University of Nevada, Reno

Invariants from Group Algebras via Topological Quantum Field Theory

A thesis submitted in partial fulfillment of the requirements for the degree of Bachelor of Arts in Mathematics with Honors

by

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Abstract

We describe a classical characterization of a Frobenius algebra A as an associative algebra equipped with a comultiplication δ which is A-linear. We use this characterization to establish the equivalence of categories between commutative Frobenius algebras and two-dimensional topological quantum field theories, a fact which is well known to experts. We then use the equivalence to derive topological invariants for closed oriented surfaces, such as the genus of a surface, using Frobenius algebras. We use the above results to provide a partial identification of those Frobenius structures on a group algebra which distinguish between closed oriented surfaces of any genus.

Contents

1 Introduction

The goal of this project is to understand the equivalence between two mathematical objects: the category of commutative, finite dimensional Frobenius algebras over a field k, and the category of two-dimensional topological quantum field theories. The study of these objects involves several fields of mathematics, including differential topology and abstract algebra.

To make this equivalence precise, we must first outline a few fundamental mathematical concepts. From there, we will define the categories of commutative, finite dimensional Frobenius algebras and 2-D topological quantum field theories (TQFTs). We will then describe the equivalence of categories and show how this equivalence can be used to reconstruct classical invariants for closed oriented surfaces, such as the genus of a surface. Lastly, we will partially classify Frobenius structures on group algebras according to their capacity to yield complete topological invariants on the corresponding TQFTs.

2 Overview and Main Results

In Sec. 3, we give several equivalent characterizations of commutative Frobenius algebras. In Sec. 4, we outline some language from category theory needed to discuss the equivalence of categories between finite dimensional commutative Frobenius algebras and two dimensional TQFTs. In Sec. 5, we discuss the category of 2-dimensional cobordisms, which is an essential part of the definition of a TQFT. In Sec. 6, we outline the proof of the equivalence of categories and then provide some applications to classifying closed oriented 2-manifolds in Sec. 7. We conclude by investigating the Frobenius structures on the group algebra of a finite abelian group in Sec. 8. We introduce the notion of a "complete invariant" associated to a Frobenius algebra. This is a purely algebraic concept, designed so that a Frobenius algebra admits a complete invariant if and only if its associated TQFT can distinguish between any two closed oriented surfaces. For a finite abelian group algebra $k[G]$, we show that the set of Frobenius structures which admit complete invariants is:

1. non-empty if $\operatorname{char} k = 0$ and,

2. not stable under the action of the group of units $k[G]^\times$ by precomposition.

We end the thesis in Sec. 9 with some concluding remarks and speculation for future work.

3 Frobenius Algebras

The defining properties of Frobenius algebras can be used to interpret topological data in the form of algebraic structures. We begin by characterizing Frobenius algebras in three equivalent ways.

Convention 3.1. Fix k to be any field. All k-algebras are taken to be associative with unit 1 and are of finite dimension. All tensor products are taken over k.

Definition 3.2. [2, Definition 2.2.1] A **Frobenius algebra** is a finite-dimensional k-algebra A equipped with a k-linear map $\varepsilon \colon A \to k$, called a **Frobenius form**, such that the nullspace of ε contains no nontrivial left ideals of A. We denote Frobenius algebra by (A, ε) to indicate the Frobenius structure.

Remark 3.3. For the purpose of relating Frobenius algebras with 2D topological quantum field theories, we will be primarily concerned with Frobenius algebras which are commutative. <u>We will assume A to be a commutative</u> k-algebra from here, unless stated otherwise.

Let $A^\times \subseteq A$ denote the group of units (i.e., invertible elements) of A. Then the left module action of A on A^* induces an obvious left action of $A^\times$ on A^* given by $u \cdot \varepsilon(a) = \varepsilon(u \cdot a)$ for all $a \in A$, where $u \in A^\times$.

Proposition 3.4. *Let $A^*_{\text{Frob}} \subseteq A^*$ denote the set of Frobenius forms on A. Then*

1. *A^*_{Frob} is invariant under the action of $A^\times$.*

2. *If A^*_{Frob} is non-empty, then the action of $A^\times$ is free and transitive.*

Proof. 1. Let $u \in A^\times$ and $\varepsilon \in A^*_{\text{Frob}}$. Let $I \trianglelefteq Null(u \cdot \varepsilon)$. Since I is an ideal, $uI = I$, and hence $0 = u \cdot \varepsilon(I) = \varepsilon(uI) = \varepsilon(I)$. Since ε is Frobenius, we conclude $I = 0$.

2. Consider the induced left A-module isomorphisms $\phi \colon A \to A^*$ defined by $1 \mapsto \varepsilon$, and $\phi' \colon A \to A^*$ defined by $1 \mapsto \varepsilon'$. Since ϕ and ϕ' are isomorphisms, there exits a unique element $x \in A$ such that $\phi(x) = \phi(x \cdot 1) = x \cdot \phi(1) = x \cdot \varepsilon = \varepsilon'$. Similarly, there exists a unique element $y \in A$ such that $\phi'(y) = \phi'(y \cdot 1) = y \cdot \phi'(1) = y \cdot \varepsilon' = \varepsilon$. Thus, the action

by $A^\times$ is transitive. Then we have that $\phi^{-1}(\varepsilon) = \phi^{-1}(y \cdot \varepsilon') = y \cdot \phi^{-1}(\varepsilon') = y \cdot x = 1$. Therefore x, y are units with $y = x^{-1}$. The uniqueness of the elements x, y ensures that the group action is free.

$\square$

It follows that we have the following proposition:

Proposition 3.5. *[2, Lemma 2.2.8] If ε is a Frobenius form on a k-algebra A, then every other Frobenius form ε' on A is of the form $\varepsilon' = u \cdot \varepsilon$, where $u \in A^\times$.*

Recall that every linear functional on a k-algebra $\varepsilon \in A^*$ naturally determines a pairing $A \otimes A \longrightarrow k$ by the assignment $x \otimes y \mapsto \varepsilon(xy)$. On the other hand, every pairing $\beta\colon A \otimes A \to k$ determines a linear functional by setting $\varepsilon(a) = \beta(1 \otimes a)$ for all $a \in A$.

Definition 3.6. [2, Section 2.1, 2.2] A pairing $\beta\colon A \otimes A \to$ is called **nondegenerate** if there exists a k-linear map $\gamma\colon k \to A \otimes A$ such that

$$(\beta \otimes id_A) \circ (id_A \otimes \gamma) = id_A = (id_A \otimes \beta) \circ (\gamma \otimes id_A)$$

We call γ the **copairing** to β. We say that β is a **Frobenius pairing** if it is nondegenerate and associative, i.e. $\beta(xy \otimes z) = \beta(x \otimes yz)$ for all $x, y, z \in A$.

Since we assume A is always finite-dimensional, β is nondegenerate in the sense above if and only if $\beta(x \otimes y) = 0$ for all $x \in A$ implies that $y = 0$, and $\beta(x \otimes y) = 0$ for all $y \in A$ implies that $x = 0$. We shall make use of both characterizations of nondegeneracy.

Proposition 3.7. *[2, Section 2.2] Let A be a finite-dimensional k-algebra.*

1. *If $\beta\colon A \otimes A \to k$ is a Frobenius pairing, then the functional $\varepsilon_\beta\colon A \to k$ defined as $\varepsilon_\beta(a) = \beta(1 \otimes a)$ is a frobenius form, and hence (A, ε) is a Frobenius algebra.*

2. *If $\varepsilon\colon A \to k$ is a Frobenius form on A, then the induced pairing $\beta_\varepsilon\colon A \otimes A \to k$, defined as $\beta_\varepsilon(x, y) = \varepsilon(xy)$ is a Frobenius pairing.*

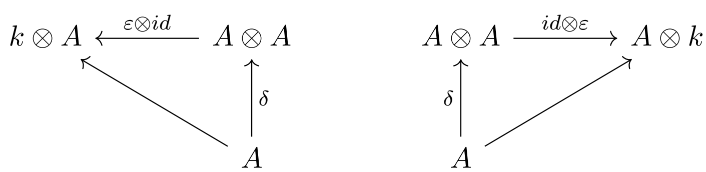
Proof. 1. Suppose $x \in A$ such that $\varepsilon_\beta(xA) = 0$. Then $\beta(1 \otimes xa) = 0$ for all $a \in A$, but since β is nondegenerate, $xa = 0$ for all $a \in A$, hence $x = 0$.

2. Suppose $x \in A$ such that $\beta_\varepsilon(x \otimes y) = 0$ for all $y \in A$. Then $\varepsilon(xy) = 0$ for all $y \in A$, or equivalently $\varepsilon(xA) = 0$. Therefore $x = 0$ since ε is a Frobenius form, so β_ε is nondegenerate. Associativity of the pairing follows from the fact that A is an associative algebra.

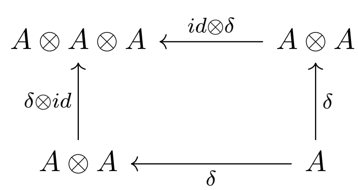
□

Remark 3.8. As a consequence of Proposition 3.7, we may either specify the Frobenius structure on an algebra by either its Frobenius form (A, ε), or equivalently, its corresponding pairing (A, β_ε).

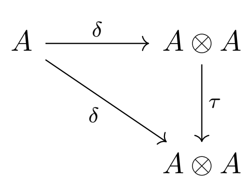
Definition 3.9. [2, Definition 2.3.1] A **coalgebra** is a k-vector space A equipped with two k-linear maps $\delta\colon A \to A \otimes A$ and $\varepsilon\colon A \to k$, called the comultiplication and counit respectively such that the following diagrams commute, called the **counit axioms**:



We say that δ is **coassociative** if the following diagram commutes:



We say that δ is **cocommutative** if the following diagram commutes:



where $\tau\colon A \otimes A \to A \otimes A$ is the twist map $\tau(x \otimes y) = y \otimes x$.

Lemma 3.10. *[2, Lemma 2.3.13] Let (A, β) be a Frobenius algebra and let $\phi\colon A \otimes A \otimes A \to k$*

be defined by $\phi = \beta \circ (\mu \otimes id_A) = \beta \circ (id_A \otimes \mu)$*. Then we have the following equality:*

$$(id_A \otimes \phi) \circ (\gamma \otimes id_A \otimes id_A) = \mu = (\phi \otimes id_A) \circ (id_A \otimes id_A \otimes \gamma)$$

Proof. We have that

$$\begin{aligned}(\phi \otimes id) \circ (id \otimes id \otimes \gamma) &= [(\beta \circ (\mu \otimes id)) \otimes id] \circ (id \otimes id \otimes \gamma) \\ &= (\beta \otimes id) \circ (\mu \otimes \gamma) \\ &= (\beta \otimes id) \circ (id \otimes \gamma) \circ \mu\end{aligned}$$

But this last expression is just μ by the nondegeneracy of β. A similar substitution shows the other equality.

□

Lemma 3.11. *[2, Lemma 2.3.15] Let* (A, β) *be a Frobenius algebra. Then we have the following equality of compositions:*

$$(\mu \otimes id_A) \circ (id_A \otimes \gamma) = (id_A \otimes \mu) \circ (\gamma \otimes id_A)$$

Proof. The result follows from substituting μ for either $(id_A \otimes \phi) \circ (\gamma \otimes id_A \otimes id_A)$ or $(\phi \otimes id_A) \circ (id_A \otimes id_A \otimes \gamma)$ from the previous lemma.

□

Theorem 3.12. *[2, Propositon 2.3.24] Let* (A, ε) *be a Frobenius algebra, and* γ *the copairing for the corresponding Frobenius pairing* β*. Then* (A, δ, ε)*, where* $\delta\colon A \to A \otimes A$ *is the linear map*

$$\delta = (id_A \otimes \mu) \circ (\gamma \otimes id_A),$$

is a cocomnutative, counital coalgebra which satisfies the condition

$$(id \otimes \mu) \circ (\delta \otimes id) = \delta \circ \mu = (\mu \otimes id) \circ (id \otimes \delta). \tag{3.1}$$

Conversely, if (A, δ, ε) *is a cocomnutative counital coalgebra satisfying* (3.1)*, then* (A, ε) *is a Frobenius algebra.*

Remark 3.13. Condition (3.1) is known as the **Frobenius condition**. By Theorem 3.12, we may either specify the Frobenius structure on an algebra by either its Frobenius form (A, ε), or equivalently, as the coalgebra (A, δ, ε) described above.

Proof. Suppose first that A is a k-algebra with multiplication $\mu\colon A \otimes A \to A$ and unit map $\eta\colon k \to A$, equipped with a coassociative cocommutative comultiplication $\delta\colon A \to A \otimes A$ and counit $\varepsilon\colon A \to k$ such that the condition

$$(id \otimes \mu) \circ (\delta \otimes id) = \delta \circ \mu = (\mu \otimes id) \circ (id \otimes \delta)$$

is satisfied. We will construct a pairing $A \otimes A \longrightarrow k$ and show that it is associative and nondegenerate. Let $\beta\colon A \otimes A \to k$ be defined by $\beta\colon = \varepsilon \circ \mu$ and let $\gamma\colon k \to A \otimes A$ be defined by $\gamma\colon = \delta \circ \eta$. Verifying the nondegeneracy of β amounts to showing that the identities $(id_A \otimes \beta) \circ (\gamma \otimes id_A) = id_A = (\beta \otimes id_A) \circ (id_A \otimes \gamma)$ hold. But since the Frobenius relation holds, we have that

$$\begin{aligned}(id \otimes \beta) \circ (\gamma \otimes id) &= (id \otimes (\varepsilon \circ \mu)) \circ ((\delta \circ \eta) \otimes id)\\ &= [(id \otimes \varepsilon) \circ (id \otimes \mu)] \circ [(\delta \otimes id) \circ (\eta \otimes id)]\\ &= (id \otimes \varepsilon) \circ [(id \otimes \mu) \circ (\delta \otimes id)] \circ (\eta \otimes id)\\ &= (id \otimes \varepsilon) \circ (\delta \circ \mu) \circ (\eta \otimes id)\\ &= [(id \otimes \varepsilon) \circ \delta] \circ [\mu \circ (\eta \otimes id)].\end{aligned}$$

By the unit and counit axioms, the last line reduces to a composition of identity maps. A similar rearrangement of parentheses and utilization of the Frobenius condition shows that $id_A = (\beta \otimes id_A) \circ (id_A \otimes \gamma)$, as well, and so we have shown that β is a nondegenerate pairing. Since μ is an associative multiplication on A, we have that $\beta(xy \otimes z) = \varepsilon(\mu(\mu(x, y), z)) = \varepsilon(\mu(x, \mu(y, z))) = \beta(x \otimes yz)$ so β is associative. Therefore (A, β) is a Frobenius algebra.

Now suppose we are given a Frobenius algebra A with associated Frobenius structure maps $\beta\colon A \otimes A \to k$ and $\varepsilon\colon A \to k$. We want to construct a compatible coalgebra structure on A which is uniquely determined by the structures we already have and which satisfies the Frobenius condition. Let $\delta / maps A \to A \otimes A$ be the map defined by $\delta = (id_A \otimes \mu) \circ (\gamma \otimes id_A)$. Note that by Lemma 3.11, we also have that $\delta = (\mu \otimes id_A) \circ (id_A \otimes \gamma)$. We first show that δ is coassociative. This follows from the associativity of μ:

$$\begin{aligned}(\delta \otimes id) \circ \delta &= [[(\mu \otimes id) \circ (id \otimes \gamma)] \otimes id] \circ [(id \otimes \mu) \circ (\gamma \otimes id)] \\ &= [id \otimes [(id \otimes \mu) \circ (\gamma \otimes id)]] \circ [(\mu \otimes id) \circ (id \otimes \gamma)] \\ &= (id \otimes \delta) \circ \delta.\end{aligned}$$

Checking that δ satisfies the Frobenius condition is easier to do graphically. We comment on this in Section 5. □

Definition 3.14. Given two Frobenius algebras, (A, ε) and (A', ε'), a k-algebra homomorphism $\varphi\colon A \to A'$ is a **Frobenius algebra homomorphism** if $\varepsilon = \varepsilon' \circ \varphi$.

Definition 3.15. The category $\mathbf{cFA}_k$ has as its objects commutative Frobenius algebras over a field k. We write the vector space A together with its structure maps by $(A, \mu, \eta, \delta, \varepsilon, \tau)$ to denote the Frobenius algebra A, although we will drop this notation and simply refer to the Frobenius algebra as A itself when it is clear what these maps are. The category $\mathbf{cFA}_k$ has as its morphisms Frobenius algebra homomorphisms

$$(A, \mu, \eta, \delta, \varepsilon, \tau) \longrightarrow (A', \mu', \eta', \delta', \varepsilon', \tau')$$

Example 3.16. We calculate some Frobenius structures on a few k-algebras. The case when A is a group algebra will be of particular importance later.

1. Let $G = \{g_0, g_1, g_2, ..., g_{N-1}\}$ be a finite abelian group (written multiplicatively), with $g_0 = e_G$ and let kG denote the group algebra of G over k. We can equip kG with a Frobenius structure and calculate each structure on the elements of G, which are the generators of the group algebra. We will start by defining a Frobenius form on kG.

Let $\varepsilon\colon kG \to k$ be given by:

$$\varepsilon(g_i) = \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{if } i \neq 0 \end{cases}$$

Using the fact that $\beta = \varepsilon \circ \mu$, it is easy to check that the Frobenius pairing $\beta\colon kG \otimes kG \to k$ is given by:

$$\beta(g_i \otimes g_j) = \begin{cases} 1, & \text{if } g_j = g_i^{-1} \\ 0, & \text{if } g_j \neq g_i^{-1} \end{cases}$$

with copairing $\gamma\colon k \to kG \otimes kG$, which is:

$$\gamma(1) = \sum_{i=0}^{N-1} g_i \otimes g_i^{-1}$$

By the theorem above, we can calculate the comultiplication δ induced by our choice of ε. Since $\delta = (id \otimes \mu) \circ (\gamma \otimes id)$, the comultiplication is given by:

$$\delta(g_n) = \sum_{i=0}^{N-1} g_n g_i \otimes g_i^{-1}$$

where $g_n g_i = \mu(g_n \otimes g_i)$.

2. Let $n \in \mathbb{N}$ and define $A\colon = k[t]/(t^n)$, where $k[t]$ is the polynomial ring in variable t with coefficients in k and $(t^n) \subseteq k[t]$ is the ideal generated by t^n. Define $\varepsilon\colon A \to k$ on the generators of A by

$$\varepsilon(t^i) = \begin{cases} 1, & \text{if } i = n-1 \\ 0, & \text{if } i \neq n-1 \end{cases}$$

Then we have Frobenius pairing $\beta\colon A \otimes A \to k$ given by

$$\beta(t^i \otimes t^j) = \begin{cases} 1, & \text{if } i+j = n-1 \\ 0, & \text{if } i+j \neq n-1 \end{cases}$$

4 Monoidal Categories and Monoidal Functors

Definition 4.1. [3, Definitions 6, 7, and 8] A **symmetric monoidal category** is a category **C** equipped with the following data:

1. A bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ called the monoidal, or tensor product.

2. A neutral object $1 \in \mathbf{C}$.

3. A natural isomorphism:

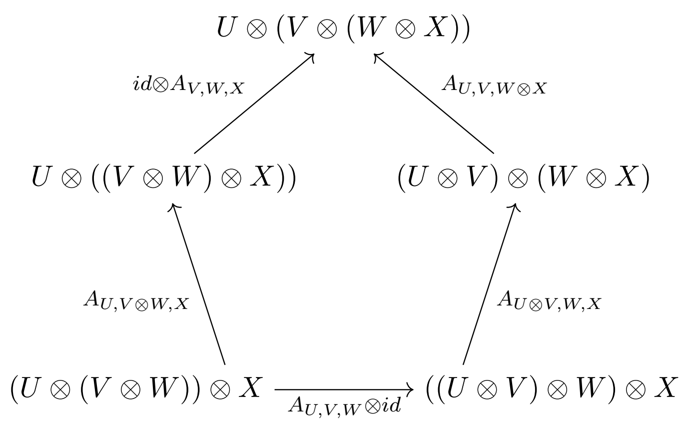
$$A_{V,W,U} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

   for objects $V, W, U \in \mathbf{C}$

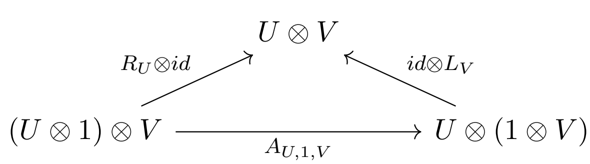
4. Natural isomorphisms $L_V : 1 \otimes V \to V$ and $R_V : V \otimes 1 \to V$ for $U \in \mathbf{C}$ called the left and right unit laws, respectively.

5. A natural isomorphism $\tau_{U,V} : U \otimes V \to V \otimes U$, called twist map, satisfying $\tau_{U,V} \circ \tau_{V,U} = id_{V \otimes U}$ for $U, V \in \mathbf{C}$

such that the **coherence constraints** are satisfied: For all objects $U, V, W, X \in \mathbf{C}$ the diagrams:



and



commute.

Definition 4.2. [3, Definition 9] Given two symmetric monoidal categories $(\mathbf{C}, \otimes, 1)$, $(\mathbf{C}', \otimes', 1')$, a **symmetric monoidal functor** $F\colon \mathbf{C} \to \mathbf{C}'$ is a functor equipped with the following data:

1. A natural isomorphism $\Phi_{U,V}\colon F(U) \otimes F(V) \to F(U \otimes V)$, and

2. An isomorphism $\phi\colon 1' \to F(1)$

which are compatible[1] with the coherence constraints.

We will also need to outline what is required for a natural transformation of symmetric monoidal functors to preserve the natural isomorphisms in order to present a well-defined characterization of the morphisms in the category **2TQFT**. Hence, we have the following definition:

Definition 4.3. [3, Definition 11] Let $(F, \Phi, \phi), (G, \Psi, \psi)\colon \mathbf{C} \to \mathbf{C'}$ be monoidal functors between symmetric monoidal categories. A natural transformation $\alpha\colon F \Rightarrow G$ is **monoidal** if the following diagrams commute:

$$\begin{array}{ccc} F(U) \otimes F(V) & \xrightarrow{\alpha(U) \otimes \alpha(V)} & G(U) \otimes G(V) \\ \Big\downarrow{\scriptstyle \Phi_{U,V}} & & \Big\downarrow{\scriptstyle \Psi_{U,V}} \\ F(U \otimes V) & \xrightarrow{\alpha(U \otimes V)} & G(U \otimes V) \end{array}$$

$$\begin{array}{ccc} 1 & \xrightarrow{\psi} & G(1) \\ \Big\downarrow{\scriptstyle \phi} & \nearrow{\scriptstyle \alpha(1)} & \\ F(1) & & \end{array}$$

The reader who is familiar with the language of monoidal categories may notice that the description above defines a *weak* symmetric monoidal category, since we only require that the maps $A_{U,V,W}$, R_U, and L_U be isomorphisms. For strict symmetric monoidal categories,

[1] See [3, Definition 9] for the precise details, which will not be needed here.

the maps $A_{U,V,W}, R_U, L_U$ are identity morphisms. However, by Mac Lane's Coherence Theorem [4, Chapter VII, Section 2], every weak symmetric monoidal category is monoidally equivalent to a strict symmetric monoidal category. Consequently we can safely assume the maps $A_{U,V,W}, R_U$, and L_U to be identity morphisms in the symmetric monoidal categories that we consider.

Example 4.4. The category of finite dimensional k-vector spaces $\mathbf{Vect}_k$ has a monoidal structure given by the tensor product of vector spaces. The neutral object then is the ground field k and the symmetric structure is given by the natural twist map:

$$\begin{aligned} \tau : U \otimes V &\to V \otimes U \\ v \otimes w &\mapsto w \otimes v. \end{aligned}$$

5 Cobordisms and Differential Topology

We now introduce **2Cob**, the category of two-dimensional cobordism classes, whose objects are the empty manifold and disjoint unions of labeled, closed 1-manifolds, i.e. copies of the circle S^1 all given the same orientation. We label the disjoint union of n copies of the circle as $\mathbf{n}$ and we label the empty manifold as $\mathbf{0}$. Therefore, we can write the object set of **2Cob** as $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, ...\}$. To describe the morphisms of **2Cob**, we need a few definitions.

Definition 5.1. [1, Sec. 4] An **oriented cobordism $\mathbf{n}$ to $\mathbf{m}$** is an oriented 2-manifold Σ together with an orientation preserving diffeomorphism $\varphi\colon \partial\Sigma \to \mathbf{n}^* \sqcup \mathbf{m}$, where $\mathbf{n}^*$ denotes $\mathbf{n}$ with the opposite orientation. We call $\mathbf{n}^*$ the 'in-boundary' of Σ and $\mathbf{m}$ the 'out-boundary' of Σ.

Two cobordisms $(\Sigma, \varphi), (\Sigma', \varphi')\colon \mathbf{n} \to \mathbf{m}$ are **equivalent** if there exists a diffeomorphism $g\colon \Sigma \to \Sigma'$ such that the following diagram commutes

$$\begin{array}{ccc} \partial\Sigma & \xrightarrow{\ \varphi\ } & \mathbf{n}^* \sqcup \mathbf{m} \\ {\scriptstyle g|_{\partial\Sigma}}\downarrow & \nearrow{\scriptstyle \varphi'} & \\ \partial\Sigma' & & \end{array}$$

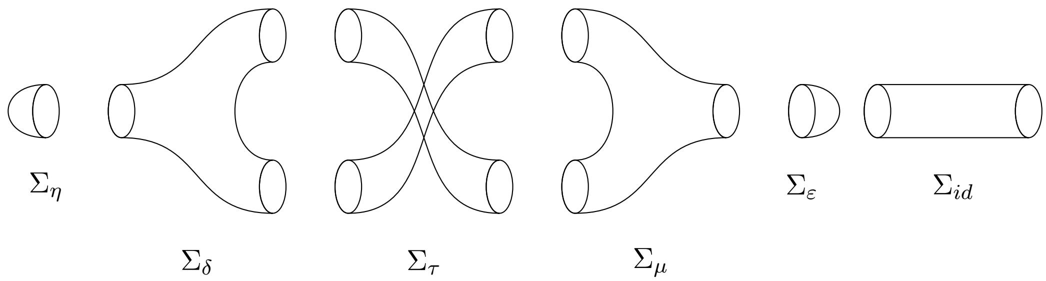
For **2Cob** to form a category, we must first define what it means to compose cobordisms. Given $(\Sigma_0, \varphi_0)\colon \mathbf{k} \to \mathbf{m}$ and $(\Sigma_1, \varphi_1)\colon \mathbf{m} \to \mathbf{n}$, we construct $\Sigma_2\colon \mathbf{k} \to \mathbf{n}$ using the diffeomorphisms $\varphi_0\colon \partial\Sigma_0 \to \mathbf{k}^* \sqcup \mathbf{m}$ and $\varphi_1\colon \partial\Sigma_1 \to \mathbf{m}^* \sqcup \mathbf{n}$ to glue Σ_0 to Σ_1 along an orientation-reversing diffeomorphism that sends the in-boundary $\mathbf{m}^*$ to the out-boundary $\mathbf{m}$, and setting $\Sigma_2 = \Sigma_0 \sqcup_{\mathbf{m}} \Sigma_1$.

In order for the definition of composing cobordisms to make sense, we need to ensure that it makes sense for the in-boundary of Σ_0 to acquire the orientation of the out-boundary of Σ_1 by first reversing the orientation of $\mathbf{m}$ is a diffeomorphic manner. Recall that there is an orientation reversing diffeomorphism $\sigma\colon \mathbf{1}^* \to \mathbf{1}$ which can be extended to a diffeomorphism $\mathbf{m}^* \longrightarrow \mathbf{m}$. It is not clear that σ induces a cobordism as we have defined. To resolve the discrepancy, we identify the cobordism induced by σ with the cobordism $\mathbf{0} \longrightarrow \mathbf{1}^* \sqcup \mathbf{1}$.

Now, we take the morphisms of **2Cob** to be oriented diffeomorphism classes of cobordisms as defined above. This definition is independent of the choice of cobordism class representative and defines an associative composition law on the object set of **2Cob**.

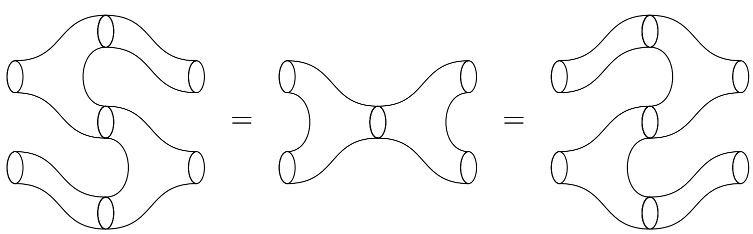
We will now illustrate the generating set morphisms in **2Cob** and the relations that they satisfy. These generators and relations play a crucial role in establishing the equivalence between 2-D TQFT's and commutative Frobenius algebras.

Let $\Sigma_\eta, \Sigma_\delta, \Sigma_\tau, \Sigma_\mu, \Sigma_\varepsilon, \Sigma_{id}$ denote the following diffeomorphism classes:



These morphisms satisfy the following relations:

(1)



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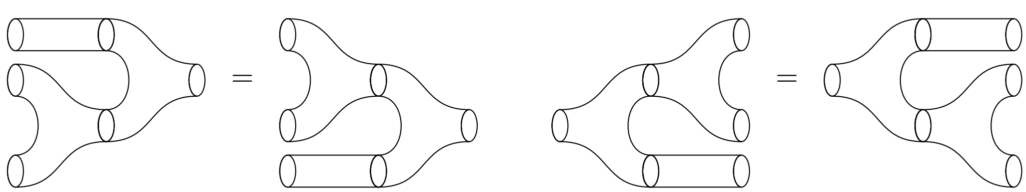

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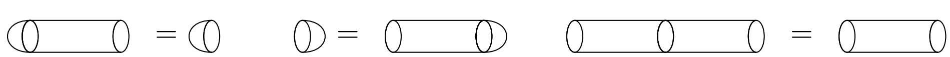

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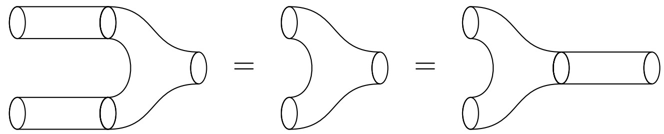

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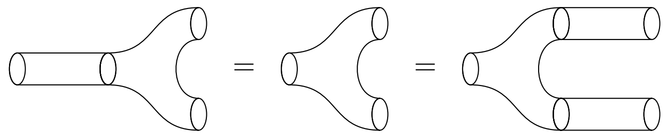
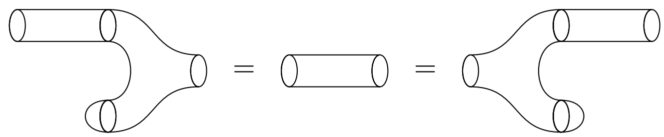
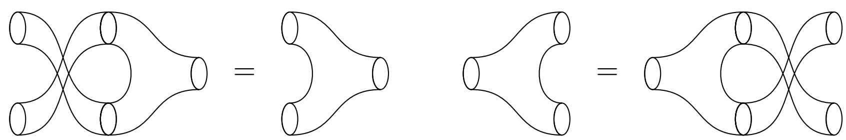


(6)



(7)



Relation (1) is the graphical expression of the Frobenius condition. Using the cobordism class Σ_δ as a representation of the comultiplication δ as defined in Theorem 3.12, Σ_μ as the multiplication μ, Σ_{id} as the identity map id_A, and so forth, it is clear graphically that δ satisfies the Frobenius condition, once relations (2) and (7) are satisfied (associativity and coassociativity, and commutativity and cocommutativity, respectively).

Proving that the morphisms above constitute a generating set for the morphisms of **2Cob** can be done by examining the critical points of Morse functions on a compact surface Σ. We will need some results form Morse theory, which are stated without here without proof.

Lemma 5.2. *[2, Corollary 1.4.21] If a cobordism Σ admits a Morse function without critical points, then Σ is diffeomorphic to a composition of twist cobordisms and identity cobordisms (cylinders).*

Lemma 5.3. *[2, Lemma 1.4.22] For compact 2-manifolds, critical points may be classified as follows, in terms of their index:*

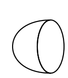
1. *index 0 critical points correspond to local minima,*
2. *index 1 critical points correspond to saddle points, and*
3. *index 2 critical points correspond to local maxima.*

Let Σ be a compact, connected, orientable surface with a Morse function $\Sigma \to [0,1]$. If there exists a unique critical point $x \in \Sigma$ that is a saddle point (i.e. has index 1), then Σ is diffeomorphic to the pair of pants, or the reverse pair of pants.

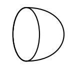
Theorem 5.4. *[2, Proposition 1.4.13] The morphisms $\Sigma_\mu, \Sigma_\eta, \Sigma_\delta, \Sigma_\varepsilon, \Sigma_\tau$ generate the set of morphisms of **2Cob**.*

Proof. To show that the morphisms outlined above generate the morphisms of **2Cob**, we must show that given a compact, connected, oriented surface (thought of as a representative of an oriented diffeomorphism class of cobordisms), we can decompose the surface into a composition of the generators.

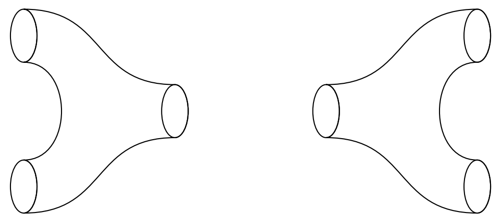
Let $\Sigma\colon \mathbf{m} \to \mathbf{n}$ be a cobordism and let $h\colon \Sigma \to [0,1]$ be a Morse function on Σ such that $h^{-1}(0) = \mathbf{m}^*$ and $h^{-1}(1) = \mathbf{n}$. By Sard's theorem, we can find a set of regular values $\{y_0, y_1, ..., y_k\} \subset [0,1]$ such that in each interval $[y_i, y_{i+1}]$, there is at most one critical value of h. Consider one of these intervals $[y_i, y_{i+1}]$ and suppose $x \in h^{-1}([y_i, y_{i+1}])$ is a critical point. If $h^{-1}([y_i, y_{i+1}])$ is not connected, then x must lie in exactly one of the connected components. Lemma 5.2 above implies that the connected components which do not contain x must be diffeomorphic to a composition of identities and twists cobordisms. Therefore we may assume that $h^{-1}([y_i, y_{i+1}])$ is connected and contains the critical point x. If x has index 0, then it is a local minimum and so $h^{-1}([y_i, y_{i+1}])$ is diffeomorphic to:



If x has index 2, then it is a local maximum and so $h^{-1}([y_i, y_{i+1}])$ is diffeomorphic to:



By Lemma 5.3, if x has index 1 then $h^{-1}([y_i, y_{i+1}])$ is diffeomorphic to one of:



$\square$

Now that we have outlined the crucial properties of **2Cob**, we can begin discussing the equivalence of categories between **2TQFT** and **cFA**$_k$.

6 The Categories 2TQFT and cFA

Definition 6.1. [2, Definition 3.2.54] The category **2TQFT** has as its objects symmetric monoidal functors

$$Z\colon \mathbf{2Cob} \to \mathbf{Vect}_k$$

which are two-dimensional topological quantum field theories, and has as its morphisms monoidal natural transformations

$$Z \Longrightarrow Z'$$

Theorem 6.2. *[2, Theorem 3.3.2] There is a canonical equivalence of categories* $\boldsymbol{2TQFT} \simeq \boldsymbol{cFA}_k$.

Proof. Let $\mathcal{F}\colon \mathbf{2TQFT} \to \mathbf{cFA}_k$ be the functor defined on objects to be

$$\mathcal{F}(Z) = (Z(\mathbf{1}), Z(\Sigma_\mu), Z(\Sigma_\eta), Z(\Sigma_\delta), Z(\Sigma_\varepsilon), Z(\Sigma_\tau))$$

where $\{\Sigma_\mu, \Sigma_\eta, \Sigma_\delta, \Sigma_\varepsilon, \Sigma_\tau\}$ is the set of generators for **2Cob**. The relations (2), (6) and (7) above guarantee that $(Z(\mathbf{1}), Z(\Sigma_\mu), Z(\Sigma_\eta))$ defines an associative, commutative k-algebra, $Z(\Sigma_\delta)$, is coassociative and cocommutative, and that $Z(\Sigma_\varepsilon)$ satisfies the counit axioms. Relation (1) implies that $Z(\Sigma_\delta)$ satisfies the Frobenius condition. Hence, the functor $\mathcal{F}$ is well defined on objects by Theorem 3.12.

We define the inverse of $\mathcal{F}$ to be the functor $\mathcal{G}\colon \mathbf{cFA}_k \to \mathbf{2TQFT}$ which sends a commutative Frobenius algebra $(A, \mu, \eta, \delta, \varepsilon, \tau)$ to the topological quantum field theory Z which has $Z(\mathbf{1}) = A$, $Z(\Sigma_\mu) = \mu$, $Z(\Sigma_\eta) = \eta$, $Z(\Sigma_\delta) = \delta$, $Z(\Sigma_\varepsilon) = \varepsilon$, and $Z(\Sigma_\tau) = \tau$. Since the Frobenius algebra conditions on the structure maps $\mu, \eta, \delta, \varepsilon, \tau$, hold and the set $\{\Sigma_\mu, \Sigma_\eta, \Sigma_\delta, \Sigma_\varepsilon, \Sigma_\tau\}$ generates the morphisms of **2Cob** by Theorem 5.4, $\mathcal{G}$ is well defined. The relations that the structure maps of a Frobenius algebra satisfy then translate into the relations (1) - (7) in the category **2Cob**. Therefore we have a one-to-one correspondence between the objects of $\mathbf{cFA}_k$ and the objects of **2TQFT**.

Now, given a morphism $\phi\colon A \to A'$ of Frobenius algebras, there is a unique morphism $\Phi\colon Z \Rightarrow Z'$ such that $\Phi(\mathbf{1}) = \phi$. The fact that TQFTs are symmetric monoidal functors and that Φ is monoidal means that Φ is determined entirely by its action on the object $\mathbf{1}$ in **2Cob**. The compatibility of Φ with the morphisms of **2Cob** is guaranteed by the fact that the maps $\delta, \mu, \eta, \varepsilon, \tau$ are (co)associative, (co)commutative and satisfy the (co)unit axioms, respectively. A similar argument shows that a monoidal natural transformation $\Phi\colon Z \Rightarrow Z'$ determines a Frobenius algebra homomorphism under $\mathcal{F}$.

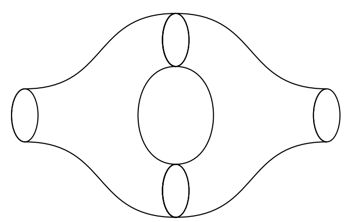
$\square$

7 Applications to Classifying Closed Oriented 2-Manifolds

Recall that the genus of a surface completely determines the diffeomorphism type of oriented closed surfaces. We can use the correspondence between Frobenius algebras and topological quantum field theories to recover classical data like the genus of a closed oriented 2-manifold and, indeed, we can also construct topological invariants which are invariant under diffeomorphism using certain Frobenius algebras. Not every Frobenius algebra will return a complete invariant, however, and the next two sections outline a few of the cases when a Frobenius algebra does give us a complete invariant.

Definition 7.1. [2, p. 128, Exercise 4] Let $(A, \mu, \eta, \delta, \varepsilon)$ be a Frobenius algebra. The **handle operator** of A is the A-linear map $\hat{\omega} = \mu \circ \delta$. The **handle element** of A is the element $\omega = \hat{\omega}(1_A) \in A$, where $1_A = \eta(1_{\Bbbk})$.

Remark 7.2. Note that since the handle operator is A-linear, it is determined entirely by the handle element. Moreover, the n-fold composition $\hat{\omega}^{\circ n} \colon A \to A$ is determined by the product $\omega^n \in A$. In **2Cob**, the composition of cobordism classes corresponding to



represents the handle operator.

In **2Cob**, the composition of cobordism classes corresponding to

Definition 7.3. Let A be a Frobenius algebra. We define the **invariant associated to** A to be the function of sets $I_A \colon \mathbb{N} \to k$ given by

$$I_A(n) = \varepsilon(\omega^n)$$

We say that I_A is a **complete invariant** if and only if I_A is injective.

Example 7.4. Here are a few cases when the invariant associated to a Frobenius algebra is not complete.

1. If A is any Frobenius algebra over a finite field $\mathbb{F}$, then the invariant I_A is not complete. Indeed, any function $\mathbb{N} \to \mathbb{F}$ cannot be injective since $|\mathbb{F}| < \infty$.

2. Let $A = k[t]/(t^2)$. Let $\varepsilon\colon A \to k$ be $1 \mapsto 0$ and $t \mapsto 1$. Then the comultiplication $\delta\colon A \to A \otimes A$ induced by this Frobenius form is

$$\begin{aligned} 1 &\longmapsto 1 \otimes t + t \otimes 1 \\ t &\longmapsto t \otimes t \end{aligned}$$

We now can calculate the handle element of A:

$$\omega = \mu \circ \delta(1) = \mu(1 \otimes t + t \otimes 1) = 2t$$

Therefore $\omega^2 = 0$, so I_A is not complete. In general, the Frobenius algebra $k[t]/(t^n)$ is not complete for all $n \in \mathbb{N}$.

8 Frobenius Structures on Group Algebras

We now turn our attention to group algebras, which always admit a Frobenius structure. We describe some cases when the Frobenius algebra structure on a group algebra can return a complete invariant.

Definition 8.1. Fix G to be a finite abelian group $G = \{g_0, g_1, ..., g_{N-1}\}$ with $g_0 = e_G$ and set $A = kG$. Let $\delta_{\text{std}} \colon A \to A \otimes A$ be the A-linear map given by

$$\delta_{\text{std}}(g) = \sum_{i=0}^{N-1} gg_i \otimes g_i^{-1} \text{ for all } g \in G$$

Let $\varepsilon_{\text{std}} \colon A \to k$ be the k-linear map given by

$$\varepsilon_{\text{std}}(g) = \begin{cases} 1, & \text{if } g = e_G \\ 0, & \text{if } g \neq e_G \end{cases}$$

We will denote kG equipped with $\delta_{\text{std}}, \varepsilon_{\text{std}}$ as kG_{std}, which we will call the **standard Frobenius structure on** kG.

Theorem 8.2. *Let G be a nontrivial finite abelian group and let $I_{kG}^{\text{std}} \colon \mathbb{N} \to k$ denote the invariant of the Frobenius algebra kG_{std}.*

1. *We have*

$$I_{kG}^{\text{std}}(n) = |G|^n$$

2. *Suppose $char(k) \nmid |G|$. Then I_{kG}^{std} is complete if and only if the integer $|G|$ is not a root of unity in k.*

3. *I_{kG}^{std} is complete if and only if $char(k) = 0$.*

Proof.

1. Observe that $|G| = N$ and that we obtain the following from the standard Frobenius

structure on kG:

$$\begin{aligned}\omega = \hat{\omega}(g_0) &= \mu(\delta(g_0)) \\ &= \mu(\sum_{i=0}^{N-1} g_i \otimes g_i^{-1}) \\ &= \sum_{i=0}^{N-1} \mu(g_i \otimes g_i^{-1}) \\ &= N \cdot g_0.\end{aligned}$$

It follows that $I_{kG}^{std}(1) = |G|$. Now, suppose $I_{kG}^{std}(n) = |G|^n$ for some $n \in \mathbb{N}$. Then $I_{kG}^{std}(n+1) = \varepsilon(\omega^{n+1}) = \varepsilon(\omega^n \cdot \omega) = \varepsilon(\omega^n \cdot N) = N \cdot \varepsilon(\omega^n) = N \cdot I_{kG}^{std}(n) = N \cdot |G|^n = |G|^{n+1}$. Therefore $I_{kG}^{std}(n) = |G|^n$ for all $n \in \mathbb{N}$.

2. Let k be such that $char(k) \nmid |G|$ and observe that this ensures $I_{kG}^{std}(n) \neq 0$ for all n. Suppose that I_{kG}^{std} is complete, but that $|G|$ is a root of unity in k. Then $I_{kG}^{std}(n) = |G|^n = 1$ for some n. Thus, for any $m \in \mathbb{N}$ we also have that $I_{kG}^{std}(mn) = |G|^{mn} = 1$, a contradiction.

   Conversely, suppose I_{kG}^{std} is not complete. Then there exist $m, n \in \mathbb{N}$ with $m \neq n$ such that $|G|^m = |G|^n$. Hence, $|G|^{|m-n|} = 1$, so $|G|$ is a root of unity in k.

3. Suppose $char(k) = p$. Since $|G|$ is an integer, Fermat's Little Theorem implies that $|G|^{p-1} = 1 \pmod{p}$. Hence, $I_{kG}^{std}(p-1) = I_{kG}^{std}(0)$. Therefore I_{kG}^{std} is not complete.

   Conversely, suppose I_{kG}^{std} is not complete. Then there exist $m, n \in \mathbb{N}$, $m \neq n$ such that $|G|^m = |G|^n$ in k. Let $\phi \colon \mathbb{Z} \to k$ be the ring homomorphism $\phi(r) = r \cdot 1$ for all $r \in \mathbb{Z}$. Then $\phi(|G|^m) = \phi(|G|^n)$, so $ker(\phi) \neq 0$, and hence $char(k) \neq 0$.

□

Now, we will examine some Frobenius structures on kG obtained via precomposing the standard Frobenius form ε_{std} with an element of G. Let $u \in G$ and let $\varepsilon_u \colon kG \to k$ be the k-linear map defined by:

$$\varepsilon_u(g) = \varepsilon_{\text{std}}(g \cdot u), \text{ for } g \in G.$$

Recall that Proposition 3.5 implies that ε_u is a Frobenius form and let $kG_u \in \mathbf{cFA}_k$ denote the corresponding Frobenius algebra.

Lemma 8.3. *The form ε_u defined above induces the following Frobenius structures on kG:*

1. *The pairing $\beta_u\colon kG \otimes kG \to k$ is given by*

$$\beta_u(g_i \otimes g_j) = \begin{cases} 1, & \text{if } g_j = g_i^{-1}u^{-1} \\ 0, & \text{if } g_j \neq g_i^{-1}u^{-1} \end{cases}$$

2. *The copairing $\gamma_u\colon \to kG \otimes kG$ is given by*

$$\gamma_u(1) = \sum_{i=0}^{N-1} g_i \otimes g_i^{-1}u^{-1}$$

3. *The comultiplication $\delta_u\colon kG \to kG \otimes kG$ is given by*

$$\delta_u(g) = \sum_{i=0}^{N-1} gg_i \otimes g_i^{-1}u^{-1}$$

Theorem 8.4. *Let kG_u be as above. Then:*

1. *The handle operator $\hat{\omega}_u\colon kG \to kG$ is determined by its action on the identity element of G, which is given by*

$$\hat{\omega}_u(g_0) = |G| \cdot u^{-1} \in kG$$

2. *For $u \neq g_0$ the invariant $I^u_{kG}\colon \mathbb{N} \to k$ associated to kG_u is given by*

$$I^u_{kG}(n) = \begin{cases} |G|, & \text{if } n = 1 \\ 0, & \text{else} \end{cases}$$

Proof. Observe that, by the preceding lemma, we have

$$\begin{aligned}\omega_u = \hat{\omega}_u(g_0) &= \mu(\delta_u(g_0)) \\ &= \mu(\sum_{i=0}^{N-1} g_i \otimes g_i^{-1}u^{-1}) \\ &= \sum_{i=0}^{N-1} \mu(g_i \otimes g_i^{-1}u^{-1}) \\ &= N \cdot g_0 \cdot u^{-1} = N \cdot u^{-1}.\end{aligned}$$

Now, by definition of ε_u, we have that for $n \in \mathbb{N}$, $I_{kG}^u(n) = \varepsilon_u(\omega_u^n) = \varepsilon_u(|G|^n \cdot u^{-n}) = |G|^n \cdot \varepsilon_u(u^{-n})$. But since $\varepsilon_u = u \cdot \varepsilon_{\text{std}}$ we have that $\varepsilon_u(u^{-n}) = \varepsilon(u^{-n+1}) = 1$ if and only if $n = 1$, and is zero otherwise. Hence $I_{kG}^u(n) = |G|$ if and only if $n = 1$, and is zero otherwise. □

Theorem 8.4 then immediately implies that if $u \in G$ is a non-identity element, I_{kG}^u is not complete.

9 Future Work

We have shown that the invariant of the standard Frobenius structure on kG is complete if and only if the characteristic of the field k is zero. Moreover, the invariant of kG_u is complete if and only the element $u \in G$ is the group identity. That is, the completeness of the invariant I_{kG}^{u} is lost when the Frobenius form is altered by a non-identity element, however it is unclear whether the completeness of the invariant would be lost when $u \in kG$ is a non-group element. Hence, for future work, we would like to determine whether using an invertible element $u \in kG - G$ to define a new Frobenius form ε_u yields a complete invariant. This would give a complete classification of invariants from Frobenius structures on group algebras over finite abelian groups. We would then expand our classification to other kinds of algebras, such as those with nilpotents. A theorem by Artin and Wedderbern [5, Chapter IX, Sec. 3] implies that every commutative semisimple algebra is isomorphic to a finite direct product of copies of the ground field k, thus commutative algebras that we are interested in fall into this category. Indeed, group alebgras over finite abelian groups are of this kind. We also wish to investigate whether there are commutative algebras which are neither nilpotent nor semisimple that admit Frobenius structures and if so, to classify their associated invariants.

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