1972

Inventory Control of Deteriorating Items

George C. Philip

INVENTORY CONTROL OF

DETERIORATING ITEMS

BY

GEORGE C. PHILIP

A thesis submitted
in partial fulfillment of the requirements for the
degree Master of Science, Major in
Mechanical Engineering
South Dakota State University

1972

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This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable for meeting the thesis requirements for this degree. Acceptance of this thesis does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser | Date

Head, Mechanical Engineering Department | Date

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CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

The problem of the control and maintenance of inventories of physical goods is common to most enterprises. In the past, inventories were considered as the 'grave yard' of American business, because surplus stock was a principal cause for business failures.[1] When companies are small and the competition is lax, a reasonably good inventory policy can be established through an intuitive understanding of the needs of the business. However, in large scale enterprises the inventory system may become too large and complex to be analysed intuitively and any deviation from the optimum inventory policy would mean substantial losses. As a result of the current small profit margins, proper inventory control has become even more important.

The basic concept in inventory control consists of striking a balance between the cost factors which increase as inventories increase and those which decrease as inventories increase. Among the costs which increase are:

1. Interest: (cost of money) Some companies use the interest paid for the capital and others use the return that could have been obtained by investing the capital elsewhere. In either case, the cost of the goods in inventory must be considered.

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[1]Thomas M. Whitin, The Theory of Inventory Management (2d ed.; Princeton, New Jersey: Princeton University Press, 1957), p. 4.

2. Deterioration costs. Deterioration costs are the losses in value due to actual deterioration, obsolescence or damage of the inventory item.

3. Insurance and taxes. Since most inventories are insured, this cost should be taken into account. Taxes are mainly property taxes.

4. Storage costs. Storage costs include rent, or its equivalent ownership costs, and heat, light and other utility costs.

It is the usual practice to combine all of these costs into a single item called the holding cost or inventory carrying cost, expressed as a percentage of the factory or purchase cost of the items being stored.

The decreasing costs are:

1. Ordering cost. The ordering cost is the internal cost incurred in placing and processing a purchase order and is usually assumed to be a constant for each order placed. It would include cost of forms, cost of preparation and, frequently, receiving inspection.

2. Set up costs. The set up costs are the costs incurred in preparing a machine for the production of an item. It is applicable to items produced internally. This cost is a constant for each set up.

3. Shortage cost. The shortage cost is the cost incurred due to the non-availability of an item in stock. This cost would include the additional cost involved in taking emergency

measures to meet the demand in time as well as the loss of customers' good will and positive loss of profit if the demand is not met in time. If a company keeps a spare parts inventory for its own use, the shortage may result in direct losses if a machine becomes inoperative due to the non-availability of spare parts in stock.

The shortage cost may depend upon the amount of the shortage, the duration of the shortage and/or the number of shortages per unit time.[2]

The earliest theoretical work in the field of inventory control was the derivation of a formula for the economic order quantity. The economic order quantity is a minimum cost relation that takes into account both the ordering and set up costs and the holding costs. Other authors call this the Economic Lot Size.

Before going into a discussion of the different models, it may be specified that the following assumptions are used for all the models unless otherwise mentioned.

(a) Lead time is known and adequate so that goods can be ordered to arrive when they are needed.

(b) Supply (production) is instantaneous.

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[2] Russell L. Ackoff and Maurice W. Sasieni, Fundamentals of Operations Research (New York: John Wiley & Sons, Inc., 1968), p. 172.

In 1915, Harris derived his 'Simple Economic Lot-Size Formula.'[3] His model has the following assumptions:

(a) Demand is continuous and is at a constant rate.

(b) No shortages are permitted, that is, the shortage cost is infinite.

Harris' formula is

$$Q = K\sqrt{\frac{PS}{C}} \tag{1.1}$$

where

Q = economic order quantity.

K = a constant which takes into account the interest and other costs such as storage cost, insurance and taxes on a daily basis.

P = ordering cost per order.

S = rate of daily usage.

C = unit cost.

About ten years later, several other investigators developed essentially the same formula for economic order quantity.[4] These took the form

$$Q = \sqrt{\frac{2YS}{IC}} \tag{1.2}$$

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[3] F. E. Raymond, Quantity and Economy in Manufacture (New York: McGraw Hill Book Co., 1931), pp. 121-122, cited by Om Prakash Goel, "Studies and Application of the Theory of Inventory and Production" (unpublished M.S. Thesis, South Dakota State University, 1966), pp. 5-6.

[4] Whitin, p. 32.

where

Q = economic order quantity.

Y = yearly sales in number of units.

S = ordering cost per order.

I = holding cost/unit time/dollar.

C = unit cost.

This 'Simple Economic Order Quantity Formula' continues to be used in the same form. An implicit assumption is that there is no deterioration during the storage period. This simplification, while true in many situations, is not universal in nature. Because almost all items degrade during storage, deterioration will be defined as decay, damage and spoilage such that the item cannot be used for its original purpose and must be destroyed or abandoned. The terms deterioration and decay will be used interchangeably. If the item can be repaired or restored and the rate of decay is constant, the cost of such repair or restoration can be included in the holding costs. Frequently this is not the case and it will be unrealistic to use this model in those cases. The effect of deterioration on inventory level is similar to that of an additional demand over the actual demand as shown in Figure 1.1. But this effect of the deterioration is not taken into account in the previous analysis.

A number of specialized inventory situations in which deterioration takes place have been reported in the literature.

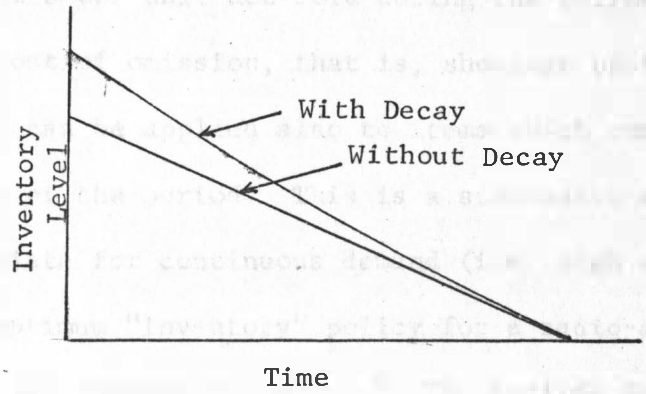

Figure 1.1. A simple inventory model.

The inventory control of style goods was studied by Whitin.[5] Hid model has the following assumptions:

(a) An order can be received only at the beginning of the single period under consideration.

(b) Demand is probabilistic in nature.

(c) The inventory at the end of the period is liquidated at a loss.

(d) The shortage cost is proportional to the quantity of goods that are short.

The condition for maximum profit is reached when the expected profit obtainable through stocking an additional unit is equal to the expected losses from stocking that unit,

i.e., $$p P + p O = (1-p)L$$

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[5]Whitin, pp. 62-72.

where

p = probability of selling an additional unit during the period

P = profit per unit

L = loss per unit not sold during the period

O = cost of omission, that is, shortage cost per unit

This model can be applied also to items which completely deteriorate at the end of the period. This is a stochastic model and might not be appropriate for continuous demand (i.e. high volume) items.

The optimum "inventory" policy for a radio-active Nuclide Generator was studied by Emmons.[6] The Nuclide Generator uses a radio-active element of long half-life which decays exponentially over time. Methods are proposed for finding the optimum replenishment quantity for such exponentially decaying substances as shown in Figure 1.2.

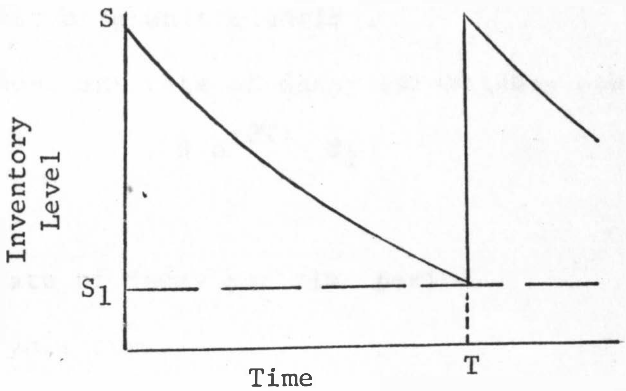

Figure 1.2. Inventory model for Nuclide Generators.

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[6] Hamilton Emmons, "A Replenishment Model for Radioactive Nuclide Generators," Management Science, January, 1968, pp. 263-274.

Specifically, a time interval T is considered during which the radioactivity of the material decreases from the initial level, S, to the final level, S_1. The following assumptions are made:

(a) Replenishment is made when the inventory level reaches the reorder point, S_1.

(b) The residual stock at the time of replenishment is discarded.

The replenishment cost and carrying cost are taken into consideration for the analysis. The replenishment cost [C(S)] is the sum of the fixed ordering cost, and the purchase cost which depends on the size of the replenishment. Note that in this model the purchase cost per unit time is a variable cost, as this is the cost of deterioration which depends on the inventory level. The carrying cost (C_1) per unit quantity per unit time is a fixed fraction (f) of the replenishment cost of a unit quantity.

The constant rate of decay establishes the relationship

$$S\, e^{-PT} = S_1 \tag{1.3}$$

where

P = rate of decay per time period.

T = cycle time.

S = order quantity.

Therefore

$$T = \frac{1}{P} \ln(S/S_1). \tag{1.4}$$

In practice, f is very small when compared with P and the carrying cost can be neglected in the calculation of the total variable cost. The total cost per unit time [K(S)] is equal to the replenishment cost per unit time [$C_2(S)$], or

$$K(S) = C_2(S)$$

$$= \frac{C(S)}{T}$$

$$= \frac{P\ C(S)}{\ln (S/S_1)} \quad (1.5)$$

Assuming that the cost function is continuous and differentiable, to minimize the cost per unit time, we set the derivative of Equation 1.5 equal to zero.

$$C'(S) \ln(S/S_1) - \frac{C(S)}{S} = 0$$

$$S = S_1\, e^{C(S)/SC'(S)}. \quad (1.6)$$

The minimum value of S which satisfies this equation can be found out by any of a number of numerical methods.

Ghare and Schrader describe a model for an item with a constant rate of deterioration.[7] The following assumptions are made:

(a) Demand is a known, regular and integrable function of time, $D_{(x)}$.

(b) No shortages are allowed.

(c) There is no repair or replacement of deteriorated inventory during a cycle time.

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[7]P. M. Ghare and G. F. Schrader, "A model for exponentially decaying inventory," Journal of Industrial Engineering, Vol. XIV (1963), pp. 238-243.

(d) Demand and decay are simultaneous and continuous.

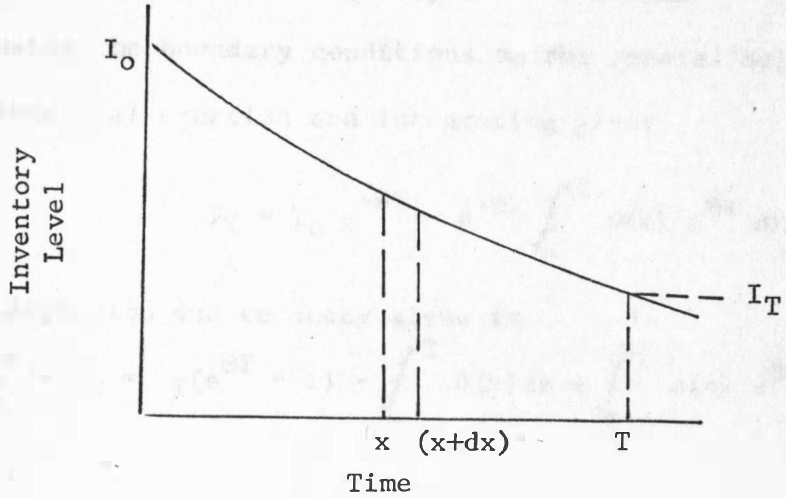

Figure 1.3. Inventory situation for Ghare's model.

Figure 1.3 indicates the generalized inventory situation as described by the assumptions. A time interval T is considered during which the initial inventory, I_o, diminishes to I_T.

The constant rate of decay of an inventory item may be represented mathematically by the differential equation

$$I_{(x + dx)} = I_x e^{-\theta dt} \tag{1.7}$$

where

I_x = inventory level at the end of time x.

$I_{(x + dx)}$ = inventory level at the end of time (x + dx).

θ = a fraction representing the rate of decay per time period.

A small interval dx after time x is considered. During the interval dx, the loss due to decay is

$$dL = I_x - I_x e^{-\theta dx}.$$

The total inventory depletion during dx is

$$-dI = I_x - I_x e^{-\theta dx} + D(x)dx$$

Imposing the boundary conditions on the general solution of this differential equation and integrating gives

$$I_T = I_o e^{-\theta T} - e^{-\theta T} \int_o^T D(x) e^{\theta x} dx . \tag{1.8}$$

The depletion due to decay alone is

$$I_T^* - I_T = I_T(e^{\theta T} - 1) - \int_o^T D(x)dx + \int_o^T D(x) e^{\theta x} dx \tag{1.9}$$

where

I_T^* = inventory level at the end of time T, if depletion were due to demand usage alone.

The exponential decay model was further simplified for constant demand, that is, $D(x) = K$, and for $I_T = 0$. Assuming that $\frac{1}{\theta}$ is considerably larger than T, Equation 1.9 can be simplified to

$$I_T^* - I_T = \frac{K\theta T^2}{2} . \tag{1.10}$$

The economic order quantity also was determined by Ghare and Schrader. The total cost per unit time is equal to the sum of the purchase cost, ordering cost and holding cost, per unit time.

$$C_t = \frac{C}{T}[KT + \frac{K\theta T^2}{2}] + \frac{A}{T} + \frac{iC}{T}[KT + \frac{K\theta T^2}{2}] \tag{1.11}$$

where

C_t = total cost/unit time.

C = unit cost.

i = a fraction such that iC is the holding cost per unit of initial inventory.

In order to minimize the total cost per unit time Equation 1.11 is differentiated and set equal to zero:

$$\frac{d}{dt}(C_t) = \frac{K\theta C}{2} - \frac{A}{T^2} + KiC + KiC\theta T = 0. \quad (1.12)$$

The value of T which satisfies this equation, T*, gives the cycle time corresponding to the economic order quantity. Finally, the economic order quantity, Q, can be computed using the equation

$$Q = KT + \frac{K\theta T^2}{2}. \quad (1.13)$$

This model can be used for a deteriorating item with a variable rate of decay if the appropriate changes are made in the basic differential equation, but this changes the entire analysis.

A dynamic deterministic lot-size model was discussed by Hadley and Whitin using dynamic programming.[8] This approach is characterized by the following assumptions:

(a) Only a fixed planning horizon or a finite period of time is considered.

(b) The planning horizon is divided into 'n' periods and orders can be placed only at the beginning of these periods. Also, there is a procurement lead time associated with each other.

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[8] G. Hadley and T. M. Whitin, Analysis of Inventory Systems (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1963), pp. 336-343.

(c) The inventory carrying cost per unit per period varies from period to period.

(d) The demand rate is deterministic.

(e) No back orders or lost sales are allowed.

(f) The unit cost of the item is a constant.

The quantities $Q_j (Q_j \geq 0, J = 1, 2, 3, \ldots n)$ to be ordered at the beginning of each of the n periods is determined by minimizing the sum of the ordering and carrying costs over the planning horizon.

Veinott has studied a multiperiod single product, non-stationary inventory situation.[9] This is a dynamic programming problem and the basic structure of the model is similar to the previous model. This has the following major differences.

(a) The author considers a constant fraction of the inventory being spoiled during each interval of time. More precisely, whenever the amount of stock on hand Y_i, after ordering in period i, is greater than the total demand D_{im} during the period, then a fraction (1-a) of the inventory on hand spoils and is not available for future use.

(b) There are several classes of demand for the product in each period.

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[9]Arthur F. Veinott, Jr., "Optimal Policy in Dynamic Single Product Non-Stationary Inventory Model With Several Demand Classes," Operations Research, March 1965, pp. 761-768.

(c) Partial or complete back logging of unfilled demand is permitted.

(d) The ordering cost is proportional to the amount of stock ordered.

(e) The holding and penalty costs vary over time.

Although both the methods discussed above are very powerful, these analyses cannot be used for items with variable rate of decay. This is due to the fact that the cost due to a variable rate of decay cannot be included in the variable carrying cost. Note that the deterioration during a particular period depends on the time at which the stock arrives, whereas the variable carrying cost is fixed for a fixed period.

Veinott's model can be used for an item with constant rate of decay subject to the above mentioned conditions.

The review of the selected articles in the literature show that deterioration has been studied under the following conditions:

(1) Constant rate of deterioration.

(2) All decay occurs at the end of the storage period.

Missing is the more general situation in which the rate of deterioration varies with time. Such generalized relations would require replacing the exponential function with one that degenerates to the exponential under specific conditions, and, if possible, degenerates to 'all deterioration at the end of the period' under a different set of specified conditions. Such a relation would be advantageous in providing a more general framework for those and other special cases.

CHAPTER 2

DEVELOPMENT OF THE MODEL

Although a number of models have been developed that provide for deterioration during an inventory cycle, they are all designed for specific situations. A generalized model is needed that will include these as special cases of the generalized model. Obviously a single general model for all possible cases is not possible, if for no other reason than having to deal with discrete and continuous functions. However, a broad generality seems possible.

A general relation that is widely used in the Reliability area is the Weibull distribution, with probability density function:

$$f(x) = \frac{B}{A} x^{B-1} e^{-x^B/A} \tag{2.1}$$

where

A = scale parameter, positive real number.

B = shape parameter, positive real number.

x = random variable, positive real number.

For inventory control work, f(x) will be the probability density function for the time to deterioration. Therefore, the probability that the item will deteriorate prior to sometime, t, is

$$F(t) = \int_o^x \frac{B}{A} t^{B-1} e^{-t^B/A} dt$$

$$= 1 - e^{-x^B/A}. \tag{2.2}$$

The instantaneous deterioration rate is

$$Z(x) = \frac{B}{A} x^{B-1} \tag{2.3}$$

These are different forms of the Weibull distribution, which can be applied to a family of decay rate functions which remain stable, or increase or decrease "smoothly" with time, that is, without any discontinuities or turning points. The constant failure rate, or exponential density function, is thus included as a special case. Experiments on certain deteriorating items have shown that their deterioration follows a Weibull distribution.[10] Hence the Weibull distribution can be advantageously used for an analysis of decay rate functions.

The inventory model using the Weibull distribution has the following assumptions:

(a) Demand is known and has a constant rate.

(b) No shortages are allowed.

(c) There is no repair or replacement of deteriorated items during a cycle time.

(d) Holding cost per unit per unit time is a constant.

(e) Ordering cost and set up cost are proportional to the number of orders and number of setups respectively.

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[10] J. N. Berrotom, "Practical Applications of Weibull distribution," Industrial Quality Control, August 1969, pp. 71-79.

(f) Unit cost is constant.

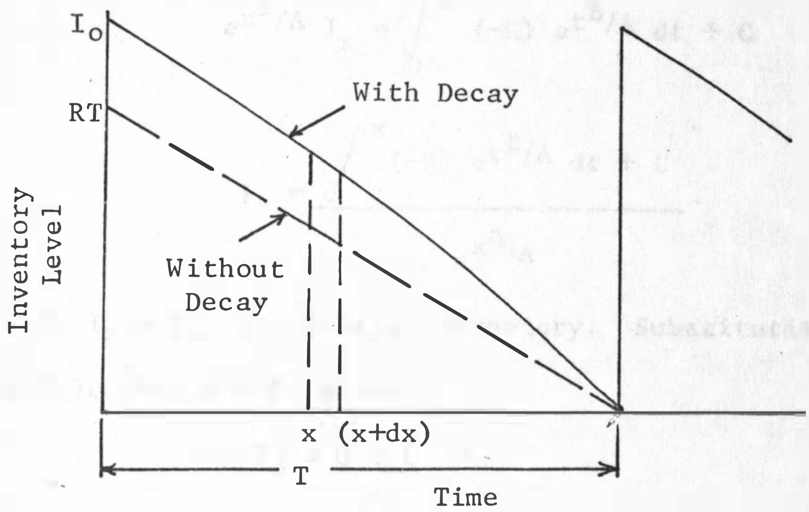

Figure 2.1. Inventory situation for a deteriorating item.

Figure 2.1 shows the inventory-time relation for the model under consideration. During the cycle time, T, the initial inventory, I_o, depletes completely due to decay and demand usage. To determine the general relation, the inventory depletion during a time dx must be expressed. That is, the change in inventory during some small time, dx, is the result of both depletion due to deterioration and depletion due to demand usage. This can be expressed as

$$-dI = I_x \text{ Rate of decay } dx + R\, dx$$

or

$$-dI = I_x \frac{B}{A} x^{B-1} dx + R\, dx \tag{2.4a}$$

where

I_x = inventory level at any time x such that $0 \leq x \leq T$.

Equation 2.4a can be rewritten as

$$\frac{dI}{dx} + \frac{B}{A} x^{B-1} I_x = -R \tag{2.4b}$$

and solved using standard procedures giving

$$e^{x^B/A} I_x = \int_o^x (-R)\, e^{t^B/A}\, dt + C \qquad (2.5a)$$

or

$$I_x = \frac{\int_o^x (-R)\, e^{t^B/A}\, dt + C}{e^{x^B/A}} . \qquad (2.5b)$$

When $x = 0$, $I_x = I_o$, the initial inventory. Substituting this into Equation 2.5b when $x = 0$, gives

$$I_o = \frac{0 + C}{1}$$

or

$$C = I_o$$

Thus

$$I_x = \frac{\int_o^x (-R)\, e^{t^B/A}\, dt + I_o}{e^{x^B/A}} \qquad (2.6)$$

At some time, T, the inventory reaches zero, that is, when

$$x = T$$

$$I_x = 0$$

These values can be substituted into Equation 2.6,

$$0 = \frac{\int^T (-R)\, e^{x^B/A}\, dx + I_o}{e^{x^B/A}} \qquad (2.7)$$

and the relation solved for I_o. Since the denominator disappears in the solution process,

$$I_o = \int_o^T R\, e^{x^B/A}\, dx . \qquad (2.8)$$

The portion of the initial inventory stocked to cover the decay process during the cycle time, T, is

$$I_D = I_o - RT$$
$$= \int_o^T R\, e^{x^B/A}\, dx - RT. \qquad (2.9)$$

Having derived a general equation for the inventories, the total variable cost for the interval, T, can be found in terms of T. The total variable cost (K_T) for the period is equal to the sum of the cost of inventory, the holding cost and the ordering cost. For holding cost calculations it is assumed that the average inventory is equal to half the initial inventory. Thus

$$K_T = C[I_o - RT] + C_1 \frac{I_o}{2} T + C_3 \qquad (2.10)$$

where

C = item cost per unit quantity.

C_1 = holding cost per unit quantity per unit time.

C_3 = ordering cost per order.

Utilizing Equations 2.8 and 2.9 provides

$$K_T = C\left[\int_o^T R\, e^{x^B/A}\, dx - RT\right] + \frac{C_1 T}{2} \int_o^T R\, e^{x^B/A}\, dx + C_3. \qquad (2.11)$$

The total variable cost per unit time (C_T) can be found by dividing K_T by the storage period T.

$$C_T = \frac{K_T}{T}$$
$$= \frac{C}{T} \int_o^T R\, e^{x^B/A}\, dx - CR + \frac{C_1}{2} \int_o^T R\, e^{x^B/A}\, dx + \frac{C_3}{T}. \qquad (2.12)$$

To minimize the total variable cost per unit time Equation 2.12 can be differentiated and set equal to zero.

That is,

$$\frac{d(C_T)}{dT} = 0.$$

Carrying out the differentiation gives

$$\frac{C}{T^2}\left[T\ R\ e^{T^B/A} - \int_o^T R\ e^{T^B/A}\,dx\right] + \frac{C_1}{2} R\ e^{T^B/A} - \frac{C_3}{T^2} = 0. \tag{2.13}$$

The terms within the bracket in Equation 2.13, which shall be designated as A, can be further simplified.

Remembering that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

it can be noted that T^B/A can replace x, giving

$$e^{T^B/A} = \sum_{n=0}^{\infty} \frac{T^{nB}}{A^n\ n!}. \tag{2.14}$$

The first term becomes

$$T\ R\ e^{T^B/A} = T\ R \sum_{n=0}^{\infty} \frac{T^{nB}}{A^n\ n!} \tag{2.15}$$

and the second term becomes

$$\int_o^T R\ e^{T^B/A}\,dx = R \int_o^T \sum_{n=0}^{\infty} \frac{x^{nB}}{A^n\ n!}\,dx\ . \tag{2.16}$$

The second term can now be integrated.

$$\int_o^T R\ e^{x^B/A}\,dx = R \sum_{n=0}^{\infty} \frac{x^{nB+1}}{(nB+1)\ A^n\ n!}\Bigg|_o^T \tag{2.17}$$

or

$$= R \sum_{n=0}^{\infty} \frac{T^{nB+1}}{(nB+1)\ A^n\ n!} \tag{2.18}$$

Substituting Equation 2.15 and 2.17 into the first term of Equation 2.13 gives

$$A = RT \sum_{n=0}^{\infty} \frac{T^{nB}}{A^n \, n!} - R \sum_{n=0}^{\infty} \frac{T^{nB+1}}{(nB+1) \, A^n \, n!} . \tag{2.19}$$

Collecting terms gives

$$A = R \sum_{n=0}^{\infty} \frac{T^{nB+1}}{A^n \, n!} \left[1 - \frac{1}{(nB+1)} \right] \tag{2.20}$$

which can be restated as

$$A = R \sum_{n=1}^{\infty} \frac{T^{nB+1}}{A^n \, n!} \; \frac{nB}{(nB+1)} \tag{2.21}$$

where the lower limit was changed to n=1 from n=0, as the expression reduces to zero for n=0.

Substituting Equation 2.21 into Equation 2.13 gives

$$\frac{C}{T^2} \quad R \sum_{n=1}^{\infty} \frac{T^{nB+1}}{A^n \, n!} \; \frac{nB}{(nB+1)} \quad + \frac{C_1}{2} R \, e^{T^B/A} - \frac{C_3}{T^2} = 0$$

or

$$C \, R \sum_{n=1}^{\infty} \frac{T^{nB-1}}{A^n \, n!} \; \frac{nB}{(nB+1)} + \frac{C_1}{2} R \, e^{T^B/A} - \frac{C_3}{T^2} = 0 . \tag{2.22}$$

If this relation is correct, when B=1, Equation 2.22 is the corresponding equation in Ghare's analysis for exponentially decaying items.

Setting B=1 gives

$$CR \sum_{n=1}^{\infty} \frac{T^{n-1}}{A^n \, n!} \; \frac{n}{(n+1)} + \frac{C_1}{2} R \, e^{T/A} - \frac{C_3}{T^2} = 0 . \tag{2.23}$$

Expanding the first quantity and using

$$e^{T/A} = [1 + T/A + T/A^2 \, 2! + . . .]$$

Equation 2.23 can be written as

$$CR\left[\frac{1}{A}\frac{1}{2} + \frac{T}{A^2\,2!}\frac{2}{3} + \cdots\right] + \frac{C_1}{2}R\left[1 + \frac{T}{A} + \frac{T^2}{A^2\,2!} + \cdot\right] - \frac{C_3}{T^2} \tag{2.24}$$

Assuming that A is considerably larger than T (which is the assumption made in Ghare's model) and neglecting terms involving $\frac{T}{A^2}$ and its higher orders in the first term in Equation 2.24 and $\frac{T^2}{A^2}$ and its higher orders in the second term, Equation 2.24 reduces to

$$\frac{CR}{2A} + \frac{C_1}{2}R + \frac{C_1\,R\,T}{2A} - \frac{C_3}{T^2} = 0. \tag{2.25}$$

The corresponding equation in Ghare's model (Equation 1.8) is

$$\frac{CK\theta}{2} + (iC)K + \frac{(iC)K\theta T}{2} - \frac{A}{T^2} = 0$$

where

$\theta = \frac{1}{A}$

$2iC = C_1$

$A = C_3$

$K = R$

Therefore Ghare's equation is a special case and the derivation is correct for at least this point.

Further, when $A \rightarrow \infty$ Equation 2.22 reduces to the equation for a non-deteriorating item. Carrying out this substitution gives

$$0 + \frac{C_1\,R}{2} - \frac{C_3}{T^2} = 0 \tag{2.26}$$

which is the corresponding equation for a non-deteriorating item. Again, this adds confidence to the derivation.

Equation 2.22 can be solved numerically for T, to obtain the cycle time, T*, corresponding to the economic order quantity. The method of solution is explained in the next chapter.

The economic order quantity, I*, is calculated from T* using Equations 2.8 and 2.17, that is

$$I^* = R \sum_{n=0}^{\infty} \frac{T^{(nB+1)}}{(nB+1)\ A^n\ n!} . \tag{2.27}$$

For the computation of the total cost per unit time to be used later for the verification of the results, Equation 2.12,

$$C_T = \frac{C}{T} \int_0^T R\ e^{x^B/A}\ dx - CR + \frac{C_1}{2} \int_0^T R\ e^{x^B/A}\ dx + \frac{C_3}{T} \tag{2.12}$$

(repeated)

can be modified, by collecting and rearranging terms, to

$$C_T = \frac{C_3}{T} - CR + \left(\frac{C}{T} + \frac{C_1}{2}\right) \int_0^T R\ e^{x^B/A}\ dx . \tag{2.28}$$

Substituting Equation 2.17

$$\int_0^T R\ e^{x^B/A}\ dx = R \sum_{n=0}^{\infty} \frac{T^{(nB+1)}}{(nB+1)\ A^n} \frac{1}{n!}$$

into the Equation 2.28 gives

$$C_T = \frac{C_3}{T} - CR + \left(\frac{C}{T} + \frac{C_1}{2}\right) R \sum_{n=0}^{\infty} \frac{T^{nB+1}}{(nB+1)\ A^n\ n!} \tag{2.29}$$

and expanding the last term gives

$$C_T = \frac{C_3}{T} - CR + \left(\frac{C}{T} + \frac{C_1}{2}\right) R \left[T + \frac{T^{B+1}}{(B+1)\ A} + \frac{T^{2B+1}}{(2B+1)\ A^2\ 2!} + \frac{T^{3B+1}}{(3B+1)\ A^3\ 3!} + \ldots\right]. \tag{2.30}$$

This equation for total cost can be solved by inserting the proper values for the variables. Thus, all of the normally desired quantities of inventory problems can be determined for inventories which are subject to a Weibull distributed decay.

CHAPTER 3

COMPUTERIZED MODEL

The solution of Equation 2.22 for the desired value of T, the optimum cycle time, cannot be carried out directly since the variable, T, cannot be separated from the other variables. The solution, however, can be found by iterative processes, which effectively try various values of T until the equation does equal zero.

The corrective method of Newton is one such method, suitable for computer solution. This method makes use of a recursion formula

$$T_n = T_{n-1} - \frac{f(T_{n-1})}{f'(T_{n-1})} \tag{3.1}$$

where

T_n = successive values of T, calculated from the preceding values T_{n-1}.

$f(T_{n-1})$ = value of the function, f(T), for $T = T_{n-1}$.

The term $\frac{f(T_{n-1})}{f'(T_{n-1})}$ is called the correction. This method can be used for the solution of Equation 2.22 using the computer.

In geometric terms, the procedure involves finding the tangent to the curve f(T) at the point T_{n-1} and determining its intersection T_n with the T axis. Figure 3.1 illustrates the method. As can be seen, the initial value of T, T_o, should be sufficiently close to T*.

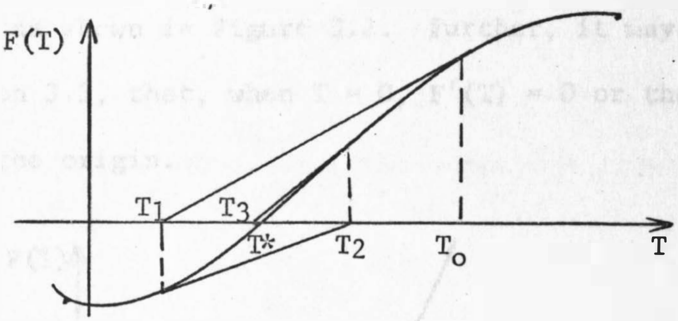

Figure 3.1. Illustration of Newton's method.

To assure convergence of T_o towards T* for all values of T_o, Equation 2.22

$$CR\sum_{n=1}^{\infty}\frac{T^{nB+1}}{A^n\, n!}\,\frac{nB}{(nB+1)} + \frac{C_1}{2} R\, e^{T^B/A} - \frac{C_3}{T^2} = 0 \qquad (2.22)\ \text{(repeated)}$$

can be modified by multiplying both sides by T^2, giving

$$CR\sum_{n=1}^{\infty}\frac{T^{nB+1}}{A^n\, n!}\,\frac{nB}{(nB+1)} + \frac{C_1}{2} R\, T^2\, e^{T^B/A} - C_3 = 0. \qquad (3.2)$$

But if the value of T is not optimum, the value of Equation 3.2 is not zero, but rather F(T), or

$$F(T) = CR\sum_{n=1}^{\infty}\frac{T^{nB+1}}{A^n\, n!}\,\frac{nB}{(nB+1)} + \frac{C_1}{2} R\, T^2\, e^{T^B/A} - C_3. \qquad (3.2a)$$

The derivative of this function, F'(T) is

$$F'(T) = CR\sum_{n=1}^{\infty}(nB+1)\frac{T^{nB+1}}{A^n\, n!}\,\frac{nB}{(nB+1)} + \frac{C_1}{2}R[T^2\, e^{T^B/A}\,\frac{B}{A}\,T^{B-1} + e^{T^B/A}\, 2T] \qquad (3.3)$$

which may be simplified to

$$F'(T) = CR\sum_{n=1}^{\infty}\frac{T^{nB}\, B}{A^n(n-1)!} + \frac{C_1}{2} R\, e^{T^B/A}\,[\frac{B}{A}\,T^{B+1} + 2T]. \qquad (3.4)$$

This expression increases steadily with an increase in the value of T. But, F'(T) is the slope of the function F(T). Hence the slope of the function F(T) increases steadily with increase in the value of T from

0 to ∞, as shown in Figure 3.2. Further, it may be noted from expression 3.3, that, when $T = 0$, $F'(T) = 0$ or the slope of $F(T)$ is zero at the origin.

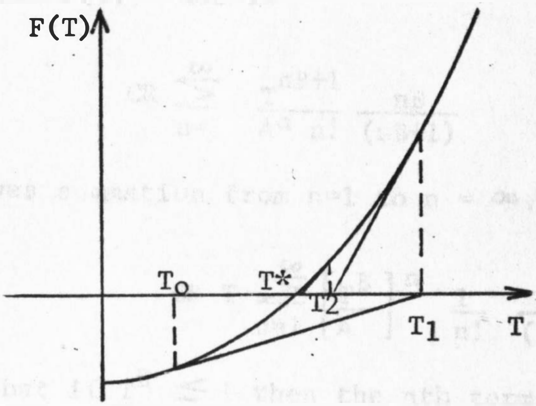

Figure 3.2. Newton's method as applied to the modified function F(T).

These conditions ensure that any positive value of T_o will bring about convergence towards T*, if Newton's method is used for solving F(T). Note that Equations 2.22 and 3.1 are forms of the same equation and will provide identical answers for T*.

The value of T_o may be chosen as the optimum cycle time for a non-deteriorating item under the same conditions as the deteriorating item. This gives a value of T_o reasonably close to T*. Thus

$$T_o = \sqrt{\frac{2C_3}{C_1 R}} \tag{3.5}$$

where

C_3, C_1, R are as previously defined.

It is intuitively obvious that in this case, T_o will be greater than T*, the optimum cycle time for a deteriorating item.

The solution of Equation 3.2 using Newton's method involves the computation of the values of F(T) and F'(T) for different values of T. It should be noted that the computation of the first term of the function F(T), that is

$$CR \sum_{n=1}^{\infty} \frac{T^{nB+1}}{A^n \; n!} \frac{nB}{(nB+1)}$$

involves summation from n=1 to n = ∞. This term may be rewritten as

$$CR \; T \sum_{n=1}^{\infty} \left[\frac{T^B}{A}\right]^n \frac{1}{n!} \frac{nB}{(nB+1)} .$$

Note that if $\frac{T^B}{A} \leq 1$ then the nth term will be less than or equal to $\left[\frac{1}{n!} \frac{nB}{(nB+1)}\right]$ times the first term. For example, even for a very low value of B = 0.5 the 7th term will be less than or equal to

$$\frac{1}{7!} \frac{7 \times 0.5}{(7 \times 0.5 + 1)} = \frac{1}{6480}$$

times the first term. In a similar manner the 8th term will be less than $\frac{1}{50,000}$ times the first term. Hence if $\frac{T^B}{A}$ could be made less than or equal to 1, then summation over the first seven terms would give sufficiently accurate values.

As $T_o \geq T^*$, due to the particular shape of the function F(T), all values of T during the iterative process will be less than T_o. Hence to satisfy the condition

$$\frac{T^B}{A} \leq 1$$

it is enough to ensure that

$$\frac{T_o^{\,B}}{A} \leq 1 .$$

$\frac{T_o^B}{A}$ can be made less than or equal to one by suitably choosing the unit of time.

For computation of F'(T) Equation 3.4

$$F'(T) = CR \sum_{n=1}^{\infty} \frac{T^{nB}\, B}{A^n (n-1)!} + \frac{C1}{2} R\, e^{T^B/A} \left[\frac{B}{A} T^{B+1} + 2T\right] \quad (3.4) \text{ (repeated)}$$

could be altered to

$$F'(T) = CR \sum_{n=1}^{\infty} \frac{T^{(n-1)B}}{A^{n-1}} \frac{T^B}{A} \frac{B}{(n-1)!} + \frac{C_1 R}{2} \frac{B}{A} T^{B+1} e^{T^B/A} + \frac{C_1 R}{2} 2T e^{T^B/A} \quad (3.6)$$

That is,

$$F'(T) = CR \frac{B}{A} T^B \sum_{n=1}^{\infty} \left(\frac{T^B}{A}\right)^{n-1} \frac{1}{(n-1)!} + \frac{C_1 R}{2} \frac{B}{A} T^{B+1} e^{T^B/A} + \frac{C_1 R}{2} 2T e^{T^B/A}. \quad (3.7)$$

Realizing that $\sum_{n=1}^{\infty} \left(\frac{T^B}{A}\right)^{n-1} \frac{1}{(n-1)!} = e^{T^B/A}$

the equation may be written as

$$F'(T) = CR \frac{B}{A} T^B e^{T^B/A} + \frac{C_1 R}{2} \frac{B}{A} T^{B+1} e^{T^B/A} + \frac{C_1 R}{2} 2T e^{T^B/A} \quad (3.8)$$

collecting terms

$$F'(T) = R\, e^{T^B/A} \left[C \frac{B}{A} T^B + \frac{C_1}{2} \frac{B}{A} T^{B+1} + C_1 T\right]. \quad (3.9)$$

Before we can proceed towards the construction of a computer program it should be noted that the expression for economic order quantity

$$I^* = R \sum_{n=0}^{\infty} \frac{T^{*\,nB+1}}{(nB+1)\, A^n\, n!}$$

also involves summation from zero to infinity.

This equation can be further simplified to

$$I^* = RT^* \sum_{n=0}^{\infty} \left(\frac{T^{*B}}{A}\right)^n \frac{1}{(nB+1)\, n!}$$

and by an argument similar to that used earlier, it can be shown that summation over the first seven terms, that is, from n=0 to n=6, will give sufficiently accurate values for I*.

Having analysed the applicability of Newton's method to the solution of Equation 3.1, and the computational aspects of the various quantities, we are now in a position to develop the computer program. The flow chart of the program for the solution of Equation 3.2 and computation of the economic order quantity is shown in Figure 3.3.

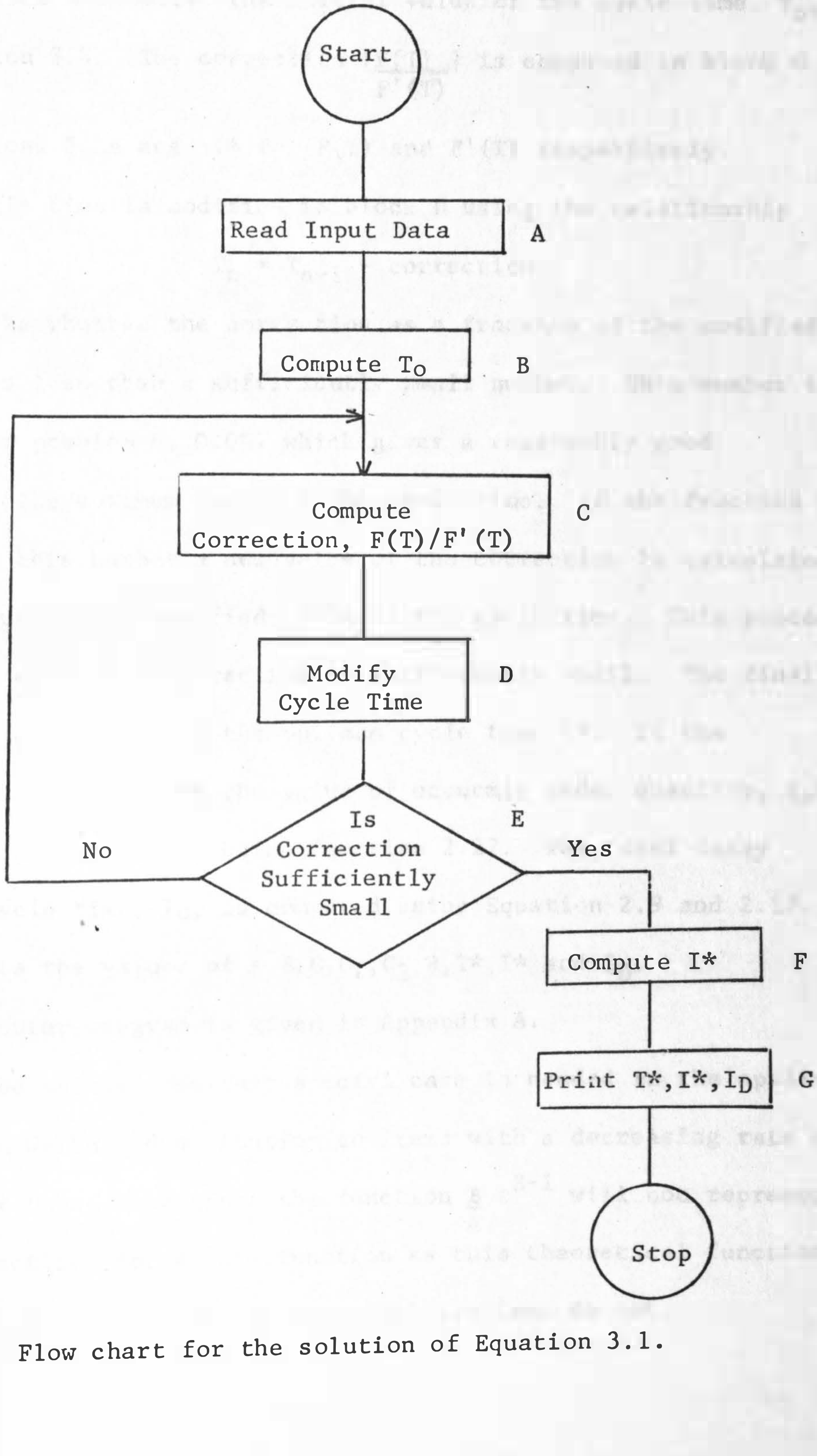

Figure 3.3. Flow chart for the solution of Equation 3.1.

In the flow diagram block A reads the input data, A, B, C, C_1, C_3, and R and block B computes the initial value of the cycle time, T_o, using Equation 3.5. The correction $[\frac{F(T)}{F'(T)}]$ is computed in block C using Equations 3.2a and 3.4 for F(T) and F'(T) respectively.

The cycle time is modified in block D using the relationship

$$T_n = T_{n-1} - \text{correction}$$

Block E checks whether the correction as a fraction of the modified cycle time is less than a sufficiently small number. This number is chosen in our problem as 0.001 which gives a reasonably good accuracy for the optimum value of the cycle time. If the fraction is greater than this number a new value of the correction is calculated, in block C, using the modified value of the cycle time. This process is repeated, until the correction is sufficiently small. The final value of the cycle time is the optimum cycle time T^*. If the correction is small enough the value of economic order quantity, I_o^*, is calculated, in block F, using Equation 2.27. The total decay during the cycle time, I_D, is computed using Equation 2.9 and 2.17. Block G prints the values of $A, B, C, C_1, C_3, R, T^*, I^*$ and I_D.

The computer program is given in Appendix A.

It may be pointed out that special care is needed in the application of the Weibull distribution to items with a decreasing rate of decay. Under these conditions the function $\frac{B}{A} t^{B-1}$ will not represent exactly a practical decay rate function as this theoretical function becomes infinite as $T \rightarrow 0$, while practical problems do not.

Figure 3.4 shows hypothetical, actual and theoretical decay rate functions. However, for large values of B, the difference between the theoretical and actual decay rate functions will be very small. Also, decreasing decay rate functions with small values of B are not of much practical importance. Hence the model developed as well as the computor program can be used in most of the practical decreasing decay rate situations with reasonably good accuracy.

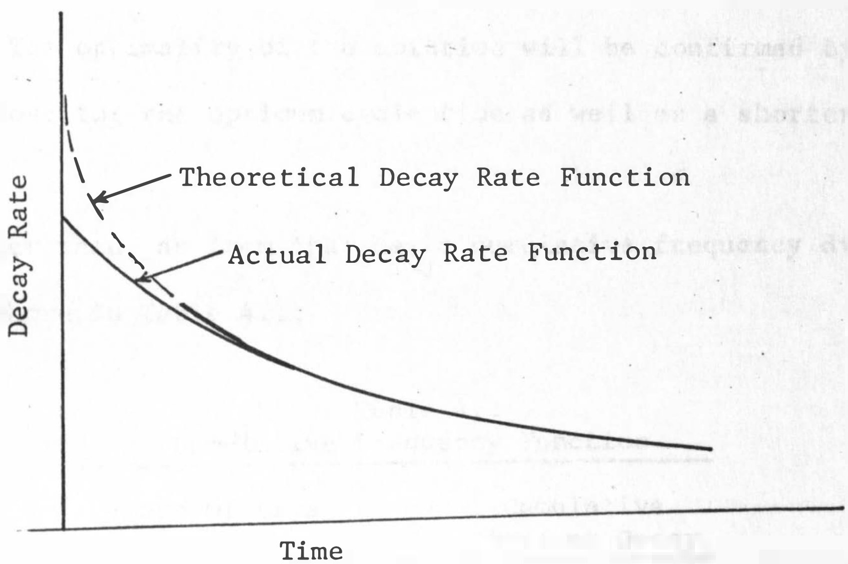

Figure 3.4. Theoretical and actual decreasing decay rate functions.

CHAPTER 4

NUMERICAL EXAMPLE

The inventory model developed in the second chapter and programmed for the digital computer in the third chapter should be demonstrated in a numerical example. A hypothetical product subject to decay will be selected and, after it has been demonstrated that the decay follows the Weibull distribution, the computer program will be used to compute the optimum cycle time and the economic order quantity. The optimality of the solution will be confirmed by finding the total cost for the optimum cycle time as well as a shorter and longer time.

Consider then, an item that has a cumulative frequency distribution as shown in Table 4.1.

Table 4.1
Cumulative Frequency Function

Number Of Days To Decay	Cumulative Percent Decay
2	0.47
5	1.9
10	5.0
15	9.5
20	14.0
25	19.0
30	24.0
35	30.0
40	35.0
45	40.0
50	45.0
55	49.0
60	54.0
65	58.0

In addition, let

C = \$4.00 per unit.

C_1 = \$0.001 per item day.

C_3 = \$20.00 per order.

R = 10 units per day.

The problem is of concern only if the distribution of the deterioration follows the Weibull distribution. A widely accepted method of confirming that the distribution is of the Weibull type that simultaneously gives the values for the parameters A and B, makes use of a special graph paper called Weibull Probability Paper (WPP). The method is derived and demonstrated in Appendix B using the data from Table 4.1. The values for A and B that are obtained are:

$$A = 600$$

$$B = 1.5$$

Utilizing these values and the normal inventory information listed above as input, a computer output is obtained as shown in Figure 4.1.

Of prime importance is the optimum cycle time of 11.64 days, the economic order quantity of 119.55 units and the total number of units that could be expected to decay, 3.15 units.

```
 A =     600.00       B =     1.5000   C =     4.0000   C1 =  0.001000
C3 =      20.00       R =    10.0000

          CYCLE TIME            CORRECTION
             63.2455              18.72205
             44.5235              14.53023
             29.9933               9.98064
             20.0126               5.73168
             14.2809               2.26224
             12.0187               0.36874
             11.6499               0.00939


OPTIMUM CYCLE TIME               =     0.11641E 02

ECONOMIC ORDER QUANTITY          =     0.11955E 03

TOTAL DECAY DURING CYCLE TIME =     0.31468E 01
```

Figure 4.1. Computer output.

The optimality of the cycle time obtained will be checked by comparing the total cost per unit time corresponding to the optimum cycle time with costs corresponding to cycles that are longer and shorter. The total cost per unit time corresponding to the optimum cycle time is obtained by substituting T = 11.64 and the values of the constants in Equation 3.2.

$$C_T = \frac{C_3}{T} - CR + \left(\frac{C}{T} + \frac{C_1}{2}\right) R \left[T + \frac{T^{B+1}}{(B+1)A} + \frac{T^{(2B+1)}}{(2B+1)\ A^2\ 2!} + \frac{T^{(3B+1)}}{(3B+1)\ A^3\ 3!} + \ldots\right] \quad (3.2)\ \text{(repeated)}$$

Substituting the numerical values gives

$$C_{11.64} = \frac{20}{11.64} - 4 \times 10 + \left[\frac{4}{11.64} + \frac{0.001}{2}\right] 10 \left[11.64 + \frac{11.64^{2.5}}{2.5 \times 600} + \frac{11.64^{4}}{4 \times 600^2 \times 2} + \frac{11.64^{5.5}}{5.5 \times 600^3 \times 6} + \ldots\right]$$

or a solution of

$$C_{11.64} = \$2.86 \text{ per day.}$$

Similarly, the value of C_T for T = 13, a slightly larger value, gives

$$C_{13} = \$2.87 \text{ per day.}$$

Finally, for a smaller value, T = 10, C_T is

$$C_{10} = \$2.91 \text{ per day.}$$

A comparison of these values indicates that the minimum cost is between T = 10 and T = 13. This is in agreement with the concept of optimality.

CHAPTER 5

CONCLUSIONS

A theoretical model has been developed for the determination of the economic order quantity for items which have Weibull decay rate function. The assumptions of constant and known rate of demand, no shortages and instantaneous supply or production are made. It has been demonstrated that under the special condition of exponential decay, this model reduces to a special model developed by Ghare and Schrader for exponential decay and under the condition of no decay, the model becomes the standard E.O.Q. model. A computer program was developed to provide the numerical solution to the model. A numerical example was used to show the solution form and to verify that the solution gives minimum total cost per unit time. Therefore, this model will provide optimal lot-sizes for deteriorating items over a wide range of practical situations.

The following recommendations for further development may be made:

1. A model may be developed using the 3-parameter Weibull distribution : $F(t) = 1 - e^{(t-r)^B/A}$
   where r = location parameter, any real number, such that $t \geq r$.
2. The Simple Weibull model should be modified to accommodate deteriorating items for which the values of A and B change during the desired cycle time.

3. More sophisticated models should be developed for the study and solution of inventory systems in which decaying items with Weibull distributions also have non-instantaneous supply or production, finite shortage cost, probabilistic demand or combinations of these attributes.

APPENDIX A

```
C     HCOST - C1     SCOST - C3    COST - C      CORR - CORRECTION
C     FT - THE FUNCTION F(T)
C     DFT - DERIVATIVE OF F(T),FUNCTION (3.4)
C     DECAY - TOTAL DECAY DURING THE CYCLE PERIOD
C     DET - THE FIRST TERM IN EQUATION (3.2)
C     HOLD - THE SECOND TERM IN EQUATION (3.2)
C     SETUP - -C3
    1 READ(11,20)A,B,COST,HCOST,SCOST,R
   20 FORMAT(F10.2,2F10.4,F10.6,2F10.4)
      I=0
      T=SQRT(2.0*SCOST/(HCOST*R))
      WRITE(12,4)A,B,COST,HCOST
    4 FORMAT(1H1,7H    A =,F10.2,3X,6H   B =,F10.4,3X,3HC =,F10.4,3X,4HC
     21 =,F10.6)
      WRITE(12,5)SCOST,R
    5 FORMAT(1H ,7H   C3 =,F10.2,3X,6H   R =,F10.4)
      WRITE(12,14)
   14 FORMAT(/11X,10HCYCLE TIME,10X,10HCORRECTION)
   40 DET=0.
      DO 45 N=1,7
      CALL NFACT(N,NFAC)
      D=(T**B/A)**N*T*N*B/((N*B+1.)*NFAC)
      DET=DET+D
   45 CONTINUE
      DET=COST*R*DET
      HOLD=HCOST*R*T**2*EXP(T**B/A)/2.
      SETUP=-SCOST
      FT=DET+HOLD+SETUP
      DFT=R*EXP(T**B/A)*(COST*B*T**B/A+HCOST*B*T**(B+1.)/(2.*A)+HCOST*T)
      CORR=FT/DFT
      WRITE(12,15)T,CORR
   15 FORMAT(1H ,10X,F10.4,10X,F10.5)
   15 FORMAT(1H ,10X,F10.4,10X,F10.5)
      T=T-FT/DFT
      I=I+1
      IF(I-10)41,9,9
   41 IF(ABS(CORR/T)-0.001)65,40,40
   65 QTY=0.
      DO 75 M=1,7
      N=M-1
      CALL NFACT(N,NFAC)
      Q=T**(N*B+1.0)/((N*B+1.)*A**N*NFAC)
   75 QTY=QTY+Q
      QTY=R*QTY
      DECAY=QTY-R*T
      WRITE(12,6)T
    6 FORMAT(/1H0,31HOPTIMUM CYCLE TIME                  =,E15.5)
      WRITE(12,7)QTY
    7 FORMAT(1H0,31HECONOMIC ORDER QUANTITY            =,E15.5)
      WRITE(12,8)DECAY
    8 FORMAT(1H0,31HTOTAL DECAY DURING CYCLE TIME =,E15.5)
      GO TO 1
    9 STOP
      END
```

APPENDIX B[11]

The Weibull cumulative frequency function is given by

$$F(t) = 1 - e^{-t^B/A}$$

For the case of deteriorating items, this represents the probability that the item will deteriorate prior to the time, t. This can be rewritten:

$$\frac{1}{1 - F(t)} = e^{t^B/A} \qquad \text{(B-1)}$$

A double logarithmic transformation will eliminate all powers of numbers. Equation B-1 is expressed as

$$\ln \ln\left[\frac{1}{1 - F(t)}\right] = - \ln A + B \ln t \qquad \text{(B-2)}$$

and provides a relation between the variables $\ln \ln\left[\frac{1}{1 - F(t)}\right]$ and $\ln t$. Therefore Equation B-2 represents a straight line with intercept $- \ln A$ and slope B.

The verification of whether an item has a decay rate function of the Weibull type reduces to the determination of whether the variables $\ln \ln\left[\frac{1}{1 - F(t)}\right]$ and $\ln t$ have a straight line relationship. This can be accomplished by plotting the values F(t) and t on the Weibull Probability Paper. If the points fall reasonably close to a straight line, it can be assumed that the deterioration adequately follows Weibull distribution. Figure B-1 illustrates the method and

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[11] Berrotoni, op. cit., pp. 77-79.

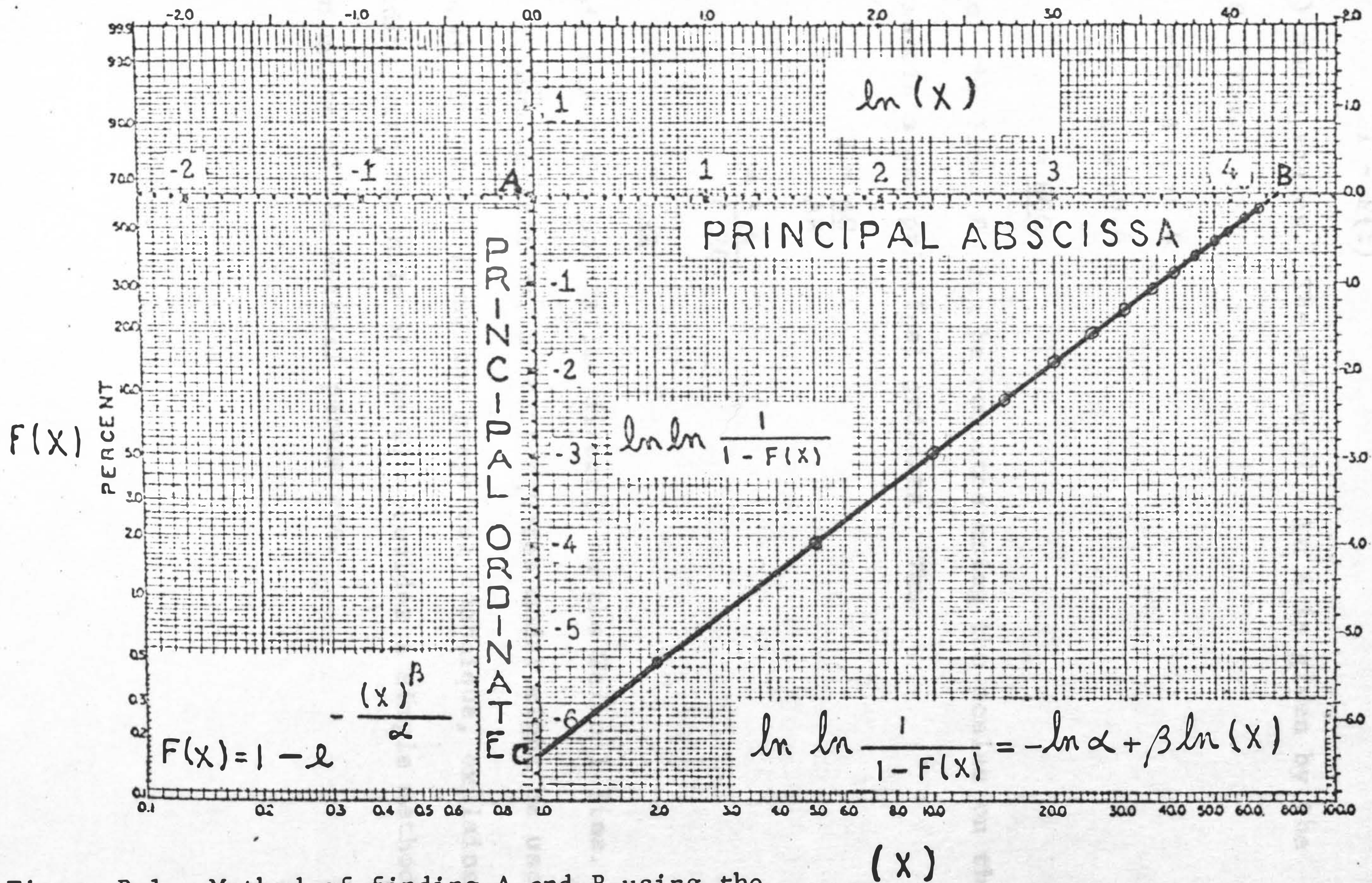

Figure B.1. Method of finding A and B using the Weibull Probability Paper.

and paper. The data used is that in Table 4.1. Note that the WPP has the $[\ln \ln (\frac{1}{1 - F(t)})]$ scale and $\ln(t)$ scale in addition to the $F(t)$ and t scales. The value of $- \ln A$ is given by the intercept C. Thus

$$- \ln A = -6.4$$

$$A = C^{6.4}$$

$$= \underline{600}$$

The slope of the line, B, can be computed using the scales on the principal abscissa and principal ordinate. Thus

$$B = \frac{AC}{AB}$$

$$= \frac{6.4}{4.26}$$

$$= \underline{1.5}$$

In this case, the points lie directly on the Weibull line. If there were random variations, a least squares model might be used to find the best fit line. Thus, the graphical technique, explained in this appendix, with the help of the WPP provides a simple method for the determination of the Weibull parameters.

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