A New Bound for the Brown–Erdős–Sós Problem

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Abstract

Let $f(n, v, e)$ denote the maximum number of edges in a 3-uniform hypergraph not containing e edges spanned by at most v vertices. One of the most influential open problems in extremal combinatorics then asks, for a given number of edges $e \geq 3$, what is the smallest integer $d = d(e)$ so that $f(n, e + d, e) = o(n^2)$? This question has its origins in work of Brown, Erdős and Sós from the early 70's and the standard conjecture is that $d(e) = 3$ for every $e \geq 3$. The state of the art result regarding this problem was obtained in 2004 by Sárközy and Selkow, who showed that $f(n, e+2+\lfloor \log_2 e \rfloor, e) = o(n^2)$. The only improvement over this result was a recent breakthrough of Solymosi and Solymosi, who improved the bound for $d(10)$ from 5 to 4. We obtain the first asymptotic improvement over the Sárközy–Selkow bound, showing that

$$f(n, e + O(\log e/ \log\log e), e) = o(n^2).$$

1 Introduction

Extremal combinatorics, and extremal graph theory in particular, asks which global properties of a graph force the appearance of certain local substructures. Perhaps the most well-studied problems of this type are Turán-type questions, which ask for the minimum number of edges that force the appearance of a fixed subgraph F. Recall that an r-uniform hypergraph (r-graph for short) is composed of a ground set V of size n (the vertices) and a collection E of subsets of V (the edges), where each edge is of size exactly r. Note that an ordinary graph is just a 2-graph. A (v, e)-*configuration* is a hypergraph with e edges and at most v vertices. Denote by $f_r(n, v, e)$ the largest number of edges in an r-graph on n vertices that contains no (v, e)-configuration. Estimating the asymptotic growth of this function for fixed integers r, e, v and large n is one of the most well-studied and influential problems in extremal graph theory. For example, when $e = \binom{v}{r}$ we get the well-known Turán problem of determining the maximum possible number of edges in an r-graph that contains no complete r-graph on v vertices. As another example, the case $r = 2$, $v = 2t$ and $e = t^2$ is essentially equivalent to the Zarankiewicz–Kővári–Sós–Turán problem, which asks for the maximum number of edges in a graph without a complete bipartite graph $K_{t,t}$.

Our focus in this paper is on a notorious question of this type, which emerged from work of Brown, Erdős and Sós [2, 3] in the early 70's and came to be named after them. A special case of this so-called Brown–Erdős–Sós conjecture (see [6, 7]) states the following:

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Conjecture 1.1 (Brown–Erdős–Sós Conjecture)**.** *For every $e \geq 3$,*

$$f_3(n, e+3, e) = o(n^2).$$

Despite much effort by many researchers, Conjecture 1.1 is wide open, having only been settled for $e = 3$ by Ruzsa and Szemerédi [14] in what is known as the $(6,3)$-theorem. To get some perspective on the significance of this special case of Conjecture 1.1, suffice it to say that the famous *triangle removal lemma* (see [4] for a survey) was devised in order to prove the $(6,3)$-theorem; that [14] was one of the first applications of Szemerédi's regularity lemma [20]; and that the $(6,3)$-theorem implies Roth's theorem [13] on 3-term arithmetic progressions in dense sets of integers. As another indication of the importance of this problem, we note that one of the main driving forces for proving the celebrated hypergraph removal lemma, obtained by Gowers [9] and Rödl et al. [10, 11, 12] (see also the paper of Tao [23]), was the hope that it would lead to a proof of Conjecture 1.1.

Since we seem to be quite far[1] from proving Conjecture 1.1, it is natural to look for approximate versions. Namely, given $e \geq 3$, find the smallest $d = d(e)$ such that $f_3(n, e+d, e) = o(n^2)$. The best result of this type was obtained 15 years ago by Sárközy and Selkow [15], who proved that

$$f_3\left(n, e+2+\lfloor \log_2 e \rfloor, e\right) = o(n^2). \tag{1}$$

Since the result of [15], the only advance was obtained by Solymosi and Solymosi [19], who improved the bound $f_3(n, 15, 10) = o(n^2)$ that follows from (1) to $f_3(n, 14, 10) = o(n^2)$.

The main result of this paper, Theorem 1, gives the first general improvement over (1). Moreover, it shows that one can replace the $\lfloor \log_2 e \rfloor$ "error term" in (1) by a much smaller, sub-logarithmic, term.

Theorem 1. *For every $e \geq 3$,*

$$f_3\left(n, e + 18 \log e / \log\log e, e\right) = o(n^2).$$

By using asymptotic estimates for the factorial (in place of cruder bounds), one can replace the multiplicative constant 18 in the above theorem by $4 + o(1)$.

Although Theorem 1 deals with 3-graphs, its proof relies on an application of the r-graph removal lemma for *all* values of r. Employing the removal lemma for arbitrary r allows us to overcome a natural barrier which stood in the way of improving the result of [15]. Since the proof of Theorem 1 is quite involved, we sketch the main new ideas underlying it in Section 2.

As we mentioned above, Conjecture 1.1 has a more general form (see [1, 16]), which states that for every $2 \leq k < r$ and $e \geq 3$ we have $f_r(n, (r-k)e + k + 1, e) = o(n^k)$. However, it is a folklore observation that this more general version is in fact equivalent to the special case stated as Conjecture 1.1 (corresponding to $k = 2$ and $r = 3$). Since this reduction does not appear in the literature, we will give its proof here. In fact, we will prove the following more general statement:

Proposition 1.2. *For every $2 \leq k < r$, $e \geq 3$ and $d \geq 1$,*

$$f_r(n, (r-k)e + k + d, e) \leq \binom{r}{3} e n^{k-2} \cdot f_3(n, e+2+d, e).$$

[1] As an indication of the difficulty of Conjecture 1.1, let us mention that the case $e = 4$ (i.e., the statement $f_3(n, 7, 4) = o(n^2)$) implies the notoriously difficult Szemerédi theorem [21, 22] for 4-term arithmetic progressions, see [7].

Setting $d = 1$ in the above proposition readily implies that Conjecture 1.1 is indeed equivalent to the general form of the Brown–Erdős–Sós conjecture stated above. The reason for stating the proposition for arbitrary d is that it allows us to infer approximate versions of the general Brown–Erdős–Sós conjecture from approximate versions of Conjecture 1.1. In particular, by combining Theorem 1 with Proposition 1.2, we immediately obtain the following corollary.

Corollary 2. *For every $2 \leq k < r$ and $e \geq 3$,*

$$f_r\left(n, (r-k)e + k - 2 + 18 \log e / \log\log e, e\right) = o(n^k).$$

The rest of the paper is organized as follows. In Section 2, we give an overview of the main ideas which go into the proof of Theorem 1. We also state the two key lemmas of the paper and explain how they imply Theorem 1. We then prove these two lemmas in Sections 3 and 4. Finally, in Section 5, we discuss an application of our results to a generalized Ramsey problem of Erdős and Gyárfás which is known to have connections to the Brown–Erdős–Sós problem. Throughout the paper, we make no effort to optimize any of the constants involved. All logarithms are natural unless explicitly stated otherwise.

2 Proof Overview and Proof of Theorem 1

Our goal in this section is fourfold. We first give an overview of the proof of Theorem 1. In doing so, we will state the two key lemmas, Lemmas 2.4 and 2.6, used in its proof. We will then proceed to show how these two lemmas can be used in order to prove Theorem 1. Finally, in Section 2.4, we prove Proposition 1.2.

2.1 Proof overview and the key lemmas

Our first simple (yet crucial) observation towards the proof of Theorem 1 is that, in order to prove the theorem, it is enough to prove the following approximate version.

Lemma 2.1. *For every $e \geq 40320 = 8!$ and $\varepsilon \in (0, 1)$, there is $n_0 = n_0(e, \varepsilon)$ such that every 3-graph H with $n \geq n_0$ vertices and at least εn^2 edges contains a (v', e')-configuration satisfying $e - \sqrt{e} \leq e' \leq e$ and $v' - e' \leq 8 \log e / \log\log e$.*

In Section 2.3 we will show how to quickly derive Theorem 1 from the above lemma. So let us proceed with the overview of the proof of Lemma 2.1. We will heavily rely on the hypergraph removal lemma, which states the following.

Theorem 3 (Hypergraph removal lemma [9, 10, 11, 12]). *For every $k \geq 2$ and $\varepsilon > 0$ there is $\gamma = \gamma(k, \varepsilon)$ such that the following holds. Let $n \geq 1$ and let J be a k-uniform n-vertex hypergraph which contains a collection of at least εn^k pairwise edge-disjoint $(k+1)$-cliques. Then J contains at least γn^{k+1} $(k+1)$-cliques.*

Let us start by describing the approach of Sárközy and Selkow [15], which roughly proceeds as follows: suppose one has already proved that every sufficiently large n-vertex 3-graph with $\Omega(n^2)$ edges contains an $(e + k, e)$-configuration. Using this fact, one then shows that every such 3-graph also contains a $(2e + k + 2, 2e + 1)$-configuration. In other words, at the price of increasing $v - e$ by 1, we multiply the number of edges by roughly 2 (and hence the term $\log_2 e$ in (1)). The proof of [15] used the graph removal lemma (at least implicitly[2]). As we mentioned before, Solymosi and Solymosi [19] improved the bound of [15] for the special case $e = 10$. The way they achieved this was by cleverly replacing the application of the graph removal lemma with an application of the 3-graph removal lemma. Roughly speaking, this allowed them to multiply a $(6, 3)$-configuration by 3, instead of by 2 as in [15].

The above discussion naturally leads one to try and extend the approach of [19] by showing that after multiplying the initial configuration by 3, one can use the 4-graph removal lemma to multiply the resulting configuration by 4, etc. Performing k such steps should (roughly) give a $(k! + k, k!)$-configuration, or equivalently, a (v, e)-configuration with $v - e = O(\log e/ \log\log e)$. There is one big challenge and two problems with this approach. The challenge is of course how to achieve this repeated multiplication process.[3] As to the problems, the first is that we do not know how to guarantee that one can indeed keep multiplying the size of the configurations. In other words, it is entirely possible that this process might get "stuck" along the way (this scenario is described in Item 1 of Lemma 2.4). More importantly, even if the process succeeds in producing a $(k! + k, k!)$-configuration for every k, it is not clear how to interpolate so as to prove Theorem 1 for values of e with $(k - 1)! < e < k!$. That is, our process only guarantees the existence of suitable configurations for a very sparse set of values of e. It it tempting to guess that the resulting $(k!+k, k!)$-configurations are "degenerate", in the sense that one can repeatedly remove from them vertices of degree 1, thus maintaining the difference $v - e$. This is however false. Having said that, we will return to this degeneracy issue after the statement of Lemma 2.6.

In what follows, it will be convenient to use the following notation.

Definition 2.2. *For a 3-graph F and $U \subseteq V(F)$, the* difference *of U is defined as $\Delta(U) := |U|-e(U)$. We will write $\Delta(F)$ for $\Delta(V(F))$, i.e., $\Delta(F) := v(F) - e(F)$, and call $\Delta(F)$ the difference of F.*

Our first key lemma, Lemma 2.4 below, comes close to achieving what is described in the paragraph above. Given an n-vertex 3-graph H with $\Omega(n^2)$ edges, the lemma almost resolves the challenge mentioned in the previous paragraph by either showing that H contains configurations with difference k and size roughly $k!$ (this is the statement of Item 2) or getting stuck in the scenario described in Item 1. The silver lining in Item 1 is that we get an arithmetic progression of values v for which we can construct (v, e)-configurations of small difference. The problem is that the common difference of this arithmetic progression might be much larger than $\sqrt{e}$, so this lemma alone cannot be used in order to prove Lemma 2.1.

The key definition in Lemma 2.4 is the notion of a nice 3-graph, which we now define. Satisfying this definition makes a 3-graph amenable to the arguments we use in the proof of Lemma 2.4.

Definition 2.3. *Let F be a 3-graph and put $k := \Delta(F) = v(F) - e(F)$. We call F* nice *if there is an independent set $A \subseteq V(F)$ of size $k + 1$ such that the following holds for every $U \subseteq V(F)$.*

1. $\Delta(U) \geq |U \cap A| - \mathbb{1}_{A\subseteq U}$.
2. *If $|U \cap A| \leq k - 1$ and $U \setminus A \neq \emptyset$, then $\Delta(U) \geq |U \cap A| + 1$.*

Lemma 2.4. *There is a sequence $(F_k)_{k\geq 3}$ of 3-graphs such that $\Delta(F_k) = v(F_k) - e(F_k) = k$, F_k is nice for each $k \geq 4$, $e(F_3) = 3$ and $e(F_k) = 5k!/12$ for each $k \geq 4$, and the following holds. For every $k \geq 4$, $r \geq 1$ and $\varepsilon \in (0, 1)$, there are $\eta = \eta(k, r, \varepsilon) \in (0, 1)$ and $n_0 = n_0(k, r, \varepsilon)$ such that every 3-graph H with $n \geq n_0$ vertices and at least εn^2 edges satisfies (at least) one of the following:*

[2]We will extend their approach in Lemma 2.6 by using the removal lemma explicitly.

[3]The special case in [19] of multiplying a $(6, 3)$-configuration by 3 proceeds by case analysis which is not generalizable.

1. *There are $3 \le j \le k-1$ and $j \le q \le v(F_j)-1$ such that, for every $1 \le i \le r$, the 3-graph H contains a (v', e')-configuration with $v'-e' \le j$ and $v' = q + i \cdot (v(F_j)-q)$.*

2. *H contains at least ηn^k copies of F_k.*

Remark 2.5. *A recurring theme in our arguments is that, given some suitable 3-graph F, we will be able to show that every sufficiently large n-vertex 3-graph H with $\Omega(n^2)$ edges contains $\Omega(n^{v(F)-e(F)})$ copies of F (unless H satisfies the assertion of Theorem 1 for some other reason). This estimate for the number of copies of F is tight, since a random hypergraph with edge density $\frac{1}{n}$ has $O(n^{v(F)-e(F)})$ copies of F w.h.p.*

The proof of Lemma 2.4 proceeds by induction on k. Namely, assuming H contains $\Omega(n^{k-1})$ copies of F_{k-1}, we show that either H contains $\Omega(n^k)$ copies of F_k or Item 1 holds. This is done as follows. Recalling that F_{k-1} is nice (for $k \ge 5$), we fix a set $A \subseteq V(F_{k-1})$ of size $|A| = k$ which witnesses this fact (see Definition 2.3). For each embedding $\varphi : V(F_{k-1}) \to V(H)$ of F_{k-1} into H, we consider the set $\varphi(A) \subseteq V(H)$. By a straightforward argument (combining an application of the multicolor Ramsey theorem with a simple cleaning procedure), we can show that either there are embeddings $\varphi_1, \ldots, \varphi_r : V(F_{k-1}) \to V(H)$ and a set $U \subseteq V(F_{k-1})$ such that $|U| \ge k-1$, $|U \cap A| \ge k-2$ and $\varphi_1|_U = \cdots = \varphi_r|_U$; or there is a family $\mathcal{F}$ of $\Omega(n^{k-1})$ embeddings $\varphi : V(F_{k-1}) \to V(H)$ such that, for any two $\varphi, \varphi' \in \mathcal{F}$, the set $U = \{u \in V(F_{k-1}) : \varphi(u) = \varphi'(u)\}$ (i.e., the set of elements on which φ and φ' agree) satisfies $|U \cap A| \le k-2$ (and $U \subseteq A$ if $|U \cap A| = k-2$). In the former case, Items 1-2 of Definition 2.3 imply that the union of the copies of F_k corresponding to $\varphi_1, \ldots, \varphi_r$ has difference at most $k-1$ (which is also the difference of F_{k-1}), from which it easily follows that Item 1 in Lemma 2.4 holds. In the latter case, we define an auxiliary $(k-1)$-uniform hypergraph by putting a $(k-1)$-uniform k-clique on the set $\varphi(A)$ for each $A \in \mathcal{F}$. The aforementioned property of $\mathcal{F}$ implies that these cliques are pairwise edge-disjoint, which allows us to apply the hypergraph removal lemma (Theorem 3) and thus infer that the number of k-cliques in our auxiliary hypergraph is at least $\Omega(n^k)$. Using again our guarantees regarding $\mathcal{F}$, we can show that most such k-cliques correspond to copies of a particular 3-graph consisting of k copies of F_{k-1} which do *not* intersect outside of the set A. This 3-graph is then chosen as F_k. One of the challenges in the proof is to then show that F_k is itself nice, thus allowing the induction to continue. The full details appear in Section 3.

We now move to our next key lemma, Lemma 2.6 below. Let us say that a 3-graph is d-degenerate if it is possible to repeatedly remove from it a set of at least d vertices which touches at most d edges. As we mentioned above, the 3-graphs F_k are not 1-degenerate, so it is not possible to take one of these 3-graphs and repeatedly remove vertices of degree at most 1 so as to obtain configurations with any desired number of edges, while not increasing the difference. One can argue, however, that since Lemma 2.1 only asks for e' to satisfy $e - \sqrt{e} \le e' \le e$, it is enough to show that the 3-graphs F_k are $\sqrt{e(F_k)}$-degenerate. Unfortunately, we cannot do even this. Instead, we will overcome the problem by using Lemma 2.6. This lemma states that if H contains many copies of some nice 3-graph G, then it also contains copies of 3-graphs $G_0 = G, G_1, G_2, \ldots$ which are all $e(G)$-degenerate and whose sizes increase. In fact, as in Lemma 2.4, we cannot always guarantee success in finding copies of $G_1, G_2, \ldots, G_\ell$ in H, since the process might get stuck in a situation analogous to the one in Lemma 2.4. Finally, the price we have to pay for the degeneracy guaranteed by Item 2 of Lemma 2.6 is that the size of the 3-graphs $G_1, G_2, \ldots, G_\ell$ only grows by a factor of roughly k at each step. Hence, just like Lemma 2.4, Lemma 2.6 also falls short of proving Lemma 2.1.

Lemma 2.6. *Let G be a nice 3-graph, put $k := \Delta(G) = v(G) - e(G)$ and assume that $k \ge 2$. Then there is a sequence of 3-graphs $(G_\ell)_{\ell \ge 0}$ having the following properties.*

1. $G_0 = G$, $\Delta(G_\ell) = v(G_\ell) - e(G_\ell) = k + \ell$ and $e(G_\ell) = \frac{k^{\ell+1}-1}{k-1} \cdot e(G)$.

2. For every $\ell \geq 0$ and every $0 \leq t \leq e(G_\ell)/e(G)$, the 3-graph G_ℓ contains a (v', e')-configuration with $v' - e' \leq k + \ell$ and $e' = t \cdot e(G)$.

3. For every $\ell \geq 0$, $r \geq 0$ and $\varepsilon \in (0, 1)$, there are $\delta = \delta(\ell, r, \varepsilon)$ and $n_0 = n_0(\ell, r, \varepsilon)$ such that, for every 3-graph H on $n \geq n_0$ vertices, if H contains at least εn^k copies of G, then (at least) one of the following conditions is satisfied:

   (a) There are $0 \leq j \leq \ell - 1$ and $k + j \leq q \leq v(G_j) - 1$ such that, for every $1 \leq i \leq r$, the 3-graph H contains a (v', e')-configuration which contains a copy of G_j, where $v' - e' \leq k + j$ and $v' = q + i \cdot (v(G_j) - q)$.

   (b) H contains at least $\delta \cdot n^{k+\ell}$ copies of G_ℓ.

Strictly speaking, we cannot apply Lemma 2.6 with G being an edge, since an edge is not a nice 3-graph (indeed, it has difference $k = 2$ but evidently contains no independent set of size $k + 1 = 3$). However, one can check that the proof also works when G is an edge (and, more generally, in any case where $k := \Delta(G) = 2$ and one can choose a (not necessarily independent) $A \subseteq V(G)$ of size 3 which satisfies Items 1-2 in Definition 2.3). By applying Lemma 2.6 with G being an edge, one recovers the construction used by Sárközy and Selkow [15] to prove (1). Generalizing this construction to other graphs G (e.g., for $k \geq 3$) presents a challenge, which we overcome by using some of the ideas from the proof of Lemma 2.4.

We now sketch the derivation of Lemma 2.1 from Lemmas 2.4 and 2.6 (the formal proof appears in the next subsection). Given e, choose k so that $(2k)! \approx e$; so $k! \approx \sqrt{e}$ and $k = O(\log e/\log\log e)$. We first apply Lemma 2.4 with k. If we are at Item 1, then we get an arithmetic progression with difference at most $v(F_k) - k \leq k! \leq \sqrt{e}$ of values v' for which we can find (v', e')-configurations of difference at most k, thus completing the proof in this case. Suppose then that we are at Item 2, implying that H contains $\Omega(n^k)$ copies of F_k. Since F_k is nice, we can apply Lemma 2.6 with $G = F_k$. Since $e(F_k) \approx k!$ and $(2k)! \approx e$, choosing, say, $\ell = 3k$ guarantees that $e(G_\ell) \approx e(F_k) \cdot k^\ell > e$ (via Item 1 of Lemma 2.6). If the application of Lemma 2.6 results in Item 3(b), then we can use Item 2 of that lemma to find a (v', e')-configuration of difference $O(k + \ell) = O(k)$ with $e - \sqrt{e} \leq e - e(G) \leq e' \leq e$, thus completing the proof. Finally, suppose that we are at Item 3(a). In this case we can find a (v', e')-configuration G' of difference $O(k + \ell) = O(k)$ with $e - e(G_j) \leq e' \leq e$. With the help of a simple trick we can also find in H a copy G^* of G_j which is *edge-disjoint* from G'. As in case 3(b) above, we use Item 2 to find a sub-configuration G'' of G^* with $e - e(G') - e(G) \leq e(G'') \leq e - e(G')$. If we now take G''' to be the union of G' and G'', we infer that G''' has difference $O(k)$ and $e - \sqrt{e} \leq e - e(G) \leq e(G''') \leq e$. So again we are done.

2.2 Deriving Lemma 2.1 from Lemmas 2.4 and 2.6

The required integer $n_0 = n_0(e, \varepsilon)$ will be chosen implicitly. Let $(F_k)_{k\geq 3}$ be the nice 3-graphs whose existence is guaranteed by Lemma 2.4. Recall that $e(F_k) = 5k!/12$ for each $k \geq 4$ and that $e(F_3) = 3$. Let $K \geq 8$ be such that $K! \leq e < (K+1)!$ and put $k := \lfloor K/2 \rfloor \geq 4$. Note that $e(F_k) \leq k! \leq (K/2)! \leq \sqrt{K!} \leq \sqrt{e}$. It is not hard to check that $K \leq 2\log e/\log\log e$ and hence $k \leq \log e/\log\log e$. We will now apply our second construction, given by Lemma 2.6. Set $G := F_k$ and let $(G_\ell)_{\ell\geq 0}$ be the sequence of 3-graphs whose existence is guaranteed by Lemma 2.6. Let ℓ be the minimal integer satisfying $e(G_\ell) \geq e$. Then $\ell \geq 1$ (because $e(G_0) = e(G) = e(F_k) < e$). We will now bound ℓ in terms of k. For our purposes, it will be enough to show that $\ell \leq 3k$. To this end, observe that

$$e(G_{3k}) = \frac{k^{3k+1}-1}{k-1} \cdot e(G) \geq k^{3k} = \lfloor K/2 \rfloor^{3\lfloor K/2 \rfloor} \geq (K+1)! > e,$$

where the first equality follows from Item 1 of Lemma 2.6 and the penultimate inequality holds for every $K \geq 8$. The fact that $e(G_{3k}) > e$ now readily implies that $\ell \leq 3k$.

Let H be a 3-graph with $n \geq n_0$ vertices and at least εn^2 edges. Partition $E(H)$ into equal-sized parts $E_1, \dots, E_{\ell+1}$ and, for each $1 \leq i \leq \ell+1$, let H_i be the hypergraph $(V(H), E_i)$. Note that $e(H_i) \geq e(H)/(\ell+1) \geq \varepsilon n^2/(\ell+1)$ for each $1 \leq i \leq \ell+1$.

Claim 2.7. *For each $1 \leq m \leq \ell+1$, either H_m satisfies the assertion of Lemma 2.1 or there exists $0 \leq j \leq \ell-1$ such that H_m contains a (v', e')-configuration which contains a copy of G_j, where $v' - e' \leq k+j$ and $e - e(G_j) \leq e' \leq e$.*

Proof. Evidently, it is enough to prove the claim for $m = 1$. We apply Lemma 2.4 to H_1 with parameters $r = e + k$ and $\varepsilon/(\ell+1)$. Suppose first that the assertion of Item 1 in Lemma 2.4 holds and let $3 \leq j \leq k-1$ and $j \leq q \leq v(F_j) - 1$ be as in that item. Let i be the maximal integer satisfying $q + i \cdot (v(F_j) - q) \leq e + j$ and note that $1 \leq i \leq e + j \leq e + k$. We may thus infer from Item 1 in Lemma 2.4 that H_1 contains a (v', e')-configuration with

$$v' = q + i \cdot (v(F_j) - q) \leq e + j, \tag{2}$$

and

$$v' - e' \leq j < k \leq \log e / \log\log e. \tag{3}$$

Note that the maximality of i guarantees that

$$v' > e + j - (v(F_j) - q). \tag{4}$$

We now observe that we can assume that $e' \leq e$. Indeed, since by (3) we have $v' - e' \leq j$, then we can remove edges until the equality $e' = v' - j$ holds. Having done that, we are guaranteed by (2) that $e' \leq e$. As to the lower bound on e', by (4) we have $e - e' = e + j - v' < v(F_j) - q \leq v(F_j) - j$. By Lemma 2.4, we have $v(F_j) - j = 5j!/12$ if $j \geq 4$ and $v(F_j) - j = 3$ if $j = 3$. In either case, we get $e - e' \leq j! \leq k! \leq \sqrt{e}$. So we see that H_1 satisfies the assertion of Lemma 2.1, as required. This completes the proof for the case that the assertion of Item 1 in Lemma 2.4 holds.

Suppose from now on that the assertion of Item 2 in Lemma 2.4 holds, namely, that H_1 contains at least ηn^k copies of $F_k = G$. This means that we may apply Lemma 2.6 to H_1. By Item 3 of Lemma 2.6, applied with $r = e + k + \ell$ and with η in place of ε, the 3-graph H_1 satisfies (at least) one of the following:

(a) There are some $0 \leq j \leq \ell - 1$ and $k + j \leq q \leq v(G_j) - 1$ such that, for every $1 \leq i \leq e + k + \ell$, H_1 contains a (v', e')-configuration which contains a copy of G_j, where $v' - e' \leq k + j$ and $v' = q + i \cdot (v(G_j) - q)$.

(b) H_1 contains a copy of G_ℓ (in fact, at least $\delta(\ell, r, \eta) \cdot n^{k+\ell}$ such copies).

Suppose first that H_1 satisfies Item (b). Let $t \geq 0$ be the maximal integer satisfying $t \cdot e(G) \leq e$ and note that $t \leq e/e(G) \leq e(G_\ell)/e(G)$, where the second inequality uses our choice of ℓ. By Item 2 of Lemma 2.6, H_1 contains a (v', e')-configuration with $v' - e' \leq k + \ell \leq 4k \leq 4 \log e / \log\log e$ and

$e' = t \cdot e(G) \le e$. By our choice of t, we have $e - e' < e(G) = 5k!/12 \le k! \le \sqrt{e}$. So in this case the assertion of Lemma 2.1 indeed holds for H_1.

From now on we assume that H_1 satisfies Item (a) and let $0 \le j \le \ell - 1$ and $k + j \le q \le v(G_j) - 1$ be as in that item. Let i be the maximal integer satisfying $q + i \cdot (v(G_j) - q) \le e + k + j$. Then $1 \le i \le e + k + j < e + k + \ell$. We may thus rely on (a) above to conclude that H_1 contains a (v', e')-configuration which contains a copy of G_j, where

$$v' = q + i \cdot (v(G_j) - q) \le e + k + j, \tag{5}$$

and

$$v' - e' \le k + j. \tag{6}$$

Note that the maximality of i guarantees that

$$v' > e + k + j - (v(G_j) - q). \tag{7}$$

We now observe that we can assume that $e' \le e$. Indeed, since by (6) we have $v' - e' \le k + j$ then we can remove edges until the equality $e' = v' - (k + j)$ holds. By (5), this would guarantee that $e' \le e$. Note (crucially) that since $e(G_j) = v(G_j) - k - j \le v' - k - j$, we can make sure that even after removing the required number of edges we still have a copy of G_j. As to the lower bound on e', by (6) and (7) we have $e - e' \le e - v' + k + j < v(G_j) - q \le v(G_j) - k - j = e(G_j)$. We conclude that H_1 indeed contains a (v', e')-configuration with the properties stated in the claim. ■

We now return to the proof of the lemma. If some H_m satisfies the assertion of Lemma 2.1 then we are done. Otherwise, Claim 2.7 implies that for each $1 \le m \le \ell + 1$ there is $0 \le j_m \le \ell - 1$ such that H_m contains a (v', e')-configuration which contains a copy of G_{j_m}, where $v' - e' \le k + j_m$ and $e - e(G_{j_m}) \le e' \le e$. By the pigeonhole principle, there are two indices $1 \le i \le \ell + 1$ whose j_m's are equal. It follows that for some $0 \le j \le \ell - 1$, H contains *edge-disjoint* subgraphs G^* and G' such that G^* is isomorphic to G_j and G' satisfies $v(G') - e(G') \le k + j$ and $e - e(G_j) \le e(G') \le e$. Let t be the maximal integer satisfying $t \cdot e(G) \le e - e(G')$ and note that $0 \le t \le e(G_j)/e(G)$. Then, by Item 2 of Lemma 2.6 (with j in place of ℓ), there is a subgraph G'' of G^* such that $v(G'') - e(G'') \le k + j$ and $e(G'') = t \cdot e(G)$. Our choice of t implies that $0 \le e - e(G') - e(G'') < e(G) \le k! \le \sqrt{e}$. Now, letting G''' be the union of G' and G'', we see that $e - \sqrt{e} \le e(G''') \le e$ and

$$v(G''') - e(G''') \le v(G') - e(G') + v(G'') - e(G'') \le 2(k + j) \le 2(k + \ell) \le 8k \le 8 \log e / \log\log e.$$

So we see that the assertion of the lemma holds with G''' as the required (v', e')-configuration.

2.3 Deriving Theorem 1 from Lemma 2.1

Our goal is to show that for every $e \ge 3$ and $\varepsilon \in (0, 1)$, there is $n_0 = n_0(e, \varepsilon)$ such that every 3-graph with $n \ge n_0$ vertices and at least εn^2 edges contains a (v, e)-configuration with $v - e \le 18 \log e / \log\log e$. As in the proof of Lemma 2.1, the required integer $n_0 = n_0(e, \varepsilon)$ will be chosen implicitly. The proof is by induction on e. Let H be a 3-graph with $n \ge n_0$ vertices and at least εn^2 edges. By (1), H contains a (v, e)-configuration with $v - e \le 2 + \lfloor \log_2 e \rfloor$. If $e \le \exp(2^{16})$, then we have $2 + \lfloor \log_2 e \rfloor \le 2 + 16 \log e / \log\log e \le 18 \log e / \log\log e$ (where the second inequality holds whenever $e \ge 3$), thus completing the proof in this case. So suppose from now on that $e > \exp(2^{16}) \ge 40320$. (The inequality $e \ge 40320$ is required to apply Lemma 2.1.)

By Lemma 2.1, H contains a (v', e')-configuration F' satisfying $e - \sqrt{e} \le e' \le e$ and $v' - e' \le 8 \log e / \log\log e$. Set $e'' := e - e'$, noting that $0 \le e'' \le \sqrt{e}$. If $e'' \le 15$, then, by adding at most 15 edges to F', one obtains a (v,e)-configuration with $v-e \le v' + 3e'' - (e'+e'') = v' - e' + 2e'' \le 8\log e/\log\log e + 30 \le 18\log e/\log\log e$, as required. (Here the last inequality is guaranteed by our assumption that e is large.) So suppose from now on that $e'' \ge 16$. Let H' be the 3-graph obtained from H by deleting the edges of F'. Since $e(H') \ge e(H) - e(F') \ge \varepsilon n^2 - e(F') \ge \frac{\varepsilon}{2}n^2$ (provided that n is large enough), we may apply the induction hypothesis to H', with parameter e'' in place of e, and thus obtain a (v'',e'')-configuration F'' which is edge-disjoint from F' (because it is contained in H') and satisfies

$$v'' - e'' \le \frac{18\log e''}{\log\log e''} \le \frac{18\log\sqrt{e}}{\log\log\sqrt{e}} = \frac{9\log e}{\log\log e - \log 2}.$$

Here, in the second inequality we used the fact that the function $x \mapsto \log x/\log\log x$ is monotone increasing for $x \ge 16$. Letting F be the union of F' and F'', we see that $e(F) = e(F') + e(F'') = e$ and $v(F) \le v(F') + v(F'')$, implying that

$$v(F) - e(F) \le v(F') - e(F') + v(F'') - e(F'') \le \frac{8\log e}{\log\log e} + \frac{9\log e}{\log\log e - \log 2} \le \frac{18\log e}{\log\log e},$$

where the last inequality holds whenever $e \ge \exp(2^{10})$. This completes the proof of the theorem.

2.4 Proof of Proposition 1.2

Let $2 \le k < r$, $e \ge 3$ and $d \ge 1$. Let H be an n-vertex r-graph with

$$e(H) \ge \binom{r}{3} e n^{k-2} \cdot f_3(n, e+2+d, e).$$

Our goal is to show that H contains a (v,e)-configuration with $v \le (r-k)e+k+d$. By averaging, there are vertices $v_1,\dots,v_{k-2}$ such that at least $\binom{r}{3}e \cdot f_3(n,e+2+d,e)$ of the edges of H contain $v_1,\dots,v_{k-2}$. Set $E_0 = \{X \setminus \{v_1,\dots,v_{k-2}\} : v_1,\dots,v_{k-2} \in X \in E(H)\}$, noting that $|E_0| \ge \binom{r}{3}e \cdot f_3(n,e+2+d,e)$ and that $|Y| = r-k+2$ for each $Y \in E_0$. We now consider two cases. Suppose first that there is a triple $T \in \binom{V(H)}{3}$ and distinct $Y_1,\dots,Y_e \in E_0$ such that $T \subseteq Y_i$ for each $1 \le i \le e$. Setting $X_i = Y_i \cup \{v_1,\dots,v_{k-2}\}$ for each $1 \le i \le e$, we observe that $|X_1 \cup \dots \cup X_e| \le (r-k-1)\cdot e + k - 2 + 3 \le (r-k)e+k$. It follows that H contains a (v,e)-configuration with $v \le (r-k)e+k$, thus completing the proof in this case.

Suppose now that for each $T \in \binom{V(H)}{3}$ it holds that $\#\{Y \in E_0 : T \subseteq Y\} \le e-1$. Then, for each $Y \in E_0$, there are at most $\binom{r}{3}(e-1)$ sets $Y' \in E_0 \setminus \{Y\}$ such that $|Y \cap Y'| \ge 3$. This means that there exists $E_1 \subseteq E_0$ of size

$$|E_1| \ge \frac{|E_0|}{\binom{r}{3}(e-1)+1} > f_3(n,e+2+d,e), \tag{8}$$

such that $|Y \cap Y'| \le 2$ for each pair of distinct $Y, Y' \in E_1$.[4] For each $Y \in E_1$, choose arbitrarily a triple $T_Y \in \binom{Y}{3}$. Let H' be the 3-graph on $V(H)$ whose edge-set is $E(H') = \{T_Y : Y \in E_1\}$. Then $e(H') = |E_1| > f_3(n,e+2+d,e)$, where the equality holds due to our choice of E_1 and the inequality due to (8). It follows that H' contains an $(e+2+d,e)$-configuration F. Now observe that the edge-set $\{Y \cup \{v_1,\dots,v_{k-2}\} : Y \in E_1 \text{ and } T_Y \in E(F)\}$ spans in H a (v,e)-configuration with $v \le v(F) + (r-k-1)e + k - 2 \le e+2+d+(r-k-1)e+k-2 = (r-k)e+k+d$, as required.

[4] To see that such an E_1 indeed exists, consider an auxiliary graph on E_0 in which Y, Y' are adjacent if and only if $|Y \cap Y'| \ge 3$ and recall the simple fact that every graph G contains an independent set of size at least $\frac{v(G)}{\Delta(G)+1}$ (where $\Delta(G)$ is the maximum degree of G). Now take E_1 to be such an independent set.

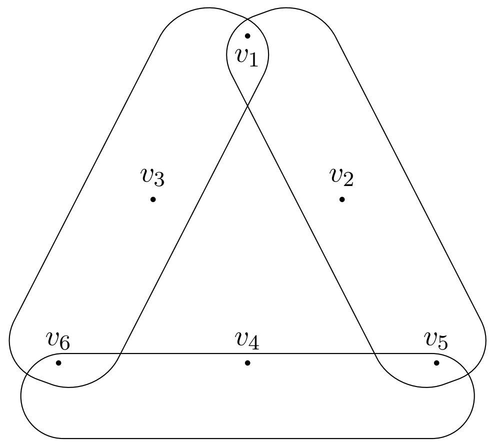

Figure 1: The 3-uniform linear 3-cycle

3 Proof of Lemma 2.4

In this section we prove Lemma 2.4. The construction of the 3-graphs F_k appearing in the statement of the lemma, as well as the proof that these 3-graphs have the required properties, is done by induction on k. The inductive step, which constitutes the main part of the proof of Lemma 2.4, is given by the following lemma.

Lemma 3.1. *Let F be a nice 3-graph, put $k = v(F) - e(F)$ and assume that $k \geq 3$. Then there exists a nice 3-graph F' such that $v(F') - e(F') = k + 1$, $e(F') = (k + 1) \cdot e(F)$ and the following holds. For every $r \geq 1$ and $\varepsilon \in (0, 1)$, there are $\delta = \delta(F, r, \varepsilon) \in (0, 1)$ and $n_0 = n_0(F, r, \varepsilon)$ such that every 3-graph H with $n \geq n_0$ vertices and at least εn^k copies of F satisfies (at least) one of the following:*

1. *There is $k \leq q \leq v(F) - 1$ such that, for every $1 \leq i \leq r$, H contains a (v', e') configuration with $v' - e' \leq k$ and $v' = q + i \cdot (v(F) - q)$.*
2. *H contains at least δn^{k+1} copies of F'.*

Ideally, we would like to start the induction by invoking Lemma 3.1 with F being an edge (so $k = \Delta(F) = 2$). As is the case with Lemma 2.6 (see the remark following this lemma), Lemma 3.1 does in fact work with F being an edge, even though an edge is not nice as per Definition 2.3. The 3-graph F' supplied by Lemma 3.1 (when applied with F being an edge) is the linear 3-cycle (see Figure 1). In fact, applying Lemma 3.1 with F being an edge recovers the proof of the (6,3)-theorem, which was discussed in Section 2.1. Unfortunately, the linear 3-cycle is not nice (this time in a meaningful way; it really cannot be used as an input to Lemma 3.1), preventing us from continuing the induction. To make matters even worse, there is in fact no 3-graph F with difference $k = 3$ which is known to be a viable input to Lemma 3.1. Indeed, note that in order for the lemma to be useful when applied with input F, we need to know that F is abundant[5] in every sufficiently large n-vertex 3-graph with $\Omega(n^2)$ edges (or at least in every such 3-graph that does not satisfy the conclusion of Theorem 1 for some other reason). Unfortunately, no such nice F (of difference 3) is known.

[5]We say that a 3-graph F is *abundant* in an n-vertex 3-graph H if H contains $\Omega(n^{v(F)-e(F)})$ copies of F. In particular, the edge is trivially abundant in every hypergraph with $\Omega(n^2)$ edges and the condition (resp. conclusion) of Lemma 3.1 can be stated as saying that F (resp. F') is abundant in H.

In light of this situation, the base step of our induction will have to involve a nice 3-graph F having difference at least 4. Fortunately, as stated in the following lemma, there does exist a nice F of difference 4 which can be shown to be abundant in every 3-graph H with n vertices and $\Omega(n^2)$ edges, unless H satisfies the assertion of Theorem 1 for a trivial reason.

Lemma 3.2. *There is a nice 3-graph F with $v(F) = 14$ and $e(F) = 10$ having the following property. For every $r \geq 1$ and $\varepsilon \in (0, 1)$, there are $\delta = \delta(r, \varepsilon) \in (0, 1)$ and $n_0 = n_0(r, \varepsilon)$ such that every 3-graph H with $n \geq n_0$ vertices and at least εn^2 edges satisfies (at least) one of the following:*

1. *For every $1 \leq i \leq r$, H contains a $(3i + 3, 3i)$-configuration.*
2. *H contains at least δn^4 copies of F.*

We note that the 3-graph F in the above lemma played a key role in the proof in [19] that $f_3(n, 14, 10) = o(n^2)$. As such, the abundance statement regarding F was already proven in [19]. Consequently, our main task in the proof of Lemma 3.2 is to show that F is nice.

The rest of this section is organized as follows. In Section 3.1, we derive Lemma 2.4 from Lemmas 3.1 and 3.2. We then prove these two lemmas in Sections 3.2 and 3.3, respectively.

3.1 Deriving Lemma 2.4 from Lemmas 3.1 and 3.2

Let F_3 be the linear 3-cycle (which has 6 vertices and 3 edges). Let F_4 be the nice 3-graph whose existence is guaranteed by Lemma 3.2. For each $k \geq 5$, let F_k be the nice 3-graph F' obtained by applying Lemma 3.1 with $F := F_{k-1}$. Then it is easy to check by induction that, for every $k \geq 4$, it holds that $v(F_k) - e(F_k) = k$, $e(F_k) = 5k!/12$ and the 3-graph F_k is nice.

Let $r \geq 1$ and $\varepsilon \in (0, 1)$. We define a sequence $(\delta_k)_{k \geq 4}$ as follows. Let $\delta_4 = \delta(r, \varepsilon)$ be defined via Lemma 3.2 and, for each $k \geq 5$, let $\delta_k = \delta\left(F_{k-1}, r, \delta_{k-1}\right)$ be given by Lemma 3.1. We now show by induction on $k \geq 4$ that the assertion of the lemma holds with $\eta = \eta(k, r, \varepsilon) := \delta_k$. For $k = 4$, Lemma 3.2 readily implies that H either satisfies the assertion of Item 2 of Lemma 2.4 or satisfies the assertion of Item 1 with $j = 3$ and $q = 3$. Let now $k \geq 5$. By the induction hypothesis, H satisfies the assertion of (at least) one of the items of Lemma 2.4 with parameter $k - 1$ (in place of k). If this is the case for Item 1, then the same item is also satisfied with parameter k and we are done. Suppose then that H satisfies the assertion of Item 2 (with parameter $k - 1$), namely, that H contains at least $\delta_{k-1} \cdot n^{k-1}$ copies of F_{k-1}. Then, by Lemma 3.1 (with parameters $F = F_{k-1}$ and δ_{k-1} in place of ε), either H satisfies the assertion of Item 1 in Lemma 2.4 (with $j = k - 1$) or it contains at least $\delta_k \cdot n^k = \eta(k, r, \varepsilon) \cdot n^k$ copies of F_k, as required by Item 2.

3.2 Proof of Lemma 3.1

Let $A \subseteq V(F)$ be as in Definition 2.3. It will be convenient to set $v := v(F)$ and to assume (without loss of generality) that $V(F) = [v]$ and $A = [k+1] \subseteq [v]$. The required nice 3-graph F' is defined as follows: take vertices $x_1, \ldots, x_{k+1}, x'_1, \ldots, x'_{k+1}$ and, for each $1 \leq i \leq k+1$, add a copy F_i of F in which x_j plays the role of $j \in V(F)$ for each $j \in [k+1] \setminus \{i\}$, x'_i plays the role of $i \in V(F)$ and all other $v(F) - k - 1$ vertices are new.

Let us calculate the number of vertices and edges in F'. First, as $A \subseteq V(F)$ is independent, the copies $F_1, \ldots, F_{k+1}$ (which comprise F') do not share edges. Hence, $e(F') = (k+1) \cdot e(F)$. Second, we have $v(F') = k + 1 + (k+1) \cdot (v(F) - k) = k + 1 + (k+1) \cdot e(F) = e(F') + k + 1$, as required.

We now show that F' is nice. We will show that F' satisfies the requirements of Definition 2.3 with respect to the set $A' := \{x'_1, \ldots, x'_{k+1}, x_1\}$. (We remark that in the definition of A' we could replace

x_1 with any other vertex among $x_1, \dots, x_{k+1}$.) For the rest of the proof, we set $X = \{x_1, \dots, x_{k+1}\}$, $X' = \{x'_1, \dots, x'_{k+1}\}$ and $A_i = (X \setminus \{x_i\}) \cup \{x'_i\}$ for each $1 \le i \le k+1$. Observe that for each $1 \le i \le k+1$, the vertices of A_i are precisely the vertices which play the roles of the vertices of $A = \{1, \dots, k+1\} \subseteq V(F)$ in the copy F_i of F.

It is evident that $|A'| = k+2$ and easy to see that A' is independent in F'. Our goal is then to show that every $U \subseteq V(F')$ satisfies the assertion of Items 1-2 in Definition 2.3 (with A' in place of A). So let $U \subseteq V(F')$ and put $U_i = U \cap V(F_i)$ for each $1 \le i \le k+1$. Since every vertex of X belongs to exactly k of the copies $F_1, \dots, F_{k+1}$ and every other vertex of F' belongs to exactly one of these copies, we have

$$|U| = \sum_{i=1}^{k+1} |U_i| - (k-1)|U \cap X|.$$

Since $F_1, \dots, F_{k+1}$ are pairwise edge-disjoint, we have

$$e(U) = \sum_{i=1}^{k+1} e(U_i).$$

It follows that

$$\Delta(U) = \sum_{i=1}^{k+1} \Delta(U_i) - (k-1)|U \cap X|. \tag{9}$$

For each $1 \le i \le k+1$, it follows from the niceness of F (and the fact that A_i plays the role of A in the copy F_i of F) that

$$\Delta(U_i) \ge |U_i \cap A_i| - \mathbb{1}_{A_i \subseteq U_i}. \tag{10}$$

Setting $s := \#\{1 \le i \le k+1 : A_i \subseteq U_i\}$, we plug (10) into (9) to obtain

$$\begin{aligned} \Delta(U) &\ge \sum_{i=1}^{k+1} |U_i \cap A_i| - (k-1)|U \cap X| - s = |U \cap X| + |U \cap X'| - s \\ &= |U \cap A'| + |U \cap \{x_2, \dots, x_{k+1}\}| - s. \end{aligned} \tag{11}$$

To see that the first equality in (11) holds, note that $A_1 \cup \dots \cup A_{k+1} = X \cup X'$ and recall that every element of X (resp. X') belongs to exactly k (resp. 1) of the sets $A_1, \dots, A_{k+1}$.

We first prove that $\Delta(U) \ge |U \cap A'| - \mathbb{1}_{A' \subseteq U}$, as required by Item 1 in Definition 2.3. If $s = 0$, then (11) readily gives $\Delta(U) \ge |U \cap A'|$. Suppose then that $s \ge 1$ and let $1 \le i \le k+1$ be such that $A_i \subseteq U_i$. Then $\{x_2, \dots, x_{k+1}\} \setminus \{x_i\} \subseteq U$, implying that $|U \cap \{x_2, \dots, x_{k+1}\}| \ge k-1$. Furthermore, if $s \ge 2$, then $\{x_2, \dots, x_{k+1}\} \subseteq U$, in which case $|U \cap \{x_2, \dots, x_{k+1}\}| = k$. Hence, it follows from (11) that $\Delta(U) \ge |U \cap A'| - \mathbb{1}_{s=k+1}$. We also note, for later use, that if $1 \le s \le k-1$ then $\Delta(U) \ge |U \cap A'| + 1$ (here we use the assumption that $k \ge 3$). Observe that if $s = k+1$, then $A_i \subseteq U_i$ for every $1 \le i \le k+1$, implying that $A' \subseteq X \cup X' \subseteq U$. So we indeed have $\Delta(U) \ge |U \cap A'| - \mathbb{1}_{A' \subseteq U}$, as required.

Next, we assume that $|U \cap A'| \le k$ and $U \setminus A' \neq \emptyset$ and show that in this case $\Delta(U) \ge |U \cap A'| + 1$ (as required by Item 2 in Definition 2.3). The assumption that $|U \cap A'| \le k$ implies that $s \le k-1$, because if $s \ge k$, then $|U \cap X'| \ge k$ and $x_1 \in U$, which means that $|U \cap A'| \ge k+1$. We already saw that $\Delta(U) \ge |U \cap A'| + 1$ if $1 \le s \le k-1$, so it remains to handle the case that $s = 0$, namely, that $A_i \not\subseteq U_i$ for each $1 \le i \le k+1$. If $U \cap \{x_2, \dots, x_{k+1}\} \neq \emptyset$, then (11) readily implies that $\Delta(U) \ge |U \cap A'| + 1$ (since $s = 0$). So suppose that $U \cap \{x_2, \dots, x_{k+1}\} = \emptyset$. Since $U \setminus A' \neq \emptyset$, there is $1 \leq i \leq k+1$ such that $U_i \setminus A' \neq \emptyset$. Our assumption that $U \cap \{x_2, \dots, x_{k+1}\} = \emptyset$ implies that $|U_i \cap A_i| \leq k-1$ and $U_i \setminus A_i \neq \emptyset$ (here we use the fact that $A_i \subseteq A' \cup \{x_2, \dots, x_{k+1}\}$ and $U_i \setminus A' \neq \emptyset$). Now it follows from the niceness of F (or, more precisely, of the copy F_i of F) that $\Delta(U_i) \geq |U_i \cap A_i| + 1$. Moreover, by (10), we have $\Delta(U_j) \geq |U_j \cap A_j|$ for each $1 \leq j \leq k+1$ (this follows from our assumption that $s = 0$). By plugging all of this into (9), in a manner similar to the derivation of (11), we obtain

$$\Delta(U) \geq |U_i \cap A_i| + 1 + \sum_{j \in [k+1] \setminus \{i\}} |U_j \cap A_j| - (k-1)|U \cap X| = |U \cap X| + |U \cap X'| + 1 \geq |U \cap A'| + 1,$$

as required.

Having proven that F' is nice, we go on to show that the assertion of the lemma holds. Given $r \geq 1$ and $\varepsilon \in (0,1)$, we set

$$\delta = \delta(F, r, \varepsilon) = \frac{1}{2}\gamma\left(k,\ 2^{-v(1+2^v r)} \cdot v^{-v} \cdot \varepsilon\right)$$

and $n_0 = n_0(F, r, \varepsilon) = 1/\delta$. Here γ is from Theorem 3 and $v = v(F)$ as before.

Let H be a 3-uniform hypergraph with $n \geq n_0$ vertices and at least εn^k copies of F. Partition the vertices of H randomly into sets $C_1, \dots, C_v$ by choosing, for each vertex $x \in V(H)$, a part C_i ($1 \leq i \leq v$) uniformly at random and independently (of the choices made for all other vertices of H) and placing x in this part. A copy of F in H will be called *good* if, for each $i = 1, \dots, v$, the vertex playing the role of i in this copy is in C_i. Since H contains at least εn^k copies of F, there are in expectation at least $v^{-v} \cdot \varepsilon n^k$ good copies of F. So fix a partition $C_1, \dots, C_v$ with at least this number of good copies of F and denote the set of these copies by $\mathcal{F}$. It will be convenient to identify each good copy of F with the corresponding embedding $\varphi : V(F) \to V(H)$ which maps each $i \in [v] = V(F)$ to a vertex in C_i. So we will assume that the elements of $\mathcal{F}$ are such mappings.

We now define an auxiliary graph $\mathcal{G}$ on $\mathcal{F}$ as follows: for each pair $\varphi_1, \varphi_2 \in \mathcal{F}$, we let $\{\varphi_1, \varphi_2\}$ be an edge in $\mathcal{G}$ if and only if the set $U := U(\varphi_1, \varphi_2) := \{i \in V(F) : \varphi_1(i) = \varphi_2(i)\}$ satisfies either $|U \cap A| \geq k$ or $|U \cap A| = k-1$ and $U \setminus A \neq \emptyset$. We distinguish between two cases. Suppose first that there is $\varphi \in \mathcal{F}$ whose degree in $\mathcal{G}$ is at least

$$d := 2^{v(1+2^v r)}.$$

Let $\varphi_1, \dots, \varphi_d$ be distinct neighbours of φ in $\mathcal{G}$. By the pigeonhole principle, there is $I_0 \subseteq [d]$ of size at least $2^{-v}d = 2^{v 2^v r}$ and a set $U_0 \subseteq V(F)$ such that, for all $i \in I_0$, it holds that $U(\varphi, \varphi_i) = U_0$. Note that by the definition of G, we have either $|U_0 \cap A| \geq k$ or $|U_0 \cap A| = k-1$ and $U_0 \setminus A \neq \emptyset$. We now consider the complete graph on I_0 and color each edge $\{i,j\} \in \binom{I_0}{2}$ of this graph with color $U(\varphi_i, \varphi_j)$. A well-known bound for multicolor Ramsey numbers (see [5]) implies that in every c-coloring of the edges of the complete graph on c^{cr} vertices, there is a monochromatic complete subgraph on r vertices. Applying this claim with $c = 2^v$, we conclude that there is $I \subseteq I_0$ of size $|I| = r$, and a set $U \subseteq V(F)$, such that $U(\varphi_i, \varphi_j) = U$ for all $\{i,j\} \in \binom{I}{2}$. Observe that for each $\{i,j\} \in \binom{I}{2}$, we have $U = U(\varphi_i, \varphi_j) \supseteq U(\varphi, \varphi_i) \cap U(\varphi, \varphi_j) = U_0$. This implies that either $|U \cap A| \geq k$ or $|U \cap A| = k-1$ and $U \setminus A \neq \emptyset$. Our choice of A via Definition 2.3 implies that in both cases $\Delta(U) \geq k$. Note also that $U \neq V(F)$ because the copies of F corresponding to $(\varphi_i : i \in I)$ are distinct.

We now show that the assertion of Item 1 in the lemma holds. Suppose without loss of generality that $I = \{1, \dots, r\}$, and write $V_i := \varphi_i(V(F) \setminus U) \subseteq V(H)$ for $1 \leq i \leq r$. Note that $V_1, \dots, V_r$ are pairwise disjoint. We also put $W := \varphi_1(U) = \cdots = \varphi_r(U)$. Now, fix any $1 \leq i \leq r$ and set $V := V_1 \cup \cdots \cup V_i \cup W$. Then $|V| = |U| + i \cdot (v(F) - |U|) = i \cdot v(F) - (i-1) \cdot |U|$ and $e_H(V) \geq e_F(U) + i \cdot (e(F) - e_F(U)) = i \cdot e(F) - (i-1) \cdot e_F(U)$. It follows that

$$\begin{aligned} |V| - e_H(V) &\leq i \cdot (v(F) - e(F)) - (i-1)(|U| - e_F(U)) = i \cdot k - (i-1) \cdot \Delta(U) \\ &\leq i \cdot k - (i-1) \cdot k = k. \end{aligned}$$

Setting $q := |U|$, we note that $q = |U| \geq \Delta(U) \geq k$ and $q \leq v(F) - 1$ (as $U \neq V(F)$). Now we see that the assertion of Item 1 of the lemma holds with this choice of q. This completes the proof in the case that $\mathcal{G}$ has a vertex of degree at least d.

From now on we assume that the maximum degree of $\mathcal{G}$ is strictly smaller than d and prove that the assertion of Item 2 in the lemma holds. Let $\mathcal{F}^* \subseteq \mathcal{F}$ be an independent set[6] of $\mathcal{G}$ of size at least $v(\mathcal{G})/d = |\mathcal{F}|/d$. Recall that we identify $V(F)$ with $[v]$ and A with $[k+1]$. We now define an auxiliary k-uniform $(k+1)$-partite hypergraph J with parts $C_1, \dots, C_{k+1}$, as follows. For each $\varphi \in \mathcal{F}^*$, put a k-uniform $(k+1)$-clique in J on the vertices $\varphi(1) \in C_1, \dots, \varphi(k+1) \in C_{k+1}$. We denote this clique by K_φ. Note that by the definition of J, every edge of J is contained in a copy of F in H, which corresponds to some embedding $\varphi \in \mathcal{F}^*$.

Our first goal is to show that the cliques $(K_\varphi : \varphi \in \mathcal{F}^*)$ are pairwise edge-disjoint. So fix any distinct $\varphi_1, \varphi_2 \in \mathcal{F}^*$ and suppose, for the sake of contradiction, that the cliques $K_{\varphi_1}, K_{\varphi_2}$ share an edge. Then there is $W \subseteq A = [k+1]$ of size $|W| = k$ such that $\varphi_1(i) = \varphi_2(i)$ for every $i \in W$. It follows that $W \subseteq U := U(\varphi_1, \varphi_2)$ and hence $|U \cap A| \geq |W| = k$. But this means that φ_1 and φ_2 are adjacent in $\mathcal{G}$, in contradiction to the fact that $\mathcal{F}^*$ is an independent set of $\mathcal{G}$.

We have thus shown that the cliques $(K_\varphi : \varphi \in \mathcal{F}^*)$ are pairwise edge-disjoint. It follows that J contains a collection of $|\mathcal{F}^*| \geq |\mathcal{F}|/d \geq 2^{-v(1+2^v r)} \cdot v^{-v} \cdot \varepsilon n^k$ pairwise edge-disjoint $(k+1)$-cliques. By Theorem 3 and our choice of $\delta = \delta(F, r, \varepsilon)$, the number of $(k+1)$-cliques in J is at least $2\delta n^{k+1}$.

A $(k+1)$-clique K in J is called *colorful* if it is not equal to K_φ for any $\varphi \in \mathcal{F}^*$. Note that all but at most n^k of the $(k+1)$-cliques in J are colorful (because the non-colorful cliques are pairwise edge-disjoint). It follows that J contains at least $2\delta n^{k+1} - n^k \geq \delta n^{k+1}$ colorful $(k+1)$-cliques (here we use our choice of n_0).

Fix any colorful $(k+1)$-clique $K = \{c_1, \dots, c_{k+1}\}$, with c_i being the unique vertex in $K \cap C_i$ for each $1 \leq i \leq k+1$. By the definition of J, for each $i \in [k+1]$ there is $\varphi_i \in \mathcal{F}^*$ such that $\varphi_i(j) = c_j$ for every $j \in [k+1] \setminus \{i\}$. We claim that $\varphi_1, \dots, \varphi_{k+1}$ are pairwise distinct. Suppose, for the sake of contradiction, that $\varphi_i = \varphi_{i'} =: \varphi$ for some $1 \leq i < i' \leq k+1$. Then, for each $1 \leq j \leq k+1$, we have $\varphi(j) = c_j$ because one of i, i' does not equal j. So we see that $K = K_\varphi$, in contradiction to the assumption that K is colorful. We conclude that $\varphi_1, \dots, \varphi_{k+1}$ are indeed pairwise distinct. It now follows that $\varphi_i(i) \neq c_i$ for each $1 \leq i \leq k+1$. Indeed, if $\varphi_i(i) = c_i$ then, fixing any $j \in [k+1] \setminus \{i\}$, we observe that $\varphi_i(\ell) = \varphi_j(\ell)$ for each $\ell \in [k+1] \setminus \{j\}$, in contradiction to the fact that K_{φ_i} and K_{φ_j} are edge-disjoint.

Recall that F' consists of vertices $x_1, \dots, x_{k+1}, x'_1, \dots, x'_{k+1}$ and copies $F_1, \dots, F_{k+1}$ of F such that the vertex playing the role of $j \in [k+1] \subseteq V(F)$ in F_i is x_j if $j \neq i$ and x'_j if $j = i$ (for every $1 \leq i, j \leq k+1$) and $F_1, \dots, F_{k+1}$ do not intersect outside of $X = \{x_1, \dots, x_{k+1}\}$. Now let $\varphi = \varphi_K : V(F') \to V(H)$ be the function which, for each $1 \leq i \leq k+1$, maps x_i to c_i and agrees with φ_i on the vertices of F_i (where we identify $V(F_i)$ with $V(F)$). Then $\varphi(x_i) = c_i$ and $\varphi(x'_i) = \varphi_i(i)$ for each $1 \leq i \leq k+1$. It is not hard to see that in order to show that φ is an embedding of F' into H it is enough to verify that $\mathrm{Im}(\varphi_i) \cap \mathrm{Im}(\varphi_j) = \{c_1, \dots, c_{k+1}\} \setminus \{c_i, c_j\}$ for each $1 \leq i < j \leq k+1$. So fix any $1 \leq i < j \leq k+1$ and consider the set $U = U(\varphi_i, \varphi_j) = \{\ell \in V(F) : \varphi_i(\ell) = \varphi_j(\ell)\}$. Then

[6] Here we use the simple fact (which was already used in Section 2.4) that every graph G has an independent set of size at least $v(G)/(\Delta(G)+1)$, where $\Delta(G)$ is the maximum degree of G.

$U \cap [k+1] = [k+1] \setminus \{i,j\}$ and, in particular, $|U \cap A| = k-1$. If $U = U \cap [k+1]$, then we are done (because in this case we would have $\mathrm{Im}(\varphi_i) \cap \mathrm{Im}(\varphi_j) = \{c_1, \dots, c_{k+1}\} \setminus \{c_i, c_j\}$, as required). On the other hand, if $U \neq U \cap [k+1]$, then $U \setminus A \neq \emptyset$, which implies that φ_i and φ_j are adjacent in $\mathcal{G}$, in contradiction to the fact that $\varphi_i, \varphi_j \in \mathcal{F}^*$ and that $\mathcal{F}^*$ is an independent set of $\mathcal{G}$. We have thus shown that each colorful $(k+1)$-clique in J gives rise to a copy of F' in H. It is also easy to see that these copies are pairwise distinct. It follows that H contains at least δn^{k+1} copies of F'.

3.3 Proof of Lemma 3.2

In the proof of Lemma 3.2, we will need the following simple claim that can be verified by exhausting all possible cases. The proof is thus omitted.

Claim 3.3. *Consider the* 3*-uniform linear* 3*-cycle on vertices* $v_1, \dots, v_6$*, as depicted in Figure 1, and let* $U \subseteq \{v_1, \dots, v_6\}$*. Then* $\Delta(U) \geq |U \cap \{v_1, \dots, v_4\}| - \mathbb{1}_{\{v_1, \dots, v_4\} \subseteq U}$*. Moreover, if* $U \setminus \{v_1, \dots, v_4\} \neq \emptyset$ *and either* $v_1 \notin U$ *or* $U \cap \{v_2, v_3\} = \emptyset$*, then* $\Delta(U) \geq |U \cap \{v_1, \dots, v_4\}| + 1$*.*

Let F denote the 3-uniform linear 3-cycle (see Figure 1). Claim 3.3 implies that F satisfies Condition 1 in Definition 2.3 with respect to $A = \{v_1, \dots, v_4\}$. However, F does *not* satisfy Condition 2 in that definition, as evidenced, e.g., by the set $U = \{v_1, v_2, v_5\}$. So the "moreover"-part of Claim 3.3 can be thought of as a (non-equivalent) variant of Condition 2 in Definition 2.3. We also note that by going over all possible choices of A, one can easily verify that F is *not* nice.

Proof of Lemma 3.2. Let F be the 3-graph depicted in Figure 2, having vertices

$$w_1, w_2, w_3, w_4, w'_1, w'_2, w'_3, w'_4, x_5, x_6, y_5, y_6, z_5, z_6,$$

and edges

$$\{w_1, w_2, x_5\}, \{x_5, w'_4, x_6\}, \{x_6, w_3, w_1\}, \{x_5, w_4, y_6\}, \{y_6, w'_3, w_1\},$$
$$\{w_1, w'_2, y_5\}, \{y_5, w_4, x_6\}, \{w'_1, w_2, z_5\}, \{z_5, w_4, z_6\}, \{z_6, w_3, w'_1\}.$$

Then $v(F) = 14$ and $e(F) = 10$. Solymosi and Solymosi [19] (implicitly) proved that for every 3-graph H with $n \geq n_0(r, \varepsilon)$ vertices and at least εn^2 edges, either H satisfies the assertion of Item 1 in the lemma or H contains at least $\delta(r, \varepsilon) \cdot n^4$ copies of F (with $\delta(r, \varepsilon)$ being roughly $\gamma(3, \varepsilon/r)$, where γ is from Theorem 3). So, in order to complete the proof, it is enough to show that F is nice.

We prove that F satisfies the requirements of Definition 2.3 with $A := \{w_4, w'_1, w'_2, w'_3, w'_4\}$. To this end, define $V_1 = \{w'_1, w_2, z_5, w_4, z_6, w_3\}$, $V_2 = \{w_1, w'_2, y_5, w_4, x_6, w_3\}$, $V_3 = \{w_1, w_2, x_5, w_4, y_6, w'_3\}$ and $V_4 = \{w_1, w_2, x_5, w'_4, x_6, w_3\}$. Observe that $F[V_i]$ is a linear 3-cycle for every $1 \leq i \leq 4$. Furthermore, considering the vertex-labeling of the linear 3-cycle in Figure 1, we see that for each $1 \leq i, j \leq 4$, the role of v_j in $F[V_i]$ is played by w_j if $j \neq i$ and by w'_j if $j = i$. Now fix any $U \subseteq V(F)$ and let us show that U satisfies Items 1-2 in Definition 2.3. For each $1 \leq i \leq 4$, define $U_i = U \cap V_i$ and $A_i := (\{w_1, \dots, w_4\} \setminus \{w_i\}) \cup \{w'_i\}$. Note that by Claim 3.3 we have $\Delta(U_i) \geq |U_i \cap A_i| - \mathbb{1}_{A_i \subseteq U_i}$.

Let us now express $\Delta(U)$ in terms of $\Delta(U_1), \dots, \Delta(U_4)$. It is easy to check that

$$|U| = \sum_{i=1}^{4} |U_i| - 2 \cdot |U \cap \{w_1, \dots, w_4\}| - |U \cap \{x_5, x_6\}| \tag{12}$$

and

$$e(U) = \sum_{i=1}^{4} e(U_i) - \mathbb{1}_{\{w_1, w_2, x_5\} \subseteq U} - \mathbb{1}_{\{w_1, w_3, x_6\} \subseteq U}. \tag{13}$$

Setting $r := \sum_{i=1}^{4} (\Delta(U_i) - |U_i \cap A_i|)$ and

$$t := |U \cap \{w_1, w_2, w_3\}| - |U \cap \{x_5, x_6\}| + \mathbb{1}_{\{w_1,w_2,x_5\} \subseteq U} + \mathbb{1}_{\{w_1,w_3,x_6\} \subseteq U},$$

we combine (12) and (13) to obtain

$$\begin{aligned}
\Delta(U) &= \sum_{i=1}^{4} \Delta(U_i) - 2 \cdot |U \cap \{w_1, \dots, w_4\}| - |U \cap \{x_5, x_6\}| + \mathbb{1}_{\{w_1,w_2,x_5\} \subseteq U} + \mathbb{1}_{\{w_1,w_2,x_6\} \subseteq U} \\
&= \sum_{i=1}^{4} |U_i \cap A_i| + r - 2 \cdot |U \cap \{w_1, \dots, w_4\}| - |U \cap \{w_1, w_2, w_3\}| + t \\
&= |U \cap A| + r + t. \qquad (14)
\end{aligned}$$

To complete the proof, it is enough to show that $r + t \geq -\mathbb{1}_{A \subseteq U}$ and that $r + t \geq 1$ if $|U \cap A| \leq 3$ and $U \setminus A \neq \emptyset$. In what follows we will frequently use the fact that $\Delta(U_i) \geq |U_i \cap A_i| - \mathbb{1}_{A_i \subseteq U_i}$ for each $1 \leq i \leq 4$, as mentioned above. We consider two cases, depending on whether $w_1 \in U$ or not. Suppose first that $w_1 \notin U$. In this case we have $t = |U \cap \{w_2, w_3\}| - |U \cap \{x_5, x_6\}|$. Furthermore, $A_i \not\subseteq U_i$ for each $2 \leq i \leq 4$, which implies that $\Delta(U_i) \geq |U_i \cap A_i|$ for these values of i. Note that if $x_5 \in U$, then $U_i \setminus A_i \neq \emptyset$ for $i = 3, 4$, so, by the "moreover"-part of Claim 3.3 (and as $w_1 \notin U$), we have $\Delta(U_i) \geq |U_i \cap A_i| + 1$ for these values of i. Similarly, if $x_6 \in U$, then $\Delta(U_i) \geq |U_i \cap A_i| + 1$ for $i = 2, 4$. Altogether, we conclude that $r \geq |U \cap \{x_5, x_6\}| + 1 - \mathbb{1}_{U \cap \{x_5,x_6\} = \emptyset} - \mathbb{1}_{A_1 \subseteq U_1}$ and hence

$$r + t \geq |U \cap \{w_2, w_3\}| + 1 - \mathbb{1}_{U \cap \{x_5,x_6\} = \emptyset} - \mathbb{1}_{A_1 \subseteq U_1}. \qquad (15)$$

If $A_1 \subseteq U_1$, then $\{w_2, w_3\} \subseteq U$ and hence $r + t \geq 1$. So we assume from now on that $A_1 \not\subseteq U_1$. It then easily follows from (15) that $r + t \geq 1$ unless $U \cap \{w_2, w_3, x_5, x_6\} = \emptyset$. Suppose then that $U \cap \{w_2, w_3, x_5, x_6\} = \emptyset$ and note that in this case $r \geq 0$ and $t = 0$, so in particular $r + t \geq 0 \geq -\mathbb{1}_{A \subseteq U}$. Furthermore, if $U \setminus A \neq \emptyset$, then $U \setminus (A_1 \cup \cdots \cup A_4) \neq \emptyset$ (because $U \cap \{w_1, w_2, w_3\} = \emptyset$), so there must be some $1 \leq i \leq 4$ such that $U_i \setminus A_i \neq \emptyset$. Now Claim 3.3 implies that $\Delta(U_i) \geq |U_i \cap A_i| + 1$ and hence $r \geq 1$. We conclude that if $U \setminus A \neq \emptyset$, then $r + t \geq 1$, as required.

Having handled the case that $w_1 \notin U$, we assume from now on that $w_1 \in U$. Here we consider several subcases, depending on the intersection of U with $\{w_2, w_3\}$. Suppose first that $U \cap \{w_2, w_3\} = \emptyset$. Then $A_i \not\subseteq U_i$ for each $1 \leq i \leq 4$, implying that $r \geq 0$. Furthermore, $t = 1 - |U \cap \{x_5, x_6\}|$. So if $U \cap \{x_5, x_6\} = \emptyset$, then $r + t \geq 1$ and we are done. On the other hand, if $U \cap \{x_5, x_6\} \neq \emptyset$, then $U_4 \setminus A_4 \neq \emptyset$, which implies, by Claim 3.3, that $\Delta(U_4) \geq |U_4 \cap A_4| + 1$. This shows that $r + t \geq 0 \geq -\mathbb{1}_{A \subseteq U}$ and in fact $r + t \geq 1$ if $|U \cap \{x_5, x_6\}| \leq 1$. So from now on we assume that $\{x_5, x_6\} \subseteq U$ and show that $r + t \geq 1$ unless $|U \cap A| \geq 4$. As $\{x_5, x_6\} \subseteq U$, we have $U_i \setminus A_i \neq \emptyset$ for $i = 2, 3$. It now follows from Claim 3.3 that for each $i = 2, 3$, if $w'_i \notin U$, then $\Delta(U_i) \geq |U_i \cap A_i| + 1$, which, combined with $\Delta(U_4) \geq |U_4 \cap A_4| + 1$, implies that $r \geq 2$ and hence $r + t \geq 1$. So, we are done unless $w'_2, w'_3 \in U$. If $w_4 \notin U$, then $U_2 = \{w_1, w'_2, y_5, x_6\}$ and hence $\Delta(U_2) = 3 = |U_2 \cap A_2| + 1$. But this implies that $r \geq 2$, again giving $r + t \geq 1$. Therefore, we may assume that $w_4 \in U$. Similarly, if $w'_4 \notin U$, then $U_4 = \{w_1, x_5, x_6\}$, from which it follows that $\Delta(U_4) = 3 = |U_4 \cap A_4| + 2$ and hence $r \geq 2$. So again, we may assume that $w'_4 \in U$. Altogether, we see that $r + t \geq 1$ unless $\{w'_2, w'_3, w_4, w'_4\} \subseteq U$, which only holds if $|U \cap A| \geq 4$.

Suppose now that $|U \cap \{w_2, w_3\}| = 1$. By symmetry, we may assume without loss of generality that $w_2 \in U$ and $w_3 \notin U$. Then $t = 2 - \mathbb{1}_{x_6 \in U}$ and $A_i \not\subseteq U_i$ for every $i \in \{1, 2, 4\}$. It follows that $r + t \geq 2 - \mathbb{1}_{x_6 \in U} - \mathbb{1}_{A_3 \subseteq U_3}$ and hence $r + t \geq 1$ unless $x_6 \in U$ and $A_3 \subseteq U_3$. Suppose then that $x_6 \in U$ and $\{w'_3, w_4\} \subseteq A_3 \subseteq U_3 \subseteq U$. As $x_6 \in U$, we have $U_2 \setminus A_2 \neq \emptyset$. Therefore, if $w'_2 \notin U$, then by

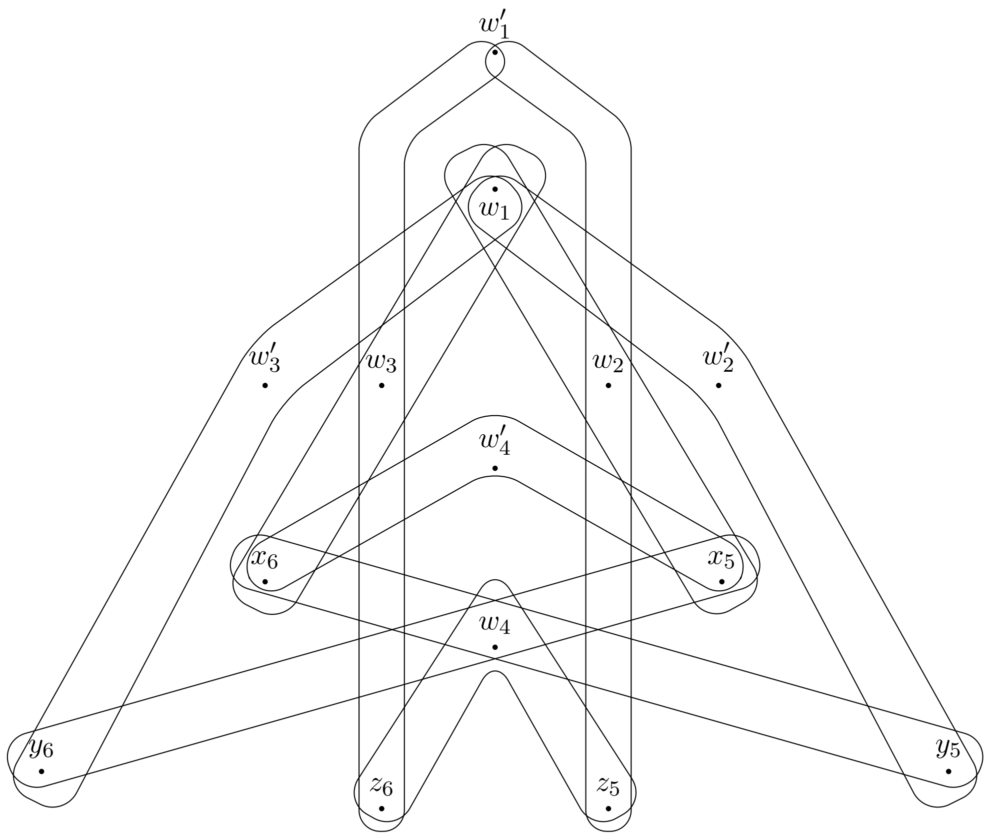

Figure 2: The (14,10)-configuration used in Lemma 3.2

Claim 3.3 we have $\Delta(U_2) \geq |U_2 \cap A_2| + 1$, which implies that $r \geq 0$ and hence $r + t \geq 1$. So we may assume that $w_2' \in U$. Similarly, if $w_4' \notin U$, then either $U_4 = \{w_1, w_2, x_6\}$ or $U_4 = \{w_1, w_2, x_5, x_6\}$. Since in both cases $\Delta(U_4) = |U_4 \cap A_4| + 1$, we infer that if $w_4' \notin U$, then $r \geq 0$ and hence $r + t \geq 1$. Overall, we see that $r + t \geq 1$ unless $\{w_2', w_3', w_4, w_4'\} \subseteq U$, as required.

It remains to handle the case that $\{w_2, w_3\} \subseteq U$. In this case, we have $t = 3$, so $r + t \geq 0$ unless $r = -4$. But if $r = -4$, then $A_i \subseteq U_i$ for each $1 \leq i \leq 4$, which implies that $A \subseteq U$. So we see that $r + t \geq -\mathbb{1}_{A \subseteq U}$, as required. Furthermore, if $|U \cap A| \leq 3$, then $\#\{1 \leq i \leq 4 : A_i \subseteq U_i\} \leq 2$ (indeed, if $A_i \subseteq U_i$ for at least 3 indices $1 \leq i \leq 4$, then $|U \cap \{w_1', \ldots, w_4'\}| \geq 3$ and $w_4 \in U$, implying that $|U \cap A| \geq 4$), so in fact we have $r \geq -2$ and hence $\Delta(U) \geq |U \cap A| + 1$. This completes the proof. ∎

4 Proof of Lemma 2.6

In this section, we prove Lemma 2.6 through a sequence of claims. We start by defining the 3-graphs $(G_\ell)_{\ell \geq 0}$ appearing in the statement of the lemma. Very roughly speaking, G_ℓ can be thought of as the 3-graph obtained by starting with a complete k-ary tree of height ℓ and replacing each of its vertices by a copy of G.

In each of the graphs G_ℓ we identity a special subset of vertices which will play a crucial role. More precisely, for every $\ell \geq 0$, the graph G_ℓ will contain a subset of vertices $A_\ell \subseteq V(G_\ell)$ which we will denote by $x_1, \ldots, x_k$ and $y_0, \ldots, y_\ell$. If G^* is a copy of some G_ℓ, then we will use $x_i(G^*)$

and $y_i(G^*)$ to denote the vertices of G^* playing the roles of x_i and y_i in G^*. We will also set $A_\ell(G^*) = \{x_1(G^*), \dots, x_k(G^*), y_0(G^*), \dots, y_\ell(G^*)\}$. When both G^* and the value of ℓ are clear from the context, we will simply write $A_\ell, x_1, \dots, x_k, y_0, \dots, y_\ell$.

Recall that G is assumed to be nice; so let $A \subseteq V(G)$ be as in Definition 2.3, noting that $|A| = k+1$ and that A is an independent set. Assuming the vertices of A are (arbitrarily) named $x_1, \dots, x_k, y_0$, we now set G_0 to be G, $y_0(G_0)$ to be y_0 and $x_i(G_0)$ to be x_i for every $1 \le i \le k$. In particular, $A_0(G_0) = A$. Proceeding by induction, we fix $\ell \ge 1$ and assume that $G_{\ell-1}$, as well as the vertices $x_i(G_{\ell-1})$ and $y_i(G_{\ell-1})$ (and thus also the set $A_{\ell-1}(G_{\ell-1})$), have already been defined. Now G_ℓ is defined as follows. Start with a set of $k+\ell+1$ vertices $x_1, \dots, x_k$ and $y_0, \dots, y_\ell$. We will set $x_i(G_\ell)$ to be x_i for every $1 \le i \le k$ and $y_i(G_\ell)$ to be y_i for every $0 \le i \le \ell$. In addition to these $k+\ell+1$ vertices, we also have k additional vertices $x'_1, \dots, x'_k$. For each $1 \le i \le k$, add a copy of $G_{\ell-1}$, denoted $G^i_{\ell-1}$, in which x_j plays the role of $x_j(G_{\ell-1})$ for each $j \in [k] \setminus \{i\}$, x'_i plays the role of $x_i(G_{\ell-1})$, y_j plays the role of $y_j(G_{\ell-1})$ for each $0 \le j \le \ell-1$ and all other $v(G_{\ell-1}) - k - \ell$ vertices are "new". As a last step, add a copy G^ℓ of G in which x_i plays the role of $x_i(G)$ for each $1 \le i \le k$, y_ℓ plays the role of $y_0(G)$ and all other $v(G) - k - 1$ vertices are "new". The resulting 3-graph is G_ℓ.

Claim 4.1. *For every $\ell \ge 0$, the set $A_\ell(G_\ell) \subseteq V(G_\ell)$ is independent and the graph G_ℓ satisfies the assertion of Item 1 of Lemma 2.6.*

Proof. We first prove by induction on ℓ that $A_\ell(G_\ell)$ is an independent set. For $\ell = 0$, this is guaranteed by our choice of $A_0(G_0) = A$. So fixing $\ell \ge 1$ and assuming the claim holds for $\ell - 1$, we now prove it for ℓ. By the definition of G_ℓ, each edge of G_ℓ belongs to one of the 3-graphs $G^1_{\ell-1}, \dots, G^k_{\ell-1}, G^\ell$. Moreover, we have $V(G^i_{\ell-1}) \cap A_\ell(G_\ell) \subseteq A_{\ell-1}(G^i_{\ell-1})$ for every $1 \le i \le k$ and similarly $V(G^\ell) \cap A_\ell(G_\ell) = A_0(G^\ell)$. So the fact that $A_\ell(G_\ell)$ is independent follows from the induction hypothesis for $\ell - 1$ and from the case $\ell = 0$.

Since $A_\ell(G_\ell)$ is independent, the subgraphs $G^1_{\ell-1}, \dots, G^k_{\ell-1}, G^\ell$, which comprise G_ℓ, are pairwise edge-disjoint. This implies that $e(G_\ell) = k \cdot e(G_{\ell-1}) + e(G)$. We now prove the two assertions of Item 1 of the lemma by induction on ℓ. The case $\ell = 0$ is immediate. As for the induction step, observe that for each $\ell \ge 1$, we have

$$e(G_\ell) = k \cdot e(G_{\ell-1}) + e(G) = \left(k \cdot \frac{k^\ell - 1}{k-1} + 1\right) \cdot e(G) = \frac{k^{\ell+1} - 1}{k-1} \cdot e(G),$$

where the second equality follows from the induction hypothesis for $\ell - 1$. Moreover, we have

$$\begin{aligned} v(G_\ell) &= 2k + \ell + 1 + k \cdot (v(G_{\ell-1}) - k - \ell) + v(G) - k - 1 \\ &= k + \ell + k \cdot (v(G_{\ell-1}) - k - \ell + 1) + v(G) - k \\ &= k + \ell + k \cdot e(G_{\ell-1}) + e(G) = k + \ell + e(G_\ell). \end{aligned}$$

Here we used the fact that $\Delta(G) = k$ and the induction hypothesis that $\Delta(G_{\ell-1}) = k + \ell - 1$. The above two expressions for $e(G_\ell)$ and $v(G_\ell)$ imply both assertions of Item 1. ■

Item 2 of Lemma 2.6 follows from the following stronger claim.

Claim 4.2. *Let $\ell \ge 1$ and $e(G_{\ell-1})/e(G) < t \le e(G_\ell)/e(G)$. Then there is a subgraph G' of G_ℓ such that $v(G') - e(G') \le k + \ell$, $e(G') = t \cdot e(G)$ and $A_\ell(G_\ell) \subseteq V(G')$.*

Before proving Claim 4.2, let us use this claim to establish the assertion of Item 2 of the lemma by induction on ℓ. The case $\ell = 0$ is trivial, so let $\ell \ge 1$ and $1 \le t \le e(G_\ell)/e(G)$. If $t > e(G_{\ell-1})/e(G)$, then the assertion of Item 2 follows from Claim 4.2 and if $t \le e(G_{\ell-1})/e(G)$, then it follows from the induction hypothesis for $\ell - 1$ and the fact that G_ℓ contains a copy of $G_{\ell-1}$.

In the proof of Claim 4.2, we will need the following simple claim. Recall that $G^1_{\ell-1}, \ldots, G^k_{\ell-1}$ are the copies of $G_{\ell-1}$ which feature in the definition of G_ℓ.

Claim 4.3. *Let $0 \le \ell' < \ell$. Then G_ℓ contains a copy G^* of $G_{\ell'}$ such that $V(G^*) \subseteq V(G^k_{\ell-1})$, $x_i(G^*) = x_i(G_\ell)$ for each $1 \le i \le k-1$ and $y_i(G^*) = y_i(G_\ell)$ for each $0 \le i \le \ell'$.*

Proof. The proof is by induction on ℓ, with the base case $\ell = 0$ holding vacuously. Let $0 \le \ell' < \ell$. If $\ell' = \ell - 1$ then $G^* = G^k_{\ell-1}$ is easily seen to satisfy the requirements of the claim. Suppose then that $\ell' \le \ell - 2$. By the induction hypothesis, $G_{\ell-1}$ contains a copy G^{**} of $G_{\ell'}$ such that $x_i(G^{**}) = x_i(G_{\ell-1})$ for each $1 \le i \le k-1$ and $y_i(G^{**}) = y_i(G_{\ell-1})$ for each $0 \le i \le \ell'$. Let G^* be the subgraph playing the role of G^{**} in the copy $G^k_{\ell-1}$ of $G_{\ell-1}$. Then it is evident that $V(G^*) \subseteq V(G^k_{\ell-1})$. Moreover, for each $1 \le i \le k-1$, we have $x_i(G^*) = x_i(G^k_{\ell-1}) = x_i(G_\ell)$, where the first equality follows from our choice of G^* and the second equality follows from the definition of G_ℓ. A similar argument shows that $y_i(G^*) = y_i(G^k_{\ell-1}) = y_i(G_\ell)$ for each $0 \le i \le \ell'$. ∎

Proof of Claim 4.2. The proof is by induction on ℓ. We start with the base case $\ell = 1$. Let $1 < t \le e(G_1)/e(G) = k+1$. Recall that $G^0_1, \ldots, G^0_k$ and G^1 are the copies of $G_0 = G$ which feature in the definition of G_1. Let G' be the subgraph of G_1 consisting of $G^1_0, \ldots, G^{t-1}_0$ and G^1. Then $e(G') = (t-1) \cdot e(G) + e(G) = t \cdot e(G)$. Moreover, $A_1(G_1) = \{x_1(G_1), \ldots, x_k(G_1), y_0(G_1), y_1(G_1)\} \subseteq V(G')$ because $\{x_1(G_1), \ldots, x_k(G_1), y_1(G_1)\} \subseteq V(G^1) \subseteq V(G')$ and $y_0(G_1) \in V(G^1_0) \subseteq V(G')$ (here we are using the fact that $t \ge 2$). Finally, note that

$$v(G') = |A_1(G_1)| + (t-1) \cdot (v(G) - k) + (v(G) - k - 1) = k + 1 + t \cdot e(G) = e(G') + k + 1,$$

as required.

Now let $\ell \ge 2$ and let t be such that

$$(k^\ell - 1)/(k-1) = e(G_{\ell-1})/e(G) < t \le e(G_\ell)/e(G) = (k^{\ell+1} - 1)/(k-1).$$

Here the equalities follow from Item 1 of the lemma. Let d be the unique integer satisfying

$$d \cdot (k^\ell - 1)/(k-1) + 1 \le t < (d+1) \cdot (k^\ell - 1)/(k-1) + 1$$

and note that $1 \le d \le k$, where the first inequality follows from the assumption $t > (k^\ell - 1)/(k-1)$ and the second inequality follows from the assumption $t \le (k^{\ell+1} - 1)/(k-1) = k \cdot (k^\ell - 1)/(k-1) + 1$. Set

$$t' = t - d \cdot (k^\ell - 1)/(k-1) - 1, \tag{16}$$

noting that $0 \le t' < (k^\ell - 1)/(k-1)$.

Suppose for now that $t' > 0$. Then there is $\ell' \ge 1$ such that $e(G_{\ell'-1})/e(G) < t' \le e(G_{\ell'})/e(G)$. Note also that $\ell' < \ell$ because $t' < (k^\ell - 1)/(k-1)$. By Claim 4.3, G_ℓ contains a copy G^* of $G_{\ell'}$ such that $V(G^*) \subseteq V(G^k_{\ell-1})$, $x_i(G^*) = x_i(G_\ell)$ for each $1 \le i \le k-1$ and $y_i(G^*) = y_i(G_\ell)$ for each $0 \le i \le \ell'$. By the induction hypothesis for ℓ' (which we apply to the copy G^* of $G_{\ell'}$), there is a subgraph G'' of G^* such that $v(G'') - e(G'') \le k + \ell'$, $e(G'') = t' \cdot e(G)$ and $A_{\ell'}(G^*) \subseteq V(G'')$. This last property of G'' implies that $x_i(G_\ell) = x_i(G^*) \in V(G'')$ for each $1 \le i \le k-1$ and $y_i(G_\ell) = y_i(G^*) \in V(G'')$ for each $0 \le i \le \ell'$. In particular, $|V(G'') \cap A_\ell(G_\ell)| \ge k + \ell'$.

Now, let G' be the subgraph of G_ℓ consisting of G^ℓ, of $G^1_{\ell-1}, \ldots, G^d_{\ell-1}$ and, in the case that $t' > 0$, of the 3-graph G'' chosen in the previous paragraph. Note that if $t' > 0$ then $d \le k-1$ (this follows from the definitions of d and t'). Combining this with the fact that $V(G'') \subseteq V(G^*) \subseteq V(G^k_{\ell-1})$, we infer that G'' is edge-disjoint from $G^1_{\ell-1}, \dots, G^d_{\ell-1}, G^\ell$ (which are themselves pairwise edge-disjoint by the definition of G_ℓ). This in turn implies that

$$e(G') = d \cdot e(G_{\ell-1}) + e(G) + e(G'') \cdot \mathbb{1}_{t'>0} = \left(d \cdot \frac{k^\ell - 1}{k-1} + t' + 1\right) \cdot e(G) = t \cdot e(G). \tag{17}$$

Here, the second equality follows from Item 1 of the lemma and from our choice of G'', while the last equality uses our choice of t' in (16). Next, we observe that $A_\ell(G_\ell) \subseteq V(G')$. Indeed, this follows from the fact that $A_\ell(G_\ell) \setminus \{x_1(G_\ell), y_\ell(G_\ell)\} \subseteq V(G^1_{\ell-1}) \subseteq V(G')$ (recall that $d \geq 1$) and that $x_1(G_\ell), y_\ell(G_\ell) \in V(G^\ell) \subseteq V(G')$. Finally, it remains to estimate $v(G') - e(G')$. To this end, note that

$$\begin{aligned} v(G') &= |A_\ell(G_\ell)| + d \cdot (v(G_{\ell-1}) - k - \ell + 1) + (v(G) - k - 1) + |V(G'') \setminus A_\ell(G_\ell)| \cdot \mathbb{1}_{t'>0} \\ &\leq k + \ell + d \cdot (v(G_{\ell-1}) - k - \ell + 1) + (v(G) - k) + (v(G'') - k - \ell') \cdot \mathbb{1}_{t'>0} \\ &\leq k + \ell + d \cdot e(G_{\ell-1}) + e(G) + e(G'') \cdot \mathbb{1}_{t'>0} = e(G') + k + \ell, \end{aligned}$$

where in the first equality we used the definition of G'; in the first inequality we used the fact that $|A_\ell(G_\ell)| = k + \ell + 1$ and $|V(G'') \cap A_\ell(G_\ell)| \geq k + \ell'$; in the second inequality we used the guarantees of Item 1 of the lemma and the fact that $v(G'') - e(G'') \leq k + \ell'$; and in the last equality we used (17). We have thus shown that $v(G') - e(G') \leq k + \ell$. This completes the proof of the claim. ■

The rest of this section is devoted to establishing Item 3 of the lemma. To this end, we first prove the following claim, which shows that the niceness of G (with respect to the set A) is carried over to some extent to all G_ℓ. From now on, we will write $A_\ell = \{x_1, \dots, x_k, y_0, \dots, y_\ell\}$ (omitting G_ℓ from the notation). We also set $X := \{x_1, \dots, x_k\}$.

Claim 4.4. *Let $\ell \geq 0$ and let $U \subseteq V(G_\ell)$ be such that $\{y_0, \dots, y_{\ell-1}\} \subseteq U$. Then*

1. *$\Delta(U) \geq |U \cap A_\ell| - \mathbb{1}_{\{x_1,\dots,x_k,y_\ell\} \subseteq U}$. In particular, if $|U \cap A_\ell| \geq k + \ell$, then $\Delta(U) \geq k + \ell$.*
2. *If $|U \cap X| \leq k - 2$ and $U \setminus A_\ell \neq \emptyset$, then $\Delta(U) \geq |U \cap A_\ell| + 1$.*
3. *If $|U \cap X| \geq k - 1$ and $U \cap V(G^\ell)$ is not contained in X, then $\Delta(U) \geq k + \ell$.*

Proof. We first prove Items 1-2 by induction on ℓ and then use these items to derive Item 3. In the base case $\ell = 0$, Items 1-2 immediately follow from the fact that $G_0 = G$ is nice and from our choice of $A_0 = A$ via Definition 2.3. Let now $\ell \geq 1$ and let $U \subseteq V(G_\ell)$. We start with Item 1. For $1 \leq i \leq k$, put $U_i := U \cap V(G^i_{\ell-1})$. Similarly, put $U_0 := U \cap V(G^\ell)$ and note that

$$|U \cap A_\ell| = |U_0 \cap \{x_1, \dots, x_k, y_\ell\}| + \ell \tag{18}$$

because $y_0, \dots, y_{\ell-1} \in U$ by assumption. Since A_ℓ is independent (see Claim 4.1), we have $e(U) = \sum_{i=0}^k e(U_i)$. Observe also that

$$|U| = \sum_{i=0}^k |U_i| - (k-1) \cdot (|U \cap X| + |U \cap \{y_0, \dots, y_{\ell-1}\}|),$$

as each element of $X \cup \{y_0, \dots, y_{\ell-1}\}$ is contained in exactly k of the sets $V(G^1_{\ell-1}), \dots, V(G^k_{\ell-1}), V(G^\ell)$ and each of the other vertices of G_ℓ is contained in exactly one of these sets. From the above formulas for $e(U)$ and $|U|$, it follows that

$$\Delta(U) = \sum_{i=0}^k \Delta(U_i) - (k-1) \cdot (|U \cap X| + \ell). \tag{19}$$

Here we used the fact that $\{y_0, \ldots, y_{\ell-1}\} \subseteq U$ by assumption. Recall that by the definition of G_ℓ, for each $1 \le i \le k$, we have

$$A_{\ell-1}(G^i_{\ell-1}) = \{x_1, \ldots, x_k, y_0, \ldots, y_{\ell-1}, x'_i\} \setminus \{x_i\}.$$

By the induction hypothesis for $\ell - 1$, applied to the copy $G^i_{\ell-1}$ of $G_{\ell-1}$, we get

$$\Delta(U_i) \ge |U_i \cap A_{\ell-1}(G^i_{\ell-1})| - \mathbb{1}_{A_{\ell-1}(G^i_{\ell-1}) \subseteq U_i} \ge |U_i \cap (A_\ell \setminus \{x_i, y_\ell\})|, \tag{20}$$

where the second inequality follows by considering whether $x'_i \in U_i$ or not. From (20), we obtain

$$\begin{aligned} \sum_{i=1}^{k} \Delta(U_i) &\ge \sum_{i=1}^{k} |U_i \cap (A_\ell \setminus \{x_i, y_\ell\})| \\ &= (k-1) \cdot |U \cap X| + k \cdot |U \cap \{y_0, \ldots, y_{\ell-1}\}| \\ &= (k-1) \cdot |U \cap X| + k\ell, \end{aligned} \tag{21}$$

where in the first equality we used the fact that each element of X belongs to exactly $k-1$ of the sets $A_\ell \setminus \{x_i, y_\ell\}$ (where $1 \le i \le k$) and each element of $\{y_0, \ldots, y_{\ell-1}\}$ belongs to all of these sets. Plugging the above into (19) gives

$$\Delta(U) \ge \Delta(U_0) + \ell. \tag{22}$$

Since G is nice and G^ℓ is a copy of G in which y_ℓ plays the role of $y_0(G)$, we have

$$\Delta(U_0) \ge |U_0 \cap \{x_1, \ldots, x_k, y_\ell\}| - \mathbb{1}_{\{x_1,\ldots,x_k,y_\ell\} \subseteq U_0}. \tag{23}$$

By combining (18), (22) and (23), we get

$$\Delta(U) \ge \Delta(U_0) + \ell \ge |U_0 \cap \{x_1, \ldots, x_k, y_\ell\}| - \mathbb{1}_{\{x_1,\ldots,x_k,y_\ell\} \subseteq U} + \ell = |U \cap A_\ell| - \mathbb{1}_{\{x_1,\ldots,x_k,y_\ell\} \subseteq U},$$

thus establishing Item 1.

Next, we prove Item 2. Suppose then that $|U \cap X| \le k-2$ and $U \setminus A_\ell \neq \emptyset$. The inequality $|U \cap X| \le k-2$ implies that $|U_0 \cap \{x_1, \ldots, x_k, y_\ell\}| \le k-1$ and that $A_{\ell-1}(G^i_{\ell-1}) \not\subseteq U_i$ for each $1 \le i \le k$. Since $U \setminus A_\ell \neq \emptyset$, there is $0 \le i \le k$ such that $U_i \setminus A_\ell \neq \emptyset$. Suppose first that $i = 0$. Then $U_0 \setminus \{x_1, \ldots, x_k, y_\ell\} \neq \emptyset$, which, combined with $|U_0 \cap \{x_1, \ldots, x_k, y_\ell\}| \le k-1$, implies that $\Delta(U_0) \ge |U_0 \cap \{x_1, \ldots, x_k, y_\ell\}| + 1$. Here we used the niceness of G (see Item 2 in Definition 2.3). By plugging our bound on $\Delta(U_0)$ into (22) and using (18), we get $\Delta(U) \ge \Delta(U_0) + \ell \ge |U_0 \cap \{x_1, \ldots, x_k, y_\ell\}| + 1 + \ell = |U \cap A_\ell| + 1$, as required. Now suppose that $1 \le i \le k$. We claim that

$$\Delta(U_i) \ge |U_i \cap (A_\ell \setminus \{x_i, y_\ell\})| + 1. \tag{24}$$

In other words, we show that the inequality bounding the leftmost term in (20) by the rightmost one is strict. If $x'_i \in U_i$, then

$$\Delta(U_i) \ge |U_i \cap A_{\ell-1}(G^i_{\ell-1})| - \mathbb{1}_{A_{\ell-1}(G^i_{\ell-1}) \subseteq U_i} = |U_i \cap A_{\ell-1}(G^i_{\ell-1})| \ge |U_i \cap (A_\ell \setminus \{x_i, y_\ell\})| + 1,$$

as required. Here, in the first inequality we used (20), in the equality we used the fact that $A_{\ell-1}(G^i_{\ell-1}) \not\subseteq U_i$ (as mentioned above) and in the last inequality we used the fact that $x'_i \in A_{\ell-1}(G^i_{\ell-1}) \setminus A_\ell$. So suppose now that $x'_i \notin U_i$ and note that in this case $U_i \setminus A_{\ell-1}(G^i_{\ell-1}) \neq \emptyset$ because $U_i \setminus A_\ell \neq \emptyset$ and $A_{\ell-1}(G^i_{\ell-1}) \subseteq A_\ell \cup \{x'_i\}$. Moreover, the intersection of U_i with the set

$\{x_1(G^i_{\ell-1}),\dots,x_k(G^i_{\ell-1})\} = \{x_1,\dots,x_k,x'_i\} \setminus \{x_i\}$ is of size at most $k-2$, because $|U \cap X| \le k-2$. So by the induction hypothesis, applied to the copy $G^i_{\ell-1}$ of $G_{\ell-1}$, we have

$$\Delta(U_i) \ge |U_i \cap A_{\ell-1}(G^i_{\ell-1})| + 1 \ge |U_i \cap (A_\ell \setminus \{x_i, y_\ell\})| + 1,$$

where the last inequality uses (20). We have thus proven (24). By repeating the calculation in (21) and plugging in (24) and (20) (which we use for each $j \in [k] \setminus \{i\}$), we obtain

$$\begin{aligned}\Delta(U) &= \sum_{i=0}^{k} \Delta(U_i) - (k-1)\cdot(|U \cap X| + \ell) \ge \Delta(U_0) + \ell + 1 \\ &\ge |U_0 \cap \{x_1,\dots,x_k,y_\ell\}| + \ell - \mathbb{1}_{\{x_1,\dots,x_k,y_\ell\}\subseteq U_0} + 1 \\ &= |U_0 \cap \{x_1,\dots,x_k,y_\ell\}| + \ell + 1 = |U \cap A_\ell| + 1.\end{aligned}$$

Here, the second inequality uses (23) and the last equality uses (18). This completes the inductive proof of Items 1-2.

It remains to deduce Item 3 from Items 1-2. Suppose then that $|U \cap X| \ge k-1$ and that $U_0 \not\subseteq X$. If $X \subseteq U$ or $y_\ell \in U$, then $|U \cap A_\ell| \ge k+\ell$, in which case Item 1 implies that $\Delta(U) \ge k+\ell$, as required. So we may assume that $|U \cap X| = k-1$ and $y_\ell \notin U$. Since U_0 is not contained in X, we must have $U_0 \setminus \{x_1,\dots,x_k,y_\ell\} \neq \emptyset$. So by the niceness of G we have $\Delta(U_0) \ge |U_0 \cap \{x_1,\dots,x_k,y_\ell\}| + 1 = k$. Plugging this into (22) gives $\Delta(U) \ge k+\ell$, as required. ∎

Item 3 of the lemma will be derived from the following claim, in a manner similar to the derivation of Lemma 2.4 from Lemma 3.1.

Claim 4.5. *For every $\ell \ge 0$, $r \ge 0$ and $\varepsilon \in (0,1)$, there are $\delta = \delta(\ell, r, \varepsilon)$ and $n_0 = n_0(\ell, r, \varepsilon)$ such that, for every 3-graph H on $n \ge n_0$ vertices, if H contains at least $\varepsilon n^{k+\ell}$ copies of G_ℓ, then (at least) one of the following conditions is satisfied:*

1. *There is $k+\ell \le q \le v(G_\ell)-1$ such that, for every $1 \le i \le r$, the 3-graph H contains a (v', e')-configuration which contains a copy of G_ℓ, where $v' - e' \le k+\ell$ and $v' = q + i\cdot(v(G_\ell) - q)$.*
2. *H contains at least $\delta \cdot n^{k+\ell+1}$ copies of $G_{\ell+1}$.*

Proof. We proceed similarly to the proof of Lemma 3.1. Fixing $\ell \ge 0$, we set $v := v(G_\ell)$,

$$\zeta := 2^{-v(1+2^v r)} \cdot v^{-v} \cdot \varepsilon,$$

$\delta = \delta(\ell, r, \varepsilon) = \frac{\zeta}{4}\cdot\gamma\left(k, \frac{\zeta}{2}\right)$ and $n_0 = n_0(\ell, r, \varepsilon) = \frac{2}{\gamma(k,\frac{\zeta}{2})}$, where γ is from Theorem 3.

Let H be a 3-graph on $n \ge n_0$ vertices, which contains at least $\varepsilon n^{k+\ell}$ copies of G_ℓ. Partition the vertices of H randomly into sets $(C_z : z \in V(G_\ell))$ by choosing, for each vertex $x \in V(H)$, a vertex $z \in V(G_\ell)$ uniformly at random and independently (of the choices made for all other vertices of H) and placing x in part C_z. A copy of G_ℓ in H will be called *good* if, for each $z \in V(G_\ell)$, the vertex playing the role of z in this copy belongs to C_z. Since H contains at least $\varepsilon n^{k+\ell}$ copies of G_ℓ, there are in expectation at least $v^{-v}\cdot\varepsilon n^{k+\ell}$ good copies of G_ℓ. So fix a partition $(C_z : z \in V(G_\ell))$ with at least this number of good copies of G_ℓ and denote the set of these copies by $\mathcal{F}$. We will identify each good copy of G_ℓ with the corresponding embedding $\varphi : V(G_\ell) \to V(H)$, while noting that $\varphi(z) \in C_z$ for each $z \in V(G_\ell)$. Recall that G^ℓ is the copy of G featured in the definition of G_ℓ. Define an auxiliary graph $\mathcal{G}$ on $\mathcal{F}$ as follows. For each pair of distinct $\varphi_1, \varphi_2 \in \mathcal{F}$, we set $U(\varphi_1, \varphi_2) := \{z \in V(G_\ell) : \varphi_1(z) = \varphi_2(z)\}$ and let $\{\varphi_1, \varphi_2\}$ be an edge in $\mathcal{G}$ if and only if $U := U(\varphi_1, \varphi_2)$ satisfies $\{y_0, \dots, y_{\ell-1}\} \subseteq U$, as well as (at least) one of the following three conditions:

(i) $|U \cap A_\ell| \geq k + \ell$.

(ii) $y_\ell \in U$ and either $|U \cap X| \geq k - 1$ or $|U \cap X| = k - 2$ and $U \setminus A_\ell \neq \emptyset$.

(iii) $|U \cap X| \geq k - 1$ and $U \cap V(G^\ell)$ is not contained in X.

Suppose first that there is $\varphi \in \mathcal{F}$ whose degree in $\mathcal{G}$ is at least

$$d := 2^{v(1+2^v r)}.$$

Let $\varphi_1, \dots, \varphi_d$ be distinct neighbours of φ in $\mathcal{G}$. By the pigeonhole principle, there is $I' \subseteq [d]$ of size at least $2^{-v}d = 2^{v2^v r}$ and a set $U' \subseteq V(G_\ell)$ such that, for all $i \in I'$, it holds that $U(\varphi, \varphi_i) = U'$. As in the proof of Lemma 3.1, we consider the coloring $\{i,j\} \mapsto U(\varphi_i, \varphi_j)$ of the pairs $\{i,j\} \in \binom{I'}{2}$ and use a bound for multicolor Ramsey numbers [5] to obtain a set $I \subseteq I'$ of size $|I| = r$ and a set $U \subseteq V(G_\ell)$ such that $U(\varphi_i, \varphi_j) = U$ for all $\{i,j\} \in \binom{I}{2}$. Observe that for each $\{i,j\} \in \binom{I}{2}$, we have $U \supseteq U(\varphi, \varphi_i) \cap U(\varphi, \varphi_j) = U'$. In particular, $\{y_0, \dots, y_{\ell-1}\} \subseteq U' \subseteq U$ (by the definition of $\mathcal{G}$). Note also that $U \neq V(G_\ell)$ because the copies $(\varphi_i : i \in I)$ of G_ℓ are distinct.

We now use Claim 4.4 to prove that $\Delta(U) \geq k + \ell$. The definition of the graph $\mathcal{G}$ implies that the set U' must satisfy one of the conditions (i)-(iii) above. Note that for each of these three conditions, if it is satisfied by U', then it is also satisfied by every superset of U' and, in particular, by U. Now, if U satisfies Condition (i) (resp. (iii)), then the bound $\Delta(U) \geq k + \ell$ immediately follows from Item 1 (resp. 3) of Claim 4.4. Suppose then that U satisfies Condition (ii). If $|U \cap X| \geq k - 1$, then $|U \cap A_\ell| \geq k + \ell$ (since Condition (ii) supposes that $y_\ell \in U$), so again we can apply Item 1 of Claim 4.4. Finally, if $|U \cap X| = k - 2$ and $U \setminus A_\ell \neq \emptyset$, then we have $\Delta(U) \geq |U \cap A_\ell| + 1 = k + \ell$, where the inequality is given by Item 2 of Claim 4.4 and the equality holds because $\{y_0, \dots, y_\ell\} \subseteq U$ and $|U \cap X| = k - 2$. We have thus shown that $\Delta(U) \geq k + \ell$ in all cases.

Suppose without loss of generality that $I = [r]$. Put $W := \varphi_1(U) = \cdots = \varphi_r(U)$ and denote $V_i := \varphi_i(V(G_\ell) \setminus U) \subseteq V(H)$ for each $1 \leq i \leq r$. Note that $V_1, \dots, V_r$ are pairwise disjoint. Now, fix any $1 \leq i \leq r$ and set $V := V_1 \cup \cdots \cup V_i \cup W$. Then

$$|V| = |U| + i \cdot (v(G_\ell) - |U|) = i \cdot v(G_\ell) - (i-1) \cdot |U|$$

and

$$e_H(V) \geq e(U) + i \cdot (e(G_\ell) - e(U)) = i \cdot e(G_\ell) - (i-1) \cdot e(U).$$

It follows that

$$\begin{aligned} |V| - e_H(V) &\leq i \cdot (v(G_\ell) - e(G_\ell)) - (i-1)(|U| - e(U)) = i \cdot (k+\ell) - (i-1) \cdot \Delta(U) \\ &\leq i \cdot (k+\ell) - (i-1) \cdot (k+\ell) = k + \ell. \end{aligned}$$

Moreover, it is evident that $H[V]$ contains a copy of G_ℓ. Finally, note that $|U| \geq \Delta(U) \geq k + \ell$ and $|U| \leq v(G_\ell) - 1$ (because $U \neq V(G_\ell)$, as mentioned above). Combining all the above, we see that the assertion of Item 1 in the claim holds with $q := |U|$. This completes the proof in the case that $\mathcal{G}$ has a vertex of degree at least d.

From now on we assume that the maximum degree of $\mathcal{G}$ is strictly smaller than d. Let $\mathcal{F}^* \subseteq \mathcal{F}$ be an independent set in $\mathcal{G}$ of size at least $v(\mathcal{G})/d = |\mathcal{F}|/d$. For each ℓ-tuple of vertices $u = (u_0, \dots, u_{\ell-1}) \in \tilde{C} := C_{y_0} \times \cdots \times C_{y_{\ell-1}}$, we denote by $\mathcal{F}^*(u)$ the set of all $\varphi \in \mathcal{F}^*$ such that $\varphi(y_i) = u_i$ for each $0 \leq i \leq \ell - 1$. Note that

$$\sum_{u \in \tilde{C}} |\mathcal{F}^*(u)| = |\mathcal{F}^*| \geq \frac{|\mathcal{F}|}{d} \geq \frac{\varepsilon n^{k+\ell}}{v^v d} = \zeta n^{k+\ell}\,. \tag{25}$$

We claim that $|\mathcal{F}^*(u)| \le n^k$ for each $u \in \tilde{C}$. To see this, fix any such u and let $\varphi_1, \varphi_2 \in \mathcal{F}^*(u)$ be distinct. If $\varphi_1(x_i) = \varphi_2(x_i)$ for each $1 \le i \le k$, then $\{x_1, \dots, x_k, y_0, \dots, y_{\ell-1}\} \subseteq U(\varphi_1, \varphi_2)$. But then U satisfies Condition (i) above, implying that $\{\varphi_1, \varphi_2\} \in E(\mathcal{G})$, in contradiction to the fact that $\mathcal{F}^*$ is an independent set in $\mathcal{G}$. So we see that for each $u \in \tilde{C}$ and for each $\varphi \in \mathcal{F}^*(u)$, the values of $\varphi(x_1), \dots, \varphi(x_k)$ determine φ uniquely. It follows that indeed $|\mathcal{F}^*(u)| \le n^k$. Now, by using (25) and averaging, we get that there are at least $\frac{\zeta}{2}n^\ell$ tuples $u \in \tilde{C}$ which satisfy $|\mathcal{F}^*(u)| \ge \frac{\zeta}{2}n^k$. Let $C \subseteq \tilde{C}$ be the set of all such tuples u. We will show that for every $u = (u_0, \dots, u_{\ell-1}) \in C$, there are at least $\frac{1}{2}\gamma(k, \frac{\zeta}{2}) \cdot n^{k+1}$ copies of $G_{\ell+1}$ in H in which u_i plays the role of $y_i(G_{\ell+1})$ for every $0 \le i \le \ell - 1$. Combining this with the fact that $|C| \ge \frac{\zeta}{2}n^\ell$, we will conclude that H contains at least $\frac{\zeta}{2}n^\ell \cdot \frac{1}{2}\gamma(k, \frac{\zeta}{2}) \cdot n^{k+1} = \delta n^{k+\ell+1}$ copies of $G_{\ell+1}$, as required.

Fix any $u \in C$. We define an auxiliary k-uniform $(k+1)$-partite hypergraph $J(u)$ with parts $C_{x_1}, \dots, C_{x_k}, C_{y_\ell}$, as follows. For each $\varphi \in \mathcal{F}^*(u)$, put a k-uniform $(k+1)$-clique in $J(u)$ on the vertices $\varphi(x_1) \in C_{x_1}, \dots, \varphi(x_k) \in C_{x_k}$, $\varphi(y_\ell) \in C_{y_\ell}$. We denote this clique by K_φ. We claim that the cliques $(K_\varphi : \varphi \in \mathcal{F}^*(u))$ are pairwise edge-disjoint. To this end, fix any pair of distinct $\varphi_1, \varphi_2 \in \mathcal{F}^*(u)$ and suppose, for the sake of contradiction, that the cliques $K_{\varphi_1}, K_{\varphi_2}$ share an edge. Then there is $Z \subseteq \{x_1, \dots, x_k, y_\ell\}$ of size $|Z| = k$ such that $\varphi_1(z) = \varphi_2(z)$ for every $z \in Z$. It follows that $Z \cup \{y_0, \dots, y_{\ell-1}\} \subseteq U(\varphi_1, \varphi_2)$. Therefore, $|U(\varphi_1, \varphi_2) \cap A_\ell| \ge k + \ell$, implying that $U(\varphi_1, \varphi_2)$ satisfies Condition (i) above. This in turn implies that $\{\varphi_1, \varphi_2\} \in E(\mathcal{G})$, which contradicts the fact that $\mathcal{F}^*(u) \subseteq \mathcal{F}(u)$ is an independent set in $\mathcal{G}$. We have thus shown that the cliques $(K_\varphi : \varphi \in \mathcal{F}^*(u))$ are indeed pairwise edge-disjoint.

It follows from the previous paragraph that $J(u)$ contains a collection of $|\mathcal{F}^*(u)| \ge \frac{\zeta}{2}n^k$ pairwise edge-disjoint $(k+1)$-cliques. By Theorem 3, the number of $(k+1)$-cliques in $J(u)$ is at least $\gamma(k, \frac{\zeta}{2}) \cdot n^{k+1}$. A $(k+1)$-clique K in $J(u)$ is called *colorful* if it is not equal to K_φ for any $\varphi \in \mathcal{F}^*(u)$. Since there are at most $|\mathcal{F}^*(u)| \le n^k$ non-colorful $(k+1)$-cliques, the number of colorful $(k+1)$-cliques in $J(u)$ is at least $\gamma(k, \frac{\zeta}{2}) \cdot n^{k+1} - n^k \ge \frac{1}{2}\gamma(k, \frac{\zeta}{2}) \cdot n^{k+1}$ (here we use our choice of n_0).

To complete the proof, it remains to show that each colorful $(k+1)$-clique in $J(u)$ corresponds to a copy of $G_{\ell+1}$ in H. Fix any colorful $(k+1)$-clique $K = \{w_1, \dots, w_k, u_\ell\}$, where u_ℓ is the unique vertex of K contained in C_{y_ℓ} and, for each $1 \le i \le k$, w_i is the unique vertex of K contained in C_{x_i}. By the definition of $J(u)$, each of the $k+1$ edges of K corresponds to an embedding of G_ℓ into H. More precisely, there are $\varphi_0, \varphi_1, \dots, \varphi_k \in \mathcal{F}^*(u)$ such that:

- For each $1 \le i \le k$, $\varphi_i(y_\ell) = u_\ell$ and $\varphi_i(x_j) = w_j$ for each $j \in [k] \setminus \{i\}$.
- $\varphi_0(x_i) = w_i$ for each $1 \le i \le k$.

We claim that $\varphi_0, \dots, \varphi_k$ are pairwise distinct. Assume, for the sake of contradiction, that $\varphi_i = \varphi_{i'} =: \varphi$ for some $0 \le i < i' \le k$. Then $\varphi(x_j) = w_j$ for each $1 \le j \le k$. Indeed, this follows from the two items above and from the (obvious) fact that one of i, i' does not equal j. Similarly, since i, i' cannot both equal 0, the first item above implies that $\varphi(y_\ell) = u_\ell$. We now see that $K = K_\varphi$, in contradiction to the assumption that K is colorful. Hence, $\varphi_0, \dots, \varphi_k$ are indeed pairwise distinct. Now the edge-disjointness of the cliques $K_{\varphi_0}, K_{\varphi_1}, \dots, K_{\varphi_k}$ implies that $w'_i := \varphi_i(x_i) \ne w_i$ for each $1 \le i \le k$ and that $u_{\ell+1} := \varphi_0(y_\ell) \ne u_\ell$.

We now show how to construct a copy of $G_{\ell+1}$ using the copies of G_ℓ corresponding to $\varphi_1, \dots, \varphi_k$ and the copy of G corresponding to $\varphi_0(G^\ell)$. In this copy of $G_{\ell+1}$, the role of $x_i(G_{\ell+1})$ will be played by w_i for every $1 \le i \le k$, the role of the vertex $x'_i \in V(G_{\ell+1})$ will be played by w'_i for every $1 \le i \le k$ (recall the definition of $G_{\ell+1}$) and the role of $y_i(G_{\ell+1})$ will be played by u_i for every $0 \le i \le \ell + 1$. (Recall that the vertices $u_0, \dots, u_{\ell-1}$ have already been fixed via the choice of u.) Note that for each

$1 \leq i \leq k$, the embedding φ_i corresponds to a copy of G_ℓ in which w_j plays the role of $x_j(G_\ell)$ for every $j \in [k] \setminus \{i\}$, w'_i plays the role of $x_i(G_\ell)$ and u_j plays the role of $y_j(G_\ell)$ for every $0 \leq j \leq \ell$. This copy of G_ℓ will play the role of G^i_ℓ in our copy of $G_{\ell+1}$. Similarly, restricting φ_0 to $V(G^\ell)$ gives a copy of G in which w_i plays the role of $x_i(G)$ for each $1 \leq i \leq k$ and $u_{\ell+1}$ plays the role of $y_0(G)$ (as $y_0(G^\ell) = y_\ell(G_\ell)$ and $\varphi_0(y_\ell(G_\ell)) = u_{\ell+1}$). By the definition of $G_{\ell+1}$, in order to show that $\mathrm{Im}(\varphi_1) \cup \cdots \cup \mathrm{Im}(\varphi_k) \cup \varphi_0(V(G^\ell))$ spans a copy of $G_{\ell+1}$, it suffices to verify that the k copies of G_ℓ given by $\varphi_1, \ldots, \varphi_k$, and the copy of G given by $\varphi_0(G^\ell)$, do not intersect outside of $\{w_1, \ldots, w_k, u_0, \ldots, u_\ell\}$. Therefore, our goal is to show that $\mathrm{Im}(\varphi_i) \cap \mathrm{Im}(\varphi_j) = \{w_1, \ldots, w_k, u_0, \ldots, u_\ell\} \setminus \{w_i, w_j\}$ for each $1 \leq i < j \leq k$ and that $\mathrm{Im}(\varphi_i) \cap \varphi_0(V(G^\ell)) = \{w_1, \ldots, w_k\} \setminus \{w_i\}$ for each $1 \leq i \leq k$. We start with the former statement. Fix any $1 \leq i < j \leq k$. Setting $U := U(\varphi_i, \varphi_j)$, note that $\mathrm{Im}(\varphi_i) \cap \mathrm{Im}(\varphi_j) = \varphi_i(U) = \varphi_j(U)$, that $y_0, \ldots, y_\ell \in U$ and that $U \cap X = X \setminus \{x_i, x_j\}$ and hence $|U \cap X| = k - 2$. If we had $U \setminus A_\ell \neq \emptyset$, then U would satisfy Condition (ii) above, which in turn would imply that $\{\varphi_i, \varphi_j\} \in E(\mathcal{G})$, thus contradicting the fact that $\mathcal{F}^*(u) \subseteq \mathcal{F}^*$ is an independent set in $\mathcal{G}$. So we see that $U \subseteq A_\ell$ and therefore $U = A_\ell \setminus \{x_i, x_j\}$. This in turn is equivalent to having $\mathrm{Im}(\varphi_i) \cap \mathrm{Im}(\varphi_j) = \{w_1, \ldots, w_k, u_0, \ldots, u_\ell\} \setminus \{w_i, w_j\}$, as required.

Let us now show that $\mathrm{Im}(\varphi_i) \cap \varphi_0(V(G^\ell)) = \{w_1, \ldots, w_k\} \setminus \{w_i\}$ holds for every $1 \leq i \leq k$. Fixing $1 \leq i \leq k$, set $U := U(\varphi_i, \varphi_0)$ and note that $A_\ell \setminus \{x_i, y_\ell\} = \{x_1, \ldots, x_k, y_0, \ldots, y_{\ell-1}\} \setminus \{x_i\} \subseteq U$. Now, if $U \cap V(G^\ell)$ were not contained in X, then U would satisfy Condition (iii) above, which would imply the false statement that $\{\varphi_i, \varphi_0\} \in E(\mathcal{G})$. So we see that $U \cap V(G^\ell) \subseteq X$. Moreover, $x_i \notin U$, because otherwise the $(k+1)$-cliques corresponding to φ_i and φ_0, respectively, would not be edge-disjoint (or, alternatively, because otherwise U would satisfy Condition (i) above). So we see that $U \cap V(G^\ell) = \{x_1, \ldots, x_k\} \setminus \{x_i\}$, which implies that $\mathrm{Im}(\varphi_i) \cap \varphi_0(V(G^\ell)) = \{w_1, \ldots, w_k\} \setminus \{w_i\}$. ■

Finally, we use Claim 4.5 in order to establish Item 3 of the lemma by induction on ℓ. The case $\ell = 0$ is trivial. Let us now fix $\ell \geq 0$, assume the validity of Item 3 for ℓ and prove the analogous statement for $\ell + 1$. It is easy to see that if the assertion of 3(a) holds for parameter ℓ, then it also holds for parameter $\ell + 1$. So we may assume that the assertion of Item 3(b) holds, namely, that H contains at least $\varepsilon' \cdot n^{k+\ell}$ copies of G_ℓ (where $\varepsilon' := \delta(\ell, r, \varepsilon)$, as given by Item 3 in the lemma). So we may apply Claim 4.5 to H (with parameter ε' in place of ε). If Item 1 of Claim 4.5 holds, then Item 3(a) of Lemma 2.6 holds with $\ell + 1$ in place of ℓ (and with $j = \ell$). If instead Item 2 of Claim 4.5 holds, then H contains at least $\delta \cdot n^{k+\ell+1}$ copies of $G_{\ell+1}$, as required by Item 3(b) in Lemma 2.6. This completes the proof of the lemma.

5 An Improved Bound for a Problem of Erdős and Gyárfás

The Brown–Erdős–Sós problem has a known connection to (a special case of) the following generalized Ramsey problem, introduced by Erdős and Gyárfás in [8]. Let $g(n, p, q)$ denote the minimum number of colors in a coloring of the edges of K_n in which every copy of K_p receives at least q colors. For a fixed $p \geq 4$, Erdős and Gyárfás [8] showed that $g(n, p, q)$ is quadratic in n if and only if $q \geq q_{\text{quad}}(p) := \binom{p}{2} - \lfloor \frac{p}{2} \rfloor + 2$ and that $\Omega(n^2) \leq g(n, p, q_{\text{quad}}(p)) \leq \binom{n}{2} - \varepsilon n^2$ for some $\varepsilon = \varepsilon(p) > 0$. They then asked for which $q_{\text{quad}}(p) < q \leq \binom{p}{2}$ it holds that $g(n, p, q) = \binom{n}{2} - o(n^2)$, observing that this question is related to the Brown–Erdős–Sós problem and using this relation to prove several partial results. The relation was further exploited by Sárközy and Selkow, who combined it with (1) (or, more precisely, with a 4-uniform analogue thereof) to show that $g(n, p, q) = \binom{n}{2} - o(n^2)$ whenever $q > q_{\text{quad}}(p) + \lceil \frac{\log_2 p}{2} \rceil$. By using our improved bound for the Brown–Erdős–Sós problem (i.e., Corollary 2), we can improve upon the result of Sárközy and Selkow [17]. For completeness, we now sketch the proof of the reduction from the above generalized Ramsey problem to the Brown–Erdős–Sós problem. This reduction has been used implicitly in [8, 17].

Proposition 5.1. *Let $p \geq 4$ and $q_{quad}(p) < q \leq \binom{p}{2}$. Set $e := \binom{p}{2} - q + 1$. If $f_4(n, p, e) = o(n^2)$, then $g(n, p, q) = \binom{n}{2} - o(n^2)$.*

Proof. Assume that $f_4(n, p, e) = o(n^2)$ and suppose, for the sake of contradiction, that (for infinitely many n) there is a coloring of the edges of K_n with $t := \binom{n}{2} - \varepsilon n^2$ colors (where $\varepsilon > 0$ is fixed) in which every copy of K_p receives at least q colors. Then at least εn^2 edges have the same color as some other edge.

Observe that each color appears fewer than $\lfloor \frac{p}{2} \rfloor$ times. Indeed, otherwise take edges $e_1, \dots, e_{\lfloor \frac{p}{2} \rfloor}$, all having the same color, and supplement them with (a suitable number of) vertices to obtain a copy of K_p which receives at most $\binom{p}{2} - \lfloor \frac{p}{2} \rfloor + 1 < q_{\text{quad}}(p) < q$, a contradiction. It follows that at least $\varepsilon n^2 / \lfloor p/2 \rfloor \geq 2\varepsilon n^2/p$ colors appear at least twice. For each such color c, fix a pair of distinct edges (e_1^c, e_2^c) which are colored with c. We claim that there are less than $(p-1)n/2$ colors c for which e_1^c and e_2^c intersect. Indeed, assign to each such intersecting pair of edges their common vertex. If the number of intersecting pairs is at least $(p-1)n/2$, then there is a vertex u which is the common vertex for at least $\lfloor \frac{p-1}{2} \rfloor$ such edge-pairs. In other words, there are distinct vertices $(x_i, y_i : 1 \leq i \leq \lfloor \frac{p-1}{2} \rfloor)$ such that the color of $\{u, x_i\}$ is the same as that of y_i for each $1 \leq i \leq \lfloor \frac{p-1}{2} \rfloor$. As before, by adding a suitable number of vertices one obtains a copy of K_p which receives at most $\binom{p}{2} - \lfloor \frac{p-1}{2} \rfloor < q_{\text{quad}}(p) < q$ colors, in contradiction to our assumption.

It follows from the above two paragraphs that there are at least $2\varepsilon n^2/p - (p-1)n/2 \geq \varepsilon n^2/p$ colors c (appearing at least twice) for which e_1^c, e_2^c are disjoint. Define an auxiliary 4-graph H on $V(K_n)$ by putting a (4-uniform) edge on $e_1^c \cup e_2^c$ for each color c for which e_1^c, e_2^c are disjoint. Since K_4 has 3 perfect matchings, we have $e(H) \geq \frac{\varepsilon n^2}{3p}$. Observe, crucially, that H contains no (p, e)-configuration. Indeed, if H contained a (p, e)-configuration, then, by the definition of H and our choice of e, the vertex set of this configuration would correspond to a copy of K_p receiving at most $\binom{p}{2} - e = q - 1$ colors, which is impossible. We thus conclude that $e(H) \leq f_4(n, p, e)$. On the other hand, $e(H) \geq \frac{\varepsilon n^2}{3p}$, implying that $f_4(n, p, e) = \Omega(n^2)$, in contradiction to our assumption. ■

By Corollary 2, applied with parameters $r = 4$, $k = 2$ and $e = \binom{p}{2} - q + 1$, the bound $f_4(n, p, e) = o(n^2)$ holds whenever $p \geq 2e + 18 \log e / \log \log e = 2(\binom{p}{2} - q + 1) + 18 \log e / \log \log e$. By rearranging, we get the inequality $q \geq \binom{p}{2} - \frac{p}{2} + 1 + 18 \log e / \log \log e$. Recalling the value of $q_{\text{quad}}(p)$ and using the (obvious) fact that $e \leq \binom{p}{2}$, we see that this inequality holds whenever $q \geq q_{\text{quad}}(p) + C \log p / \log \log p$ for some suitable absolute constant C. By combining this with Proposition 5.1, we obtain the following improvement upon the aforementioned result from [17].

Theorem 4. *There is an absolute constant C such that $g(n, p, q) = \binom{n}{2} - o(n^2)$ for every $p \geq 4$ and $q \geq q_{quad}(p) + C \log p / \log \log p$.*

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