Non-diffusive pitch-angle scattering of a distribution of energetic particles by coherent whistler waves

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Key Points:

- A much improved condition is provided for how coherent whistler waves scatter the pitch-angle of energetic particles.
- A significant fraction of energetic, thermally distributed particles undergo this scattering.
- The theory reveals a critical mechanism not contained in the widely-used second-order trapping theory.

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Abstract

Whether or not coherent magnetospheric whistler waves play important roles in the pitch-angle scattering of energetic particles is a crucial question in magnetospheric physics. The interaction of a thermal distribution of energetic particles with coherent whistler waves is thus investigated. The distribution is prescribed by the Maxwell-Jüttner distribution, which is a relativistic generalization of the Maxwell-Boltzmann distribution. Coherent whistler waves are modeled by circularly polarized waves propagating parallel to the background magnetic field. It is shown that for parameters relevant to magnetospheric chorus, a significant fraction (1-5%) of the energetic particle population undergoes drastic, non-diffusive pitch-angle scattering by coherent chorus. The scaling of this fraction with the wave amplitude may also explain the association of relativistic microbursts to large-amplitude chorus. A much improved condition for large pitch-angle scattering is presented that is related to, but may or may not include the exact resonance condition depending on the particle's initial conditions. The theory reveals a critical mechanism not contained in the widely-used second-order trapping theory.

Plain Language Summary

A certain class of plasma waves called whistler waves is abundant in the Earth's magnetosphere. The interaction between whistler waves and energetic particles trapped in the Earth's magnetic field can cause the particles to escape the trap and cause pulsating auroras or damage spacecraft. Although previous studies have mostly focused on diffusive mechanisms, we show that a significant fraction of the energetic particles interacts non-diffusively or coherently with the wave. We also show that a widely-used condition for such interaction is incomplete and provide a more accurate alternative.

1 Introduction

Whistler waves are right-handed circularly polarized electromagnetic plasma waves that are ubiquitous in the Earth's magnetosphere (Gurnett & O'Brien, 1964; Burtis & Helliwell, 1969; Russell et al., 1969; Tsurutani & Smith, 1974), Jupiter's magnetosphere (Sentman & Goertz, 1978; Leubner, 1982; Tsurutani et al., 1993), and Saturn's magnetosphere (Barbosa & Kurth, 1993; Akalin et al., 2006; Hospodarsky et al., 2008). These waves are also important in the solar wind (Coroniti et al., 1982; Vocks et al., 2005), fast magnetic reconnection (Mandt et al., 1994; Bellan, 2014; Chai et al., 2016; Yoon & Bellan, 2017, 2018; Haw et al., 2019), and helicon plasma sources (Boswell, 1984; Chen & Boswell, 1997). In particular, the interaction between energetic charged particles and magnetospheric whistler waves is important since the interaction can change the pitch-angle of the particles, potentially scattering them into the loss cone of a magnetic mirror configuration such as the Earth's dipole magnetic field. Because the escaped energetic particles can cause pulsating auroras at the Earth's poles and energetic particles in general can damage spacecraft, this interaction has been the focus of many studies for decades (Kennel & Petschek, 1966; Lyons et al., 1971; Helliwell & Crystal, 1973; Lyons, 1974; Summers et al., 1998; Horne & Thorne, 2003; Albert, 2005; Omura & Summers, 2006; Tsurutani et al., 2013; A. V. Artemyev et al., 2013; A. Artemyev et al., 2016).

Relativistic wave-particle resonance has been known to be an important element of particle energization and pitch-angle scattering. Resonant interaction arises when

$$\omega - kv_z = \frac{\Omega}{\gamma}. \tag{1}$$

Here, ω is the wave frequency, k is the wavenumber parallel to the background magnetic field B_0 which is oriented in the z direction, v_z is the parallel particle velocity, and $\Omega = qB_0/m$ is the cyclotron frequency of the particle with charge q and mass m. Also, $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$ is the particle Lorentz factor where v is the particle speed and c is the speed of light. Kennel and Petschek (1966) first quantified the scattering mechanism by which incoherent whistler waves lead to velocity space diffusion, and numerous studies have further developed this mechanism (Lyons et al., 1971; Lyons, 1974; Albert, 2005; Tsurutani et al., 2009). However, recent spacecraft measurements indicate that the observed chorus bursts are, in fact, extremely coherent and that these waves, especially large-amplitude ones ($\delta B/B_0 \sim 0.01$ where δB is the wave magnetic field), are directly linked to electron energization, loss, and microbursts (Anderson & Milton, 1964; Cattell et al., 2008; Tsurutani et al., 2009, 2013; Gao et al., 2014; Breneman et al., 2017). This linkage suggests that a non-diffusive process could be governing what is observed.

There has thus been a continuing and substantial theoretical effort to investigate the dynamics of energetic particles under coherent whistler waves. Bortnik et al. (2008) numerically investigated *ad hoc* the coherent interaction between large-amplitude whistler waves and relativistic particles. Lakhina et al. (2010) showed via calculations of pitch-angle diffusion coefficients that coherent chorus subelements can cause rapid pitch angle scattering, although Lakhina et al. (2010) used diffusion coefficients calculated from incoherent whistler waves (Kennel & Petschek, 1966) and used non-relativistic equations of motion whereas the actual wave-particle interaction involves relativistic particles (10 keV to MeV (Tsurutani et al., 2013; Breneman et al., 2017)). Bellan (2013) presented an exact analytical calculation involving a relativistic particle in a right-handed circularly polarized electromagnetic wave. This calculation showed that a certain class of particles undergo quick, drastic pitch-angle scattering depending on whether the individual particle's initial conditions meet a certain criterion, which will be discussed in the next section. Also note that other studies have investigated this single-particle problem via various methods (Roberts & Buchsbaum, 1964; Ginet & Heinemann, 1990; Qian, 2000; Bourdier & Gond, 2000). However, an analysis of the importance of this mechanism for a distribution of particles has not yet been done. To demonstrate importance, one must show that a significant fraction of the particles in the distribution experiences this drastic scattering. If this can be demonstrated, then the particle interaction with coherent whistler waves will be a dominant pitch-angle scattering mechanism.

We extend in this paper the analysis presented in Bellan (2013) to the relativistic generalization of a thermal distribution of particles; the generalization is prescribed by the Maxwell-Jüttner distribution (Jüttner, 1911). It is found that for parameters relevant to magnetospheric chorus, coherent right-handed circularly polarized waves propagating parallel to the background magnetic field trigger large, non-diffusive pitch-angle scatterings for a significant fraction ($1\% - 5\%$) of the energetic particles. The scaling of this fraction with the wave amplitude may also explain the association of relativistic microbursts to large-amplitude chorus (Breneman et al., 2017). A new condition for large pitch-angle scattering is also presented; this condition specifies a certain range related to Eq. 1, but may or may not include exact resonance depending on the particle initial conditions. Test-particle simulations corroborate the predictions made by this analysis. It is also demonstrated that the widely-used second-order trapping theory (Sudan & Ott, 1971; Nunn, 1974; Omura et al., 1991, 2007, 2008) is a simplified approximation of the theory presented in this paper and that this simplified approximation effectively misses critical details of the wave-particle interaction. The present study illustrates that coherent whistler waves are an important cause of non-diffusive pitch-angle scattering and provides an accurate condition for this scattering.

2 Two-Valley Motion Review

Let us begin with a brief review of the large pitch-angle scattering mechanism presented in Bellan (2013). A thorough comprehension of this single-particle mechanism is essential for understanding the ensuing analysis presented here. It is assumed that the wave is right-handed circularly polarized and travels parallel to a uniform background magnetic field, so the total magnetic field can be expressed as $\mathbf{B} = B_0\hat{z} + \tilde{\mathbf{B}}$ where

$$\tilde{\mathbf{B}} = \kappa B_0 \left[\hat{x}\sin\left(kz - \omega t\right) + \hat{y}\cos\left(kz - \omega t\right)\right]. \tag{2}$$

Here κ is the wave amplitude relative to the background B_0. Faraday's law determines the wave electric field to be:

$$\tilde{\mathbf{E}} = -\frac{\omega}{k}\hat{z} \times \tilde{\mathbf{B}} = \frac{\omega}{k}\tilde{B}\left[\hat{x}\cos\left(kz - \omega t\right) - \hat{y}\sin\left(kz - \omega t\right)\right]. \tag{3}$$

The relativistic Lorentz force equation determines the motion of a charged particle:

$$\frac{d}{dt}\left(\gamma\boldsymbol{\beta}\right) = \frac{q}{m}\left(\frac{\tilde{\mathbf{E}}}{c} + \boldsymbol{\beta} \times \mathbf{B}\right) \tag{4}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ and $\gamma = \left(1 - \beta^2\right)^{-1/2}$.

In Bellan (2013), a left-handed circularly polarized wave was used although the study was intended for right-handed waves. However, the result therein is unaffected by this apparent error because the sign of the particle charge was unspecified. Although it was not explicitly stated, the analysis was carried out assuming that the charge is positive, e.g., for positrons or ions. If the charge is assumed to be negative, the same wave-particle interaction arises when the wave is assumed to have a right-handed polarization. Therefore, the theory in Bellan (2013) describes wave-particle interactions between positively charged particles and left-handed waves, and equivalently between negatively charged particles and right-handed waves — or electrons and right-handed whistler waves. This equivalence can also be seen using charge-parity-time symmetry, which is a fundamental law of any Lorentz-invariant system (Greenberg, 2002); making the changes $z \to -z$ and $t \to -t$ in Eqs. 2 and 3 changes the sense of rotation of the wave, and the relevant physics must be equivalent when the change $q \to -q$ is made.

In this paper, the analysis in Bellan (2013) with the left-handed wave and positively charged particles will be used for two reasons. First, the analysis can then be kept general for any particle with any sign of charge. Second, the derivation of a separate theory for negatively charged particles will merely be a matter of some sign changes and is not worth the additional complexity in understanding the core points of this paper.

In Bellan (2013), a "frequency mismatch" parameter

$$\xi = 1 + \alpha\gamma\left(n\beta_z - 1\right) \tag{5}$$

was defined, where $\alpha = \omega/\Omega$ is the normalized frequency, $\beta_z = v_z/c$ is the normalized parallel velocity, and $n = ck/\omega$ is the refractive index. Equation 1 is satisfied when $\xi = 0$, so ξ is a measure of the departure from resonance. An exact rearrangement of Eq. 4 leads to an equation of motion for a particle moving in ξ-space (Bellan, 2013):

$$\frac{1}{\Omega'}\frac{d^2\xi}{dt'^2} = -\frac{\partial\psi}{\partial\xi} \tag{6}$$

where

$$\psi\left(\xi\right) = \frac{1}{8}\xi^4 + \left(\kappa'^2 - \frac{\xi_0^2}{2} - s\kappa'\sin\phi_0\right)\frac{\xi^2}{2} - \kappa'^2\xi \tag{7}$$

is the pseudo-potential for ξ-space motion. Here the primed quantities are calculated in the wave frame, i.e., a frame moving with a velocity $\hat{z}\omega/k$. The subscript 0 refers to the value at the initial time $t = t' = 0$ and there are two parameters, namely s and ϕ_0. The parameter s is defined as

$$s = \frac{\alpha n\beta_{\perp 0}\gamma_0}{\gamma_T} = \frac{k\rho_0}{\gamma_T} \tag{8}$$

where $\gamma_T = \left(1 - n^{-2}\right)^{-1/2}$ is the Lorentz factor of the wave, and ρ is the relativistic Larmor radius. The parameter ϕ_0 is defined as the initial angular orientation of the perpendicular velocity in the $x - y$ plane, i.e., the angle between $\beta_{\perp 0}$ and $\tilde{E}(t = 0, z = 0)$. The shape of the pseudo-potential is entirely determined by the initial conditions of the particle with respect to the wave as prescribed by ξ_0, s, and ϕ_0. Note that s is an initial condition of the particle because α and n are fixed parameters in the present analysis.

Multiplying Eq. 6 by $d\xi/dt'$ and integrating with respect to t' yields the particle pseudo-energy,

$$W = \frac{1}{2\Omega'^2}\left(\frac{d\xi}{dt'}\right)^2 + \psi\left(\xi\right), \tag{9}$$

which is a constant of the motion. For certain initial conditions, $\psi\left(\xi\right)$ consists of two valleys separated by a hill in between. If the initial particle pseudo-energy is sufficiently large to go over the hill between the two valleys, then the particle undergoes two-valley motion in ξ-space. This motion involves large changes in ξ and thus in β_z, $\beta_\perp$ and the pitch-angle $\theta_{\mathrm{pitch}} = \tan^{-1}\beta_\perp/\beta_z$.

3 Two-Valley Motion Condition

Let us now derive the conditions for two-valley motion for a given particle. The conditions consist of two parts: $\psi(\xi)$ must first be two-valleyed, and the particle must have sufficient pseudo-energy to overcome the hill between the two-valleys. The initial particle kinetic pseudo-energy can be expressed as (Bellan, 2013)

$$\frac{1}{2\Omega'^2}\left(\frac{d\xi}{dt'}\right)^2_{t'=t=0} = \frac{1}{2}s^2\kappa'^2\cos^2\phi_0, \tag{10}$$

so the total pseudo-energy is

$$W = \frac{1}{2}s^2\kappa'^2\cos^2\phi_0 - \frac{\xi_0^4}{8} + \frac{\xi_0^2}{2}\kappa'\left(\kappa' - s\sin\phi_0\right) - \kappa'^2\xi_0. \tag{11}$$

We write Eq. 7 as $\psi\left(\xi\right) = \xi^4/8 + b\xi^2/2 - \kappa'^2\xi$ where $b = \kappa'^2 - \xi_0^2/2 - s\kappa'\sin\phi_0$. Then $d\psi/d\xi = \xi^3/2 + b\xi - \kappa'^2$, so one extremum is at small $\xi \simeq \kappa'^2/b$ and two extrema are at large $\xi \simeq \pm\sqrt{-2b}$. Since $d^2\psi/d\xi^2 = 3\xi^2/2 + b$, for $b < 0$ the large extrema are local minima (two valleys) and the small extremum is a local maximum (a hill). For $b \geq 0$, the large extrema are undefined, so there is a minimum at $\xi \simeq \kappa'^2/b$. Figure 1a shows an example of a two-valley $\psi\left(\xi\right)$ for which $b < 0$, and Fig. 1b shows a one-valley $\psi\left(\xi\right)$ for which $b \geq 0$.

We now make the assumption

$$\kappa' \ll s, \tag{12}$$

which will be shown in Section 5 to be appropriate for relevant magnetospheric situations. Then, $b \simeq -\xi_0^2/2 - s\kappa'\sin\phi_0$ is negative for

$$\xi_0^2 \geq -2s\kappa'\sin\phi_0. \tag{13}$$

All particles having $\sin\phi_0 > 0$ satisfy this equation because ξ_0^2 is non-negative. Particles having $\sin\phi_0 \leq 0$ satisfy Eq. 13 only if they are a certain distance away from exact resonance ($\xi = 0$).

Now, inserting $\xi = \kappa'^2/b$ in Eq. 7, we have the height of the hill to be $\psi_{max} \simeq -\kappa'^4/\left(2b\right)$. Therefore, the particle has enough pseudo-energy to cross over the hill if

$$\frac{1}{2}s^2\kappa'^2\cos^2\phi_0 - \frac{\xi_0^4}{8} + \frac{\xi_0^2}{2}\kappa'\left(\kappa' - s\sin\phi_0\right) \geq \kappa'^2\xi_0 - \frac{\kappa'^4}{2b}. \tag{14}$$

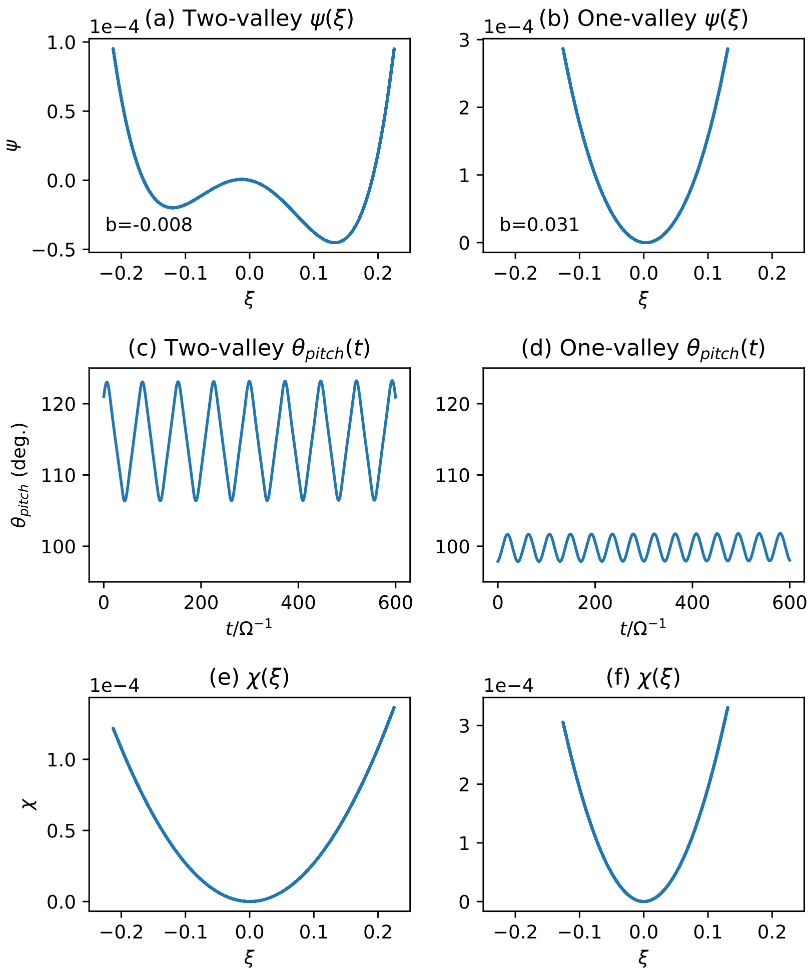

Figure 1. (a) An example of a two-valley $\psi(\xi)$ for which $b = -0.008 < 0$. (b) An example of a one-valley $\psi(\xi)$ for which $b = 0.031 \geq 0$. (c) The time-dependent pitch-angle of the particle undergoing two-valley motion, and (d) that of the particle undergoing one-valley motion. The wave parameters were $\kappa = 0.01$, $\alpha = 0.25$, and $n(\alpha) = 18$ from Eq. 28. (e), (f) The approximated pseudo-potentials χ obtained by keeping only the term involving $s\kappa'$ in Eq. 7 for the respective particles.

We now assume and justify later that the terms on the right-hand side of Eq. 14 are much smaller than those on the left-hand side. Using Eq. 12, Eq. 14 becomes

$$\frac{1}{2}s^2\kappa'^2\cos^2\phi_0 - \frac{\xi_0^4}{8} - \frac{\xi_0^2}{2}\kappa' s\sin\phi_0 \geq 0, \tag{15}$$

whose solution is

$$\xi_0^2 \leq 2s\kappa'\left(1-\sin\phi_0\right). \tag{16}$$

Now we derive the conditions for which the assumptions regarding Eq. 14 are valid. This is done by using the solution (i.e., Eq. 16) obtained under the assumptions and deriving the conditions for which the terms on the right-hand side of Eq. 14 are indeed small compared to those on the left-hand side. Using Eq. 16 as an equality, it is seen that each term on the left-hand side of Eq. 14 is $O\left(s^2\kappa'^2\right)$ except for the $\kappa'^2\xi_0^2/2$ term which is ignored by Eq. 12. On the right-hand side, $\kappa'^2\xi_0 = O\left(\sqrt{s\kappa'^5}\right)$ so it can be ignored if $\kappa' \ll s^3$. Examining the second term, $\kappa'^4/2b = O\left(\kappa'^3/s\right)$ because $b = O\left(s\kappa'\right)$, so it can be ignored if $\kappa' \ll s$. Since $\kappa' \ll 1$ for linear waves, $\kappa' \ll s^3$ and $\kappa' \ll s$ are both true for $s \geq 1$, and $\kappa' \ll s^3$ is a stronger statement than $\kappa' \ll s$ if $s < 1$. Therefore, for $\kappa' \ll s^3$ – which will later be demonstrated to be valid for relevant magnetospheric parameters – the following gives the condition for which a particle undergoes two-valley motion and thus a large pitch-angle scattering:

$$-2s\kappa'\sin\phi_0 \leq \xi_0^2 \leq 2s\kappa'\left(1-\sin\phi_0\right). \tag{17}$$

Equation 17 is one of the main results of this paper. For $\phi_0 \geq 0$, Eq. 17 becomes Eq. 16 and specifies a certain range around $\xi_0 = 0$. However, for $\phi_0 < 0$ that statistically represents half of the particle population, Eq. 17 does not include $\xi_0 = 0$, which means that particles further away from exact resonance undergo two-valley motion and thus large pitch-angle scattering. Therefore, Eq. 17 specifies the exact range of the initial distance from resonance that leads to two-valley motion.

Figure 1c shows the time-dependent pitch-angle $\theta_{\mathrm{pitch}}(t)$ of the particle that has enough pseudo-energy to undergo two-valley motion in the two-valley pseudo-potential in Fig. 1a. Figure 1d shows $\theta_{\mathrm{pitch}}(t)$ of the particle moving in the one-valley pseudo-potential. The particle in Fig. 1c experiences a much larger change in pitch-angle than that in Fig. 1d. The rate of change of the pitch-angle in Fig. 1c is also very large; the wave period is $T_{\mathrm{wave}}\Omega = 2\pi/\alpha \simeq 25$, so the pitch-angle changes by $\sim 15^\circ$ in $t\Omega \simeq 40$ or in about one to two wave periods.

4 Distribution of $\boldsymbol{\xi}$

The initial particle distribution in ξ-space will now be derived. The subscript zero will henceforth be dropped because only the initial conditions are being examined. The thermal distribution is assumed to be the Maxwell-Jüttner distribution (Jüttner, 1911) with an isotropic temperature. This is the relativistic generalization of the Maxwell-Boltzmann distribution and can be expressed in terms of the Lorentz factor γ as

$$f_\gamma = \frac{\gamma^2\sqrt{1-1/\gamma^2}}{\Theta K_2\left(1/\Theta\right)}\exp\left(-\frac{\gamma}{\Theta}\right), \tag{18}$$

where $\Theta = k_B T/mc^2$ is the normalized temperature and K_n is the modified Bessel function of the second kind of order n. This distribution is a considerable simplification, and repercussions of this simplification and possible remedies will be discussed in Section 7. Using $\gamma = \sqrt{1+p^2/m^2c^2} = \sqrt{1+\bar{p}^2}$ where $\bar{\mathbf{p}} = \mathbf{p}/mc$ is the normalized particle momentum, Eq. 18 can be expressed as

$$f_{\bar{\mathbf{p}}} = \frac{1}{4\pi\Theta K_2\left(1/\Theta\right)}\exp\left(-\frac{\sqrt{1+\bar{p}^2}}{\Theta}\right). \tag{19}$$

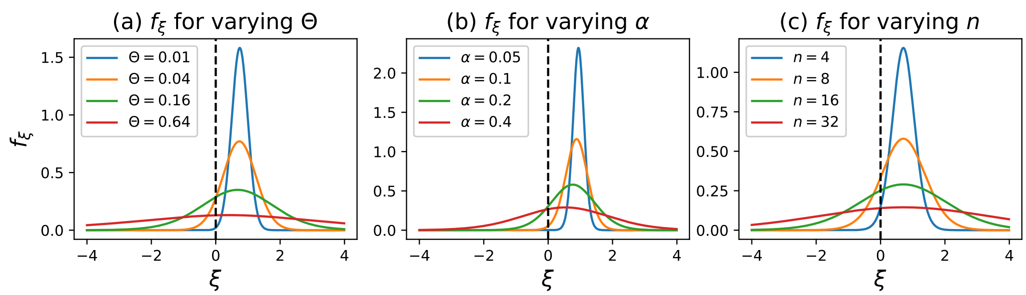

Figure 2. f_ξ for different (a) Θ, (b) α and (c) n values. The default values are $\Theta = 0.1$, $\alpha = 0.25$, and $n = 10$. The black dashed line is the resonant condition $\xi = 0$.

Integrating Eq. 19 in $\bar{p}_z$ and over all angles gives $f_{\bar{p}_\perp}$:

$$f_{\bar{p}_\perp} = \frac{\bar{p}_\perp \sqrt{1+\bar{p}_\perp^2}}{\Theta K_2\left(1/\Theta\right)} K_1 \left(\frac{\sqrt{1+\bar{p}_\perp^2}}{\Theta} \right). \tag{20}$$

Note that $f_{\bar{\mathbf{p}}}$ is defined in 3D $\bar{\mathbf{p}}$-space so that $\int f_{\bar{\mathbf{p}}} d^3\bar{\mathbf{p}} = 1$, whereas $f_{\bar{p}_\perp}$ is defined in 1D $\bar{p}_\perp$-space so that $\int f_{\bar{p}_\perp} d\bar{p}_\perp = 1$. Integrating Eq. 19 in $\bar{p}_x$ and $\bar{p}_y$ gives $f_{\bar{p}_z}$:

$$f_{\bar{p}_z} = \frac{\Theta}{2K_2\left(1/\Theta\right)} \left(1 + \frac{\sqrt{1+\bar{p}_z^2}}{\Theta} \right) \exp\left(-\frac{\sqrt{1+\bar{p}_z^2}}{\Theta} \right), \tag{21}$$

where $\int f_{\bar{p}_z} d\bar{p}_z = 1$. The details of the derivations of $f_{\bar{p}_\perp}$ and $f_{\bar{p}_z}$ are given in Appendix A and Appendix B, respectively.

Now, noting that $\gamma\boldsymbol{\beta} = \gamma\mathbf{v}/c = \mathbf{p}/mc = \bar{\mathbf{p}}$, the mismatch parameter (Eq. 5) can be expressed as

$$\xi = 1 + \alpha\left(n\bar{p}_z - \gamma\right). \tag{22}$$

The probability distribution of having a specific ξ is obtained by multiplying the probability distribution of having a certain γ by that of having the corresponding $\bar{p}_z$ which yields the specified ξ, and then integrating over all γ (full derivation given in Appendix C). The solution is

$$f_\xi = \int_1^\infty \frac{\gamma^2\sqrt{1-1/\gamma^2}}{2\alpha n K_2^2\left(1/\Theta\right)} \left(1 + \frac{\sqrt{1+\bar{p}_z^2\left(\gamma,\xi\right)}}{\Theta} \right) \exp\left(-\frac{\gamma + \sqrt{1+\bar{p}_z^2\left(\gamma,\xi\right)}}{\Theta} \right) d\gamma, \tag{23}$$

where $\bar{p}_z\left(\gamma,\xi\right) = \left[\left(\xi-1\right)/\alpha + \gamma\right]/n$ is a rearrangement of Eq. 22 and $\int f_\xi d\xi = 1$. Given Θ, α and n, Eq. 23 is an integral solution for f_ξ.

Figure 2 shows f_ξ for different (a) Θ, (b) α and (c) n values. The default values are $\Theta = 0.1$, $\alpha = 0.25$, and $n = 10$,where $\alpha = 0.25$ and $\Theta = 0.1$ are relevant values for the dayside outer magnetosphere (Tsurutani et al., 2009), and $n = 18 \sim 10$ from the whistler dispersion relation (Eq. 28). The black dashed vertical line represents the resonant condition $\xi = 0$ (or equivalently Eq. 1). As Θ, α and n increase from zero, f_ξ broadens and more particles are resonant. After a certain threshold, however, too much broadening leads to the decrease of the local magnitude of $f_\xi\left(\xi = 0\right)$ and reduces the number of resonant particles. Increasing α significantly changes the mean value of ξ as well, raising this threshold higher.

5 Fraction of Particles Undergoing Two-Valley Motion

Before calculating the fraction of particles undergoing two-valley motion, the probability distribution of the limits of integration (Eq. 17) must first be derived. Again, the subscript zero will be dropped. From Eq. 8, $s = \alpha n \beta_\perp \gamma/\gamma_T = \alpha\sqrt{n^2-1}\bar{p}_\perp$, so the relevant distribution is that of $\bar{p}_\perp$ and $\sin\phi$. Equation 20 prescribes $f_{\bar{p}_\perp}$, and assuming that ϕ is isotropic, the probability distribution of $\Phi = \sin\phi$ is the Arcsine(-1,1) distribution,

$$f_\Phi = \frac{1}{\pi\sqrt{1-\Phi^2}}, \tag{24}$$

for $\Phi \in (-1, 1)$.

We now have all the ingredients to calculate the fraction of particles that undergo two-valley motion in ξ-space and thus experience large pitch-angle scattering. This fraction can be found by calculating the probability that both Eqs. 13 and 16 (i.e., Eq. 17) are satisfied. In the case $\Phi > 0$ when Eq. 13 is always met, after defining a numerical factor $a = 2\alpha\kappa'\sqrt{n^2-1}$ so that $2s\kappa'\sin\phi = a\bar{p}_\perp\Phi$ and $2s\kappa'(1-\sin\phi) = a\bar{p}_\perp(1-\Phi)$, the probability of two-valley motion is

$$p_+ = \int_{\bar{p}_\perp=0}^{\infty}\int_{\Phi=0}^{1} f_{\bar{p}_\perp} f_\Phi \int_{\xi=-\sqrt{a\bar{p}_\perp(1-\Phi)}}^{\sqrt{a\bar{p}_\perp(1-\Phi)}} f_\xi d\xi d\Phi d\bar{p}_\perp. \tag{25}$$

In the opposite case where $\Phi \leq 0$, the probability of two-valley motion is,

$$p_- = \int_{\bar{p}_\perp=0}^{\infty}\int_{\Phi=-1}^{0} f_{\bar{p}_\perp} f_\Phi \left(\int_{-\sqrt{a\bar{p}_\perp(1-\Phi)}}^{-\sqrt{-a\bar{p}_\perp\Phi}} f_\xi d\xi + \int_{\sqrt{-a\bar{p}_\perp\Phi}}^{\sqrt{a\bar{p}_\perp(1-\Phi)}} f_\xi d\xi\right) d\Phi d\bar{p}_\perp. \tag{26}$$

The total fraction of particles undergoing two-valley motion is then $p_{tv} = p_+ + p_-$.

There are four degrees of freedom when calculating p_{tv}: Θ, α, n and κ. However, one degree of freedom can be eliminated by linking α and n through the whistler wave dispersion relation, which, for parallel propagation and $\Omega_p/\Omega \gg 1$ where Ω_p is the electron plasma frequency, is

$$\frac{c^2k^2}{\omega^2} = \frac{\Omega_p^2/\omega^2}{|\Omega|/\omega - 1}. \tag{27}$$

In terms of the dimensionless variables used in this paper, this becomes

$$n = \frac{\Omega_p/\Omega}{\sqrt{\alpha(1-\alpha)}}, \tag{28}$$

which can be used to express $n(\alpha)$ if Ω_p/Ω is specified. Using parameters in Tsurutani et al. (2009) ($n_e \simeq 10$ cm^{-3},$B_0 \simeq 125$ nT), we obtain $\Omega_p/\Omega \simeq 8$; this value will be used throughout the rest of the analysis.

Let us now calculate p_{tv} for the parameters in the range $0.0001 \leq \kappa \leq 0.01$, $0.1 \leq \alpha \leq 0.8$ and $0.01 \leq \Theta \leq 10$ (corresponding to electron thermal energies from 5.11 keV to 5.11 MeV). Since the parameter range is determined, the conditions for which the assumption $\kappa' \ll s^3$ that was used to derive Eq. 17 is true can now be determined. Because $n \gg 1$, $\kappa' = \kappa/\gamma_T = \kappa\sqrt{1-1/n^2} \simeq \kappa$ and $s = \alpha\sqrt{n^2-1}\bar{p}_\perp \simeq \alpha n\bar{p}_\perp$. From Eq. 28 it follows that $\alpha n = (\Omega_p/\Omega)\sqrt{\alpha/(1-\alpha)}$. We now compare the largest value of κ to the lowest value of s^3, which involves the smallest values of α and Θ. For $\Theta \ll 1$, the most likely $\bar{p}_\perp$ is $\sqrt{\Theta}$ (see Appendix D). Thus, the condition $\kappa' \ll s^3$ can be expressed as $\kappa \ll \left((\Omega_p/\Omega)\sqrt{\alpha\Theta/(1-\alpha)}\right)^3$, or $\Theta \gg \kappa^{2/3}/\left[(\Omega_p/\Omega)^2(\alpha/(1-\alpha))\right]$. Inserting $\alpha = 0.1$ and $\kappa = 0.01$ shows that $\kappa' \ll s^3$ is valid if $\Theta \gg 0.0066$. Thus, $0.01 \leq \Theta \leq 10$ is consistent with $\kappa' \ll s^3$.

Figure 3 shows contours of p_{tv} as a function of α and Θ for different κ values. For $\kappa \geq 0.001$, which is typical for magnetospheric chorus (Tsurutani et al., 2009; Macúšová et al., 2015), a significant fraction (1%−5%) of particles undergo two-valley motion and thus large pitch-angle scattering. However, p_{tv} decreases at high Θ ($\Theta \gtrsim 1$), and this phenomenon is related to the decrease of the local magnitude of $f_\xi(\xi = 0)$ if there is too

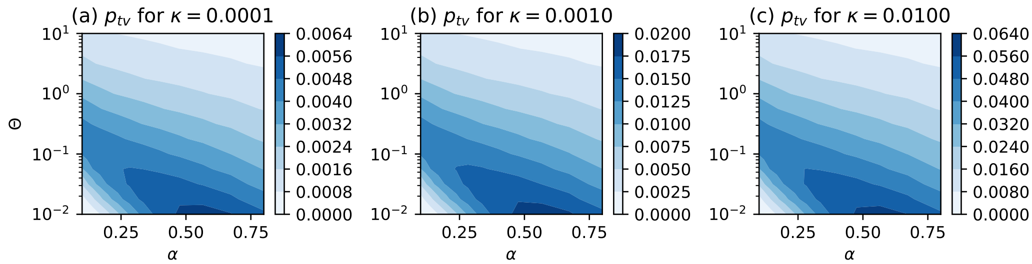

Figure 3. p_{tv} as a function of α and Θ for different κ values.

much broadening of f_ξ, as shown in Fig. 2. Figure 3 also shows that $p_{tv}(\alpha, \Theta)$ has more or less the same shape across a wide range of κ but its magnitude is proportional to $\sqrt{\kappa}$. This is because the limits of the ξ integrals in Eqs. 25 and 26 scale as $\sqrt{a} \sim \sqrt{\kappa}$, so if the integration range is sufficiently small so that the integrand may be approximated by a linear function, it follows that $p_{tv} \propto \sqrt{\kappa}$.

6 Numerical Verification

The analytical predictions presented in this paper will now be verified via numerical simulations. A computer code was written which solves Eq. 4 and $d\mathbf{x}/dt = c\boldsymbol{\beta}$ using the fully implicit Runge-Kutta method of the Radau IIA family of order 5 (Hairer & Wanner, 1991) in the `scipy.integrate.solve_ivp` package in Python 3.7. This particular method was used because it yielded the smallest numerical error out of the available methods in the Python package, measured by the drift of the average value of the pitch-angle over the full simulation time. This drift should be zero in principle because the coefficients of $\psi(\xi)$ are time-independent, but numerical error introduces a small drift. For example, the simulations in Figs. 1c and 1d show that the particles' pitch-angles undergo oscillatory motion, but there are ever-so-slight, almost unnoticeable drifts ($\lesssim 0.1^\circ$) of the average values. The error was quantified by using the statistics of the 10,000 particles in Fig. 5c. The Radau method with a time step $\Delta t = 0.2$ yielded a median value for the pitch-angle drift of 0.07° with a standard deviation of 0.14°, which is far smaller than the pitch-angle oscillation of a vast majority of the particles. The simulation time was set long enough for every particle to undergo at least several oscillations in the pitch-angle. The electromagnetic fields were prescribed by Eqs. 2 and 3, which is a simplified model of a whistler wave. The code was parallelized with the `multiprocessing` package.

It will first be verified that particles which satisfy Eq. 17 and thus undergo two-valley motion experience large pitch-angle scattering. 2,500 particle trajectories were numerically integrated, and the initial particle momenta were scanned in the range $\bar{p}_\perp \in [0, 2]$, $\bar{p}_z \in [-0.5, 0]$, and $\phi = \pi/4, -\pi/4$. The wave amplitude was $\kappa = 0.005$, and the wave frequency was $\alpha = 0.25$, which gives $n = 18$ using Eq. 28.

Figure 4a shows the regions of initial momentum space (dark green) that satisfy the unapproximated two-valley criteria (Eqs. 14 and $b < 0$) for $\phi = \pi/4$. Figure 4b shows regions of this space that satisfy the approximated criterion (Eq. 17). The regions are virtually identical except for $\bar{p}_\perp \lesssim 0.1$ because for sufficiently large $\bar{p}_\perp$, the $\kappa \ll s^3$ approximation holds. Figures 4a and 4b are effectively predictions of large pitch-angle scattering. The colors in Fig. 4c show the pitch-angle range that a particle undergoes for each point in $(\bar{p}_\perp, \bar{p}_z)$ space; this pitch-angle range is defined by the absolute difference between the maximum and minimum pitch-angles along the particle trajectory. For example, the particle in Fig. 1c has a pitch-angle range of $\sim 15^\circ$, and that in Fig. 1d

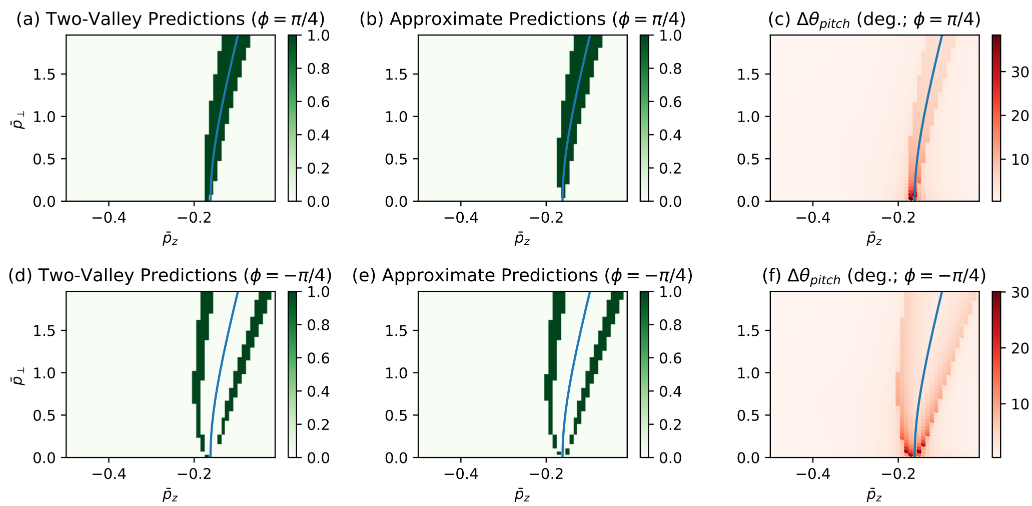

Figure 4. (a) Regions of initial momentum space (dark green) that satisfy the unapproximated two-valley criteria (Eqs. 14 and $b < 0$) for $\phi = \pi/4$. (b) Regions of this space that satisfy the approximated criterion (Eq. 17) for $\phi = \pi/4$. (c) Pitch-angle range (in degrees) within a single particle trajectory for a range of initial particle momenta for $\phi = \pi/4$. (d-f) are the same as (a-c) except for $\phi = -\pi/4$. Blue lines represent the resonance condition (Eq. 1; $\xi = 0$). The wave parameters were $\alpha = 0.25$, $\kappa = 0.005$, and $n = 18$ from Eq. 28.

has a pitch-angle range of $\sim 3°$. Figures 4d-f are the same as Figs. 4a-c except for $\phi = -\pi/4$. It can be clearly seen that if a particle's initial momentum satisfies the two-valley criteria, it undergoes a large pitch-angle scattering.

The blue curves in Fig. 4 represent the resonance condition (Eq. 1; $\xi = 0$). The curve is found by solving $\xi = 1 + \alpha(n\bar{p}_z - \gamma) = 1 + \alpha(n\bar{p}_z - \sqrt{1 + \bar{p}_\perp^2 + \bar{p}_z^2}) = 0$ for $\bar{p}_\perp(\bar{p}_z)$ and restricting the domain of $\bar{p}_z$ to be consistent with $\gamma = \alpha^{-1} + n\bar{p}_z \geq 1$. In Figs. 4d-f, the blue lines do not pass through regions of two-valley motion and large scattering. This fact is consistent with Eq. 17 which qualitatively states that for $\phi < 0$, the condition for two-valley motion and large scattering does not include $\xi = 0$.

Next, the analytical prediction for p_{tv} will be verified via the Monte-Carlo method. The trajectories of 10,000 particles whose initial momenta were randomly sampled from Eq. 19 were respectively integrated for $\kappa = 0.0001, 0.001$ and 0.01. Other parameters were $\alpha = 0.25$, $n = 18$, and $\Theta = 0.1$.

Figure 5 shows the pitch-angle range (in degrees) of the randomly sampled particles for different κ values. Red points represent particles that meet the two-valley criterion (Eq. 17), and the text inside represents the percentage of red particles. Figure 3 shows that for $\alpha = 0.25$ and $\Theta = 0.1$, the predicted percentage ranges are $0.4-0.48\%$, $1.25 - 1.5\%$ and $4.00 - 4.80\%$ for $\kappa = 0.0001, 0.001$ and 0.01, respectively, which approximately agree with the results in Fig. 5. Red points generally experience significantly larger pitch-angle scattering than other particles, as can be seen from the median value of the red points (red horizontal lines). However, it can be seen that there are blue points that also experience large scattering; examining the pseudo-potential $\psi(\xi)$ for these points shows that these particles have pseudo-energies that are just short of overcoming the two-valley hill, so they "almost" undergo two-valley motion and experience substantial pitch-angle scattering. Therefore, we conclude that p_{tv} is a lower-bound for the fraction of particles with large pitch-angle scattering.

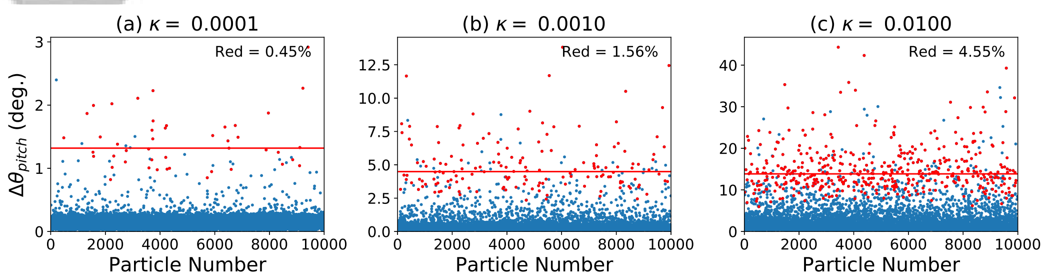

Figure 5. Pitch-angle range of 10,000 particles whose initial momenta were randomly sampled from Eq. 19 for different κ values. Red points represent particles that meet the two-valley criterion (Eq. 17), and the text inside represents the percentage of red particles. The red horizontal lines represents the median $\Delta\theta_{pitch}$ of the red particles in degrees.

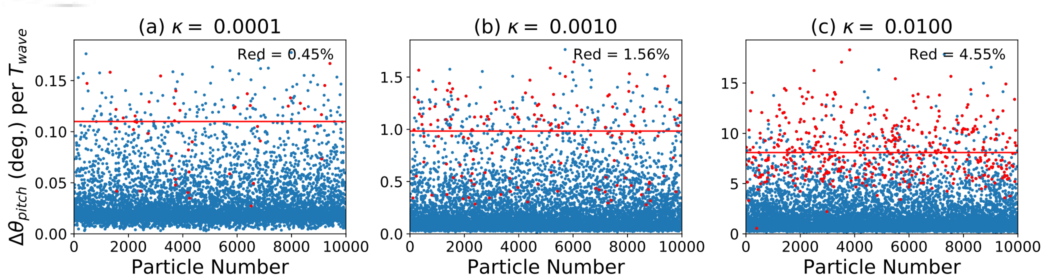

Figure 6. Pitch-angle change per wave period of the respective simulations in Fig. 5. The red horizontal lines respectively represent the median value of the pitch-angle change per wave period of the red particles.

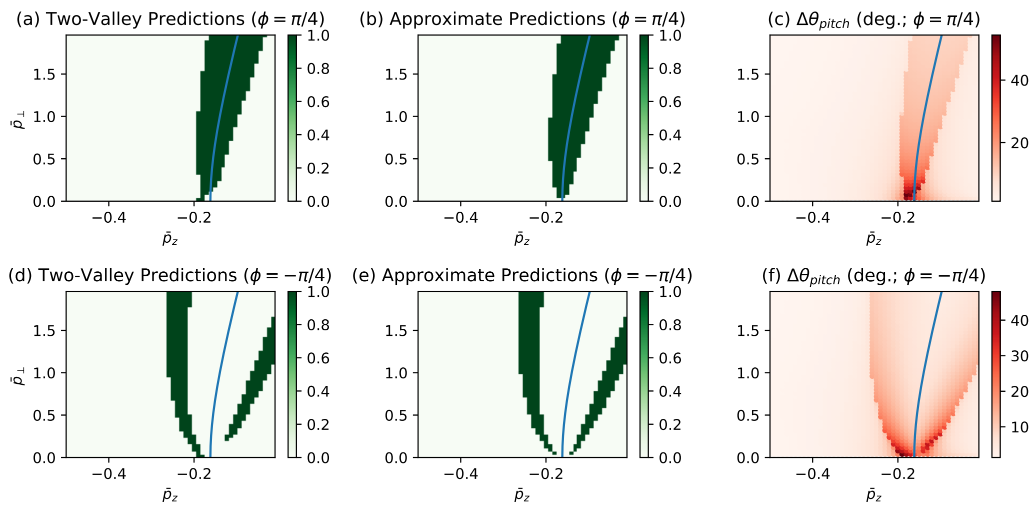

Figure 7. Same as Fig. 4, but for $\kappa = 0.02$.

Even if two-valley motion were to cause large pitch-angle scattering, the mechanism would not be significant if this scattering could not occur within a short enough time. Thus, it is necessary to show that the coherent wave lasts sufficiently long for two-valley motion to occur. Figure 6 shows the pitch-angle change within a single wave period for the respective simulations in Fig. 5. Tsurutani et al. (2009) observed in the outer magnetosphere coherent chorus elements with amplitudes $\kappa \simeq 0.0016$ ($B_0 \simeq 125$ nT and wave field $\tilde{B} \simeq 200$ pT) that are 0.1 $\sim$ 0.5 s long with a frequency of $\sim$ 700Hz. These elements consisted of subelements or packets lasting 5 $\sim$ 10 ms, corresponding to about 3.5 to 7 wave periods. $\kappa \simeq 0.0016$ approximately corresponds to Fig. 6b, which shows that red particles can reach their median pitch-angle range ($\sim 5°$ from Fig. 5b) in five wave periods on average. For $\kappa = 0.01$ (Fig. 6b), this rate is even faster as the red particles can reach their median pitch-angle range of $\sim 15°$ (Fig. 5c) in about two wave periods.

7 Discussion

The results presented here may help explain the association of large-amplitude whistler waves to relativistic microbursts ($\sim$ 1 MeV) (Breneman et al., 2017) and may explain the lack of such energetic microbursts in small-amplitude chorus (Tsurutani et al., 2013). Particle energization is not a subject of this paper and thus will not be discussed; it will be assumed that the particles are first energized by some mechanism that yields a relativistic distribution, and then the ensuing pitch-angle dynamics are studied in order to concentrate on one topic. It should be noted, however, that energization and pitch-angle scattering may occur simultaneously.

In Fig. 3, for small amplitudes ($0.0001 \leq \kappa \leq 0.001$), only up to 0.5% of particles in a distribution with a temperature of $\sim$ 1 MeV (corresponding to $\Theta \simeq 2$) interact with the wave, whereas for large amplitudes ($\kappa \simeq 0.01$), ~2% of such particles do. This is because the range of the two-valley condition in Eq. 17 scales with the wave amplitude κ; i.e., as the wave amplitude increases, more particles, including energetic particles, satisfy the two-valley condition.

The interaction of large-amplitude waves with relativistic particles is further explained in Fig. 7, which is the same as Fig. 4 but for a larger wave amplitude ($\kappa = 0.02$).

It can clearly be seen that the predictions of large scattering in Fig. 7 are much broader in phase space than those in Fig. 4. This is important because in Fig. 4, relativistic particles with $\bar{p} \gtrsim 1$ must have large initial pitch-angles to interact with the wave since the two-valley condition is a narrow range related to the exact resonance condition, and thus these particles must undergo extremely large pitch-angle scatterings in order to jump into the loss cone. However, in Fig. 7, the range for two-valley motion is much increased, allowing for relativistic particles with smaller initial pitch-angles to interact with the wave. The deviation of the two-valley condition from the exact resonance condition is because the range in Eq. 17 scales with κ. Furthermore, the pitch-angle range itself is significantly increased in Fig. 7. Therefore, a larger wave amplitude allows for relativistic particles with lower initial pitch-angles to interact with the wave, while simultaneously increasing the amount of pitch-angle scattering; these two effects lead to more relativistic particles being pitch-angle scattered into the loss-cone.

There are a few limitations to the present analysis that may be subject to future work. First, the Maxwell-Jüttner distribution is a simplification and should not be considered as a distribution representing the entire electron population. The actual distribution is a sum of these Maxwellians or other functions such as the kappa distribution (Pierrard & Lazar, 2010). If the actual distribution can be expressed as a weighted sum of Maxwelll-Jüttner distributions, then the total fraction of particles that undergo two-valley motion is the sum of the partial fractions for each distribution. On the other hand, if the actual distribution is another sufficiently simple function, then an analysis similar to that in Sections 4 and 5 may be conducted by replacing Eq. 18 by the actual distribution. However, depending on the complexity of the actual distribution, its transition to Eq. 23 may be more complicated.

Second, the particle temperature is assumed for simplicity to be isotropic, whereas observations indicate that electron temperature in the magnetosphere in general is anisotropic and electron distribution functions can be more complex than simple anisotropic distributions (Li et al., 2010). The transition to an anisotropic Maxwell-Jüttner distribution is outlined in Livadiotis (2016) and Treumann and Baumjohann (2016).

Third, the wave is assumed to have exact parallel propagation, whereas many instances of magnetospheric chorus involve oblique propagation (Santolík et al., 2009; A. Artemyev et al., 2016). Also, chorus typically exhibits frequency and amplitude changes over a short time period (Tsurutani et al., 2009), but the model presented here is based on a plane wave with a fixed frequency and wavenumber (Eqs. 2 and 3). However, including obliquity and variable frequency makes the analysis considerably more complicated and so would be inappropriate for an inaugural analysis.

8 Comparison to Second-order Trapping Theory

A popular theory describing wave-particle interactions is the second-order trapping effect presented in, e.g., Sudan and Ott (1971), Nunn (1974) and Omura et al. (1991). Omura et al. (2007) and Omura et al. (2008) present relativistic generalizations of the theory. However, it will now be shown that this previous theory is an approximation of the theory presented here; this approximation effectively misses the critical two-valley nature of the pseudo-potential.

Omura et al. (1991) use the following coupled equations for non-relativistic speeds:

$$\frac{d\zeta}{dt} = k(v_z - V_R), \tag{29}$$

$$\frac{d}{dt}(v_z - V_R) = \frac{\omega_t^2}{k}(\sin\zeta + S), \tag{30}$$

where

$$V_R = \frac{\omega - \Omega}{k}, \tag{31}$$

ζ is the angle between $\mathbf{v}_\perp$ and $\tilde{\mathbf{B}}$, $\omega_t = \sqrt{kv_\perp \Omega\kappa}$ is the trapping frequency, and S is a parameter that is equal to zero when the background magnetic field is spatially uniform and ω is a constant. Therefore, setting $S = 0$, differentiating Eq. 30 in time, and using Eq. 29,

$$\frac{d^2}{dt^2}(v_z - V_R) = \frac{\omega_t^2}{k}\cos\zeta\frac{d\zeta}{dt}, \tag{32}$$
$$= \omega_t^2(v_z - V_R)\cos\zeta. \tag{33}$$

Letting $\gamma = 1$ in Eq. 5 and rearranging shows that

$$\xi = \frac{k}{\Omega}\left(v_z - V_R\right), \tag{34}$$

so Eq. 33 becomes

$$\frac{d^2\xi}{dt^2} = \xi\omega_t^2\cos\zeta, \tag{35}$$
$$= -\frac{\partial}{\partial\xi}\left(-\frac{\xi^2}{2}\omega_t^2\cos\zeta\right), \tag{36}$$
$$\frac{1}{\Omega^2}\frac{d^2\xi}{dt^2} = -\frac{\partial\chi(\xi)}{\partial\xi}, \tag{37}$$

where $\chi(\xi) = -\xi^2\omega_t^2\cos\zeta/2\Omega^2$ is the pseudo-potential of this system.

Now, let us examine the term involving $s\kappa'$ in Eq. 7 assuming $\gamma_0 = \gamma_T = 1$;

$$-s\kappa\sin\phi_0\frac{\xi^2}{2} = -\alpha n\kappa\beta_{\perp 0}\sin\phi_0\frac{\xi^2}{2}, \tag{38}$$
$$= -\frac{\omega}{\Omega}\frac{ck}{\omega}\kappa\frac{v_{\perp 0}}{c}\sin\phi_0\frac{\xi^2}{2}, \tag{39}$$
$$= -\frac{\omega_{t0}^2}{\Omega^2}\sin\phi_0\frac{\xi^2}{2}, \tag{40}$$
$$= \chi_0(\xi), \tag{41}$$

because ζ and ϕ are related by $\zeta = \phi - \pi/2$, so $\cos\zeta = \sin\phi$. $\chi_0(\xi)$ is $\chi(\xi)$ except that $v_{\perp 0}$ and ϕ_0 are used instead of $v_\perp$ and ϕ, and the relationship is similar for ω_{t0} and ω_t. Therefore, $\chi(\xi)$ results from keeping only the $s\kappa'$ term in $\psi(\xi)$. This is important because $\chi(\xi)$ only describes either a trapping or a non-trapping potential but not a two-valley potential.

Figure 1e and 1f plot the approximated pseudo-potentials $\chi(\xi)$ for the particles in Fig. 1a and 1b, respectively. For both particles, $\chi(\xi)$ is clearly a one-valley potential, whereas the unapproximated $\psi(\xi)$ is two-valleyed for the particle in Fig. 1a and thus it undergoes much larger pitch-angle scattering than the particle in Fig. 1b . Therefore, if the theory in Omura et al. (1991) were to be used, it would be impossible to distinguish between the two particles which clearly have an extremely large difference in the amount of pitch-angle scattering.

Another important problem with the second-order trapping theory is that the time-dependence of the variables is ambiguous at best. Omura et al. (1991) imply that $v_\perp$ and thus ω_t are time-dependent but then treat $v_\perp$ as a constant when they state that combining Eqs. 29 and 30 gives a pendulum equation. Sudan and Ott (1971) admit that $v_\perp$ is time-dependent, but then argue that it can be treated as a constant, as specified in the sentence after their Eq. 10. In the present theory, however, we explicitly differentiate between the initial variables and the time-dependent ones, so no approximation regarding time-dependence needs to be made. This is an extremely important point because this time-dependence of $v_\perp$ gives the two-valley potential whereas treating it as a constant does not. This fact can be more explicitly illustrated by examining Eq. 26

in Bellan (2013) which is an equation for the parallel velocity (recall that $\beta_z = v_z/c$ and prime refers to the wave frame):

$$\frac{1}{\Omega'}\frac{d^2\beta_z'}{dt'^2} = \xi\boldsymbol{\beta}_\perp' \cdot \frac{\tilde{\mathbf{B}}_\perp'}{B_0} - \beta_z'\frac{\tilde{\mathbf{B}}_\perp'}{B_0}\cdot\frac{\tilde{\mathbf{B}}_\perp'}{B_0}. \tag{42}$$

The second-order trapping theory effectively drops the last term in Eq. 42 and ignores the time-dependence of the first term on the right-hand side. This leads to

$$\frac{1}{\Omega'}\frac{d^2\xi}{dt'^2} = \alpha n\frac{\gamma'}{\gamma_T}\xi\boldsymbol{\beta}_\perp' \cdot \frac{\tilde{\mathbf{B}}_\perp'}{B_0}, \tag{43}$$

which is equivalent to Eq. 37 if $\gamma' = \gamma_T = 1$ is assumed. However, Eq. 35 of Bellan (2013) states that

$$\boldsymbol{\beta}_\perp' \cdot \frac{\tilde{\mathbf{B}}_\perp'}{B_0} = \boldsymbol{\beta}_{\perp 0}' \cdot \frac{\tilde{\mathbf{B}}_{\perp 0}'}{B_0} - \frac{\gamma_T}{2\alpha n\gamma'}\left(\xi^2 - \xi_0^2\right), \tag{44}$$

which means that treating $v_\perp$ as a constant effectively misses the ξ-dependence in Eq. 44, which is the reason for the two-valley shape of the pseudo-potential.

For example, neglecting the ξ_0^2 term in Eq. 44 leads to erroneous conclusions regarding the shape of the potential near $\xi = 0$. In Fig. 1e, $\chi(\xi)$ is a valley because $-s\kappa'\sin\phi_0$ is positive in this case. However, the correct pseudo-potential $\psi(\xi)$ in Fig. 1a is a hill near $\xi = 0$ because $-\xi_0^2/2 - s\kappa'\sin\phi_0$ in Eq. 7 is negative in this case. Also, the ξ^2 term in Eq. 44, which leads to the positive ξ^4 term in Eq. 7, prevents the pseudo-potential from diverging to $-\infty$ as $\xi \to \pm\infty$. This prevents the particle ξ from veering off to infinity; this phenomenon is unphysically allowed if the approximated $\chi(\xi)$ is used and $\sin\phi_0 > 0$. The term linear in ξ in Eq. 7 which affects the asymmetry of the two-valleys is also neglected in $\chi(\xi)$. The fact that $v_\perp$ is not constant can be explicitly seen in Fig. 5g of Bellan (2013), where $v_\perp$ of a particle undergoing two-valley motion varies in time by over a factor of three.

It should be noted, however, that for a non-uniform background field and/or time-dependent wave frequencies, S is finite in Eq. 30 and this may have an important role in the system additional to the effects described in the present paper. In fact, many studies that use the approximated second-order trapping theory focus on the non-local processes where effects due to finite S are significant (e.g., in Omura et al. (2007)). The present study focuses on local scattering that happens over only a few wave periods, so S can be presumed to be small, and $\psi(\xi)$ instead of $\chi(\xi)$ must be used.

A simple way to see that S is locally negligible is to consider the physical length of the wave for the duration of the pitch-angle scattering. From Figs. 5 and 6, maximum deflection happens within 5 wave periods. For $B_0 = 125$ nT, $\Omega = 2.2\times10^4$ rad/s. Using wave parameters that have been used so far, $\alpha = 0.25$ gives $\omega = 5.5\times10^3$ rad/s, and $n = 18$ gives the wavelength to be $\lambda = 19$ km. Therefore, 5 wave periods corresponds to about 100 km, which is a minuscule distance compared to the length scale of the magnetosphere at $L \simeq 5$ where plentiful amounts of relativistic microbursts occur (Tsurutani et al., 2013). The time-dependence of the wave frequency is also negligible because a single chorus element lasts for around 0.1-0.5 s while its frequency increases by about 50%, but five wave periods corresponds to less than 0.01 s (Tsurutani et al., 2009). Therefore, S can be considered to be negligible during the local scattering process.

9 Conclusion

The interaction of a relativistically-consistent thermal distribution of particles with a coherent right-handed circularly polarized wave has been investigated. Departure from wave-particle resonance for each particle is expressed by a frequency mismatch parameter ξ, where $\xi = 0$ represents perfect resonance. An exact rearrangement of the relativistic particle equation of motion shows that ξ follows pseudo-Hamiltonian dynamics with an associated pseudo-potential $\psi(\xi)$. If $\psi(\xi)$ has two-valleys separated by a hill, and the particle has enough pseudo-energy to overcome the hill, then the particle undergoes two-valley ξ-space motion that produces a large, non-diffusive pitch-angle scattering.

An accurate condition for two-valley motion and thus for large pitch-angle scattering has been derived; this condition is related to but may or may not include the exact resonance condition (Eq. 1), and the range of this condition scales with the wave amplitude. Assuming that the particle distribution is Maxwell-Jüttner, which is a relativistic generalization of the Maxwell-Boltzmann distribution, for typical magnetospheric parameters a significant fraction $(1-5\%)$ of the particles undergoes two-valley motion. The pertinent analysis can potentially be used for the actual local electron distribution, which may not be exactly Maxwellian. Numerical simulations confirm the analytical results. The scaling of the fraction of interacting particles with the wave amplitude may also explain the association of relativistic microbursts to large-amplitude chorus. The present theory is more accurate and exact than the widely-used second-order trapping theory as second-order trapping theory fails to take into account two-valley motion.

Appendix A Derivation of $f_{\bar{p}_\perp}$

In cylindrical coordinates, Eq. 19 is equivalent to

$$f_{\bar{\mathbf{p}}} d^3\bar{\boldsymbol{p}} = \frac{1}{4\pi\Theta K_2(1/\Theta)} \exp\left(-\frac{\sqrt{1+\bar{p}^2}}{\Theta}\right) \bar{p}_\perp d\bar{p}_\perp d\phi d\bar{p}_z. \tag{A1}$$

Integrating in $\bar{p}_z$ gives

$$\int_{\bar{p}_z=-\infty}^{\infty} f_{\bar{\mathbf{p}}} \bar{p}_\perp d\bar{p}_\perp d\phi d\bar{p}_z = \int_{\bar{p}_z=-\infty}^{\infty} \frac{1}{4\pi\Theta K_2(1/\Theta)} \exp\left(-\frac{\sqrt{1+\bar{p}_\perp^2+\bar{p}_z^2}}{\Theta}\right) \bar{p}_\perp d\bar{p}_\perp d\phi d\bar{p}_z \tag{A2}$$

$$= \bar{p}_\perp d\bar{p}_\perp d\phi \int_{\bar{p}_z=0}^{\infty} \frac{1}{2\pi\Theta K_2(1/\Theta)} \exp\left(-\frac{\sqrt{1+\bar{p}_\perp^2+\bar{p}_z^2}}{\Theta}\right) d\bar{p}_z, \tag{A3}$$

where $\bar{p}^2 = \bar{p}_\perp^2 + \bar{p}_z^2$. Defining

$$a^2 = \frac{1+\bar{p}_\perp^2}{\Theta^2} \tag{A4}$$

and

$$t = \frac{\bar{p}_z}{\Theta}, \tag{A5}$$

the $\bar{p}_z$-integral in Eq. A3 becomes

$$\frac{1}{2\pi K_2(1/\Theta)} \int_0^\infty \exp\left(-\sqrt{a^2+t^2}\right) dt. \tag{A6}$$

Now we define

$$t = a\sinh z, \tag{A7}$$

so $\sqrt{a^2+t^2} = a\sqrt{1+\sinh^2 z} = a\cosh z$ and $dt = a\cosh z dz$. Equation A6 is now

$$\frac{a}{2\pi K_2(1/\Theta)} \int_0^\infty \cosh z \exp(-a\cosh z)\, dz. \tag{A8}$$

The z-integral in Eq. A8 evaluates to $K_1(a)$ where K_n is the modified Bessel function of the second kind of order n (Zwillinger, 2015, Section 8.432, 1.).

Therefore, Eq. A3 is now

$$\frac{\sqrt{1+\bar{p}_\perp^2}}{2\pi\Theta K_2(1/\Theta)} K_1\left(\frac{\sqrt{1+\bar{p}_\perp^2}}{\Theta}\right) \bar{p}_\perp d\bar{p}_\perp d\phi. \tag{A9}$$

Integrating in ϕ yields the final result:

$$f_{\bar{p}_\perp} d\bar{p}_\perp = \frac{\bar{p}_\perp\sqrt{1+\bar{p}_\perp^2}}{\Theta K_2(1/\Theta)} K_1\left(\frac{\sqrt{1+\bar{p}_\perp^2}}{\Theta}\right) d\bar{p}_\perp, \tag{A10}$$

which is Eq. 20.

Appendix B Derivation of $f_{\bar{p}_z}$

In cylindrical coordinates, Eq. 19 is equivalent to

$$f_{\bar{\mathbf{p}}} d^3\bar{\boldsymbol{p}} = \frac{1}{4\pi\Theta K_2(1/\Theta)} \exp\left(-\frac{\sqrt{1+\bar{p}^2}}{\Theta}\right) \bar{p}_\perp d\bar{p}_\perp d\phi d\bar{p}_z. \tag{B1}$$

Integrating in all ϕ and $\bar{p}_\perp$, Eq. B1 becomes

$$f_{\bar{p}_z} d\bar{p}_z = \int_{\bar{p}_\perp=0}^{\infty} \frac{1}{2\Theta K_2(1/\Theta)} \exp\left(-\frac{\sqrt{1+\bar{p}_\perp^2+\bar{p}_z^2}}{\Theta}\right) \bar{p}_\perp d\bar{p}_\perp d\bar{p}_z. \tag{B2}$$

Letting $\eta^2 = 1+\bar{p}_\perp^2+\bar{p}_z^2$ while keeping $\bar{p}_z$ constant so that

$$\eta d\eta = \bar{p}_\perp d\bar{p}_\perp, \tag{B3}$$

we have

$$f_{\bar{p}_z} d\bar{p}_z = \int_{\eta=\sqrt{1+\bar{p}_z^2}}^{\infty} \frac{1}{2\Theta K_2(1/\Theta)} \exp\left(-\frac{\eta}{\Theta}\right) \eta d\eta d\bar{p}_z. \tag{B4}$$

Using the integral formula (Zwillinger, 2015, Section 3.351, 2.)

$$\int_u^\infty x^n e^{-\mu x} dx = e^{-u\mu} \sum_{k=0}^{n} \frac{n!}{k!} \frac{u^k}{\mu^{n-k+1}}, \tag{B5}$$

where $x=\eta$, $u=\sqrt{1+\bar{p}_z^2}$, $\mu = 1/\Theta$, and $n=1$ in this case, we have

$$f_{\bar{p}_z} d\bar{p}_z = \frac{1}{2\Theta K_2(1/\Theta)} \left(\Theta^2 + \Theta\sqrt{1+\bar{p}_z^2}\right) \exp\left(-\frac{\sqrt{1+\bar{p}_z^2}}{\Theta}\right), \tag{B6}$$

$$= \frac{\Theta}{2K_2(1/\Theta)} \left(1+\frac{\sqrt{1+\bar{p}_z^2}}{\Theta}\right) \exp\left(-\frac{\sqrt{1+\bar{p}_z^2}}{\Theta}\right), \tag{B7}$$

which is Eq. 21.

Appendix C Derivation of f_{ξ}

ξ is defined as

$$\xi = 1+\alpha(n\bar{p}_z - \gamma) = 1+\alpha\zeta, \tag{C1}$$

where $\zeta = n\bar{p}_z - \gamma$.

f_ζ will first be derived. Defining $R = n\bar{p}_z$ (so $d\bar{p}_z = dR/n$ and $\zeta = R - \gamma$), we have

$$f_{\bar{p}_z}(\bar{p}_z)d\bar{p}_z = f_{\bar{p}_z}(R/n)\frac{dR}{n} \tag{C2}$$

$$= \frac{\Theta}{2nK_2(1/\Theta)}\left(1 + \frac{\sqrt{1+R^2/n^2}}{\Theta}\right)\exp\left(-\frac{\sqrt{1+R^2/n^2}}{\Theta}\right)dR \tag{C3}$$

$$= f_R(R)dR. \tag{C4}$$

Now, in order for $\zeta = R - \gamma$ to be true, the value of R has to equal $\zeta + \gamma$ for a given value of γ. The probability distribution of this occurrence integrated over all values of γ gives f_ζ:

$$f_\zeta(\zeta) = \int_1^\infty f_\gamma(\gamma)\, f_R(\zeta+\gamma)\, d\gamma \tag{C5}$$

$$= \int_1^\infty \frac{\gamma^2\sqrt{1-1/\gamma^2}}{2nK_2^2(1/\Theta)}\left(1 + \frac{\sqrt{1+(\zeta+\gamma)^2/n^2}}{\Theta}\right)\exp\left(-\frac{\gamma+\sqrt{1+(\zeta+\gamma)^2/n^2}}{\Theta}\right)d\gamma. \tag{C6}$$

Finally, rearranging Eq. C1 yields $\zeta(\xi) = (\xi - 1)/\alpha$ so that $d\zeta = d\xi/\alpha$. It follows that

$$f_\zeta(\zeta)d\zeta = f_\zeta([\xi-1]/\alpha)\frac{d\xi}{\alpha} \tag{C7}$$

$$= \int_1^\infty \frac{\gamma^2\sqrt{1-1/\gamma^2}}{2\alpha nK_2^2(1/\Theta)}\left(1 + \frac{\sqrt{1+(\zeta(\xi)+\gamma)^2/n^2}}{\Theta}\right)\exp\left(-\frac{\gamma+\sqrt{1+(\zeta(\xi)+\gamma)^2/n^2}}{\Theta}\right)d\gamma d\xi \tag{C8}$$

$$= f_\xi(\xi)d\xi. \tag{C9}$$

Writing $\bar{p}_z(\gamma,\xi) = (\zeta(\xi)+\gamma)/n$ yields a more compact expression:

$$f_\xi(\xi) = \int_1^\infty \frac{\gamma^2\sqrt{1-1/\gamma^2}}{2\alpha nK_2^2(1/\Theta)}\left(1 + \frac{\sqrt{1+\bar{p}_z^2(\gamma,\xi)}}{\Theta}\right)\exp\left(-\frac{\gamma+\sqrt{1+\bar{p}_z^2(\gamma,\xi)}}{\Theta}\right)d\gamma, \tag{C10}$$

which is Eq. 23.

Appendix D Derivation of non-relativistic $f_{\bar{p}_\perp}$

From Eq. A10,

$$f_{\bar{p}_\perp} = \frac{\bar{p}_\perp\sqrt{1+\bar{p}_\perp^2}}{\Theta K_2(1/\Theta)}K_1\left(\frac{\sqrt{1+\bar{p}_\perp^2}}{\Theta}\right). \tag{D1}$$

For $\bar{p}_\perp \ll 1$,

$$\sqrt{1+\bar{p}_\perp^2} \simeq 1 + \frac{\bar{p}_\perp^2}{2} \tag{D2}$$

For $\Theta \ll 1$, it is seen that (Watson, 1995, Section 7.23)

$$K_2(1/\Theta) \simeq \sqrt{\frac{\pi\Theta}{2}}e^{-1/\Theta}, \tag{D3}$$

and for small $\Theta \ll 1$ and $\bar{p}_\perp \ll 1$,

$$K_1\left(\frac{\sqrt{1+\bar{p}_\perp^2}}{\Theta}\right) \simeq K_1\left(\frac{1}{\Theta}+\frac{\bar{p}_\perp^2}{2\Theta}\right) \tag{D4}$$

$$\simeq \sqrt{\frac{\pi\Theta}{2+\bar{p}_\perp^2}}\exp\left(-\frac{1}{\Theta}-\frac{\bar{p}_\perp^2}{2\Theta}\right), \tag{D5}$$

so to lowest order,

$$f_{\bar{p}_\perp} \simeq \frac{\bar{p}_\perp}{\Theta}\sqrt{1+\frac{\bar{p}_\perp^2}{2}}\exp\left(-\frac{\bar{p}_\perp^2}{2\Theta}\right) \tag{D6}$$

$$\simeq \frac{\bar{p}_\perp}{\Theta}\exp\left(-\frac{\bar{p}_\perp^2}{2\Theta}\right). \tag{D7}$$

The most likely $\bar{p}_\perp$ value given this probability distribution function is

$$\bar{p}_{\perp,\mathrm{ML}} = \sqrt{\Theta}. \tag{D8}$$

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