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## Asset Price Volatility and Price Extrema

### Comments

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## ASSET PRICE VOLATILITY AND PRICE EXTREMA

CAREY CAGINALP AND GUNDUZ CAGINALP

**ABSTRACT.** The relationship between price volatility and a market extremum is examined using a fundamental economics model of supply and demand. By examining randomness through a microeconomic setting, we obtain the implications of randomness in the supply and demand, rather than assuming that price has randomness on an empirical basis. Within a very general setting the volatility has an extremum that precedes the extremum of the price. A key issue is that randomness arises from the supply and demand, and the variance in the stochastic differential equation governing the logarithm of price must reflect this. Analogous results are obtained by further assuming that the supply and demand are dependent on the deviation from fundamental value of the asset.

### 1. INTRODUCTION

**1.1. Overview.** In financial markets two basic entities are the expected relative price change and volatility. The latter is defined as the standard deviation of relative price change in a specified time period. The expected relative price change is, of course, at the heart of finance, while volatility is central to assessing risk in a portfolio. Volatility plays a central role in the pricing of options, which are contracts whereby the owner acquires the right, but not the obligation, to buy or sell at a particular price within a specified time interval.

In classical finance, it is generally assumed that relative price change is random, but volatility is essentially constant for a particular asset [1].

In this way, price change and volatility are essentially decoupled in their treatment. In particular, the relative price change per unit time  $P^{-1}dP/dt = d \log P/dt$  is given by a sum of a deterministic term that expresses the long term estimate for the growth, together with a stochastic term given by Brownian motion.

Hence, the basic starting point for much of classical finance, particularly options pricing (see e.g., [2, 3]), is the stochastic equation for  $\log P$  as a function of  $\omega \in \Omega$  (the sample space) and  $t$  given by

$$(1.1) \quad d \log P = \mu dt + \sigma dW.$$

where  $W$  is Brownian motion, with  $\Delta W := W(t) - W(t - \Delta t) \sim \mathcal{N}(0, \Delta t)$ , so  $W$  is normal with variance  $\Delta t$ , mean 0, and independent increments (see [4, 5]). While  $\mu$  and  $\sigma$  are often assumed to be constant, one can also stipulate deterministic and time dependent or stochastic  $\mu$  and  $\sigma$ . The stochastic differential equation above is

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short for the integral form (suppressing  $\omega$  in notation) for arbitrary  $t_1 < t_2$

$$(1.2) \quad \log P(t_2) - \log P(t_1) = \int_{t_1}^{t_2} \mu dt + \int_{t_1}^{t_2} \sigma dW$$

For  $\mu, \sigma$  constant, and  $\Delta t := t_2 - t_1$ , one can write

$$(1.3) \quad \Delta \log P := \log P(t_2) - \log P(t_1) = \mu \Delta t + \sigma \Delta W.$$

The classical equation (1.1) can be regarded as partly an empirical model based on observations about volatility of prices. It also expresses the theoretical construct of infinite arbitrage that eliminates significant distortions from the expected return of the asset as a consequence of rational comparison with other assets such as risk free government (i.e., Treasury) bonds. Hence, this equation can be regarded as a limiting case (as supply and demand approach infinity) of other equations involving finite supply and demand [6] (Appendix A). Thus, it does not lend itself to modification based upon random changes in finite supply and demand. An examination of the relationship between volatility and price trends, tops and bottoms requires analysis of the more fundamental equations involving price change. A suitable framework for analyzing these problems is the asset flow approach based on supply/demand that have been studied in [7, 8, 9, 10], and references therein.

An intriguing question that we address is the following. Suppose there is an event that is highly favorable for the fundamentals of an asset. There is the expectation that there will be a peak and a turning point, but no one knows when that will occur. By observing the volatility of price, can one determine whether, and when, a peak will occur in the future? In general, our goal is to delve deeper into the price change mechanism to understand the relationship between relative price change and volatility.

Our starting point will be the basic supply/demand model of economics (see e.g., [11, 12, 13]). We argue that there is always randomness in supply and demand. However, for a *given* supply and demand, one cannot expect nearly the same level of randomness in the resulting price. Indeed, for actively traded equities, there are many market makers whose living consists of exploiting any price deviations from the optimal price determined by the supply/demand curves at that moment. While there will be no shortage of different opinions on the long term prospects of an investment, each particular change in the supply/demand curve will produce a clear, repeatable short term change in the price.

Given the broad validity of the Central Limit Theorem, one can expect that the randomness in supply and demand of an actively traded asset on a given, small time interval will be normally distributed. Thus, supply and demand can be regarded as bivariate normally distributed random variables, with a correlation that will be close to  $-1$  since the random factors that increase demand tend to decrease supply.

In Sections 2 and 3 we explore the implications of this basic price equation that involves the ratio of demand/supply. By assuming that the supply and demand are normally distributed with a ratio of means that are characterized by a maximum, we prove that an maximum in the expectation of the price is preceded by an extremum in the price volatility. This means that given a situation in which one expects a market bottom based on fundamentals, the variance or volatility can be a forecasting tool for the extremum in the trading price. Furthermore, in pricing options, this approach shows that the assumption of constant volatility can be

improved by understanding the relationship between the variance in price and the peaks and nadirs of expected price.

Subsequently, in Section 3, we generalize the dependence on demand/supply in the basic model, and find that under a broad set of conditions one has nevertheless the result that the extremum in variance precedes the expected price extremum.

In Section 4 we introduce the concept of price change that depends on supply and demand through the fundamental value. The trader motivations are assumed to be classical in that they depend only on fundamental value; however, the price equation involves the finiteness of assets, which is a non-classical concept. Without introducing non-classical concepts such as the dependence of supply and demand on price trend, we obtain a similar relationship between the volatility and the expected price.

In Section 5, we prove that within the assumptions of this model and generalizations, the peak of the expected log price occurs after the peak in volatility.

**1.2. General Supply/Demand model and stochastics.** We write the general price change model in terms of the price,  $P$ , the demand,  $\tilde{D}$ , and supply,  $\tilde{S}$ . In particular, the relative price change is equal to a function of the excess demand,  $(\tilde{D} - \tilde{S})/\tilde{S}$  (see e.g., [11], [12]). That is, we have

$$(1.4) \quad P^{-1}dP/dt = G\left(\tilde{D}/\tilde{S}\right)$$

where  $G : \mathbb{R}^+ \rightarrow \mathbb{R}$  satisfies (a)  $G(1) = 0$ , (b)  $G'(x) > 0$  for all  $x \in \mathbb{R}^+$ . If symmetry between  $\tilde{D}$  and  $\tilde{S}$  is assumed, then one can also impose (c)  $G(1/x) = -G(x)$ . A prototype function with properties (a) – (c) is given by  $G(x) := x - 1/x$ .

A basic stochastic process based on (1.4) for log  $P$  is defined by

$$(1.5) \quad d \log P(t, \omega) = a(t, \omega) dt + b(t, \omega) dW(t, \omega)$$

for some functions  $a$  and  $b$  in  $H_2[0, T]$ , the space of stochastic processes with a second moment integrable on  $[0, T]$  (see [5]). The terms  $a(t, \omega)$  and  $b(t, \omega)$  can be identified from  $G$  and the nature of randomness that is assumed. In any time interval  $\Delta t$ , there is a random term in  $\tilde{D}$  and  $\tilde{S}$ . The assumption is that there are a number of agents who are motivated to place buy orders. The relative fraction is subject to randomness so that the deterministic demand,  $D(t)$ , multiplied by  $1 + \frac{\sigma}{2}R(t; \omega)$  for some random variable  $R(t; \omega)$ . Likewise, one has the deterministic supply,  $S(t)$ , by  $1 - \frac{\sigma}{2}R(t; \omega)$ . This yields, for sufficiently small  $\sigma$ , the approximation

$$(1.6) \quad \frac{D(t; \omega)}{S(t; \omega)} - 1 \rightarrow \frac{D(t) \{1 + \frac{\sigma}{2}R\}}{S(t) \{1 - \frac{\sigma}{2}R\}} - 1 \doteq \frac{D(t)}{S(t)} - 1 + \frac{D(t)}{S(t)} \sigma R,$$

with  $\sigma$  being either constant, time dependent or stochastic. We can then write

$$G\left(\tilde{D}/\tilde{S}\right) \doteq G(D/S) + G'(D/S) \left(\sigma \frac{D}{S} R\right)$$

and thereby identify  $a(t; \omega) = G(D/S)$  and  $b(t; \omega) = \sigma \frac{D}{S} G'(D/S)$ . Note that we view the randomness as arising only from the  $\sigma R$  term, so we can assume that  $D$  and  $S$  are deterministic functions of  $t$  at this point. Later on in this paper we consider additional dependence on  $D$  and  $S$ . By assuming that the random variable  $R$  is normal with variance  $\Delta t$  and  $w(t + \Delta t) - w(t)$  is independent of

$w(t) - w(t + \Delta t)$ , one obtains the stochastic process below (in which  $D(t)$  and  $S(t)$  are deterministic).

By differentiating (c), we note

$$\frac{1}{x}G' \left( \frac{1}{x} \right) = xG'(x),$$

and thereby write the stochastic differential equation

$$d \log P(t, \omega) = G(D/S) dt + \frac{1}{2} \left\{ \frac{D}{S} G' \left( \frac{D}{S} \right) + \frac{S}{D} G' \left( \frac{S}{D} \right) \right\} dW(t, \omega).$$

In particular, for  $G(x) := x - 1/x$  one has

$$d \log P = \left( \frac{D}{S} - \frac{S}{D} \right) dt + \sigma \left\{ \frac{D}{S} + \frac{S}{D} \right\} dW.$$

We are interested in the relationship between volatility and market extrema, and focus on market tops by using the simpler equation for the function  $G(x) := x - 1$  for which (c) holds only approximately near  $D/S = 1$ . The equation is then (see Appendix)

$$(1.7) \quad d \log P(t, \omega) = \left( \frac{D(t)}{S(t)} - 1 \right) dt + \sigma(t, \omega) \frac{D(t)}{S(t)} dW(t, \omega).$$

For market bottoms, one can obtain similar results (see Appendix).

We will specialize to  $\sigma$  deterministic or even constant below. If we were to assume that the supply and demand have randomness that is not necessarily the negative of one another, then we can write instead,

$$(1.8) \quad \frac{D(1 + \sigma_a R_a)}{S(1 - \sigma_b R_b)} \cong (1 + \sigma_a R_a + \sigma_b R_b) \frac{D}{S} - 1.$$

yielding the analogous stochastic process,

$$(1.9) \quad d \log P(t, \omega) = \left( \frac{D(t)}{S(t)} - 1 \right) dt + \frac{D(t)}{S(t)} \{ \sigma_a dW_a + \sigma_b dW_b \}.$$

**1.3. Derivation of the stochastic equation.** We make precise the ideas above by starting again with (1.4) where  $D(t; \omega)$  and  $S(t; \omega)$  are random variables that are anticorrelated bivariate normals with means  $\mu_D(t)$  and  $\mu_S(t)$  and both have variance  $\sigma_1^2$ . We can regard the means as the deterministic part of the supply and demand at any time  $t$ , so that with  $\Sigma$  as the covariance matrix [14], we write

$$(1.10) \quad (D(t; \omega), S(t; \omega)) \sim \mathcal{N}(\vec{\mu}(t), \Sigma) \quad \text{with} \quad \vec{\mu} := (\mu_D, \mu_S), \quad \Sigma := \begin{pmatrix} \sigma_1^2(t) & -1 \\ -1 & \sigma_1^2(t) \end{pmatrix}.$$

For any fixed  $t$ , one can show that the density of  $D/S$  is given by

$$(1.11) \quad f_{D/S}(x) = \frac{1 + \mu_D/\mu_S}{\sqrt{2\pi} \frac{\sigma_1}{\mu_S} (x+1)^2} e^{-\frac{1}{2} \left( \frac{\sigma_1}{\mu_S} \right)^2 \frac{(x - \mu_D/\mu_S)^2}{(x+1)^2}}.$$

Other approximations in different settings have been studied in [15, 16, 17] and references therein.

For values of  $x$  near the mean of  $D/S$ , one has

$$(1.12) \quad (x+1)^2 \doteq \left( \frac{\mu_D}{\mu_S} + 1 \right)^2.$$

We can use this to approximate the density, using  $\sigma_{R_q}^2 := \frac{\sigma_1^2}{\mu_S^2} \left( \frac{\mu_D}{\mu_S} + 1 \right)^2$  as the approximate variance of  $D/S$ , as

$$(1.13) \quad f_{D/S}(x) \doteq \frac{1}{\sqrt{2\pi\sigma_{R_q}}} e^{-\frac{(x-\mu_D/\mu_S)^2}{2\sigma_{R_q}^2}}; \quad f_{\frac{D}{S}-1}(x) \doteq \frac{1}{\sqrt{2\pi\sigma_{R_q}}} e^{-\frac{(x-\mu_D/\mu_S+1)^2}{2\sigma_{R_q}^2}}.$$

With this expression for the density of  $R_1 := D/S - 1$ , we can write the basic supply/demand price change equation as

$$(1.14) \quad \frac{\Delta \log P}{\Delta t} \doteq R_1 \sim \mathcal{N} \left( \frac{\mu_D}{\mu_S} - 1, \sigma_{R_q}^2 \right),$$

where each variable depends on  $t$  and  $\omega$ . Subtracting out the  $\frac{\mu_D}{\mu_S} - 1$ , defining  $R_0 \sim \mathcal{N} \left( 0, \sigma_{R_q}^2 \right)$ , and noting that  $R_0$  depends on  $t$  through  $\sigma_{R_q}^2$ , we write

$$(1.15) \quad \Delta \log P \doteq \left( \frac{\mu_D}{\mu_S} - 1 \right) \Delta t + \sigma_{R_q} R_0 \Delta t.$$

By definition of Brownian motion, we can write

$$(1.16) \quad \Delta \log P \doteq \left( \frac{\mu_D}{\mu_S} - 1 \right) \Delta t + \sigma_{R_q} \Delta W.$$

With  $\sigma_1$  and  $\mu_D$  held constant, an increase in  $\mu_S$  leads to a decrease in the variance  $\sigma_{R_q}$ . We would like to approximate this under the condition that  $\mu_D/\mu_S \approx 1$ . By rescaling the units of  $\mu_D, \mu_S, \sigma_1$  together and assuming that each of  $\mu_D$  and  $\mu_S$  are sufficiently close to 1 that we can consider the leading terms in a Taylor expansion, and write

$$(1.17) \quad \mu_D = 1 + \delta_D, \quad \mu_S = 1 + \delta_S.$$

Note that  $\mu_D$  and  $\mu_S$  will be nearly equal unless one is far from equilibrium. Ignoring the terms higher than first order one has

$$(1.18) \quad \begin{aligned} \sigma_{R_q}^2 &= \frac{\sigma_1^2}{(1 + \delta_S)^2} \left( 1 + \frac{1 + \delta_D}{1 + \delta_S} \right)^2 \\ &\doteq 4\sigma_1^2 (1 - 3\delta_S + \delta_D). \end{aligned}$$

We are considering  $-\delta_S = \delta_D =: \delta$  so that

$$(1.19) \quad \sigma_{R_q}^2 = 4\sigma_1^2 (1 + 4\delta).$$

Using Taylor series approximation, one has

$$(1.20) \quad \left( \frac{\mu_D}{\mu_S} \right)^2 = \left( \frac{1 + \delta}{1 - \delta} \right)^2 \doteq 1 + 4\delta.$$

We can thus write the stochastic equation above as

$$(1.21) \quad \Delta \log P \doteq \left( \frac{\mu_D}{\mu_S} - 1 \right) \Delta t + 2\sigma_1 \frac{\mu_D}{\mu_S} \Delta W,$$

so that the differential form is given in terms of  $f := \mu_D/\mu_S - 1$  by

$$(1.22) \quad d \log P(t) = f(t) dt + \sigma(f(t) + 1) dW(t)$$

This is in agreement with the heuristic derivation above, with  $\sigma = 2\sigma_1$  and  $\sigma_1^2$  as the variance of each of  $S$  and  $D$ .

## 2. LOCATION OF MAXIMA OF SUPPLY/DEMAND VERSUS PRICE

**2.1. The deterministic model.** We will show that if  $D/S - 1$  is given by a deterministic function  $f$ , then the stochastic equation above will imply that the variance over a small time interval  $\Delta t$  will have an extremum before the price has its extremum.

Once we do this simplest case, it will generalize it to the situation where  $f := D/S - 1$  is also stochastic, and show that the same result holds.

To this end, first consider the simple, purely deterministic case:

$$(2.1) \quad P^{-1} \frac{dP}{dt} = \frac{D}{S} - 1 =: f, \text{ i.e., } \frac{d}{dt} \log P(t) = f(t)$$

Assume that  $f$  is a prescribed function of  $t$  that is  $C^1(I)$  for  $I \supset (t_0, \infty) \supset (t_a, t_b)$  satisfying:

- (i)  $f(t) > 0$  on  $(t_a, t_b)$ ,  $f(t) < 0$  on  $I \setminus (t_a, t_b)$  and  $f + 1 > 0$  on  $I$ ;
- (ii)  $f'(t) > 0$  if  $t < t_m$ ,  $f'(t) < 0$  if  $t > t_m$ ,  $f'(t_m) = 0$ ;
- (iii)  $f''(t) < 0$  if  $t \in (t_a, t_b)$ .

Then  $\log P(t)$  is increasing on  $t \in (t_a, t_b)$  and decreasing on  $t \in (t_b, \infty)$  and has a maximum at  $t_b$ .

In other words, the peak of  $f$  occurs at  $t_m$  while the peak of  $\log P$  is attained at  $t_b > t_m$ . This demonstrates the simple idea that price peaks some time after the peak in demand/supply. In fact, during pioneering experiments Smith, Suchanek and Williams [18] observed that bids tend to dry up shortly before a market peak. Also, the important role of the ratio of cash to asset value in a market bubble that was predicted in [7] was confirmed in experiments starting with [8].

**2.2. The stochastic model.** Recall that  $\mu_D$  and  $\mu_S$  are deterministic functions of time only. We model the problem as discussed above so the only randomness below is in the  $dW$  variable. The stochastic equation given by (1.22) for a continuous function  $f := \mu_D/\mu_S - 1$ , in the integral form, for any  $t_1 < t_2$  and  $\Delta \log P := \log P(t_2) - \log P(t_1)$  is

$$(2.2) \quad \Delta \log P = \int_{t_1}^{t_2} f(z) dz + \int_{t_1}^{t_2} \sigma(z) (f(z) + 1) dW(z).$$

Note that for the time being we are assuming that  $\sigma$  and  $f$  may depend on time but are deterministic. We compute the expectation<sup>1</sup> and variance of this quantity:

$$(2.3) \quad E[\Delta \log P] = \int_{t_1}^{t_2} f(z) dz$$

since  $f$  is deterministic and  $E[dW] = 0$ ;

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<sup>1</sup>We let  $\mathbb{E}[Y]^2$  denote  $\mathbb{E}[(Y^2)]$ .



$$(2.4) \quad \begin{aligned} \text{Var} [\Delta \log P] &= E \left[ \int_{t_1}^{t_2} f(z) dz + \int_{t_1}^{t_2} \sigma(z) \{f(z) + 1\} dW(z) \right]^2 \\ &\quad - \left( E \left[ \int_{t_1}^{t_2} f(z) dz + \int_{t_1}^{t_2} \sigma(z) \{f(z) + 1\} dW(z) \right] \right)^2. \end{aligned}$$

The  $\int f(z) dz$  term is deterministic and vanishes when its expectation is subtracted. The expectation of the  $dW$  and the  $dzdW$  terms vanishes also. We are left with

$$(2.5) \quad \begin{aligned} \text{Var} [\Delta \log P] &= E \left[ \int_{t_1}^{t_2} \sigma(z) \{f(z) + 1\} dW(z) \right]^2 \\ &= \int_{t_1}^{t_2} \sigma^2(z) \{f(z) + 1\}^2 dz \end{aligned}$$

using the standard result ([5], p. 68).

We want to consider a small interval  $(t, t + \Delta t)$  so we set  $t_1 \rightarrow t$  and  $t_2 \rightarrow t + \Delta t$ . We have

$$(2.6) \quad \begin{aligned} V(t, t + \Delta t) &:= \text{Var} [\log P(t + \Delta t) - \log P(t)] \\ &= \int_t^{t+\Delta t} \sigma^2(z) \{f(z) + 1\}^2 dz. \end{aligned}$$

$$(2.7) \quad \begin{aligned} \mathbb{V}(t) &:= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} V(t, t + \Delta t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int_t^{t+\Delta t} \sigma^2(z) \{f(z) + 1\}^2 dz \\ &= \sigma^2(t) \{f(t) + 1\}^2. \end{aligned}$$

**Example 2.1.** For  $\sigma := 1$ , the maximum variance of  $\Delta \log P$  will be when  $\{f(z) + 1\}^2$  is at a maximum, which is when  $f$  has its maximum, i.e., at  $t_m$ .

$$(2.8) \quad \frac{d}{dt} \mathbb{V}(t) = \frac{d}{dt} \{f(t) + 1\}^2 = 2 \{f(t) + 1\} \frac{d}{dt} f(t)$$

Since  $1 + f(t) > 0$  in all cases, we see that the derivative of  $\mathbb{V}(t)$  is of the same sign as the derivative of  $f$ , so the limiting variance  $\mathbb{V}(t)$  is increasing when  $f$  is increasing and vice-versa. Recall that  $\log P$  increases so long as  $f > 0$ , and decreases when  $f < 0$ . In other words, for the peak case, one has  $f(t) > 0$  if and only if  $t \in (t_a, t_b)$  with a maximum at  $t_m$ . When  $f$  has a peak, the maximum of  $\mathbb{V}(t)$  will be at  $t_m$  when  $f(t)$  has its maximum.

To summarize, if the coefficient of  $dW$  is  $\sigma \{1 + f(t)\}$  with  $\sigma$  constant and  $f$  has a maximum at  $t_m$  then  $\mathbb{V}(t)$  will also have a maximum at  $t_m$  so that the maximum in  $E \log P$  will occur after the maximum in  $\mathbb{V}(t)$  since  $\partial_t E \log P(t) = f(t)$ .

*Remark 2.2.* We have shown that  $E \log P(t)$  has a maximum, at some time  $t_m$  that is preceded by a maximum in  $\mathbb{V}(t)$ . We can use this together with Jensen's inequality to show that  $E[P(t_m)/P(t)] \geq 1$  for arbitrary  $t$ . Indeed, since  $E \log P(t_m) \geq E \log P(t)$  we can write

$$(2.9) \quad E \log \frac{P(t_m)}{P(t)} \geq 0.$$

Let  $Y := P(t_m)/P(t_1)$  and  $g(x) := e^x$  in Jensen's inequality,  $Eg(Y) \geq g(E[Y])$ , we have

$$(2.10) \quad EY = Ee^{\log Y} \geq e^{E \log Y} \geq 1.$$

Hence, the expected ratio of price at  $t_m$  to the price at any other point  $t$  is greater than 1.

*Remark 2.3.* The conclusion above can be contrasted with the standard model (1.1) adjusted so that  $\mu(t) := \frac{\mu_D(t)}{\mu_S(t)}$  has the same property of a peak at some time  $t_m$ . Performing the same calculation of (2.4)-(2.8) for this model yields the result  $\mathbb{V}(t) = \sigma^2$  so that it provides no information on the expected peak of prices.

### 3. ADDITIONAL RANDOMNESS IN SUPPLY AND DEMAND

**3.1. Stochastic Supply and Demand.** Let  $f := D/S - 1$  be a stochastic function such that  $-1 \leq Ef$  and  $E|f| \leq C_1$ . With  $X(t) := \log P(t)$  and  $\Delta X := X(t + \Delta t) - X(t)$ , we write the SDE in differential and integral forms as

$$(3.1) \quad dX = f dt + \sigma(1 + f) dW$$

$$(3.2) \quad X(t + \Delta t) - X(t) = \int_t^{t+\Delta t} f(s) ds + \int_t^{t+\Delta t} \sigma(s)(1 + f(s)) dW(s).$$

where we will assume  $\sigma$  is a continuous, deterministic function of time, though we can allow it to be stochastic in most of the sequel.

One has since  $EdW = 0$  and  $E[dsdW] = 0$  one obtains again the identities

$$(3.3) \quad E\Delta X = \int_t^{t+\Delta t} Ef(s) ds,$$

$$(3.4) \quad \begin{aligned} \text{Var}[\Delta X] &= E \left[ \int f ds + \int \sigma(1 + f) dW \right]^2 \\ &\quad - \left( E \left[ \int f ds + \int \sigma(1 + f) dW \right] \right)^2 \\ &= \text{Var} \left[ \int f ds \right] + 2E \left[ \int f ds \int \sigma(1 + f) dW \right] + E \left[ \int \sigma(1 + f) dW \right]^2 \end{aligned}$$

where all integrals are taken over the limits  $t$  and  $t + \Delta t$ .

**Lemma 3.1.** *Let  $\sup_{[0,T]} E|f|^2 \leq C^2$ . Then for some  $C$  depending on this bound, one has*

$$(3.5) \quad \left| E \int_t^{t+\Delta t} f(s') ds' \int_t^{t+\Delta t} \sigma(s) \{1 + f(s)\} dW(s) \right| \leq C (\Delta t)^{3/2}.$$

*Proof.* We apply the Schwarz inequality to obtain

$$(3.6) \quad \begin{aligned} &\left| E \int_t^{t+\Delta t} f(s') ds' \int_t^{t+\Delta t} \sigma(s) \{1 + f(s)\} dW(s) \right| \\ &\leq \left\{ E \left( \int_t^{t+\Delta t} f(s') ds' \right)^2 \right\}^{1/2} \left\{ E \left( \int_t^{t+\Delta t} \sigma(s) \{1 + f(s)\} dW(s) \right)^2 \right\}^{1/2}. \end{aligned}$$

We bound each of these terms. Using the Schwarz inequality on the  $\int ds$  integral, we obtain using generic  $C$  throughout,

$$(3.7) \quad E \left( \int_t^{t+\Delta t} f(s') ds' \right)^2 \leq C (\Delta t)^2.$$

The second term is bounded using the fact that  $\sigma$  is deterministic,

$$(3.8) \quad E \left( \int_t^{t+\Delta t} \sigma(s) \{1 + f(s)\} dW(s) \right)^2 = \int_t^{t+\Delta t} \sigma^2(s) E \{1 + f(s)\}^2 ds \leq C \Delta t.$$

Taking the square roots of (3.7) and (3.8), and combining with (3.6) proves the lemma.  $\square$

**Lemma 3.2.** *Let  $\sigma$  be a continuous, deterministic function and assume  $\sup_{[0,T]} E |f|^2 \leq C^2$ . Then*

$$(3.9) \quad \left| \text{Var} [\Delta X] - \int_t^{t+\Delta t} \sigma^2(s) E \{1 + f(s)\}^2 ds \right| \leq C (\Delta t)^{3/2}$$

*Proof.* Basic stochastic analysis yields

$$(3.10) \quad E \left( \int_t^{t+\Delta t} \sigma^2(s) \{1 + f(s)\} dW \right)^2 = \int_t^{t+\Delta t} \sigma^2(s) E \{1 + f(s)\}^2 ds.$$

Thus, using (3.4) and  $f \in H_2[0, T]$  we have the result (3.9).  $\square$

Now, we would like to determine the maximum of  $\mathbb{V}(t)$  and show that it precedes the maximum of the expected log price. From the calculations above, one has

**Lemma 3.3.** *In the general case, assuming  $E |f(t)|^2 < C^2$  on  $t \in [0, T]$  but allowing stochastic  $\sigma$  such that  $E\sigma^2 < C$  one has*

$$(3.11) \quad \mathbb{V}(t) := \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} V(t, t + \Delta t) = E \left[ \sigma^2(t) (1 + f(t))^2 \right].$$

**Lemma 3.4.** *Suppose  $\sup_{[0,T]} E |f(t)|^2 < C^2$  and  $\sigma$  is a deterministic continuous function on  $t \in [0, T]$  then one has*

$$(3.12) \quad \mathbb{V}(t) = \sigma^2 \{1 + Ef\}^2 + \sigma^2 \text{Var} f.$$

and the extrema of  $\mathbb{V}(t)$  occur at  $t$  such that

$$(3.13) \quad 2\sigma\sigma' \left\{ [1 + Ef]^2 + \text{Var} f \right\} + \sigma^2 \left\{ 2[1 + Ef] (Ef)' + (\text{Var} f)' \right\} = 0.$$

*Proof.* Using Lemma 3.3, we write

$$(3.14) \quad \begin{aligned} \mathbb{V}(t) &= \sigma^2 E [1 + 2f + f^2] = \sigma^2 \left\{ 1 + 2Ef + (Ef)^2 + Ef^2 - (Ef)^2 \right\} \\ &= \sigma^2 (1 + Ef)^2 + \sigma^2 \text{Var} f. \end{aligned}$$

Differentiation implies the second assertion.  $\square$

**Lemma 3.5.** *Suppose  $E|f(t)|^2 < C^2$  on  $t \in [0, T]$ , while  $\sigma$  and  $\text{Var}[f(t)]$  are constant in  $t$ . Then the extremum of  $\mathbb{V}(t)$  occur for  $t$  such that*

$$(3.15) \quad \frac{d}{dt} E f(t) = 0.$$

*Proof.* From the previous Lemma, we have  $V(t, t + \Delta t) := \int_t^{t+\Delta t} \sigma^2(s) E[1 + f(s)]^2 ds$ , yielding

$$(3.16) \quad \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} V(t, t + \Delta t) = \sigma^2 (1 + E f(t))^2 + \text{Var}[f(t)]$$

Since we are assuming that  $\text{Var}[f(t)]$  is constant in time, we obtain

$$(3.17) \quad \begin{aligned} \frac{\partial}{\partial t} \lim_{\Delta t \rightarrow 0} V(t, t + \Delta t) &= \frac{\partial}{\partial t} \left\{ \sigma^2 (1 + E f(t))^2 \right\} \\ &= 2\sigma^2 (1 + E f(t)) \frac{d}{dt} E f(t). \end{aligned}$$

Thus, the right-hand side vanishes if and only if  $\frac{d}{dt} E f(t) = 0$ , i.e., at  $t_m$  (by definition of  $t_m$ ). Note that we have  $1 + f > 0$  so that  $1 + E f > 0$ .  $\square$

**3.2. Properties of  $f$ .** The condition  $E|f|^2 < C^2$  is easily satisfied by introducing randomness in many forms. For the Lemma above, we would also like to satisfy  $\text{Var}[f(t)] = \text{const}$ .

Another way of attaining this (up to exponential order) is to define  $f$  as the stochastic process

$$(3.18) \quad df(t) = \mu_f(t) dt + \sigma_f(t) dW(t)$$

where  $\mu_f$  and  $\sigma$  are both time dependent but deterministic.

We can assume that  $f(t_0)$  is a given, fixed value, and obtain (see e.g., [4], [5])

$$(3.19) \quad \text{Var}[f(t)] = E \left[ \int_{t_0}^t \sigma_f(s) dW(s) \right]^2 = \int_{t_0}^t \sigma_f^2(s) ds$$

since  $\sigma_f(s)$  is deterministic.

In particular, if one has  $\sigma_f(s) := e^{-s/2}$ , then  $\text{Var}[f(t)] \leq e^{-t_0}$  while  $\int_0^{t_0} \sigma(s) ds = 1 - e^{-t_0}$  so one has approximately constant variance for  $t \geq t_0$  for large  $t_0$ . In particular, one has

$$(3.20) \quad \frac{d}{dt} \text{Var}[f(t)] = \frac{d}{dt} \int_{t_0}^t \sigma_f^2(s) ds = \sigma_f^2(t) = e^{-t}.$$

**3.3. General coefficient of  $dW$ .** The stochastic differential equation (3.1) entails a coefficient of  $dW$  that is proportional to  $D/S$ . One can also consider the implications of a coefficient that is proportional to the excess demand  $D/S - 1$  or a monomial of it. More generally, we can write  $h(t) := g(f(t))$  for an arbitrary continuous function  $g$  leading to the stochastic differential equation

$$(3.21) \quad d \log P = f dt + \sigma h dW,$$

where  $\sigma$  can also be stochastic or deterministic function of time.

From this stochastic equation one has immediately

$$(3.22) \quad \frac{dE[\log P]}{dt} = E f$$

similar to the completely deterministic model, except that  $f$  is replaced by  $E f$ .

From the integral version of the stochastic model, we can write the expectation and variance as

$$(3.23) \quad E[\Delta \log P] = \int_t^{t+\Delta t} E f(s) ds$$

$$(3.24) \quad V(t, t + \Delta t) := \text{Var}[\Delta \log P] = \text{Var} \left[ \int_t^{t+\Delta t} f(s) ds \right] + 2\mathbb{E} \left[ \int_t^{t+\Delta t} \sigma(s) h(s) dW(s) \right] \\ + \int_t^{t+\Delta t} E[\sigma(s) h(s)]^2 ds.$$

The middle term on the right-hand side vanishes while the first term is of order  $(\Delta t)^2$ , yielding the following relation for  $\mathbb{V}(t)$ .

**Lemma 3.6.** *Let  $h(t) := g(f(t))$  and  $\sigma$  satisfy  $Eh^2 < C$ ,  $E\sigma^2 < C$ . Then one has*

$$(3.25) \quad \mathbb{V}(t) := \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} V(t, t + \Delta t) = E[\sigma(t) h(t)]^2.$$

Next, we examine whether  $\mathbb{V}(t)$  occurs prior to the maximum of  $\log P(t)$  in several examples.

**Example 3.7.** Consider the function  $g(z) = z^q$  where  $q \in \mathbb{N}$ . Let  $\sigma := 1$  and  $f \in L^2[0, t]$  be deterministic. From the Lemma above, we obtain

$$(3.26) \quad \mathbb{V}(t) = h(t)^2 = f(t)^{2q}, \quad \frac{d}{dt} \mathbb{V}(t) = 2q f(t)^{2q-1} \frac{d}{dt} f(t).$$

When  $f := D/S - 1$  has a maximum, note that on some interval  $(t_a, t_b)$  it is positive (as demand exceeds supply) and  $f$  has its maximum for some value  $t_m \in (t_a, t_b)$ . The identity above implies that  $\mathbb{V}(t)$  has a maximum when  $f$  has a maximum. Also, the defining stochastic equation above implies  $E \log P$  has its maximum at  $t_b > t_m$ .

**Example 3.8.** (Symmetry between  $D$  and  $S$  and more general coefficients) If we hypothesize that the level of noise is proportional essentially to the magnitude (or its square) of the difference between  $D$  and  $S$  divided by the sum (which is a proxy for trading volume), then we can write that coefficient as

$$(3.27) \quad \sigma \frac{(D - S)^2}{(D + S)^2}.$$

We can consider a more general case in which we write, for example, for  $\sigma = \text{const}$ ,

$$(3.28) \quad d \log P(t) = \left( \frac{D}{S} - 1 \right) dt + \sigma \left( \frac{D - S}{D + S} \right)^p dW$$

where  $p \in \mathbb{N}$  can be either even or odd. Note that we can write all terms as functions of  $f := D/S - 1$ , so  $f + 2 = D/S + 1 > 0$  since  $D$  and  $S$  are positive, and we have

$$(3.29) \quad d \log P(t) = f dt + \sigma \left( \frac{f}{f + 2} \right)^p dW.$$

We write

$$(3.30) \quad \mathbb{V}(t) := \lim_{\Delta t \rightarrow 0} \frac{V(t, t + \Delta t)}{\Delta t} = E \left[ \sigma(t) \left( \frac{f(t)}{f(t) + 2} \right)^p \right]^2$$

If  $f$  is deterministic and  $\sigma$  is constant, we have upon differentiation,

$$(3.31) \quad \frac{d}{dt} \mathbb{V}(t) = 4p\sigma^2 \frac{f^{2p-1}}{[f+2]^{2p+1}} \frac{df}{dt}$$

Recalling  $f+2 > 0$  the sign of  $\frac{d}{dt} \mathbb{V}$  depends only on  $f^{2p-1} df/dt$ . Notice that it makes no difference whether  $p$  is even or odd.

If  $f$  has a single maximum at  $t_m \in (t_a, t_b)$  such that  $f(t) > 0$  iff  $t \in (t_a, t_b)$ , and  $f < 0$  iff  $t \notin [t_a, t_b]$  then we have a relative maximum in  $\mathbb{V}$  at  $t_m$ .

Hence, we see that if the coefficient of  $dW$  is a deterministic term of the form  $((D-S)/(D+S))^p$  and  $f$  has a maximum, whether  $p$  is even or odd (i.e., the coefficient increases or decreases with excess demand), then the limiting volatility  $\mathbb{V}$  also has a maximum.

**Example 3.9.** Generalizing this concept further, we define a function  $H(z)$  such that  $H(z) > 0$  for all  $z \in \mathbb{R}$  and

$$(3.32) \quad \text{sgn} H'(z) = \text{sgn}(z).$$

We consider the stochastic equation, with  $f$  deterministic

$$(3.33) \quad d \log P = f dt + \sigma \left\{ H \left( \frac{f}{f+2} \right) \right\}^{1/2} dW$$

so that  $\mathbb{V}(t) = \sigma^2 H \left( \frac{f(t)}{f(t)+2} \right)$  with  $\sigma = \text{const}$ .

While in principle,  $f(t) := D(t)/S(t) - 1 \in (-1, \infty)$ , except under conditions that are very far from equilibrium, one can assume  $f(t) \in (-a/2, a/2)$  for some small  $a$ , at least  $a \in (0, 1]$ .

We compute

$$(3.34) \quad \begin{aligned} \sigma^{-2} \frac{d}{dt} \mathbb{V} &= \frac{d}{dt} H \left( \frac{f}{f+2} \right) \\ &= H' \left( \frac{f}{f+2} \right) \frac{2}{(f+2)^2} \frac{df}{dt}. \end{aligned}$$

Based on this calculation, one concludes if  $f$  has a maximum, recalling that  $f := D/S - 1$  is positive near the maximum, then  $d\mathbb{V}/dt$  has the same sign as  $df/dt$ . So a maximum in  $\mathbb{V}$  corresponds to a maximum in  $f$ , while  $\log(P)$  has its maximum at  $t_b > t_m$ .

#### 4. SUPPLY AND DEMAND AS A FUNCTION OF VALUATION

We consider the basic model (1.4) now with the excess demand, i.e.,  $D/S - 1$ , depending on the valuation,  $P_a(t)$ , which can be regarded either as a stochastic or deterministic function. It is now commonly accepted in economics and finance that the trading price will often stray from the fundamental valuation [18, 19]. We write the price equation for the time evolution as

$$(4.1) \quad \frac{d}{dt} \log P(t) = \frac{D}{S} - 1 = \log \frac{P_a(t)}{P(t)}.$$

The right hand side of equation (4.1) is a linearization (as discussed in Section 1.3) and the right hand side of has the same linearization as  $(P_a - P)/P$ . The equation simply expresses the idea that undervaluation is a motivation to buy, while overvaluation is a motivation to sell, as one assumes in classical finance. The

non-classical feature is the absence of infinite arbitrage. Analogous to Section 1.3, we write the stochastic version of (4.1) as

$$(4.2) \quad d \log P(t, \omega) = \log \frac{P_a(t, \omega)}{P(t, \omega)} dt + \sigma(t, \omega) \left( 1 + \log \frac{P_a(t, \omega)}{P(t, \omega)} \right) dW(t, \omega).$$

At this point we allow both  $P_a$  and  $\sigma$  to be stochastic, with  $EP_a^2 < C$  and  $E\sigma^2 < C$  but will specialize to given and deterministic  $P_a$  and  $\sigma$  after the first result. We also assume  $1 + \log(P_a/P) > 0$ , i.e.,  $P_a/P > e^{-1}$ , i.e., the fundamental value,  $P_a$ , and trading price,  $P$ , do not differ drastically.

**Notation 1.** Let  $X := \log P$ ,  $X_a := \log P_a$ ,  $y := E \log P$ ,  $y_a := E \log P_a$ ,  $z := E(\log P)^2$ . When  $\log P_a$  and  $\log P$  are deterministic, we write lower case  $x_a$  and  $x$ , respectively.

The equation (4.2) is short for the integral form (using the notation above) for any  $t_2 > t_1 > t_0$ ,

$$(4.3) \quad X(t_2) - X(t_1) = \int_{t_1}^{t_2} X_a - X ds + \int_{t_1}^{t_2} \sigma(s, \omega) (1 + X_a - X) dW(s).$$

Noting that  $E \int f(t) dW = 0$ , we find the expectation of (4.3) as

$$(4.4) \quad y(t_2) - y(t_1) = \int_{t_1}^{t_2} y_a(s) ds - \int_{t_1}^{t_2} y(s) ds$$

i.e., one has the ODE, with  $y_0 := y(t_0) := E[\log P(t_0)]$ ,

$$(4.5) \quad \frac{d}{dt} y(t) = y_a(t) - y(t), \quad y(t_0) := y_0$$

This has the unique solution, for known  $y_a(t)$ ,

$$(4.6) \quad y(t) = e^{t_0-t} y(t_0) + e^{-t} \int_{t_0}^t y_a(s) e^s ds.$$

Note that if we eliminate randomness altogether, i.e.,  $\sigma := 0$  and deterministic  $P_a(t)$ ,

$$(4.7) \quad \frac{d}{dt} \log P(t) = \log \frac{P_a(t)}{P(t)},$$

with solution

$$(4.8) \quad x(t) = e^{t_0-t} x(t_0) + e^{-t} \int_{t_0}^t e^s x_a(s) ds.$$

where  $x(t) := \log P(t)$  and  $x_a(t) := \log P_a(t)$ . We note that the solution of  $y(t) = E \log P(t)$  of  $\log P$  in terms of  $y_a(t) = E \log P_a(t)$  is the same as  $\log P(t)$  in terms of  $\log P_a(t)$ , i.e. both expected value and deterministic  $\log P$  satisfy the same equation.

**4.1. The stochastic problem.** We write the SDE (4.2) as

$$(4.9) \quad dX = (X_a - X) dt + \sigma (1 + X_a - X) dW.$$

We say  $X$  is a solution to a SDE if  $X \in H_2[0, T]$  and solves the integral version of the SDE for almost every  $\omega \in \Omega$ . The solution to the stochastic equation (4.2),  $X(t, \omega)$  is unique, belongs to  $H_2[0, T]$  and is continuous in  $t \in [0, T]$  for almost every  $\omega \in \Omega$  ([5] p. 94). We denote the remaining set  $\Gamma$ , so that  $X(t, \omega)$  is continuous

in  $t$  for all  $\omega \in \Omega \setminus \Gamma$ . One has by basic measure theory (e.g., [20]), that for any measurable function such as  $X$  or  $X^2$  one has

$$\begin{aligned} E \int_t^{t+\Delta t} X(s, \omega) ds &= \int_{\Omega} \int_t^{t+\Delta t} X(s, \omega) ds dP(\omega) \\ (4.10) \qquad &= \int_{\Omega \setminus \Gamma} \int_t^{t+\Delta t} X(s, \omega) ds dP(\omega) + \int_{\Gamma} \int_t^{t+\Delta t} X(s, \omega) ds dP(\omega). \end{aligned}$$

Thus from here on we can ignore the set  $\Gamma$  and assume that  $X(t, \omega)$  is continuous when the expectation value is computed.

Next, using (4.4) we compute the variance, of  $\Delta X := X(t + \Delta t, \omega) - X(t, \omega)$  and later we will determine the terms that vanish upon dividing by  $\Delta t$ ,

$$\begin{aligned} V(t, t + \Delta t) &:= E[X(t + \Delta t) - EX(t)]^2 - (E[X(t + \Delta t) - X(t)])^2 \\ &= E \left[ \int_t^{t+\Delta t} X_a - X ds + \int_t^{t+\Delta t} \sigma(1 + X_a - X) dW(s) \right]^2 \\ (4.11) \qquad &- \left( E \left[ \int_t^{t+\Delta t} X_a - X ds + \int_t^{t+\Delta t} \sigma(1 + X_a - X) dW(s) \right] \right)^2. \end{aligned}$$

Note that with  $\Delta X := X(t + \Delta t) - X(t)$  we have

$$\begin{aligned} V(t, t + \Delta t) &= Var[X(t + \Delta t) - X(t)] = Var \left[ \log \frac{P(t + \Delta t)}{P(t)} \right] \\ (4.12) \qquad &= Var \left[ \log \left( \frac{\Delta P}{P} + 1 \right) \right] \simeq Var \left[ \frac{\Delta P}{P} \right]. \end{aligned}$$

so that  $V(t, t + \Delta t)$  is essentially a measure of the variance of relative price change. Since  $E \int_t^{t+\Delta t} \sigma(1 + X_a - X) dW(s) = 0$  one has

$$\begin{aligned} V(t, t + \Delta t) &= E \left[ \int_t^{t+\Delta t} X_a - X ds + \int_t^{t+\Delta t} \sigma(1 + X_a - X) dW(s) \right]^2 \\ &- \left( E \int_t^{t+\Delta t} X_a - X ds \right)^2 \\ (4.13) \qquad &= V_1(t, t + \Delta t) + V_2(t, t + \Delta t) + V_3(t, t + \Delta t) \end{aligned}$$

where

$$\begin{aligned} V_1(t, t + \Delta t) &:= E \left( \int_t^{t+\Delta t} X_a - X ds \right)^2 - \left( E \int_t^{t+\Delta t} X_a - X ds \right)^2, \\ V_2(t, t + \Delta t) &:= 2E \left[ \int_t^{t+\Delta t} X_a - X ds \int_t^{t+\Delta t} \sigma(1 + X_a - X) dW(s) \right] \\ V_3(t, t + \Delta t) &:= E \left( \int_t^{t+\Delta t} \sigma(1 + X_a - X) dW(s) \right)^2 \\ (4.14) \qquad &= \int_t^{t+\Delta t} E[\sigma(1 + X_a - X)]^2 ds \end{aligned}$$



**Lemma 4.1.** *Let  $X$  be a solution to the SDE (4.9) with  $\sigma(t, \omega)$  and  $X_a(t, \omega)$  continuous for all  $t \in [0, T]$  and all  $\omega \in \Omega$ , with bounded second moments. Then*

- (i)  $|V_1(t, t + \Delta t)| \leq C(\Delta t)^2$  so  $\lim_{\Delta t \rightarrow 0} V_1(t, t + \Delta t)/\Delta t = 0$ , and,
- (ii)  $|V_2(t, t + \Delta t)| \leq C(\Delta t)^{3/2}$  so  $\lim_{\Delta t \rightarrow 0} V_2(t, t + \Delta t)/\Delta t = 0$ .

*Proof.* (i) (a) We consider the first term in  $V_1$ , namely,

$$\begin{aligned} E \left( \int_t^{t+\Delta t} X_a - X ds \right)^2 &= \int_{\Omega} \left( \int_t^{t+\Delta t} X_a - X ds \right)^2 dP(\omega) \\ (4.15) \qquad \qquad \qquad &= \int_{\Omega \setminus \Gamma} \left( \int_t^{t+\Delta t} X_a - X ds \right)^2 dP(\omega) \end{aligned}$$

where we have omitted the set of measure zero,  $\Gamma$ , outside of which  $X$  is continuous in  $t$  on a closed bounded interval. Hence, one can bound the integrand by  $C(\Delta t)^2$ . Thus we have

$$(4.16) \qquad \qquad \qquad E \left( \int_t^{t+\Delta t} X_a - X ds \right)^2 \leq C(\Delta t)^2.$$

(i) (b) Similarly the second term can be bounded as

$$\begin{aligned} \left( E \int_t^{t+\Delta t} X_a - X ds \right)^2 &= \left( \int_{\Omega \setminus \Gamma} \left( \int_t^{t+\Delta t} X_a - X ds \right) dP(\omega) \right)^2 \\ (4.17) \qquad \qquad \qquad &\leq C(\Delta t)^2. \end{aligned}$$

Hence, part (i) of the lemma has been proven.

(ii) Using the Schwarz inequality on the second term we have

$$\begin{aligned} \frac{1}{2} V_2(t, t + \Delta t) &= E \left\{ \left( \int_t^{t+\Delta t} X_a - X ds \right) \left( \int_t^{t+\Delta t} \sigma(1 + X_a - X) dW(s) \right) \right\} \\ (4.18) \qquad \qquad \qquad &\leq \left\{ E \left( \int_t^{t+\Delta t} X_a - X ds \right)^2 \right\}^{1/2} \left\{ E \left( \int_t^{t+\Delta t} \sigma(1 + X_a - X) dW(s) \right)^2 \right\}^{1/2}. \end{aligned}$$

Using continuity properties, we have the following bound on the first term,

$$(4.19) \qquad \qquad \qquad \left\{ E \left( \int_t^{t+\Delta t} X_a - X ds \right)^2 \right\}^{1/2} \leq C(\Delta t).$$

For the second we use the basic property used above,

$$\begin{aligned} \left\{ E \left( \int_t^{t+\Delta t} \sigma(1 + X_a - X) dW(s) \right)^2 \right\}^{1/2} &= \left\{ \int_t^{t+\Delta t} E[\sigma(1 + X_a - X)]^2 ds \right\}^{1/2} \\ &= \left\{ \int_{\Omega \setminus \Gamma} \int_t^{t+\Delta t} E[\sigma(1 + X_a - X)]^2 ds \right\}^{1/2} \\ (4.20) \qquad \qquad \qquad &\leq C(\Delta t)^{1/2}. \end{aligned}$$

Hence, the proof of the second part of the lemma follows from the following bound:

$$(4.21) \quad V_2(t, t + \Delta t) \leq \left\{ E \left( \int_t^{t+\Delta t} X_a - X ds \right)^2 \right\}^{1/2} \left\{ E \left( \int_t^{t+\Delta t} \sigma (1 + X_a - X) dW(s) \right)^2 \right\}^{1/2} \\ \leq C (\Delta t)^{3/2}$$

This proves the second part of the Lemma.  $\square$

Thus, Lemma 4.1 indicates that in analyzing  $V(t, t + \Delta t) / \Delta t$  in the limit of  $\Delta t \rightarrow 0$  amounts to analyzing  $V_3(t, t + \Delta t) / \Delta t$ .

At this point we assume that both  $P_a$  and  $\sigma$  are deterministic but need not be constant in time, and we now use lower case,  $x_a := \log P_a$ .

**Lemma 4.2.** *Let  $\sigma$  and  $P_a$  be deterministic, and  $X(t)$  as solution to the SDE (4.9). Then*

$$(4.22) \quad V_3(t, t + \Delta t) = \int_t^{t+\Delta t} \sigma^2 [1 + x_a(s) - EX(s)]^2 ds + \int_t^{t+\Delta t} \sigma^2 Var[X(s)] ds.$$

*Proof.* Using the expression (4.14) above, the identity follows upon adding and subtracting  $EX^2(s)$  in the integrand.  $\square$

**Lemma 4.3.** *Let  $\sigma$  and  $P_a$  be deterministic and continuous. Then*

$$(4.23) \quad \mathbb{V}(t) := \lim_{\Delta t \rightarrow 0} \frac{V(t, t + \Delta t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{V_3(t, t + \Delta t)}{\Delta t} \\ = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left\{ \int_t^{t+\Delta t} \sigma^2 [1 + x_a - y]^2 + Var[X] ds \right\} \\ = \sigma^2 [1 + x_a - y]^2 + Var[X].$$

Next, we will compute  $Var[X]$  starting with  $E[X^2]$  and assuming that  $P_a$  and  $\sigma$  are deterministic.

**Lemma 4.4.** *Let  $\sigma$  and  $x_a$  be deterministic and continuous. Then  $z(t) := EX^2(t)$  satisfies the ODE*

$$(4.24) \quad \frac{dz}{dt} = (\sigma^2 - 2)z + (2 - 2\sigma^2)x_a y - 2\sigma^2 y + \sigma^2(1 + x_a)^2 \\ z(t_0) = y(t_0)^2 =: y_0^2.$$

*Proof.* The stochastic process for  $X(t)$ , i.e., (4.9) can be written

$$A(t, \omega) := (x_a - X), \quad B(t, \omega) := \sigma(1 + x_a - X)$$

$$(4.25) \quad dX(t, \omega) = A(t, \omega) dt + B(t, \omega) dW(t)$$

Ito's formula provides the differential for a smooth function of  $X$  as

$$(4.26) \quad df(X(t), t) = \left[ \frac{\partial f(X(t), t)}{\partial t} + A(t) \frac{\partial f(X(t), t)}{\partial x} + \frac{B^2(t)}{2} \frac{\partial^2 f(X(t), t)}{\partial x^2} \right] dt + B(t) \frac{\partial f(X(t), t)}{\partial x} dW(t).$$

For  $f(x) := x^2$  we have then from Ito's formula,

$$(4.27) \quad dX^2 = \left[ (\sigma^2 - 2) X^2 + (2 - 2\sigma^2) x_a X - 2\sigma^2 X + \sigma^2 (1 + x_a)^2 \right] dt + \sigma(1 + x_a - X) (2X) dW$$

Hence, we can write in the usual way, as  $EdW$  vanishes:

$$(4.28) \quad E[X^2(t) - X^2(t_0)] = \int_{t_0}^t (\sigma^2 - 2) EX^2 + (2 - 2\sigma^2) x_a EX - 2\sigma^2 EX + \sigma^2 (1 + x_a)^2 ds$$

Using the notation  $y(t) := E(\log P)$  and  $z(t) := E(\log P)^2$  we have

$$(4.29) \quad z(t) - z(t_0) = \int_{t_0}^t (\sigma^2 - 2) z(t) + (2 - 2\sigma^2) x_a y(t) - 2\sigma^2 y(t) + \sigma^2 (1 + x_a)^2 ds.$$

Differentiation with respect to  $t$  yields the result and proves the lemma.  $\square$

In the sequel, we assume for simplicity that  $\sigma$  is constant in time, and  $x_a(t)$  is deterministic and smooth. We can solve for  $z$  directly but it will be more illuminating if we write the solution in the following form.

**Lemma 4.5.** *Let  $x_a$  be a continuous function. The unique solution to*

$$(4.30) \quad \frac{dz_0}{dt} = -2z_0 + 2x_a y$$

$$(4.31) \quad z_0(t_0) := y(t_0)^2$$

is given by  $z_0(t) = y(t)^2$ .

*Proof.* Note that  $x_a = y_a = EX_a$  since  $X_a$  is deterministic under our current assumption. We know that  $y(t)$  is a solution to the equation

$$(4.32) \quad \frac{d}{dt} y(t) = y_a(t) - y(t), \quad y(t_0) := y_0$$

so we can substitute  $x_a = y' + y$  into (22) and obtain

$$(4.33) \quad z_0' + 2z_0 = 2yy' + 2y^2 = 2y(y' + y) = 2x_a y.$$

Hence,  $z_0(t) := y(t)^2$  solves (4.34), (4.31) and from basic ODE theory, the solution is unique so long as  $x_a$  is continuous.  $\square$

**Lemma 4.6.** *The unique solution to (4.24) is given by*

$$(4.34) \quad z(t) := z_0(t) + \sigma^2 z_1(t) = y(t)^2 + \sigma^2 z_1(t)$$

with  $z_1(t)$  defined by

$$(4.35) \quad z_1(t) = \int_{t_0}^t e^{(2-\sigma^2)(s-t)} [y(s) - (1+x_a(s))]^2 ds.$$

*Proof.* Let  $z_1$  be defined by  $z(t) = z_0(t) + \sigma^2 z_1(t) = y(t)^2 + \sigma^2 z_1(t)$ . Substituting into (4.24) yields

$$(4.36) \quad \begin{aligned} z'_0 + \sigma^2 z'_1 &= (\sigma^2 - 2)(z_0 + \sigma^2 z_1) + (2 - 2\sigma^2)x_a y - 2\sigma^2 y + \sigma^2(1+x_a)^2 \\ &= \sigma^2 z_0 - 2z_0 + (\sigma^2 - 2)\sigma^2 z_1 + (2 - 2\sigma^2)x_a y - 2\sigma^2 y + \sigma^2(1+x_a)^2 \end{aligned}$$

so that the terms  $z'_0$  and  $-2z_0 + 2x_a y$  vanish from both sides.. Using  $z_0 = y^2$  we have left, upon dividing by  $\sigma^2$ , the equation for  $z_1$

$$(4.37) \quad z'_1 + (2 - \sigma^2)z_1 = [y - (1+x_a)]^2,$$

and elementary methods yield the solution (4.34 - 4.35).  $\square$

Note that although  $\sigma \in \mathbb{R}$  was used in this proof, comparable result can be obtained in the general case in which  $\sigma$  is a continuous and deterministic function.

Thus, Lemmas 4.5 and 4.6 yield the following identity for  $Var[X(t)]$ .

**Theorem 4.7.** *Let  $\sigma \in \mathbb{R}$  and  $x_a(t)$  be deterministic and continuous. Let  $c := (2 - \sigma^2)$  and*

$$(4.38) \quad \sigma^2 I(t, t + \Delta t) := Var[X(t + \Delta t)] - Var[X(t)].$$

$$(4.39) \quad w(s) := [1 + x_a(s) - y(s)]^2.$$

*Then one has the following identities:*

$$(4.40) \quad Var[X(t)] = \sigma^2 \int_{t_0}^t e^{c(s-t)} [y(s) - (1+x_a(s))]^2 ds$$

$$(4.41) \quad I(t, t + \Delta t) = \int_t^{t+\Delta t} e^{c(s-t)} w(s) ds.$$

*Proof.* The identities follow immediately from Lemma 4.6 and the definition of variance in terms of  $z$  and  $y$ . I.e.,

$$(4.42) \quad \begin{aligned} Var[X(t)] &= E[X(t)]^2 - [EX(t)]^2 \\ &= z(t) - y(t)^2 = \sigma^2 z_1(t) \\ &= \sigma^2 \int_{t_0}^t e^{(2-\sigma^2)(s-t)} [1 + x_a(s) - y(s)]^2 ds. \end{aligned}$$

$\square$

*Remark 4.8.* The maximum value of  $Var[X(t + \Delta t)] - Var[X(t)]$  occurs for  $t$  such that the average weighted value of  $w(s)$  with exponential weighting of  $(2 - \sigma^2)$  is maximal on  $(t, t + \Delta t)$ .

Using the lemmas above, we obtain directly the following result.

**Theorem 4.9.** *Let  $x_a$  be continuous. Then we have the identities,*

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \sigma^{-2} (\Delta t)^{-1} V(t, t + \Delta t) &= \lim_{\Delta t \rightarrow 0} \sigma^{-2} (\Delta t)^{-1} V_3(t, t + \Delta t) \\ &= w(t) + \text{Var}[X(t)] \\ (4.43) \quad \text{i.e.,} \quad \sigma^{-2} \mathbb{V}(t) &= w(t) + \sigma^2 \int_{t_0}^t e^{(2-\sigma)^2(s-t)} w(s) ds \end{aligned}$$

$$\begin{aligned} Q(t) &:= \frac{d}{dt} \lim_{\Delta t \rightarrow 0} \sigma^{-2} \frac{V(t, t + \Delta t)}{\Delta t} = \sigma^{-2} \frac{d}{dt} \mathbb{V}(t) \\ (4.44) \quad &= w'(t) + \sigma^2 w(t) - \sigma^2 (2 - \sigma^2) \int_{t_0}^t e^{(2-\sigma)^2(s-t)} w(s) ds. \end{aligned}$$

## 5. MARKET EXTREMA

The main objective of this section is to apply the results above understand the temporal relationship between the extrema of the (log) fundamental value,  $x_a(t)$ , and the expected (log) trading price,  $y(t)$ .

### 5.1. Price Maxima.

**Notation 2.** *Let  $t_0$  be the initial time, and  $t_m$  be defined by  $x'_a(t_m) = 0$ , i.e., the time at which the fundamental value,  $x_a$ , attains its maximum. The time  $t_*$  is defined as the first time at which  $y'(t_*) = x_a(t_*) - y(t_*)$  vanishes, and the curves  $x_a(t)$  and  $y(t)$  first intersect.*

**Notation 3.** *Let  $\hat{x}_a(t) := e^t x_a(t)$ ,  $\hat{y}(t) := e^t y(t)$ ,  $\hat{y}_0 := e^{t_0} \hat{y}(t_0)$ .*

**Condition  $\sigma$ .** Let  $\sigma \in (0, 1)$  be a constant, so  $c := 2 - \sigma^2 \in (1, 2)$ .

We will assume this condition throughout, though some results are valid without it.

**Condition C.** (i) The function  $x_a : [t_0, \infty) \rightarrow (0, \infty)$  has the property that for some  $t_m \in (0, \infty)$  one has

- (i)  $x'_a(t) > 0$  if  $t < t_m$ ;  $x'_a(t_m) = 0$ ;  $x'_a(t) < 0$  if  $t > t_m$ .
- (ii) Let  $y(t_0) =: y_0 \in (0, \infty)$  one has

$$(5.1) \quad x_a(t_0) - x'_a(t_0) < y_0 < x_a(t_0).$$

(iii) For some  $\delta, m_1 \in (0, \infty)$  one has

$$(5.2) \quad -x'_a(t) > m_1 > 0 \quad \text{if } t > t_m + \delta.$$

**Remarks.** We set  $y_0 =: y(t_0)$ , so the two inequalities in Condition C (ii) state that initially (i.e., at  $t_0$ ) the price is below the fundamental value, i.e., undervaluation ( $y(t_0) = y_0 < x_a(t_0)$ ). Using the ODE  $y' = x_a - y$  one has that the first inequality in Condition C (ii) is equivalent to  $x'_a(t_0) > y'(t_0) > 0$  stipulating that the valuation has begun to increase relative to trading price. Condition C (iii) can be relaxed to some extent although the condition then appears more technical.

**Condition E.** With  $t_*$  be defined as above, assume  $2x'_a(t_*) + \sigma^2 e^{c(t_0 - t_*)} < 0$ .

**Remarks.** Note that this condition is satisfied automatically if  $t_0 \rightarrow -\infty$ . So long as there is an interval  $(t_m, t_*)$  on which  $x'_a(t_*) < -\sigma^2 e^{c(t_0-t_*)}$  (the latter is exponentially small if  $t_* - t_0 \gg 1$ ) the Condition  $E$  will be satisfied.

Recalling that  $y(t)$  is given by (4.6), i.e.,

$$(5.3) \quad \hat{y}(t) = \hat{y}(t_0) + \int_{t_0}^t \hat{x}_a(s) ds.$$

since  $y_a = x_a$  as the latter is deterministic.

Initially, we have from  $C(ii)$  that  $x_a(t_0) > y(t_0)$ . We want to first prove that  $y$  intersects with  $x_a$  at some value  $t_*$  and that this value  $t_*$  occurs after  $t_m$  (i.e., the time at which  $x_a$  has its peak).

**Theorem 5.1.** *Assume that  $C$  holds. Then there exists a least value  $t_* \in (t_m, \infty)$  such that for  $t < t_*$  one has  $y(t) < x_a(t)$ , and,  $y(t_*) < x_a(t_*)$ , i.e.,*

$$(5.4) \quad \hat{y}(t_*) = \hat{y}_0 + \int_{t_0}^{t_*} \hat{x}_a(s) ds = \hat{x}_a(t_*).$$

Since  $y' = x_a - y$ , the maximum of  $y$  is attained at  $t_*$ .

*Proof.* Let  $I(t) := \hat{x}_a(t) - \hat{y}_0 - \int_{t_0}^t \hat{x}_a(s) ds$ , so  $I(t_0) > 0$  by condition  $C(ii)$ . Computing the derivative and using Condition  $C(i)$  yields

$$(5.5) \quad I'(t) = \hat{x}'_a(t) - \hat{x}_a(t) = e^t x'_a(t) > 0 \text{ if } t < t_m.$$

Hence, one has  $I(t) < 0$  if  $t < t_m$ . On the other hand, by Condition  $C(iii)$ , when  $t > t_m + \delta$  one has

$$(5.6) \quad I'(t) = e^t x'_a(t) \leq e^{t_m} (-m_1)$$

so that  $I(t_*) = 0$  for some finite  $t_* > t_m$ .  $\square$

**Lemma 5.2.** *Under  $C(i), (ii)$  there exists a  $t_1 \in (t_0, t_m)$  such that  $w'(t_1) = 0$ ,  $w'(t) > 0$  if  $t \in [t_0, t_1)$ , and  $w'(t) < 0$  if  $t_m < t < t_*$ . Consequently, we have*

$$(5.7) \quad t_0 < t_1 < t_m < t_*.$$

*Proof.* Recall (4.43) and note  $w' = 2[1 + x_a - y](x'_a - y')$ , whose sign is determined by

$$(5.8) \quad S(t) := x'_a(t) - y'(t) = x'_a(t) - x_a(t) + y(t)$$

when  $t < t_*$  [i.e., when  $x_a(t) > y(t)$ ]. For  $t_0$  we have from  $C(ii)$  that  $S(t_0) > 0$ .

For  $t_m < t < t_*$  we have from  $C(i)$  that  $x'_a(t) < 0$  while  $y'(t) = x_a(t) - y(t) > 0$  as noted earlier in the proof, yielding

$$(5.9) \quad S(t) = x'_a(t) - x_a(t) + y(t) < 0.$$

By continuity, there exists a  $t_1 \in (t_0, t_m)$  such that  $S(t_1) = 0$  and  $S(t) > 0$  for  $t < t_1$ . I.e.,  $t_1$  is the first crossing for  $S(t)$  and hence for  $w(t)$ . The ordering (5.7) thus follows.  $\square$

**Lemma 5.3.** *Assuming Condition  $C$ , one has  $Q(t_1) > 0$ .*

*Proof.* Since  $t_1 < t_m < t_*$  one has  $x_a(t_1) > y(t_1)$  and consequently  $w(t_1)$  exceeds 1 and is thus positive. Hence, we can replace  $w(s)$  by  $w(t_1)$  in the integral, and factor, in order to obtain the inequality

$$\begin{aligned}
 Q(t_1) &\geq 0 + \sigma^2 w(t_1) - \sigma^2 c \int_{t_0}^{t_1} e^{c(s-t_1)} w(t_1) ds \\
 &= \sigma^2 w(t_1) \left\{ 1 - \left( 1 - e^{c(t_0-t_1)} \right) \right\} \\
 (5.10) \quad &= \sigma^2 w(t_1) e^{c(t_0-t_1)} > 0.
 \end{aligned}$$

□

**Lemma 5.4.** *If Conditions C and E hold, then  $Q(t_*) < 0$ .*

*Proof.* We write

$$(5.11) \quad Q(t_*) = w'(t_*) + \sigma^2 w(t_*) - \sigma^2 c \int_{t_0}^{t_*} e^{c(s-t_*)} w(s) ds,$$

and note that for any  $t \leq t_*$  one has  $x_a(t) > y(t)$  by Thm 5.1. Consequently, we have the inequality

$$(5.12) \quad w(t) = [1 + x_a(t) - y(t)]^2 \geq 1 = w(t_*).$$

By using this minimum value of  $w$  that is subtracted, we have

$$(5.13) \quad Q(t_*) \leq w'(t_*) + \sigma^2 w(t_*) - \sigma^2 c \int_{t_0}^{t_*} e^{c(s-t_*)} 1 ds.$$

Also, from Thm 5.1, we have  $y'(t_*) = x_a(t_*) - y(t_*) = 0$ , so a computation yields

$$(5.14) \quad w'(t_*) = 2[1 + x_a(t_*) - y(t_*)](x'_a(t_*) - 0) = 2x'_a(t_*).$$

Using  $w(t_*) = 1$ , and evaluating the integral, one obtains

$$(5.15) \quad Q(t_*) \leq 2x'_a(t_*) + \sigma^2 e^{c(t_0-t_*)} < 0.$$

The last inequality follows from Condition E. □

Hence, recalling that  $t_0 < t_1 < t_m < t_*$ , we obtain the result that the maximum of  $Q$ , the limiting volatility precedes the peak of  $y(t)$ , which occurs at  $t_*$ .

**Theorem 5.5.** *There exists a  $t_v \in (t_1, t_*)$  such that  $Q'(t_v) = 0$ .*

In summary, the derivative of  $y$  catches up to  $x_a$  at  $t_1$ . Recalling (4.44), we see that  $Q(t_v) = \sigma^{-2} d\mathbb{V}(t_v)/dt = 0$  corresponds to a maximum in  $\mathbb{V}$ , and this occurs after  $t_1$  and before  $t_m$  where  $x_a$  has its peak. The peak of  $x_a$  precedes the peak of  $y$  at  $t_*$ . Thus,  $\mathbb{V}$  has a maximum prior to the maxima of  $x_a$  and  $y$ .

In conclusion, we have shown that the limiting volatility  $\mathbb{V}(t)$  attains its maximum prior to that of the expected logarithm of the price,  $y(t)$ .

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## APPENDIX

We start with

$$d \log P(t, \omega) = G(D/S) dt + \frac{1}{2} \left\{ \frac{D}{S} G' \left( \frac{D}{S} \right) + \frac{S}{D} G' \left( \frac{S}{D} \right) \right\} \sigma dW(t, \omega).$$

and set  $G(x) := x - 1/x$ , so the model is

$$(5.16) \quad d \log P = \left( \frac{D}{S} - \frac{S}{D} \right) dt + \left( \frac{D}{S} + \frac{S}{D} \right) \sigma dW$$

in which the supply and demand are on a symmetric footing. In other words, when supply exceeds demand, the price moves down in the same way as it moves up when demand exceeds supply. The coefficient for  $dW$  is symmetric in  $S$  and  $D$ .

In order to study market tops and bottoms, we would like to simplify this expression. We consider the regimes: (i)  $D$  and  $S$  deviate by a small amount, and (ii)  $D$  and  $S$  deviate by a large amount.

(i) Suppose that  $D = q + \delta'$  and  $S = q - \varepsilon'$  where  $q > 0$  and  $\delta'$  and  $\varepsilon'$  are small in magnitude, i.e., one is not far from equilibrium. Then we have with  $\delta := \delta'/q$



and  $\varepsilon := \varepsilon'/q$

$$\begin{aligned}\frac{D}{S} - 1 &= \frac{1 + \delta}{1 - \varepsilon} - 1 \simeq \delta + \varepsilon, \\ 1 - \frac{S}{D} &= 1 - \frac{1 - \varepsilon}{1 + \delta} \simeq \delta + \varepsilon, \\ \frac{1}{2} \left( \frac{D}{S} - \frac{S}{D} \right) &= \frac{1}{2} \left( \frac{1 + \delta}{1 - \varepsilon} - \frac{1 - \varepsilon}{1 + \delta} \right) \simeq \delta + \varepsilon\end{aligned}$$

So all three terms are equal up through  $O(\delta, \varepsilon)$ . Thus, when one is not too far from equilibrium, these terms are approximately equal and one can use any of them in the deterministic part of the price equation.

Similarly, under these near equilibrium conditions, the terms  $D/S$ ,  $S/D$  and  $(D/S + S/D)/2$  are all equal through  $O(1)$ .

(ii) Next suppose that we are far from equilibrium, and note that

$$\frac{D}{S} \simeq \frac{D}{S} + \frac{S}{D} \text{ and } \frac{D}{S} - 1 \simeq \frac{D}{S} - \frac{S}{D} \text{ if } D/S \gg 1.$$

Similarly, one has

$$\frac{S}{D} \simeq \frac{D}{S} + \frac{S}{D} \text{ and } 1 - \frac{S}{D} \simeq -\frac{S}{D} - \frac{D}{S} \text{ if } S/D \gg 1.$$

Applying these approximations to (5.16) we see that for market tops (when  $D \geq S$ ) we can use the model

$$d \log P = \left( \frac{D}{S} - 1 \right) dt + \frac{D}{S} \sigma dW,$$

and analogously, for market bottoms, (when  $S \geq D$ ) we use

$$d \log P = \left( 1 - \frac{S}{D} \right) dt + \frac{S}{D} \sigma dW.$$

Note that for the coefficient of  $\sigma dW$ , we are essentially approximating  $G(x) := x + 1/x$  with  $x$  when  $x \geq 1$  and by  $1/x$  when  $x \leq 1$ . Near  $x = 1$ , of course, this introduces a factor of 2 that can be incorporated into  $\sigma$ .

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