it in the following. We will estimate its influence below. The second term, together with S_Φ term in Eq.(12), combine to our final result for the action in the adiabatic approximation:

$$S_{eff} = -\frac{1}{2}\int G_\Phi(\xi, \varphi|\xi', \varphi')\psi_g^2(\xi', \varphi')\psi_g^2(\xi, \varphi)\frac{d\varphi' d\varphi}{\varepsilon(\varphi')\varepsilon(\varphi)}d\xi' d\xi = -\frac{g}{2}\int \frac{\kappa-1}{\kappa}\left(\frac{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}{\varphi_{12}^2}\right)^{1/2}\frac{d\varphi_1 d\varphi_2}{\varepsilon(\varphi_1)\varepsilon(\varphi_2)} \tag{14}$$

For the applicability of our adiabatic approximation strong inequality $\kappa \gg 1$ is needed, thus $\frac{\kappa-1}{\kappa} \approx 1$.

B. Main non-adiabatic correction

The aim of this Section is to find the first non-adiabatic correction to the action. This correction is due to virtual transitions between the levels of the 1D quantum mechanical problem with the Hamiltonian (7) which describes motion along coordinate ξ. General form of such a correction to S_{eff} is

$$\delta S_{eff} = \left[\sum_n \int_0^\beta d\tau \frac{(\partial_\tau H)_{ng}(\partial_\tau H)_{gn}}{(E_n(\tau) - E_g(\tau))^3}\right]_{saddle} \tag{15}$$

Here E_n is an energy of the excited state n which adiabatically depends on τ and $(\partial_\tau H)_{ng}$ is a matrix element of the operator $\partial_\tau H$ between ground state and n-th state. Equation (15) can be obtained applying quantum-mechanical perturbation theory with respect to time-dependent terms in the Hamiltonian. The expression (15) comes in the next order after the Berry phase term.

To employ general form (15) for our purpose, it is convenient to introduce the following notations:

$$M_{n\alpha} = \int_0^\infty \psi_n(\chi)\psi_g(\chi)\chi^\alpha \frac{2d\chi}{\chi} = \frac{1}{\sqrt{\frac{\Gamma(n+1)\Gamma(\kappa-n)\Gamma(\kappa-1)}{\kappa-2n-1}}} \frac{\Gamma(-1-n+\kappa+\alpha)\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{16}$$

In the limit $\kappa \gg 1$ we have: $M_{n\alpha} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}\kappa^{\alpha-n/2}$. Time derivative $\partial H/\partial\tau$ can be written in the form

$$\partial_\tau H = 2\gamma\partial_\tau\lambda e^{-\xi} - \frac{\kappa\sqrt{\gamma\lambda}\partial_\tau\lambda}{4\lambda} e^{-\xi/2} = \frac{\partial_\tau\lambda}{32\lambda}\left(\chi^2 - \kappa\chi\right) \tag{17}$$

Using Eq.(17) and notations (16) we write:

$$(\partial_\tau H)_{gn} = \frac{1}{32}\frac{\partial_\tau\lambda}{\lambda}(M_{n2} - \kappa M_{n1}) = \frac{1}{32}\frac{\partial_\tau\lambda}{\lambda} n\kappa^{2-n/2}\sqrt{\Gamma(n+1)} \tag{18}$$

Here the limit of large κ was used to obtain the last result. As $E_n = -\frac{1}{32}(-\kappa+2n+1)^2$ and $\kappa \gg 1$ the leading contribution to the S_{eff} comes from the first term in the sum. It brings us to the following expression:

$$\delta S_{eff} = \frac{1}{2}\int_0^\beta \left(\frac{\partial_\tau\lambda}{\lambda}\right)^2 d\tau = \frac{1}{2}\int_0^{2\pi} \frac{d\varphi}{\varepsilon(\varphi)} \left(\partial_\varphi\varepsilon(\varphi)\right)^2 \tag{19}$$

The last expression follows from the expression for λ in (9).

Now we recall the last term in the action (6), which was not taken into account in the adiabatic approximation. In the limit of large κ the contribution of this term into the ground-state energy can be evaluated as $-2\gamma^2\int d\xi\psi_g^2(\xi)e^{-2\xi}$. Thus its contribution to the effective action is

$$\delta S = -\frac{1}{2}\int_0^\beta \int d\xi\psi_g^2(\xi)(2\gamma e^{-\xi})^2 \approx -\frac{1}{2}\int_0^{2\pi}\frac{d\varphi}{\varepsilon(\varphi)}\varepsilon^2(\varphi) \tag{20}$$

Combining the terms in Eqs.(19,20) we find total non-adiabatic contribution to the action

$$\delta S_{eff} = -\int_0^\beta Sch\left\{e^{i\varphi(\tau)},\tau\right\}d\tau \tag{21}$$

which exactly reproduces the Schwarzian action known for the SYK$_4$ theory. Full action is given by the sum of Eq.(21) and Eq.(14):

$$S_{eff} = \frac{1}{2}\int_0^{2\pi}\frac{d\varphi}{\varepsilon(\varphi)}\left((\partial_\varphi\varepsilon(\varphi))^2 - \varepsilon(\varphi)^2\right) - \frac{g}{2}\int\left(\frac{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}{\varphi_{12}^2}\right)^{1/2}\frac{d\varphi_1 d\varphi_2}{\varepsilon(\varphi_1)\varepsilon(\varphi_2)} \tag{22}$$

In the next Section we will evaluate fluctuations of $\varepsilon(\varphi)$ controlled by the action (22).

II. FLUCTUATION CORRECTIONS

In the Section we analyze Gaussian fluctuations of the function $\varepsilon(\varphi)$ using the action provided in Eq.(22), and estimate corrections to the fermion Green function related to these fluctuations.

A. Gaussian fluctuations of the $\varepsilon(\varphi)$ function

Consider the 2nd-order expansion of the action over Fourier-components $\delta\varepsilon_m$ defined as

$$\varepsilon(\theta) = \varepsilon_0 + \frac{1}{2\pi}\sum_m \delta\varepsilon_m e^{im\theta} \tag{23}$$

We will assume $\delta\varepsilon(\theta) \ll \varepsilon_0$; equivalently, we write $\varphi = \theta + u(\theta)$ and $u(\theta) \ll 1$. Do derive the action up to quadratic terms in fluctuations, we need to expand $\varepsilon(\varphi)$ up to a second order:

$$\varepsilon(\varphi) = \varepsilon_0 \frac{d\varphi}{d\theta} = \varepsilon_0(1 + u'(\theta)) \approx \varepsilon_0(1 + u'(\varphi) - u(\varphi)u''(\varphi)) \tag{24}$$

The first term in Eq.(22) leads to:

$$\frac{1}{2}\int_0^{2\pi} \frac{d\varphi}{\varepsilon(\varphi)} \left((\partial_\varphi \varepsilon(\varphi))^2 - \varepsilon(\varphi)^2\right) \approx \frac{\varepsilon_0}{2}\int_0^{2\pi} d\varphi \left((u'')^2 - (1 + u'u')\right) = \frac{1}{4\pi\varepsilon_0}\sum_m \delta\varepsilon_m \delta\varepsilon_{-m}(m^2-1) \tag{25}$$

The second term in Eq.(22) is not quite trivial to handle, since the integral over $(\varphi_1 - \varphi_2)$ formally diverges, so some regularization is needed. Explicit regularization with invariant short-scale cut-off $\varphi_{12}^2/\varepsilon(\varphi_1)\varepsilon(\varphi_2) > l$ can be used to demonstrate that higher harmonics ε_m are free from this log-divergence. Since this calculation is relatively cumbersome, we present here simpler derivation based on dimensional regularization. Namely, we replace power $\frac{1}{2}$ in the 2-nd term in (22) by some $d < \frac{1}{2}$ and then take the limit $d \to \frac{1}{2} - 0$. At $d < \frac{1}{2}$ straightforward Fourier-transformation leads to (with the accuracy up to terms quadratic in ε_m):

$$\frac{g}{4\gamma}\int \left(\frac{\varepsilon(\varphi)\varepsilon(\varphi')}{\varphi_{12}}\right)^d \frac{d\varphi' d\varphi}{\varepsilon(\varphi')\varepsilon(\varphi)} = \frac{1}{2}\frac{g}{4\gamma}\sum_{m\neq 0}\frac{u_m u_{-m}}{m^2}\int_0^{2\pi}\frac{d\varphi}{2\pi} 2(d-1)\varepsilon_0^{2d-4}\left(\frac{1}{4\sin^2(\varphi)}\right)^d ((d-1)\cos(2m\varphi) + d) \tag{26}$$

Then last integral in Eq.(26) can be calculated using the following formula:

$$\int_0^{2\pi}\frac{d\varphi}{2\pi}\left(\frac{1}{4\sin^2(\varphi)}\right)^d e^{2im\varphi} = \frac{1}{2\cos(\pi d)}\frac{\Gamma(m+d)}{\Gamma(2d)\Gamma(1+m-d)} \tag{27}$$

where m is any integer number. We are interested in the m-dependent coefficients which are obtained by derivative of the ratio $\Gamma(m+d)/\Gamma(m+1-d)$ over d, evaluated in the limit $d \to \frac{1}{2}$. The result reads

$$S_{eff} \approx \frac{1}{4\pi\varepsilon_0}\sum_m \delta\varepsilon_m\delta\varepsilon_{-m}(m^2-1) + \frac{g}{2}\sum_m \frac{\tilde{\psi}(m)}{4\pi\varepsilon_0^3}\delta\varepsilon_m\delta\varepsilon_{-m} \tag{28}$$

Here $\tilde{\psi}(x) = \Psi(x+1/2) - \Psi(-1/2)$ and $\Psi(x) = (\ln\Gamma(x))'$ is the digamma function. This action leads to the following correlation function:

$$\langle\delta\varepsilon_m\delta\varepsilon_{-m}\rangle = \frac{2\pi\varepsilon_0^3}{\varepsilon_0^2(m^2-1) + \frac{g}{2}\tilde{\psi}(m)} \tag{29}$$

We use it below for calculations of the corrections to fermion Green function.

B. Estimation of the fluctuations of the kinetic term

The contribution to the action from the kinetic term has the form:

$$S_{kin} = \int \frac{\kappa}{32}d\tau \qquad \kappa^2 = 32g\ln\left(\frac{\kappa}{8\varepsilon(\varphi)}\right) \tag{30}$$

Assuming smallness of fluctuations we can write $\kappa = \kappa_0 + \delta\kappa$ where κ_0 is defined by $\varepsilon(\varphi) = \varepsilon_0$. We will also define a parameter $\alpha = \frac{32g}{\kappa_0^2} \ll 1$. The connection between $\delta\kappa$ and $\delta\varepsilon$ can be obtained from the definition of κ and has the form:

$$\delta\kappa = \frac{\kappa_0}{2}\left(\frac{\alpha}{2}\left(\frac{\delta\varepsilon}{\varepsilon_0}\right)^2 - \alpha\frac{\delta\varepsilon}{\varepsilon_0}\right) \tag{31}$$

This expression leads to the following form of the above action:

$$S_{kin} = \frac{1}{2\pi}\frac{g}{2\kappa_0}\sum_n \frac{\delta\varepsilon_n \delta\varepsilon_{-n}}{\varepsilon_0^2} \tag{32}$$

One can see smallness of this part due to the factor $\frac{1}{\kappa\varepsilon_0} \ll 1$ with respect to the second term in the (28)

C. Correction to the Green function

Fermion Green function can be obtained as an average of the field $\hat{G}(\theta_1, \theta_2)$, evaluated with the effective action (22), where

$$\hat{G}(\theta_1, \theta_2) = -\left(b\gamma^2 \frac{\varepsilon(\theta_1)\varepsilon(\theta_2)}{4\sin^2\left(\frac{\varphi(\theta_1)-\varphi(\theta_2)}{2}\right)}\right)^\Delta \tag{33}$$

The saddle point approximation ($\varphi(\theta) = \theta$) leads to $\langle \hat{G}(\theta_1, \theta_2)\rangle = G_c = -\left(b\gamma^2 \frac{\varepsilon_0^2}{\theta_{12}^2}\right)^\Delta$. We are interested in the quadratic correction to the Green's function. So we need to find the second-order correction by $\delta\varepsilon$ to $\hat{G}$:

$$\frac{\delta \hat{G}(\theta_1,\theta_2)}{G_c(\theta_1,\theta_2)} = \frac{1}{2}\sum_{m\neq \pm 1,0} \langle \delta\varepsilon_m \delta\varepsilon_{-m}\rangle O_m(\theta_1 - \theta_2)$$

$$O_m(\theta) = -\frac{\Delta}{(2\pi)^2 \sin^2\left(\frac{\theta}{2}\right)\varepsilon_0^2 m^2}\Big((\Delta(1-m^2)+1)\cos(m\theta) + \cos(\theta)\left((\Delta-1)m^2 - \Delta + \Delta\left(m^2+1\right)\cos(m\theta)\right)$$
$$-\Delta\left(m^2+1\right) + m^2 + 2\Delta m \sin(\theta)\sin(m\theta) - 1\Big)$$

For large κ only terms with large m will be important. In this case: $O_m(\theta) = \frac{2\Delta}{(2\pi\varepsilon_0)^2}(\Delta - 1 + \Delta\cos(m\theta)) \sim \frac{2\Delta}{(2\pi\varepsilon_0)^2}$ so we can write

$$\frac{\delta \hat{G}(\theta_1,\theta_2)}{G_c(\theta_1,\theta_2)} \sim \frac{1}{2}\frac{2\Delta}{(2\pi\varepsilon_0)^2}\sum_{m\neq\pm 1,0}\langle \delta\varepsilon_m\delta\varepsilon_{-m}\rangle = \frac{1}{2}\frac{2\Delta}{2\pi}\sum_{m\neq\pm1,0}\frac{\varepsilon_0}{\varepsilon_0^2(m^2-1)+\frac{g}{2}\tilde{\psi}(m)} \sim \frac{\Delta}{\pi}\frac{1}{\varepsilon_0 m_*} \tag{34}$$

Here m_* is defined us $\varepsilon_0^2(m_*^2 - 1) = \frac{g}{2}\tilde{\psi}(m_*)$. For large κ we can write, using Eq.(9): $\varepsilon_0 m_* = \frac{\kappa}{8}$, thus corrections to fermion Green function are small at any θ.

III. HIGHER ORDERS OF THE FERMIONIC GREEN FUNCTION.

The major object of our theory is the Majorana Green function $G(\tau)$ averaged over disorder variables which enter the Hamiltonian, Eq.(1) of the main text. However, local Majorana Green function $G_i(\tau,\tau') = -\langle\chi_i(\tau)\chi_i(\tau')\rangle$ contains more information about system's dynamics.

One of the methods to extract this additional information is to consider higher-order Green functions, defined below:

$$G^{(p)}(\tau,\tau') \equiv \langle\left(-\frac{1}{N}\sum_i \chi_i(\tau)\chi_i(\tau')\right)^p\rangle \tag{35}$$

Here we restrict ourselves by the region of moderately high $p \ll N$, where it is easy to show that

$$G^{(p)}(\tau_1,\tau_2) = (-1)^p \left\langle \left[b\frac{e^{-\xi_1-\xi_2}}{\sin^2(\frac{1}{2}(\varphi_1-\varphi_2))}\right]^{\Delta p}\right\rangle = (-1)^p C^2_{\Delta p}\left\langle\left[\frac{b}{4\gamma}\frac{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}{\sin^2(\frac{1}{2}(\varphi_1-\varphi_2))}\right]^{\Delta p}\right\rangle_{S_\varphi} \tag{36}$$

Angular brackets in the middle formula of the above equation mean averaging over quantum action S_{eff}, see Eq.(11) of the main text. Formula in the R.H.S. of (36) is obtained after we take average over fluctuations of ξ_1 and ξ_2 over the polaron ground state $\psi_g(\xi)$, where C_α is defined below:

$$C_\alpha = \left(\frac{2\gamma}{\varepsilon(\varphi)}\right)^\alpha \int e^{-\alpha\xi}\psi_g^2(\xi,\varphi)d\xi = \frac{\Gamma(\kappa+2\alpha-1)}{\Gamma(\kappa-1)}\kappa^{-\alpha}(\kappa-1)^{-\alpha} \approx \exp\left(\frac{2\alpha^2}{\kappa}\right) \tag{37}$$

We used assumption $\alpha \ll \kappa$ to make the last approximation. Final averaging over S_φ in the R.H.S. of Eq.(36) should be done with the full phase-dependent action given by Eq.(22). Last expression in Eq.(37) is valid in the main order of approximation for $\kappa \gg 1$ and $\alpha \gg 1$.

Consider now the effect of integration over fluctuations of angular modes $\varepsilon(\varphi)$ and define relevant measure for these fluctuations

$$g_p(\tau_1,\tau_2) = \frac{\langle G^{(p)}(\tau_1,\tau_2)\rangle}{C^2_{\Delta p} G^p_c(\tau_1,\tau_2)} = \langle \exp\left[\Delta p \delta g(\theta_1,\theta_2)\right]\rangle = \exp\left(\frac{(\Delta p)^2}{2}\langle(\delta g(\theta_1,\theta_2))^2\rangle\right) \tag{38}$$

where $G_c(\tau_1,\tau_2)$ is the conformal saddle-point Green function, while the function $\delta g(\theta_1,\theta_2)$ is defined via the relation

$$\frac{\varepsilon(\varphi_1)\varepsilon(\varphi_2)}{4\sin^2(\frac{\varphi_1-\varphi_2}{2})}\cdot\left[\frac{\varepsilon_0\varepsilon_0}{4\sin^2(\frac{\theta_1-\theta_2}{2})}\right]^{-1} \equiv 1+\delta g(\theta_1,\theta_2) = 1+u'(\theta_1)+u'(\theta_2)+\cot\left(\frac{\theta_1-\theta_2}{2}\right)(u(\theta_2)-u(\theta_1)) \tag{39}$$

We use here definitions $\varphi = \theta + u(\theta)$ and $\varepsilon(\varphi) = \varepsilon_0 \frac{d\varphi}{d\theta}$. To calculate the average in the R.H.S. of Eq.(38) we need to expand the R.H.S. of Eq.(39) up to linear terms in $u(\theta)$ and then use Fourier series:

$$\delta g(\theta_1,\theta_2) = \frac{1}{2\pi}\sum_m \left(ime^{im\theta_1} + ime^{im\theta_1} + \cot\left(\frac{\theta_1-\theta_2}{2}\right)\left(e^{im\theta_2}-e^{im\theta_1}\right)\right)u_m \tag{40}$$

Now we can average R.H.S. of Eq.(38) in the Gaussian approximation, using representation (40) and correlation function defined in (29). Correlation function in the θ-representation is (below $\theta = \theta_1 - \theta_2$):

$$\begin{aligned}\langle\delta g^2(\theta_1,\theta_2)\rangle &= \frac{1}{(2\pi)^2}\sum_m\left(2m\cos\left(\frac{m\theta}{2}\right) - 2\cot\left(\frac{\theta}{2}\right)\sin\left(\frac{m\theta}{2}\right)\right)^2\langle u_m u_{-m}\rangle \\ &\approx \frac{1}{2\pi\varepsilon_0}\Re\sum_{m\neq 0,\pm 1}\frac{1}{m^2}\frac{1}{m^2+m_*^2}\left[2m^2\left(1+e^{im\theta}\right)+4im\cot\left(\frac{\theta}{2}\right)e^{im\theta}+2\cot^2\left(\frac{\theta}{2}\right)\left(1-e^{im\theta}\right)\right] \\ &= \frac{1}{\varepsilon_0}\frac{(2+m_*\theta)}{m_*^3\theta^2}\left[2m_*\theta\cosh\left(\frac{m_*\theta}{2}\right)-4\sinh\left(\frac{m_*\theta}{2}\right)\right]\exp\left\{-\frac{m_*\theta}{2}\right\} \equiv \frac{8}{\kappa}f(\theta)\end{aligned} \tag{41}$$

where $\varepsilon_0 m_* = \kappa/8$ and last equality just defines a convenient notation. Asymptotic limits for the function $f(\theta)$ are given by

$$f(\theta) = \begin{cases} 1 & m_*\theta \gg 1 \\ \frac{\theta m_*}{3} & m_*\theta \ll 1\end{cases} \tag{42}$$

Finally, combining Eqs.(36,37,38,41) and replacing $\Delta \to \frac{1}{4}$ we obtain

$$\frac{G^{(p)}(\tau_1,\tau_2)}{[G(\tau_1,\tau_2)]^p} = \exp\left[\frac{p^2}{4\kappa}\left(1+f(\theta_{12})\right)\right] \tag{43}$$

IV. THE GREEN FUNCTION OF THE BOSON FIELD ON THE HYPERBOLIC PLANE.

The action of the bosonic field is

$$S_\Phi = \frac{1}{2g}\int d\mu\Phi(x)(-L-\frac{1}{4}+\delta^2)\Phi(x) \tag{44}$$

Here L is the Laplace operator and $d\mu$ is an invariant measure on the hyperbolic plane and $\delta \to 0$. We use the Poincaré disk model. The Green function of the bosonic field satisfy the following equation:

$$(-L-\frac{1}{4}+\delta^2)G(z_1,z_0) = g\frac{\delta(z_1-z_0)}{\sqrt{g(x_0)}} \tag{45}$$

All objects here are invariant under $SL(2,R)$ transformations so let us use transforms which maps $z_0 \mapsto 0$ in this case $z_1 \mapsto \frac{z_1 - z_0}{1 - z_1 \bar{z}_0}$. In new coordinates the form of equation will be the same but δ function will be localized in the origin of the hyperbolic plane so we expect the rotation invariant solution. It leads us to the equation:

$$\left[-(1-u)^2(u\partial_u^2 + \partial_u) - \frac{1}{4} + \delta^2 \right] G(z) = g\frac{\delta(u)}{4\pi} \tag{46}$$

Here $u = |z|^2$. This equation can be written as the homogeneous equation with boundary conditions: the Green function should decay faster than $(1-u)^{1/2}$ at $u \to 1$, while at $u \ll 1$ it should behave as $G(u) \to -\frac{\ln(u)}{4\pi}$. Then we come to the following result:

$$G(u) = g\frac{1}{4}(1-u)^{\frac{1}{2}+\delta} {}_2F_1\left(\frac{1}{2}+\delta, \frac{1}{2}+\delta, 1+2\delta, 1-u\right) \tag{47}$$

Here ${}_2F_1(a,b,c;x)$ is a hypergeometric function. In the limit $\delta \to 0$

$$G(z_1, z_0) = g\frac{\sqrt{w}K(w)}{2\pi} \quad \text{where} \quad w = \frac{(1-|z_1|^2)(1-|z_0|^2)}{(1-z_1\bar{z}_0)(1-z_0\bar{z}_1)} \tag{48}$$

Here $K(w)$ is the complete elliptic integral of the first kind. In the limit $w \to 0$ we have:

$$G_\Phi(z_1, z_0) \approx \frac{g}{4} w^{1/2} \tag{49}$$

It is the last form (49) for the Bose field Green function G_Φ, which we use in the main text and in Sec.I above.