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Reproducing Kernel Krein Spaces of Analytic Functions and Inverse Scattering

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REPRODUCING KERNEL KREIN SPACES OF ANALYTIC FUNCTIONS AND INVERSE SCATTERING

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ABSTRACT

The purpose of this thesis is to study certain reproducing kernel Krein spaces of analytic functions, the relationships between these spaces and an inverse scattering problem associated with matrix valued functions of bounded type, and an operator model.

Roughly speaking, these results correspond to a generalization of earlier investigations on the applications of de Branges' theory of reproducing kernel Hilbert spaces of analytic functions to the inverse scattering problem for a matrix valued function of the Schur class.

The present work considers first a generalization of a portion of de Branges' theory to Krein spaces. We then formulate a general inverse scattering problem which includes as a special case the more classical inverse scattering problem of finding linear fractional representations of a given matrix valued function of the Schur class and use the theory alluded to above to obtain solutions to this problem.

Finally, we give a model for certain hermitian operators in Pontryagin spaces in terms of multiplication by the complex variable in a reproducing kernel Pontryagin space of analytic functions.

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INTRODUCTION

The present thesis studies the links between various inverse scattering problems and reproducing kernel Krein spaces of analytic functions.

We recall that a $p \times p$ valued Schur function is a $p \times p$ valued function, analytic and contractive in the open upper half plane $\mathbb{C}_+$ (or in the open unit disk $\mathbb{D}$). For such a function S, the inverse scattering problem consists of finding representations of the form

$$S = (AW + B)(CW + D)^{-1} \tag{1}$$

where W is itself a $p \times p$ valued Schur function and the function $\Theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is J_0-contractive in Ω_+, where Ω_+ denotes either $\mathbb{D}$ or $\mathbb{C}_+$, i.e.

$$\Theta(\lambda) J_0 \Theta^*(\lambda) \leq J_0 \tag{2}$$

for λ in Ω_+ and

$$J_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} . \tag{3}$$

This problem encompasses a number of classical problems, in particular the inverse spectral problem (see e.g. [DMk]) and various interpolation problems (see [D1] or [DD]). The last paper treated the case where $p = 1$ and where Θ is rational. The objective of this thesis was to obtain more general solutions to (1); to that purpose we were first led to the study of de Branges' theory of reproducing kernel Hilbert spaces of analytic functions and to study the links between this theory and representations of Schur functions as in (1).

A method was developed to get solutions to (1) using de Branges' theory. This method, as well as its applications, is described in two papers ([AD1], [AD2]) and is now briefly reviewed.

We will suppose $\Omega_+ = \mathbb{C}_+$. By multiplying S by a constant of modulus 1 if need be, we can always suppose that $\det(I + S) \not\equiv 0$ in Ω_+ and thus that the function

$$\Phi = (I + S)^{-1}(I - S)$$

is well defined in $\mathbb{C}_+$; it is a Caratheodory function, i.e. its real part is positive in $\mathbb{C}_+$, and, by the Herglotz representation formula, may be written as

$$\Phi(\lambda) = A - iB\lambda + \frac{1}{i\pi} \int d\mu(\gamma) \left\{ \frac{1}{\gamma - \lambda} - \frac{\gamma}{\gamma^2 + 1} \right\}$$

where A and B are $p\times p$ valued, subject to $A+A^*=0$, $B+B^*\geq 0$ (A^* is the adjoint of A) and μ is a $p\times p$ valued function defined on $\mathbb{R}$, increasing ($t\leq t' \Rightarrow \mu(t')-\mu(t)$ is a positive $p\times p$ matrix) and subject to: (Tr denotes the trace)

$$\mathrm{Tr}\int \frac{d\mu(\gamma)}{1+\gamma^2} < \infty \ .$$

To simplify the analysis, we suppose $A=0$ and $B=0$.

Let $\mathcal{M}$ be a reproducing kernel Hilbert space of $p\times 1$ valued functions, analytic in some open Δ, which sits isometrically in $L_p^2(d\mu)$, and which is closed under the operators R_w

$$R_w f = \frac{f(\lambda)-f(w)}{\lambda - w}$$

where w and λ are in Δ. To $\mathcal{M}$ we associate the space $\mathcal{M}^\square$ of $2p\times 1$valued functions

$$\mathcal{M}^\square = \left\{ \begin{pmatrix} f \\ f_- \end{pmatrix},\ f\in\mathcal{M} \right\}$$

with

$$f_-(\lambda) = \frac{-1}{i\pi}\int d\mu(\gamma)\left\{\frac{f(\lambda)-f(\gamma)}{\lambda-\gamma} - \frac{f(\lambda)\gamma}{\gamma^2+1}\right\} \tag{4}$$

and with norm

$$\left\|\begin{bmatrix} f \\ f_- \end{bmatrix}\right\|^2_{\mathcal{M}^\square} = 2\|f\|^2_\mu \ .$$

When Δ intersects both $\mathbb{C}_+$ and the open lower half plane $\mathbb{C}_-$, $\mathcal{M}^\square$ is a reproducing kernel Hilbert space of $2p\times 1$ valued functions, the reproducing kernel of which is of the form

$$\frac{J_1 - U(\lambda)J_1U^*(w)}{-2\pi i(\lambda-\overline{w})}$$

where U is a $2p\times 2p$ valued function analytic in Δ and where

$$J_1 = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix} . \tag{5}$$

This result follows from an application of a theorem of de Branges ([dB3]) on the structure of certain reproducing kernel Hilbert spaces. Moreover, by a theorem of de Branges and Rovnyak ([dBR]), the $p\times 2p$ function $[\Phi, I_p]U$ is (Δ, J_1) admissible, i.e.

$$\sum c_j^*[\Phi(w_j), I_p]\left(\frac{U(w_j)J_1U^*(w_i)}{-2\pi i(\overline{w}_j - w_i)}\right)[\Phi(w_i), I_p]^* c_i \geq 0 \tag{6}$$

for any choice $c_1, \dots, c_r$ in $\mathbb{C}_{p\times 1}$ and $w_1, \dots, w_r$ in Δ and any integer r.

It follows further from relation (6) that (1) is satisfied with

$$\Theta = MUM^*$$

where

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} I_p & I_p \\ I_p & -I_p \end{pmatrix} .$$

Thus, to any subspace $\mathcal{M}$ of analytic functions with the above-mentioned properties, there corresponds a solution Θ to the inverse scattering problem associated to S.

A byproduct of this analysis is that the reproducing kernel of $\mathcal{M}$ must be of the form

$$\frac{E_+(\lambda)E_+^*(w) - E_-(\lambda)E_-^*(w)}{-\pi i(\lambda - \overline{w})} \tag{7}$$

where E_+ and E_- are $p \times p$ valued functions analytic in Δ. (For additional discussion, see Section 9.2.)

We also remark that "kernels" of the form (7) are characteristic of densely defined hermitian operators. Every densely defined simple hermitian operator with deficiency indices (p, p) is unitarily equivalent to multiplication by the complex variable in a reproducing kernel Hilbert space with a reproducing kernel of the form (6) (see [AD2]).

(To avoid confusion, we mention that conversely multiplication by the complex variable in a reproducing kernel Hilbert space with a reproducing kernel of the form (7) may fail to be densely defined.)

The method was described for the case $\Omega_+ = \mathbb{C}_+$, but is also valid in the case $\Omega_+ = \mathbb{D}$. One has to use the Herglotz representation theorem for functions which are analytic and have positive real part in $\mathbb{D}$.

We finally remark that (1) holds if and only if the map τ

$$F \longrightarrow [I_p, -S]F$$

maps the reproducing kernel Hilbert space $\mathcal{H}(\Theta)$ with reproducing kernel

$$\frac{J_0 - \Theta(\lambda)J_0\Theta^*(w)}{\rho_w(\lambda)}$$

contractively into the reproducing kernel Hilbert space $H(S)$ with reproducing kernel

$$\frac{I_p - S(\lambda)S^*(w)}{\rho_w(\lambda)}$$

where

$$\begin{aligned} \rho_w(\lambda) &= 1-\lambda\overline{w} & if\ \Omega_+ = \mathbb{D} \\ &= -2\pi i(\lambda-\overline{w}) & if\ \Omega_+ = \mathbb{C}_+ \end{aligned}$$

and where I_p denotes the $p \times p$ unit matrix. (This is discussed in detail in Section 8.1.)

This remark allows us to extend the definition of the inverse scattering problem to a wider class of functions. Let J be a $m \times m$ signature matrix, i.e. a matrix which is subject to

$$J = J^* = J^{-1} .$$

We will consider $k \times m$ valued functions X analytic in some open subset Δ_X of $\mathbb{C}$, and such that there exists a reproducing kernel Krein space $\mathcal{B}(X)$ of $k \times 1$ valued functions analytic in Δ_X with reproducing kernel

$$\frac{X(\lambda)JX^*(w)}{\rho_w(\lambda)} . \tag{8}$$

The inverse scattering problem for X is now, by definition, to find $m \times m$ valued functions Θ analytic in a subset of Δ_X, such that a reproducing kernel Krein space $\mathcal{K}(\Theta)$ of $m \times 1$ valued analytic functions with reproducing kernel

$$\frac{J - \Theta(\lambda)J\Theta^*(w)}{\rho_w(\lambda)}$$

exists and such that map

$$F \longrightarrow XF$$

sends $\mathcal{K}(\Theta)$ contractively into $\mathcal{B}(X)$, i.e.

$$[XF, XF]_{\mathcal{B}(X)} \leq [F, F]_{\mathcal{K}(\Theta)} .$$

We remark that the case of $p \times p$ Schur functions corresponds to $X = [I_p, -S]$ and $J = J_0$.

To study this problem, we were led to generalize part of de Branges' theory to the Krein space framework. Reproducing kernel Krein spaces with reproducing kernels of the form (8) for various classes of X and J are studied in Sections 4, 5, 6 and 7, which form the second chapter of the thesis. Of particular interest is the case where $J = \begin{pmatrix} J_0 & 0 \\ 0 & -J_0 \end{pmatrix}$, $X = [I_m, \Theta]$ and the function Θ is of bounded type in Ω_+ and J_0-unitary a.e. on the boundary of Ω_+. These functions Θ generalize the J_0-inner functions and the corresponding spaces $\mathcal{K}(\Theta)$ with reproducing kernel

$$\frac{J_0 - \Theta(\lambda)J_0\Theta^*(w)}{\rho_w(\lambda)}$$

are studied in Section 6. Even the case where the space $\mathcal{K}(\Theta)$ is finite dimensional is of interest and differences with the J-inner case appear. This is explained in Section 6.5.

Sections 4 and 5 deal with the cases:

$X = [I_p, -S]$, $J = J_0$ and S, $p \times q$ valued of bounded type in Ω_+

$X = [\Phi, I_p]$, $J = J_1$ and Φ, $p \times p$ valued of bounded type in Ω_+.
In the latter case, special attention is paid to the case where $\mathcal{B}(X)$ is a Pontryagin space. This will happen if and only if additional restrictions are placed on Φ.

Section 7 links $\mathcal{K}(\Theta)$ spaces to the spaces considered in Section 4, in the Pontryagin space framework.

The third chapter of the thesis deals with the inverse scattering problem defined above and operator models. It is broken into three sections.

Section 8 exhibits rational solutions while more general solutions are considered in Section 9. In Section 9 we also mention a Riccati equation related to this circle of ideas. Finally, the ideas developed in the thesis are of use in the study of hermitian operators in a Pontryagin space, and a model for certain hermitian operators is given in Section 10.

The contributions of the research which led to this thesis are:

a) The development of the method discussed above to solve (1) when S is in the Schur class [see Sections 6.5 and 9.2].

b) The generalization of a part of de Branges' theory to the Krein space framework (see Chapter II).

c) The reformulation of the inverse scattering problem and the application of b) to the study of its solutions (see Sections 8 and 9).

d) A model for π-hermitian operators (see Section 10).

Finally, a word on notation. $\mathbb{N}$ denotes the positive integers and $\mathbb{C}$ the complex numbers; $\mathbb{C}_{p\times q}$ denotes the $p \times q$ matrices with complex entries and $\mathbb{C}_{p\times 1}$ will usually be denoted by $\mathbb{C}_p$.

The symbols $\rho_w(\lambda)$, Ω_+, $\mathbb{D}$, $\mathbb{C}_+$, $\mathbb{C}_-$ have already been introduced. $\mathbb{E}$ will denote $\mathbb{C} \setminus \overline{\mathbb{D}}$ and Ω_- either $\mathbb{E}$ or $\mathbb{C}_-$, and ∂ will denote the boundary of Ω_+, i.e. either the real line $\mathbb{R}$ or the unit circle $\mathbb{T}$.

$w \to w'$ will denote the symmetry with respect to ∂:

$$\begin{aligned} w' &= \overline{w} \quad && if\ \partial = \mathbb{R} \\ w' &= \overline{w}^{-1} \quad && if\ \partial = \mathbb{T} \end{aligned}$$

and a subset of $\mathbb{C} \cup \{\infty\}$ is said to be symmetric if it is closed under $w \to w'$.

For a $\mathbb{C}_{p\times q}$ valued function A defined in a symmetric subset, we define $A^{\#}(w) = A^*(w')$ where the adjoint of an operator or of a matrix A (with respect to the standard inner product) will be denoted by A^*.

H_r^2 (L_r^2) denotes the usual Hardy (Lebesgue) space of $r \times 1$ valued functions with entries in the scalar Hardy (Lebesgue) space H^2 (L^2) associated with ∂, and $\underline{p}$ denotes the orthogonal projection from L_r^2 onto H_r^2. To ease the typography, the dependence upon r is not indicated.

Finally, the symbol I denotes the identity operator in any considered space.

CHAPTER I
PRELIMINARIES

Introduction

The first chapter of this thesis is devoted to various preliminary facts needed in the sequel; it is divided into three sections. In Section 1 we review the part of the theory of Krein spaces relevant to our purposes, while Section 2 deals with reproducing kernel Krein spaces and various problems associated with reproducing kernels. In Section 3, reproducing kernel Krein spaces of analytic functions are defined, and the various spaces to be studied in the second chapter are introduced.

Two theorems are proved in this chapter, which will be needed in the sequel; in Section 2 we generalize to the Pontryagin framework the theorem of Aronszajn which associates to a positive kernel a reproducing kernel Hilbert space. In Section 3 we study the structure of resolvent invariant finite dimensional vector spaces of analytic functions; though simple in proof, this theorem will give some insight into the finite dimensional aspects of the theory to be developed in the second chapter of this thesis, and this will also throw additional light on the finite dimensional versions of various theorems of de Branges.

Moreover, in Section 2, we make some connections between the notion of complementary subspaces (due to de Branges) and a theorem of Aronszajn on the sum of two positive kernels. We feel that these connections make the notion of complementary subspaces more accessible and help shed light on a number of important facts in the theory of Hilbert spaces of analytic functions.

1. Elementary facts on Krein spaces and Pontryagin spaces

In this section we review the part of the theory of indefinite inner product spaces needed for our purposes. Indefinite inner product spaces are studied for instance in Bognar's book [Bo] and the main facts and much of the relevant literature may also be found in the review paper of Azizov and Iohvidov [AI].

An indefinite inner product space V is a vector space over the complex numbers with a hermitian form $[\ ,\ ]$. Two vectors x and y will be said to be orthogonal (with respect to $[\ ,\ ]$) if $[x,y]=0$. We will denote orthogonality by $[\perp]$: $(x[\perp]y)$. Two subspaces $\mathcal{L}_1$ and $\mathcal{L}_2$ will be orthogonal: $\mathcal{L}_1[\perp]\mathcal{L}_2$, if for every ℓ_1 in $\mathcal{L}_1$ and ℓ_2 in $\mathcal{L}_2$, we have $[\ell_1,\ell_2]=0$.

A subspace $\mathcal{L}$ will be positive (negative) (neutral) if the form $[\ ,\ ]$ is positive (negative) (null) on $\mathcal{L}$, i.e., if for every x in $\mathcal{L}$, $[x,x]\geq 0$ $([x,x]\leq 0)$ $([x,x]=0)$. $\mathcal{L}$ will be positive definite if $x\neq 0$ implies $[x,x]>0$; one defines similarly negative definite subspaces.

We remark that the Cauchy-Schwarz inequality holds on positive (negative) (neutral) subspaces. The subspace $\mathcal{L}$ will be called non-degenerate if $\mathcal{L}\cap\mathcal{L}^{[\perp]}=\{0\}$. $[+]$ will denote the sum of two orthogonal subspaces. Such a sum may not be direct. When it is direct, we will write $[\dot{+}]$.

The general theory of indefinite product spaces is rather involved, and some hypotheses are made to simplify the situation. The space will be said to be decomposable if it admits a decomposition

$$V=V_+[\dot{+}]V_0[\dot{+}]V_-$$

where V_+ is positive definite, V_- is negative definite, and V_0 is neutral. Not every space is decomposable, and even when it exists, such a decomposition is not usually unique. On the other hand, Krein spaces and Pontryagin spaces are two very useful kinds of decomposable subspaces.

A Krein space is a decomposable space which admits a decomposition for which $V_0=0$ and $(V_+,[\ ,\ ])$ and $(V_-,-[\ ,\ ])$ are Hilbert spaces. A Pontryagin space is a Krein space for which one of the spaces $V_\pm$ is finite dimensional. In this thesis, we shall always take the finite dimensional space to be V_-. The number $k=dim\ V_-$ is the dimension of any maximal negative subspace of V and it is independent of the decomposition of V. To emphasize on this number, we will usually denote Pontryagin spaces by the symbol Π_k, and we will refer to k as being the rank of indefiniteness (or for simplicity, the rank) of V. k is also the number of negative squares of the hermitian form $(u,v)\mapsto[u,v]$, and, for simplicity, we will also refer to k as being the number of negative squares of a space Π_k.

We remark that the new hermitian form

$$<x,y>=[x_+,y_+]-[x_-,y_-] \tag{1.1}$$

transforms the Krein space V into a Hilbert space. We notice that the norm (1.1) depends on the decomposition of V, but ([Bo], p.102) that all such norms (1.1) are equivalent.

Krein spaces may also be defined in the following equivalent way. Let $(V, <,>)$ be a Hilbert space, let P be any orthogonal projection in V and let $[x,y] =< x, Jy >$, where $J = P - Q$ and $Q = I - P$. It is easy to check that $(V, [\ ,\ ])$ is a Krein space and that, conversely, any Krein space may be gotten in such a way. The operator J satisfies

$$J = J^* = J^{-1} \tag{1.2}$$

and will be called a signature operator.

The topology associated to a Krein space will be the topology of any of the associated Hilbert spaces; in particular, the space will be separable if, by definition, one of the associated Hilbert spaces is separable.

In the Pontryagin space case, the following equivalent way to define the topology is convenient (see [KL2].

DEFINITION 1.1. Let Π_k be a Pontryagin space. The sequence (f_n) converges to f if both the following conditions are satisfied:

1) $\lim[f_n, f_n] = [f, f]$
2) for all g in Π_k , $\lim[f_n, g] = [f, g]$.

We now mention two facts which exhibit fundamental differences between Krein spaces and Hilbert spaces. First, a closed non-degenerate subspace $\mathcal{L}$ of a Krein space Vneed not be a Krein space itself (see e.g. [Bo], p.104). Nevertheless a closed non-degenerate subspace of a Pontryagin space is itself a Pontryagin space ([Bo], p.186).

Secondly, for a Krein (or even a Pontryagin space) $V, [\ ,\ ])$, a subspace V^1 may be itself a Krein space with respect to $[\ ,\ ]$ without being closed in V (see [Bo], p.186 and p. 111).

Both facts are linked to the notion of orthocomplemented subspaces, as we now explain. For a non-degenerate subspace $\mathcal{L}$ of a Krein space V, the sum $\mathcal{L}[\dot{+}]\mathcal{L}^{[\perp]}$ is in general only dense in V. The space is orthocomplemented if $\mathcal{L}[\dot{+}]\mathcal{L}^{[\perp]} = V$ (orthocomplemented spaces can be defined in general inner product spaces; see [Bo], p.18).

The next theorem indicates the relevance of this concept.

THEOREM 1.1. ([Bo], p.104). A subspace $\mathcal{L}$ of the Krein space V is orthocomplemented if and only if $\mathcal{L}$ is closed and is a Krein space.

In spite of the differences between Krein spaces and Hilbert spaces, the Riesz representation theorem continues to hold in Krein spaces, as is easily verified.

THEOREM 1.2. Let $(V, [\ ,\ ])$ be a Krein space and let $x \mapsto Lx$ be a continuous linear functional defined on V. Then, there is a unique element y in V such that

$$Lx = [x, y]$$

for all x in V.

The Riesz representation theorem allows us to define the adjoint of an operator between two Krein spaces as in the Hilbert space case. Indeed, let T be a linear operator between two Krein spaces $(V, [\ ,\ ])$ and $(V', [\ ,\ ]')$ with dense domain $\mathcal{D}(T) \subset V$. The adjoint T^+ of T is defined by:

$$\mathcal{D}(T^+) = \{x \in V';\ y \mapsto [Ty, x]' \text{ is a continuous linear functional}\}\ .$$

T^+x is then the uniquely defined operator such that

$$[Ty, x]' = [y, T^+x]\ ,\ y \in \mathcal{D}(T)\ .$$

The various classes of operators are defined as in the Hilbert space case. For instance, a self adjoint operator is a densely defined operator from a Krein space into itself and such that $T = T^+$. The theory is much more involved than in the Hilbert space case (see e.g. [Bo] and [AI]).

In the case of Pontryagin spaces, the adjoint is called π–adjoint to emphasize that the context is that of Pontryagin spaces; similarly one will then speak of π–unitary, π-isometric and π-selfadjoint operators, rather than unitary, isometric or self adjoint operators.

We conclude this section with a discussion of π-contractions between Pontryagin spaces. In the theory of Hilbert spaces, the fact that the adjoint of a contraction is a contraction is recurrently used. The situation for Pontryagin spaces is more delicate. We first focus on π–isometries.

DEFINITION 1.2. Let $(\Pi_k, [\ ,\ ])$ and $(\Pi'_{k'}, [\ ,\ ]')$ be two Pontryagin spaces. A map τ from Π_k into $\Pi'_{k'}$ is a π isometry if, for all f and g in Π_k, we have:

$$[f, g] = [\tau f, \tau g]'\ .$$

We mention that a π-isometry may fail to be continuous (see [Bo], pp.124 & 188). The following lemma is the analogue for π-isometries of the above mentioned property for contractions between Hilbert spaces. It is more than probable that the result is known, but we were unable to extract it from the current literature and thus present a proof.

LEMMA 1.1. Let τ be a continuous π-isometry between $(\Pi_k, [\ ,\ ])$ and $(\Pi'_{k'}, [\ ,\ ]')$. The hermitian form

$$(u, v) \mapsto [u, v]' - [\tau^+ u, \tau^+ v]$$

has $k' - k$ negative squares.

PROOF. We first remark that the range of τ, *Ran* τ, is a closed non-degenerate subspace of $\Pi'_{k'}$, and thus orthocomplemented ([Bo], p.186). Indeed, let $v = \tau f$ be an element of *Ran* τ orthogonal to all *Ran* τ.Then

$$[v, \tau u]' = 0 \text{ for all } u \text{ in } \Pi_k$$

and so too

$$[f,u] = 0 \text{ for all } u \text{ in } \Pi_k$$

which forces $f = 0$, and hence $v = 0$. Next, let $v_n = \tau f_n$ be a converging sequence of elements in *Ran* τ, and let v be its limit. We want to show that v belongs to *Ran* τ. By Definition 1.1 we know that:

$$\lim[v_n, v_n]' = [v,v]'$$

$$\lim[v_n, w]' = [v,w]' \text{ for all } w \text{ in } \Pi'_{k'} \ .$$

Let w be of the special form $w = \tau f$ with f in Π_k. We have (from Definition 1.1):

$$\lim[f_n, f_n] = [v,v]'$$

$$\lim[f_n, f] = [v, \tau f]'$$

from which follows that $\lim f_n$ exists, and hence $v = \tau \lim f_n$ is in *Ran* τ. (We used here the continuity of τ.) Let now $\Pi''_{k''}$ denote the orthocomplement of *Ran* τ. Then

$$\Pi'_{k'} = Ran\ \tau[\dot{+}]\Pi''_{k''} \ .$$

By the isometry property, *Ran* τ is a Pontryagin space with k negative squares and thus, $\Pi''_{k''}$ is a Pontryagin space ([Bo], p.186, Theorem 2.2 and Corollary 2.3) with k'' negative squares. Moreover, k'' is easily seen to be equal to $k - k'$, and we may write every element u in $\Pi'_{k'}$ as $u = \tau h + u_0$ for some u_0 in $\Pi''_{k''}$.

From $[\tau f, u]' = [f, h]$ for any f in Π_k, we see that $\tau^+ u = h$. Let $v = \tau g + v_0$ be another element of $\Pi'_{k'}$. We have:

$$[u,v]' = [\tau^+ u, \tau^+ v] + [u_0, v_0]$$

i.e. $(u,v) \mapsto [u,v]' - [\tau^+ u, \tau^+ v]$ has $k - k'$ negative squares.

REMARK. When $k' = k$, the lemma says that the π-adjoint of a π-isometry is a π-contraction, i.e. for any x in Π_k we have:

$$[\tau^+ x, \tau^+ x]' \le [x,x] \ .$$

We will need an analogue of Lemma 1.1 for π-contractions. When Π_k is different from $\Pi'_{k'}$, we are unable to say much. However, if $\Pi_k = \Pi'_{k'}$, then Krein and Smulian proved the following fact ([KS], p.106):

LEMMA 1.2. Let τ be a continuous π-contraction from the Pontryagin space Π_k into itself. Then τ^+ is also a π-contraction.

Lemma 1.1 will be needed to generalize various factorization theorems in our context, as will be shown in Section 6. It seems rather difficult to generalize these lemmas to the Krein space context.

2. Reproducing Kernel Krein Spaces

Let Z be a subset of $\mathbb{C}$ and let V be a Krein space of $m \times 1$ valued functions, defined on Z, with inner product $[\ ,\ ]$. V is a reproducing kernel Krein space if there exists an $m \times m$ valued function $K_z(w)$, defined on $Z \times Z$, such that for any z in Z and c in $\mathbb{C}_{m\times 1}$, we have:

(1) The function $K_z c : w \mapsto K_z(w)c$ belongs to V.

(2) For any f in V, $c^* f(z) = [f, K_z c]$.

The function K is called the reproducing kernel of the space and is easily seen to be unique. We remark that the linear span of the $K_z c$, for z in Z and c in $\mathbb{C}_{m\times 1}$ is dense in V.

When the space V is a Pontryagin space of rank k rather than a Krein space, K uniquely determines V (as will be seen from Theorem 2.1) and $K_z(w)$ has k negative squares, i.e. (see [KL3], p.202), for any number p, any choice of $z_1, \ldots, z_p$ in Z, and $c_1, \ldots, c_p$ in $\mathbb{C}_{m\times 1}$, the self adjoint $p \times p$ matrix with ij entry

$$c_j^* K_{z_i}(z_j) c_i \tag{2.1}$$

has at most k negative eigenvalues and exactly k strictly negative eigenvalues for some choice of $p, z_1, \ldots, z_p, c_1, \ldots, c_p$. This fact is easily checked when V is finite dimensional; when V is infinite dimensional, the proof follows from the fact ([Bo], p.185, Theorem 1.4) that the linear span of the $K_z c$, being dense in V, contains a k dimensional negative definite subspace.

When $k = 0$, the matrix-valued function $K_z(w)$ will be termed positive when all the matrices defined by (2.1) have positive or zero eigenvalues. Moore (see [Ar]) showed how to associate to any positive function K a reproducing kernel Hilbert space with reproducing kernel K. In case K has k negative squares, we have (see also [B2]):

THEOREM 2.1. Let Z be a subset of $\mathbb{C}$ and let $K_z(w)$ be a $\mathbb{C}_{m\times m}$ valued function defined on $Z \times Z$, which has k negative squares. Then there exists a unique reproducing kernel Pontryagin space of rank k $(\Pi(K), [\ ,\ ])$ such that, for any z in Z and c in $\mathbb{C}_{m\times 1}$

1) $K_z c$ belongs to $\Pi(K)$

2) for all f in $\Pi(K)$, $c^* f(z) = [f, K_z c]$.

PROOF. We first outline the proof, and then fill in the details in a number of steps. Let $\Pi^\circ(K)$ be the set of finite linear combinations of the functions $K_w c$, for w in Z and c in $\mathbb{C}_{m\times 1}$. On $\Pi^\circ(K)$, we define a hermitian form via the rule

$$[K_\alpha c, K_\beta d] = d^* K_\alpha(\beta) c\ .$$

We have to show that $[\ ,\ ]$ is well defined, and that $(\Pi^\circ(K), [\ ,\ ])$ has no subspaces of dimension $k+1$ which are negative with respect to $[\ ,\ ]$. $\Pi(K)$ will then be the completion of $\Pi^\circ(K)$.

STEP 1. $[\ ,\ ]$ is well defined.

PROOF OF STEP 1. Let f and g be two elements of $\Pi^\circ(K)$, and suppose that:

$$f = \Sigma K_{\alpha_i}\varsigma_i = \Sigma K_{\alpha'_i}\varsigma'_i$$

$$g = \Sigma K_{\beta_j}\xi_j = \Sigma K_{\beta'_j}\xi'_j \ .$$

We want to show that:

$$[\Sigma K_{\alpha_i}\varsigma_i, \Sigma K_{\beta_j}\xi_j] = [\Sigma K_{\alpha'_i}\varsigma'_i, \Sigma K_{\beta'_j}\xi'_j] \ .$$

Indeed

$$\begin{aligned}
[\Sigma K_{\alpha_i}\varsigma_i, \Sigma K_{\beta_j}\xi_j] &= \sum_i\sum_j \xi_j^* K_{\alpha_i}(\beta_j)\varsigma_i \\
&= \sum_j \xi_j^*(\sum_i K_{\alpha_i}(\beta_j)\varsigma_i) \\
&= \sum_j \xi_j^* f(\beta_j) \\
&= \sum_j \xi_j^*(\sum_i K_{\alpha'_i}(\beta_j)\varsigma'_i) \\
&= \sum_i\sum_j \xi_j^* K_{\alpha'_i}(\beta_j)\varsigma'_i \\
&= (\sum_i\sum_j \varsigma_i'^* K_{\beta_j}(\alpha'_j)\xi_j)^* \\
&= (\sum_i \varsigma_i'^* g(\alpha'_i))^* \\
&= (\sum_i \varsigma_i'^*(\sum_j K_{\beta'_j}(\alpha'_i)\xi'_j))^* = [\Sigma K_{\alpha'_j}\xi_i, \Sigma K_{\beta'_j}\xi'_j] \ .
\end{aligned}$$

We further remark that $\Pi^\circ(K)$ has no negative definite subspaces of dimension bigger than k. Indeed, if there is such a space, then it is included in the linear span of $K_{w_1}c_1, \ldots, K_{w_r}c_r$ for some choice of $w_1, \ldots w_r$ in Z and $c_1, \ldots, c_r$ in $\mathbb{C}_{m\times 1}$. Thus we see that the $r \times r$ matrix with ij entry

$$c_j^* K_{w_i}(w_j)c_j$$

has more than k strictly negative eigenvalues, contradicting the fact that K has k negative squares.

Before turning to Step 2, we remark that Step 1 easily implies that for all f in $\Pi^\circ(K)$, β in Z and c in $\mathbb{C}_{m\times 1}$, we have:

$$c^* f(\beta) = [f, K_\beta c] \ . \tag{2.2}$$

In particular, (2.2) will imply that $\Pi^\circ(K)$ has no isotropic subspaces.

STEP 2. Let $\mathcal{M}$ be a negative definite subspace of $\Pi^\circ(K)$ of dimension k. Then, $\mathcal{M}^{[\perp]}$ is a preHilbert space and $\mathcal{M}[\dot{+}]\mathcal{M}^{[\perp]} = \Pi^\circ(K)$.

PROOF OF STEP 2. Such a space $\mathcal{M}$ exists, since the function K has k negative squares. $\mathcal{M}$ is non-degenerate and finite dimensional and thus is orthocomplemented (Corollary 11.9, p.26 of [Bo]). Moreover, since $\Pi^\circ(K)$ has no negative definite subspaces of dimension greater than k, $\mathcal{M}$ is maximal negative definite. Thus, by Lemma 6.4, p.13 of [Bo], $\mathcal{M}^{[\perp]}$ is positive. Finally, $\mathcal{M}^{[\perp]}$ is a preHilbert space since $\Pi^\circ(K)$ has no isotropic subspaces. Indeed, let $x \in \mathcal{M}^{[\perp]}$ be such that $[x,x] = 0$. Then, by the Cauchy-Schwarz inequality

$$|[x,y]|^2 \leq [x,x][y,y] = 0$$

for all y in $\mathcal{M}^{[\perp]}$. Thus $[x,z] = 0$ for all z in $\Pi^\circ(K)$, and so by (2.2), $x = 0$.

Let H be the completion of $\mathcal{M}^{[\perp]}, [\ ,\ ]$, and let $\Pi(K)$ be the set of elements of the form

$$F = h + m$$

with h in H, m in $\mathcal{M}$, and inner product

$$[F,F]_{\Pi(K)} = [m,m]_{\Pi^\circ(K)} + [h,h]_H \ .$$

It is easy to check that $[\ ,\ ]$ is well defined.

STEP 3. $\Pi(K)$ satisfies the conditions of the theorem.

PROOF OF STEP 3. $\Pi(K)$ is clearly a Pontryagin space, and $K_\beta c$ belongs to $\Pi(K)$ for β in Z and c in $\mathbb{C}_{m\times 1}$. For (f_n) a sequence of elements in $\Pi^\circ(K)$, we know that

$$c^* f_n(\beta) = [f_n, K_\beta c] \ .$$

Every element in $\Pi(K)$ is a limit of elements in $\Pi^\circ(K)$ and so we see that for $f = \lim f_n$, $\lim(f_n(\beta))$ exists (and is independent of the sequence (f_n)) and so

$$c^*(\lim\ f_n)(\beta) = [\lim\ f_n, K_\beta c] \ .$$

We conclude by identifying $\lim\ f_n$ and the function defined on Z by $(\lim\ f_n)(\beta)$.

The uniqueness of the construction is ensured by the following step.

STEP 4. A reproducing kernel Pontryagin space is uniquely determined by its reproducing kernel.

PROOF OF STEP 4. We could prove this fact as a consequence of Theorem 2.1, p. 102 and Corollary 6.3, p. 92 of [Bo] but present a direct argument.

Let $(\mathcal{K}, [\ ,\ ]_{\mathcal{K}})$ and $(\mathcal{K}', [\ ,\ ]_{\mathcal{K}'})$ be two reproducing kernel Pontryagin spaces of $m \times 1$ valued functions, defined on the same set Z and with the same reproducing kernel K. We first remark that $\Pi^\circ(K)$ is densely included in both $\mathcal{K}$ and $\mathcal{K}'$ and that, for f, g in $\Pi^\circ(K)$

$$[f, g]_{\mathcal{K}} = [f, g]_{\mathcal{K}'}$$

where $\Pi^\circ(K)$ is as in Step 1. This serves to show that the construction of Step 2 is independent of the particular choice of $\mathcal{K}$ or $\mathcal{K}'$, and that the completion of the orthogonal of $\mathcal{M}$ is the same in $\mathcal{K}$ and $\mathcal{K}'$, from which it follows that $\mathcal{K}$ and $\mathcal{K}'$ coincide. □

We did not succeed to prove a Krein space version of this theorem. Indeed, the problem of characterizing functions $K_y(x)$ such that a reproducing kernel Krein space with reproducing kernel K exists seems to be much more involved.

We now turn to the problem of decomposition of a reproducing kernel. If $K = K_1 + K_2$ where K_1 and K_2 are both positive kernels, then, the reproducing kernel Hilbert space H_i associated with K_i is contractively included in the reproducing kernel Hilbert space H associated with K, for $i = 1, 2$, i.e.

$$x \in H_i \Longrightarrow x \in H$$

$$\|x\|_H \leq \|x\|_{H_i} \ .$$

This fact is due to Aronsjan ([Ar]) and follows easily from Theorem 2.3 below, which is also due to him. As a matter of fact, the spaces H_1 and H_2 are complementary subspaces, as we now explain. We first recall the definition of complementary subspaces. This notion is due to de Branges (see [dB8]) and generalizes the notion of orthogonal subspaces in Hilbert space. We begin with a definition.

DEFINITION 2.1. Let H and H_1 be two Hilbert spaces with norm $\|\ \|$ and $\|\ \|_1$. H_1 is said to be contractively included in H if

a) $H_1 \subset H$

b) The injection i_1 from H_1 into H is a contraction, i.e. for any x in H_1,

$$\|x\| \leq \|x\|_1 \ .$$

When i_1 is an isometry, we can write $H = H_1 \oplus H_2$, H_2 being the orthogonal complement of H_1 in H. The next theorem considers the case where i_1 is not necessarily an isometry.

THEOREM 2.2. ([dB8]): If $(H_1, \|\ \|_1)$ is a Hilbert space which is contained contractively in a Hilbert space $(H, \|\ \|)$, then there exists a unique Hilbert space $(H_2, \|\ \|_2)$ which is contained contractively in H, such that

$$H = H_1 + H_2$$

and the inequality

$$\|x\|^2 \leq \|x_1\|_1^2 + \|x_2\|_2^2 \tag{2.4}$$

holds when $x = x_1 + x_2$, with x_1 in H_1 and x_2 in H_2 and such that every element of H has a decomposition for which equality holds.

We mention that the pair (x_1, x_2) for which there is equality in (2.4) is given by

$$x_1 = i_1^*(x)$$
$$x_2 = x - i_1^*(x)$$

where i_1 is the injection from H_1 into H.

We further remark that the norm $\| \ \|_2$ may be expressed as

$$\|x\|_2^2 = \sup_{a \in H_1} (\|x + a\|^2 - \|a\|_1^2) \tag{2.5}$$

and that H_2 is exactly the subspace of elements x in H for which $\|x\|_2$ is finite ([dB8]).

The space H_2 whose existence and uniqueness is guaranteed by Theorem 2.2 is termed the complementary space to H_1. It is clear that H_1 is also the complementary space to H_2. Accordingly, we shall refer to H_1 and H_2 as complementary subspaces.

It is of interest to compare this theorem with the following theorem of Aronszajn on the sums of positive kernels.

THEOREM 2.3. ([Ar]): Let K_1, K_2 and $K = K_1 + K_2$ be positive kernels defined on some common set Z, and let H_1, H_2 and H be the corresponding reproducing kernel Hilbert spaces. Then H is equal to the set of elements of the form

$$x = x_1 + x_2 \qquad x_i \in H_i$$

and

$$\|x\|^2 = \inf_{x = x_1 + x_2} (\|x_1\|_1^2 + \|x_2\|_2^2) \ . \tag{2.6}$$

There is a unique choice (x_1, x_2) for which the infimum is indeed achieved.

We mention that (2.5) and (2.6) are formally dual formulas. We were not able to make this idea more precise, but we remark that H_1 and H_2 in Theorem 2.3 are complementary subspaces in H.

In the Krein space context, the situation is more complicated, and if K is the reproducing kernel of the RKKS $\mathcal{K}$ and $K = K_1 + K_2$, each K_i being the reproducing kernel of a RKKS $\mathcal{K}_i$, it may be false that $\mathcal{K}_i \subset \mathcal{K}$, as the example

$$0 = K + (-K)$$

clearly exhibits. This fact leads us to introduce the following definitions.

DEFINITION 2.2. Let $K = K_1 + K_2$ be a decomposition of the reproducing kernel K into two reproducing kernels K_i, with associated Pontryagin spaces $\mathcal{K}$, $\mathcal{K}_1$ and $\mathcal{K}_2$. The decomposition is said to be good if $\mathcal{K}_1 \subset \mathcal{K}$. The decomposition is said to be regular if $\mathcal{K}_1$ is orthocomplemented in $\mathcal{K}$, i.e. if $\mathcal{K}_1$ is closed in $\mathcal{K}$, and $\mathcal{K}_1$ is isometrically included in $\mathcal{K}$, i.e., (in view of [Bo], Theorem 3.4, p. 104), for any x, y in $\mathcal{K}_1$ we have

$$[x,y]_{\mathcal{K}_1} = [x,y]_{\mathcal{K}} \ .$$

We remark that in this particular case, closedness is a consequence of the isometric inclusion.

LEMMA 2.1. Let $\mathcal{K}_1$ and $\mathcal{K}$ be two reproducing kernel Pontryagin spaces of $m \times 1$ valued functions defined on some set Z and suppose that $\mathcal{K}_1$ is included isometrically in $\mathcal{K}$. Then, $\mathcal{K}_1$ is closed in $\mathcal{K}$.

PROOF. Let $K_\beta(z)$ (resp. $K^1_\beta(z)$) denote the reproducing kernel of $\mathcal{K}$ (resp. $\mathcal{K}_1$) where β and z belong to Z, and let (f_n)be a converging sequence of elements in $\mathcal{K}$. We want to show that if $f_n \in \mathcal{K}_1$ for all n, then lim f_n belongs to $\mathcal{K}_1$. Let $f = \lim\ f_n$. For any c in $\mathbb{C}_{m\times 1}$ and w in Z, we have:

$$\lim [f_n, K_w c] = c^* f(w)$$

and, since f_n belongs to $\mathcal{K}_1$,

$$c^* f_n(w) = [f_n, K^1_w c]_1$$

where $[\ ,\ ]$ (resp. $[\ ,\ ]_1$) denotes the inner product in $\mathcal{K}$ (resp. $\mathcal{K}_1$).

We further denote by σ (resp. σ_1) any signature operator from $\mathcal{K}$ into itself (resp. $\mathcal{K}_1$ into itself) such that the form $< u, v >= [u, \sigma v]$ (resp. $< u, v >_1= [u, \sigma_1 v]_1$) makes $\mathcal{K}$ (resp. $\mathcal{K}_1$) into a Hilbert space. For u in $\mathcal{K}_1$,

$$[f_n, u]_1 =< f_n, \sigma_1 u >_1= [f_n, u] =< f_n, \sigma u > \ .$$

Thus $| < f_n, \sigma_1 u >_1 |^2 \leq < f_n, f_n >< \sigma u, \sigma u >$. Since $\lim\ f_n = f$ in $\mathcal{K}$, $\sup_n < f_n, f_n >< \infty$, and an application of the Banach Steinhaus theorem shows that $\sup_n < f_n, f_n >_1 < \infty$. Thus, f_n has a weakly converging subsequence in $\mathcal{K}_1$ which we still denote by (f_n). Let g be the weak limit of f_n. g belongs to $\mathcal{K}_1$. Moreover, for w in Z and c in $\mathbb{C}_{m\times 1}$,

$$< g, \sigma_1 K^1_w c >_1= \lim < f_n, \sigma_1 K^1_w c >_1= \lim [f_n, K^1_w c]_1 = \lim\ c^* f_n(w)$$

and so

$$c^* g(w) = \lim\ c^* f_n(w) \ .$$

Since $\lim\ c^* f_n(w) = c^* f(w)$, we see that $g = f$ and thus $\mathcal{K}^1$ is closed in $\mathcal{K}$. □

3. Reproducing Kernel Krein Spaces of Analytic Functions

In this section we define reproducing kernel Krein spaces of analytic functions and set some more notations. In particular, we briefly introduce the various Krein spaces to be studied in the next sections. Finally, we prove a theorem on the structure of certain finite dimensional spaces of analytic functions.

A reproducing kernel Krein space of analytic functions is a reproducing kernel Krein space, the elements of which are vector valued functions analytic in some open set U. Usually, the set U will be of the form $\mathbb{C}\setminus\partial \cup A$ or $\Omega_+\setminus A$, where A is a Nevanlinna zero set, i.e. a subset of $\mathbb{C}\setminus\partial$ of points $\{w_j\}$ subject to

$$\begin{array}{lll} \Sigma\frac{|w_j-w_j^*|}{1+|w_j|^2}<\infty & if & \partial=\mathbb{R} \\ \Sigma|(1-|w_j|)|<\infty & if & \partial=\mathbb{T}\ . \end{array}$$

More general cases where U is an arbitrary open set will also be considered.

An illustrative example of RKKS of analytic function is the space H_J^2: Let J be a $m\times m$ signature matrix, i.e. an element of $\mathbb{C}_{m\times m}$ subject to

$$J^* = J^{-1} = J \tag{3.1}$$

and consider the usual Hardy space H_m^2 of $m\times 1$ vectors with entries in H^2 and let $[\ ,\ ]$ be the hermitian form defined by

$$[f,g] = < f, Jg >_{H_m^2} \tag{3.2}$$

where $< \ , \ >_{H_m^2}$ denotes the usual inner product in H_m^2. Then H_m^2 endowed with $[\ ,\ ]$ is a reproducing kernel Krein space of functions analytic in Ω_+ with reproducing kernel

$$K_w(\lambda) = \frac{J}{\rho_w(\lambda)}\ .$$

This reproducing kernel Krein space is denoted by H_J^2.

We also mention that reproducing kernel Pontryagin spaces of analytic functions appear in the study of univalent functions (see Chapter 6 of [dB8]).

A vector space V of functions analytic in some subset U will be said to be resolvent invariant if, for any w in U and f in V, the function defined by

$$(R_w f)(\lambda) = \frac{f(\lambda)-f(w)}{\lambda - w}\ ,\ \lambda\in U\ , \tag{3.3}$$

belongs to V. We shall refer to the operator R_w as the resolvent-like operator since, as is well known (and easily checked), the operators $R_w: w \in U$, satisfy the resolvent identity

$$R_w - R_v = (w - v)R_w R_v$$

for w and v in U.

We will often say that a space of analytic functions is resolvent invariant without referring to the domain of analyticity, when it is clear from the context.

This definition enables us to present another example of RKKS of analytic functions; to ease future reference, we present this example in the form of a theorem.

THEOREM 3.1. ([D1], Section 5): Let V be a non-degenerate, orthocomplemented closed and resolvent invariant subspace of H_J^2. Then, V is a reproducing kernel Krein space of $m \times 1$ valued functions analytic in Ω_+, and the reproducing kernel of V may be written as

$$K_w(\lambda) = \frac{J - \Theta(\lambda)\hat{J}\Theta^*(w)}{\rho_w(\lambda)} \tag{3.4}$$

where $\hat{J}$ is a $m' \times m'$ signature matrix (with $m' \leq m$) and Θ is an $m \times m'$ valued function analytic in Ω_+, which has non-tangential limits a.e. on Ω_+ and such that

$$\hat{J} = \Theta^*(\gamma)J\Theta(\gamma) \text{ a.e. on } \partial \ .$$

Moreover,when V is a Hilbert space, and if $J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ and $\hat{J} = \begin{pmatrix} I_{p'} & 0 \\ 0 & -I_{q'} \end{pmatrix}$, then $p \geq p'$ and $q \geq q'$.

This theorem, specialized to $J = I_m$, is essentially equivalent to the Beurling-Lax Theorem, while, in its general form, it is essentially equivalent to a result of Ball and Helton [BH]. Under certain circumstances, $\hat{J} = J$, and Theorem 3.1 is then a particular case of a Krein space version of a result of de Branges. RKKS with a reproducing kernel of the form (3.4) with $J = \hat{J}$ will play a very important role in this thesis. They will be called $\mathcal{K}(\Theta)$ spaces or $\mathcal{K}_J(\Theta)$ spaces (depending upon whether the matrix J is understood from the context or not). Note that $\mathcal{K}(\Theta)$ spaces are not necessarily included in a H_J^2 space. The more general case will be studied in Section 6. Spaces for which $J = I_p$, $\hat{J} = I_q$ and Θ is of bounded type in Ω_+ (not necessarily with isometric value on ∂) will be studied in Section 4. For notational purposes, it is convenient to write S rather than Θ in this case. The reproducing kernel is then

$$\frac{I_p - S(\lambda)S^*(w)}{\rho_w(\lambda)} \ . \tag{3.5}$$

Section 5 will deal with another important example of RKKS of analytic functions wherein the reproducing kernel cannot be put in the form (3.4) but will be of the form

$$\frac{\Phi(\lambda) + \Phi^*(w)}{\rho_w(\lambda)} \tag{3.6}$$

for some $m \times m$ valued function of bounded type in Ω_+, extended to Ω_- via $\Phi + \Phi^{\#} = 0$.

We may mention that RKKS with reproducing kernel of the form (3.5) (resp. (3.6)) will be called $K(S)$ (resp. $\mathcal{L}(\Phi)$) spaces. Following de Branges' notations, $K(S)$ will be denoted $H(S)$ when it is a Hilbert space.

In [B2] and a series of papers by Krein and Langer ([KL1], [KL2], [KL3]), kernels of the form (3.4), (3.5), (3.6) which have a finite number of negative squares are considered. We now introduce three related families of functions; the latter two generalize the usual Schur and Caratheodory classes and (up to a multiplicative constant for the last) have been introduced by Krein and Langer.

DEFINITION 3.1. Let J be an $m \times m$ signature matrix. Let $\mathcal{A}_J^k(\Delta)$ denote the class of $m \times m$ valued functions Θ which are analytic in the open subset Δ, and are such that $\frac{J-\Theta(\lambda)J\Theta^*(w)}{\rho_w(\lambda)}$ has k negative squares for λ and w in Δ.

.. $S_{p\times q}^k$ denotes the class of $p \times q$ valued functions meromorphic in Ω_+, and such that $\frac{I_p-S(\lambda)S^*(w)}{\rho_w(\lambda)}$ has k negative squares.

... C_p^k denotes the class of $p \times p$ valued functions which are meromorphic in $\mathbb{C} \setminus \partial$, and are such that $\frac{\Phi(\lambda)+\Phi^*(w)}{\rho_w(\lambda)}$ has k negative squares.

We remark that, by Theorem 2.1, to any element of $\mathcal{A}_J^k(\Delta)$ (resp. $S_{p\times q}^k$), (resp. C_p^k) there corresponds a reproducing kernel Pontryagin space. Moreover, when $k = 0$, $S_{p\times q}^k$, denoted also by $S_{p\times q}$ is the usual Schur class and C_p^k, denoted by C_p, is the usual Caratheodory class. Finally, when Δ is understood from the context, we will write $\mathcal{A}_J^k$ rather than $\mathcal{A}_J^k(\Delta)$.

We conclude this section with a remark on invariance under the resolvent-like operators R_w.

THEOREM 3.2. Let V be a finite dimensional vector space of $m \times 1$ valued functions which are analytic in some open set U, and suppose that V is resolvent invariant. Then a basis of V consists of a finite union of chains of the form

$$\begin{aligned} f_1(\lambda) &= \frac{c_1}{\lambda-w} \\ f_2(\lambda) &= \frac{c_1}{(\lambda-w)^2} + \frac{c_2}{\lambda-w} \\ &\vdots \\ f_n(\lambda) &= \frac{c_1}{(\lambda-w)^n} + \frac{c_2}{(\lambda-w)^{n-1}} + \ldots + \frac{c_n}{(\lambda-w)} \end{aligned}$$

or

$$\begin{aligned} p_0(\lambda) &= c_1 \\ p_1(\lambda) &= c_1\lambda + c_2 \\ p_2(\lambda) &= c_1\lambda^2 + c_2\lambda + c_3 \\ &\vdots \\ p_n(\lambda) &= c_1\lambda^n + c_2\lambda^{n-1} + \ldots + c_n \end{aligned}$$

where $c_1, \ldots, c_n \in \mathbb{C}_{m\times 1}$, $w \in \mathbb{C}$ and $n \in \mathbb{N}$.

PROOF. We proceed in a number of steps. We fix some w_0 where the elements of V are analytic and consider the operatorR_{w_0}. Since the vector space V is finite dimensional, there exists a basis of V which consists of generalized eigenelements of R_{w_0}, i.e. of functions f such that

$$(R_{w_0} - \alpha)^n f = 0$$

for some α in $\mathbb{C}$ and some integer n.

Step 1. The eigenelements of R_{w_0} are of the form $f(\lambda) = \frac{c}{\lambda - w}$ (resp. c) when the corresponding eigenvalue is non-zero (resp. zero).

Proof of Step 1. Let α be an eigenvalue and let f be a corresponding eigenelement. Then $R_{w_0} f = \alpha f$ implies

$$\frac{f(\lambda) - f(w_0)}{\lambda - w_0} = \alpha f(\lambda)$$

i.e.

$$f(\lambda) = \frac{f(w_0)}{1 - \alpha(\lambda - w_0)}$$

from which the claim follows.

When R_{w_0} has only simple eigenvalues, the theorem is thus proved. In general, this is not the case and to conclude, we need to compute the form of the generalized eigenelements of V.

Step 2. Suppose that f_n belongs to V. Then f_k belongs to V for all integers k between 1 and n, and $f_1, \ldots f_n$ is a basis of the Jordan chain corresponding to the eigenvector $\frac{c}{\lambda - w}$ and the eigenvalue $\alpha = -\frac{1}{w_0 - w}$.

Proof of Step 2. Let $\alpha_1, \ldots, \alpha_n$ be n different points of U. We have

$$R_{\alpha_1} f_n = -f_n \frac{1}{\alpha_1 - w} - f_{n-1} \frac{1}{(\alpha_1 - w)^2} \cdots - f_1 \frac{1}{(\alpha_1 - w)^n} . \tag{3.7}$$

Thus

$$-\begin{bmatrix} R_{\alpha_1} f_n \\ \vdots \\ R_{\alpha_n} f_n \end{bmatrix} = \begin{bmatrix} \frac{1}{(\alpha_1 - w)} & \cdots\cdots & \frac{1}{(\alpha_1 - w)^n} \\ & \cdots\cdots & \\ \frac{1}{(\alpha_n - w)} & \cdots\cdots & \frac{1}{(\alpha_n - w)^n} \end{bmatrix} \begin{bmatrix} f_n \\ \vdots \\ f_1 \end{bmatrix} . \tag{3.8}$$

The $n \times n$ matrix with jk entry $\frac{1}{(\alpha_j - w)^k}$ is invertible since it is a Vandermonde matrix and the α_j are distinct. Thus from (3.8) we get the f_k, $k \leq n$, as a linear combination of the $R_{\alpha_j} f_n$ and therefore, because of the resolvent invariance of V, they must belong to V as claimed. The second claim follows from formula (3.7). The proof is completed.

Step 3. Let α be a non-zero eigenvalue of R_w. Then for any $n \geq 1$, the vector space $\mathrm{Ker}(R_{w_0} - \alpha)^n$ is included in the space of finite linear combinations of $\frac{\xi}{(1 - \alpha(\lambda - w_0))^\ell}$, for $\ell = 1, \ldots, n$ and ξ in $\mathbb{C}_{m \times 1}$.

Proof of Step 3. We proceed by induction on n; for $n = 1$ the claim is true by Step 1. Next; suppose the claim is true for n and consider f to be a solution of

$$(R_{w_0} - \alpha)^{n+1} f = 0 \ .$$

Then,

$$(R_{w_0} - \alpha)^n (R_{w_0} - \alpha) f = 0$$

and therefore, by the induction hypothesis, there exists a choice of complex constants $\beta_{j\ell}$ such that

$$(R_{w_0} - \alpha) f = \sum_{j=1}^{m} \sum_{\ell=1}^{n} \beta_{j\ell} \frac{c_j}{(1 - \alpha(\lambda - w_0))^\ell}$$

where $c_1, \ldots, c_m$ is a basis of $\mathbb{C}_{m\times 1}$. Thus

$$\frac{f(\lambda)(1 - \alpha(\lambda - w_0)) - f(w_0)}{\lambda - w_0} = \sum_{j=1}^{m} \sum_{\ell=1}^{n} \frac{\beta_{j\ell} c_j}{(1 - \alpha(\lambda - w_0))^\ell}$$

i.e.

$$f(\lambda) = \frac{(\lambda - w_0)}{1 - \alpha(\lambda - w_0)} \sum_{j=1}^{n} \sum_{\ell=1}^{n} \frac{\beta_{j\ell} c_j}{(1 - \alpha(\lambda - w_0))^\ell} + \frac{f(w_0)}{1 - \alpha(\lambda - w_0)} \ .$$

Since

$$\frac{\lambda - w_0}{1 - \alpha(\lambda - w_0)} = \frac{1}{\alpha} \left(\frac{1}{1 - \alpha(\lambda - w_0)} - 1 \right)$$

it follows that

$$f(\lambda) = \frac{f(w_0)}{1 - \alpha(\lambda - w_0)} + \frac{1}{\alpha} \sum_{j=1}^{m} \sum_{\ell=1}^{n} \frac{\beta_{j\ell} c_j}{(1 - \alpha(\lambda - w_0))^{\ell+1}}$$
$$- \frac{1}{\alpha} \sum_{j=1}^{m} \sum_{\ell=1}^{n} \frac{\beta_{j\ell} c_j}{(1 - \alpha(\lambda - w_0)^\ell}$$

which finishes the proof of Step 3.

Steps 4 and 5 are analogues of Steps 2 and 3 for the case $\alpha = 0$ and their proofs will be omitted.

Step 4. If $p_j(\lambda)$ belongs to V so does $p_k(\lambda)$ for $k = 0, \ldots, j$, and $\{p_0, \ldots, p_j\}$ is a Jordan chain corresponding to the eigenvector p_0 and eigenvalue 0.

Step 5. Ker $R_{w_0}^n$ is included in the space of linear combinations of the $\xi\lambda^j$, $j = 1, \ldots, n$ and ξ in $\mathbb{C}_{m\times 1}$.

We now conclude the proof of the theorem. A basis of V consists of maximal Jordan chains corresponding to the eigenvalues of R_{w_0}. By Step 3, we know where the generalized eigenspace corresponding to a nonzero eigenvalue sits, and by Step 2 we can compute an associated Jordan chain. The case of the zero eigenvalue is treated by Steps 4 and 5 and thus the proof is complete.

CHAPTER II: THEORY OF REPRODUCING KERNEL KREIN SPACES OF ANALYTIC FUNCTIONS

Introduction

In the present chapter we generalize part of de Branges' theory to the framework of Krein spaces. The main effort is made in the section on $\mathcal{K}(\Theta)$ spaces(Section 6), but it proves convenient to first study $\mathcal{K}(S)$ spaces (Section 4) and $\mathcal{L}(\Phi)$ spaces (Section 5). The interested reader can proceed directly to Section 6. Finally, in Section 7, we study linear fractional transformations linking these spaces.

In order to generalize de Branges' theory to the present framework, it was first necessary to see which tools from Hilbert space theory are used in de Branges' papers and which of these tools are still valid in the Krein framework.

A careful analysis shows that the following results are recurrently used in the theory of reproducing kernel Hilbert spaces of analytic functions.

1) The Riesz representation theorem.
2) The fact that the adjoint of a contraction between two Hilbert spaces is still a contraction.
3) The theorem on the sum of two positive kernels (Aronsjan [Ar]) (in a non-explicit way).

1) is used to define the reproducing kernel, while 2) is used to prove various inclusion theorems; 3) is not explicitly used in de Branges' papers, but this theorem helps to understand various notions, in particular that of overlapping subspace, and helps one to see why contractive inclusion is a more natural notion than isometric inclusion in the theory.

While there seems to be no reasonable version of 3) in the Krein framework, 1) and 2) are more easily dealt with. The Riesz representation theorem is still valid in the Krein space framework, as remarked in Section 1, and versions of 2) for operators between Pontryagin spaces were also mentioned in that section.

4. $K(S)$ Spaces

In this section we study $K(S)$ spaces, i.e. reproducing kernel Krein spaces with a reproducing kernel of the form

$$k_w(\lambda) = \frac{I_p - S(\lambda)S^*(w)}{\rho_w(\lambda)} \tag{4.1}$$

where S is $p \times q$ valued and meromorphic in Ω_+. We first prove that, to every S of bounded type in Ω_+, there corresponds such a space $K(S)$. This seems to be a new result and is the main result of this section. We then focus on a number of special cases.

It is convenient to first prove some preliminary lemmas. Let us consider a Hilbert space $(\mathcal{H}, <,>)$, and let Γ be a bounded self-adjoint operator from $\mathcal{H}$ into itself. Let $\lambda \to E_\lambda$ be the resolution of the identity associated with Γ, so that

$$\Gamma = \int \lambda dE_\lambda$$

and recall that $|\Gamma|$ and $sgn\ \Gamma$, the absolute value and the sign of Γ, are defined by

$$|\Gamma| = \int |\lambda| dE_\lambda$$

$$sgn\ \Gamma = \int_{\lambda \neq 0} \frac{\lambda}{|\lambda|} dE_\lambda$$

and that

$$\Gamma = |\Gamma| sgn\ \Gamma = sgn\ \Gamma |\Gamma|\ . \tag{4.2}$$

Next, we introduce two hermitian forms, $< >_\Gamma$ and $[\ ,\ ]_\Gamma$, on $Ran\ \Gamma$, the range of Γ, by

$$\begin{aligned} <\Gamma u, \Gamma v>_\Gamma &= <|\Gamma|u, v> \quad \text{for } u,v \text{ in } \mathcal{H} \\ [\Gamma u, \Gamma v]_\Gamma &= <\Gamma u, v> \quad \text{for } u,v \text{ in } \mathcal{H}\ . \end{aligned} \tag{4.3}$$

We first check that $<\ ,\ >_\Gamma$ and $[\ ,\ ]_\Gamma$ are well defined in $Ran\ \Gamma$. Indeed, let $\Gamma u = \Gamma u'$ and $\Gamma v = \Gamma v'$ for elements u, u', v, v' in $\mathcal{H}$. Then, since $|\Gamma|u = |\Gamma|u'$ if and only if $\Gamma u = \Gamma u'$,

$$\begin{aligned} <|\Gamma|u, v> &= <|\Gamma|u', v> \\ &= <u', |\Gamma|v> \\ &= <u', |\Gamma|v'> \\ &= <|\Gamma|u', v'>\ . \end{aligned}$$

Thus, $<\ ,\ >_\Gamma$ is well defined, as is $[\ ,\ ]_\Gamma$ by a similar argument.

Let now $F = \Gamma u$ be an element of $Ran\ \Gamma$ such that $<F, F>_\Gamma = 0$. Then, $<|\Gamma|u, u> = 0$, i.e. $|\Gamma|u = 0$ and so $\Gamma u = 0$, i.e. $F = 0$. This shows that $(Ran\ \Gamma, <\ ,\ >_\Gamma)$ is a preHilbert space, since $Ran\ \Gamma$ is positive with respect to $<\ ,\ >_\Gamma$.

LEMMA 4.1. Let $\mathcal{K}$ be the completion of $(Ran\ \Gamma, <\ ,\ >_\Gamma)$. Then, $[\ ,\ ]_\Gamma$ is continuously extendable to $\mathcal{K}$ and $(\mathcal{K}, [\ ,\ ]_\Gamma)$ is a Krein space. Moreover, $sgn\ \Gamma$ maps $\mathcal{K}$ into $\mathcal{K}$, and,

$$sgn\ \Gamma = (sgn\ \Gamma)^* = (sgn\ \Gamma)^{-1} \tag{4.4}$$

on $\mathcal{K}$ (where $(sgn\ \Gamma)^*$ denotes the adjoint of the operator $sgn\ \Gamma$ in the Hilbert space $\mathcal{K}$).

PROOF. We begin by proving (4.4). Let $F = \Gamma u$ and $G = \Gamma v$ be two elements in $Ran\ \Gamma$. From $(sgn\ \Gamma)F = \Gamma(sgn\ \Gamma)u$, it is clear that

$$sgn\ \Gamma(Ran\ \Gamma) \subset Ran\ \Gamma\ .$$

Since

$$< F, G >_\Gamma = < |\Gamma| u, v >_{\mathcal{H}} = < |\Gamma|\ sgn\ \Gamma u,\ sgn\ \Gamma v >_{\mathcal{H}} = < sgn\ \Gamma F,\ sgn\ \Gamma G >_\Gamma$$

for $F = \Gamma u$, $G = \Gamma v$, it is clear that $sgn\ \Gamma$ is an isometry on $\mathcal{K}$, and hence continuous. Moreover, $sgn\ \Gamma$ is selfadjoint, since

$$< sgn\ \Gamma F, G >_\Gamma = < u, \Gamma v >_{\mathcal{H}} = < F, sgn\ \Gamma G >_\Gamma\ .$$

Since a self adjoint isometry is unitary, it is clear that $sgn\ \Gamma$ is invertible in $\mathcal{K}$, and thus (4.4) holds.

We now turn to the proof that $[\ ]_\Gamma$ is continuously extendable to $\mathcal{K} \times \mathcal{K}$. We first prove the following estimate: For $F = \Gamma u$ and $G = \Gamma v$ in $Ran\ \Gamma$,

$$|[F, G]_\Gamma|^2 \leq < F, F >_\Gamma < G, G >_\Gamma\ . \tag{4.5}$$

Indeed,

$$\begin{aligned} |[F, G]_\Gamma| &= |< \Gamma u, v >| \\ &= |< |\Gamma| sgn\ \Gamma u, v >| \\ &= |< |\Gamma|^{\frac{1}{2}} sgn\ \Gamma u, |\Gamma|^{\frac{1}{2}} v >| \\ &\leq \||\Gamma|^{\frac{1}{2}} sgn\ \Gamma u\| \cdot \||\Gamma|^{\frac{1}{2}} v\| \\ &= \|F\|_\Gamma \cdot \|G\|_\Gamma\ . \end{aligned}$$

This estimate shows that $[F, G]_\Gamma$ is extendable to all of $\mathcal{K} \times \mathcal{K}$ continuously with respect to the topology induced by $<\ ,\ >_\Gamma$ on $\mathcal{K} \times \mathcal{K}$. We furthermore remark that, because of (4.4), $sgn\ \Gamma$ is the difference of two complementary projections; thus, $(\mathcal{K}, [\ ,\ ]_\Gamma)$ is a Krein space, since

$$[F, G]_\Gamma = < F, sgn\ \Gamma G >_\Gamma$$

which holds for any F and G in $\mathcal{K}$. □

It is not difficult to prove the estimate

$$< F, F > \leq \||\Gamma|^{\frac{1}{2}}\|^2 < F, F >_\Gamma \tag{4.6}$$

for F in $Ran\ \Gamma$, from which it follows that a Cauchy sequence with respect to $< , >_\Gamma$ is a Cauchy sequence in $\mathcal{H}$. We could infer, on general grounds, that $\mathcal{K} \subset \mathcal{H}$; it suffices for present purposes to prove it under a supplementary hypothesis which serves to simplify the proof.

LEMMA 4.2. With the notations of Lemma 4.1, suppose that $\mathcal{H}$ is a reproducing kernel Hilbert space of $p \times 1$ valued functions, defined on some subset Z of $\mathbb{C}$, with reproducing kernel $k_w(\lambda)$. Then, $\mathcal{K}$ is a reproducing kernel Krein space, and $\mathcal{K} \subset \mathcal{H}$ (as sets).

PROOF. Let $F = \Gamma u$ be an element of $Ran\ \Gamma$, w be in Z and c in $\mathbb{C}_{p\times 1}$. Then

$$\begin{aligned} c^*(\Gamma u)(w) &= < \Gamma u, k_w c >_{\mathcal{H}} \\ &= < |\Gamma| u, sgn\ \Gamma k_w c >_{\mathcal{H}} \\ &= < \Gamma u, sgn\ \Gamma \cdot \Gamma k_w c >_\Gamma \end{aligned}$$

which shows that $F \to c^*F(w)$ is continuous in the Hilbert space $(\mathcal{K}, < , >_\Gamma)$; thus $(\mathcal{K}, [\ ,\ ]_\Gamma)$ is a reproducing kernel Krein space.

To show that $\mathcal{K} \subset \mathcal{H}$, let (Γu_n) be a converging sequence in $\mathcal{K}$, with limit G, and let F be its limit in $\mathcal{H}$ which exists by (4.6). We want to show that $F = G$. Indeed, for w in Z and c in $\mathbb{C}_{p\times 1}$,

$$\begin{aligned} c^*F(w) = \lim c^*F_n(w) &= \lim < F_n, k_w c >_{\mathcal{H}} \\ &= \lim < \Gamma u_n, k_w c >_{\mathcal{H}} \\ &= \lim < \Gamma u_n, sgn\ \Gamma\ \Gamma(k_w c) >_\Gamma \\ &= c^*G(w) \end{aligned}$$

i.e. $F = G$ and thus $\mathcal{K} \subset \mathcal{H}$. □

We now turn to the construction of the space $K(S)$. Let S be a $p \times q$ matrix valued function of bounded type in Ω_+. S may be thus written as $S = S_1^{-1}S_2$, where S_1 and S_2 are in $S_{p\times p}$ and $S_{p\times q}$ respectively, det S_1 being non-identically zero. Such a choice is of course not unique, but we do not require that S_1 and S_2 be coprime. Moreover, S is not presumed to be contractive a.e. on ∂. We will denote by $\Omega(S)$ the largest open subset of Ω_+ where S is analytic and where S_1 is invertible, and by Γ the bounded self-adjoint operator from H_p^2 into itself which is defined by

$$\Gamma = S_1 \underline{p} S_1^*|_{H_p^2} - S_2 \underline{p} S_2^*|_{H_p^2} \ . \tag{4.7}$$

(We recall that $\underline{p}$ denotes the orthogonal projection from L_p^2 into H_p^2.)

To define $K(S)$, we proceed in a number of steps. We first apply Lemma 4.1 to the space $\mathcal{H} = H_p^2$ and the operator Γ defined by (4.7) to get a Krein space $(\mathcal{K}, [\ ,\ ]_\Gamma)$. We then let

$$\mathcal{K}_1 = \{S_1^{-1}f\ ;\ f \in \mathcal{K}\}$$

with inner product $[\ ,\ ]$ defined by

$$[S_1^{-1}f, S_1^{-1}g] = [f, g]_{\mathcal{K}} \ .$$

The elements of $\mathcal{K}_1$ are $p \times 1$ valued functions which are meromorphic in Ω_+; indeed, from Lemma 4.2, $\mathcal{K}$ is included in H_p^2 and thus $S_1^{-1}f$ for an element f in $\mathcal{K}$ is meromorphic in Ω_+. Moreover, $\mathcal{K}_1$ is clearly a Krein space, since $\mathcal{K}$ is a Krein space. We claim that $(\mathcal{K}_1, [\ ,\ ])$ is a reproducing kernel Krein space with reproducing kernel $k_w(\lambda) = \frac{I_p - S(\lambda)S^*(w)}{\rho_w(\lambda)}$. We first check that $k_w(\lambda)c$ belongs to $\mathcal{K}_1$ for w in $\Omega(S)$ and c in $\mathbb{C}_{p\times 1}$. Indeed, let

$$u(\lambda) = \frac{S_1^{-*}(w)c}{\rho_w(\lambda)} \ . \tag{4.8}$$

An easy computation leads to $S_1^{-1}\Gamma u = k_w(\lambda)c$, and thus $k_w(\lambda)c$ belongs to $\mathcal{K}_1$. We now verify the reproducing kernel property. Let f be an element of $\mathcal{K}_1$ of the special form $S_1^{-1}\Gamma v$, for some v in H_p^2. Then,

$$[f, k_w c] = [S_1^{-1}\Gamma v, k_w c] = [\Gamma v, \frac{\Gamma S_1^{-*}(w)c}{\rho_w}]_{\mathcal{K}} = <\Gamma v, \frac{S_1^{-*}(w)c}{\rho_w}>$$

and thus

$$[f, k_w c] = c^* S_1^{-1}(w)(\Gamma v)(w)$$

i.e.

$$[f, k_w c] = c^* f(w) \ . \tag{4.9}$$

Since such functions f are dense in $\mathcal{K}_1$, formula (4.9) extends to all of $\mathcal{K}_1$ by continuity, and we have thus proved the following theorem.

THEOREM 4.1. Let $S = S_1^{-1}S_2$ be a $p \times q$ matrix valued function of bounded type in Ω_+, with S_1 in $\mathcal{S}_{p\times p}$ and S_2 in $\mathcal{S}_{p\times q}$. Furthermore, let Γ be the bounded self-adjoint operator from H_p^2 into itself defined by $S_1\underline{p}S_1^* - S_2\underline{p}S_2^*$. Then, the closure of the set of functions of the form

$$f = S_1^{-1}\Gamma u = (\underline{p} - S\underline{p}S^*)S_1^* u \ , \ \ u \text{ in } H_p^2$$

with respect to the norm $< f, f > = < |\Gamma| u, u >_{H_p^2}$ is a reproducing kernel Krein space with reproducing kernel $\frac{I_p - S(\lambda)S^*(w)}{\rho_w(\lambda)}$ when endowed with the hermitian form

$$[f, f] = < f, S_1^* u >_{L_p^2}$$

which may also be written as

$$[f, f] = < \Gamma u, u >_{H_p^2} \ .$$

In Theorems 4.4 and 4.5, we will give other descriptions of $K(S)$ under a supplementary hypothesis.

We remark that the construction of $K(S)$ depends on the representation of S as $S_1^{-1}S_2$. When $K(S)$ is a Pontryagin space, Step 4 of Theorem 2.1 permits us to conclude that $K(S)$ is independent of the choice of S_1, S_2; in general the proposed construction presents one RKKS with reproducing kernel (4.1), but we cannot conclude that the constructed space is the only one with such a reproducing kernel. We also note that the map

$$\begin{pmatrix} \underline{p}S_1^*u \\ \underline{p}S_2^*u \end{pmatrix} \longrightarrow [I_p, -S] \begin{pmatrix} \underline{p}S_1^*u \\ \underline{p}S_2^*u \end{pmatrix}$$

maps a subspace of the space H_J^2 (which was introduced in Section 3) into a dense subset of $K(S)$.

In the construction of $K(S)$, we supposed S to be of bounded type. This hypothesis was basic to the proof. We conjecture that this condition is in fact necessary.

CONJECTURE. Let S be a $p \times q$ valued function which is meromorphic in Ω_+. Then there exists a RKKS with kernel (4.1) if and only if S is of bounded type.

This conjecture is reinforced by the following result, related to the Pontryagin space case.

THEOREM 4.2. ([KL1]): Let S be in the class $S_{p\times q}^k$. Then, S may be written as $B_0^{-1}S_0$, where B_0 is a $p \times p$ finite Blaschke product and S_0 is in $S_{p\times q}$.

Thus, since (by Theorem 2.1) $K(S)$ is a reproducing kernel Pontryagin space if and only if S belongs to $S_{p\times q}^k$ for some choice of k, Theorem 4.2 implies that if $K(S)$ is a reproducing kernel Pontryagin space, then S is of bounded type. In the Pontryagin space case, we also notice that S is contractive a.e. on ∂. In the Krein framework, it should be of interest to study what the hypothesis $\|S\| \leq 1$ a.e. in ∂ imposes on the structure of $K(S)$.

In the remainder of this section, Theorem 4.1 is applied successively to four special cases:

a) $S \in S_{p\times q}$

b) $p = q$ and S inner

c) $S = S_1^{-1}S_2$ with $H(S_1) \cap H(S_2) = \{0\}$ (see below for the notation $H(S_1)$).

d) S unitary a.e. on ∂.

When S belongs to the class $S_{p\times q}$, we may choose $S_1 = I_p$, and thus $|\Gamma| = \Gamma$ and $K(S)$ may be described as the closure of the elements in H_p^2 of the form

$$v = u - S\underline{p}S^*u \tag{4.10}$$

in the norm

$$\|v\|_{K(S)}^2 = \|u\|_{H_p^2}^2 - \|\underline{p}S^*u\|_{H_p^2}^2 \ . \tag{4.11}$$

We remark that Theorem 4.1 guarantees that (4.11) is a genuine norm on the space $K(S)$. This may also be confirmed directly by the observation that, if $\|v\| = 0$, then $<(I - \underline{p}S\underline{p}S^*)u, u>_{H_p^2} = 0$, and hence $v = (I - \underline{p}S\underline{p}S^*)u = 0$.

(As mentioned in Section 3, the spaces $K(S)$ will usually be denoted by $H(S)$ in the Hilbert space case, following de Branges' notation.)

In this case, there is an alternative definition of $H(S)$, due to deBranges, which makes use of the notion of complementary subspaces (see Section 2 for the definition), as we now explain. Suppose that for some λ in Ω_+

$$Rank\ S(\lambda) = q \ .$$

This forces $p \geq q$, and the rank condition holds for all λ in Ω_+ with the possible exception of a Nevanlinna zero set. We let $H = H_p^2$, the usual Hardy space of $p \times 1$ vectors with entries in H^2, and define H_+ by

$$H_+ = \{Sh \ ; \ h \in H_q^2\}$$

with norm $\|Sh\|_+ = \|h\|_{H_q^2}$.

The rank condition ensures that the norm is well defined. It is readily seen that H_+ is a Hilbert space. Moreover, H_+ is clearly included contractively in H, and thus it has a complementary subspace $(H_-, \| \ \|_-)$, which we now compute and identify with $H(S)$.

We first compute the minimal decomposition of an element in H_p^2. Let i_+ be the injection from H_+ into H, and let x be in H_p^2. Clearly, $i_+^* x = Su$ for some u in H_q^2 which remains to be determined. Let v be in H_p^2. We have the following equalities

$$< i_+^* x, Sv >_{H_+} = < x, i_+ Sv >_H = < x, Sv >_{H_p^2}$$

and thus

$$< Su, Sv >_{H_+} = < S^* x, v >_{H_p^2}$$

i.e.

$$< u, v >_{H_p^2} = < S^* x, v >_{H_p^2}$$

and so $u = \underline{p}S^* x$.

The decomposition of x in H_p^2 is thus:

$$x = S\underline{p}S^* x + x - S\underline{p}S^* x \ .$$

Therefore the complementary subspace of H_+ in H_p^2 is (see Theorem 2.2)

$$H_- = \{x - S\underline{p}S^* x, \ \ x \in H_p^2\}$$

with norm $\|x\|^2 - \|x_+\|_+^2$, where $x_+ = S\underline{p}S^* x$, i.e.

$$\|x\|^2 - \|\underline{p}S^* x\|^2$$

which exhibits H_- as $H(S)$ by virtue of (4.10) and (4.11). Thus, by the remarks following Theorem 2.2, $H(S)$ may be defined, for S of full rank, as

$$\{x \in H_p^2, \ \sup_{v \in H_q^2} (\|x + Sv\|_{H_p^2}^2 - \|v\|_{H_q^2}^2) < \infty\}$$

and then

$$\|x\|^2_{H(S)} = \sup_{v \in H^2_q} (\|x + Sv\|^2_{H^2_p} - \|v\|^2_{H^2_q}) \ .$$

This characterization of $H(S)$ still holds even if the rank condition is not fulfilled (see [dB8]). We also mention that $H(S)$ is resolvent invariant for S in $\mathcal{S}_{p\times q}$, a fact we could not prove in general. In the case $\partial = \mathbb{T}$, deBranges proves that, for f in $H(S)$,

$$\|R_0 f\|^2 \leq \|f\|^2 - |f(0)|^2 \ . \tag{4.12}$$

It is of interest to give characterizations of $H(S)$ spaces, i.e. to find necessary and sufficient conditions for a Hilbert space to be a $H(S)$ space. When $\partial = \mathbb{T}$, a $H(S)$ space has the following properties:

1) It is a reproducing kernel Hilbert space of $p \times 1$ valued functions which are analytic in $\mathbb{D}$.
2) It is resolvent invariant.
3) The inequality (4.12) holds.

In general, these three conditions are not sufficient to ensure that a Hilbert space is a reproducing kernel Hilbert space with reproducing kernel of the form (4.1). We refer to [dB8] for a detailed account of this problem; apparently, to give necessary and sufficient conditions on a Hilbert space for it to be a $H(S)$ space, is an open problem. Nevertheless, a slight strengthening of 1), 2) and 3) permits one to give some partial answers. Indeed, let us consider a Hilbert space $\mathcal{H}$ satisfying the following three conditions:

1') Elements in $\mathcal{H}$ are $p \times 1$ valued functions, analytic in $\mathbb{D}$ and in some neighborhood V of some point δ on $\mathbb{T}$, and $\mathcal{H}$ is a reproducing kernel Hilbert space, i.e. the functionals $f \mapsto c^* f(w)$ are continuous for c in $\mathbb{C}_{p\times 1}$ and w in $\mathbb{D} \cup V$.

2') $\mathcal{H}$ is resolvent invariant: $R_\alpha \mathcal{H} \subset \mathcal{H}$ for α in $\mathbb{D} \cup V$.

3') For any f and g in $\mathcal{H}$ and α, β in $\mathbb{D} \cup V$,

$$< f, g > + \alpha < R_\alpha f, g > + \overline{\beta} < f, R_\beta g > + (1 - \alpha\overline{\beta}) < R_\alpha f, R_\beta g > = g^*(\beta) f(\alpha) \ . \tag{4.13}$$

Then, the reproducing kernel of $\mathcal{H}$ is of the form (4.1) for some function S in $\mathcal{S}_{p\times p}$ which is analytic in $\mathbb{D} \cup V$, and is such that *det* $S \not\equiv 0$ in $\mathbb{D} \cup V$.

This result is a particular case of a disk version due to Ball ([B1]) of a theorem of de Branges, in the technically improved form due to Rovnyak. More general conditions may be imposed, to get to the desired conclusions, and will be discussed in Section 6. For the moment, it suffices to mention that the hypothesis of analyticity at a point of $\mathbb{T}$ may be weakened in various manners, but holds in all practical cases we know of including in particular the finite dimensional case. We further remark that (4.13) specialized to $f = g$ and $\alpha = \beta = 0$ leads to (4.12) with equality rather than inequality.

To ease future reference, we gather the above discussion in the following theorem.

THEOREM 4.3. (deBranges-Rovnyak): Let $\mathcal{H}$ be a reproducing kernel Hilbert space of $p \times 1$ valued functions which are analytic in $\mathbb{D}$ and at some point δ of $\mathbb{T}$, which is resolvent invariant, and for which equality (4.13) holds. Then, the reproducing kernel for $\mathcal{H}$ is of the form (4.1) where S is analytic at the point δ. In particular, S belongs to $S_{p\times p}$ and $det\ S \not\equiv 0$ in $\mathbb{D}$.

There is an analogous theorem for the line case, with (4.13) replaced by

$$< R_\alpha f, g > - < f, R_\beta g > -(\alpha - \overline{\beta}) < R_\alpha f, R_\beta g >= 2\pi i g^*(\beta) f(\alpha) \ . \qquad (4.14)$$

When S is $p \times p$ valued and inner, it is not difficult, using (4.10) and (4.11), to see that $H(S)$ coincides with $H_p^2 \ominus SH_p^2$. Spaces $H(S)$ corresponding to square inner functions are easier to handle then general $H(S)$ spaces. For instance, if S_1 and S_2 are two elements in $S_{p\times q}$, then $H(S_1) \cap H(S_2)$ is not necessarily a $H(S)$ space, as is the case when both S_1 and S_2 are $p \times p$ valued and inner.

LEMMA 4.3. Let S_1 and S_2 be two $p \times p$ inner functions. Then,

$$H(S_1) \cap H(S_2) = H(S_3)$$

where S_3 is a $p \times p$ valued inner function. Moreover $S_1' = S_3^{-1}S_1$ and $S_2' = S_3^{-1}S_2$ are inner and

$$H(S_1') \cap H(S_2') = \{0\} \ .$$

PROOF. $H(S_1) \cap H(S_2)$ is a resolvent invariant closed subspace of H_p^2, and by the Beurling–Lax theorem, there is a $p \times k$ valued inner function S_3 such that $H(S_1) \cap H(S_2) = H_p^2 \ominus S_3 H_k^2$. The reproducing kernel for $H_p^2 \ominus S_3 H_k^2$ is $\frac{I_p - S_3(\lambda)S_3^*(w)}{\rho_w(\lambda)}$, and thus, we get

$$I_p - S_3(w)S_3^*(w) \leq I_p - S_1(w)S_1^*(w)$$

for w in Ω_+, from the inclusion $H_p^2 \ominus S_3 H_k^2 \subset H(S_1)$. Therefore $S_1(w)S_1^*(w) \leq S_3(w)S_3^*(w)$. Taking w where $det\ S_1(w) \neq 0$, we see that $k = p$ and thus S_3 is square. Finally, let $\mathcal{M} = H(S_1') \cap H(S_2')$; $S_3\mathcal{M}$ is easily seen to be both in $H(S_1) \cap H(S_2)$ and $[H(S_1) \cap H(S_2)]^\perp$ and thus $\mathcal{M} = \{0\}$.

An easy consequence of Lemma 4.3 is

LEMMA 4.4. Let S be a $p \times p$ matrix valued function of bounded type in Ω_+, which is unitary a.e. on the boundary. Then, $S = S_1^{-1}S_2$ where S_1 and S_2 are $p \times p$ inner functions which may be chosen such that $H(S_1) \cap H(S_2) = \{0\}$.

PROOF. By hypothesis, we may write S as $\frac{H}{h}$, where H (resp. h) is in $S_{p\times p}$ (resp. S). Let $h = s_1 o_1$ be the inner outer factorization of h. By the unitarity of S on ∂, we get:

$$H^*H = o_1^* o_1 I_p \quad \text{a.e. on } \partial$$

from which follows, by an argument similar to ([SNF], Proposition 4.1, p.200) that

$$H = S_2 o_1$$

for some $p \times p$ inner function S_2. Indeed, the map X

$$o_1 v \longrightarrow Hv$$

where v is in H_p^2 is an isometry from H_p^2 into itself and is equal to multiplication by an inner function S_2, since it commutes with multiplication by the complex variable when $\partial = \mathbb{T}$ and with multiplication by exponentials ($\lambda \mapsto e^{i\lambda t}$) when $\partial = \mathbb{D}$. Thus

$$X(o_1 v) = S_2 o_1 v$$

for some $p \times p$ inner function S_2, and from which we have $H = S_2 o_1$ and thus $S = S_1^{-1} S_2$ with $S_1 = s_1 I_p$.

The second part of the lemma is a corollary of Lemma 4.3. □

We now turn to the third special case, namely the case where S is of bounded type, and $H(S_1) \cap H(S_2) = \{0\}$. This condition is always achievable for S unitary, as mentioned in Lemma 4.4, and also for $S = S_1^{-1} S_2$ with a finite dimensional $H(S_1)$ or $H(S_2)$. We do not know if the condition is achievable for general S. The condition is convenient as the next theorem shows.

THEOREM 4.4. Let $S = S_1^{-1} S_2$ be a $p \times q$ valued function of bounded type in Ω_+ and suppose that $H(S_1) \cap H(S_2) = \{0\}$. Then, the set of functions of the form

$$F = S_1^{-1}(u_1 + u_2) \tag{4.15}$$

where $u_1 \in H(S_1)$, $u_2 \in H(S_2)$, and the inner product

$$[F, F] = < u_2, u_2 >_{H(S_2)} - < u_1, u_1 >_{H(S_1)} \tag{4.16}$$

is a RKKS with reproducing kernel (4.1).

PROOF. Let $\mathcal{K}$ denote the set of functions of the form (4.15) with the hermitian form (4.16), and let $\Omega(S)$ be the subset of Ω_+ where $det\ S \neq 0$ and where S_1 is invertible. Since $H(S_1) \cap H(S_2) = \{0\}$, the decomposition of an element F in $\mathcal{K}$ as $S_1^{-1}(u_1 + u_2)$ is unique, and thus, the form (4.16) is well defined. We check now that for w in $\Omega(S)$ and c in $\mathbb{C}_{p\times 1}$, we have the reproducing kernel property, i.e.

$$[F, k_w c] = c^* F(w)$$

for F in $\mathcal{K}$. Indeed,

$$k_w c = \left\{ \frac{I - S(\lambda) S(w)^*}{\rho_w(\lambda)} \right\} c = S_1^{-1}(-k_w^1 + k_w^2) S_1^{-*}(w) c \tag{4.17}$$

where $k_w^i = \frac{I_p - S_i(\lambda)S_i^*(w)}{\rho_w(\lambda)}$ for $i = 1, 2$. Thus

$$\begin{aligned}[F, k_w c] &= [S_1^{-1}(u_1 + u_2), S_1^{-1}(k_w^2 - k_w^1)S_1^*(w)c] \\ &= < u_1, k_w^1 S_1^{-*}(w)c >_{H(S_1)} + < u_2, k_w^2 S_1^*(w)c >_{H(S_2)} \\ &= c^* S_1^{-1}(w)(u_1(w) + u_2(w)) \\ &= c^* F(w) \ .\end{aligned}$$

In order to check that $\mathcal{K}$ endowed with $[\ ,\ ]$ is a Krein space, it suffices to notice that the hermitian form

$$< F, F > = < u_1, u_1 >_{H(S_1)} + < u_2, u_2 >_{H(S_2)}$$

makes $\mathcal{K}$ into a Hilbert space and that

$$[F, F] = < F, \Sigma F >$$

where Σ is the operator from $\mathcal{K}$ into $\mathcal{K}$ defined by

$$\Sigma S_1^{-1}(u_1 + u_2) = S_1^{-1}(-u_1 + u_2) \ .$$

□

By Lemma 4.4, we notice that Theorem 4.4 is applicable to functions S which are unitary a.e. on ∂; spaces $K(S)$ with a unitary S have supplementary properties, which we now discuss under the additional assumption that S is analytic at some point of ∂. We first treat the case $\partial = \mathbb{R}$, and consider a function $S = S_1^{-1}S_2$ unitary a.e. on $\mathbb{R}$. We suppose moreover that S_1 and S_2 are inner and analytic in a neighborhood V of a point δ of $\mathbb{R}$. Without loss of generality, V may be supposed symmetric with respect to $\mathbb{R}$. Now let α be a point in $\Omega(S) \cap V$. We check that, for any c in $\mathbb{C}_{p\times 1}$, the function $k_{\overline{\alpha}}c$ belongs to $K(S)$ and reproduces at $\overline{\alpha}$. Indeed, S_i satisfies in V the equation $S_i S_i^{\#} = I_p$ and is inner; it is then not difficult to see that $k_{\overline{\alpha}}^i c$ belongs to $H(S_i)$ and that

$$< f, k_{\overline{\alpha}}^i c > = c^* f(\overline{\alpha}) \tag{4.18}$$

for f a finite linear combination of terms of the form $k_w^i c$, w in Ω_+ and c in $\mathbb{C}_{p\times 1}$. Such f are well defined at $\overline{\alpha}$, and from equation (4.18) one can define $f(\overline{\alpha})$ for any f in $H(S_i)$ by continuity arguments. The functions f of $H(S_i)$ thus extended to all of V are in fact analytic in V, as is easily checked. Hence, the desired property follows for $k_{\overline{\alpha}}c$, by (4.17) and the definition of the inner product in $K(S)$.

We denote by $\dot{\mathcal{K}}$ the linear span of the $k_\alpha c$, α in $V \cap \Omega(S)$ and c in $\mathbb{C}_{p\times 1}$. From the reproducing kernel property, $\dot{\mathcal{K}}$ is dense in $K(S)$. Moreover, $\dot{\mathcal{K}}$ is resolvent invariant, i.e. $R_\alpha \dot{\mathcal{K}} \subset \dot{\mathcal{K}}$ for α in V, and (4.13) holds in $\dot{\mathcal{K}}$; the proof is as in [AD1], Theorem 2.3, and relies on the formula

$$2\pi i(\beta - \gamma^*)(R_\beta k_\gamma c)(\lambda) = \{k_{\beta^*}(\lambda)S(\beta)S^*(\gamma) - k_\gamma(\lambda)\}c \tag{4.19}$$

which holds for any β,γ in V and c in $\mathbb{C}_{p\times 1}$. (4.19) permits one to evaluate the various terms in (4.13) for elements in $\dot{\mathcal{K}}$ and check that (4.13) indeed holds.

The case $\partial = \mathbb{T}$ is treated similarly and omitted. This discussion is gathered in the next theorem:

THEOREM 4.5. Let $S = S_1^{-1}S_2$ be a $p \times p$ valued function, unitary a.e. on ∂, and suppose that S_1 and S_2 are analytic in some common open set V symmetric with respect to ∂ and intersecting ∂. Then, $\dot{\mathcal{K}}$, the linear span of the functions $k_w c$, w in V and c in $\mathbb{C}_{p\times 1}$ is such that

1) $\dot{\mathcal{K}}$ is dense in $\mathcal{K}(S)$.
2) $\dot{\mathcal{K}}$ is resolvent invariant: $R_\alpha \dot{\mathcal{K}} \subset \dot{\mathcal{K}}$ for α in V.
3) (4.13) (resp. (4.14)) holds for f,g in $\dot{\mathcal{K}}$ and α,β in V when $\partial = \mathbb{R}$ (resp. $\mathbb{T}$).

5. $\mathcal{L}(\Phi)$ Spaces

The present section is devoted to the study of $\mathcal{L}(\Phi)$ spaces, i.e. to the study of reproducing kernel Krein spaces of $p \times 1$ valued functions, meromorphic in $\mathbb{C} \setminus \partial$, which admit a reproducing kernel of the form $\frac{\Phi(\lambda)+\Phi^*(w)}{\rho_w(\lambda)}$, Φ being meromorphic in $\mathbb{C} \setminus \partial$, $p \times p$ valued, and subject to $\Phi + \Phi^\# = 0$. Such spaces were introduced by de Branges in the Hilbert space case, and played an important role in the study of self adjoint operators. Moreover, de Branges gave a characterization of such spaces ([dB2], [dBR]).

Here, we first generalize to the Pontryagin framework the characterization of $\mathcal{L}(\Phi)$ spaces; we also discuss the case of Krein spaces, but the results are much less complete. Then, we turn to the Hilbert space case and make various comments.

Before stating the Pontryagin version of de Branges' result, we recall a definition due to Krein and Langer [KL3]. Recall that C_p^k was defined to be the class of $p \times p$ valued functions Φ, meromorphic in $\mathbb{C} \setminus \partial$, such that the kernel $\frac{\Phi(\lambda)+\Phi^*(w)}{\rho_w(\lambda)}$ has k negative squares in $\mathbb{C} \setminus \partial$ and $\Phi + \Phi^\# = 0$.

DEFINITION 5.1. ([KL3]): Let $\partial = \mathbb{R}$ and suppose $\Phi \in C_p^k$. Then, Φ satisfies $\mathcal{D}$ if there exists a sequence of points (z_n) in $\mathbb{C}_+$ such that

$$\lim Im\, z_n = \infty \quad \text{and}$$

$$\lim \frac{\Phi(z_n)}{Im\, z_n} = 0 \ .$$

We now state the theorems to be proved; there are two theorems, one for the case $\partial = \mathbb{T}$ and one for the case $\partial = \mathbb{R}$.

THEOREM 5.1. Let π_k be a reproducing kernel Pontryagin space of $p \times 1$ valued functions, meromorphic in $\mathbb{C} \setminus \mathbb{R}$. Then π_k has a reproducing kernel of the form $\frac{\Phi(\lambda)+\Phi^*(w)}{-2\pi i(\lambda-\overline{w})}$ with a function Φ in C_p^k satisfying $(\mathcal{D})$ if and only if

1) π_k is resolvent invariant,

2) $\lim_{\rho \to +\infty} F(\rho e^{i\theta}) = 0$ for all F in π_k and all $\theta \in (0, \pi)$,

3) for all α, β where the elements of π_k are analytic and all F, G in π_k

$$[R_\alpha F, G] - [F, R_\beta G] - (\alpha - \overline{\beta})[R_\alpha F, R_\beta G] = 0 \ . \tag{5.1}$$

THEOREM 5.2. Let π_k be a reproducing kernel Pontryagin space of $p \times 1$ valued functions, meromorphic in $\mathbb{C} \setminus \mathbb{T}$, and analytic at 0. Then, π_k has a reproducing kernel of the form $\frac{\Phi(\lambda)+\Phi^*(w)}{1-\lambda\overline{w}}$ for a function Φ in C_p^k analytic at 0 if and only if

1) π_k is resolvent invariant,

2) for all α and β where the elements of π_k are analytic and all F, G in π_k,

$$[F,G] + \alpha[R_\alpha F, G] + \overline{\beta}[F, R_\beta G] - (1 - \alpha\overline{\beta})[R_\alpha F, R_\beta G] = 0 \ . \tag{5.2}$$

The proofs of Theorems 5.1 and 5.2 rely on two representation formulas for elements of C_p^k, due to Krein and Langer, which we now list:

THEOREM 5.3. ([KL3], Satz 4.1, p.212): Let $\partial = \mathbb{R}$ and let Φ be an element of C_p^k satisfying $(\mathcal{D})$. Then, there is a Pontryagin space of rank k, Π_k, a π-self adjoint operator A from Π_k into itself and a bounded operator Γ from $\mathbb{C}_{p\times 1}$ into Π_k such that, for non-real λ in the resolvent set $\rho(A)$ of A, we have:

$$\Phi(\lambda) = +iS - y_0\Gamma^+\Gamma - i(\lambda - \overline{z}_0)\Gamma^+(A - z_0)(A - \lambda)^{-1}\Gamma \tag{5.3}$$

where $z_0 = x_0 + iy_0$ is some fixed point of $\mathbb{C}_+$, not in the spectrum of A, where S is a $p \times p$ matrix such that $S = S^*$, and where Γ^+ is the π-adjoint of Γ. Moreover, A and Γ are such that:

$$\Pi_k = \bigvee_{\lambda \in \rho(A)} Ran(A - \lambda)^{-1}\Gamma \ . \tag{5.4}$$

THEOREM 5.4.([KL1]). Let $\partial = \mathbb{T}$ and let Φ be an element of C_p^k which is analytic at 0. Then, there is a Pontryagin space of rank k, Π_k, a π unitary operator U from Π_k into itself, and an operator Γ from $\mathbb{C}_{p\times 1}$ into Π_k such that, for λ in the resolvent set of U, not on $\mathbb{T}$, we have:

$$\Phi(\lambda) = i \ Im \ \Phi(0) + \Gamma^+(U + \lambda)(U - \lambda)^{-1}\Gamma \ . \tag{5.5}$$

Moreover, U and Γ are such that

$$\Pi_k = \bigvee_{\lambda \in \rho(U)} Ran(U - \lambda)^{-1}\Gamma \ . \tag{5.6}$$

We recall that a π-self adjoint operator in a Pontryagin space has possibly a symmetrically displaced non-real spectrum, which consists of a finite number of non-real points, and which are eigenvalues of finite multiplicity. Similarly, a π-unitary operator may have a part of its spectrum not on $\mathbb{T}$; this part consists of at most a finite number of points, each of which are eigenvalues of finite multiplicity.

For Φ in C_p^k satisfying $(\mathcal{D})$ when $\partial = \mathbb{R}$ or analytic at 0 when $\partial = \mathbb{T}$, we have the following formulas which are easily gotten from (5.3) and (5.5), for λ, w in $\rho(A)$ or $\rho(U)$ and not on ∂:

$$\frac{\Phi(\lambda) + \Phi^*(w)}{-2\pi i(\lambda - \overline{w})} = \frac{1}{2\pi}\Gamma^+(A - \overline{z}_0)(A - \lambda)^{-1}(A - \overline{w})^{-1}(A - z_0)\Gamma \text{ if } \partial = \mathbb{R} \tag{5.7}$$

$$\frac{\Phi(\lambda) + \Phi^*(w)}{1 - \lambda\overline{w}} = \Gamma^+(U^+ - \overline{w})^{-1}(U - \lambda)^{-1}\Gamma \text{ if } \partial = \mathbb{T} \ , \tag{5.8}$$

(for (5.7), we refer to (3.10) of [KL3] and for (5.8) we refer to (2.11) of [KL1]).

Theorem 2.1 ensures that there is a reproducing kernel Pontryagin space of rank k with kernel

$$\frac{\Phi(\lambda)+\Phi^*(w)}{\rho_w(\lambda)} .$$

The objective of Theorems 5.1 and 5.2 is to give an analytic description of this space. We begin with two lemmas.

LEMMA 5.1. Let $\partial = \mathbb{R}$ and let $\Phi \in \mathcal{C}_p^k$ satisfy $(\mathcal{D})$. Then,

$$\mathcal{L}(\Phi) = \{F : F(\lambda) = \frac{1}{2\pi}\Gamma^+(A-\overline{z}_0)(A-\lambda)^{-1}f;\ f \in \Pi_k\}$$

with inner product

$$[F,F]_{\mathcal{L}(\Phi)} = \frac{1}{2\pi}[f,f]_{\Pi_k} .$$

LEMMA 5.2. Let $\partial = \mathbb{D}$ and let $\Phi \in \mathcal{C}_p^k$ be analytic at 0. Then,

$$\mathcal{L}(\Phi) = \{F : F(\lambda) = \Gamma^+(U-\lambda)^{-1}f;\ f \in \Pi_k\}$$

with inner product

$$[F,F]_{\mathcal{L}(\Phi)} = [f,f]_{\Pi_k} .$$

We will prove Lemma 5.1 only since Lemma 5.2 has a similar proof. From these lemmas, it is readily seen that $\mathcal{L}(\Phi)$ is resolvent invariant. For example, if α is not in $(\mathbb{T} \cup \sigma(U))$, then $F(\lambda) = \Gamma^+(U-\lambda)^{-1}f$ and

$$(R_\alpha F)(\lambda) = \Gamma^+(U-\lambda)^{-1}(U-\alpha)^{-1}f \tag{5.9}$$

which clearly implies that $R_\alpha\mathcal{L}(\Phi) \subset \mathcal{L}(\Phi)$.

PROOF OF LEMMA 5.1. We first remark that a space $\mathcal{L}(\Phi)$ exists by Theorem 2.1. For non-real $w_1,\ldots,w_\ell$ in $\rho(A)$ and $p\times 1$ vectors $c_1,\ldots,c_\ell$ let $f = \Sigma_1^\ell(A-z_0)(A-\overline{w}_j)^{-1}\Gamma c_j$. Then, from equation (5.7), the function F defined by

$$F(\lambda) = \frac{1}{2\pi}\Gamma^+(A-\overline{z}_0)(A-\lambda)^{-1}f$$

is equal to

$$F(\lambda) = \sum_1^\ell \frac{\Phi(\lambda)+\Phi^*(w_j)}{-2\pi i(\lambda-\overline{w}_j)}c_j$$

and thus belongs to $\mathcal{L}(\Phi)$. Moreover,by (5.7),

$$\frac{1}{2\pi}[f,f]_{\Pi_k} = \Sigma c_j^*\frac{\Phi(w_j)+\Phi^*(w_k)}{-2\pi i(w_j-\overline{w}_k)}c_k = [F,F]_{\mathcal{L}(\Phi)} .$$

By (5.4), such f are dense in Π_k, and to conclude, we need to show that the isometry $f \to F$ is continuous, to extend it to every $f \in \Pi_k$. From ([Bo,], p.188, Theorem 3.1), an isometric operator in a Pontryagin space, the closure of the range and the closure of the domain of which are non-degenerate, is continuous. This fact is easily extended to the present context where the isometry is between two different Pontryagin spaces of the same rank which permits us to conclude the proof.

We now turn to the proof of Theorem 5.1.

PROOF OF THEOREM 5.1. We first prove the necessity. Let F and G be elements of $\mathcal{L}(\Phi)$; by Lemma 5.1 we have

$$F(\lambda) = \frac{1}{2\pi}\Gamma^+(A-\overline{z}_0)(A-\lambda)^{-1}f$$

$$G(\lambda) = \frac{1}{2\pi}\Gamma^+(A-\overline{z}_0)(A-\lambda)^{-1}g$$

for some f and g in Π_k. It is thus clear that lim $F(\rho e^{i\theta}) = 0$ when ρ goes to infinity and $\theta \in (0,\pi)$. Thus since, as was already remarked, $\mathcal{L}(\Phi)$ is resolvent invariant, it remains only to check (5.1). We have, from the analogue of (5.9) and Lemma 5.1$(\alpha, \beta \in \rho(A))$,

$$[R_\alpha F, G]_{\mathcal{L}(\Phi)} = \frac{1}{2\pi}[(A-\alpha)^{-1}f, g]_{\Pi_k}$$

$$[F, R_\beta g]_{\mathcal{L}(\Phi)} = \frac{1}{2\pi}[f, (A-\beta)^{-1}g]_{\Pi_k}$$

$$[R_\alpha F, R_\beta G]_{\mathcal{L}(\Phi)} = \frac{1}{2\pi}[(A-\alpha)^{-1}f, (A-\beta)^{-1}g]_{\Pi_k}$$

from which (5.1) follows, since $A = A^+$.

Conversely, we first check that a space with the properties of the theorem has a kernel of the form $\frac{\Phi(\lambda)+\Phi^*(w)}{-2\pi i(\lambda-\overline{w})}$ for some $p \times p$ valued function Φ, meromorphic in $\mathbb{C} \setminus \mathbb{R}$, and is subject to $\Phi + \Phi^{\#} = 0$. The proof follows the Hilbert space case ([dB6], Theorem 6) but is outlined for completeness.

Let α_0 be a non-real number such that the elements of π_k are analytic both at α_0 and $\overline{\alpha}_0$ and let us apply identity (5.1) to $\alpha = \overline{\beta} = \alpha_0$, and $F(\lambda) = K_\mu(\lambda)a$, $G(\lambda) = K_\nu(\lambda)b$, where a and b are in $\mathbb{C}_{p\times 1}$ and ν, μ are points of analyticity of the elements of π_k. We then get

$$\left[\frac{K_\mu(\lambda)a - K_\mu(\alpha_0)a}{\lambda-\alpha_0}\ ,\ K_\nu(\lambda)b\right]_{\pi_k} = \left[K_\mu(\lambda)a\ ,\ \frac{K_\nu(\lambda)b - K_\nu(\overline{\alpha}_0)b}{\lambda-\overline{\alpha}_0}\right]_{\pi_k}$$

from which it follows, after some computations, that

$$(\overline{\mu}-\nu)K_\mu(\nu) = (\overline{\mu}-\alpha_0)K_\mu(\alpha_0) - (\nu-\alpha_0)K_{\overline{\alpha}_0}(\nu)\ . \tag{5.10}$$

Let us set

$$\Phi(\nu) = -2\pi i(\alpha_0 - \nu)K_{\overline{\alpha}_0}(\nu) - i\pi(\overline{\alpha}_0 - \alpha_0)K_{\alpha_0}(\alpha_0) \ . \tag{5.11}$$

Then,

$$\Phi^*(\mu) = 2\pi i(\overline{\alpha}_0 - \overline{\mu})K_\mu(\overline{\alpha}_0) - i\pi(\overline{\alpha}_0 - \alpha_0)K_{\alpha_0}(\alpha_0) \ .$$

Hence,

$$\begin{aligned}&\Phi^*(\mu) + \Phi(\nu)\\ &= -2\pi i(\alpha_0 - \nu)K_{\overline{\alpha}_0}(\nu) + 2\pi i(\overline{\alpha}_0 - \overline{\mu})K_\mu(\overline{\alpha}_0) - 2\pi i(\overline{\alpha}_0 - \alpha_0)K_{\alpha_0}(\alpha_0) \ .\end{aligned} \tag{5.12}$$

But from (5.10) we get, putting $\mu = \alpha_0$,

$$(\overline{\alpha}_0 - \nu)K_{\alpha_0}(\nu) = (\overline{\alpha}_0 - \alpha_0)K_{\alpha_0}(\alpha_0) - (\nu - \alpha_0)K_{\overline{\alpha}_0}(\nu) \ . \tag{5.13}$$

Combining (5.12) and (5.13) we get to

$$\Phi^*(\mu) + \Phi(\nu) = 2\pi i(-(\overline{\alpha}_0 - \nu)K_{\alpha_0}(\nu) + (\overline{\alpha}_0 - \overline{\mu})K_\mu(\overline{\alpha}_0)) \ .$$

The left side of (5.10) is independent of the choice of α_0. Thus using (5.10) with α_0 replaced by $\overline{\alpha}_0$ leads then to

$$K_\mu(\nu) = \frac{\Phi(\nu) + \Phi^*(\mu)}{-2\pi i(\nu - \overline{\mu})} \ . \tag{5.14}$$

We now show that the function Φ satisfies the $(\mathcal{D})$ assumption; indeed, we can choose a family of points (λ_n) in $\mathbb{C}_+$ such that $Re\ \lambda_n$ is zero, $Im\ \lambda_n$ goes to infinity, and the elements of π_k are analytic at λ_n. Then, for some fixed w where Φ is analytic, we have

$$\lim_{n\to\infty} \frac{\Phi(\lambda_n) + \Phi^*(w)}{\lambda_n - w} = 0 \tag{5.15}$$

and thus

$$\lim_{n\to\infty} \frac{\Phi(\lambda_n)}{Im\ \lambda_n} = 0 \ .$$

Hence Φ satisfies $(\mathcal{D})$, and the proof of Theorem 5.1 is finished since Φ belongs to $\mathcal{C}_p^k$. □

The proof of Theorem 5.2 is similar and omitted.

We remark that, from Lemma 5.1, elements of $\mathcal{L}(\Phi)$ are analytic in $\mathbb{C}\setminus\partial$, with the possible exception of a finite set of points. As mentioned earlier, Theorems 5.1 and 5.2 are due to de Branges in the Hilbert space case.

To verify that our result indeed generalizes de Branges' result, some more analysis is needed: In the Hilbert space case, the $(\mathcal{D})$ hypothesis is not imposed, and we must see what restrictions on functions Φ in $\mathcal{C}_p$ this hypothesis forces. By the Herglotz representation theorem, we can write

$$\Phi(\lambda) = A - iB\lambda + \frac{1}{i\pi}\int d\mu(\gamma)\left\{\frac{1}{\gamma - \lambda} - \frac{\gamma}{\gamma^2 + 1}\right\} \quad \lambda \in \mathbb{C}_\pm \tag{5.16}$$

where A and B are $p \times p$ matrices subject to $A + A^* = 0$, $B = B^* \geq 0$ and μ is a $p \times p$ valued increasing function, (i.e. $\gamma \leq \gamma' \Rightarrow \mu(\gamma') - \mu(\gamma)$ is a positive matrix), subject to the trace condition $\int \frac{d\, Tr\, \mu(\gamma)}{\gamma^2+1} < \infty$. It is not difficult to show that Φ satisfies $(\mathcal{D})$ if and only if $B = 0$.

Thus, we generalize de Branges' result only for functions Φ for which $B = 0$. To get the result for any Φ is not difficult, as we now briefly explain. First, from (5.16) we get, for any λ and w in $\mathbb{C}\setminus\mathbb{R}$

$$\frac{\Phi(\lambda) + \Phi^*(w)}{-2\pi i(\lambda - \overline{w})} = \frac{B}{2\pi} + \frac{1}{2\pi^2}\int \frac{d\mu(\gamma)}{(\gamma - \lambda)(\gamma - \overline{w})} \ . \tag{5.17}$$

(5.17) is the Hilbert space version of (5.7), at least when $B = 0$ and it is not difficult (see for instance [dB6]) to show that $\mathcal{L}(\Phi)$ may be identified as

THEOREM 5.5. Let Φ be an element of $\mathcal{C}_p$. Then,

$$\mathcal{L}(\Phi) = \{F; F(\lambda) = \frac{Bx}{2\pi} + \frac{1}{\pi}\int \frac{d\mu(\gamma)f(\gamma)}{\gamma - \lambda},\ \lambda \in \mathbb{C}\setminus\mathbb{R}\}$$

where x is in $\mathbb{C}_{p\times 1}$, and f in $L^2_{p\times 1}(d\mu)$. Moreover,

$$\|F\|^2_{\mathcal{L}(\Phi)} = \frac{1}{2\pi}\|B^{\frac{1}{2}}x\|^2_{\mathbb{C}_{p\times 1}} + 2\|f\|^2_{L^2(d\mu)}$$

from which follows that

$$\mathcal{L}(\Phi) = \mathcal{L}(\Phi_0) \oplus \mathcal{L}(\Phi_1)$$

with

$$\Phi_0(\lambda) = -iB\lambda$$
$$\Phi_1(\lambda) = \Phi - \Phi_0(\lambda)$$

and it is easy to conclude that Theorem 5.1 holds for any function Φ in $\mathcal{C}_p$, even without hypothesis $(\mathcal{D})$; the assumption 2) of the theorem on the behavior of the functions at ∞ must also be removed, and we have:

THEOREM 5.6. ([dB6]): Let H be a reproducing kernel Hilbert space of $p \times 1$ valued functions analytic in $\mathbb{C}\setminus\mathbb{R}$, and resolvent invariant. Then, the reproducing kernel of H is of the form $\frac{\Phi(\lambda)+\Phi^*(w)}{-2\pi i(\lambda-\overline{w})}$ for a $p \times p$ valued function Φ of Caratheodory class if and only if, for any F, G in H and α, β non-real, we have

$$< R_\alpha F, G > - < F, R_\beta G > -(\alpha - \overline{\beta}) < R_\alpha F, R_\beta G >= 0 \ .$$

For Krein spaces, the analysis is less complete because of the lack of representation formulas like (5.3) and (5.5). Nevertheless, we are able to prove a number of results which are now sketched briefly.

THEOREM 5.7. Let $\mathcal{K}$ be a reproducing kernel Krein space of $p \times 1$ valued analytic functions, meromorphic in $\mathbb{C} \backslash \mathbb{R}$, which is resolvent invariant and for which (5.1) holds. Then the reproducing kernel of $\mathcal{K}$ is of the form $\frac{\Phi(\lambda)+\Phi^*(w)}{-2\pi i(\lambda-\overline{w})}$, where Φ is $p \times p$ valued, meromorphic in $\mathbb{C} \backslash \partial$ and satisfies $\Phi + \Phi^{\#} = 0$.

The proof of this theorem is essentially the same as the proof of Theorem 6 of [dB6] which was recalled in the proof of Theorem 5.1. We cannot prove a converse theorem since we lack information on the class of functions Φ for which a reproducing kernel Krein space $\mathcal{L}(\Phi)$ exists. We conjecture that necessary and sufficient conditions on Φ are that Φ is of bounded type in Ω_+ and that $\Phi + \Phi^{\#} = 0$.

In connection with this conjecture we mention the following result, which is an easy consequence of a theorem of Krein and Langer mentioned in the previous section (Theorem 4.2).

THEOREM 5.8. If Φ belongs to C_p^k, then Φ is of bounded type in Ω_+.

Moreover, we now show that to any $p \times p$ valuedfunction Φ, and of bounded type in Ω_+ we can associate a reproducing kernel Krein space with kernel $\frac{\Phi(\lambda)+\Phi^*(w)}{\rho_w(\lambda)}$.

THEOREM 5.9. Let Φ be a $p \times p$ matrix valued function of bounded type in Ω_+. Then there exists a reproducing kernel Krein space of $p \times 1$ valued functions meromorphic in Ω_+ with reproducing kernel $\frac{\Phi(\lambda)+\Phi^*(w)}{\rho_w(\lambda)}$ with w in Ω_+.

The space defined by Theorem 5.9 will be called $\mathcal{L}_+(\Phi)$; beware that the elements of $\mathcal{L}_+(\Phi)$ are defined only in Ω_+, whereas elements in spaces $\mathcal{L}(\Phi)$ are defined in $\mathbb{C} \backslash \partial$. Theorem 5.9 is not that useful since we do not know much about $\mathcal{L}_+(\Phi)$ in general; in particular, we do not know for which functions Φ, $\mathcal{L}_+(\Phi)$ is resolvent invariant.

PROOF OF THEOREM 5.9. We first remark that one can find a positive number k different from zero, such that $det(I_p + k\Phi(\lambda)) \not\equiv 0$ in Ω_+. Let S be defined by $(I_p + k\Phi(\lambda))^{-1}(I_p - k\Phi(\lambda))$. S is of bounded type in Ω_+ and hence by Theorem 4.1 there exists a reproducing kernel Krein space with kernel $\frac{I_p - S(\lambda)S^*(w)}{\rho_w(\lambda)}$.

From the identity

$$I_p - S(\lambda)S^*(w) = k[I_p + k\Phi(\lambda)]^{-1}[\Phi(\lambda) + \Phi^*(w)][I_p + k\Phi(w)]^{-*}$$

from which it is not difficult to see that $\mathcal{L}_+(\Phi)$ exists, and that

$$\mathcal{L}_+(\Phi) = \{F = \frac{I_p + k\Phi(\lambda)}{\sqrt{k}}G, \quad G \in K(S)\}$$

with inner product

$$[F, F]_{\mathcal{L}_+(\Phi)} = [G, G]_{K(S)} \ .$$

The reproducing kernel property is easily checked. □

We conclude this section with some remarks on equations (5.1) and (5.2). These may look strange at first, or even at second glance. Nevertheless, they may be very helpful in certain situations, as we now illustrate.

LEMMA 5.3. Let $\partial = \mathbb{R}$ and let α, β be in $\mathbb{C}$ and c, d be in $\mathbb{C}_{p\times 1}$. Let Φ be in the class C_p^k, and suppose that Φ satisfies $(\mathcal{D})$. Suppose further that $f_1, \ldots, f_n$, $g_1, \ldots, g_m$ belong to $\mathcal{L}(\Phi)$, with $f_j(\lambda) = \frac{c}{(\lambda-\alpha)^j}$ and $g_j(\lambda) = \frac{d}{(\lambda-\beta)^j}$. Then, we have the following relationships

$$(\overline{\beta} - \alpha)[f_n, g_m] = -[f_n, g_{m-1}] + [f_{n-1}, g_m] \quad if \quad n > 1\,,\ m > 1 \tag{5.18}$$

$$(\overline{\beta} - \alpha)[f_n, g_1] = [f_{n-1}, g_1] \quad if \quad n > 1 \tag{5.19}$$

$$(\overline{\beta} - \alpha)[f_1, g_m] = -[f_1, g_{m-1}] \quad if \quad m > 1 \tag{5.20}$$

$$(\overline{\beta} - \alpha)[f_1, g_1] = 0\ .$$

REMARK. Similar formulas hold for $\partial = \mathbb{T}$.

PROOF. We prove relationship (5.18); the other relations are easily deduced by similar arguments. The proof relies on the formula

$$R_w f_n = -\sum_1^n f_j \cdot \frac{1}{(w-\alpha)^{n+1-j}}\ .$$

Suppose $n > 1$, $m > 1$, and let $\alpha = \overline{\beta} = w$ in formula (5.1), where w is such that the functions of $\mathcal{L}(\Phi)$ are analytic at w and $\overline{w}$. Applying formula (5.1) for $F = f_n$ and $G = g_m$, we get

$$-\sum_1^n [f_j, g_m] \frac{1}{(w-\alpha)^{n+1-j}} + \sum_1^m [f_n, g_{j'}] \frac{1}{(w-\overline{\beta})^{m+1-j'}} = 0$$

which may be written as

$$[f_n, g_m]\left(\frac{1}{w-\overline{\beta}} - \frac{1}{w-\alpha}\right) = -\frac{1}{(w-\overline{\beta})^2}[f_n, g_{m-1}] + \frac{1}{(w-\alpha)^2}[f_{n-1}, g_m] + r(w) \tag{5.21}$$

where the function $r(w)$ is easily seen to satisfy $\lim\ w^2 r(w) = 0$, as $|w|$ goes to infinity. Multiplying (5.21) by w^2 and letting $|w|$ go to ∞, we get the desired formula.

These formulas allow us to make a number of observations on the structure of $\mathcal{L}(\Phi)$ spaces. For instance, when $\overline{\beta} - \alpha$ is different from 0, the linear span of $\{f_1, \ldots, f_n\}$ is orthogonal to $\{g_1, \ldots, g_m\}$ if $[f_1, g_1] = 0$. In particular, specializing these formulas to $n = m$, $f_i = g_i$, $i = 1, \ldots, n$, and α non-real, we see that $\{f_1, \ldots, f_n\}$ is neutral if $[f_1, f_1] = 0$, and thus is of dimension smaller than k.

When the two chains are equal, $n > 1$, and β is real, (5.20) gives $[f_1, f_1] = 0$, and thus we have the following lemma.

LEMMA 5.4. Let $\partial = \mathbb{R}$ and let β belong to $\mathbb{R}$. Then a $\mathcal{L}(\Phi)$ reproducing kernel Hilbert space cannot contain a chain of the form

$$\frac{c}{\lambda-\beta}, \ldots, \frac{c}{(\lambda-\beta)^n} \tag{5.22}$$

with $n > 1$.

Thus, chains of the form (5.22) with real β seem characteristic of the indefinite case.

Finally, we mention that the structure of elements in C_p^k is given in the paper [KL3]; to apply this structure to the decomposition of the space $\mathcal{L}(\Phi)$ as a direct sum of various smaller spaces seems of interest, but will not be touched upon here; similarly, the study of finite dimensional $\mathcal{L}(\Phi)$ spaces seems of interest. This will, we hope, be the subject of a future paper, but will not be touched upon here.

6. $\mathcal{K}(\Theta)$ Spaces

Let J be an $m \times m$ signature matrix and Θ an $m \times m$ matrix valued function which is meromorphic and J contractive in Ω_+, i.e.,

$$\Theta(\lambda)J\Theta^*(\lambda) \le J \tag{6.1}$$

at every point λ of Ω_+ at which Θ is analytic. The kernel

$$K_w(\lambda) = \frac{J - \Theta(\lambda)J\Theta^*(w)}{\rho_w(\lambda)} \tag{6.2}$$

is then a positive kernel, and, by a theorem of Aronszajn, we can associate to it a reproducing kernel of $\mathbb{C}_{m\times 1}$ valued functions, meromorphic in Ω_+ with reproducing kernel (6.2). Notice that this space coincides with the space $H(S)$ defined in Section 4 when $J = I_m$. From (6.1) it follows that the function Θ is of bounded type in Ω_+ (see [AD1]), and thus has non-tangential limits a.e. on ∂. When these satisfy

$$\Theta(\gamma)J\Theta^*(\gamma) = J \quad \text{a.e. on } \partial \tag{6.3}$$

the function Θ is called J-inner and may be extended to Ω_- via

$$\Theta J \Theta^{\#} = J \ . \tag{6.4}$$

The kernel (6.2) is still positive for λ and w points of analyticity of Θ in $\mathbb{C}\backslash\partial$; the correspondingreproducing kernel Hilbert space is denoted by $\mathcal{H}(\Theta)$. $\mathcal{H}(\Theta)$ spaces have been defined and studied by de Branges in [dB3], [dB4] and later publications (for the case of J-inner functions refer to [AD1], and to the lecture notes [D1]).

The aim of the present section is to study $\mathcal{K}(\Theta)$ spaces, i.e. reproducing kernel Krein spaces with reproducing kernel of the form (6.2), when hypothesis (6.1) is removed. In Section 6.1 we will show that such a space exists when the function Θ satisfies:

1) Θ is of bounded type in Ω_+

2) Θ is J-unitary a.e. on ∂

3) Θ is extended to Ω_- via $\Theta J \Theta^{\#} = J$.

In a number of cases in the present section, we will consider spaces $\mathcal{K}(\Theta)$ for which the condition 2) is weakened to

2′) Θ is J-unitary at some point δ of ∂ (i.e. the non-tangential limit $\Theta(\delta_+)$ exists and satisfies

$$\Theta(\delta_+)J\Theta^*(\delta_+) = J \ .$$

The section is divided into subsections as follows: In 6.1, as already mentioned, an existence theorem is proved. In 6.2 various converse theorems, i.e. theorems giving sufficient (and in certain cases necessary and sufficient) conditions on a reproducing kernel Krein space to have a kernel of the form (6.2) are proved. In 6.3 we prove various inclusion theorems. 6.4 is devoted to the finite dimensional case. And finally, in 6.5, we consider two more examples; in particular, Section 6.5 contains an example which is the core of our study of the Inverse Scattering Problem for square Schur functions ([AD1], [AD2]). A review of these two papers and their relationship with the present thesis will be discussed in Section 9.

6.1. A Structure Theorem.

The aim of this subsection is to prove:

THEOREM 6.1. Let Θ be an $m \times m$ valued function of bounded type in Ω_+ which is J-unitary a.e. on ∂ and is extended to Ω_- via $\Theta J \Theta^{\#} = J$. Moreover, let $\Omega(\Theta)$ be the set of points of $\mathbb{C} \backslash \partial$ where Θ is both analytic and invertible. Then, there exists a reproducing kernel Krein space of $m \times 1$ valued functions analytic in $\Omega(\Theta)$ with reproducing kernel

$$K_w(\lambda) = \frac{J - \Theta(\lambda) J \Theta^*(w)}{\rho_w(\lambda)} .$$

Let $\mathcal{K}_0$ denote the set of finite linear combinations of the $K_w c$ for w in $\Omega(\Theta)$ and c in $\mathbb{C}_{m\times 1}$. Then, $R_\alpha \mathcal{K}_0 \subset \mathcal{K}_0$ for α in $\Omega(\Theta)$ and, for any f, g in $\mathcal{K}_0$ and α, β in $\Omega(\Theta)$, we have:

$$[R_\alpha f, g] - [f, R_\beta g] - (\alpha - \overline{\beta})[R_\alpha f, R_\beta g] = 2\pi i \; g^*(\beta) J f(\alpha) \quad \text{if } \partial = \mathbb{R} \tag{6.5}$$

$$[f, g] - \alpha[R_\alpha f, g] - \overline{\beta}[f, R_\beta g] + (1 - \alpha\overline{\beta})[R_\alpha f, R_\beta g] = g^*(\beta) J f(\alpha) \quad \text{if } \partial = \mathbb{T} . \tag{6.6}$$

PROOF. We first suppose that $J = I_m$ and that Θ is inner. Theorem 6.1 reduces then to Theorem 4.3 of [AD1]; in particular $\mathcal{K}(\Theta)$ is equal to $H_m^2 \ominus \Theta H_m^2$, the elements of which are extended to Ω_- via the formula

$$c^* f(w) = < f, \frac{I_m - \Theta(\lambda)\Theta^*(w) c}{\rho_w(\lambda)} >_{H_m^2}$$

for w in $\Omega(\Theta)$, c in $\mathbb{C}_{m\times 1}$ and f in $H_m^2 \ominus \Theta H_m^2$. Still in the case for which $J = I_m$, we suppose next that Θ is unitary a.e. on the boundary. Then, as shown in Section 4, Θ may be written as $\Theta_1^{-1}\Theta_2$ where the Θ_i are $m \times m$ inner functions and $H(\Theta_1) \cap H(\Theta_1) = \{0\}$. We define $\mathcal{K}(\Theta)$ to be the set of functions of the form

$$F = \Theta_1^{-1}(u_1 + u_2)$$

where u_i belongs to $\mathcal{H}(\Theta_i)$, and with hermitian form

$$[F, G] = < u_2, v_2 >_{\mathcal{H}(\Theta_2)} - < u_1, v_1 >_{\mathcal{H}(\Theta_1)} ,$$

where $G = \Theta_1^{-1}(v_1 + v_2)$. Since $H(\Theta_1) \cap H(\Theta_2) = \{0\}$, $\mathcal{H}(\Theta_1) \cap \mathcal{H}(\Theta_2)$ is also zero and the hermitian form is well defined. Moreover, consider the operator σ from $\mathcal{K}(\Theta)$ into itself defined by

$$\sigma F = \Theta_1^{-1}(-u_1 + u_2) \ .$$

From $[\sigma F, F] = < u_2, u_2 > + < u_1, u_1 >$ it is plain that $(\mathcal{K}(\Theta), [\sigma, \ ])$ is a Hilbert space and thus that $(\mathcal{K}(\Theta), [\ ,\ ])$ is a Krein space. Finally, the reproducing property is verified using the decomposition

$$\frac{I_m - \Theta(\lambda)\Theta^*(w)}{\rho_w(\lambda)} = \Theta_1^{-1}(\lambda) \left\{ \frac{-I_m + \Theta_1(\lambda)\Theta_1^*(w)}{\rho_w(\lambda)} - \frac{I_m - \Theta_2(\lambda)\Theta_2^*(w)}{\rho_w(\lambda)} \right\} \Theta_1^{-*}(w) \ .$$

We remark that although this proof follows almost word for word the proof of Theorem 4.4, there is no redundancy. Theorem 4.4 dealt with functions meromorphic in Ω_+ while here we consider spaces of functions meromorphic in $\mathbb{C} \setminus \partial$.

The case $J = -I_m$ may be dealt with in a similar way, and therefore it remains only to consider J to be of the form

$$\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

with p and q non-zero integers. Moreover, following the notations of ([AD1], p.613), let us denote by P (resp. Q) the $m \times m$ matrix

$$\begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} 0 & 0 \\ 0 & I_q \end{pmatrix}\right) \ .$$

From the J-unitarity of Θ on $\mathbb{R}$, it is easy to see that the $m \times m$ valued function $(P - \Theta Q)$ is invertible in $\mathbb{C} \setminus \mathbb{R}$, with the possible exception of a Nevanlinna zero set, and that the function $\Sigma = (P - \Theta Q)^{-1}(\Theta P - Q)$ is J-unitary on $\mathbb{R}$ a.e., and of bounded type in $\mathbb{C}$. Moreover, as in ([AD1], formula (4.5)), we see that

$$I_m - \Sigma(\lambda)\Sigma^*(w) = (P - \Theta(\lambda)Q)^{-1}\{J - \Theta(\lambda)J\Theta^*(w)\}(P - Q\Theta^*(w))^{-1} \ .$$

By the above discussion for the case $J = I_m$, a reproducing kernel Krein space $\mathcal{K}(\Sigma)$ exists, and it is then easy to check that a space $\mathcal{K}(\Theta)$ exists, which may be identified as

$$\mathcal{K}(\Theta) = \{F; (P - \Theta Q)^{-1}F \in \mathcal{K}(\Sigma)\}$$

with inner product

$$[F, G]_{\mathcal{K}(\Theta)} = [(P - \Theta Q)^{-1}F, (P - \Theta Q)^{-1}G]_{\mathcal{K}(\Sigma)} \ .$$

The reproducing kernel of $\mathcal{K}(\Theta)$ is $\frac{J - \Theta(\lambda)J\Theta^*(w)}{\rho_w(\lambda)}$, and the present theorem is thus a generalization of Theorem 4.4 of [AD1] to the present framework.

When $\partial = \mathbb{R}$, the resolvent invariance of $\mathcal{K}_0$ and formula (6.5) relies on the analogue of formula (4.18) which appears in [AD1] (see [AD1], p.598/599)

$$2\pi i(\beta - \gamma^*)(R_\beta K_\gamma \eta)(\lambda) = \{K_{\beta^*}(\lambda) J\Theta(\beta) J\Theta^*(\gamma) - K_\gamma(\lambda)\}\eta \tag{6.7}$$

where K is the reproducing kernel of $\mathcal{K}(\Theta)$ and β, γ are in $\Omega(\Theta)$. As already mentioned in Section 4, such a formula enables us to evaluate terms such as $[R_\beta u, v]_{\mathcal{K}(\Theta)}$ for u and v in $\mathcal{K}_0$ and thus prove (6.5). The case $\partial = \mathbb{T}$ and formula (6.6) is treated similarly, and this concludes the proof of Theorem 6.1. □

In the Hilbert space case, formulas (6.5) and (6.6) may be extended to all of $\mathcal{H}(\Theta)$ as we now explain. We were unable to show that (6.5) and (6.6) hold on all of $\mathcal{K}(\Theta)$ for general Θ. We focus on the case $\partial = \mathbb{R}$ and prove that, for α in $\Omega(\Theta)$, R_α is bounded. Indeed, from (6.5) specialized to $\alpha = \beta$ and $f = g$, we get

$$[R_\alpha f, f] - [f, R_\alpha f] + (\alpha - \overline{\alpha})[R_\alpha f, R_\alpha f] = 2\pi i \; f^*(\alpha) J f(\alpha) \; .$$

Hence, since in the Hilbert space case, there is a constant k_α such that $\|f(\alpha)\|_{\mathbb{C}^{m\times 1}} \leq k_\alpha \|f\|_{\mathcal{H}(\Theta)}$,

$$|(\alpha - \overline{\alpha})| \; \|R_\alpha f\|^2 \leq 2\|R_\alpha f\| \; \|f\| + 2\pi \; k_\alpha^2 \|f\|^2_{\mathcal{H}(\Theta)} \tag{6.8}$$

from which it is easy to see that $\|R_\alpha\| < \infty$. The operators R_α being bounded, (6.5) extends to all $\mathcal{H}(\Theta)$, and a similar argument takes care of (6.6).

As for $\mathcal{L}(\Phi)$ and $K(S)$ spaces, we lack necessary and sufficient conditions on a function Θ for a Krein space $\mathcal{K}(\Theta)$ to exist. We mention the following theorem, the proof of which follows the first part of the proof of Theorem 6.1, and can be regarded as the "one-sided" version of Theorem 6.1.

THEOREM 6.2. Let

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \quad \text{and} \quad \Theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a $m \times m$ valued function of bounded type in Ω_+, where A and D are respectively $p \times p$ and $q \times q$ valued, and suppose *det* $D \not\equiv 0$ in Ω_+. Then, there exists a reproducing kernel Krein space $\mathcal{K}_+(\Theta)$of $p \times 1$ valued functions meromorphic in Ω_+ with reproducing kernel

$$\frac{J - \Theta(\lambda) J \Theta^*(w)}{\rho_w(\lambda)} \; .$$

We remark that elements of the space $\mathcal{K}_+(\Theta)$ do not generally satisfy (6.5) and (6.6) for α, β in Ω_+, and cannot be extended to Ω_-.

Finally, we remark that condition *det* $D \not\equiv 0$ will be satisfied in the following particular case: At some point δ of ∂, Θ has a non-tangential limit such that $\Theta(\delta_+) J \Theta^*(\delta_+) = J$.

6.2. Converse Theorems.

In this subsection we study theorems which are converses of Theorem 6.1, i.e. theorems which give sufficient (and in certain cases, necessary and sufficient) conditions for a reproducing kernel Krein space to have a reproducing kernel of the form (6.2). In these theorems,the functions are supposed to have a non-tangential limit at some point of ∂. This hypothesis is not very strong, and in many cases of interest, much more will be true: the functions will be analytic at some point of ∂. For ease of future reference, we introduce the following definition.

DEFINITION 6.1. A function f will have property $(\mathcal{B})$ if there is a point δ of ∂ such that for every sequence (δ_n) which approaches δ non-tangentially, $f(\delta_n)$ approaches a limit independent of the sequence (we will denote the limit by $f(\delta)$). In particular, this means that f has non-tangential limits in Ω_+ and in Ω_- which coincide.

A space $\mathcal{K}$ will have property $\mathcal{B}$ at a point δ of ∂ if all its elements have property $(\mathcal{B})$ at the point δ.

The next two theorems are no real surprise and their proofs follow the Hilbert space case.

THEOREM 6.3. Let $\mathcal{K}$ be a reproducing kernel Krein space of $m \times 1$ valued functions which are analytic in some open set U which is symmetric about $\mathbb{R}$. Suppose that the elements of $\mathcal{K}$ satisfy $(\mathcal{B})$ at some point δ of $\mathbb{R} \cap \overline{U}$ ($\overline{U}$ denotes the closure of U) and that $\mathcal{K}$ is resolvent invariant. Suppose further that there is a signature matrix J such that, for any α, β in U and any F, G in $\mathcal{K}$,

$$[R_\alpha F, G] - [F, R_\beta G] - (\alpha - \overline{\beta})[R_\alpha F, R_\beta G] = 2\pi i G^*(\beta) J F(\alpha) \ .$$

Then there exists a $m \times m$ valued function Θ, analytic in U, such that $\Theta J \Theta^{\#} = J$, and such that the reproducing kernel of $\mathcal{K}$ is of the form

$$\frac{J - \Theta(\lambda) J \Theta^*(w)}{-2\pi i(\lambda - \overline{w})} \ .$$

Moreover, Θ is J-unitary at the point δ.

THEOREM 6.4. Let $\mathcal{K}$ be a reproducing kernel Krein space of $m \times 1$ valued functions which are analytic in some open subset U of $\mathbb{C}$ which is symmetric about $\mathbb{T}$. Suppose that the elements of $\mathcal{K}$ satisfy $(\mathcal{B})$ at some point δ of $\mathbb{T}$ and that $\mathcal{K}$ is resolvent invariant. Suppose further that there is a signature matrix J such that, for any α, β in U and any F, G in $\mathcal{K}$,

$$[F, G] + \alpha[R_\alpha F, G] + \overline{\beta}[F, R_\beta G] - (1 - \alpha\overline{\beta})[R_\alpha F, R_\beta G] = G^*(\beta) J F(\alpha) \ .$$

Then there exists a $m \times m$ valued function Θ analytic in U such that $\Theta J \Theta^{\#} = J$, and such that the reproducing kernel of $\mathcal{K}$ is of the form

$$\frac{J - \Theta(\lambda) J \Theta^*(w)}{1 - \lambda\overline{w}} \ .$$

Moreover, Θ is J-unitary at the point δ.

Two differences should be noted with the Hilbert space case. First, in the Hilbert space case, the theorems are true for any symmetric open set U, without the hypothesis $(\mathcal{B})$ on the elements of $\mathcal{K}$; in particular, U may be at some positive distance of ∂. Secondly, if $\mathcal{K}$ is a reproducing kernel Hilbert space, then, the stated conditions are necessary as well as sufficient for the space $\mathcal{K}$ to have such a reproducing kernel.

To prove these theorems, we follow the strategy of [AD1]. We first prove a theorem which encompasses both theorems; the proof of the two special cases is then just the same as in the Hilbert space case and will be omitted.

THEOREM 6.5. Let $\mathcal{K}$ be a reproducing kernel Krein space of $m \times 1$ valued functions which are analytic in some open symmetric subset U of $\mathbb{C}$ and which satisfy $(\mathcal{B})$ at some point δ of ∂ ($\delta \in \overline{U}$). Then, the reproducing kernel of $\mathcal{K}$ is of the form

$$K_w(\lambda) = \frac{J - \Theta(\lambda)J\Theta^*(w)}{\rho_w(\lambda)}$$

where w and λ are in U and where Θ is $m \times m$ valued, analytic in U, and satisfy $J = \Theta J \Theta^{\#}$ for some signature matrix J if and only if the numerator

$$N_w(\lambda) = \rho_w(\lambda)K_w(\lambda)$$

of the reproducing kernel of $\mathcal{K}$ satisfies the identity

$$-N_w(\lambda) + N_w(\mu) + N_{\mu'}(\lambda) = N_{\mu'}(\lambda)JN_w(\mu) \tag{6.9}$$

for any w, λ, μ in U, where $\mu' = \overline{\mu}$ if $\partial = \mathbb{R}$ and $\overline{\mu}^{-1}$ if $\partial = \mathbb{T}$. Moreover, the function Θ is J-unitary at the point δ.

PROOF. One direction is clear; (6.9) holds for any kernel of the form (6.1) independent of the $(\mathcal{B})$ assumption. Conversely, let $\{\delta_r\}$ be a sequence of points of $U \cap \Omega_+$ such that δ_r converges non-tangentially to δ. Putting $\mu = \delta_r$ in (6.9) we get (since $N_w^*(\lambda) = N_\lambda(w)$)

$$-N_w(\lambda) + N_w(\delta_r) + N_\lambda^*(\delta_r') = N_\lambda^*(\delta_r')JN_w(\delta_r) \ .$$

Letting r go to infinity, we get

$$-N_w(\lambda) + N_w(\delta) + N_\lambda^*(\delta) = N_\lambda^*(\delta)JN_w(\delta)$$

i.e.,

$$J - N_w(\lambda) = (N_\lambda^*(\delta) - J)J(N_w(\delta) - J) \ . \tag{6.10}$$

To conclude, it suffices to set $\Theta(w) = N_w^*(\delta) - J$. We now check that Θ is J-unitary at the point δ, and is analytic in U. We first check that *det* Θ is non-vanishing in U. Indeed from

$$\rho_w(\lambda)K_w(\lambda) = J - \Theta(\lambda)J\Theta^*(w)$$

putting $\lambda = w'$, we get (since $K_w(w')$ is well defined)

$$J = \Theta(w')J\Theta^*(w) \tag{6.11}$$

therefore $det\ \Theta \neq 0$ in U. From (6.10) we may thus write, for some fixed w in U

$$\Theta(\lambda) = (J - N_w(\lambda))J\Theta^*(w)^{-1} \tag{6.12}$$

from which it is plain that $\lambda \to \Theta(\lambda)$ is analytic in U and has a non-tangential limit at δ, since $\lambda \to N_w(\lambda)$ has such properties. Let $\Theta(\delta)$ be the limit of $\Theta(\lambda)$ at δ. (6.11) readily implies that $\Theta(\delta)$ is J-unitary.

This concludes the proof of the theorem. As mentioned earlier, Theorems 6.3 and 6.4 are easily obtained from Theorem 6.5 just as in the Hilbert space case which is treated in Section 2 of [AD1]. Their proofs will therefore be omitted.

We conclude this section with a lemma on the behavior of the function Θ at the point δ.

LEMMA 6.1. With the notations and hypothesis of Theorem 6.5

a) The function $\lambda \to \frac{J-\Theta(\lambda)J\Theta^*(\delta)}{\rho_\delta(\lambda)}c$ belongs to $\mathcal{K}(\Theta)$ for all c in $\mathbb{C}_{m\times 1}$, where λ belongs to U.

b) For any sequence (δ_n) in U converging non-tangentially to δ the limit of

$$\frac{J - \Theta(\delta_n)J\Theta^*(\delta)}{\rho_\delta(\delta_n)}$$

as $n \uparrow \infty$, exists and is independent of the choice of the sequence (δ_n).

c) The sequence of matrices $\frac{J-\Theta(\delta_n)J\Theta^*(\delta_n)}{\rho_{\delta_n}(\delta_n)}$ is bounded in norm.

PROOF. From the existence of $\lim[f, K_{\delta_n}c]_{\mathcal{K}(\Theta)}$, we conclude that a subsequence of $K_{\delta_n}c$ converges weakly in $\mathcal{K}(\Theta)$ by an application of the Banach Steinhaus theorem (see the proof of part c) below for more details on this kind of argument in Krein spaces). Let g be the weak limit

$$\begin{aligned} d^*g(w) &= [g, K_w d] \\ &= \lim[K_{\delta_n}c, K_w d] \end{aligned}$$

from which we see that

$$g(w) = \frac{J - \Theta(w)J\Theta^*(\delta)}{\rho_\delta(w)}c\ ,$$

g belongs to $\mathcal{K}$.This proves a) and therefore also b) since every $g \in \mathcal{K}$ satisfies $(\mathcal{B})$ at δ.

Now to prove c), let σ designate a signature operator from $\mathcal{K}$ into $\mathcal{K}$, which transforms $\mathcal{K}$ into a Hilbert space when $[f, g]$ is replaced by $[f, \sigma g] = \langle f, g \rangle$. An application of the Banach Steinhaus theorem to the operators

$$L_n(f) = [f, K_{\delta_n}d]_{\mathcal{K}(\Theta)}$$

shows that $\sup_n \|L_n\| < \infty$, where $\|L_n\|$ designates the norm of the operator L_n in the Hilbert space $(\mathcal{K}, < \ , \ >)$. But

$$\|L_n\|^2 = < \sigma K_{\delta_n} d, \sigma K_{\delta_n} d >$$

and thus we get c) since

$$\begin{aligned} |d^* K_{\delta_n}(\delta_n) d| &= | < \sigma K_{\delta_n} d, K_{\delta_n} d > | \\ &\leq \|\sigma K_{\delta_n} d\| \|K_{\delta_n} d\| = \|L_n\|^2 \ . \end{aligned}$$

We next show that if Θ is an $m \times m$ valued function such that a space $\mathcal{K}(\Theta)$ exists and is a Hilbert space and such that Θ has a non-tangential limit at a point δ and satisfies condition c) of Lemma 6.1, then, the elements of $\mathcal{K}(\Theta)$ have property $(\mathcal{B})$ at the point δ.

Indeed, let $\Omega(\Theta)$ be the set of points in $\mathbb{C} \backslash \partial$ where Θ is analytic and invertible. Since Θ has a non-tangential limit at δ, we see that the elements which are finite linear combinations of $K_w d$, d in $\mathbb{C}_{m \times 1}$ and w in $\Omega(\Theta)$ satisfy hypothesis $(\mathcal{B})$ at δ, i.e., for f such a finite linear combination,

$$\lim [f, K_{\delta_n} d]$$

exists and is independent of (δ_n).

Since $\mathcal{K}(\Theta)$ is a Hilbert space and since the function Θ satisfies the condition c) of Lemma 6.1, we have

$$\sup_n |[f, K_{\delta_n} d]|^2 \leq \sup_n [f, f] d^* K_{\delta_n}(\delta_n) d < \infty \ .$$

Since the set of such f is dense in $\mathcal{K}(\Theta)$ an application of Theorem 6, p.60 of [DS] then finishes the proof: The limit $[f, K_{\delta_n} d]$ exists for all f in $\mathcal{K}(\Theta)$, and the operator

$$f \mapsto (\lim \ f(\delta_n))$$

is bounded for any sequence (δ_n) converging non-tangentially to δ.

We were unable to extend the above discussion to the Krein space framework.

REMARK. One could think of more general kernels of the form

$$K_w(\lambda) = \frac{J_1 - \Theta(\lambda) J_2 \Theta^*(w)}{\rho_w(\lambda)}$$

where J_1 and J_2 are different $m \times m$ signature matrices. Under $(\mathcal{B})$ this is not a more general situation. Indeed, let δ be a point in ∂ at which limits exist. Since

$$\Theta(\lambda) J_2 \Theta^*(\lambda') = J_1$$

we get

$$\Theta(\delta) J_2 \Theta^*(\delta) = J_1 \ .$$

Therefore $\Theta(\delta)$ is invertible and

$$K_w(\lambda) = \frac{J_1 - \Theta(\lambda)\Theta(\delta)^{-1}J_1[\Theta(w)\Theta(\delta)^{-1}]^*}{\rho_w(\lambda)} .$$

6.3. Factorizations and Inclusions.

In this subsection we consider the links between factorizations of J-unitary functions of bounded type and inclusions of $\mathcal{K}(\Theta)$ spaces. In the Hilbert space case, we recall that factorization and contractive inclusion are equivalent to each other; if Θ is J-inner and can be written as $\Theta_1\Theta_2$, with Θ_i being J-inner, $i = 1, 2$, then $\mathcal{K}(\Theta_1)$ is contractively included in $\mathcal{K}(\Theta)$, and conversely (see e.g. [AD1], Theorems 5.1 and 5.3). The question of deciding when this inclusion is isometric is much more involved and is related to the notion of overlapping subspace. In the Krein space, there is a priori no relationship between factorizations and inclusions, as the example $I_m = \Theta\Theta^{-1}$ with Θ a J-unitary function of bounded type exhibits. We are thus led to the notions of good factorizations and of regular factorizations which are linked to the related notions introduced in Section 2 about the decompositions of kernels and focus on the Pontryagin space case.

DEFINITION 6.2. $\Theta = \Theta_1\Theta_2$ is a good factorization of the J-unitary function Θ of bounded type into the product of two such functions if $\mathcal{K}(\Theta_1)$ is included in $\mathcal{K}(\Theta)$.

DEFINITION 6.3. $\Theta = \Theta_1\Theta_2$ is a regular factorization of the J-unitary function Θ of bounded type in Ω_+ into the product of two such functions if $\mathcal{K}(\Theta_1)$ is orthocomplemented in $\mathcal{K}(\Theta)$, i.e. (see [Bo], Theorem 3.4, p.104), if $\mathcal{K}(\Theta_1)$ is isometrically included in $\mathcal{K}(\Theta)$ and closed in $\mathcal{K}(\Theta)$.

We recall that $\mathcal{K}(\Theta_1)$ is isometrically included in $\mathcal{K}(\Theta)$ if it is included in $\mathcal{K}(\Theta)$ and if, moreover, for any f and g in $\mathcal{K}(\Theta_1)$, we have

$$[f, g]_{\mathcal{K}(\Theta_1)} = [f, g]_{\mathcal{K}(\Theta)} .$$

We also recall that from Lemma 2.1, $\mathcal{K}(\Theta_1)$ is automatically closed in $\mathcal{K}(\Theta)$, if it is isometrically included in $\mathcal{K}(\Theta)$.

The notion of regular factorization is useful in a number of cases, in particular in the finite dimensional case; our first result related to regular factorizations is the following theorem which is a generalization to the Pontryagin framework of a result of the theory of Hilbert spaces of analytic functions (see e.g. Theorem 5.2 of [AD1]).

THEOREM 6.6. Let $\Theta = \Theta_1\Theta_2$ be a regular factorization of the J-unitary function Θ and suppose that $\mathcal{K}(\Theta)$ is a Pontryagin space. Then,

$$\mathcal{K}(\Theta) = \mathcal{K}(\Theta_1)[\dot{+}]\Theta_1\mathcal{K}(\Theta_2) .$$

PROOF. We first outline the proof. $\Theta = \Theta_1\Theta_2$ is a regular factorization, and so $\mathcal{K}(\Theta_1)$ and $\mathcal{K}(\Theta_2)$ exists since Θ_1 and Θ_2 are J-unitary of bounded type in Ω_+. Moreover, $\mathcal{K}(\Theta_1)$ is orthocomplemented, i.e. we may write

$$\mathcal{K}(\Theta) = \mathcal{K}(\Theta_1)[\dot{+}]\mathcal{K}(\Theta_1)^{[\perp]} \ . \tag{6.13}$$

The imbedding operator i is continuous since both its domain and range are orthocomplemented (see [Bo], Theorem 3.8, p.126, or Theorem 3.10, p.127). Thus its adjoint i^+ is well defined and continuous. We verify that the decomposition of an element f in $\mathcal{K}(\Theta)$ along (6.13) is given by

$$f = ii^+f + (f - ii^+f)$$

and then we identify $\{f - ii^+f, f \in \mathcal{K}(\Theta)\}$ with $\Theta_1\mathcal{K}(\Theta_2)$.

We denote by K (resp. K^i)the reproducing kernel of $\mathcal{K}(\Theta)$ (resp. of $\mathcal{K}(\Theta_i)$), and let $\Omega(\Theta), \Omega(\Theta_i)$ have the same meaning as in Theorem 6.1. For c in $\mathbb{C}_{m\times 1}$ and for w in $\Omega(\Theta) \cap \Omega(\Theta_1)$, we have

$$ii^+K_wc = K_w^1c \ .$$

From this equality we see that for f in $\mathcal{K}(\Theta)$, $f - ii^+f$ belongs to $\mathcal{K}(\Theta_1)^{[\perp]}$. Indeed,

$$\begin{aligned}[f - ii^+f, K_w^1c] &= [f - ii^+f, ii^+K_wc] \\ &= [ii^+f - ii^+ii^+f, K_wc] = 0\end{aligned}$$

since $ii^+ii^+ = ii^+$. On the other hand, ii^+f belongs to $\mathcal{K}(\Theta_1)$ for any f in $\mathcal{K}(\Theta)$ and thus we have the inclusions

$$\begin{aligned}\{ii^+f; f \in \mathcal{K}(\Theta)\} &\subset \mathcal{K}(\Theta_1) \\ \{f - ii^+f; f \in \mathcal{K}(\Theta)\} &\subset \mathcal{K}(\Theta_1)^{[\perp]} \ .\end{aligned}$$

These inclusions are in fact equalities as is seen from

$$\begin{aligned}\mathcal{K}(\Theta) &= \{ii^+f; f \in \mathcal{K}(\Theta)\}[\dot{+}]\{f - ii^+f; f \in \mathcal{K}(\Theta)\} \\ &\subseteq \mathcal{K}(\Theta_1)[\dot{+}]\mathcal{K}(\Theta_1)^{[\perp]} = \mathcal{K}(\Theta) \ .\end{aligned}$$

$\mathcal{K}(\Theta_1)^{[\perp]}$ is easily checked to be a RKKS with reproducing kernel $K - K^1$. But we have

$$\Theta_1(\lambda)\left\{\frac{J - \Theta_2(\lambda)J\Theta_2^*(w)}{\rho_w(\lambda)}\right\}\Theta_1^*(w) = K_w(\lambda) - K_w^1(\lambda) \ ,$$

and thus, the space $\Theta_1\mathcal{K}(\Theta_2)$ with hermitian form

$$[\Theta_1 f, \Theta_1 g] = [f, g]_{\mathcal{K}(\Theta_2)}$$

for f and g in $\mathcal{K}(\Theta_2)$ has also reproducing kernel $K - K^1$, and so

$$\Theta_1\mathcal{K}(\Theta_2) = \mathcal{K}(\Theta_1)^{[\perp]} \ .$$

The next theorem relates the ranks of $\mathcal{K}(\Theta), \mathcal{K}(\Theta_1)$ and $\mathcal{K}(\Theta_2)$.

THEOREM 6.7. Let $\Theta = \Theta_1\Theta_2$ be a regular factorization of the J-unitary function Θ and suppose that $\mathcal{K}(\Theta)$ is a Pontryagin space of rank k and $\mathcal{K}(\Theta_1)$ is a Pontryagin space of rank k_1. Then, the function Θ_2 is in the class $\mathcal{A}_J^{k-k_1}$.

We first recall that $\mathcal{A}_J^k$ is defined to be the class of functions meromorphic in $\mathbb{C}\setminus\partial$ and such that the kernel $\frac{J-\Theta(\lambda)J\Theta^*(w)}{\rho_w(\lambda)}$, where λ and w are points of analyticity of Θ, has k negative squares.

PROOF. Let i be the inclusion from $\mathcal{K}(\Theta_1)$ into $\mathcal{K}(\Theta)$ and let K^1 (resp. K)be the reproducing kernel of $\mathcal{K}(\Theta_1)$ (resp. $\mathcal{K}(\Theta)$). As already remarked in the proof of Theorem 6.6, i is continuous. We can thus apply Lemma 1.1. The hermitian form

$$(u,v) \mapsto [u,v]_{\mathcal{K}(\Theta)} - [i^+u, i^+v]_{\mathcal{K}(\Theta_1)}$$

has $k-k_1$ negative squares.

We apply this fact to $u = v = \Sigma_1^r K_{w_j}c_j$, where the w_j are in $\Omega(\Theta)\cap\Omega(\Theta_1)$ and the c_j are in $\mathbb{C}_{m\times 1}$ ($\Omega(\Theta)$ and $\Omega(\Theta_1)$ have the same meaning as in Theorem 6.1). We get the kernel

$$\frac{\Theta_1(\lambda)J\Theta_1^*(w) - \Theta(\lambda)J\Theta^*(w)}{\rho_w(\lambda)}$$

where λ and w are in $\Omega(\Theta)\cap\Omega(\Theta_1)$ has $k-k_1$ negative squares, from which the claim follows.

6.4. $\mathcal{K}(\Theta)$ Spaces: The Finite Dimensional Case.

In this section we focus on the important case of finite dimensional $\mathcal{K}(\Theta)$ spaces. We first show that for a function Θ of bounded type in Ω_+ and J-unitarya.e. on ∂ the $\mathcal{K}(\Theta)$ space is finite dimensional if and only if Θ is rational. We then study the factorization of such rational functions. Finally, we give some formulas for inner products of elements in a finite dimensional $\mathcal{K}(\Theta)$ space.

We first need some terminology and a preliminary result. Let $\mathcal{M}$ be a v-dimensional subspace of a Pontryagin space with inner product $[\ ,\ ]$, and let $p_1,\ldots,p_v$ be a basis of $\mathcal{M}$. The Pick matrix associated with $\mathcal{M}$ is the $v\times v$ hermitian matrix with ij entry $[p_j,p_i]$. It is easy to show that the Pick matrix is invertible if and only if $\mathcal{M}$ is non-degenerate. Furthermore, the identity

$$[\sum \alpha_j p_j, \sum \beta_i p_i] = <\mathbb{P}\underline{\underline{\alpha}}, \underline{\underline{\beta}}>_{\mathbb{C}_{v\times 1}}$$

with

$$\underline{\underline{\alpha}} = \begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_v\end{bmatrix}\quad,\quad \underline{\underline{\beta}} = \begin{bmatrix}\beta_1\\ \vdots\\ \beta_v\end{bmatrix}\quad,$$

makes it clear that the rank of $\mathcal{M}$ is equal to the number of negative eigenvalues of $\mathbb{P}$. Moreover when $\mathcal{M}$ is reproducing kernel space, its reproducing kernel is closely related to $\mathbb{P}$.

LEMMA 6.2. Let $\mathcal{M}$ be a v-dimensional reproducing kernel Pontryagin space of $m \times 1$ valued functions defined in some subset Z of $\mathbb{C}$, with basis $p_1(\lambda), \ldots, p_v(\lambda)$. Then the reproducing kernel of $\mathcal{M}$, $K_w(\lambda)$, is given by the formula

$$K_w(\lambda) = \sum_{i,j=1}^{v} p_i(\lambda)(\mathbb{P}^{-1})_{ij} p_j^*(w) \tag{6.14}$$

where λ and w are in Z.

PROOF. Let w be in Z and a vector c in $\mathbb{C}_{m\times 1}$. The function $\lambda \rightarrow K_w(\lambda)c$ belongs to $\mathcal{M}$ and thus there exist coefficients $\alpha_1(w,c), \ldots, \alpha_v(w,c)$ such that

$$K_w(\lambda)c = \sum_{1}^{v} p_i(\lambda)\alpha_i(w,c) \ .$$

It is not difficult to see that $\alpha_i(w,c) = \alpha_i(w)c$ for some $1 \times m$ vector $\alpha_i(w)$. Thus

$$K_w(\lambda)c = \sum_{1}^{v} p_i(\lambda)\alpha_i(w)c \ .$$

From $c^* p_j(w) = [p_j, K_w c]$, we get

$$c^* p_j(w) = \sum_{i=1}^{v} (\alpha_i(w)c)^* [p_j, p_i]$$

i.e.

$$p_j(w) = \sum_{i=1}^{v} [p_j, p_i]\alpha_i^*(w)$$

from which we get

$$\alpha_i^*(w) = \sum_{s=1}^{v} p_s(w)(\mathbb{P}^{-1})_{si}$$

and hence the result. □

When the kernel is of the form $\frac{J-\Theta(\lambda)J\Theta^*(w)}{\rho_w(\lambda)}$, we can thus set(so that Θ satisfies normalization $\Theta(w_0) = I_m$)

$$\Theta(\lambda) = I_m - \rho_{w_0}(\lambda)\left\{\sum_{i,j=1}^{v} p_i(\lambda)(\mathbb{P}^{-1})_{ij} p_j^*(w_0)\right\} J \tag{6.15}$$

where w_0 is any point on ∂ where the elements of $\mathcal{K}(\Theta)$ are analytic; in the Hilbert space context, (6.15) appears in [AD2] (formula (10.2)). Of course Θ depends on w_0 but recall that all such Θ differ by multiplication on the right by some J-unitary constant.

We now turn to the characterization of finite dimensional $\mathcal{K}(\Theta)$ spaces.

THEOREM 6.8. Let Θ be a J-unitary function of bounded type in Ω_+. Then, $\mathcal{K}(\Theta)$ is finite dimensional if and only if Θ is rational.

PROOF. One direction is clear; indeed, if $\mathcal{K}(\Theta)$ is finite dimensional, it is then *resolvent invariant* by Theorem 6.1 and thus, by Theorem 2.2, the elements of $\mathcal{K}(\Theta)$ are rational, from which Θ is easily checked to be rational by formula (6.15) of Lemma 6.2. The proof of the converse is related to Theorem 6.1 and in fact follows the proof of that theorem. Indeed, we first suppose that $J = I_m$. Θ is then of the form $\Theta_1^{-1}\Theta_2$ where Θ_1 and Θ_2 are rational $m \times m$ valued inner functions. Using Theorem 4.1, we see that

$$\mathcal{K}(\Theta) = \{\underline{p}\Theta_1^* u - \Theta \underline{p}\Theta_2^* u; \quad u \in H_m^2\}$$

and to conclude that $\mathcal{K}(\Theta)$ is finite dimensional, it suffices to remark that the operators $\underline{p}\Theta_1^*$ and $\underline{p}\Theta_2^*$ restricted to H_m^2 have finite dimensional range for rational inner Θ_1 and Θ_2. A similar argument takes care of the case $J = -I_m$, and, without loss of generality, we may suppose

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

but this case is easily reduced to the case $J = I_m$ as in Theorem 6.1.

We now present an important example of a finite dimensional $\mathcal{K}(\Theta)$ space.

THEOREM 6.9. Let J be an $m \times m$ signature matrix and let $c_1, \ldots, c_p$ (resp. $w_1, \ldots, w_p$) be p elements of $\mathbb{C}_{m\times 1}$(resp. Ω_+), and suppose that the span of $\frac{c_1}{\rho - w_1}, \ldots, \frac{c_p}{\rho - w_p}$ is non-degenerate in H_J^2. Then this span is a reproducing kernel Pontryagin space with a reproducing kernel of the form

$$\frac{J - \Theta(\lambda) J \Theta^*(w)}{\rho_w(\lambda)}$$

where Θ is the rational J-unitary matrix valued function given by formula (6.15). The choice of Θ is unique up to multiplication by a J-unitary constant matrix on the right. Moreover, the rank of $\mathcal{K}(\Theta)$ is equal to the number of negative eigenvalues of the Pick matrix $\mathbb{P}$ with ij coefficient

$$\frac{c_i^* J c_j}{\rho_{w_j}(w_i)}.$$

We mention that the function Θ appearing in Theorem 6.9 satisfies

$$c_j^* \Theta(w_j) = 0 \quad j = 1, \ldots, p \tag{6.16}$$

and that Theorem 6.9 is linked to various interpolation problems (see [D1]).

To prove Theorem 6.9, we first need a lemma:

LEMMA 6.3. Let F, G be two elements in H_J^2, analytic at α and β respectively, and suppose that α and β belong to $\mathbb{C} \setminus \partial$. Then $R_\alpha F$ and $R_\beta G$ belong to H_J^2 and equalities (6.5) (resp. (6.6)) hold when $\partial = \mathbb{R}$ (resp. $\partial = \mathbb{T}$).

The proof of this lemma is elementary and omitted.

PROOF OF THEOREM 6.9. Let us denote by $\mathcal{M}$ the linear span of the functions $\frac{c_1}{\rho_{w_1}}, \ldots, \frac{c_p}{\rho_{w_p}}$. Since the matrix $\mathbb{P}$ is invertible, $\mathcal{M}$ has no isotropic part and thus is a reproducing kernel Pontryagin space. Moreover, $\mathcal{M}$ is resolvent invariant, and, by the preceding lemma, equality (6.5) (resp. (6.6)) is satisfied for F, G in $\mathcal{M}$ and α, β in $\mathbb{C} \setminus \{\partial \cup \{w_1, \ldots, w_p\}\}$ when $\partial = \mathbb{R}$ (resp. $\partial = \mathbb{T}$). The reproducing kernel of $\mathcal{M}$ is then of the asserted form by Theorem 6.3 (resp. Theorem 6.4) when $\partial = \mathbb{R}$ (resp. $\partial = \mathbb{T}$). The J-unitary function Θ may be chosen as in formula (6.15) from the discussion following Lemma 6.2. Finally, the assertion on the rank of the Pontryagin space $\mathcal{M}$ follows from the remarks on the properties of the Pick matrix made at the beginning of the subsection. □

The spaces introduced in Theorem 6.9 allow us to point out an interesting difference between J-inner and J-unitary functions. Indeed, when the space $\mathcal{M}$ defined in the proof of Theorem 6.9 is positive, the function Θ is J-inner Blaschke product, and the structure of Θ is then well understood: One can write Θ as a product of J-inner rational functions θ_j

$$\Theta = \prod_1^N \theta_j$$

in such a way that with $\Theta_j = \theta_1 \ldots \theta_j$, dim $\mathcal{K}(\Theta_j) = j$ and

$$\mathcal{K}(\Theta_1) \subset \mathcal{K}(\Theta_2) \subset \ldots \subset \mathcal{K}(\Theta_{i+1}) \subset \ldots \subset \mathcal{K}(\Theta)$$

the inclusions being moreover isometric.

Such a result is false in general for a rational J-unitary function as is shown by the next example. Let $\partial = \mathbb{R}$, w_1, w_2 two distinct points of $\mathbb{C}_-$, and c_1, c_2 two vectors of $\mathbb{C}_{m \times 1}$ such that

$$c_1^* J c_2 \neq 0$$

and

$$c_1^* J c_1 = c_2^* J c_2 = 0$$

and consider $\mathcal{M}$ to be the two dimensional subspace of H_J^2 spanned by $\frac{c_1}{\lambda - w_1}$ and $\frac{c_2}{\lambda - w_2}$. $\mathcal{M}$ is non-degenerate in H_J^2, and by Theorem 6.9, $\mathcal{M}$ is a reproducing kernel Pontryagin space with reproducing kernel of the form $\frac{J - \Theta(\lambda) J \Theta^*(w)}{-2\pi i(\lambda - \overline{w})}$; Θ is rational and we show that it cannot be factorized as

$$\Theta = \Theta_1 \Theta_2 \ ,$$

Θ_1 and Θ_2 being J-unitary and

$$0 \subsetneqq \mathcal{K}(\Theta_1) \subsetneqq \mathcal{K}(\Theta) \ .$$

Indeed, suppose such a factorization exists; then, $\mathcal{K}(\Theta_1)$ is of dimension 1 and thus, a basis of $\mathcal{K}(\Theta_1)$ will be of the form

$$v = \alpha_1 \frac{c_1}{\lambda - w_1} + \alpha_2 \frac{c_2}{\lambda - w_2}$$

for some complex numbers α_1 and α_2. Since $\mathcal{K}(\Theta_1)$ is resolvent invariant, $R_\alpha^n v$ belongs to $\mathcal{K}(\Theta_1)$ for every integer $n \geq 1$ and every point α different from w_1 and w_2; we easily deduce that both $\frac{\alpha_1 c_1}{\lambda - w_1}$ and $\frac{\alpha_2 c_2}{\lambda - w_2}$ belong to $\mathcal{K}(\Theta)$ and hence that either $\alpha_1 = 0$ or $\alpha_2 = 0$, since dim $\mathcal{K}(\Theta) = 1$. Thus, $\mathcal{K}(\Theta_1)$ is spanned by either $\frac{\alpha_1 c_1}{\lambda - w_1}$ or $\frac{\alpha_2 c_2}{\lambda - w_2}$. Since both vectors are neutral in H_J^2, this is impossible.

The above discussion seems to warrant the following definition:

DEFINITION 6.4. A rational J-unitary function Θ will be termed an elementary section if $\mathcal{K}(\Theta)$ is finite dimensional and if it does not contain isometrically any reproducing kernel Pontryagin space of the form $\mathcal{K}(\Theta_1)$ of lower dimension. We will call dim $\mathcal{K}(\Theta)$ the degree of the elementary section.

In the case of a J-inner function Θ, the elementary sections are of degree 1, and, by an application of Theorem 3.2, it is seen that a basis for the corresponding space consists of

$$f(\lambda) = c$$

or

$$f(\lambda) = \frac{c}{\lambda - w}$$

where c is in $\mathbb{C}_{m \times 1}$ and w is in $\mathbb{C}$. Some computations show that when $\partial = \mathbb{R}$, Θ is of three possible forms:

a) Blaschke section ($\partial = \mathbb{R}$):

$$\Theta(\lambda) = I_m + \left(\frac{\lambda - w}{\lambda - \overline{w}} - 1 \right) u(u^* J u)^{-1} u^* J \tag{6.17}$$

where u is in $\mathbb{C}_{m \times 1}$ and $u^* J u > 0$ (resp. < 0) if $w \in \mathbb{C}_+$ (resp. $\mathbb{C}_-$).

b) Brune section ($\partial = \mathbb{R}$):

$$\Theta(\lambda) = I_m - \frac{i\alpha}{\lambda - a} u u^* J \tag{6.18}$$

where u is in $\mathbb{C}_{m \times 1}$, $u^* J u = 0$, and α is real and α is > 0.

c) Polynomial section ($\partial = \mathbb{R}$):

$$\Theta(\lambda) = I_m + i\alpha\lambda uu^* J \tag{6.19}$$

where α and u are as in b).

When $\partial = \mathbb{T}$, only two cases are to be considered, Blaschke and Brune sections, which are then of the form:

d) Blaschke section ($\partial = \mathbb{T}$)

$$\Theta(\lambda) = I_m + \left(\frac{\lambda - w}{1 - \lambda\overline{w}} - 1\right) u(u^* J u)^{-1} u^* J \tag{6.20}$$

where u is in $\mathbb{C}_{m\times 1}$ and $u^* J u > 0$ (resp. < 0) if $w \in \mathbb{D}$ (resp. $\mathbb{E}$).

e) Brune section ($\partial = \mathbb{T}$)

$$\Theta(\lambda) = I_m + \alpha\frac{\lambda + a}{\lambda - a} uu^* J \tag{6.21}$$

where $u^* J u = 0$, $a \in \mathbb{T}$, and α is strictly positive.

To carry out these computations, one takes advantage of formula (6.15) and it is necessary to compute the inner product $[f, f]$. This is done using formulas (6.5) and (6.6). Such and more general computations appear at the end of the subsection.

Moreover, any rational J-inner function is a finite product of elementary sections of degree 1. Such a statement is false for general J-unitary functions as was already demonstrated. The problem of factorizing J-unitary rational functions is of much interest. We first thought that a J-unitary rational function can always be factorized as a finite product of elementary sections of degree 1 or 2. However, the next example shows that there are elementary sections of degree larger than 2.

Let c be a $m \times 1$ vector such that $c^* J c = 0$, let a be a real number and let $\mathcal{M}$ be the linear span of $f_1 = \frac{c}{\lambda - a}$, $f_2 = \frac{c}{(\lambda - a)^2}$ and $f_3 = \frac{c}{(\lambda - a)^3}$. On $\mathcal{M}$ we define an inner product by

$$[f_3, f_1] = [f_2, f_2] = [f_1, f_3] = 1$$

$$[f_i, f_j] = 0 \text{ for all other choices of } i \text{ and } j \ .$$

$\mathcal{M}$ is clearly a resolvent invariant reproducing kernel Pontryagin space of functions analytic in $\mathbb{C} \setminus [a]$. Its kernel is of the form

$$\frac{J - \Theta(\lambda) J \Theta^*(w)}{-2\pi i(\lambda - \overline{w})}$$

for some rational J-unitary function Θ. In order to prove this one has to check equality (6.5) in $\mathcal{M}$. This leads to easy but somewhat tedious computations. Similarly, it is not difficult to show that if $\mathcal{M}' \subset \mathcal{M}$ and is resolvent invariant, then $\mathcal{M}'$ is spanned by either f_1 or f_1 and f_2, and thus, by definition of [,], $\mathcal{M}$ cannot be a Pontryagin space, which shows that Θ is an elementary section of degree 3.

This example can easily be modified to exhibit elementary sections of arbitrary degree. Let $\mathcal{M}$ be the span of the $f_j = \frac{c}{(\lambda-a)^j}$, $j = 1,\ldots,N$, and define [,] on $\mathcal{M}$ by

$$\begin{aligned}[f_i, f_j] &= 1 \quad \text{if } i+j = N+1 \\ &= 0 \quad \text{otherwise} .\end{aligned}$$

Then $\mathcal{M}$ is a $\mathcal{K}(\Theta)$ space for some rational J-unitary Θ and degree(Θ) $= N$.

A general theory of J-unitary functions should answer the following questions: What are the J-unitary elementary sections, and is a J-unitary rational function a product of J-unitary elementary sections? We have only partial results in these directions, but they seem to be of use when considering Pontryagin space versions of the classical Schur algorithm.

THEOREM 6.10. Let Θ be the $m \times m$ J-unitary function appearing in Theorem 6.9. Then, Θ is a product of elementary sections of degree 1 or 2.

In order to prove this result, we first need to recall a number of definitions and results related to the theory of hermitian matrices, which are taken from Perlis' book [Pe]. They apparently originate in [Di].

DEFINITION.6.5. ([Pe], p.94): A submatrix of a square matrix A is called principal if it is obtained by deleting certain rows and the like numbered columns. The determinant of a principal submatrix is called a principal subdeterminant.

When the submatrix is obtained by keeping only the first i rows and columns, it will be denoted by $A_{[i]}$. We remark that for a positive $p \times p$ matrix

$$A = A_{[p]} \text{ invertible } \Rightarrow A_{[i]} \text{ invertible for all } i \leq p \ .$$

This fact is false for a mere hermitian matrix as the following example clearly shows.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} .$$

The following result is basic to the proof of Theorem 6.10.

THEOREM 6.11. ([Pe], p.103, Theorem 5.20): Let A be symmetric or hermitian. Then A has rank r if and only if A has a non-singular $r \times r$ principal submatrix S such that every principal submatrix of A containing S and one or two additional rows and columns is singular.

We now explain how this theorem will be used. Let A be hermitian, $p \times p$ valued and invertible, and suppose that for some i smaller than $(p-2)$, the principal submatrix $A_{[i]}$

is invertible. Should all the principal submatrices containing $A_{[i]}$ and one or two additional rows and columns be singular, we would have rank $A = i$ by Theorem 6.11, which contradicts the invertibility of A. This leads easily to the following conclusion:

THEOREM 6.12. Let A be $p \times p$ valued hermitian invertible, and suppose that for some $i \le p-2$, det $A_{[i]} \neq 0$. Then, there exists a principal submatrix containing $A_{[i]}$ and one or two additional rows and columns which is non-singular.

In order to prove Theorem 6.10, we first prove the following result which is a corollary of Theorem 6.11.

THEOREM 6.13. Let $\mathbb{P}$ be a $p \times p$ hermitian matrix which is invertible. Then there is a permutation matrix U such that, with

$$\mathbb{P}' = U\mathbb{P}U^* \ ,$$

we have

$$\det \ \mathbb{P}'_{[i_j]} \neq 0$$

for a sequence $i_1 < i_2 < \ldots < i_r = p$ such that $i_j - i_{j-1} \le 2$.

PROOF. We first remark that there exists a principal submatrix of dimension 1×1 or 2×2 which is invertible. Indeed, let $\mathbb{P} = (P_{ij})$; if $P_{ii} \neq 0$ for some index i, then the claim is proved. If on the contrary, $P_{ii} = 0$ for all i, then det $\mathbb{P} \neq 0$ forces $P_{1j} \neq 0$ for some j and the submatrix corresponds to the first and j-th columns and corresponding rows.

We denote by $\mathbb{P}_1$ the matrix obtained from $\mathbb{P}$ by interchanging the first and i-th columns and the first and i-th rows in the first case (some $P_{ii} \neq 0$) and the second and j-th column and second and j-th rows in the second case. We set $i_1 = 1$ in the first case and $i_1 = 2$ in the second case (all $P_{ii} = 0$). Clearly,

$$\det(\mathbb{P}_1)_{[i_1]} \neq 0$$

and there is a permutation matrix U_1 such that

$$\mathbb{P}_1 = U_1\mathbb{P}U_1^* \ .$$

If $i_1 = p$, then the proof is finished. If not, then since det $\mathbb{P}_1 \neq 0$, an application of Theorem 6.12 implies that there exists a principal submatrix containing $(\mathbb{P}_1)_{[i_1]}$ and one or two additional rows and columns (say the r-th, or the r-th and the s-th rows and columns) which is invertible. If not, we would have Rank $\mathbb{P}_1 = i_1$, which is impossible if $i_1 < p$.

Let $\mathbb{P}_2$ be the matrix obtained by interchanging the (i_1+1)-th and the r-th rows (and corresponding columns) or the (i_1+1)-th and the r-th rows and the (i_1+2)-th and the s-th rows (and corresponding columns), and let $i_2 = i_1+1$ or i_1+2, according as if you interchange one or two rows (and correspondingly one or two columns). Then, $[\mathbb{P}_2]_{[i_1]} = [\mathbb{P}_1]_{[i_1]}$ and thus is invertible; $[\mathbb{P}_2]_{[i_2]}$ is invertible by construction, and there is a permutation matrix U_2 such that

$$\mathbb{P}_2 = U_2\mathbb{P}_1U_2^* \ .$$

If $i_2 = p$, the proof is finished. If not, we reiterate the procedure. $\mathbb{P}_2$ is invertible and has principal submatrix $(\mathbb{P})_{[i_2]}$ which is invertible. If all the principal submatrices obtained fror $(\mathbb{P}_2)_{[i_2]}$ by adding one or two additional rows and columns were singular, the matrix $\mathbb{P}$ would itself be of rank i_2, by Theorem 6.11, and non-invertible. Thus there exists a matri $\mathbb{P}_3$ such that

1) $(\mathbb{P}_3)_{[i_2]} = (\mathbb{P}_2)_{[i_2]}$.
2) $(\mathbb{P}_3)_{[i_3]}$ is invertible with $i_3 = i_2 + 1$ or $i_2 + 2$.
3) $\mathbb{P}_3$ is obtained by interchanging one or two rows (and the corresponding columns) c $\mathbb{P}_2$ which are of index higher than i_2. Hence, there is a permutation U_3 such that

$$\mathbb{P}_3 = U_3\mathbb{P}_2U_3^* = U_3U_2U_1\mathbb{P}(U_3U_2U_1)^* \ .$$

It is then clear that, after a finite number of steps, one gets a permutation U such tha $U\mathbb{P}U^*$ satisfies the requirements of the theorem. □

We now turn to the proof of Theorem 6.10. Let $\mathcal{M}$ be the linear span of the $\frac{c_j}{\rho_{w_j}}$ $j = 1, \ldots, p$. Without loss of generality, we may suppose that they are indexed in such a wa; that

$$\det \ \mathbb{P}_{[i_j]} \neq 0$$

for a sequence $i_1 < i_2 < \ldots < i_r$ and $1 \leq |i_j - i_{j-1}| \leq 2$, where $\mathbb{P}$ is the Pick matri associated to $\mathcal{M}$.

Let $\mathcal{M}_j$ denote the linear span of $\frac{c_1}{\rho_{w_1}}, \ldots, \frac{c_{i_j}}{\rho_{w_j}}$. We have

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots \subset \mathcal{M}_{j-1} \subset \mathcal{M}_j \subset \ldots \subset \mathcal{M}_r = \mathcal{M}$$

and each $\mathcal{M}_j$ is a Pontryagin subspace of $\mathcal{M}$ which is resolvent invariant and for which (6.1) (resp. (6.2)) hold if $\partial = \mathbb{R}$ (resp. $\partial = \mathbb{T}$). From $\mathcal{M}_{j-1} \subset \mathcal{M}_j$ we get that there exists J-unitary functions Ψ_j and Θ_j such that $\Psi_0 = I_m$, $\Psi_r = \Theta$ and

1) $\mathcal{K}(\Psi_j) = \mathcal{M}_j \ , \ j = 1, \ldots, r$
2) $\Psi_{j+1} = \Psi_j \cdot \Theta_j$
3) $\dim \ \mathcal{K}(\Theta_j) = 1$ or 2.

The last assertion is a consequence of Theorem 6.7 which asserts that

$$\mathcal{K}(\Psi_{j+1}) = \mathcal{K}(\Psi_j)[\dot{+}]\Psi_j\mathcal{K}(\Theta_j) \ .$$

In particular Θ may be factorized as

$$\Theta = \Theta_r \cdot \Theta_{r-1}, \ldots, \Theta_2 \cdot \Theta_1 \ .$$

From the construction it is clear that the Θ_j are elementary sections. Indeed, if one of the Θ_j, say Θ_{j_0}, is such that

$$\dim \ \mathcal{K}(\Theta_{j_0}) = 2$$

and Θ_{j_0} admits a factorization, there would be a $\mathcal{K}(\Theta)$ space between $\mathcal{M}_{j_0}$ and $\mathcal{M}_{j_0+1}$, which is impossible by the construction.

In view of Theorem 6.10 it is of interest to characterize elementary sections of degree 1 or 2. We turn now to this problem and begin with a lemma.

LEMMA 6.3. Let $\mathcal{M} = \mathcal{K}(\Theta)$ be a space corresponding to an elementary section Θ and suppose that $\mathcal{M}$ is included isometrically in H_J^2. Then, if dim $\mathcal{M} = 1$, $\mathcal{M}$ is spanned by $\{c\}$ or $\{\frac{c}{\rho_w}\}$, where c is in $\mathbb{C}_{m\times 1}$ and w in Ω_+, and if dim $\mathcal{M} = 2$, $\mathcal{M}$ is spanned by $\{\frac{c_1}{\rho_{w_1}}, \frac{c_2}{\rho_{w_2}}\}$, or $\{c_1, c_2\}$, or $\{c_1, \frac{c_2}{\rho_{w_2}}\}$, where w_1, w_2 are in Ω_+ and c_1, c_2 are subject to $c_1^* J c_1 = c_2^* J c_2 = 0$, $c_1^* J c_2 \neq 0$, or by $\{\frac{c_1}{\rho_{w_1}}, \frac{c_1}{\rho_{w_1}^2} + \frac{c_2}{\rho_{w_1}}\}$ or $\{c_1, \lambda c_1 + c_2\}$ where c_1, c_2 are now subject to $c_1^* J c_1 = 0$, $c_1^* J c_2 \neq 0$.

We only outline the proof of Lemma 6.3. By Theorem 2.2, $\mathcal{M}$ has one of the following forms: If dim $\mathcal{M} = 1$ then either $\mathcal{M}$ =span$\{c\}$ or $\mathcal{M}$ =span$\{\frac{c}{\lambda - w}$, where w is in $\mathbb{C}$, c in $\mathbb{C}_{m\times 1}$,and if dim $\mathcal{M} = 2$ then $\mathcal{M}$ has one of the following forms:

$$\text{span}\{\tfrac{c_1}{\lambda - w_1}, \tfrac{c_2}{\lambda - w_2}, \quad \text{span}\{\tfrac{c}{\lambda - w}, \tfrac{c}{(\lambda - w)^2} + \tfrac{d}{\lambda - w}\}$$
$$\text{span}\{c_1, \lambda c_1 + c_2\}, \ \text{span}\{c, \tfrac{c_1}{\lambda - w_2}\}$$

and one has to check, conversely, which of these cases, and on which conditions on c, c_1, c_2, w, w_1, w_2, is $\mathcal{M}$ a $\mathcal{K}(\Theta)$ space, and where moreover Θ is an elementary section.

We will compute the matrix Θ where $\mathcal{M}$ is of the special form $\mathcal{M} = \text{span}\{\frac{c_1}{\rho_{w_1}}, \frac{c_2}{\rho_{w_2}}\}$ with w_1, w_2 in Ω_+ and $c_1^* J c_1 = c_2^* J c_2 = 0$, $c_1^* J c_2 \neq 0$. (The assumption $\mathcal{M} \subset H_J^2$ permits us to evaluate the inner products easily.)

The associated Pick matrix P is then equal to

$$\mathbb{P} = \begin{pmatrix} 0 & \frac{c_1^* J c_2}{\rho_{w_2}(w_1)} \\ \frac{c_2^* J c_1}{\rho_{w_1}(w_2)} & 0 \end{pmatrix}$$

and thus, using Lemma 6.2 and formula (6.15), we can choose

$$\Theta(\lambda) = I_m - \rho_{w_0}(\lambda)\left\{\frac{c_1 c_2^*}{\rho_{w_1}(\lambda)\rho_{w_0}(w_2)}\ \frac{\rho_{w_1}(w_2)}{c_2^* J c_1} + \frac{c_2 c_1^*}{\rho_{w_2}(\lambda)\rho_{w_0}(w_1)}\ \frac{\rho_{w_2}(w_1)}{c_1^* J c_2}\right\} J\ .$$

Let us focus on the special case $m = 2$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We can then suppose that $c_1 = \rho_i \begin{pmatrix} 1 \\ k_i \end{pmatrix}$ with $|k_i| = 1$, $i = 1, 2$, and we get to

$$\Theta(\lambda) = I_2 - \rho_{w_0}(\lambda)\times$$
$$\left\{\frac{\begin{pmatrix} 1 \\ k_1 \end{pmatrix}^{(1,\overline{k}_2)}}{1 - k_1\overline{k}_2}\ \frac{\rho_{w_1}(w_2)}{\rho_{w_1}(\lambda)\rho_{w_0}(w_2)} + \frac{\begin{pmatrix} 1 \\ k_2 \end{pmatrix}^{(1,\overline{k}_1)}}{1 - k_2\overline{k}_1}\ \frac{\rho_{w_2}(w_2)}{\rho_{w_2}(w_1)\rho_{w_0}(\lambda)}\right\} J\ .$$

Such Θ seem to be generalizations of Blaschke sections. In particular, for $m = 2$, the k_i should be analogues of the Schur coefficients. This is under investigation.

We conclude this section with some computations of inner products in a $\mathcal{K}(\Theta)$ space. Since a finite dimensional $\mathcal{K}(\Theta)$ space is resolvent invariant, we know from Theorem 2.2 the form of a basis of $\mathcal{K}(\Theta)$. Thus, this section seems the appropriate place to compute inner products such as

$$[\lambda^n c, \lambda^m d]_{\mathcal{K}(\Theta)}, [\lambda^n c, \frac{d}{(\lambda-\beta)^m}]_{\mathcal{K}(\Theta)} \text{ and } \left[\frac{c}{(\lambda-\alpha)^n}, \frac{d}{(\lambda-\beta)^m}\right]_{\mathcal{K}(\Theta)} .$$

Rather than give formulas for all possible cases, we will treat a number of illustrative examples. The interested reader should be able to apply the method used to treat the other cases.

To ease the notation, we denote the function $\frac{c}{(\lambda-\alpha)^n}$ by f_n and the function $\frac{d}{(\lambda-\beta)^m}$ by g_m. Our first result is:

LEMMA 6.4. With the above notation, the following identities hold if $\partial = \mathbb{R}$:

$$(\overline{\beta}-\alpha)[f_n, g_m] = [f_{n-1}, g_m] - [f_n, g_{m-1}] \text{ for } n > 1 \text{ and } m > 1 \tag{6.22}$$

$$(\overline{\beta}-\alpha)[f_1, g_m] = -[f_1, g_{m-1}] \text{ for } m > 1 \tag{6.23}$$

$$(\overline{\beta}-\alpha)[f_n, g_1] = [f_{n-1}, g_1] \text{ for } n > 1 \tag{6.24}$$

$$(\overline{\beta}-\alpha)[f_1, g_1] = 2\pi i d^* J c \ .$$

Before proving the lemma, we mention a number of remarks. The reader may find of interest to compare these formulas calculated in $\mathcal{K}(\Theta)$ to the formulas (5.18)-(5.20) calculated in $\mathcal{L}(\Phi)$. The results do not depend on the finite dimensionality of $\mathcal{K}(\Theta)$; moreover, similar formulas hold for $\partial = \mathbb{T}$, and the following consequences are of interest.

1) If $\alpha \neq \overline{\alpha}$, then $[f_1, f_1] = 0 \Rightarrow [f_j, f_k] = 0$ for all j, k such that f_j and f_k belong to $\mathcal{K}(\Theta)$, i.e. the space spanned by the f_j is neutral.
2) If $\alpha \neq \overline{\beta}$ then, if $[f_1, g_1] = 0$, $[f_j, g_k] = 0$ for all j and k such that f_j and g_k are in $\mathcal{K}(\Theta)$. The space spanned by the f_j and the space spanned by g_j are orthogonal.
3) If $\alpha = \overline{\beta}$, the formulas reduce to

$$[f_n, g_{m-1}] = [f_{n-1}, g_m] \text{ for } n > 1 \text{ and } m > 1$$

$$[f_1, g_{m-1}] = 0 \text{ for } m > 1$$

$$[f_{n-1}, g_1] = 0 \text{ for } n > 1 \ .$$

In particular, $[f_1, g_1] = 0$, and so, a reproducing kernel Hilbert space of the form $\mathcal{K}(\Theta)$ cannot contain a chain of the form

$$\frac{c}{\lambda-\alpha}, \ldots, \frac{c}{(\lambda-\alpha)^n}$$

for $n > 1$ and $\alpha \in \mathbb{R}$.

These relations will be computed by using relationships (6.5) and (6.6) and by taking advantage of the formula

$$R_w \frac{c}{(\lambda-\alpha)^n} = -\sum_{j=1}^{n} \frac{c}{(\lambda-\alpha)^{n+1-j}} \cdot \frac{1}{(w-\alpha)^j} .$$

Indeed, when $\partial = \mathbb{R}$, applying (6.5) to $f = f_n$, $g = g_m$ with the α and β of (6.5) set equal to w and $\overline{w}$, we get

$$\sum_1^m [f_n, g_{m+1-j}] \frac{1}{(w-\overline{\beta})^j} - \sum_1^n \frac{1}{(w-\alpha)^j} [f_{n+1-j}, g_m] = \frac{2\pi i\ d^* Jc}{(w-\alpha)^n (w-\overline{\beta})^m}$$

i.e., if n and m are strictly bigger than 1,

$$\begin{aligned} &[f_n, g_m] \frac{(\overline{\beta}-\alpha)}{(w-\overline{\beta})(w-\alpha)} \\ &= \frac{2\pi i\ d^* Jc}{(w-\alpha)^n (w-\overline{\beta})^m} + [f_{n-1}, g_m] \frac{1}{(\alpha-w)^2} - \frac{[f_n, g_{m-1}]}{(\overline{\beta}-w)^2} + r(w) \end{aligned} \tag{6.25}$$

where $r(w)$ is a rational function such that

$$\lim_{|w|\to\infty} w^2 r(w) = 0 .$$

Multiplying identity (6.25) by w^2 and letting w go to infinity, we obtain (6.22); the other cases are treated similarly.

We conclude this subsection with the following two lemmas.

LEMMA 6.5. Let $\partial = \mathbb{T}$, and suppose that the functions $\lambda^n c$ and $\lambda^n d$ belong to $\mathcal{K}(\Theta)$. Then, for $s \le n$, $s' \le n$,

$$\begin{aligned} [\lambda^s c, \lambda^{s'} d] &= 0 && \text{if } s \neq s' \\ &= d^* Jc && \text{if } s = s' . \end{aligned}$$

Before proving this lemma, we notice the following corollaries:

1) If $d^* Jc = 0$, then, $\text{span}\{c, \lambda c, \ldots, \lambda^n c\}$ is orthogonal to $\text{span}\{d, \lambda d, \ldots, \lambda^n d\}$.

2) If $d^*Jd = 0$, then, span $\{d, \lambda d, \ldots, \lambda^n d\}$ is a neutral subspace.

We notice that such remarks are of interest in the study of finite dimensional $\mathcal{K}(\Theta)$ spaces. In particular, if $d^*Jd = 0$, then n may not be too big; more precisely, $n \leq \text{rank } \mathcal{K}(\Theta)$.

PROOF OF THE LEMMA. Let us specialize equality (6.6) to $\alpha = \beta = 0$. Then for F, G in $\mathcal{K}(\Theta)$,

$$[F, G] - [R_0F, R_0G] = G^*(0)JF(0)$$

from which we see that

$$[c, d] = d^*Jc$$

for $s \geq 1$ and $s' \geq 1$,

$$[\lambda^s c, d] = [c, \lambda^{s'} d] = 0$$

and, for $s > 1$ and $s' > 1$,

$$[\lambda^s c, \lambda^{s'} d] = [\lambda^{s-1} c, \lambda^{s'-1} d] \ .$$

From these identities the desired result follows easily.

LEMMA 6.6. Let $\partial = \mathbb{T}$ and suppose that $f = \frac{\xi}{1-\lambda\overline{w}}$ belongs to $\mathcal{K}(\Theta)$, where w is in $\mathbb{D}$. Then,

$$[f, f]_{\mathcal{K}(\Theta)}(1 - |w|^2) = \xi^*J\xi \ .$$

In particular

$$[f, f]_{\mathcal{K}(\Theta)} = [f, f]_{H_J^2} \ .$$

The proof is left to the reader.

6.5. Some More Examples.

Theorem 6.1 dealt with the existence of $\mathcal{K}(\Theta)$ spaces for functions Θ which are J-unitary a.e. on ∂ and of bounded type in Ω_+, and the preceding section dealt with the important example of finite dimensional $\mathcal{K}(\Theta)$ spaces. The aim of this subsection is to present two other examples of $\mathcal{K}(\Theta)$ spaces. First we present an example relying on Azizov's work [A], which is linked to operator models and system theory. Then, we present a result of our paper [AD1], which exhibits a class of $\mathcal{K}(\Theta)$ spaces which are useful in the solution of the Inverse Scattering Problem for Caratheodory functions.

Let T be a $m \times m$ valued function analytic in a neighborhood of zero, V, and let J be a signature matrix. Azizov ([A]) proves that T may be represented as

$$T(\mu) = T_{22} + \mu T_{21}(I - \mu T_{11})^{-1}T_{12} \ , \ \mu \in V \ , \tag{6.26}$$

where the operator T_{11} is from some auxiliary Hilbert space H_1 into itself, T_{12} (resp. T_{21}) is from $\mathbb{C}_{m\times 1}$ into H_1 (resp. from H_1 into $\mathbb{C}_{m\times 1}$), and T_{22} maps $\mathbb{C}_{m\times 1}$ into itself. Moreover the operator $\underline{\underline{T}} = (T_{ij})$ from $H_1 \oplus \mathbb{C}_{m\times 1}$ into itself is $\begin{pmatrix} J_1 & 0 \\ 0 & J \end{pmatrix}$ unitary for some signature operator J_1 from H_1 into itself.

$$\underline{\underline{T}}\hat{J}\underline{\underline{T}}^* = \underline{\underline{T}}^*\hat{J}\underline{\underline{T}} = \hat{J} \tag{6.27}$$

with $\hat{J} = \begin{pmatrix} J_1 & 0 \\ 0 & J \end{pmatrix}$.

It is implicit in formula (6.26) that the non-zero μ are such that $\frac{1}{\mu}$ is in the resolvent set of T_{11}.

We wish to emphasize that under rather general conditions, any $m \times m$ valued function analytic at 0 may be represented in the form (6.26). The new ingredient in Azizov's result is condition (6.27). In the case of a J-inner rational function T, it is not difficult to see that (6.27) is forced by (6.26) and by the J-innerness, and moreover that $J_1 = I$. The work of Genin et.al. (see [G]) is linked to the present circle of ideas, but the connections will be explored elsewhere.

It follows readily from the representation formula (6.26) that, for μ, v in V,

$$\frac{J - T(\mu)JT^*(v)}{1-\mu\overline{v}} = T_{21}(I-\mu T_{11})^{-1}J_1(I-\overline{v}T_{11}^*)^{-1}T_{21}^* \ . \tag{6.28}$$

Formula (6.28) has a flavor of reproducing kernel, and under supplementary assumptions, it will follow from (6.28)that there is a reproducing kernel Krein space with reproducing kernel

$$K_v(\mu) = \frac{J - T(\mu)JT^*(v)}{1-\mu\overline{v}} \ .$$

Let

$$\mathcal{M}_1 = \bigvee_{\mu\in V} \operatorname{Ran}(I-\overline{\mu}T_{11}^*)^{-1}T_{21}^*$$

and let $\mathcal{K}$ be the space of $m \times 1$ valued functions of the form

$$F(\mu) = T_{21}(I-\mu T_{11})^{-1}J_1 f$$

where f is in $\mathcal{M}_1$. It is easy to see that such F are analytic in V and that for any v in V and c in $\mathbb{C}_{m\times 1}$, the function $K_v c$ belongs to $\mathcal{K}$. Moreover, if F is identically zero in V, it follows that f is in $\mathcal{N}_1 = (J_1\mathcal{M}_1)^{\perp}$. $\mathcal{N}_1$ can also be written as

$$\mathcal{N}_1 = \mathcal{M}_1^{[\perp]}$$

where $[\perp]$ denotes orthogonality in the Krein space $(H_1, < \ , J_1 \ >_1)$, with $< \ , \ >_1$ the inner product of H_1. Thus, under the assumption

$$\mathcal{M}_1 \cap \mathcal{M}_1^{[\perp]} = \{0\} \tag{6.29}$$

the mapping $f \to F$ is one to one from $\mathcal{M}_1$ into $\mathcal{K}$, and the hermitian form

$$[F,G]_{\mathcal{K}} =< f, J_1 g >_1$$

is then well defined in $\mathcal{K}$ where $G(\mu) = T_{21}(I-\mu T_{11})^{-1}J_1 g$ and g is in $\mathcal{M}_1$. Moreover,

$$\begin{aligned}[F,K_v c]_{\mathcal{K}} &=< f, J_1(I-\bar{v}T_{11}^*)^{-1}T_{21}^* c >_1 \\ &=< T_{21}(I-vT_{11})^{-1}J_1 f, c >_{\mathbb{C}^{m\times 1}} \\ &= c^* F(v)\end{aligned}$$

and thus $K_v(\mu)$ has the reproducing kernel property in $\mathcal{K}$.

We cannot prove in general, even under assumption (6.29), that $\mathcal{K}$ is a Krein space. It will be a Krein space (and hence by the above analysis, the RKKS of $m \times 1$ valued functions analytic in V with reproducing kernel K) in at least one of the following circumstances.

a) $\mathcal{M}_1 = H_1$, i.e. the pair (T_{11}, T_{12}) is observable.

b) $J_1 = I_{H_1}$ i.e. $T(\mu)$ is J contractive in V (for $|\mu| \leq 1$).

c) One of the operators $I + J_1$ or $I - J_1$has finite rank.

Indeed, in the first case the operator σ_1 defined by

$$(\sigma_1 \; F)(\mu) = T_{21}(I-\mu T_{11})^{-1} f \tag{6.30}$$

maps $\mathcal{K}$ into $\mathcal{K}$, and, for any F, G in $\mathcal{K}$,

$$[F, \sigma_1 G]_{\mathcal{K}} =< f, g >_1 \tag{6.31}$$

which clearly exhibits $(\mathcal{K}, [\;,\;])$ as a Krein space.

In the other two cases, (6.28) implies that the function $K_v(\mu)$ has a finite number of negative squares (or of positive squares) in V, and thus a space $\mathcal{K}(T)$ exists by Theorem 2.1.

We mention that the spaces obtained this way need not be spaces of functions defined in the whole disk. This property will depend on the spectrum of T_{11}, and we thus get a wider class of functions than the functions considered up to now. The above example seems to imply various links between system theory and reproducing kernel Krein spaces of the form $\mathcal{K}(\Theta)$. While such links are known in the Hilbert space case, it seems to us that a lot is still to be done in the Krein space framework.

The second example is taken, as already mentioned, from [AD1]. We present only a simplified version. The reader is referred to the paper for more general situations.

Let μ be a $p \times p$ valued function defined on the line, and increasing, (i.e. $\mu(t') - \mu(t)$ is a positive $p \times p$ matrix if $t \leq t'$). Suppose moreover that

$$\int Tr \; d\mu(t) < \infty \tag{6.32}$$

where Tr denotes the trace, and let $L_p^2(d\mu)$ denote the associated Lebesgue space of $p \times 1$ valued functions f such that

$$\int f^*(t)d\mu(t)f(t) < \infty \ .$$

We consider in $L_p^2(d\mu)$ a space $\mathcal{M}$ which has the following properties

1) Elements of $\mathcal{M}$ are analytic in some open set Δ which is symmetric with respect to $\mathbb{R}$
2) $\mathcal{M}$ is resolvent invariant
3) $\mathcal{M}$ is a reproducing kernel Hilbert space.

(The simplest example of such a space $\mathcal{M}$ is the set of constant multiples of $(\lambda - w)^{-1}\xi$, for fixed $w \notin \mathbb{R}$ and $\xi \in \mathbb{C}_{p\times 1}$).

Next, to each element f in $\mathcal{M}$ we associate the function f_- via the formula

$$f_-(\lambda) = \frac{1}{\pi i}\int_{-\infty}^{\infty} d\mu(t)\frac{f(t) - f(\lambda)}{t - \lambda} \ . \tag{6.33}$$

We remark that because of condition (6.32), $f_-(\lambda)$ is well defined and is also analytic in Δ. Elementary computations show that for α, β in Δ, and f, g in $\mathcal{M}$, we have

$$\begin{aligned} &< R_\alpha f, g >_\mu - < f, R_\beta g >_\mu -(\alpha - \overline{\beta}) < R_\alpha f, R_\beta g > \\ &= \pi i\{g^*(\beta)f_-(\alpha) + g_-^*(\beta)f(\alpha)\} \end{aligned} \tag{6.34}$$

and moreover that the right side of (6.34) may be written as

$$\pi i G^*(\beta)J_1F(\alpha)$$

with $J_1 = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ and $F(\alpha) = \begin{pmatrix} f(\alpha) \\ f_-(\alpha) \end{pmatrix}$.

Thus, applying a representation theorem of de Branges (see subsection 6.2), it is not difficult to get to the following theorem (a more general version, valid for $\partial = \mathbb{R}$ and $\partial = \mathbb{T}$ with (6.32)replaced by $\int \frac{Tr\ d\mu(t)}{1+t^2} < \infty$ is presented in Theorem 3.1, p.600 of [AD1]):

THEOREM 6.14. Let μ be an increasing $p \times p$ valued function subject to (6.32), and let $\mathcal{M}$ be a reproducing kernel Hilbert space of $p \times 1$ valued functions, analytic in some symmetric open set Δ, resolvent invariant, and sitting isometrically in $L_p^2(d\mu)$. Then, the set of functions $F = \begin{pmatrix} f \\ f_- \end{pmatrix}$ with f in $\mathcal{M}$ and f_- as in (6.33) is a reproducing kernel Hilbert space with respect to the inner product

$$\|F\|^2 = 2\|f\|_\mu^2$$

and its reproducing kernel is of the form

$$\frac{J_1 - U(\lambda)J_1U^*(w)}{-2\pi i(\lambda - \overline{w})}$$

for a $2p \times 2p$ valued function U which is analytic in Ω, andis subject to

$$UJ_1U^{\#} = J_1$$

in Δ.

Finally, the reproducing kernel for $\mathcal{M}$ is

$$k_w(\lambda) = 2[I_p, 0]\left\{\frac{J_1 - U(\lambda)J_1U^*(w)}{-2\pi i(\lambda - \overline{w})}\right\}\begin{bmatrix} I_p \\ 0 \end{bmatrix} \ . \tag{6.35}$$

An important byproduct of the stronger form of Theorem 6.14 is the following result (which is valid for $\partial = \mathbb{R}$ or $\partial = \mathbb{T}$).

THEOREM 6.15. Let $\mathcal{M}$ be a resolvent invariant reproducing kernel Hilbert space of $p \times 1$ valued functions analytic in a symmetric set Δ, and suppose that $\mathcal{M}$ sits isometrically in $L_p^2(d\mu)$, where μ is an increasing function on ∂, subject to

$$\int_{\partial} \mathrm{Tr} \frac{d\mu(t)}{1+t^2} < \infty \ .$$

Then the reproducing kernel of $\mathcal{M}$ is of the form

$$k_w(\lambda) = 2 \cdot \frac{A(\lambda)B^*(w) + B(\lambda)A^*(w)}{\rho_w(\lambda)} \tag{6.36}$$

for $p \times p$ valued functions A and B analytic in Δ.

PROOF. The result is a corollary of Theorem 6.14. It suffices to set $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where all the blocks are $p \times p$ valued, and to explicitly evaluate formula (6.35). The details are left to the reader. □

Reproducing kernels of the form (6.35) are thus closely related to $\mathcal{H}(\Theta)$ spaces, with signature matrix $J_1 = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$. Such kernels, with entire functions A, B, appear in the theory of stationary gaussian processes, (see [DMk] for the scalar case). They also appear in the model theory of hermitian operators (see [dBR], and the last section of the present thesis), in the theory of canonical differential equations (see [DI]) and in the study of the Szegö's formula (see [D2]).

Finally, the links between Theorem 6.14 and the linear fractional representations of Caratheodory functions will be explained in Section 9.2.

7. Linear Fractional Transformations

Let $J_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ and let Θ be a function which is J_0-unitary on ∂ and of bounded type in Ω_+, such that the associated Krein space $\mathcal{K}(\Theta)$ is a Pontryagin space of rank k_Θ. Let S be an element of $S_{p\times q}^{k_s}$ for some integer k_s so that the associated Krein space $K(S)$ is a Pontryagin space of rank k_s. The aim of this section is to study linear fractional representations of S as a function of Θ when the map τ

$$F \to [I_p, -S]F \tag{7.1}$$

maps the Pontryagin space $\mathcal{K}(\Theta)$ contractively into the Pontryagin space $K(S)$, i.e.,

$$[\tau F, \tau F]_{K(S)} \le [F, F]_{\mathcal{K}(\Theta)} \ .$$

Such mappings τ will appear in our study of the inverse scattering problem which is presented in Section 8. A simple example of such a mapping also occurs in Section 4, wherein it was noted that the map

$$\begin{pmatrix} pS_1^* u \\ \underline{p}S_2^* u \end{pmatrix} \longrightarrow [I_p, -S] \begin{pmatrix} pS_1^* u \\ \underline{p}S_2^* u \end{pmatrix} \quad u \in H_p^2$$

maps the subspace $\mathcal{M}$ of H_J^2

$$\mathcal{M} = \left\{ \begin{pmatrix} pS_1^* u \\ \underline{p}S_2^* u \end{pmatrix} ; \quad u \in H_p^2 \right\}$$

contractively into $K(S)$.

Although in this last example $\mathcal{M}$ itself need not be a $\mathcal{K}(\Theta)$ space, it will typically contain subspaces which are $\mathcal{K}(\Theta)$ spaces and are mapped contractively into the Pontryagin space $K(S)$ by the map τ.

Before stating the theorems to be proved in this section, we introduce some notation. The function Θ will be written as

$$\Theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is $p \times p$ valued and, for a $p \times q$ function W, we define

$$T_\Theta(W) = (AW + B)(CW + D)^{-1} \ ,$$

whenever the indicated inverse exists.

The two results of this section are:

THEOREM 7.1. Let $\mathcal{K}(\Theta)$ and $\mathcal{K}(S)$ be two Pontryagin spaces of rank k_Θ and k_S respectively, and suppose that the map τ

$$F \rightarrow [I_p, -S]F$$

is a π-isometry from $\mathcal{K}(\Theta)$ into $\mathcal{K}(S)$. Then,

$$S = T_\Theta(W)$$

for an element W in the class $S^k_{p\times q}$, with $k = k_S - k_\Theta$.

THEOREM 7.2. With the notation of Theorem 7.1, suppose $k_\Theta = k_S$. Then, if τ is a continuous π-contraction from $\mathcal{K}(\Theta)$ into $\mathcal{K}(S)$,

$$S = T_\Theta(W)$$

for some element W in the Schur class $S_{p\times q}$.

These theorems and their proofs are modelled on an earlier result of de Branges and Rovnyak ([dBR], [dB6]), who established Theorem 7.2 and its converse in the special case that $k_s = k_\Theta = 0$ (so that $\mathcal{K}(\Theta)$ and $\mathcal{K}(S)$ are Hilbert spaces) and $p = q$.

Before proving Theorem 7.1, we find it convenient to recall that, in a Pontryagin space $(\Pi_k, [\ ,\ ])$, a sequence of elements (f_n) converges to an element f if and only if

a) for any u in Π_k, $\lim\ [f_n, u] = [f, u]$

b) $\lim\ [f_n, f_n] = [f, f]$

PROOF OF THEOREM 7.1. Let $k_w(\lambda)$ (resp. $K_w(\lambda)$) denote the reproducing kernel of $\mathcal{K}(S)$ (resp. $\mathcal{K}(\Theta)$), and let $\Omega(S)$ (resp. $\Omega(\Theta)$) denote the domain of analyticity of the elements of $\mathcal{K}(S)$ (resp. of $\mathcal{K}(\Theta)$). The proof proceeds in a number of steps, the first two of which serve to show that τ is a continuous map from $\mathcal{K}(\Theta)$ into $\mathcal{K}(S)$.

Step 1. The closure of Ran τ is a Pontryagin space.

Proof of Step 1. In view of Corollary 2.3, p.186 of [Bo], it suffices to show that the closure of Ran τ is non-degenerate. Let g be an element in $\overline{\text{Ran}\ \tau}$, the closure of Ran τ, and suppose that

$$[g, h] = 0 \quad \text{for all } h \text{ in } \overline{\text{Ran}\ \tau}\ .$$

Since $g = \lim\ \tau f_n$ for some sequence (f_n) of elements in $\mathcal{K}(\Theta)$ and since the inner product is continuous,

$$\lim[\tau f_n, h] = 0$$

which, for h of the form τu, leads to

$$\lim[f_n, u]_{\mathcal{K}(\Theta)} = 0\ ,$$

for every $u \in \mathcal{K}(\Theta)$.

Similarly,

$$0 = [g, g]_{K(S)} = \lim[\tau f_n, \tau f_n]_{K(S)} = \lim[f_n, f_n]_{K(\Theta)} \ .$$

Thus the sequence (f_n) converges to 0 in $\mathcal{K}(\Theta)$. In particular, for any w in $\Omega(\Theta)$,

$$\lim \ f_n(w) = 0 \ .$$

But, from $g = \lim \ \tau f_n$,it follows that

$$g(w) = \lim[I, -S(w)]f_n(w)$$

for w in $\Omega(\Theta) \cap \Omega(S)$, and thus $g = 0$.

Step 2. τ is continuous (adapted from [Bo], Theorem 3.1, p.188).

Proof of Step 2. Let $\mathcal{K}(\Theta) = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$ be a decomposition of $\mathcal{K}(\Theta)$ into two orthogonal subspaces, where $\mathcal{K}_-$ is a negative subspace of dimension k_Θ. Moreover, in $\mathcal{K}(\Theta)$, let $< \ , \ >_\Theta$ be the hermitian form defined by

$$< f, f >_\Theta = [f_+, f_+]_{K(\Theta)} - [f_-, f_-]_{K(\Theta)} \tag{7.2}$$

where $f = f_+ + f_-$, $f_\pm \in \mathcal{K}_\pm$. Then, $(\mathcal{K}(\Theta), < \ , \ >_\Theta)$ is a Hilbert space. Since τ is a π-isometry,

$$\text{Ran } \tau = \tau\mathcal{K}_+[\dot{+}]\tau\mathcal{K}_- \ . \tag{7.3}$$

On the other hand, $\tau\mathcal{K}_-$ is non-degenerate finite dimensional, and so is orthocomplemented in the Pontryagin space $\overline{\text{Ran } \tau}$, i.e.

$$\overline{\text{Ran } \tau} = (\tau\mathcal{K}_-)^{[\perp]}[\dot{+}](\tau\mathcal{K}_-)$$

and thus,

$$\tau\mathcal{K}_+ \subset (\tau\mathcal{K}_-)^{[\perp]} \ . \tag{7.4}$$

Let us denote by $< \ , \ >_\tau$ the positive inner product defined on $\overline{\text{Ran } \tau}$ by

$$< g, g >_\tau = [g_+, g_+]_{K(S)} - [g_-, g_-]_{K(S)}$$

where g_- (resp. g_+) belongs to $\tau\mathcal{K}_-$ (resp. $(\tau\mathcal{K}_-)^{[\perp]}$).

It is clear that $(\overline{\text{Ran } \tau}, < \ , \ >_\tau)$ is a Hilbert space and that

$$[u, v]_{K(S)} = < u, \sigma v >_\tau$$

for u, v in $\overline{\text{Ran } \tau}$, where σ is defined by

$$\sigma(g_+ + g_-) = g_+ - g_- \ .$$

Now, let $f = f_+ + f_-$ be an element of $\mathcal{K}(\Theta)$. Using inclusion (7.4), τf_+ is in $(\tau\mathcal{K}_-)^{[\perp]}$, and thus

$$\begin{aligned} < \tau f, \tau f >_\tau &= [\tau f_+, \tau f_+]_{\mathcal{K}(S)} - [\tau f_-, \tau f_-]_{\mathcal{K}(S)} \\ &= [f_+, f_+]_{\mathcal{K}(\Theta)} - [f_-, f_-]_{\mathcal{K}(\Theta)} \\ &=< f, f >_\Theta \end{aligned}$$

and thus τ, as an isometry between the two exhibited Hilbert spaces, is bounded.

Step 3. Let τ^+ be the adjoint of τ. Then, for c in $\mathbb{C}_{p\times 1}$ and w in $\Omega(\Theta) \cap \Omega(S)$,

$$\tau^+ k_w c = K_w \begin{pmatrix} I \\ -S^*(w) \end{pmatrix} c \ .$$

Proof of Step 3. This is a typical argument from the theory of Hilbert spaces of analytic functions which is still valid in the present context.

We first remark that τ^+ is continuous and everywhere defined, since τ is ([Bo], Theorem 2.2, p. 122). Let v now be in $\Omega(\Theta)$ and d be in $\mathbb{C}_{m\times 1}$ $(m = p + q)$. Then,

$$[\tau^+ k_w c, K_v d]_{\mathcal{K}(\Theta)} = d^*(\tau^+ k_w c)(v) \ .$$

On the other hand,

$$\begin{aligned} [\tau^+ k_w c, K_v d]_{\mathcal{K}(\Theta)} &= [k_w c, \tau K_v d]_{\mathcal{K}(S)} \\ &= [\tau K_v d, k_w c]^*_{\mathcal{K}(S)} \\ &= (c^*[I_p, -S(w)]K_v(w)d)^* \\ &= d^* K_w(v) \begin{pmatrix} c \\ -S^*(w)c \end{pmatrix} \ . \end{aligned}$$

This completes the proof of Step 3. The rest of the proof of the theorem relies on Lemma 1.1.

Step 4. Let X be the $p \times m$ valued function defined by

$$X = [I_p, \quad -S]\Theta = [A - SC, B - SD] \ .$$

Then, the function $\frac{X(\lambda)JX^*(w)}{\rho_w(\lambda)}$ has $(k_s - k_\Theta)$ negative squares for λ and w in $\Omega(S) \cap \Omega(\Theta)$.

Proof of Step 4. τ is continuous and thus Lemma 1.1 may be used. The step is then a direct consequence of Lemma 1.1 applied to $u = v = \Sigma k_{w_j} c_j$, w_j in $\Omega(S) \cap \Omega(\Theta)$ and c_j in $\mathbb{C}_{p\times 1}$. Indeed, easy computations lead to

$$[u, v]_{\mathcal{K}(S)} - [\tau^+ u, \tau^+ v]_{\mathcal{K}(\Theta)} = \frac{\Sigma c_i^* X(w_i) J X^*(w_j) c_j}{\rho_{w_i}(w_j)}$$

and the conclusion comes from the fact that the hermitian form $(u,v) \mapsto [u,v] - [\tau^+ u, \tau^+ v]$ has $k_s - k_\Theta$ negative squares.

To conclude the proof of the theorem, it is sufficient to prove that, in Ω_+, det $(A - SC) \not\equiv 0$. Then setting

$$W = (A - SC)^{-1}(SD - B)$$

we have $S = T_\Theta(W)$ and W is in $S_{p\times q}^{(k_s - k_\Theta)}$.

Step 5. det $(A - SC) \not\equiv 0$ in Ω_+.

Proof of Step 5. From the fact that Θ is J-unitary a.e. on ∂, it is easy to get to

$$\|CA^{-1}\| < 1 \text{ a.e. on } \partial$$

since, a.e. on ∂,

$$A^*A - C^*C = I_p \ .$$

On the other hand, since S is in $S_{p\times q}^{k_s}$, Theorem 4.2 implies that $S = B_0^{-1}S_0$ where S_0 is in $S_{p\times q}$ and B_0 is a $p \times p$ Blaschke product. Thus

$$\|S\| \leq 1 \quad \text{a.e. on } \partial \ .$$

Writing $A - SC = (I_p - SCA^{-1})A$, we conclude that det $A - SC \not\equiv 0$ a.e. on ∂. Since this function is of bounded type in Ω_+,

$$\det(A - SC) \not\equiv 0 \text{ in } \Omega_+ \ .$$

This concludes the proof of Step 5 and of the theorem. □

We now briefly outline the proof of Theorem 7.2. We denote by k the common value of k_Θ and k_S. Indeed, let K_- be a k dimensional negative definite subspace of $K(\Theta)$; τK_- is then negative definite, and we choose a decomposition of $K(S)$:

$$K(S) = K_+ [\dot{+}] K_-$$

with $\tau K_- \subset K_-$. The matrix of τ is then

$$\begin{bmatrix} \tau_{11} & 0 \\ \tau_{21} & \tau_{22} \end{bmatrix}$$

where τ_{22} is from K_- into K_-, τ_{11} from K_+ into K_+ and τ_{21} from K_+ into K_-. τ_{11}, τ_{21} and τ_{22} are clearly bounded operators. The π contraction property leads to

$$[\tau_{22} f_-, \tau_{22} f_-] \leq [f_-, f_-] \ .$$

Since $k_\Theta = k_s$, this forces τ_{22}^{-1} to exist, and as in ([KS], p. 106, Theorem 3.1), this fact serves to show that τ^+ is still a π-contraction. The proof is then finished as Steps 3-5 of the proof of Theorem 7.1.

CHAPTER III
INVERSE SCATTERING PROBLEMS AND RELATED TOPICS

Introduction

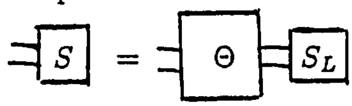
This chapter deals with two applications of the theory developed in the second chapter. First, we define an inverse scattering problem (ISP) for the class of $k \times m$ matrix valued functions X which are analytic in some open set Δ and are such that, for some signature matrix J, a reproducing kernel Krein space with reproducing kernel $\frac{X(\lambda)JX^*(w)}{\rho_w(\lambda)}$ exists where λ and w are in Δ. This is done in Section 8 where some solutions to the ISP are also given, while more general solutions are given in Section 9.

Finally, Section 10 gives a model for π-hermitian operators with equal and finite deficiency indices.

8. The Inverse Scattering Problem for Functions of Bounded Type

In this section we define an inverse scattering problem for a class of matrix valued functions X of bounded type in Ω_+. The term inverse scattering is best explained by the fact that for

$X = [I_p, -S]$, where S is in the Schur class, $S_{p\times p}$, the inverse scattering problem associated with X consists of finding a representation for S as the cascade



of a passive $2p$-port with cascade scattering function Θ and a passive load with scattering function S_L(see e.g. [DD]).

The outline of this section is as follows. In 8.1 we give the definition of the inverse scattering problem, while 8.2 is devoted to finding certain solutions to it. More general solutions for the case $X = [I_p, -S]$, S $p \times q$ valued and of bounded type in Ω_+ will be presented in the next section.

8.1. Definition of the Inverse Scattering Problem.

Let X be a $k \times m$ valued function, analytic in some open subset Δ of $\mathbb{C}$, and let J be an $m \times m$ signature matrix. The function X will be called (Δ, J) admissible if

$$\sum_1^r \frac{c_i^* X(w_i) J X^*(w_j) c_j}{\rho_{w_i}(w_j)} \geq 0$$

for all choices of r, $w_1, \ldots, w_r$ in Δ and $c_1, \ldots, c_r$ in $\mathbb{C}_{k\times 1}$, i.e. if the "kernel"

$$\frac{X(\lambda) J X^*(w)}{\rho_w(\lambda)} \tag{8.1}$$

is positive. We will denote by $\mathcal{B}(X)$ the associated reproducing kernel Hilbert space of $k \times 1$ valued functions which are analytic in Δ, with reproducing kernel (8.1).

We remark that strictly speaking the notion of admissibility depends on the function ρ. In this thesis, $\rho_w(\lambda)$ is either $1 - \lambda\overline{w}$ (disk case), or $-2\pi i(\lambda - \overline{w})$ (line case), but other choices could be thought of, and we hope to come back to this issue in the future.

If Φ belongs to C_p, then $[\Phi, I_p]$ is (Ω_+, J_1) admissible for $J_1 = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$, while if S belongs to $S_{p\times q}$, then $[I_p, -S]$ is (Ω_+, J_0) admissible for $J_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$.

The inverse scattering problem for X consists of finding all $m \times m$ valued functions U which are analytic in an open subset Δ_1 of Δ, and such that XU is (Δ_1, J) admissible. In general, Δ_1 will differ from Δ by at most a Nevanlinna zero set.

For a function U such that a reproducing kernel Hilbert space $\mathcal{H}(U)$ with reproducing kernel $\frac{J-U(\lambda)JU^*(w)}{\rho_w(\lambda)}$ exists, the inverse scattering problem has the following equivalent formulation

THEOREM 8.1. Let U be $m \times m$ valued, analytic in Δ_1, and suppose that a reproducing kernel Hilbert space $\mathcal{H}(U)$ exists. Then XU is (Δ_1, J) admissible if and only if $\tau: F \to XF$ maps $\mathcal{H}(U)$ contractively into $\mathcal{B}(X)$.

PROOF. Suppose first that u maps $\mathcal{H}(U)$ contractively into $\mathcal{B}(X)$. Let $k_w(\lambda)$ (resp. $K_w(\lambda)$) denote the reproducing kernel of the space $\mathcal{B}(X)$ (resp. $\mathcal{H}(U)$). As in Step 3 of the proof of Theorem 7.1,

$$\tau^* k_w c = K_w X^*(w) c$$

where w is in Δ_1 and c is in $\mathbb{C}_{k\times 1}$. τ^* is a contraction and thus for any integer ℓ and any choice of $w_1, \ldots, w_\ell$ in Δ_1 and $c_1, \ldots, c_\ell$ in $\mathbb{C}_{k\times 1}$

$$\| \sum_{j=1}^{\ell} \tau^* k_{w_j} c_j \|^2_{\mathcal{H}(U)} \leq \| \sum_{j=1}^{\ell} k_{w_j} c_j \|^2_{\mathcal{B}(X)}$$

from which it follows that

$$\sum_{i,j=1}^{\ell} c_i^* k_{w_j}(w_i) c_j \geq \sum_{i,j=1}^{\ell} c_i^* X(w_i) K_{w_j}(w_i) X^*(w_j) c_j$$

and hence, upon writing out K explicitly, it follows that

$$\sum_{i,j=1}^{\ell} c_i^* X(w_i) \frac{U(w_i) J U^*(w_j)}{\rho_{w_j}(w_i)} X^*(w_j) c_j \geq 0$$

i.e., XU is (Δ_1, J) admissible.

Conversely, the admissibility of XU implies that the map

$$\kappa : \sum_{j=1}^{\ell} k_{w_j} c_j \mapsto \sum_{j=1}^{\ell} K_{w_j} X^*(w_j) c_j$$

is well defined and extends to all of $\mathcal{B}(X)$. Moreover, κ is a contraction from $\mathcal{B}(X)$ into $\mathcal{H}(U)$; κ^* is easily seen to be equal to τ. Thus, τ maps $\mathcal{H}(U)$ contractively into $\mathcal{B}(X)$. The details are much the same as for the special case $X = [\Phi, I_p]$ and $J = J_1$, which is treated in [AD1], p.628-629. □

We refer the reader to Section 6 of [AD1] for a discussion of the cases

$$X = [I_p, -S] \text{ and } J = J_0 \left(= \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix} \right)$$

and

$$X = [\Phi, I_p] \text{ and } J = J_1 \left(= \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix} \right)$$

and of the relationship between them.

Theorem 8.1 shows how to rephrase the inverse scattering-problem in terms of reproducing kernel Hilbert spaces. This suggests the following generalization to reproducing kernel Krein spaces: Given a $k \times m$ valued function X analytic in an open set Δ, and such that a RKKS with reproducing kernel (8.1) exists, find $m \times m$ functions U such that a RKKS $\mathcal{K}(U)$exists and such that the map $\tau : \; F \to XF$ maps $\mathcal{K}(U)$ contractively into $\mathcal{B}(X)$, i.e., for any F in $\mathcal{K}(U)$,

$$[\tau F, \tau F]_{\mathcal{B}(X)} \leq [F, F]_{\mathcal{K}(U)} \; . \tag{8.2}$$

In the sequel we will be interested in cases where τ is an isometry, i.e. for which there is equality in (8.2). When $X = [I_p, -S]$ and $\mathcal{K}(U)$ and $\mathcal{B}(X)$ are Pontryagin spaces rather than Krein spaces, the isometry will force fractional linear representations of S in terms of U, as explained in Section 7.

8.2. On Rational Solutions to the Inverse Scattering Problem.

Given a $k \times m$ valued function X, which is analytic in an open set Δ included in Ω_+ and is such that a RKKS with reproducing kernel (8.1) exists for some signature matrix J, we show how to get rational solutions to the inverse scattering problem for X. More general solutions for the special cases $X = [I_p, -S]$ and $X = [\Phi, I_p]$, where S is $p \times q$ valued of bounded type in Ω_+ and where Φ is in $\mathcal{C}_p$ will be considered in Sections 9.1 and 9.2.

Let $\mathcal{M}(X)$ be the closure in the Krein space H_J^2 of the elements of the form $(\frac{X^*(w)c}{\rho_w})$, where w is in Δ and c is in $\mathbb{C}_{k \times 1}$ (recall that here $\Delta \subset \Omega_+$). In general, $\mathcal{M}(X)$ is degenerate, i.e.

$$\mathcal{M}(X) \cap \mathcal{M}(X)^{[\perp]} \neq \{0\}$$

and therefore is not a reproducing kernel Krein space. However, typically, $\mathcal{M}(X)$ has subspaces which are Krein spaces. For instance, the linear span of the elements

$$\frac{JX^*(w_1)c_1}{\rho_{w_1}}, \ldots, \frac{JX^*(w_r)c_r}{\rho_{w_r}} \tag{8.3}$$

where $w_1, \ldots, w_r$ are in Δ and $c_1, \ldots, c_r$ are in $\mathbb{C}_{k \times 1}$, is a reproducing kernel Krein space if the $r \times r$ matrix with ij entry

$$\frac{c_i^* X(w_i) J X^*(w_j) c_j}{\rho_{w_j}(w_i)} \tag{8.4}$$

is invertible. Then the linear span of the elements (8.3) is a finite dimensional Krein subspace of $\mathcal{M}(X)$, and an application of Theorem 6.3 or of Theorem 6.4 (according as $\Omega_+ = \mathbb{D}$ or $\mathbb{C}$) shows that this subspace is a reproducing kernel Pontryagin space of $m \times 1$ valued functions analytic in $\mathbb{C} \setminus \{w_1', \ldots, w_r'\}$, with reproducing kernel of the form

$$\frac{J - U(\lambda) J U^*(w)}{\rho_w(\lambda)}$$

for an $m \times m$ valued function U, analytic in $\mathbb{C} \setminus \{w_1', \ldots, w_r'\}$.

We show that $\tau : F \to \tau F$ is an isometry from $\mathcal{K}(U)$ into $\mathcal{B}(X)$. Indeed, an element of $\mathcal{K}(U)$ may be written as

$$F(\lambda) = \sum_1^r \alpha_i \frac{J X^*(w_i) c_i}{\rho_{w_i}}$$

for some choice of complex numbers $\alpha_1, \ldots, \alpha_r$, and thus

$$XF = \sum_1^r \alpha_i \frac{X(\lambda) J X^*(w_i) c_i}{\rho_{w_i}(\lambda)}$$

clearly belongs to $\mathcal{B}(X)$. Moreover, since $\mathcal{K}(U)$ is isometrically included in H_J^2,

$$[F, F]_{\mathcal{K}(U)} = \sum \alpha_j^* \alpha_i \frac{c_j^* X(w_j) J X^*(w_i) c_i}{\rho_{w_i}(w_j)}$$

and hence

$$[F, F]_{\mathcal{K}(U)} = [\tau F, \tau F]_{\mathcal{B}(X)} \ .$$

We have thus exhibited a solution $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to the inverse scattering problem associated with X for every choice of $w_1, \ldots, w_r$ and $c_1, \ldots, c_r$ for which the matrix (8.4) is invertible. These are not the only possibilities. As already mentioned, when $J = J_0 = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ and $X = [I, -S]$, U and S will be linked by

$$S = (AW + B)(CW + D)^{-1}$$

where W is in $S_{p \times q}^{k_s - k_U}$, if $\mathcal{B}(X)$ (resp. $\mathcal{K}(U)$) has k_s (resp. k_U) negative squares.

For the links with interpolation problems (in the Hilbert space case), the reader is referred to ([AD2], Section 10) and to [D1].

9. Non-Rational Solutions

In the preceding section we defined the inverse scattering problem (ISP) and exhibited a class of rational solutions. In this section we exhibit a wider class of solutions for two specific choices of X:

1) $X = [I_p, -S]$, $J = J_0$, $\Delta = \Omega_+ \backslash A$, where S is $p \times q$ valued of bounded type in Ω_+ and A is a Nevanlinna zero set.

2) $X = [\Phi, I_p]$, $J = J_1$, $\Delta = \Omega_+$, where Φ is in C_p.

The first choice which is presented in 9.1, is essentially equivalent to the case where the function $X = [X_1, X_2]$ is of bounded type in Ω_+ and the determinant of the $p \times p$ matrix X_1 does not vanish identically. This includes a fairly wide class of cases.

The second choice is treated by rather different methods in 9.2, and the class of solutions obtained is wider. The methods and their applications were already reported on in [AD1], [AD2].

Finally, Section 9.3 treats a Riccati equation related to certain solutions Θ.

9.1. The Case $X = [I_p, -S]$.

We first briefly outline the method to be used to obtain solutions to the ISP associated with $X = [I_p, -S]$, when S is $p \times q$ valued and of bounded type in Ω_+. Let $S = S_1^{-1} S_2$, where S_1 is in $S_{p\times p}$ and S_2 is in $S_{p\times q}$. We will consider subspaces of $H^2_{J_0}$ of the form

$$\mathcal{M} = \left\{ F = \begin{pmatrix} \underline{p} S_1^* u \\ \underline{p} S_2^* u \end{pmatrix} ; \ u \in H_p^2 \ominus \Xi H_p^2 \right\} \tag{9.1}$$

where Ξ is a $p \times p$ inner function and where the elements of $\mathcal{M}$ are suitably extended to Ω_-. Then, when Ξ has a point of analyticity on ∂ and the operator

$$\Gamma_\Xi = P_\Xi \Gamma|_{H_p^2 \ominus \Xi H_p^2} \tag{9.2}$$

is invertible, where

$$\Gamma = S_1 \underline{p} S_1^* - S_2 \underline{p} S_2^*$$

and P_Ξ is the orthogonal projection from H_p^2 onto $H_p^2 \ominus \Xi H_p^2$, the space $\mathcal{M}$ is a reproducing kernel Krein space with reproducing kernel of the form

$$\frac{J_0 - \Theta(\lambda) J_0 \Theta^*(w)}{\rho_w(\lambda)} .$$

It is then not difficult to check that Θ is a solution to the inverse scattering problem associated with $X = [I_p, -S]$: the map $F \to [I_p, -S]F$ maps $\mathcal{K}(\Theta)$ isometrically into $\mathcal{B}(X)$, $\mathcal{B}(X)$ is readily seen to be the space $K(S)$ defined in Section 4, i.e. the closure of the set of

$$f = S_1^{-1}\Gamma u\ , \quad u \in H_p^2 \tag{9.3}$$

with indefinite inner product

$$[f, f]_{K(S)} = < \Gamma u, u > \quad . \tag{9.4}$$

We now explain in more detail the method sketched above. We suppose, as already mentioned, the function Ξ to be analytic at some point δ of ∂. Indeed, this need not be the case, even for a Blaschke product. Nevertheless, this hypothesis does not seem to be too restrictive.

Under this hypothesis, the elements of $H_p^2 \ominus \Xi H_p^2$ have an analytic extension at the point δ which is such that the space $H_p^2 \ominus \Xi H_p^2$ is resolvent invariant in a neighborhood of δ. (The proof of this fact was outlined at the end of of Section 4.)

The function Ξ is extended to Ω_- via

$$\Xi\Xi^{\#} = I_p \tag{9.5}$$

where $\Xi^{\#}(w) = \Xi^*(w')$, $w' = \overline{w}$ (resp. $\frac{1}{\overline{w}}$) when $\partial = \mathbb{R}$ (resp. $\partial = \mathbb{T}$), and let $\Omega(\Xi)$ be the subset of $\mathbb{C}$ where Ξ thus extended is analytic. We remark that $\Omega(\Xi)$ contains a neighborhood of δ.

We now show that the elements of $\mathcal{M}$ are analytic in $\Omega(\Xi)$ and that $\mathcal{M}$ is resolvent invariant in $\Omega(\Xi)$. We first remark that for an element S in $H_{p\times q}^{\infty}$, such that $S(0) = 0$ if $\partial = \mathbb{T}$ and $S(i) = 0$ if $\partial = \mathbb{R}$, we have for λ in Ω_+ and u in H_p^2

$$(\underline{p}S^*u)(\lambda) = \frac{1}{2\pi i}\int S^*(t)\left\{\frac{u(t)-u(\lambda)}{t-\lambda} - \frac{u(\lambda)t}{t^2+1}\right\}dt \quad \text{if } \partial = \mathbb{R} \tag{9.6}$$

$$(\underline{p}S^*u)(\lambda) = \frac{1}{2\pi}\int S^*(e^{it})\frac{u(e^{it})-u(\lambda)}{e^{it}-\lambda}e^{it}dt \quad \text{if } \partial = \mathbb{T} \tag{9.7}$$

and thus, applying these formulas to $S_1^* - S_1^*(i)$ and $S_2^* - S_2^*(i)$ if $\partial = \mathbb{R}$, we get to, writing $S_1 = S_1 - S_1(i) + S_1(i)$,

$$(\underline{p}S_1^*u)(\lambda) = S_1^*(i)u(\lambda) + \frac{1}{2\pi i}\int (S_1^*(t) - S_1^*(i))\left\{\frac{u(t)-u(\lambda)}{t-\lambda} - \frac{u(\lambda)t}{t^2+1}\right\}dt \tag{9.8}$$

$$(\underline{p}S_2^*u)(\lambda) = S_2^*(i)u(\lambda) + \frac{1}{2\pi i}\int (S_2^*(t) - S_2^*(i))\left\{\frac{u(t)-u(\lambda)}{t-\lambda} - \frac{u(\lambda)t}{t^2+1}\right\}dt$$

and similarly for $\partial = \mathbb{T}$,

$$(\underline{p}S_1^*u)(\lambda) = S_1^*(0)u(\lambda) + \frac{1}{2\pi}\int (S_1^*(e^{it}) - S_1^*(0))\frac{u(e^{it})-u(\lambda)}{e^{it}-\lambda}e^{it}dt \tag{9.9}$$

$$(\underline{p}S_2^*u)(\lambda) = S_2^*(0)u(\lambda) + \frac{1}{2\pi}\int (S_2^*(e^{it}) - S_2^*(0))\frac{u(e^{it}) - u(\lambda)}{e^{it} - \lambda}e^{it}dt \ .$$

When the function u is in $H_p^2 \ominus \Xi H_p^2$ and extended analytically to $\Omega(\Xi)$ as above, then formula (9.8) (when $\partial = \mathbb{R}$) and formula (9.9) (when $\partial = \mathbb{T}$) provide an extension of the elements of $\mathcal{M}$ to $\Omega(\Xi)$. Since $\Omega(\Xi)$ contains a neighborhood of δ, the functions $\mathcal{M}$ are thus extended analytically in a neighborhood of that point. Moreover, for any α in $\Omega(\Xi)$ and u in $H_p^2 \ominus \Xi H_p^2$,

$$\underline{p}S_1^* R_\alpha u = R_\alpha \underline{p}S_1^* u$$

where $\underline{p}S_1^* u$ and $\underline{p}S_1^* R_\alpha u$ are extended to $\Omega(\Xi)$ via (9.8) and (9.9). Thus $\mathcal{M}$ is resolvent invariant, and by Lemma 6.3, equality (6.5) (resp. (6.6)) holds for f, g in $\mathcal{M}$ and α, β first in Ω_+ and then, by analytic continuation, to all of $\Omega(\Xi)$ when $\partial = \mathbb{R}$ (resp. $\partial = \mathbb{T}$).

To conclude, we need to show that $\mathcal{M}$ is a Krein space. Then, an application of Theorem 6.3 if $\partial = \mathbb{R}$, or Theorem 6.4 if $\partial = \mathbb{T}$, will give the asserted form of the reproducing kernel of $\mathcal{M}$. The proof that $\mathcal{M}$ is a Krein space uses the presumed invertibility of the operator defined by (9.2).

Under this assumption, we first remark that an element of $\mathcal{M}$ is uniquely defined by its associated element u in $H_p^2 \ominus \Xi H_p^2$. Indeed, if

$$\begin{pmatrix} \underline{p}S_1^* u \\ \underline{p}S_2^* u \end{pmatrix} = 0$$

then,

$$\Gamma_\Xi u = P_\Xi [S_1, -S_2] \begin{pmatrix} \underline{p}S_1^* u \\ \underline{p}S_2^* u \end{pmatrix} = 0 \ .$$

Therefore, by the presumed invertibility of Γ_Ξ, $u = 0$. This observation allows us to show that $\mathcal{M}$ is a RKKS. Indeed, let

$$F = \begin{pmatrix} \underline{p}S_1^* u \\ \underline{p}S_2^* u \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} \underline{p}S_1^* v \\ \underline{p}S_2^* v \end{pmatrix} \ .$$

Then,

$$[F, G]_{H_{J_0}^2} = < \Gamma_\Xi u, v >_{H_p^2} \ . \tag{9.11}$$

We deduce from this equality that $\mathcal{M}$ is a Krein space. Indeed, let σ be the operator from $\mathcal{M}$ into $\mathcal{M}$ defined by

$$\sigma \begin{pmatrix} \underline{p}S_1^* u \\ \underline{p}S_2^* u \end{pmatrix} = \begin{pmatrix} \underline{p}S_1^* (sgn\ \Gamma_\Xi) u \\ \underline{p}S_2^* (sgn\ \Gamma_\Xi) u \end{pmatrix}$$

where $sgn\ \Gamma_\Xi$ denotes the signum of the bounded self-adjoint operator Γ_Ξ; see e.g. Section 4 for a more detailed discussion of the signum operator. Clearly,

$$\sigma^2 = I \tag{9.12}$$

and for any F and G in $\mathcal{M}$,

$$[F, \sigma G]_{H^2_{J_0}} = [\sigma F, G]_{H^2_{J_0}} = < |\Gamma_\Xi| u, v >_{H^2_p} \tag{9.13}$$

where $|\Gamma_\Xi|$ denotes the absolute value of the operator Γ_Ξ. From (9.13) it follows that $\mathcal{M}$ with the inner product

$$< F, G >_\sigma = [F, \sigma G]_{H^2_{J_0}} \tag{9.14}$$

is a Hilbert space. Indeed, since $|\Gamma_\Xi|$ is bounded and boundedly invertible, there are two constants k_1 and k_2 such that

$$k_2 < u, u >_{H^2_p} \leq < F, F >_\sigma \leq k_1 < u, u >_{H^2_p} \quad , \tag{9.15}$$

and hence $\mathcal{M}$ is readily seen to be complete in the norm induced by (9.14). To conclude that $\mathcal{M}$ is a Krein space, it suffices to check that the operator σ is a signature operator in the Hilbert space $(\mathcal{M}, < \ , \ >_\sigma)$ i.e. satisfies

$$\sigma = \sigma^{-1} = \sigma^* \ .$$

But this is self-evident. Then the relationship

$$[F, G]_{H^2_{J_0}} = < F, \sigma G >_\sigma$$

expresses the fact that $\mathcal{M}$ is a Krein space when endowed with the inner product $[\ ,\ ]_{H^2_{J_0}}$.

We now check that the functional $F \to c^* F(\alpha)$ is bounded for c in $\mathbb{C}_{(p+q)\times 1}$ and α in $\Omega(\Xi)$. Indeed, we first suppose α not on ∂; then writing $c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$,

$$|c^* F(\alpha)| \leq |c_1^* (\underline{p} S_1^* u)(\alpha)| + |c_2^* (\underline{p} S_2^* u)(\alpha)| \ .$$

Using (9.8) and (9.9) we see that there are constants d_1, d_2 (depending on α) such that

$$|(\underline{p} S_i^* u)(\alpha)| \leq d_i \|u\|_{H^2_p} \quad i = 1, 2$$

and hence we conclude that for some constant d (depending on α),

$$|c^* F(\alpha)| \leq d \|u\|_{H^2_p} \ .$$

Using (9.15), we conclude that $F \to c^* F(\alpha)$ is continuous for such α. The case of α on ∂ is treated by an application of the uniform boundedness principle. Since $\Omega(\Xi)$ intersects ∂, the functions of $\mathcal{M}$ are analytic at some point of ∂ and in particular satisfy the hypothesis $(\mathcal{B})$ (see Definition 6.1). Theorem 6.3 (resp. Theorem 6.4) permits us to conclude that $\mathcal{M}$ is a $\mathcal{K}(\Theta)$ space when $\partial = \mathbb{R}$ (resp. $\partial = \mathbb{T}$).

It remains to show that this function Θ is a solution to the inverse scattering problem for $[I_p, -S]$. Let F be in $\mathcal{M}$. Then,

$$[I_p, -S]F = S_1^{-1}\Gamma u$$

and thus by the description of the space $\mathcal{B}(X)$ which is equal to $K(S)$ (see Section 4), $[I_p, -S]F$ belongs to $K(S)$. Moreover, with $\tau F = [I_p, -S]F$,

$$[\tau F, \tau F]_{K(S)} = [F, F]_{\mathcal{M}} = < \Gamma u, u >_{H_p^2} \tag{9.16}$$

and (9.16) demonstrates that τ is isometric. Thus, Θ is a solution to the ISP associated with $[I_p, -S]$.

We collect the preceding discussion in a theorem.

THEOREM 9.1. Let $S = S_1^{-1}S_2$ be a $p \times q$ valued function of bounded type in Ω_+. Let moreover Ξ be a $p \times p$ inner function which is analytic at a point $\delta \in \partial$ and is such that the operator Γ_Ξ defined by (9.2) is invertible. Then the subspace of $H_{J_0}^2$ which consists of functions of the form

$$\begin{pmatrix} \underline{p S_1^* u} \\ \underline{p S_2^* u} \end{pmatrix} \quad u \text{ in } H_p^2 \ominus \Xi H_p^2$$

is a reproducing kernel Krein space of $(p+q) \times 1$ valued functions which are analytic both in Ω_+, and in a neighborhood of δ. Its reproducing kernel is of the form

$$\frac{J_0 - \Theta(\lambda) J_0 \Theta^*(w)}{\rho_w(\lambda)}$$

for a $(p+q) \times (p+q)$ valued function Θ which is a solution to the ISP associated with $[I_p, -S]$.

We notice that all solutions Θ obtained by Theorem 9.1 are such that:

1) The corresponding space $\mathcal{K}(\Theta)$ is isometrically included in H_J^2.

2) The elements of $\mathcal{K}(\Theta)$ have a common point of analyticity on ∂.

It is not yet clear if we get all solutions satisfying 1) and 2) by this method. The particular solutions exhibited in Section 8 correspond to the special case where $H_p^2 \ominus \Xi H_p^2$ has a basis of the form

$$\frac{\xi_1}{\rho_{w_1}}, \ldots, \frac{\xi_r}{\rho_{w_r}} .$$

By using limiting arguments we could get more general solutions, i.e. solutions for which $\mathcal{K}(\Theta)$ is not included in $H_{J_0}^2$, but we will not go into these considerations. Finally we remark that when the operator Γ_Ξ is strictly positive, then $\mathcal{M}$ isa Hilbert space; thus the hypothesis of analyticity at a point of ∂ may be removed since Theorems 6.3 and 6.4 are still valid without this hypothesis.

9.2. The Special Case $X = [\Phi, I_p]$, where Φ is in $\mathcal{C}_p$.

We now present a second method for obtaining a wider class of solutions to the ISP problem for $[I_p, -S]$ when S is restricted to be in the Schur class $S_{p\times p}$. The solutions Θ thus obtained do not necessarily satisfy $\mathcal{K}(\Theta) \subset H^2_{J_0}$. These results have been presented in [AD1] and therefore will only be sketched here.

Rather than $[I_p, -S]$, it is more convenient to consider $X = [\Phi, I_p]$ and $J = J_1$, where $\Phi = (I_p+S)^{-1}(I_p - S)$ (by multiplying S by a constant of modulus 1, we can always suppose that det $(I_p + S)$ does not vanish identically). As explained in [AD1], $[\Phi, I_p]U$ is (Ω_+, J_1) admissible if and only if $[I_p, -S]\Theta$ is (Ω_+, J_0) admissible, where Θ and U are linked by

$$\Theta = MUM^*$$

where $M = \frac{1}{\sqrt{2}}\begin{pmatrix} I_p & I_p \\ I_p & -I_p \end{pmatrix}$ (see Theorem 6.1 of [AD1]).

Let $\partial = \mathbb{R}$. We suppose that Φ may be written as

$$\Phi(\lambda) = \frac{1}{\pi i}\int d\mu(\gamma)\{\frac{1}{\gamma-\lambda} - \frac{\gamma}{\gamma^2+1}\} \ , \ \lambda \in \mathbb{C}_+ \ ,$$

where $d\mu$ is a summable $p \times p$ valued positive measure.

As in Section 6.5, let $\mathcal{M}$ be a reproducing kernel Hilbert space of $p \times 1$ valued functions analytic in some symmetric subspace Δ of $\mathbb{C}$, resolvent invariant, and included isometrically in $L^2_p(d\mu)$, and for f in $\mathcal{M}$ let f_- be defined by

$$f_-(\lambda) = \frac{-1}{\pi i}\int d\mu(\gamma)\{\frac{f(\lambda)-f(\gamma)}{\lambda-\gamma} - \frac{f(\lambda)\gamma}{\gamma^2+1}\} \ , \ \lambda \in \Delta \ . \tag{9.17}$$

For λ in Δ,

$$(\Phi f + f_-)(\lambda) = \frac{1}{\pi i}\int \frac{d\mu(\gamma)f(\gamma)}{\gamma-\lambda} \ .$$

Using the description of $\mathcal{L}(\Phi)$ in terms of $L^2_p(d\mu)$ (see Theorems 5.5), we conclude that $(\Phi f + f_-)$ belongs to $\mathcal{L}(\Phi)$ and that

$$\|\Phi f + f_-\|^2_{\mathcal{L}(\Phi)} = 2\|f\|^2_\mu \ .$$

On the other hand we know (Theorem 6.14) that the space of functions

$$F = \begin{pmatrix} f \\ f_- \end{pmatrix}$$

with the norm

$$\|F\|^2 = 2\|f\|^2_\mu$$

is a $\mathcal{H}_{J_1}(U)$ space for some function U in the class $\mathcal{A}_{J_1}(\Delta)$. Hence the map

$$F \to [\Phi, I_p]F \tag{9.18}$$

maps $\mathcal{H}_{J_1}(U)$ isometrically into $\mathcal{L}(\Phi)$, from which it follows, by Theorem 6.2 of [AD1], that

$$[\Phi, I_p]U \text{ is } (\Delta, J_1) \text{ admissible} .$$

The reader will note a difference of a factor of $\frac{1}{\sqrt{2}}$ between the map (9.18) and the map (6.7), p.626 of [AD1]. This difference arises from a difference of a factor of 2 in the definition of the reproducing kernel for $\mathcal{L}(\Phi)$ in [AD1], and should not be a burden.

Section 10 of [AD1] considers various examples including the case

$$\mathcal{M} = \left\{ \frac{c}{\gamma - a}, \dots, \frac{c}{(\gamma - a)^n} \right\}$$

where a is in $\mathbb{R}$. In this instance the corresponding Θ is a product of n elementary Brune sections (as defined in (6.21)) and we no longer have $\mathcal{H}(\Theta) \subset H^2_{J_0}$.

Thus this method is more general than the method developed in Section 9.1 (but of course is restricted to the case $S \in \mathcal{S}_{p\times p}$). The scope of the method is discussed in Section 7 of [AD1], and the reader is in particular referred to Theorem 7.1 of that section. Finally, we mention that for a function Φ in $\mathcal{C}_p$ of the general form

$$\Phi(\lambda) = A - iB\lambda + \frac{1}{\pi i} \int d\mu(\gamma) \left\{ \frac{1}{\gamma - \lambda} - \frac{\gamma}{\gamma^2 + 1} \right\}$$

the function f_- shall be defined by

$$f_-(\lambda) = -(A - iB\lambda)f(\lambda) + \frac{1}{\pi i} \int d\mu(\gamma) \left\{ \frac{f(\gamma) - f(\lambda)}{\gamma - \lambda} - \frac{f(\lambda)\gamma}{\gamma^2 + 1} \right\} .$$

The reader will note the analogy with formula (9.8).

9.3. A Riccati Equation.

The aim of this subsection is to mention a Riccati equation associated with certain solutions Θ of the ISP for $X = [I_p, -S]$, where S is in $\mathcal{S}_{p\times q}$. We consider the case $\partial = \mathbb{R}$ and will suppose that we have a solution Θ of the form

$$\Theta(\lambda) = \int_0^{\hookrightarrow t} exp\{-\varphi(\lambda)H_u du\} \tag{9.19}$$

where

1) φ is in the Caratheodory class $\mathcal{C}$.
2) $0 < \ell < \infty$ and $\int_0^{\curvearrowright \ell}$ denotes a right multiplicative integral (see [Br], p.143).
3) $u \mapsto H_u$ is a $\mathbb{C}_{m\times m}$ valued measurable function $(m = p+q)$ subject to $H_uJ_0 \geq 0$ and $Tr\ H_uJ_0 = 1$ a.e. on $[0,\ell]$. (The trace condition is a normalization and may be removed.)

Such a function Θ occurs for example in the theory of canonical differential equations (see e.g. [GK]). In fact it follows from a fundamental theorem of Potapov that every J-inner entire Θ admits a representation of the form (9.19) with $\varphi(\lambda) = -i\lambda$ (see [P]). This example also serves to show that Θ does not determine the function H_u uniquely (for an account of this problem when Θ is entire, see [GK], Appendix, Section 4).

The Brune sections considered in Section 6 may also be written this way by choosing $\varphi(\lambda) = \frac{i}{\lambda-\gamma_0}$ for a real number γ_0 and $H_u = H$, constant on $[0,\ell]$ and subject to

$$HJ_0H^* = 0\ .$$

We will denote by $\begin{pmatrix} \alpha_u & \beta_u \\ \gamma_u & \delta_u \end{pmatrix}$ the function H_u where α_u is $p\times p$ valued and will call such a solution Θ a φ-section. The terminology is motivated from network theory. To carry out the indicated calculations it is useful to note that $HJ_0H^* = H^2J_0$ since $HJ_0 = J_0H^*$ (in view of the presumed positivity of HJ_0). Actually the Brune sections of Section 6 are obtained by choosing $H = vv^*J_0$, where v is a J_0 neutral vector.

For $t \in (0,\ell)$, let

$$\Theta_t(\lambda) = \int_0^{\curvearrowright t} exp\{-\varphi(\lambda)H_u du\} = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix}(\lambda)\ .$$

Then, by definition of the multiplicative integral,

$$\frac{\partial}{\partial t}\Theta_t(\lambda) = -\varphi(\lambda)\Theta_t(\lambda)H_t \tag{9.20}$$

from which one can see that if $X\Theta$ is (Ω_+, J_0) admissible, then so is $X\Theta_t$ for any t in $(0,\ell)$. Hence

$$S = T_{\Theta_t}(W_t) \quad t \in (0,\ell)$$

for functions W_t in $S_{p\times q}$. The function $t \to W_t$ will satisfy a Riccati equation; but we first need some more notations. We introduce

$$\begin{aligned} L_t^{(1)} &= A_tW_t + B_t & \qquad L_t^{(2)} &= W_tR_t - P_t \\ R_t^{(1)} &= C_tW_t + D_t & \qquad R_t^{(2)} &= Q_t - W_tU_t \end{aligned}$$

where

$$\Theta_t^{-1} = \begin{pmatrix} P_t & Q_t \\ R_t & U_t \end{pmatrix} .$$

Then

$$S(\lambda) = L_t^{(1)}(R_t^{(1)})^{-1} = (L_t^{(2)})^{-1} R_t^{(2)}$$

for $t \in (0, \ell)$, and we have the following result.

THEOREM 9.2.

a) The function W_t satisfies

$$\frac{\partial W_t}{\partial t}(\lambda) = -\varphi(\lambda)\{-\beta_t + W_t(\lambda)\delta_t - \alpha_t W_t(\lambda) + W_t(\lambda)\gamma_t W_t(\lambda)\} . \tag{9.21}$$

b) The functions $R_t^{(1)}$ and $L_t^{(1)}$ satisfy

$$\frac{\partial Y_t}{\partial t}(\lambda) = -\varphi(\lambda) Y_t(\lambda)\{\gamma_t W_t(\lambda) + \delta_t\} . \tag{9.22}$$

c) The functions $R_t^{(2)}$ and $L_t^{(2)}$ satisfy

$$\frac{\partial Y_t}{\partial t}(\lambda) = -\varphi(\lambda)(W_t(\lambda)\delta_t - \alpha_t) Y_t(\lambda) . \tag{9.23}$$

PROOF. To ease the typography, we remove the notation indicating the dependence upon t and λ. (9.20) leads to:

$$\frac{\partial A}{\partial t} = -\varphi(A\alpha + B\gamma) \tag{9.24}$$

$$\frac{\partial B}{\partial t} = -\varphi(A\beta + B\delta) \tag{9.25}$$

$$\frac{\partial C}{\partial t} = -\varphi(C\alpha + D\gamma) \tag{9.26}$$

$$\frac{\partial D}{\partial t} = -\varphi(C\alpha + D\delta) . \tag{9.27}$$

From $S = (AW + B)(CW + D)^{-1}$, we have

$$S(CW + D) = (AW + B)$$

and therefore, upon differentiating with respect to t, we obtain

$$S\left(\frac{\partial C}{\partial t} W + C\frac{\partial W}{\partial t} + \frac{\partial D}{\partial t}\right) = \left(\frac{\partial A}{\partial t} W + A\frac{\partial W}{\partial t} + \frac{\partial B}{\partial t}\right) .$$

Using $S = (WR - P)^{-1}(Q - WU)$, we get to

$$(Q - WU)\left(\frac{\partial C}{\partial t}W + C\frac{\partial W}{\partial t} + \frac{\partial D}{\partial t}\right) = (WR - P)\left(\frac{\partial A}{\partial t}W + A\frac{\partial W}{\partial t} + \frac{\partial B}{\partial t}\right) .$$

Taking advantage of (9.24)-(9.27) and of

$$\begin{pmatrix} P & Q \\ R & U \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$$

one easily gets the equation (9.21). Equations (9.22) and (9.13) then follow readily, upon taking advantage of (9.21) and (9.24)-(9.27). □

Conversely, equation (9.21) makes sense for every choice of φ, H_t and initial condition $W_0(\lambda) = S(\lambda)$, with S in $\mathcal{S}_{p\times q}$. In general, $W_t(\lambda)$ is not in $\mathcal{S}_{p\times q}$ for $t > 0$. In other words, the φ-section need not be a solution to the ISP associated to $[I_p, -S]$. The condition $W_t(\lambda) \in \mathcal{S}_{p\times q}$ forces relationships between H, φ and S which are under investigation.

For other connections between linear fractional transformations and Riccati equations [Bru] and [O] are suggested.

10. A Model for π-Hermitian Operators

In this section we utilize the techniques developed earlier to establish a model for a class of π-hermitian operators. In particular we shall show, that under suitable restrictions, every π-hermitian operator with deficiency indices (p,p) can be modeled by the operator of multiplication by the complex variable λ in a reproducing kernel Pontryagin space of $p \times 1$ valued meromorphic functions. The precise statement is given in Theorem 10.1 (the definitions of simple hermitian operators, canonical extensions and regular points which appear there, are given in (10.1) and (10.2). In the Hilbert space case, more complete theorems which include the case of isometric operators and hermitian operators, have been developed and presented in [AD2] (Theorems 8.2, 8.3 and 8.4).

It seems to us that the model presented in Theorem 10.1 gives additional insight into the series of papers of Krein and Langer on extension problems (see [KL1],...,[KL4]). In that series, the study of the π-self adjoint extensions (resp. π-unitary extensions) of a simple π-hermitian (resp. π-isometric) operator with equal deficiency indices is shown to provide a unifying framework to the study of various interpolation problems.

THEOREM 10.1. Let H be a densely defined simple π-hermitian operator in a Pontryagin space Π_k, which has a regular real point a and has equal and finite deficiency indices (p,p). Suppose that H admits a π-self adjoint extension H_0 with the point a in its resolvent set. Then there is a function U in $\mathcal{A}^k_{J_1}(\Delta)$, where Δ is a subset of $\mathbb{C}$, symmetric with respect to $\mathbb{R}$ and where J_1 denotes the signature matrix $\begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$ such that H is a π-unitarily equivalent to multiplication by the complex variable in the reproducing kernel Pontryagin space with reproducing kernel

$$K_w(\lambda) = \frac{A(\lambda)B^*(w) + B(\lambda)A^*(w)}{-2\pi i(\lambda - \overline{w})}$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is the decomposition of U into four $p \times p$ valued blocks.

We first review a few facts on π-hermitian operators and π-self adjoint operators. An operator H in a Pontryagin space Π_k is said to be π-hermitian if it is densely defined and if, for any x and y in its domain,

$$[Hx, y] = [x, Hy]$$

where $[\ ,\ ]$ denotes the inner product in Π_k. This can also be written $H \subset H^+$ in the graph sense where H^+ denotes the π-adjoint of H.

Let $\mathcal{M}_z = Ran(H - zI)$ and $\mathcal{N}_{\overline{z}} = \mathcal{M}_z^{[\perp]}$ for z in $\mathbb{C}$. The π-hermitian operator H will be called simple if

$$\Pi_k = \bigvee_{z \in U} \mathcal{N}_z \tag{10.1}$$

for any open subspace U of $\mathbb{C}$ which intersects both $\mathbb{C}_+$ and $\mathbb{C}_-$ ([KL3], p.194).

A point a in $\mathbb{C}$ is said to be regular for H if there is a constant $k(a) > 0$ such that

$$\|(H-a)f\| \geq k(a)\|f\| \tag{10.2}$$

for every f in the domain of H, where $\| \ \|$ denotes any norm associated to $[\ ,\ ]$ as in (1.1). (Recall that all such norms are equivalent.) Regular points are defined and studied in [KL4]. In particular, the set of regular points is open ([KL4], p.391).

The spaces $\mathcal{M}_z$ and $\mathcal{N}_{\bar{z}}$ have a number of properties which are gathered in the next lemma:

LEMMA 10.1. Let H be a π-hermitian operator in a Pontryagin space Π_k. Then,

a) There exists a constant $h > 0$ such that for $|Im\ z| > h$, $\mathcal{N}_z$ is a positive subspace.

b) dim $\mathcal{N}_z$ is constant in $\mathbb{C}_+$ and in $\mathbb{C}_-$ (separately), with the exception of the points which are eigenvalues of H. These are in finite number in $\mathbb{C}_+$ and in $\mathbb{C}_-$.

c) $\mathcal{M}_z$ is closed for z in $\mathbb{C}_+ \cup \mathbb{C}_- \backslash \sigma(H)$.

d) A simple π-hermitian operator has no eigenvalues.

PROOF OF LEMMA 10.1. a) is Theorem 3.2 of [KL2], while b) and c) are proved on page 139 of that same paper. d) is proved in [KL3], p. 197.

The numbers $\mathcal{N}_z$ are called the deficiency indices of H. When dim $\mathcal{N}_z = \{0\}$ for z in $\mathbb{C}_+ \cup \mathbb{C}_- \backslash \sigma(H)$, then $H = H^+$, i.e. the operator is π-self adjoint.

A π-self adjoint operator defined in some Pontryagin space $\tilde{\Pi}_{k'}$ will be an extension of the π-hermitian operator H if its graph is included in the graph of H. If $k = k'$ the extension is said to be regular, while it is called canonical if $\tilde{\Pi}_{k'} = \Pi_k$, i.e. the extension does not go beyond the original space.

Lemma 10.2 gathers a number of facts on π-self adjoint extensions that we will need in the sequel.

LEMMA 10.2.

a) Every π-hermitian operator admits regular π-self adjoint extensions.

b) The spectrum of each regular π-self adjoint extension lies in the strip $|Im\ z| \leq h$, where h is as in Lemma 10.1.

c) A π-hermitian operator has canonical π-self adjoint extensions if and only if it has equal deficiency indices.

PROOF OF LEMMA 10.2. a) and b) are respectively Lemma 2.1 and Theorem 2.1 of [KL2]. c) is in Lemma 2.1; the term "canonic extension" is introduced in [KL3].

The next lemma suggests that our hypothesis on the existence of a self adjoint extension H_0 with the point a in its resolvent set is superfluous.

LEMMA 10.3. Let a be a regular point of the π hermitian operator, operator with deficiency indices $(1,1)$. Then, there is a canonical π-self adjoint extension of H which has the point a in its resolvent set.

PROOF. See [KL4], Behauptung 1.2.

We finally point out that the set of z for which $(\mathcal{N}_z, [\ ,\ ])$ is degenerate, has an involved structure (see [KL2], [KL3]). This set, denoted by Δ_H, has no interior points and $\mathbb{C}_+ \cup \mathbb{C}_- \backslash \Delta_H$ is an open set. Moreover, the signature of $\mathcal{N}_z$ is constant on each component of $\mathbb{C}_+ \cup \mathbb{C}_- \backslash \Delta_H$ (see Theorem 3.3 of [KL2]).

We now outline the proof of Theorem 10.1; the details are then expanded in a number of lemmas. We denote by p the common value of the deficiency indices of H, and we suppose $p < \infty$. The first part of the proof is directly inspired by [K1]; it consists of establishing the existence of a p-dimensional subspace M of Π_k such that

$$M \dot{+} \mathcal{M}_z = \Pi_k \tag{10.3}$$

for z in $\mathbb{C}_+ \cup \mathbb{C}_-$, with the exception of a set of points not accumulating in $\mathbb{C}_+ \cup \mathbb{C}_-$. We will denote the set of non-real z for which (10.3) holds by $\Omega(H)$. Since H is supposed simple, $\mathcal{M}_z$ is closed in $\mathbb{C}_+ \cup \mathbb{C}_-$ by Lemma 10.1 (H has no non-real spectrum), and thus, $\mathcal{M}_z$ is closed for z in $\Omega(H)$. Thus, to any f in Π_k, we can associate a pair of functions $u_f(z)$ and $v_f(z)$, with values in M and in the domain of H, such that

$$f = u_f(z) + (H - z)v_f(z)$$

and it is not difficult to show that, if furthermore f is in the domain H, then

$$u_{Hf}(z) = z u_f(z) \quad z \in \Omega(H) \tag{10.4}$$

from which it is possible (as done in [K1] for the Hilbert space case and in [KL4] for the special case $p = 1$ and $\Omega(H) = \mathbb{C}$) to see that H is unitarily equivalent to multiplication by the complex variable in a resolvent invariant reproducing kernel Pontryagin space of $p \times 1$ valued analytic functions.

The rest of the proof of Theorem 10.1, which differs from Krein's work, is to identify the underlying reproducing kernel Pontryagin space in terms of a $\mathcal{K}(U)$ space, using the tools developed in Section 6.

Let $m_1, \ldots, m_p$ be a basis of M and let us denote by $f_+(z)$, $f_-(z)$ the $p \times 1$ valued functions defined by

$$\xi^* f_+(z) = [u_f(z), \underline{\underline{\xi}}] \tag{10.5}$$

$$\xi^* f_-(z) = [v_f(z), \underline{\underline{\xi}}] \tag{10.6}$$

where $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_p \end{pmatrix}$ is in $\mathbb{C}_{p\times 1}$ and $\underline{\underline{\xi}} = \Sigma \xi_i m_i$.

We will consider the space $\Pi(H)$ of such functions $\begin{pmatrix} f_+ \\ f_- \end{pmatrix}$, and will define an inner product on $\Pi(H)$ via

$$\left[\begin{pmatrix} f_+ \\ f_- \end{pmatrix}, \begin{pmatrix} f_+ \\ f_- \end{pmatrix}\right]_{\Pi(H)} = 2[f,f]_{\Pi_k} \ . \tag{10.7}$$

It turns out that $\Pi(H)$ is a resolvent invariant reproducing kernel Pontryagin space of $2p \times 1$ valued functions analytic in $\Omega(H)$, with a kernel of the form

$$\frac{J_1 - U(\lambda)J_1U^*(w)}{-2\pi i(\lambda - \overline{w})}$$

where the function U is the one alluded to in the statement of the theorem. Finally, we will identify the set of functions f_+ as a reproducing kernel Pontryagin space with the required properties.

This procedure was first developed for the Hilbert space case. That work is reported in [AD2] which will be referred to in a number of places when there are no essential changes in argument.

We first prove the existence of the subspace M and need three lemmas for that purpose. To this end we choose a canonical extension H_0 of H and a point z_+ in $\mathbb{C}_+$ such that with $z_- = \overline{z}_+$

1) $\mathcal{N}_{\overline{z}_+}$ and $\mathcal{N}_{\overline{z}_-}$ are positive definite
2) z_+ and z_- are in the resolvent set of H_0
3) The regular point of H is in the resolvent set of H_0.

Such a choice is possible by the preceding lemmas if $p = 1$ and from the hypothesis on H if p is different from 1. Indeed, we choose H_0 which satisfies 3) either by Lemma 10.3 or by the hypothesis on H. Then we take z_+ outside the strip $|Im\ z| \leq h$ where h is as in Lemma 10.1.

LEMMA 10.4. Let $\varphi_0^1, \ldots, \varphi_0^p$ be a basis of $\mathcal{N}_{\overline{z}_+}$. Then $\varphi^1(z), \ldots, \varphi^p(z)$ is a basis of $\mathcal{N}_{\overline{z}}$ for $\overline{z}$ in $\mathbb{C}_+ \cup \mathbb{C}_- \backslash \sigma(H_0)$, where

$$\varphi^j(z) = (H_0 - \overline{z})^{-1}(H_0 - \overline{z}_+)\varphi_0^j \quad j = 1, \ldots, p \ .$$

PROOF. It is clear that $\varphi^j(z)$ belongs to $\mathcal{N}_{\overline{z}}$ for $j = 1, \ldots, p$. Let us now consider $\alpha_1, \ldots, \alpha_p$ such that

$$\Sigma \alpha_j \varphi^j(z) = 0 \ .$$

Since $\overline{z}$ belongs to $\rho(H_0)$, this forces

$$\Sigma \alpha_j (H_0 - \overline{z}_+)\varphi_0^j = 0$$

and therefore, since $\bar{z}_+$ belongs to $\rho(H_0)$,

$$\Sigma \alpha_j \varphi_0^j = 0 \ .$$

Thus $\alpha_j = 0$, and the p elements $\varphi^1(z), \ldots, \varphi^p(z)$ are a basis of the p-dimensional subspace $\mathcal{N}_{\bar{z}}$. □

LEMMA 10.5. Let $m_j(\varepsilon) = \varphi^j(z_+) + \varepsilon \varphi^j(z_-)$ and let

$$d_\varepsilon(z) = \det([m_j(\varepsilon), \varphi^k(z)])_{j,k} \ .$$

Then there exists an ε_0 such that $d_{\varepsilon_0}(z) \neq 0$ in $\mathbb{C}_+ \cup \mathbb{C}_-$, with the possible exception of a set of isolated points without accumulation points in $\mathbb{C}_+ \cup \mathbb{C}_-$.

PROOF. We first check that one can choose an ε_0 such that

$$d_{\varepsilon_0}(z_+) \neq 0 \quad \text{and} \quad d_{\varepsilon_0}(z_-) \neq 0 \ .$$

Indeed, $d_\varepsilon(z_\pm)$ are two polynomials in ε; $d_\varepsilon(z_+)$ does not vanish at $\varepsilon = 0$, and thus does not vanish identically. On the other hand,

$$\varepsilon^n d_{\frac{1}{\varepsilon}}(z_-) = \det([\varphi^j(z_-) + \varepsilon \varphi^j(z_+), \varphi^k(z_-)])$$

hence

$$\lim_{\varepsilon \to 0} \varepsilon^n d_{\frac{1}{\varepsilon}}(z_-) > 0 = \det([\varphi^j(z_-), \varphi^k(z_-)]) > 0$$

i.e. $\varepsilon^n d_{\frac{1}{\varepsilon}}(z_-)$ is a non-identically vanishing polynomial, and thus so is $d_\varepsilon(z_-)$.

The two polynomials $d_\varepsilon(z_\pm)$ do not vanish identically and so have a common non-zero point, i.e. we can find an ε_0 as required.

Consider now the function $z \to d_{\varepsilon_0}(z)$. It is analytic (in $\bar{z}$) in $\mathbb{C}_+ \cup \mathbb{C}_- \backslash \sigma(H_0)$ and does not vanish in $z_\pm$. Hence it does not vanish identically in $\mathbb{C}_+ \cup \mathbb{C}_- \backslash \sigma(H_0)$; more precisely, it vanishes on a set not accumulating in $\mathbb{C}_+ \cup \mathbb{C}_- \backslash \sigma(H_0)$.

The third lemma needed is taken from Bognar's book [Bo].

LEMMA 10.6. Let X and Y be two p-dimensional subspaces of a Pontryagin space Π_k, and let $\{x_1, \ldots, x_p\}$, (resp. $\{y_1, \ldots, y_p\}$) be a basis of X (resp. Y), and suppose that

$$\det([x_i, y_j]) \neq 0 \ .$$

Then,

1) $X \cap Y^{[\perp]} = X^{[\perp]} \cap Y = \{0\}$.
2) $\Pi_k = X^{[\perp]} \dot{+} Y = X \dot{+} Y^{[\perp]}$.

PROOF. The first claim is clear. The spaces X and Y are then a dual pair of spaces ([Bo], p.20) and the second claim then holds in any inner product space ([Bo], p.23, Lemma 10.8).

We are now in a position to prove the existence of the subspace M. With the given choice of H_0 and z_+, we take a basis $\varphi_0^1, \ldots, \varphi_0^p$ of $\mathcal{N}_{\bar{z}_+}$ and form the functions $m_j(\varepsilon)$ as in Lemma 10.5. We fix an ε_0 for which $d_{\varepsilon_0}(z) \not\equiv 0$. Then, for all the points z at which $d_{\varepsilon_0}(\bar{z}) \neq 0$, the spaces $M = \text{span}\{m_j(\varepsilon_0)\}$ and $\mathcal{N}_{\bar{z}}$ are dual subspaces and therefore, by Lemma 10.4,

$$\Pi_k = M \dot{+} \mathcal{N}_{\bar{z}}^{[\perp]} .$$

To conclude we remark that $\mathcal{N}_{\bar{z}}^{[\perp]} = \mathcal{M}_z^{[\perp][\perp]}$, and thus,

$$\mathcal{N}_{\bar{z}}^{[\perp]} = \mathcal{M}_z$$

since, by Theorem 3.6, p.104 of [Bo], $\mathcal{M}_z$ being closed is equal to $\mathcal{M}_z^{[\perp][\perp]}$. □

The reader will notice that the existence of the specified π-self adjoint extension with a in its resolvent is not needed to prove the existence of M. Any π-self adjoint extension, such that z_+ and $\bar{z}_+$ are in its resolvent set, would do.

The specific choice of H_0 is used now, when we study the space $\Pi(H)$.

From the simplicity of H it is readily checked that $f_+ \equiv 0$ if and only if $f = 0$ and thus, the inner product (10.7) is well defined and $(\Pi(H), [\ ,\ ]_{\Pi(H)})$ is a Pontryagin space.

By the choice of H_0, $(H_0 - z)^{-1}$ is analytic in a neighborhood of the point a and thus, $d_{\varepsilon_0}^{-1}(z)$ is analytic at some real point b possibly different from a. The elements of $\Pi(H)$ are then analytic at b. Indeed, for z in $\Omega(H)$, $f - u_f(z)$ is in $\mathcal{N}_{\bar{z}}^{[\perp]}$ and thus, for $j = 1, \ldots, p$ (provided the m_s are supposed an orthonormal basis of M),

$$[f, \varphi^j(z)] = [u_f(z), \varphi^j(z)] = \sum_{s=1}^{p} [m_s, \varphi^j(z)][u_f(z), m_s] . \tag{10.8}$$

(10.8) serves to establish that f_+ is analytic at b, and similarly for f_-, since $v_f(z)$ is given by the formula

$$v_f(z) = (H_0 - z)^{-1}(f - u_f(z)) . \tag{10.9}$$

Formulas (10.8) and (10.9) also serve to prove that the functionals

$$\begin{pmatrix} f_+ \\ f_- \end{pmatrix} \mapsto c^* \begin{pmatrix} f_+(z) \\ f_-(z) \end{pmatrix}$$

are continuous for c in $\mathbb{C}_{2p\times 1}$ and z in $\Omega(\Xi)$ in a neighborhood of b. Hence $\Pi(H)$ is a reproducing kernel Pontryagin space of $\mathbb{C}_{2p\times 1}$ valued functions which are analytic in $\Omega(H)$ and in a neighborhood of some real point b. It remains to show that $\Pi(H)$ is resolvent invariant and that identity (6.5) with $J = J_1$ holds in $\Pi(H)$. This is done as in ([AD2], Theorem 8.3), and, applying Theorem 6.5, we conclude that $\Pi(H)$ has a reproducing kernel of the form

$$\frac{J_1 - U(\lambda)J_1U^*(w)}{-2\pi i(\lambda - \overline{w})} .$$

The computations are briefly outlined here for completeness.

To prove the resolvent invariance, we first remark that the decompositions

$$\begin{aligned} f &= u_f(\lambda) + (H - \lambda I)v_f(\lambda) \\ &= u_f(w) + (H - wI)v_f(w) \end{aligned}$$

imply that

$$v_f(w) = \frac{u_f(\lambda) - u_f(w)}{\lambda - w} + (H - \lambda I)(\frac{v_f(\lambda) - v_f(w)}{\lambda - w}) \tag{10.10}$$

and hence, that

$$\begin{aligned} [v_f(w)]_+(\lambda) &= (R_w f_+)(\lambda) \\ [v_f(w)]_-(\lambda) &= (R_w f_-)(\lambda) . \end{aligned}$$

Thus, $v_f(w)$ is associated to $R_w\binom{f_+}{f_-}$ via the decomposition (10.3) and $\Pi(H)$ is resolvent invariant. It remains to see that identity (6.5) holds in $\Pi(H)$. We first observe that

$$\begin{aligned} [R_\alpha\binom{f_+}{f_-}, \binom{g_+}{g_-}] &= 2[v_f(\alpha), g] \\ &= 2[v_f(\alpha), g - u_g(\beta)] + 2[v_f(\alpha), u_g(\beta)] . \end{aligned}$$

Moreover,

$$\begin{aligned} [v_f(\alpha), g - u_g(\beta)] &= [v_f(\alpha), (H - \beta I)v_g(\beta)] \\ &= [H - \beta^* I)v_f(\alpha), v_g(\beta)] \\ &= [(H - \alpha I + (\alpha - \beta^*)I)v_f(\alpha), v_g(\beta)] \\ &= [f - u_f(\alpha), v_g(\beta)] + (\alpha - \beta^*)[v_f(\alpha), v_g(\beta)] . \end{aligned}$$

Thus

$$\begin{aligned} [R_\alpha\binom{f_+}{f_-}, \binom{g_+}{g_-}]_{\Pi(H)} &= 2[f - u_f(\alpha), v_g(\beta)] + 2(\alpha - \beta^*)][v_f(\alpha), v_g(\beta)] + 2[v_f(\alpha), u_g(\beta)] \\ &= 2[v_f(\alpha), u_g(\beta)] - 2[u_f(\alpha), v_g(\beta)] \\ &\quad + 2(\alpha - \beta^*)[v_f(\alpha), v_g(\beta)] + 2[f, v_g(\beta)] . \end{aligned}$$

But

$$2[f, v_g(\beta)] = [\binom{f_+}{f_-}, R_\beta\binom{g_+}{g_-}]_{\Pi(H)}$$

and

$$2[v_f(\alpha), v_g(\beta)] = [R_\alpha\binom{f_+}{f_-}, R_\beta\binom{g_+}{g_-}]_{\Pi(H)}$$

and it is not difficult to get identity (6.5) upon expanding $[v_f(\alpha), u_g(\beta)] - [u_f(\alpha), v_f(\beta)]$.

It finally remains to check that $\Pi_+(H)$, the set of functions f_+ when f is in $\Pi(H)$, with inner product

$$[f_+, f_+] = [f, f]_{\Pi_k}$$

is a reproducing kernel Pontryagin space with the required kernel. The proof of these facts are as in Theorems 8.2 and 8.3 of [AD2], and the basis is formula (10.4) and is briefly recalled here.

From (10.4) we know that if f is in the domain of H,

$$(Hf)_+(\lambda) = \lambda f_+(\lambda) \ .$$

Conversely, let (f_+, g_+) be a pair of elements of $\Pi_+(H)$ such that

$$g_+(\lambda) = \lambda f_+(\lambda) \ . \tag{10.11}$$

We want to show that f belongs then to the domain of H. But (10.11) is equivalent to the assumption that

$$\lambda u_f(\lambda) = u_g(\lambda)$$

for some $g \in \Pi_k$ and so the decompositions

$$g - u_g(w) = (H - wI)t \text{ and } f - u_f(w) = (H - wI)q$$

with $t = v_g(w)$ and $q = v_f(w)$ both in the domain of H imply that

$$\begin{aligned}(\lambda - w)u_t(\lambda) &= u_g(\lambda) - u_g(w) \\ &= \lambda u_f(\lambda) - wu_f(w) \\ &= (\lambda - w)u_f(\lambda) + w\{u_f(\lambda) - u_f(w)\} \\ &= (\lambda - w)\{u_f(\lambda) + wu_q(\lambda)\} \ .\end{aligned}$$

Thus

$$t = f + wq$$

which clearly exhibits f as an element of the domain of H. □

REMARK. Theorem 6.5 could be invoked since the elements of $\Pi(H)$ are analytic at the real point b, and in particular satisfy the $(\mathcal{B})$ hypothesis. In the Hilbert space case, it is not necessary to have a real regular point, since the analogue of Theorem 6.5 is valid without assuming the $(\mathcal{B})$ hypothesis.

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מרחבי קריין של פונקציות אנאליסיות
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הוגש למועצה המדעית של מכון וייצמן למדע
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תקציר

מסרת עבודה זו היא לחקור מרחבי קריין מסויימים של פונקציות אנאליסיות עם גרעין משחזר, היחסים בין מרחבים אלה לבין בעייה מסויימת של פיזור הפוך הקשורה בפונקציות מרומורפיות מטיפוס חסום.

באופן כללי, תוצאות אלה מהוות הכללה של מחקר קודם של יישומי תיאוריית דה-בראנז' לבעיית הפיזור ההפוך לפונקציה מטיפוס שור.

בעבודה זו אני מכליל תחילה חלק מתיאוריית דה-בראנז' למרחבי קריין. אחר כך אני מגדיר בעיית פיזור הפוך כללית, הכוללת כמקרה פרטי את בעיית הפיזור ההפוך לפונקציה מטיפוס שור. אני משתמש בתיאוריה המוזכרת לעיל לצורך קבלת פתרונות לבעייה זו.

לבסוף אני מציע מודל לאופראטורים הרמיטיים מסויימים במרחבי פונטריאגין במונחי כפל במשתנה מורכב במרחב פונטריאגין עם גרעין משחזר.

הבעת תודה.

המחקר שהביא לעבודה זו נעשה בהדרכתו של פרופ' הרי דים. לא די במשפטים אחדים כדי להביע את מלוא הכרת תודתי לעזרתו ולהדרכתו התמידיים. פרופ' דים לא חסך מזמנו לדיונים פוריים מאוד שאיפשרו עבודה זו.

כמו כן ברצוני להודות לפרופ' ישראל גוכברג על עזרתו במשך עבודת הדוקטור שלי. הרצאותיו על תורת האופראטורים במכון וייצמן היו מעוררות השראה.

תודה מיוחדת חייב אני לפרופ' לואי דה-בראנז'. עבודה זו חייבת רבות לתיאוריה שלו על מרחבי הילברט של פונקציות אנאליטיות.

לבסוף ברצוני להודות לגב' רובי מצרי על עבודת ההדפסה המעולה שעשתה במשך תקופת הדוקטוראט שלי בכלל ובהדפסת עבודה זו בפרט, לעתים קרובות תחת לחצי זמן גדולים.