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EXACT AND STRONGLY EXACT FILTERS

M.A. MOSHIER, A. PULTR, AND A.L. SUAREZ

Abstract. A meet in a frame is exact if it join-distributes with every element, it is strongly exact if it is preserved by every frame homomorphism. Hence, finite meets are (strongly) exact which leads to the concept of an exact resp. strongly exact filter, a filter closed under exact resp. strongly exact meets. It is known that the exact filters constitute a frame $\mathrm{Filt}_{\mathsf{E}}(L)$ somewhat surprisingly isomorphic to the frame of joins of closed sublocales. In this paper we present a characteristic of the coframe of meets of open sublocales as the dual to the frame of strongly exact filters $\mathrm{Filt}_{\mathsf{sE}}(L)$.

Introduction

The concept of an exact meet in a distributive lattice is fairly intuitive. Think of the lattice of open sets of a topological space; finite meets coincide with intersections, infinite meets typically do not. Those that do so exhibit special behavior, in particular they distribute over joins (that is, $(\bigwedge_i a_i)\vee b = \bigwedge_i(a_i\vee b)$), as the finite ones do. The history of exact meets goes back to MacNeille's dissertation (1935) published in [10]; in Bruns and Lakser [6] they played a role in the study of injective hulls of meets semilattices. The fact that they generalize finite meets naturally leads to the notion of an *exact filter*, an up-set (increasing subset) closed under all exact meets.

The system $\mathrm{Filt}_{\mathsf{E}}(L)$ of all exact filters in a frame L is a frame (a quotient of the frame $\mathfrak{U}(L)$ of all up-sets of L). It turned out ([2]), rather surprisingly, that it was isomorphic to the frame $\mathsf{S}_{\mathsf{c}}(L)$ of the joins of closed sublocales, an important device in studying various phenomena in point-free topology (like e.g. scatteredness, relation of subspaces and sublocales, modelling discontinuity).

The concept of exact meet has a stronger modification, that of a *strongly exact* meet. It appeared (probably) first in 1993 in [16] and

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was later studied e.g. in [4] (see also[11]). The definition will be given below in 2.2, here it suffices to state that the strongly exact meets are precisely those that are preserved by all frame homomorphisms (recall that frame homomorphisms are explicitly requested to preserve finite meets; thus, again, strongly exact meets are a generalization of finite ones). And because $x \mapsto x \vee b$ is a frame homomorphism into the closed sublocale b, these indeed satisfy a stronger condition than exactness. Again, as above, this leads to a natural notion of a *strongly exact filter*. In this paper we prove that the system of all strongly exact filters is also naturally isomorphic to an important system of sublocales, namely the system $\mathsf{S}_{\mathfrak{o}}(L)$ of the *fitted* ones. It should be noted that the question of representation of $\mathsf{S}_{\mathfrak{o}}(L)$ by filters was also opened in [2] (there was found a simple ad hoc characteristic, and a not quite so simple characteristic using a transfinite procedure, none of them pointing in the direction of strong exactness).

Furthermore, we show that the frame of exact filters $\mathrm{Filt}_{\mathsf{E}}(L)$ is a sublocale of the frame of strongly exact ones, $\mathrm{Filt}_{\mathsf{sE}}(L)$. Now this is somewhat strange: The translation of the former to sublocales is covariant while the translation of the latter is contravariant. In the conclusion of this paper we also present an analysis and explanation of this phenomenon.

1. Preliminaries

1.1. We use the standard notation for meets (infima) and joins (suprema) in posets (partially ordered sets): $a \wedge b$, $\bigwedge A$ or $\bigwedge_{a\in A} a$, $a \vee b$, $\bigvee A$ or $\bigvee_{a\in A} a$. Our posets will typically be complete lattices, but we will use the symbols also for the infima and suprema in more general posets in case they exist.

The least resp. largest element (if it exists) will be denoted by 0 resp. 1. Further, we write

$$\uparrow a \text{ for } \{x \mid x \geq a\} \quad\text{and}\quad \uparrow A = \{x \mid \exists\, a \in A,\ x \geq a\}.$$

The subsets $A \subseteq (X, \leq)$ such that $\uparrow A = A$ will be referred to as *up-sets*.

1.2. If X, Y are posets we say that monotone maps $f : X \to Y$ and $g : Y \to X$ are *(Galois) adjoint, f to the left and g to the right*, and write $f \dashv g$, if

$$f(x) \leq y \iff x \leq g(y),$$

equivalently, if $fg \leq \mathrm{id}$ and $gf \geq \mathrm{id}$. It is standard that

(1) left adjoints preserve all existing suprema and right adjoints preserve all existing infima,

(2) and on the other hand, if X, Y are complete lattices then each $f : X \to Y$ preserving all suprema is a left adjoint (has a right adjoint), and each $g : Y \to X$ preserving all infima is a right adjoint.

1.3. A *frame* (*coframe*) is a complete lattice L satisfying the distributivity law

$$\text{(frm)} \qquad \big(\bigvee A\big) \wedge b = \bigvee\{a \wedge b \mid a \in A\}$$

$$\text{(cofrm)} \qquad \Big(\big(\bigwedge A\big) \vee b = \bigwedge\{a \vee b \mid a \in A\}\Big)$$

for all $A \subseteq L$ and $b \in L$. A *frame homomorphism* preserves all joins and all finite meets.

The rule (frm) makes each map $(-) \wedge b$ a left adjoint; consequently a frame has *Heyting structure* with the Heyting operation $\to$ satisfying

$$a \wedge b \le c \quad \text{iff} \quad a \le b \to c$$

(note that, however, a frame homomorphism is not necessarily a Heyting one). Similarly, a coframe has *co-Heyting structure* with the *difference* $c \smallsetminus b$ satisfying

$$a \vee b \ge c \quad \text{iff} \quad a \ge c \smallsetminus b.$$

In particular, in a frame resp. coframe we have the *pseudocomplements* $x^* = x \to 0$ resp. *supplements* $x^\# = 1 \smallsetminus x$ with the De Morgan laws

$$\big(\bigvee a_i\big)^* = \bigwedge a_i^* \quad \text{resp.} \quad \big(\bigwedge a_i\big)^\# = \bigvee a_i^\#.$$

Recall that a *complement* of x in a distributive lattice, that is, a y such that $x \vee y = 1$ and $x \wedge y = 0$, is both a pseudocomplement and a supplement. for this (fairly exceptional) case we will also use the symbol x^*.

1.3.1. The following is a well-known (although seldom explicitly mentioned) fact.

Proposition. *Let b be a complemented element in a distributive lattice L. Then we have $\big(\bigvee A\big) \wedge b = \bigvee\{a \wedge b \mid a \in A\}$ and $\big(\bigwedge A\big) \vee b = \bigwedge\{a \vee b \mid a \in A\}$) for any $A \subseteq L$.*

Proof Immediately follows from the fact that $(-) \wedge b$ has a right adjoint $b^* \vee (-)$, and $(-) \vee b$ has a left adjoint $b^* \wedge (-)$. □

1.4. A typical frame is the lattice $\Omega(X)$ of open sets of a topological space X. For continuous maps $f : X \to Y$ there are frame homomorphisms $\Omega(f) = (U \mapsto f^{-1}[U]) : \Omega(Y) \to \Omega(X)$. Thus we have a contravariant functor

$$\Omega : \mathbf{Top} \to \mathbf{Frm},$$

where **Top** is, of course, the category of topological spaces, and **Frm** that of frames. To make it covariant one considers the category $\mathbf{Frm}^{\mathrm{op}}$, calls it the category of locales, denoted **Loc**, and speaks of the reversed frame homomorphisms as *localic maps*. On the important subcategory of *sober spaces*, the functor $\Omega : \mathbf{Sob} \to \mathbf{Loc}$ is a full embedding. This justifies thinking of frames (locales) as representing generalized spaces, and of localic maps as representing the continuous maps between them.

1.4.1. The category **Loc** is more natural than it first seems. A localic map $f : M \to L$ opposite to a frame homomorphism $h : L \to M$ can be taken concretely to be the right Galois adjoint of h. In this way, **Loc** is also a concrete category.

1.5. Sublocales. In the category **Frm** of frames, extremal epimorphisms are precisely the onto frame homomorphisms (while general epimorphisms are not very transparent). Hence, the extremal monomorphisms in **Loc** (viewed as in 1.4.1) are the one-to-one localic maps. This leads to the following natural approach to subobjects of frames (locales).

A *sublocale* of a frame L is a subset $S \subseteq L$ such that the embedding map $j : S \subseteq L$ is a localic one. It turns out (see e.g. [12]) that such subsets are characterized by the requirements that

(S1) for every $M \subseteq S$ the meet $\bigwedge M$ lies in S, and
(S2) for every $s \in S$ and every $x \in L$, $x \to s$ lies in S.

The system $\mathsf{S}(L)$ of all sublocales of L, ordered by inclusion, is a complete lattice with fairly transparent structure:

$$\bigwedge S_i = \bigcap S_i \quad \text{and} \quad \bigvee S_i = \{\bigwedge M \mid M \subseteq \bigcup S_i\}.$$

(Note that sublocales of L are quotients with the quotient maps adjoint to the embeddings, not subframes: the joins in S typically differ from those in L). The least sublocale $\bigvee \emptyset = \{1\}$ is designated by O and referred to as the *void sublocale*[1]. It is a fundamental fact that

the lattice $\mathsf{S}(L)$ *is a coframe.*

1.5.1. With each element $a \in L$ there is associated an *open sublocale*

$$\mathfrak{o}(a) = \{x \mid x = a \to x\} = \{a \to x \mid x \in L\}$$

[1]This may sound odd but it makes good sense; if L happens to have points, they are sublocales of the form $\{a, 1\}$ with prime $a \neq 1$. So O is even smaller than a point.

and a *closed sublocale*

$$\mathfrak{c}(a) = \uparrow a.$$

In case of a space X (represented as the frame $\Omega(X)$) these precisely correspond to the open and closed subspaces (and to the *open* and *closed parts* from the pioneering article [8]).

One has the following identities (see e.g. [12]):

$$\mathfrak{o}(0) = \mathsf{O},\ \mathfrak{o}(1) = L,\ \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b) \text{ and } \mathfrak{o}(\bigvee_i a_i) = \bigvee_i \mathfrak{o}(a_i),$$

$$\mathfrak{c}(0) = L,\ \mathfrak{c}(1) = \mathsf{O},\ \mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b) \text{ and } \mathfrak{c}(\bigvee_i a_i) = \bigcap_i \mathfrak{c}(a_i).$$

Thus in particular one has a frame embedding $\mathfrak{c} = (a \mapsto \mathfrak{c}(a)) : L \to \mathsf{S}(L)^{\mathrm{op}}$.

1.5.2. The adjoints $h : L \to S$ to the embeddings $j : S \subseteq L$ give rise to two other representations of subobjects of frames. There are the *congruences* $E_j = \{(a,b) \mid h(a) = h(b)\}$ and the *nuclei* $\nu = j \cdot h$. The nuclei $\nu : L \to L$ are monotone maps characterized by the rules

$$\text{(nucl)} \qquad x \le \nu(x), \quad \nu(x \wedge y) = \nu(x) \wedge \nu(y), \quad \text{and} \quad \nu(\nu(x)) = \nu(x).$$

Nuclei (in the natural order) are in a one-to-one antitone relation with sublocales given by

$$S \mapsto \nu_S(x) = \Big(x \mapsto \bigwedge (S \cap \uparrow x)\Big) \quad \text{and} \quad \nu \mapsto S_\nu = \{x \mid x = \nu(x)\}.$$

Thus we have the *frames* $\mathcal{N}(L)$ resp. $\mathfrak{C}(L)$ of nuclei resp. of congruences constituting alternative representations of the system of subobjects of L. Although they are geometrically not as intuitive as the $S(L)$ (where the order is the natural inclusion, and the coframe structure in its coHeyting aspect provides the natural operation $S \smallsetminus T$ modelling the difference of subspaces) they are often very useful.

Open and closed subobjects in these frames are as follows: the open and closed nuclei associated with the open and closed sublocales are

$$\nu_{\mathfrak{o}(a)}(x) = a \to x \quad \text{and} \quad \nu_{\mathfrak{c}(a)}(x) = a \vee x,$$

and in $\mathfrak{C}(L)$ we have the open resp. closed congruences

$$\Delta_a = \{(x,y) \mid x \wedge a = y \wedge a\} \quad \text{resp.} \quad \nabla_a = \{(x,y) \mid x \vee a = y \vee a\}.$$

1.5.3. Proposition. *For every $s \in S$ and every $x \in L$,*

$$x \to s = \nu_S(x) \to s.$$

Proof. Use (S2): $y \le x \to s$ iff $x \le y \to s$ iff $\nu(x) \le y \to s$ iff $y \le \nu(x) \to s$. □

1.6. As introduced in [8], a frame L is said to be *subfit* if every open sublocale is a join of closed sublocales, and *fit* if every sublocale is a meet of open ones.[2] Subfitness and fitness are useful separation axioms; in spaces, the condition (sfit) is slightly weaker than T_1. It is not hereditary, and fitness is precisely its hereditary modification, that is, L is fit iff each of its sublocales is subfit.

For more about frames, in particular more details about sublocales, the reader can consult [9] or [12].

2. Exact and strongly exact

2.1. The technical definition of an exact meet of a subset A of a lattice the reader may know from [3] (the concept goes back to [10]; in [6] it played - under the name of *admissible meet* - a crucial role in the study of injective hulls) is formally different from follows. We use an equivalent, and more transparent characteristic: A meet $\bigwedge A$ in a frame L is *exact* if it distributes over join, that is, if

$$\text{for every } b \in L, \quad (\bigwedge A) \vee b = \bigwedge \{x \vee b \mid x \in A\}.$$

2.1.1. Note. The notion is fairly intuitive. For T_D spaces X, exact meets in $\Omega(X)$ are the intersections of systems of open sets that are open ([4]). See also [15].

2.2. We speak of a *strongly exact meet* $\bigwedge A$ in a frame L if the meet (intersection) $\bigwedge_{x\in A} \mathfrak{o}(x) = \bigcap_{x\in A} \mathfrak{o}(x)$ in $\mathsf{S}(L)$ is an open sublocale.

2.2.1. Notes. 1. We have mentioned the fact that in a wide class of spaces, exact meets are the systems of open sets with open intersections. Here the assumption concerns *sublocales*; although open sublocales correspond precisely to open subsets, their intersections as sublocales do not necessarily correspond to the intersections of subsets.

2. Strongly exact meets appeared in [16] (1994) under the name of *free meets.*

3. The characteristic property of the strongly exact meets is that they are preserved by all frame homomorphisms, that is, $h(\bigwedge A) = \bigwedge h[A]$, although they may be infinite. Though this characterization was one of the main motivations for the notion, and justified the earlier

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[2] This can be rewritten in first order formulas as follows:

(sfit) $\qquad a \not\leq b \implies \exists c,\ a \vee c = 1 \neq b \vee c, \quad \text{and}$

(fit) $\qquad a \not\leq b \implies \exists c,\ a \vee c = 1 \text{ and } c \rightarrow b \neq b.$

name of *free meets*, we wish to emphasize their relation to exact meets, hence the preference for our terminology. We refer the reader to [16] (1994) for a proof of this and several other equivalent characterizations.

2.3. Lemma. *If $a = \bigwedge_{i\in J} a_i$ is strongly exact then $\bigcap_{i\in J} \mathfrak{o}(a_i) = \mathfrak{o}(a)$.*
Proof. We have $\bigcap_{i\in J} \mathfrak{o}(a_i) = \mathfrak{o}(b)$ for some b. Then $\mathfrak{o}(b) \subseteq \mathfrak{o}(a_i)$ for every i, hence $b \leq a_i$ for every i, so that $b \leq a$. On the other hand, $a \leq a_i$ and hence $\mathfrak{o}(a) \subseteq \mathfrak{o}(a_i)$ for every i, so $\mathfrak{o}(a) \subseteq \bigcap \mathfrak{o}(a_i) = \mathfrak{o}(b)$, and $a \leq b$. □

2.4. Proposition. *If $a = \bigwedge_{i\in J} a_i$ is a strongly exact meet then for every b the meet $\bigwedge_{i\in J}(a_i \vee b)$ is also strongly exact.*
Moreover

$$\bigwedge_{i\in J}(a_i \vee b) = (\bigwedge_{i\in J} a_i) \vee b.$$

Consequently, each strongly exact meet is exact.
Proof. We have

$$\bigcap_i \mathfrak{o}(a_i \vee b) = \bigcap_i (\mathfrak{o}(a_i) \vee \mathfrak{o}(b)) = (\bigcap_i \mathfrak{o}(a_i)) \vee \mathfrak{o}(b) =$$
$$= \mathfrak{o}(\bigwedge_i a_i) \vee \mathfrak{o}(b) = \mathfrak{o}((\bigwedge_i a_i) \vee b)$$

open, hence the meet $\bigwedge_i(a_i \vee b)$ is strongly exact. Using 2.3 we obtain that $\bigwedge_i(a_i \vee b) = (\bigwedge_i a_i) \vee b$. □

2.5. Each finite meet is, trivially, strongly exact. Thus we have, for meets in frames, the implications

$$\text{finite} \;\Rightarrow\; \text{strongly exact} \;\Rightarrow\; \text{exact}.$$

A *filter* in L is said to be *exact* resp. *strongly exact* if it is closed under all exact meets resp. all strongly exact ones. Thus, if we denote, in this order, $\mathrm{Filt}_{\mathsf{E}}(L)$, $\mathrm{Filt}_{\mathsf{sE}}(L)$ and $\mathrm{Filt}(L)$, the set of all exact filters, strongly exact filters and general filters, we have that

$$\mathrm{Filt}_{\mathsf{E}}(L) \subseteq \mathrm{Filt}_{\mathsf{sE}}(L) \subseteq \mathrm{Filt}(L).$$

3. Strongly exact filters and fitted sublocales

3.1. Fitted sublocales and fitting. A sublocale is said to be *fitted* if it is an intersection of open sublocales (recall that a frame is fit iff all of its sublocales are subfit iff all of its sublocales are fitted – see [8], or [9, 12]) define the *fitting* by setting

$$S^\circ = \bigcap\{\mathfrak{o}(a) \mid S \subseteq \mathfrak{o}(a)\} = \bigcap\{T \mid T \text{ fit and } S \subseteq T\}$$

Fitting is an operator of a Kuratowski closure type, that is,

$$\mathsf{O}^\circ = \mathsf{O},\ \ S \subseteq T \ \Rightarrow\ S^\circ \subseteq T^\circ,\ \ (S^\circ)^\circ = S^\circ, \quad \text{and} \quad (S \vee T)^\circ = S^\circ \vee T^\circ$$

(for more details on this operator see [7]). Consequently, the system of all fitted sublocales,

$$\mathsf{S}_\mathfrak{o}(L) = \{S \mid S = S^\circ\}$$

is a sub-coframe of $\mathsf{S}(L)$.

3.2. An adjunction. Let L be a frame. Consider the system

$$\mathfrak{U}(L) = \{A \subseteq L \mid \emptyset \neq A = {\uparrow}A\}.$$

In this systems of subsets of L, the joins (with the exception of the void one, which is $\{0\}$) are the unions, and the meets are the intersections. Consequently, it is a frame, and its dual $\mathfrak{U}(L)^{\mathrm{op}}$ is a coframe. Define

$$U : \mathsf{S}(L) \to \mathfrak{U}(L)^{\mathrm{op}} \quad \text{and} \quad M : \mathfrak{U}(L)^{\mathrm{op}} \to \mathsf{S}(L)$$

by setting

$$U(S) = \{a \mid S \subseteq \mathfrak{o}(a)\} \quad \text{and}$$
$$M(A) = \bigwedge\{\mathfrak{o}(a) \mid a \in A\} = \bigcap\{\mathfrak{o}(a) \mid a \in A\}$$

It is easy to check that $S \subseteq MU(S)$ and $A \subseteq UM(A)$ so that we have an adjunction

$$\mathfrak{U}(L)^{\mathrm{op}} \underset{M}{\overset{U}{\leftrightarrows}} \mathsf{S}(L). \quad (\bot)$$

3.2.1. Notes. 1. In the language of congruences we can represent this adjunction as

$$\mathfrak{U}(L) \underset{U}{\overset{M}{\rightleftarrows}} \mathfrak{C}(L) \quad (\bot)$$

with

$$U(C) = \{a \mid aC1\} \quad \text{and}$$
$$M(A) = \bigvee\{\Delta(a) \mid a \in A\}$$

The fixpoints of UM are the upsets that are top equivalence classes of some congruence, while the fixpoints of MU are the congruences determined by their top equivalence class, namely the fitted ones.

2. Since $\mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b)$ each up-set $U(S)$ is a filter. Thus we actually have an adjunction

$$\mathrm{Filt}(L)^{\mathrm{op}} \underset{M}{\overset{U}{\leftrightarrows}} \mathsf{S}(L). \quad (\bot)$$

3.3. The isomorphism induced by this adjunction is between the system of the S such that $MU(S) \subseteq S$ on the one side, and on the other, those $A \in \mathfrak{U}(L)^{\mathrm{op}}$ such that $UM(A) \subseteq A$. The former, more explicitly, consists of the S such that

$$S = \bigcap\{\mathfrak{o}(a) \mid S \subseteq \mathfrak{o}(a)\}.$$

In other words, this is the set of all S satisfying $S = S^{\circ}$, the coframe $\mathsf{S}_{\mathfrak{o}}(L)$ of the fitted sublocales.

3.4. For the up-sets (filters) satisfying $UM(A) \subseteq A$ we obtain the explicit formula

$$\bigcap\{\mathfrak{o}(a) \mid a \in A\} \subseteq \mathfrak{o}(b) \;\Rightarrow\; b \in A. \tag{3.3.1}$$

In [2] such filters were called *fitted* and characterized by a transfinite procedure. One of the main objects of this note is to prove that they are the strongly exact filters.

3.5. Theorem. *A filter satisfies (3.3.1) if and only if it is strongly exact. Thus, the adjunction from 3.1 yields an isomorphism*

$$\iota : \mathrm{Filt}_{\mathsf{sE}}(L)^{\mathrm{op}} \cong \mathsf{S}_{\mathfrak{o}}(L)$$

given by $\iota(A) = \bigcap_{a\in A} \mathfrak{o}(a)$ *and* $\iota^{-1}(S) = \{a \mid S \subseteq \mathfrak{o}(a)\}$.

Proof. $\Rightarrow$: Let A satisfy (3.3.1) and let $a = \bigwedge_i a_i$ be a strongly exact meet with $a_i \in A$. Then for all i, $\bigcap_{a\in A} \mathfrak{o}(a) \subseteq \mathfrak{o}(a_i)$, hence $\bigcap_{a\in A} \mathfrak{o}(a) \subseteq \bigcap_i \mathfrak{o}(a_i) = \mathfrak{o}(a)$, and hence $a \in A$.

$\Leftarrow$: Let $\bigcap_{a\in A} \mathfrak{o}(a) \subseteq \mathfrak{o}(b)$. Then $\mathfrak{o}(b) = (\bigcap_{a\in A} \mathfrak{o}(a)) \vee \mathfrak{o}(b) = \bigcap_{a\in A}(\mathfrak{o}(a) \vee \mathfrak{o}(b)) = \bigcap_{a\in A} \mathfrak{o}(a \vee b)$ and hence the meet $b = \bigwedge_{a\in A}(a \vee b)$ is exact. Since all the $a \vee b$ with $a \in A$ are in A, $b \in A$. ☐

3.6. Corollary. $\mathrm{Filt}_{\mathsf{sE}}(L)$ *is a frame.*

(We will learn more in the next section.)

4. Comparing $\mathrm{Filt}_{\mathsf{sE}}(L)$ with $\mathrm{Filt}_{\mathsf{E}}(L)$

4.1. The Heyting operation in $\mathfrak{U}(L)$. It is given by

$$B \to C = \{x \mid \forall b \in B,\ b \vee x \in C\}.$$

It is a folklore, and straightforward, but since we do not know from where to quote it we will present a three-line proof.

If $A \cap B \subseteq C$ and if $a \in A$ then for every $b \in B$, $a \vee b \geq a, b$, hence in $A \cap B$ and consequently in C, and $a \in B \to C$. If $A \subseteq B \to C$ and if $a \in A \cap B$ then $a \in B \to C$ and $a = a \vee a \in C$.

4.2. Proposition. $\mathrm{Filt}_{\mathsf{sE}}(L)$ *is a sublocale of* $\mathfrak{U}(L)$.

Proof. Obviously $\mathrm{Filt}_{\mathsf{sE}}(L)$ is closed under meets (intersections) in $\mathfrak{U}(L)$. Now let $B \in \mathfrak{U}(L)$ be arbitrary and $C \in \mathrm{Filt}_{\mathsf{sE}}(L)$. Let $a = \bigwedge_{i\in J} a_i$ be a strongly exact meet with all a_i in $B \to C$. Thus,

$$\forall i \in J,\ \forall b \in B,\ a_i \vee b \in C,$$

and $\bigcap_i \mathfrak{o}(a_i) = \mathfrak{o}(a)$. Consider an arbitrary $b \in B$. By 2.4 also $\bigwedge_i(a_i \vee b)$ is strongly exact, and it is a strongly exact meet in $C \in \mathrm{Filt}_{\mathsf{sE}}(L)$, and by 2.4 again this is the same as $a \vee b$ so that $a \vee b = \bigwedge_i(a_i \vee b)$ is in C and since $b \in B$ was arbitrary, a is in $B \to C$. $\square$

4.3. In [5] and [14], and from another point of view in [2], one exploited the adjunction

$$\mathfrak{U}(L) \underset{U'}{\overset{J}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{S}(L)$$

with

$$U' : \mathsf{S}(L) \to \mathfrak{U}(L) \quad \text{and} \quad J : \mathfrak{U}(L) \to \mathsf{S}(L)$$

defined by

$$U'(S) = \{a \mid \uparrow a \subseteq S\} \quad \text{and}$$
$$J(A) = \bigvee\{\uparrow a \mid \uparrow a \subseteq A\} = \{\bigwedge B \mid B \subseteq A\}.$$

If was shown that $U'J$ was a nucleus (for scattered L in [5], then generally in [14]) leading to a proof that the $\mathsf{S}_{\mathfrak{c}}(L) = JU'(L)$ was a frame. One has

$$\mathsf{S}_{\mathfrak{c}}(L) = \{S \in \mathsf{S}(L) \mid S = \bigvee\{\mathfrak{c}(a) \mid \mathfrak{c}(a) \subseteq S\}\}$$

and this extension of L (for subfit L, a Boolean one) plays a role in the study of various phenomena like scatteredness, relation of subspaces and sublocales, or modelling discontinuity – see, e.g. [5, 14, 13]).

4.3.1. The frame $\mathrm{Filt}_{\mathsf{E}}(L)$. Though showing that $JU'(L)$ is the lattice of the joins of closed sublocales (the lattice that was the original motivation of [5]) is straightforward, identifying $U'J(L)$ with the lattice

$$\mathrm{Filt}_{\mathsf{E}}(L)$$

of exact filters is more subtle. It is one of the main results of [2].

4.3.2. Once one has this one can use the nucleus from [5, 14] to conclude that $\mathrm{Filt}_{\mathsf{E}}(L)$ is a sublocale of $\mathfrak{U}(L)$ ([5, 14] and, also using 4.2 (and the trivial fact that if $S \subseteq T \subseteq L$ and S, T are sublocales of L then S is a sublocale of T) conclude that

$$\mathrm{Filt}_{\mathsf{E}}(L) \textit{ is a sublocale of } \mathrm{Filt}_{\mathsf{sE}}(L).$$

4.3.2. Remarks. 1. The last conclusion, however, we can easily make directly.

Note that 4.2 is a direct and very simple proof (independent on 3.5 resp. 3.6) that $\mathrm{Filt}_{\mathsf{sE}}(L)$ is a frame. Similarly we can prove very easily that $\mathrm{Filt}_{\mathsf{E}}(L)$ is a sublocale of $\mathfrak{U}(L)$ and hence a frame. First, we see that we have the following, simpler, parallel to Proposition 2.4: $(\bigwedge(a_i \vee b)) \vee c = ((\bigwedge a_i) \vee b) \vee c) = (\bigwedge a_i) \vee (b \vee c) = \bigwedge(a_i \vee (b \vee c)) = \bigwedge((a_i \vee b) \vee c)$. Next, we can repeat the proof of 4.2, not mentioning the $\bigcap \mathfrak{o}(a_i)$. From this $\mathrm{Filt}_{\mathsf{E}}(L)$ is indeed a sublocale of $\mathfrak{U}(L)$.

2. The reader may wonder whether this may not provide a proof of the fact that $\mathsf{S}_{\mathfrak{c}}(L)$ is a frame simpler than that from [5] resp. [14]. But one should not forget that to finish such proof we have to combine the just made observation with the isomorphism $\mathrm{Filt}_{\mathsf{E}}(L) \cong \mathsf{S}_{\mathfrak{c}}(L)$ which is not quite so easy ([2]).

4.4. The construction in the previous section produced an isomorphism of $\mathrm{Filt}_{\mathsf{sE}}(L)$ with $\mathsf{S}_{\mathfrak{o}}(L)^{\mathrm{op}}$ while the isomorphism in 4.3 is between $\mathrm{Filt}_{\mathsf{E}}(L)$ and $\mathsf{S}_{\mathfrak{c}}(L)$. Both the lattices $\mathsf{S}_{\mathfrak{c}}(L)$ and $\mathsf{S}_{\mathfrak{o}}(L)$ are naturally covariantly embedded into $\mathsf{S}(L)$; the question naturally arises how the sublocale embedding $\mathrm{Filt}_{\mathsf{E}}(L) \subseteq \mathrm{Filt}_{\mathsf{sE}}(L)$ reflects in an embedding of $\mathsf{S}_{\mathfrak{c}}(L)$ into $\mathsf{S}_{\mathfrak{o}}(L)^{\mathrm{op}}$.

4.4.1. The supplement as a map $\mathsf{S}_{\mathfrak{o}}(L)^{\mathrm{op}} \to \mathsf{S}_{\mathfrak{c}}(L)$. Let us realize that, because of the coframe De Morgan law

$$(\bigwedge S_i)^{\#} = \bigvee S_i^{\#}$$

we have a mapping

$$\# = (S \mapsto S^{\#}) : \mathsf{S}_{\mathfrak{o}}(L)^{\mathrm{op}} \to \mathsf{S}_{\mathfrak{c}}(L)$$

and that this mapping preserves all joins.

Since it preserves all joins it has to have a right adjoint h with

$$S^{\#} \subseteq T \quad \text{iff} \quad S \supseteq h(T).$$

For this h we easily derive a formula (S° is "the other closure", the fitting, see 3.1):

We have (in $\mathsf{S}(L)$) $S^{\#} \subseteq T$ iff $S \vee T = L$ iff $T^{\#} \subseteq S$ and hence

$$h(T) = \bigcap\{S \in \mathsf{S}_{\mathfrak{o}}(L) \mid T^{\#} \subseteq S\} = (T^{\#})^{\circ}.$$

We will show that this is what corresponds to the embedding from 4.3.1.

4.4.2. Proposition. *The sublocale embedding* $k : \mathsf{S}_{\mathfrak{c}}(L) \to \mathsf{S}_{\mathfrak{o}}(L)^{\mathrm{op}}$ *corresponding to the embedding* $j : \mathrm{Filt}_{\mathsf{E}}(L) \subseteq \mathrm{Filt}_{\mathsf{sE}}(L)$ *is given by the* *formula*

$$k(S) = (S^{\#})^{\circ}.$$

Proof. Consider the diagram

$$\begin{array}{ccc} \mathrm{Filt}_{\mathsf{sE}}(L) & \xrightarrow[\cong]{\alpha} & \mathsf{S}_{\mathfrak{o}}(L)^{\mathrm{op}} \\ {\scriptstyle j=\subseteq}\uparrow & & \uparrow{\scriptstyle k} \\ \mathrm{Filt}_{\mathsf{E}}(L) & \xleftarrow[\cong]{\beta} & \mathsf{S}_{\mathfrak{c}}(L) \end{array}$$

in which $\beta(S) = \{a \mid \uparrow a \subseteq S\}$, $\alpha(F) = \bigcap\{\mathfrak{o}(a) \mid a \in F\}$, and k is the localic embedding we would like to determine. It is given by

$$k(S) = \beta(\alpha(S)) = \bigcap\{\mathfrak{o}(a) \mid \uparrow a \subseteq S\} = \bigcap\{\mathfrak{o}(a) \mid \mathfrak{o}(a) \vee S = L\}.$$

For $S \in \mathsf{S}_{\mathfrak{c}}(L)$ let $T = S^{\#}$ be the supplement in $\mathsf{S}(L)$ and let $\mathfrak{o}(x) \supseteq T$. Then $\mathfrak{o}(x) \vee S = L$ and hence $\mathfrak{o}(x) \supseteq k(S)$. On the other hand let $\mathfrak{o}(x) \supseteq k(S)$. Then by the coframe distributivity, $\mathfrak{o}(x) \vee S = \bigcap\{\mathfrak{o}(a) \vee S \mid \mathfrak{o}(a) \vee S = L\} = L$ and hence $\mathfrak{o}(a) \supseteq T$. Thus, for open $\mathfrak{o}(x)$, $\mathfrak{o}(x) \supseteq T$ iff $\mathfrak{o}(x) \supseteq k(S)$ so that the two sublocales have the same fitting. Since one of them is already fitted we have $T^{\circ} = k(S)^{\circ} = k(S)$. □

5. A survey of the situation

5.1. From 4.3.1 and 4.2. we see that we have a sequence of sublocales

$$\mathrm{Filt}_{\mathsf{E}}(L) \subseteq \mathrm{Filt}_{\mathsf{sE}}(L) \subseteq \mathrm{Filt}(L) \subseteq \mathfrak{U}(L). \tag{5.1.1}$$

5.1.1. Notes. 1. For $\mathrm{Filt}(L)$ the sublocale embedding proof is straightforward and for $\mathrm{Filt}_{\mathsf{sE}}(L)$ it is based on direct reasoning. The direct proof known from the literature for $\mathrm{Filt}_{\mathsf{E}}(L)$ ([5, 14]) is not quite so easy. Having factored this via $\mathrm{Filt}_{\mathsf{sE}}(L)$ simplifies things.

2. Another distinction between the frames in the sequence is that while it is easy to see that $\mathrm{Filt}(L)$ and $\mathrm{Filt}_{\mathsf{sE}}(L)$ are in general not coframes (for fit L, $\mathrm{Filt}_{\mathsf{sE}}(L)^{\mathrm{op}}$ is isomorphic to the whole of $\mathsf{S}(L)$, typically not a frame, for $\mathrm{Filt}(L)$ see the example below), the question whether $\mathrm{Filt}_{\mathsf{E}}(L)$ is not a coframe is an open problem: for subfit L, $\mathrm{Filt}_{\mathsf{E}}(L)$ is a Boolean algebra (in fact, precisely for the subfit L), and in other examples we know it is non-Boolean frame-and-coframe.

3. An example of a non-coframe $\mathrm{Filt}(L)$ is provided already by $L = \Omega(\mathbb{I})$ where $\mathbb{I}$ is the compact unit interval $[0, 1]$. Consider the filters

$$\mathcal{F}_n = \{U \mid U \supseteq (\frac{1}{n}, 1]\}, \quad n = 1, 2, \ldots,$$
$$\mathcal{G} = \{U \mid \exists k, U \supseteq [0, \frac{1}{k})\}.$$

Then $\bigcap_n \mathcal{F}_n = \{(0, 1], \mathbb{I}\}$ and hence

$$(\bigcap_n \mathcal{F}_n) \vee \mathcal{G} = \mathcal{G} \neq L$$

while for each n, $\mathcal{F}_n \vee \mathcal{G} \ni \emptyset$, hence $\mathcal{F}_n \vee \mathcal{G} = L$ and hence

$$\bigcap_n (\mathcal{F}_n \vee \mathcal{G}) = L.$$

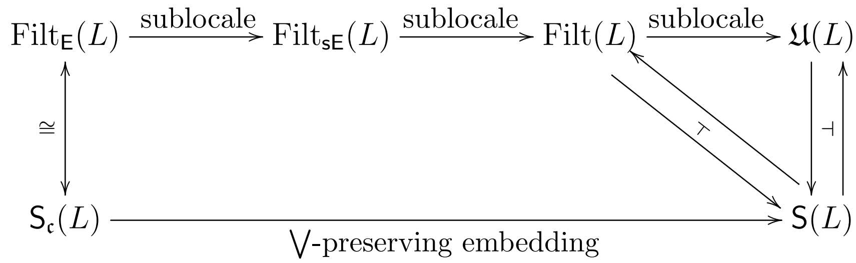
5.2. While the "algebraic line" of the constructions goes in (5.1.1) smoothly from very special up-sets to more and more general ones (the upper line in the following diagram), the geometric line, the lower one in the diagram, is not quite so transparent.

$$\begin{array}{ccccccc}
\mathrm{Filt}_{\mathsf{E}}(L) & \xrightarrow{\text{sublocale}} & \mathrm{Filt}_{\mathsf{sE}}(L) & \xrightarrow{\text{sublocale}} & \mathrm{Filt}(L) & \xrightarrow{\text{sublocale}} & \mathfrak{U}(L) \\
\cong \updownarrow & & \updownarrow \cong & & & & \\
\mathsf{S}_{\mathfrak{c}}(L) & \xrightarrow[\text{sublocale}]{} & \mathsf{S}_{\mathfrak{o}}(L)^{\mathrm{op}} & & \xrightarrow[\text{subframe}]{} & & \mathsf{S}(L)^{\mathrm{op}}.
\end{array}$$

Note that while the upper embeddings are sublocalic, in the lower line we have a sublocale and a subframe, and also that the relation of the ends is in a way peculiar: both $\mathfrak{U}(L)$ and $\mathsf{S}(L)^{\mathrm{op}}$ are frames, but the orders are opposite to each other when both are regarded as sub-posets of the powerset of L.

It may be of some interest to look for a point-free interpretation of $\mathrm{Filt}(L)$. The $\mathrm{Filt}_{\mathsf{E}}(L)$ and $\mathrm{Filt}_{\mathsf{sE}}(L)$ have been interpreted in [2] and this article and $\mathfrak{U}(L)$ is the free frame on the semilattice $(L, \vee)$. For $\mathrm{Filt}(L)$ we do not know, but as we have seen in 3.2.1 and will recall in the next subsection, it does somehow naturally fit into the picture.

5.3. In the adjunction $J \vdash U'$ from [2] (the adjunction leading to the natural isomorphism $\mathrm{Filt}_{\mathsf{E}}(L) \cong \mathsf{S}_{\mathfrak{c}}(L)$ in [14], the right side of the diagram below) we have the lower line of the diagram from 5.2 replaced by a *covariant* join-preserving embedding $\mathsf{S}_{\mathfrak{c}}(L) \to \mathsf{S}(L)$ (this embedding has some useful properties, in particular in the subfit case – see [14, 13]).



The other adjunction indicated is the reduction to $\mathrm{Filt}(L)$ as in 3.2.1.

All the frames and coframes indicated in the two diagrams are various extensions of the original frame L. Some of them have a geometric interpretation ($\mathsf{S}_{\mathfrak{c}}(L)$, $\mathsf{S}_{\mathfrak{o}}(L)$, $S(L)$, $\mathsf{S}(L)^{\mathrm{op}}$ – the two last ones have different role, the geometry of subobjects resp. space of subobjects), some both an algebraic and a geometric one ($\mathrm{Filt}_{\mathsf{E}}(L)$, $\mathrm{Filt}_{\mathsf{sE}}(L)$), and $\mathfrak{U}(L)$ as a free algebra has an algebraic one. How the $\mathrm{Filt}(L)$ is to be interpreted (and used) is not quite clear.

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