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Convergence of positive series and ideal convergence*

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Abstract

Let $\mathcal{I} \subsetneq 2^{\mathbb{N}}$ be an admissible ideal, we say that a sequence (x_n) of real numbers $\mathcal{I}-$converges to a number L, and write $\mathcal{I} - \lim x_n = L$, if for each $\varepsilon > 0$ the set $A_\varepsilon = \{n : |x_n - L| \geq \varepsilon\}$ belongs to the ideal $\mathcal{I}$. In this paper we discuss the relation ship between convergence of positive series and the convergence properties of the summand sequence. Concretely, we study the ideals $\mathcal{I}$ having the following property as well:

$$\sum_{n=1}^{\infty} a_n^\alpha < \infty \text{ and } 0 < \inf_n \frac{n}{b_n} \leq \sup_n \frac{n}{b_n} < \infty \Rightarrow \mathcal{I} - \lim a_n b_n^\beta = 0,$$

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where $0 < \alpha \leq 1 \leq \beta \leq \frac{1}{\alpha}$ are real numbers and (a_n), (b_n) are sequences of positive real numbers. We characterize $T(\alpha, \beta, a_n, b_n)$ the class of all such admissible ideals $\mathcal{I}$.

This accomplishment generalized and extended results from the papers [4, 7, 12, 16], where it is referred that the monotonicity condition of the summand sequence in so-called Olivier's Theorem (see [13]) can be dropped if the convergence of the sequence (na_n) is weakend. In this paper we will study $\mathcal{I}$-convergence mainly in the case when $\mathcal{I}$ stands for $\mathcal{I}_{<q}$, $\mathcal{I}_c^{(q)}$, $\mathcal{I}_{\leq q}$, respectively.

Keywords: $\mathcal{I}$-convergence, convergence of positive series, Olivier's theorem, admissible ideals, convergence exponent

MSC: 40A05, 40A35

1. Introduction

We recall the basic definitions and conventions that will be used throughout the paper. Let $\mathbb{N}$ be the set of all positive integers. A system $\mathcal{I}$, $\emptyset \neq \mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal, provided $\mathcal{I}$ is additive ($A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$), and hereditary ($A \in \mathcal{I},\ B \subset A$ implies $B \in \mathcal{I}$). The ideal is called nontrivial if $\mathcal{I} \neq 2^{\mathbb{N}}$. If $\mathcal{I}$ is a nontrivial ideal, then $\mathcal{I}$ is called admissible if it contains the singletons ($\{n\} \in \mathcal{I}$ for every $n \in \mathbb{N}$). The fundamental notation which we shall use is $\mathcal{I}-$convergence introduced in the paper [11] (see also [3] where $\mathcal{I}-$convergence is defined by means of filter-the dual notion to ideal). The notion $\mathcal{I}-$convergence corresponds to the natural generalization of the notion of statistical convergence (see [5, 17]).

Definition 1.1. Let (x_n) be a sequence of real (complex) numbers. We say that the sequence $\mathcal{I}-$converges to a number L, and write $\mathcal{I} - \lim x_n = L$, if for each $\varepsilon > 0$ the set $A_\varepsilon = \{n : |x_n - L| \geq \varepsilon\}$ belongs to the ideal $\mathcal{I}$.

In the following we suppose that $\mathcal{I}$ is an admissible ideal. Then for every sequence (x_n) we have immediately that $\lim_{n\to\infty} x_n = L$ (classic limit) implies that (x_n) also $\mathcal{I}-$converges to a number L. Let $\mathcal{I}_f$ be the ideal of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$–convergence coincides with the usual convergence. Let $\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$, where $d(A)$ is the asymptotic density of $A \subseteq \mathbb{N}$ ($d(A) = \lim_{n\to\infty} \frac{\#\{a \leq n : a \in A\}}{n}$, where $\#M$ denotes the cardinality of the set M). Usual $\mathcal{I}_d-$convergence is called statistical convergence. For $0 < q \leq 1$ the class

$$\mathcal{I}_c^{(q)} = \{A \subset \mathbb{N} : \sum_{a \in A} a^{-q} < \infty\}$$

is an admissible ideal and whenever $0 < q < q' < 1$, we get

$$\mathcal{I}_f \subsetneq \mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_c^{q'} \subsetneq \mathcal{I}_c^{(1)} \subsetneq \mathcal{I}_d.$$

The notions the admissible ideal and $\mathcal{I}-$convergence have been developed in several directions and have been used in various parts of mathematics, in particular in number theory, mathematical analysis and ergodic theory, for example [1, 2, 5, 6, 9–11, 15, 17–19].

Let λ be the convergence exponent function on the power set of $\mathbb{N}$, thus for $A \subset \mathbb{N}$ put

$$\lambda(A) = \inf\Big\{t > 0 : \sum_{a\in A} a^{-t} < \infty\Big\}.$$

If $q > \lambda(A)$ then $\sum_{a\in A} \frac{1}{a^q} < \infty$, and $\sum_{a\in A} \frac{1}{a^q} = \infty$ when $q < \lambda(A)$; if $q = \lambda(A)$, the convergence of $\sum_{a\in A} \frac{1}{a^q}$ is inconclusive. It follows from [14, p. 26, Examp. 113, 114] that the range of λ is the interval $[0, 1]$, moreover for $A = \{a_1 < a_2 < \cdots < a_n < \dots\} \subseteq \mathbb{N}$ the convergence exponent can be calculate by using the following formula

$$\lambda(A) = \limsup_{n\to\infty} \frac{\log n}{\log a_n}.$$

It is easy to see that λ is monotonic, i.e. $\lambda(A) \le \lambda(B)$ whenever $A \subseteq B \subset \mathbb{N}$, furthermore, $\lambda(A \cup B) = \max\{\lambda(A), \lambda(B)\}$ for all $A, B \subset \mathbb{N}$.

2. Overwiew of known results

In this section we mention known results related to the topic of this paper and some other ones we use in the proofs of our results. Recently in [19] was introduced the following classes of subsets of $\mathbb{N}$:

$$\begin{aligned}
\mathcal{I}_{<q} &= \{A \subset \mathbb{N} : \lambda(A) < q\}, \text{ if } 0 < q \le 1,\\
\mathcal{I}_{\le q} &= \{A \subset \mathbb{N} : \lambda(A) \le q\}, \text{ if } 0 \le q \le 1, \text{ and}\\
\mathcal{I}_0 &= \{A \subset \mathbb{N} : \lambda(A) = 0\}.
\end{aligned}$$

Clearly, $\mathcal{I}_{\le 0} = \mathcal{I}_0$. Since $\lambda(A) = 0$ when $A \subset \mathbb{N}$ is finite, then $\mathcal{I}_f = \{A \subset \mathbb{N} : A \text{ is finite}\} \subset \mathcal{I}_0$, moreover, there is proved [19, Th.2] that each class $\mathcal{I}_0$, $\mathcal{I}_{<q}$, $\mathcal{I}_{\le q}$, respectively forms an admissible ideal, except for $\mathcal{I}_{\le 1} = 2^{\mathbb{N}}$.

Proposition 2.1 ([19, Th.1])**.** *Let $0 < q < q' < 1$. Then we have*

$$\mathcal{I}_0 \subsetneq \mathcal{I}_{<q} \subsetneq \mathcal{I}_c^{(q)} \subsetneq \mathcal{I}_{\le q} \subsetneq \mathcal{I}_{<q'} \subsetneq \mathcal{I}_c^{(q')} \subsetneq \mathcal{I}_{\le q'} \subsetneq \mathcal{I}_{<1} \subsetneq \mathcal{I}_c^{(1)} \subsetneq \mathcal{I}_{\le 1} = 2^{\mathbb{N}},$$

and the difference of successive sets is infinite, so equality does not hold in any of the inclusions.

The claim in the following proposition is a trivial fact about preservation of the limit.

Proposition 2.2 ([11, Lemma])**.** *If $\mathcal{I}_1 \subset \mathcal{I}_2$, then $\mathcal{I}_1 - \lim x_n = L$ implies $\mathcal{I}_2 - \lim x_n = L$.*

In [13] L. Olivier proved results so-called Olivier's Theorem about the speed of convergence to zero of the terms of convergent positive series with nonincreasing terms. Precisely, if (a_n) is a nonincreasing positive sequence and $\sum_{n=1}^{\infty} a_n < \infty$, then $\lim_{n\to\infty} na_n = 0$ (see also [8]). In [16], T. Šalát and V. Toma made the remark that the monotonicity condition in Olivier's Theorem can be dropped if the convergence the sequence (na_n) is weakened by means of the notion of $\mathcal{I}$-convergence (see also [7]). In [12], there is an extension of results in [16] with very nice historical contexts of the object of our research.

Since $0 = \lim_{n\to\infty} na_n = \mathcal{I}_f - \lim na_n$, then the above mentioned Olivier's Theorem can be formulated in the terms of $\mathcal{I}$-convergence as follows:

$$(a_n) \text{ nonincreasing and } \sum_{n=1}^{\infty} a_n < \infty \;\Rightarrow\; \mathcal{I} - \lim na_n = 0,$$

holds for any admissible ideal $\mathcal{I}$ (this assertion is a direct corollary of the facts $\mathcal{I}_f \subseteq \mathcal{I}$ and Proposition 2.2), and providing (a_n) to be a sequence of positive real numbers.

The following simple example

$$a_n = \begin{cases} \frac{1}{n}, & \text{if } n = k^2, (k = 1, 2, \dots) \\ \frac{1}{2^n}, & \text{otherwise,} \end{cases}$$

shows that monotonicity condition of the positive sequence (a_n) can not be in general omitted. This example shows that $\limsup_{n\to\infty} na_n = 1$, thus the ideal $\mathcal{I}_f$ does not have for positive terms the following property

$$\sum_{n=1}^{\infty} a_n < \infty \;\Rightarrow\; \mathcal{I} - \lim na_n = 0. \tag{2.1}$$

The previous example can be strengthened taking $a_n = \frac{\log n}{n}$ if n is square, in such case the sequence (na_n) is not bounded yet. In [16], T. Šalát and V. Toma characterized the class $S(T)$ of all admissible ideals $\mathcal{I} \subset 2^{\mathbb{N}}$ having the property (2.1), for sequences (a_n) of positive real numbers.

They proved that

$$S(T) = \{\mathcal{I} \subset 2^{\mathbb{N}} : \mathcal{I} \text{ is an admissible ideal such that } \mathcal{I} \supseteq \mathcal{I}_c^{(1)}\}.$$

J. Gogola, M. Mačaj, T. Visnyai in [7] introduced and characterized the class $S_q(T)$ of all admissible ideals $\mathcal{I} \subset 2^{\mathbb{N}}$ for $0 < q \le 1$ having the property

$$\sum_{n=1}^{\infty} a_n^q < \infty \;\Rightarrow\; \mathcal{I} - \lim na_n = 0, \tag{2.2}$$

providing (a_n) be a positive real sequence. The stronger condition of convergence of positive series requirest the stronger convergence property of the summands as well. They proved

$$S_q(T) = \{\mathcal{I} \subset 2^{\mathbb{N}} : \mathcal{I} \text{ is an admissible ideal such that } \mathcal{I} \supseteq \mathcal{I}_c^{(q)}\}.$$

Of course, if $q = 1$ then $S_1(T) = S(T)$.

In [12], C. P. Niculescu, G. T. Prăjitură studied the following implication, which is general as (2.1):

$$\sum_{n=1}^{\infty} a_n < \infty \text{ and } \inf_n \frac{n}{b_n} > 0 \ \Rightarrow \ \mathcal{I} - \lim a_n b_n = 0, \tag{2.3}$$

for sequences (a_n), (b_n) of positive real numbers.

They proved that the ideal $\mathcal{I}_d$ fulfills (2.3). In the next section we are going to show that $\mathcal{I}_c^{(1)}$ is the smallest admissible ideal partially ordered by inclusion which also fulfills (2.3).

3. $\mathcal{I}_c^{(q)}$– convergence and convergence of positive series

In this part we introduce and characterize the class of such ideals that fulfill the following implication (3.1). Obviously this class will generalize the results of (2.2) and (2.3). On the other hand, we define the smallest admissible ideal partially ordered by inclusion which fulfills (3.1).

In the sequel we are going to study the ideals $\mathcal{I}$ having the following property:

$$\sum_{n=1}^{\infty} a_n^{\alpha} < \infty \text{ and } 0 < \inf_n \frac{n}{b_n} \le \sup_n \frac{n}{b_n} < \infty \Rightarrow \mathcal{I} - \lim a_n b_n^{\beta} = 0, \tag{3.1}$$

where $0 < \alpha \le 1 \le \beta \le \frac{1}{\alpha}$ are real numbers and (a_n), (b_n) are positive sequences of real numbers.

We denote by $T(\alpha, \beta, a_n, b_n)$ the class of all admissible ideals $\mathcal{I} \subset 2^{\mathbb{N}}$ having the property (3.1). Obviously $T(1, 1, a_n, n) = S(T)$ and $T(q, 1, a_n, n) = S_q(T)$.

Theorem 3.1. *Let $0 < \alpha \le 1 \le \beta \le \frac{1}{\alpha}$ be real numbers. Then for every positive real sequences (a_n), (b_n) such that*

$$\sum_{n=1}^{\infty} a_n^{\alpha} < \infty \quad \text{and} \quad \inf_n \frac{n}{b_n} > 0$$

we have

$$\mathcal{I}_c^{(\alpha\beta)} - \lim a_n b_n^{\beta} = 0.$$

Proof. Let $\varepsilon > 0$, put $A_\varepsilon = \{n \in \mathbb{N} : a_n b_n^{\beta} \ge \varepsilon\}$. We proceed by contradiction. Then there exists such $\varepsilon > 0$ that $A_\varepsilon \notin \mathcal{I}_c^{(\alpha\beta)}$, thus

$$\sum_{n \in A_\varepsilon} \frac{1}{n^{\alpha\beta}} = \infty. \tag{3.2}$$

For $n \in A_\varepsilon$ we have

$$a_n^\alpha \geq \varepsilon^\alpha \frac{1}{b_n^{\alpha\beta}} = \varepsilon^\alpha \left(\frac{n}{b_n}\right)^{\alpha\beta} \frac{1}{n^{\alpha\beta}} \geq \varepsilon^\alpha \left(\inf_n \frac{n}{b_n}\right)^{\alpha\beta} \frac{1}{n^{\alpha\beta}},$$

and so

$$\sum_{n=1}^{\infty} a_n^\alpha \geq \sum_{n\in A_\varepsilon} a_n^\alpha \geq \varepsilon^\alpha \left(\inf_n \frac{n}{b_n}\right)^{\alpha\beta} \sum_{n\in A_\varepsilon} \frac{1}{n^{\alpha\beta}}.$$

Using this and the assumption for a sequence (b_n) and (3.2) we get

$$\sum_{n=1}^{\infty} a_n^\alpha = \infty,$$

which is a contradiction. □

If in Theorem 3.1 we put $\alpha = q$ and $\beta = 1$, we can obtain the following corollary.

Corollary 3.2. *For every positive real sequences (a_n), (b_n) such that*

$$\sum_{n=1}^{\infty} a_n^q < \infty \quad \textit{and} \quad \inf_n \frac{n}{b_n} > 0$$

we have

$$\mathcal{I}_c^{(q)} - \lim a_n b_n = 0.$$

Already in the case when $q = 1$ in Corollary 3.2, we get a stronger assertion than given in [12] for the ideal $\mathcal{I}_d$, because of $\mathcal{I}_c^{(1)} \subsetneq \mathcal{I}_d$.

Remark 3.3. Let (a_n), (b_n) be positive real sequences. For special choices α and (b_n) in Corollary 3.2, we can obtain the following:

i) Putting $\alpha = 1$. Then we get: If $\sum_{n=1}^{\infty} a_n < \infty$ and $\inf_n \frac{n}{b_n} > 0$ then $\mathcal{I}_c^{(1)} - \lim a_n b_n = 0$ (which is stronger result as [12, Theorem 5]).

ii) Putting $\alpha = 1$ and $b_n = n$. Then we get: If $\sum_{n=1}^{\infty} a_n < \infty$ then $\mathcal{I}_c^{(1)} - \lim a_n n = 0$ (see [16, Theorem 2.1]).

iii) Putting $\alpha = q$ and $b_n = n$. Then we get: If $\sum_{n=1}^{\infty} a_n^q < \infty$ then $\mathcal{I}_c^{(q)} - \lim a_n n = 0$ (see [7, Lemma 3.1]).

Theorem 3.4. *Let $0 < \alpha \leq 1 \leq \beta \leq \frac{1}{\alpha}$ be real numbers. If for some admissible ideal $\mathcal{I}$ holds*

$$\mathcal{I} - \lim a_n b_n^\beta = 0$$

for every sequences (a_n), (b_n) of positive numbers such that

$$\sum_{n=1}^{\infty} a_n^\alpha < \infty \quad \textit{and} \quad \sup_n \frac{n}{b_n} < \infty,$$

then

$$\mathcal{I}_c^{(\alpha\beta)} \subseteq \mathcal{I}.$$

Proof. Let us assume that for some admissible ideal $\mathcal{I}$ we have $\mathcal{I} - \lim a_n b_n^\beta = 0$ and take an arbitrary set $M \in \mathcal{I}_c^{(\alpha\beta)}$. It is sufficient to prove that $M \in \mathcal{I}$. Since $\mathcal{I} - \lim a_n b_n^\beta = 0$ we have for each $\varepsilon > 0$ the set $A_\varepsilon = \{n \in \mathbb{N} : a_n b_n^\beta \geq \varepsilon\} \in \mathcal{I}$. Since $M \in \mathcal{I}_c^{(\alpha\beta)}$ we have $\sum_{n\in M} \frac{1}{n^{\alpha\beta}} < \infty$. Now we define the sequence a_n as follows:

$$a_n = \begin{cases} \frac{1}{n^\beta}, & \text{if } n \in M, \\ \frac{1}{2^n}, & \text{if } n \notin M. \end{cases}$$

Obviously the sequence (a_n) fulfills the premises of the theorem as $a_n > 0$ and

$$\sum_{n=1}^{\infty} a_n^\alpha = \sum_{n\in M} \left(\frac{1}{n^\beta}\right)^\alpha + \sum_{n\notin M} \left(\frac{1}{2^n}\right)^\alpha \leq \sum_{n\in M} \frac{1}{n^{\alpha\beta}} + \sum_{n=1}^{\infty} \left(\frac{1}{2^\alpha}\right)^n < \infty.$$

Hence $a_n n^\beta = 1$ for $n \in M$ and so for each $n \in M$ we have

$$a_n b_n^\beta = a_n n^\beta \left(\frac{b_n}{n}\right)^\beta = \left(\frac{b_n}{n}\right)^\beta \geq \frac{1}{\left(\sup_n \frac{n}{b_n}\right)^\beta} > 0.$$

Denote by $\varepsilon(\beta) = \left(\sup_n \frac{n}{b_n}\right)^{-\beta} > 0$ and preceding considerations give us

$$M \subset A_{\varepsilon(\beta)} \in \mathcal{I}.$$

Thus $M \in \mathcal{I}$, what means $\mathcal{I}_c^{(\alpha\beta)} \subseteq \mathcal{I}$. □

The characterization of the class $T(\alpha, \beta, a_n, b_n)$ is the direct consequence of Theorem 3.1 and Theorem 3.4.

Theorem 3.5. *Let $0 < \alpha \leq 1 \leq \beta \leq \frac{1}{\alpha}$ be real numbers and (a_n), (b_n) be sequences of positive real numbers. Then the class $T(\alpha, \beta, a_n, b_n)$ consists of all admissible ideals $\mathcal{I} \subset 2^{\mathbb{N}}$ such that $\mathcal{I} \supseteq \mathcal{I}_c^{(\alpha\beta)}$.*

For special choices α, β and (b_n) in Theorem 3.5 we can get the following.

Corollary 3.6. *Let $0 < q \leq 1$ be a real number and (a_n) be positive real sequences having the properties*

$$\sum_{n=1}^{\infty} a_n^q < \infty.$$

Then we have

i) $T(q, 1, a_n, n) = \{\mathcal{I} \subset 2^{\mathbb{N}} : \mathcal{I}$ *is admissible ideal such that* $\mathcal{I} \supseteq \mathcal{I}_c^{(q)}\} = S_q(T)$,

ii) $T(1, 1, a_n, n) = \{\mathcal{I} \subset 2^{\mathbb{N}} : \mathcal{I}$ *is admissible ideal such that* $\mathcal{I} \supseteq \mathcal{I}_c^{(1)}\} = S(T)$.

4. $\mathcal{I}_{<q}-$ and $\mathcal{I}_{\leq q}-$convergence and convergence of series

In this section we will study the admissible ideals $\mathcal{I} \subset 2^{\mathbb{N}}$ having the special property (4.1) and (4.3), respectively.

$$\sum_{n=1}^{\infty} a_n^{q_k} < \infty \text{ for every } k \text{ and } 0 < \inf_n \frac{n}{b_n} \leq \sup_n \frac{n}{b_n} < \infty \Rightarrow \mathcal{I} - \lim a_n b_n = 0, \quad (4.1)$$

where (q_k) is a strictly decreasing sequence which is convergent to q, $0 \leq q < 1$ and (a_n), (b_n) are sequences of positive real numbers.

Denote by $T_q^{q_k}(a_n, b_n)$ the class of all admissible ideals $\mathcal{I}$ having the property (4.1).

Theorem 4.1. *Let $0 \leq q < 1$ and (q_k) be a strictly decreasing sequence which is convergent to q. Then for positive real sequences (a_n), (b_n) such that holds*

$$\sum_{n=1}^{\infty} a_n^{q_k} < \infty, \text{ for every } k \text{ and } \inf_n \frac{n}{b_n} > 0,$$

we have

$$\mathcal{I}_{\leq q} - \lim a_n b_n = 0.$$

Proof. Again, we proceed by contradiction. Put $A_\varepsilon = \{n \in \mathbb{N} : a_n b_n \geq \varepsilon\}$. Then there exists such $\varepsilon > 0$ that $A_\varepsilon \notin \mathcal{I}_{\leq q}$, thus $\lambda(A_\varepsilon) > q$. Hence there exists such $i \in \mathbb{N}$, that $q < q_{k_i} < \lambda(A_\varepsilon)$, and so we get

$$\sum_{n \in A_\varepsilon} \frac{1}{n^{q_{k_i}}} = \infty. \quad (4.2)$$

For $n \in A_\varepsilon$ we have

$$a_n^{q_{k_i}} \geq \varepsilon^{q_{k_i}} \frac{1}{b_n^{q_{k_i}}} = \varepsilon^{q_{k_i}} \left(\frac{n}{b_n}\right)^{q_{k_i}} \frac{1}{n^{q_{k_i}}} \geq \varepsilon^{q_{k_i}} \left(\inf_n \frac{n}{b_n}\right)^{q_{k_i}} \frac{1}{n^{q_{k_i}}},$$

therefore

$$\sum_{n=1}^{\infty} a_n^{q_{k_i}} \geq \sum_{n \in A_\varepsilon} a_n^{q_{k_i}} \geq \varepsilon^{q_{k_i}} \left(\inf_n \frac{n}{b_n}\right)^{q_{k_i}} \sum_{n \in A_\varepsilon} \frac{1}{n^{q_{k_i}}}.$$

Using this and the assumption for a sequence (b_n) and (4.2) we get

$$\sum_{n=1}^{\infty} a_n^{q_{k_i}} = \infty,$$

what is a contradiction. $\square$

Theorem 4.2. *Let $0 \leq q < 1$ and (q_k) be a strictly decreasing sequence which is convergent to q. If for some admissible ideal $\mathcal{I}$ holds*

$$\mathcal{I} - \lim a_n b_n = 0$$

for every sequences (a_n), (b_n) of positive numbers such that

$$\sum_{n=1}^{\infty} a_n^{q_k} < \infty, \text{ for every } k \text{ and } \sup_n \frac{n}{b_n} < \infty,$$

then

$$\mathcal{I}_{\leq q} \subseteq \mathcal{I}.$$

Proof. Let us assume that for any admissible ideal $\mathcal{I}$ we have $\mathcal{I} - \lim a_n b_n = 0$ and take an arbitrary set $M \in \mathcal{I}_{\leq q}$. It is sufficient to prove that $M \in \mathcal{I}$. Since $M \in \mathcal{I}_{\leq q}$ we have $\lambda(M) \leq q$ and so for each $q_k > q$ we get

$$\sum_{n \in M} \frac{1}{n^{q_k}} < \infty.$$

Moreover $\mathcal{I} - \lim a_n b_n = 0$ and so for each $\varepsilon > 0$ the set $A_\varepsilon = \{n \in \mathbb{N} : a_n b_n \geq \varepsilon\} \in \mathcal{I}$. Define the sequence (a_n) as follows:

$$a_n = \begin{cases} \frac{1}{n}, & \text{if } n \in M, \\ \frac{1}{2^n}, & \text{if } n \notin M. \end{cases}$$

The sequence (a_n) fulfills the premises of the theorem, $a_n > 0$ and for each q_k we obtain

$$\sum_{n=1}^{\infty} a_n^{q_k} = \sum_{n \in M} \frac{1}{n^{q_k}} + \sum_{n \notin M} \left(\frac{1}{2^n}\right)^{q_k} \leq \sum_{n \in M} \frac{1}{n^{q_k}} + \sum_{n=1}^{\infty} \left(\frac{1}{2^{q_k}}\right)^n < \infty.$$

Now $a_n n = 1$ for $n \in M$. Therefore for each $n \in M$ we have

$$a_n b_n = a_n n \Big(\frac{b_n}{n}\Big) = \frac{b_n}{n} \geq \frac{1}{\sup_n \frac{n}{b_n}} > 0.$$

Denote by $\varepsilon = \left(\sup_n \frac{n}{b_n}\right)^{-1} > 0$ we have

$$M \subset A_\varepsilon \in \mathcal{I}.$$

Thus $M \in \mathcal{I}$, what means $\mathcal{I}_{\leq q} \subseteq \mathcal{I}$. □

The above mentioned results (Theorem 4.1 and Theorem 4.2) allow us to give a characterization for the class $T_q^{q_k}(a_n, b_n)$.

Theorem 4.3. *Let $0 \leq q < 1$ and (q_k) be a strictly decreasing sequence which converges to q. Let (a_n), (b_n) be positive real sequences. Then the class $T_q^{q_k}(a_n, b_n)$ consists of all admissible ideals $\mathcal{I} \subset 2^{\mathbb{N}}$ such that $\mathcal{I} \supseteq \mathcal{I}_{\leq q}$.*

Let us consider the following property and pronounce for it analogical results as above.

$$\sum_{n=1}^{\infty} a_n^{q_k} < \infty \text{ for some } k \text{ and } 0 < \inf_n \frac{n}{b_n} \le \sup_n \frac{n}{b_n} < \infty \Rightarrow \mathcal{I} - \lim a_n b_n = 0, \quad (4.3)$$

where (q_k) is a strictly increasing sequence of positive numbers which is convergent to q, $0 < q \le 1$ and (a_n), (b_n) are sequences of positive real numbers.

Denote by $T_{q_k}^{q}(a_n, b_n)$ the class of all admissible ideals $\mathcal{I}$ having the property (4.3).

Theorem 4.4. *Let $0 < q \le 1$ and (q_k) be a strictly increasing sequence of positive numbers which is convergent to q. Then for positive real sequences (a_n), (b_n) such that holds*

$$\sum_{n=1}^{\infty} a_n^{q_{k_0}} < \infty, \text{ for some } k_0 \in \mathbb{N} \text{ and } \inf_n \frac{n}{b_n} > 0,$$

we have

$$\mathcal{I}_{<q} - \lim a_n b_n = 0.$$

Proof. Again, we proceed by contradiction. Then there exists $\varepsilon > 0$ such that $A_\varepsilon = \{n \in \mathbb{N} : a_n b_n \ge \varepsilon\} \notin \mathcal{I}_{<q}$, thus $\lambda(A_\varepsilon) \ge q$. For each $k \in \mathbb{N}$ (as well for k_0) we have $q_k < q \le \lambda(A_\varepsilon)$, and so

$$\sum_{n \in A_\varepsilon} \frac{1}{n^{q_k}} = \infty. \quad (4.4)$$

Further the proof continues by the same way as it was outlined in Theorem 4.1. □

Theorem 4.5. *Let $0 < q \le 1$ and (q_k) be a strictly increasing sequence of positive numbers which is convergent to q. If for some admissible ideal $\mathcal{I}$ holds*

$$\mathcal{I} - \lim a_n b_n = 0$$

for every sequences (a_n), (b_n) of positive numbers such that

$$\sum_{n=1}^{\infty} a_n^{q_{k_0}} < \infty \text{ for some } k_0 \in \mathbb{N} \text{ and } \sup_n \frac{n}{b_n} < \infty,$$

then

$$\mathcal{I}_{<q} \subseteq \mathcal{I}.$$

Proof. Let us assume that for any admissible ideal $\mathcal{I}$ we have $\mathcal{I} - \lim a_n b_n = 0$ and take an arbitrary $M \in \mathcal{I}_{<q}$. It is sufficient to prove that $M \in \mathcal{I}$. Since $M \in \mathcal{I}_{<q}$ we have $\lambda(M) < q$ and so there exists a sufficiently large $k_0 \in \mathbb{N}$ such that $\lambda(M) < q_{k_0} < q$. So

$$\sum_{n \in M} \frac{1}{n^{q_{k_0}}} < \infty.$$

Again, the proof continues by the same way as it was outlined in Theorem 4.2. □

The above results (Theorem 4.4 and Theorem 4.5) allow us to give a characterization for the class $T^q_{q_k}(a_n, b_n)$.

Theorem 4.6. *Let $0 < q \leq 1$ and (q_k) be a strictly increasing sequence of positive numbers which converges to q. Let (a_n), (b_n) be positive real sequences. Then the class $T^q_{q_k}(a_n, b_n)$ consists of all admissible ideals $\mathcal{I} \subset 2^{\mathbb{N}}$ such that $\mathcal{I} \supseteq \mathcal{I}_{<q}$.*

5. Summary and scheme of main results

Let (a_n), (b_n) be fix sequences of positive real numbers having the appropriate property (3.1), (4.1) and (4.3), respectively. Denote in short classes given above $T(\alpha, \beta, a_n, b_n) = T(\alpha, \beta)$, $T^{q_k}_q(a_n, b_n) = T^{q_k}_q$ and $T^q_{q_k}(a_n, b_n) = T^q_{q_k}$. Then we have

i) for $0 < \alpha \leq 1 \leq \beta \leq \frac{1}{\alpha}$,

$$T(\alpha, \beta) = \{\mathcal{I} \subset 2^{\mathbb{N}} : \mathcal{I} \text{ is admissible ideal such that } \mathcal{I} \supseteq \mathcal{I}^{(\alpha\beta)}_c\},$$

ii) for $1 \geq q_k > q \geq 0$ $(k = 1, 2 \dots)$, $q_k \downarrow q$ as $k \to \infty$,

$$T^{q_k}_q = \{\mathcal{I} \subset 2^{\mathbb{N}} : \mathcal{I} \text{ is admissible ideal such that } \mathcal{I} \supseteq \mathcal{I}_{\leq q}\},$$

iii) for $0 < q_k < q \leq 1$ $(k = 1, 2 \dots)$, $q_k \uparrow q$ as $k \to \infty$,

$$T^q_{q_k} = \{\mathcal{I} \subset 2^{\mathbb{N}} : \mathcal{I} \text{ is admissible ideal such that } \mathcal{I} \supseteq \mathcal{I}_{<q}\}.$$

For special cases the following scheme shows the smallest(minimal) admissible ideals partially ordered by inclusion which belong to the classes in the second line.

$$\begin{array}{ccccccccccccc}
\mathcal{I}_0 & \subsetneq & \mathcal{I}^{(\alpha\beta)}_c & \subsetneq & \mathcal{I}_{<q} & \subsetneq & \mathcal{I}^{(q)}_c & \subsetneq & \mathcal{I}_{\leq q} & \subsetneq & \mathcal{I}_{<1} & \subsetneq & \mathcal{I}^{(1)}_c \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
T^{q_k}_0 & \supsetneq & \underset{if\ \alpha\beta<q}{T(\alpha,\beta)} & \supsetneq & T^q_{q_k} & \supsetneq & \underset{if\ \alpha\beta=q}{T(\alpha,\beta)} & \supsetneq & T^{q_k}_q & \supsetneq & T^1_{q_k} & \supsetneq & \underset{if\ \alpha\beta=1}{T(\alpha,\beta)}
\end{array}$$

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