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Robust Appointment Scheduling for Random Service Time Using Min-Max Optimization

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Robust Appointment Scheduling for Random Service Time Using Min-Max Optimization

By

Tasmia Jannat Tumpa

A Thesis
Submitted to the Faculty of Graduate Studies
through the Department of Mechanical, Automotive and Materials Engineering
in Partial Fulfillment of the Requirements for
the Degree of Master of Applied Science
at the University of Windsor

Windsor, Ontario, Canada

2020

Robust Appointment Scheduling for Random Service Time Using Min-max Optimization

by

Tasmia Jannat Tumpa

APPROVED BY:

_____

B. Chaouch
Odette School of Business

_____

E. Kim
Department of Mechanical, Automotive and Materials Engineering

_____

F. Baki, Co-Advisor
Odette School of Business

_____

A. Azab, Co-Advisor
Department of Mechanical, Automotive and Materials Engineering

May 15, 2020

DECLARATION OF CO-AUTHORSHIP / PREVIOUS PUBLICATION

I. Co-Authorship

I hereby declare that this thesis incorporates materials that are result of joint research of the author and her supervisors, Prof. Fazle Baki and Prof. Ahmed Azab. Chapter 4, 5 and 6 of this thesis were co-authored with Prof. Fazle Baki and Prof. Ahmed Azab. In all cases, the key ideas, primary contributions, experimental designs, data analysis, interpretation, and writing were performed by the author; Prof. Fazle Baki and Prof. Ahmed Azab provided feedback the on refinement of ideas, overall coordination, improvements and editing of the manuscript.

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This thesis includes 1 conference paper that has been previously submitted, as follows:

Thesis Chapter	Publication title/full citation	Publication status*
Chapter 4, 5 and 6	Tumpa, T., Baki, M.F., Azab, A., "Robust Appointment Scheduling Using Min-max Optimization", Canadian Operational Research Society (CORS) 62nd Annual Conference, 2020.	*Submitted*

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ABSTRACT

Appointment Scheduling is an increasingly challenging problem for service-centers, healthcare, production and transportation sector. Challenges include meeting growing demand and high expectation of service level among the customers and ensuring an efficient service system which reduces the expenditure related to idle times and underutilization of the system. The problem becomes more complicated in the presence of processing time uncertainties. In this study, a Robust Appointment Scheduling model is developed using Min-max Optimization to provide appointment dates for a system with a single processor. The objective is to minimize the cost of the worst-case scenario under any realization of the processing time of the jobs. The proposed methodology requires less information regarding the uncertain parameters and can provide optimal solution while only considering the extreme bounds of the uncertain parameters. Therefore, it is applicable to any probability distribution of the uncertain parameters. The model is well suited for any general case appointment scheduling problem regardless of the application field. Since the problem is NP-hard, an Iterative Solution Procedure and a Dynamic Programming model are developed for solving larger instances of problem in polynomial time. In addition, propositions that support the robust model are provided along with theoretical proofs. Appointment scheduling of two case studies, a Dentist's clinic and VIA Rail Canada are performed. Both case studies exhibit high performance of the proposed robust model in terms of cost savings and computational efforts. This work will contribute both to the literature related to uncertainty handling in decision making and to the industries, which aim to achieve an efficient service system.

DEDICATION

In the name of Allah, the most gracious, the most merciful.

I dedicate this work to my parents, Shahida Begum and MD. Azizul Hoque and my life partner MD. Hafizur Rahman. I am grateful to my sisters Dr. Arifa Shabiha Huq, Dr. Armana Shabiha Huq, Dr. Aysa Siddika and Azmary Jannat Aurin for their continuous support and encouragement.

ACKNOWLEDGEMENTS

I would like to acknowledge my co-advisors, Prof. Fazle Baki and Prof. Ahmed Azab, for allowing me to work on this research. Furthermore, I would like to thank them for their guidance, suggestions and continuous support. I would also like to acknowledge my committee members, Dr. Ben Chaouch and Dr. Eunsik Kim, for providing invaluable constructive criticisms.

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INTRODUCTION

1.1 Overview

Appointment Scheduling is an increasingly challenging problem in many service areas (Ahmadi-Javid et al., 2017). Challenges include meeting increasing demand and expectation of high service level among the customers and ensuring an efficient service system which reduces the expenditure related to idle times and underutilization of the system, etc. The problem is intensified in the presence of processing time uncertainty varying from one customer to another. In this regard, Appointment Scheduling can improve the system efficiency drastically and significant savings can be achieved. Specially, in developed countries where expenditures are huge for service centres, a good appointment schedule is crucial (Laan et al., 2018).

Appointment scheduling is needed where appointment dates are to be determined in advance for the jobs, which will be processed sequentially in a highly utilized processor when the processing duration is uncertain and varies from one job to another. Jobs are not available prior to their appointment dates. In other words, a static-class of appointment scheduling approach is considered, in which a finite number of appointments are scheduled prior to the beginning of the actual service (Cayirli and Veral, 2009). Appointment schedule determines a time slot for each job specifying their appointment dates when the jobs, processor, and the associated resources become available. However, since the processing time is uncertain, some jobs take more time than assigned to it whereas some jobs finish earlier than the next jobs appointment time. When a job takes more time than assigned, the next jobs have to wait for its completion and will start later than their original appointment time. This results in waiting of the subsequent jobs and may result in overtime for the processor and the associated resources at the end of the schedule. On the other hand, if a job finishes earlier than the next jobs appointment date, the processor and the associated recourses remain idle until the next job's appointment time. This results in underutilization of the system. Therefore, an optimal appointment schedule is required considering the

trade-off between the underutilization and the overtime of both the processor and the jobs. The objective is to minimize the worst cost of the appointment schedule by minimizing earliness and tardiness of both the jobs and the processor for any realization of the processing durations.

Figure 1.1 describes an appointment schedule of three jobs whose processing durations are uncertain and need to be scheduled sequentially. Since the processing durations are uncertain, some job may finish earlier than its assigned time whereas some jobs may finish later than its assigned time. Let, A_1, A_2, A_3 denote the given appointment time of job 1, job 2 and job 3 respectively and P_1, P_2, P_3 denote their actual processing time respectively. For job 1, processing time P_1 is less than the next job's appointment time A_2. As a result, the processor and the associated resources face some idle time there causing earliness of the system. For job 2, since processing time P_2 is greater than the next job's appointment time A_3 , the next job will start at a later time than the assigned appointment time A_3 and make the next jobs to wait until the completion of the previous job.
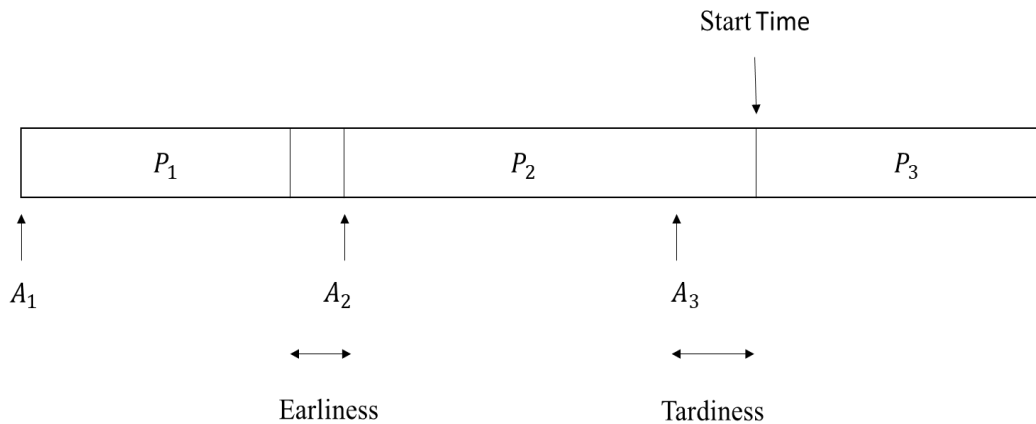
Figure 1: Appointment Schedule of 3 jobs under uncertainty

1.2 Application Area

The most impactful application area of appointment systems can be healthcare services such as doctor's clinic, surgical scheduling, radiation therapy clinics for cancer patients, dentist's clinic, physical therapy center, healthcare diagnostics operations (CAT Scans,

MRI, etc.) and so on. Besides these, the problem is also encountered while scheduling container, vessel and terminal operations at sea ports, gateway and runway scheduling of aircrafts in airports, automobile service centers, consulting professionals such as lawyer's and accountant's office and so on. Apart from service industries, appointment systems are also used in project management, production, manufacturing and transportation (Begen and Queyranne, 2011).

For example, in a doctor's clinic, patients are the jobs that need to be scheduled and the doctor and associated resources are the processor. Processing time of each patient are uncertain and varies from one patient to another patient. Some patients take more time than the assigned time causing overage time and waiting for the next patients. Whereas some patients take less time than the assigned time resulting in idle time of the processor and the resources and hence, underutilization of the system. Therefore, it is necessary to optimize the trade-off between underutilization and overtime for both the jobs and the processor.

Similarly, for railway transportation system, stations are the jobs that need to be scheduled. The train and the resources are the processor. In this scenario, processing time means the time for a train to travel from one station to the next station, which is subject to uncertainty. For some stations, arrival of the train happens after the scheduled time causing overage time and waiting for the passengers of the later stations. Whereas, for some stations, arrival time of the train is less than the scheduled time causing idle time for the processor and underutilization of the system. Hence, it is required to optimize between underutilization and overtime for both the processor and the jobs.

1.3 Thesis Statement

In this study, uncertainty of processing time is considered for providing an appointment schedule that minimize the total underage and overage cost of both the jobs and the processor for the worst case under any realization of the processing durations. A Robust Appointment Scheduling model under processing time uncertainty has been developed using Min-max Optimization which will allow handling processing time uncertainty

without distributional information of the processing durations and provide a schedule that perform well for all possible realization of the scenarios and hedge against the worst-case scenario.

This study aims to examine if Robust Appointment Scheduling model performs better while subject to uncertain processing time compared to the other modelling approaches in the literature. Through numerical computations it is proved that the proposed robust model performs better both in terms of minimizing the worst case performance and also computational efforts.

1.4 Research Objectives

The ultimate goal of this thesis is to incorporate processing time uncertainty while decision making regarding appointment scheduling. The objective is to develop a Robust Appointment Scheduling model for any probability distribution of the uncertain processing durations, which minimizes the total cost of the worst case scenario. It is also kept into consideration that the proposed appointment scheduling model should generally be applicable to many areas. This robust model can be used in software packages for the purpose of appointment scheduling. It is in fact applicable in Microsoft Excel for small instances. This work aims to contribute to the literature related to uncertainty handing in decision making in the absence of distributional information of the uncertain parameters. The overall objectives of this thesis are as follows:

1. Developing a Nonlinear Robust Appointment Scheduling model that minimizes the total underage and overage cost of the worst-case scenario for any realization of the processing time.
2. Developing a Mixed Integer Linear Programming model for Robust Appointment Scheduling that minimizes the total underage and overage cost of the worst-case scenario for any realization of the processing time.
3. Providing theoretical proofs for the proposed propositions that supports the robust model.

4. Providing a solution approach using Iterative Search Procedure and Dynamic Programming for solving large instances of the appointment scheduling problem in polynomial time.

5. Conducting two case studies, one for a Dentist's Clinic and another for VIA Rail Canada.

1.5 Outline of the Thesis

In this thesis, there are seven chapters in total, which are organized as follows. In chapter 1, we provide an overview of the appointment scheduling problem under processing time uncertainty, possible application areas of our appointment scheduling model and overall objective of the proposed work. Chapter 2 consists of literature review, analysis of the existing modelling approaches for appointment scheduling, and research gaps. Chapter 3 explains Robust Optimization methodology, how data uncertainty is structured in Robust Optimization and the Min-max criteria in robust decision making. Chapter 4 contains the formulation of the proposed Nonlinear model and Mixed Integer Linear Programming model along with the theoretical proofs of the propositions. In addition, a Stochastic Programming version of the robust model is provided to compare with the proposed robust model. In the last, some special instances of the appointment scheduling problem have been discussed. Chapter 5 explains the Iterative Search Procedure and the Dynamic Programming Model which will allow to solve the larger instances of the Robust Appointment Scheduling model in polynomial time. Chapter 6 provides the case study results for VIA Rail Canada and for a local Dentist's Clinic in Montreal, Canada. Finally, chapter 7 contains conclusions and a discussion about the future research scopes.

CHAPTER 2
LITERATURE REVIEW

This chapter provides the most relevant literature for appointment scheduling under processing time uncertainty. It emphasizes on the decision considerations, application areas, modelling approaches and solution approaches covered in the literature for appointment scheduling. This chapter is concluded by providing a research gap in the context of processing time uncertainty for solving appointment scheduling problem.

2.1 Overview

Appointment scheduling is a well-studied topic in literature. Although appointment system is applicable to a large number of settings where appointment times are scheduled for a set of customers and a service provider (for instance, patients and medical practitioner, doctors and operating room, clients and consulting professional - lawyer or accountant, automobile service centers, tractor trailers and receiving bay, legal cases and a court room, students and a professor, etc.), the literature mostly covers healthcare appointment systems for it being most challenging in terms of uncertainty, importance, high demand and expenditure compared to the other application areas (Ahmadi-Javid et al., 2017), (Robinson and Chen, 2003). However, some studies acknowledge the possible application of their scheduling models in other areas such as project management, production, manufacturing and transportation sector (Begen and Queyranne, 2011).

Sabria and Daganzo, (1989) consider scheduling of arrival of container vessels at a seaport employing a single server queuing system. Bendavid and Golany, (2009) consider project scheduling with stochastic activity durations with the objective to minimize total expected cost. Elhafsi, (2002) studies a production system of multiple stages with stochastic lead times. The objective was to determine planned lead times so that the expected total cost related to inventory, tardiness, and earliness is minimized.

Healthcare is one of the largest industries in the developed countries and the need to improve its efficiency is of utmost value considering the expenditure in healthcare sector

and increasing demand for healthcare services (Ahmadi-Javid et al., 2017). Appointment scheduling in healthcare sector has great importance on efficiency and service quality to the patients (Laan et al., 2018). Appointment scheduling in healthcare can improve both the medical outcome and patient satisfaction (Denton et al., 2007). In recent years, outpatient clinics have become more popular among people due to shorter hospital stays, preventive medical concerns and service (Ahmadi-Javid et al., 2017). Appointment systems are important components for efficient care delivery in outpatient clinics.

There are many studies done in healthcare appointment scheduling considering various factors, however, this review focuses on studies that perform appointment scheduling considering service time uncertainty. Ahmadi-Javid et al., (2017) provides a comprehensive review of analytical and numerical optimization studies, modelling approach and solution methods for outpatient appointment system. They arrange the recent literature from strategic, tactical and operational decision levels and present future research scopes for outpatient appointment schedule. They find that the uncertainty in appointment systems is mostly handled by stochastic optimization and stochastic dynamic programming (Markov Decision Process). In a related paper, Cayirli and Veral, (2009) perform an extensive survey on healthcare appointment system. They study about problem formulation and modelling approaches for outpatient appointment scheduling in previous literature. They conclude that the existing literature mostly incorporate service time uncertainty in their models using Stochastic Programming, Queuing Theory or Markov Decision processes.

2.2 Modelling Approaches

In this section, a detailed discussion about the decision considerations, application areas, modelling approaches, and solution approaches is presented for appointment scheduling in the literature.

2.2.1 Stochastic Programming

Appointment scheduling problem is largely modeled using stochastic programing approach for incorporating uncertainty of processing time. Begen and Queyranne, (2011) incorporate service time uncertainty using single stage stochastic model and apply joint discrete probability distribution for integer processing durations to obtain optimal integer appointment schedule in polynomial time. Berg et al., (2014) address both booking, scheduling and sequencing issue of a single stochastic server in outpatient procedure centers by formulating a two-stage stochastic programming model. Castaing et al., (2016) formulate a two stage stochastic integer program model for designing patient appointment schedule under uncertainty in chemotherapy infusion treatment time. Denton and Gupta, (2003) present a two stage stochastic programming model for scheduling patients under service time uncertainty and a given processing sequence considering independent and identically distributed continuous service time distribution. Robinson and Chen, (2003) model appointment scheduling problem in a doctor's clinic for a given sequence using stochastic linear programming approach. They propose heuristics and use Monte Carlo simulation for solving large instances of the problem. Chen and Robinson, (2014) propose a stochastic linear programming model for patient appointment scheduling and sequencing considering random and heterogeneous service time. They use Benders Decomposition method for solving the problem. Choi and Banerjee, (2016) model outpatient appointment scheduling system using stochastic dynamic programming and propose a stochastic integer programming version of the problem. They use branch and bound and news vendor heuristics to solve the problem. Creemers et al., (2012) model operations room appointment scheduling using stochastic programming with the objective to minimize patient waiting time and use queuing theory to obtain expected patient waiting time to be used later in the stochastic programming model. For solving the model for large number of patients, they propose a heuristic approach. Erdogan et al., (2015) develop a two stage stochastic linear programming model for appointment scheduling problem under service time and demand uncertainty when they consider no shows. On the other hand, they develop a multistage stochastic linear programming model to consider dynamic appointment scheduling where the customers are scheduled one at a time as they request for appointment dates. They

incorporate a decomposition-based algorithms to solve the problem efficiently. Qu et al., (2013) model appointment scheduling problem using stochastic programming approach for random service duration for an eye clinic.

Some studies use sample average approximation when distribution of the service time is not known and only sample data are available to handle service time uncertainty while performing appointment scheduling (Oh et al., 2013). Sample average approximation tackles data uncertainty by approximating the expected value of a stochastic program with random sample average obtained from Monte Carlo Simulation (Begen et al., 2012). Begen and Queyranne, (2011) use single stage stochastic programming approach for appointment scheduling problem with discrete random durations. They assume that the distribution of the service times are not known and use sample average approximation method to solve the appointment scheduling problem under service time uncertainty. Mancilla and Storer, (2012) formulate a stochastic integer programming model for appointment scheduling and sequencing problem using sample average approximation and develop a heuristic solution approach based on Bender's decomposition to compare results with the exact models.

2.2.2 Queuing Theory

Another popular way to model the appointment scheduling problem under processing time uncertainty is through queuing theory. Outpatient clinics can be regarded as queuing systems and appointment system for that are designed using queuing theory (Cayirli and Veral, 2009). Zacharias and Yunes, (2018) develop queuing model for appointment scheduling in healthcare under stochastic environment to optimize between resource utilization and short waiting time. Hassin and Mendel, (2008) design appointment scheduling of healthcare clinics as queuing system with the objective to minimize expected waiting time and idle time for exponential service time distribution. Liu and Ziya, (2014) incorporate queuing theory to model appointment scheduling problem for outpatient clinics. Tang et al., (2014) propose a queuing model for appointment scheduling in healthcare clinics with no shows considering exponential distribution of the service time. Wang and Gupta, (2011) develop a queuing model to find appointment dates of jobs in a

single-server system to minimize expected customer delay and server completion time with identical jobs, identical costs, and exponentially distributed processing durations. In their numerical studies, the optimal allocated time for each job shows a "dome" structure; i.e., it increases first and then decreases. In a related paper, Kaandorp and Koole, (2007) study outpatient appointment scheduling with exponential processing durations and no-shows.

2.2.3 Markov Decision Process

Soltani et al., (2019) model appointment scheduling problem for stochastic service time and multiple provider system with identical providers in a service center (counseling center). They develop a discrete time Markov chain model and apply heuristic methods to achieve better solution than the existing models. Anderson et al., (2015) address appointment scheduling in an outpatient clinic considering stochastic service time using Monte Carlo simulation. Here, service time is considered to follow uniform distribution. Saremi et al., (2013) address the appointment scheduling problem of outpatient multistage operations room using simulation based Tabu Search method for stochastic service time and heterogeneous patients. They propose a mixed integer linear programming model for solving deterministic version of the problem and use it as an initial solution for the simulation model. Lin et al., (2011) develop a Markov Decision Process (MDP) model for sequential clinical scheduling that books patients to optimize the performance of the clinic operations. They consider overbooking to compensate no show situations and use Dynamic Programming to find schedules for larger problems.

2.3 Summary and Research Gaps

While performing appointment scheduling, service time is considered to be either deterministic or stochastic (Denton et al., 2007). Service time uncertainty in appointment scheduling is mostly handled using Stochastic Programming, Queuing Theory and Markov Decision Process (Ahmadi-Javid et al., 2017). Cayirli and Veral, (2009) state that majority of the studies assume patients or service types to be homogenous for scheduling purpose

and use independently and identically distributed (i.i.d) service time distribution. Whereas some studies acknowledge service type heterogeneity and use independently and distinctly distributed service time distribution (i.d.d). A variety of service time distribution can be found in the literature (Cayirli and Veral, 2009). However, it is found that most studies use exponential distribution to make their model tractable (Zeng et al., 2010). Chakraborty et al., (2010) perform appointment scheduling of patients using general service distribution and show that assumption of gamma distribution for service time can significantly reduce computational effort. In most of the studies, it is assumed that the distribution of the service time is known beforehand which may not always hold in practice (Turkcan et al., 2012), (Mancilla and Storer, 2012), (Begen and Queyranne, 2011).

It is found that most stochastic programming models assume identical service time distribution for all patients in their appointment scheduling models (Cayirli and Veral, 2009). However, for heterogeneous case patients, service time distribution will depend on the patient and the service type (Ahmadi-Javid et al., 2017). In such cases, assuming identical service distribution for heterogeneous patients or service types will lead to faulty assumption. For example, in healthcare services, in the cases of patient heterogeneity, where different patients have different health issues and hence, require different level of service and care, the assumption of an identical service time distribution will negatively affect a stochastic scheduling approach. Therefore, some recent studies used independent and distinctly distributed service time distribution (i.d.d) for considering patient heterogeneity. In this regard, the proposed Robust Appointment Scheduling model incorporates Min-max Optimization, which does not require to know the probability distribution of the uncertain parameters. Hence, the proposed Robust Appointment Scheduling model will be well suited for both homogenous and heterogeneous service types because it will eliminate the need to identify different service time distributions for heterogeneous service types to achieve an optimal schedule.

Although the stochastic programming approach can handle service time uncertainty, stochastic models rapidly enlarge as a function of stages and scenarios. Since appointments are to be given beforehand, the scope for applying recourse is very limited when the actual processing time doesn't match the assigned time slot. There is very limited opportunity to

adjust the schedule once it is provided to all the patients since the patients are usually not available before the given appointment dates. Moreover, it is sometimes necessary to know the uncertain service time distribution. One of the main problems regarding stochastic programming in handling data uncertainty is the fact that it requires to assign probability to various data instances. Fitting a probability distribution requires to have access to sufficient amount of data and is not a trivial exercise for decision makers (Kouvelis and Yu, 2013). Reliable estimation of event probabilities is also extremely difficult in many cases (Aissi et al., 2009). Similarly, for applying queuing theory to appointment systems, it is necessary to know the distribution of the uncertain parameters. Simulation approaches are used to solve complex models to obtain solutions that perform better although they don't ensure optimality of the solution.

It is common in literature to assume that probability distribution of the processing time is known to the decision maker. This is only true when enough data is available to fit a distribution. Due to a lack of data, finding the probability distribution of service time can be difficult (Denton et al., 2007). There are lack of studies in literature, which can handle service time uncertainty while appointment scheduling in the absence of distributional information. This gap is also reflected in other studies (Mak et al., 2015).

Stochastic programming gives optimal solution for the expected or most likely scenario, however other realization of the scenarios are strongly neglected in stochastic programming approach. Most likely or expected scenarios are just subsets of the potentially realizable scenarios. Stochastic programming optimizes expected system performance over all potential scenarios or just performance of the most likely scenario but fails to hedge against the poor system performance for some other realization of the data scenarios (Aissi et al., 2009). Dealing with the expected value may not ensure enough protection against potentially high impact but low probable events. Therefore, stochastic decision-making approaches will fail to protect against high impact events that might have low probability. As a result, stochastic programming approaches do not consider the risk aversion nature of the decision maker properly and therefore has limited application to many areas for instance, handling high impact events with low probability. This approach is inappropriate for moderate and high risk decision making under uncertainty (Mulvey et al., 2016).

Data uncertainty is sometimes handled with sensitivity analysis. However, sensitivity analysis is a reactive approach towards data uncertainty. It doesn't incorporate any mechanism to deal with uncertain parameters proactively. It fails to produce solutions that are insensitive to data uncertainty and to control the sensitive parameters (Mulvey et al., 2016).

Despite the plethora of literature available in appointment scheduling, there are lack of studies which handle service time uncertainty in the absence of distributional information. Also, the current literature fails to propose generally applicable guidelines for appointment scheduling because they focus only on situation specific problems or models complex environment considering intricacies of case specific factors (such as no show, fairness, overbooking, capacity and demand constraint, emergency arrival, service interruption, processor lateness, etc.). Such studies are only applicable to those situation specific problems and not for the general appointment scheduling problem in other application fields. Moreover, there is a lack of efficient and effective appointment scheduling methods that are easy to implement and do not require to assign probabilities to future uncertain instances. In this regard, the proposed robust model can provide optimal solution while considering only the extreme bounds of the uncertain processing times. As a result, the robust model eliminates the need of assigning probability to the uncertain parameters or the need to know the probability distribution of the uncertain parameters.

2.4 Contribution of This Research

In this thesis, a Robust Appointment Scheduling model for random service time is developed using Min-max Optimization. For incorporating processing time uncertainty, the model requires less information regarding uncertainty unlike other approaches based on probability. It is not necessary to know the distribution of the uncertain processing duration for applying the proposed Robust Appointment Scheduling model. To incorporate processing time uncertainty in the robust model, it is only required to know the extreme bounds of the uncertain parameter. As a result, the model will be well suited for both homogeneous and heterogeneous service types. The proposed model is also applicable to

any probability distribution of the processing time. The scenario realization and hence, the computational effort of the proposed model is significantly lower than that of stochastic programming approach. The model will be applicable and well suited for any general case appointment scheduling problem regardless of application field. It reduces scenario realization and hence is easy to implement. It can be used in software packages for appointment scheduling purpose. It is even applicable in Microsoft Excel for small instances. For solving the larger instances of Robust Appointment Scheduling problem in polynomial time, an Iterative Solution Procedure and a Dynamic Programming model is proposed. This work aims to contribute both to the literature related to uncertainty handling in decision making and to the industries which aim to achieve an efficient service system.

CHAPTER 3
PRELIMINARY

Decision making process becomes complicated in the presence of uncertainty of parameters in many dynamic problems in technology and business. For example, in scheduling, uncertainty of the processing time makes scheduling decisions complicated. The best way to deal with uncertainty is to incorporate it in the decision making process, understand it, and then structure it accordingly.

There are many ways in the literature to handle uncertainty in decision making for a system including stochastic programming, robust optimization, sensitivity analysis, queuing models, discrete event simulation, etc. In this work, Robust Optimization is applied for incorporating processing time uncertainty while decision making regarding appointment scheduling.

This chapter contains a detailed discussion about the Robust Optimization, its advantages over other decision making approaches and how data uncertainty can be expressed in this approach.

3.1 Robust Optimization Overview

Robust optimization is a comprehensive mathematical programming framework for robust decision making. It enables the decision maker to take into account uncertainty in order to produce decisions that will behave reasonable under any likely input data (Aissi et al., 2009). Robust optimization allows decision making in the presence of inadequate knowledge of the uncertain parameter and provides a solution that minimizes the worst case performance for any realization of the uncertain parameter over the given set of realizable scenarios (Kouvelis and Yu, 2013). Applications of robust optimization include but is not limited to financial planning, appointment scheduling, power capacity expansion, structural design and so on. Robust optimization is suitable for various real world decision making situations including unique and non-repetitive decisions, decisions with precautions (like safety system) and for decisions to meet predefined goals. In the proposed

robust optimization framework, Min-max criterion is applied where the robust decision is to minimize the worst case performance for any possible realization of the input data. In the Min-max approach, the worst-case performance is recorded as the robustness indicator of the decision.

Robust optimization has many advantages for handling data uncertainty compared to other modelling approaches based on probability. Advantages of Robust Optimization over other decision-making approaches are:

- It acknowledges uncertainty of the variables and acts proactively to deal with any possible realization of the uncertain parameter,
- It is applicable in the presence of uncertain, or incomplete, or noisy data
- Robust optimization is more applicable than other decision making approaches for unique and non-repetitive decision-making, which is very much common in real world application.
- It accounts for the risk aversion nature of the decision maker since it accounts for the high impact events that has low probabilities.
- Robust optimization performs better than other decision making approaches in situations where dealing with worst case scenario is crucial.
- It provides a solution that minimizes the worst-case performance for any realization of the uncertain parameter over the given set of realizable scenarios.
- It is simple to use and requires less information regarding uncertainty unlike other approaches based on probability.
- It allows handling data uncertainty with upper and lower bounds of the uncertain parameter instead of knowing its distribution.
- It can significantly reduce scenario realization compared to other decision making approaches based on probability (Mulvey et al., 2016).

3.2 Robust Optimization Framework

The proposed methodology initially identifies potentially realizable input data instances for the decision model that is appropriate for the decision situation, without attempting to assign probabilities to various instances, and then proceeds to find the decision that performs well even in the worst case of the identified input data instance, or in other words, it performs well for all realizable input data instances (Aissi et al., 2009).

In this study, the Robust Optimization framework is structured into three important steps presented in Figure 2. These steps are:

1. Structuring of data uncertainty with the use of scenarios. This constructs the input data instances.

2. Incorporating robustness in decision making using the Min-Max criteria. Here, the objective is to minimize the maximum cost among all the input data instances.

3. The formal development of the Decision model and providing output of the robust decisions.
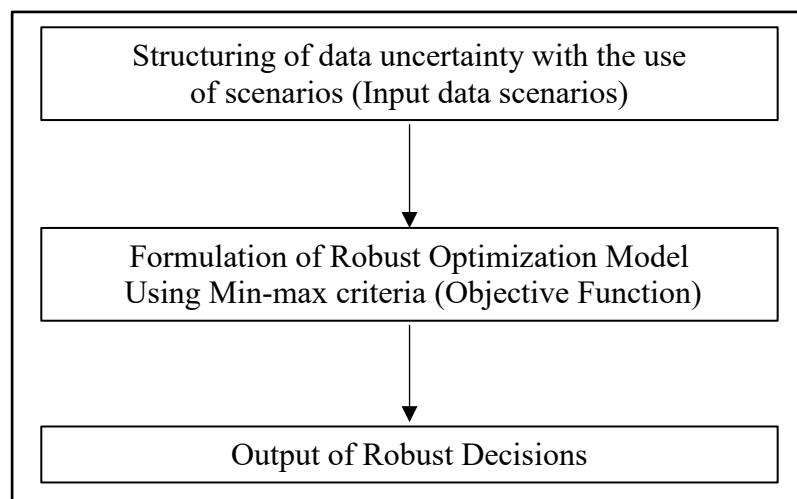
Figure 2: Robust Optimization Framework

3.2.1 Structuring Data Uncertainty through Scenario Planning Approach

One of the important tools for structuring data uncertainty in decision making is Scenario Planning. Scenarios will represent several contrasting features, which represent possibilities of the future and which are generated by using the decision maker's own model of the system and its realities. In robust optimization, no probability is needed to be attached to the various outcomes. It would allow the decision maker to be prepared for any unconventional but still potentially realizable outcomes and will help to cope satisfactorily in all cases. Scenario planning requires to generate and evaluate all potential scenarios. It is the decision maker's mental image of the current system's decision situation and the future that will generate the scenarios, and subsequently the robust decision that can cope satisfactorily with all of them. As a result, scenarios should be structured based on sound analysis.

In the scenario-based approach, each scenario corresponds to an assignment of values for the uncertain input parameter. There are two different ways to represent all scenarios for the uncertain parameters; discrete scenario case and interval scenario case.

Discrete Scenario Case

For a discrete scenario case, uncertainty of an input parameter is presented as a set of discrete value instances. In this approach, input data uncertainty is structured as a finite set Ω of scenarios where each scenario, $\omega \in \Omega$ is presented as a vector, $A^\omega = (a_1^\omega, \dots, a_n^\omega)$ where, $a_i^\omega \in \mathbb{R}^+$, $a_i^\omega \in A^\omega$, $\omega \in \Omega$, $i = 1, 2, \dots, n$.

Interval Scenario Case

In interval scenario case, uncertainty of a parameter is denoted as a range bounded by its extreme values and the parameter can have any value within that range. Input parameter a_i can have any value in the interval $[\underline{a_i}, \overline{a_i}]$, where, $\underline{a_i}$ is the lowest value of that intval and $\overline{a_i}$ is the highest value of that range and $0 \le \underline{a_i} \le \overline{a_i}$. In this approach, the total scenario

set, Ω is the Cartesian product of all values within the intervals $[\underline{a_i}, \overline{a_i}]$, for $i = 1, 2, ..., n$. In interval case scenario, Ω is an infinite vector set.

Extreme Scenario Case

An extreme scenario case is the case where all the input parameters correspond to either of their extreme values, $a_i = \underline{a_i}$ or $\overline{a_i}$ for $i = 1, 2, ..., n$. In other words, input parameter, a_i can have either of the extreme value of the interval $[\underline{a_i}, \overline{a_i}]$. The total scenario set, Ω is the Cartesian product of the extreme values of the intervals $[\underline{a_i}, \overline{a_i}]$ for $i = 1, 2, ..., n$. As a result, total scenario set, Ω is a finite vector set having a total of 2^n scenarios.

3.2.2 Min-Max Approach

In the Min-max approach, the worst case performance is the robustness indicator of the decision model. The objective is to minimize the maximum cost that can occur across all possible input data scenarios.

Let, Ω denote the set of all potentially realizable input data scenarios. Let, Y be the set of our decision variables such that $y_1, y_2, ... \in Y$ and A, be the set of our input data instances which is subjected to uncertainty, meaning, $A^\omega = (a_1^\omega, ..., a_n^\omega), \omega \in \Omega$. The notation A^ω is used to represent the instance of the input data for scenario ω such that $\omega \in \Omega$. Let, $f(Y, A^\omega)$ denote the cost function that evaluates the quality of the decision variable vector set, A. It is to be noted that the cost function is dependent of both the decision variable vector, Y and the input data instance, A^ω which is subjected to uncertainty, i.e. $\omega \in \Omega$. Now, if Z^* denote the maximum cost among all the scenarios then we can write,

$$Z^* = \max_{\omega \in \Omega} f(Y, A^\omega) \qquad (1)$$

Robust optimization aims to minimize the worst cost among all possible scenarios. Hence, we write,

$$min \, Z^* \quad \forall \omega \in \Omega \qquad (2)$$

Or,

$$\min_{y \in Y} \max_{a \in A} f(y, a^{\omega}) \ \forall \omega \in \Omega \tag{3}$$

In the next chapter, we formulate the Robust Appointment Scheduling model using the Min-mix criteria.

CHAPTER 4

PROBLEM DESCRIPTION AND MODEL FORMULATION

In this chapter, a formal definition of the appointment scheduling problem and how it is modeled using the Robust Optimization are explained. It also contains some properties of the problem. Then, a comparison of the robust model with its stochastic counterpart and an analysis of the results obtained from both the approaches are discussed.

4.1 Problem Description

We have $(n + 1)$ jobs that need to be sequentially processed on a single processor. The processing sequence is given. When a job finishes earlier than the next job's appointment date, the system experiences some cost due to under-utilization. This cost is referred as the underage cost. On the other hand, if a job finishes later than the next job's appointment date, the system experiences overage cost due to the overtime of the current job and the waiting of the next jobs. All the cost coefficients and processing durations are assumed to be nonnegative, and job 1 starts on time; i.e., the start time for the first job is zero. The $n + 1^{st}$ job is a dummy job with processing duration equal to 0. The dummy job is used to compute the overage or underage cost of the n −th job. The processing duration of job i is denoted by p_i. Let, l_i and k_i denote the minimum and maximum value of processing duration p_i, respectively. The underage cost rate, u_i of job i is the unit cost (per unit time) incurred when job i is completed at a time C_i before the appointment date of the next job A_{i+1}. The overage cost rate, o_i of job i is the unit cost incurred when job i is completed at a date C_i after the appointment date of the next job A_{i+1}. Thus the total cost due to job i completing at date C_i is,

$$u_i(A_{i+1} - C_i)^+ + o_i(C_i - A_{i+1})^+ \qquad (4)$$

Where $(x)^+$ is the positive part of real number x. Our decision variables are $\{A_i\}$ and the objective is to minimize the cost defined above for the worst $\{p_i\}$ possible. Formally, we define our problems as:

$$min_{\{A_i\}}max_{\{l_i \le p_i \le k_i\}} \sum_{i=1}^{n}[u_i(A_{i+1} - C_i)^+ + o_i(C_i - A_{i+1})^+] \tag{5}$$

4.1.1 Notations

We use the following notations in our mathematical models:

Parameters

u_i	Underage cost of job i
o_i	Overage cost of job i
p_i^ω	Processing time of job i in scenario ω; $l_i \le p_i^\omega \le k_i$

Index Sets

i	Job, $i = 1, 2, \ldots, n+1$.
ω	Scenario, $\omega \in \Omega$.

Variables

A_i	Appointment date of job i; $A_1 = 0$.
x_i^ω	1, if job i finishes after the next jobs appointment time in scenario ω; 0, otherwise
C_i^ω	Completion time of job i in scenario ω, $C_i^\omega = max\,(A_i, C_{i-1}^\omega) + p_i^\omega$ for $2 \le i \le n+1$, $C_i^\omega = p_i^\omega$ for $i = 1$.
$d_i^{-\omega}$	Underutilization of the facility prior to the completion of job i in scenario ω; $d_i^{-\omega} = max(0, A_{i+1} - C_i^\omega) = (A_{i+1} - C_i^\omega)^+$
$d_i^{+\omega}$	Overtime of job i in scenario ω; $d_i^{+\omega} = max(0, C_i^\omega - A_{i+1}) = (C_i^\omega - A_{i+1})^+$

4.1.2 Assumptions

There are few assumptions that define the context of the appointment scheduling problem considered in this study. They are stated here:

- Appointment dates are to be given before any processing starts
- Jobs are not available before their appointment dates.
- There is a single processor, which processes job one at a time and is always available during the scheduling time.
- Jobs arrive on time and walk-ins are not allowed.
- There are no missing appointments or interruptions.
- The sequence of the schedule is known.

4.2 Mathematical Models

Initially a Nonlinear Programming model is developed. Then the nonlinearity is removed by developing a Mixed Integer Linear Programming model, making it our second model.

4.2.1 Model 1 (Nonlinear)

$$Min\ Z \tag{6}$$

Subject to,

$$Z \geq \sum_{i=1}^{n}(u_i d_i^{-\omega} + o_i d_i^{+\omega})\ \forall \omega \in \Omega \tag{7}$$

$$C_1^{\omega} = p_1^{\omega}\ \forall \omega \in \Omega \tag{8}$$

$$C_i^{\omega} + d_i^{-\omega} - d_i^{+\omega} = A_{i+1}\ \forall \omega \in \Omega, 1 \leq i \leq n \tag{9}$$

$$C_i^{\omega} = C_{i-1}^{\omega} + d_{i-1}^{-\omega} + p_i^{\omega}\ \forall \omega \in \Omega, 2 \leq i \leq n \tag{10}$$

$$d_i^{-\omega} d_i^{+\omega} = 0\ \forall \omega \in \Omega, 1 \leq i \leq n \tag{11}$$

$$C_i^\omega, d_i^{-\omega}, d_i^{+\omega}, A_{i+1} \geq 0 \ \forall \omega \in \Omega, 1 \leq i \leq n \tag{12}$$

Detailing the structure of the model, the objective function 6, minimizes the maximum of total underage and overage cost across all the scenarios. Constraint 7 ensures that Z is the maximum underage and overage cost over all scenarios. Constraint 8 sets the completion time in a scenario equal to the processing time in that scenario. This holds true for the first job because the first job starts at time 0, i.e., $A_1 = 0$. Constraint 9 calculates the underutilization of the facility and overtime for each job in each scenario. Constraint 10 calculates the completion time of the jobs other than the first job. Note that,

$$C_i^\omega = max(C_{i-1}^\omega, A_i) + p_i^\omega = C_{i-1}^\omega + d_{i-1}^{-\omega} + p_i^\omega \ \forall \omega \in \Omega, 2 \leq i \leq n \tag{13}$$

Constraint 11 ensures that when $d_i^{-\omega} > 0$, $d_i^{+\omega} = 0$ and when $d_i^{+\omega} > 0, d_i^{-\omega} = 0$. It means that both underage time and overage time cannot happen at the same time for a job in a scenario. Constraint 12 is the non-negativity constraint. It is to be noted that the only nonlinear constraint is constraint 11.

4.2.2 Model 2 (Mixed Integer Linear Programming)

Next, the nonlinearity from constraint 11 of Model 1 is removed by developing a Mixed Integer Linear Programming model. For that, a binary variable is introduced and the nonlinear constraint is replaced with two linear constraints as follows:

$$Min \ Z \tag{14}$$

Subject to,

$$Z \geq \sum_{i=1}^{n}(u_i d_i^{-\omega} + o_i d_i^{+\omega}) \ \forall \omega \in \Omega \tag{15}$$

$$C_1^\omega = p_1^\omega \ \forall \omega \in \Omega \tag{16}$$

$$C_i^\omega + d_i^{-\omega} - d_i^{+\omega} = A_{i+1} \ \forall \omega \in \Omega, 1 \leq i \leq n \tag{17}$$

$$C_i^\omega = C_{i-1}^\omega + d_{i-1}^{-\omega} + p_i^\omega \ \forall \omega \in \Omega, 2 \leq i \leq n \tag{18}$$

$$d_i^{+\omega} \leq x_i^{\omega} \sum_{j=1}^{n} k_j \quad \forall \omega \in \Omega, 1 \leq i \leq n \tag{19}$$

$$d_i^{-\omega} \leq (1 - x_i^{\omega}) \sum_{j=1}^{n} k_j \quad \forall \omega \in \Omega, 1 \leq i \leq n \tag{20}$$

$$C_i^{\omega}, d_i^{-\omega}, d_i^{+\omega}, A_{i+1} \geq 0 \quad \forall \omega \in \Omega, 1 \leq i \leq n \tag{21}$$

$$x_i^{\omega} \in \{0,1\} \quad \forall \omega \in \Omega, 1 \leq i \leq n \tag{22}$$

Like Model 1, the objective function 14 minimizes Z. Constraint 15 ensures that Z is the maximum underage and overage cost over all the scenarios. Constraint 16 sets the completion time in a scenario equal to the processing time in that scenario. This holds true for the first job because the first job starts at time 0; i.e., $A_1 = 0$. Constraint 17 calculates the underutilization of the facility and overtime for each job in each scenario. Constraint 18 calculates the completion time of the jobs other than the first job. Constraints 19 and 20 replaces the nonlinear constraint 11 of Model 1 by binary variables and these two linear constraints. With these linear constraints, it is ensured that when $d_i^{-\omega} > 0$, $d_i^{+\omega} = 0$ and when $d_i^{+\omega} > 0$, $d_i^{-\omega} = 0$. It means that both underage time and overage time do not happen at the same time for a job in a scenario. Constraint 21 is the non-negativity constraint. Constraint 22 defines the binary variables.

4.3 Scenario Planning

For structuring uncertainty of processing time in the model, at first interval scenario case for the processing time is considered, where processing time of a job can have any value within an interval range. It is considered because when jobs have a large number of processing time instances, it is more convenient to work with the range of the uncertain parameters rather than determining each instance individually.

If processing time of job i is p_i, we consider that the processing time instances of job i fall within the range bounded by its extreme values, i.e. it's lowest possible time, $\underline{p_i} = l_i$ and highest possible time, $\overline{p_i} = k_i$ or $p_i \in [l_i, k_i]$. Specifying processing time instances by their range implies an infinite set of total scenarios (the infinite set is denoted as Ω). As it

is mentioned earlier that for interval case scenario, the total scenario set Ω is the Cartesian product of all values within the intervals $[\underline{p_i}, \overline{p_i}]$, for $i = 1, 2, ..., n$, resulting in an infinite vector set Ω.

4.3.1 Limiting to a Finite Scenario Set

Even when processing times are specified as independent ranges for each job, attentions can be restricted to an appropriately selected, finite set of discrete scenarios to determine the worst-case scenario for any given schedule.

Kouvelis and Yu, (2013) prove that for any sequence and given the makespan performance criteria, both for one machine flow shop and two machine flow shop problem with interval processing data, worst case scenario for robust scheduling belongs to the set of extreme point instances, i.e., the worst case scenario belongs to the set of extreme bounds of the uncertain processing times of each job.

For the proposed Robust Appointment Scheduling model, it is proved that for interval processing time scenario, the worst case scenario belongs to the set of extreme points (proved in Proposition 3). Therefore, scenario realization is limited to the extreme points of the interval range without loss of optimality. As a result processing time instances for each job correspond to either of their extreme values only, i.e. lower and upper bounds, $p_i = \underline{p_i}$ or $p_i = \overline{p_i}$ or $p_i^{\omega^*} \in \{l_i, k_i\}$. The total scenario set, Ω is the Cartesian product of only the extreme values of the intervals $[\underline{p_i}, \overline{p_i}]$ for $i = 1, 2, ..., n$. Therefore, for n number of jobs each with two possible instances of processing time, there will be a total of 2^n number of scenarios. As a result, total scenario set Ω is a finite vector set having 2^n scenarios.

4.3.2 Scenario Labelling

An n−digit binary number representation is used to label each scenario ω such that the i−th digit (from left) is 0 if processing time of job i is equal to its lower limit, i.e. $p_i^\omega = l_i$ and the i−th digit (from left) is 1 if processing time of job i is equal to its upper limit, i.e. $p_i^\omega = k_i, i = 1, 2, \dots, n$.

For instance, for a 3-job scheduling problem, there is a total of 2^3 or 8 scenarios which is represented in Table 1 using the proposed binary representation approach.

<p align="center">Table 1: Binary representation of scenarios for 3 jobs</p>

Scenarios	Scenario label	Job 1	Job 2	Job 3
1	000	$p_1^1 = l_1$	$p_2^1 = l_2$	$p_3^1 = l_3$
2	001	$p_1^2 = l_1$	$p_2^2 = l_2$	$p_3^2 = k_3$
3	010	$p_1^3 = l_1$	$p_2^3 = k_2$	$p_3^3 = l_3$
4	011	$p_1^4 = l_1$	$p_2^4 = k_2$	$p_3^4 = k_3$
5	100	$p_1^5 = k_1$	$p_2^5 = l_2$	$p_3^5 = l_3$
6	101	$p_1^6 = k_1$	$p_2^6 = l_2$	$p_3^6 = k_3$
7	110	$p_1^7 = k_1$	$p_2^7 = k_2$	$p_3^7 = l_3$
8	111	$p_1^8 = k_1$	$p_2^8 = k_2$	$p_3^8 = k_3$

Here, scenario 7 or "110" corresponds to the following set of processing times: $\{p_1^7 = k_1, p_2^7 = k_2, p_3^7 = l_3\}$. This means that in scenario 7 or "110", processing time of job 1 is equal to its upper limit, k_1, processing time of job 2 is equal to its upper limit, k_2 and processing time of job 3 is equal to its lower limit, l_3 respectively.

Figure 3 shows a tree representation of the scenarios for 3 jobs. For the 2 scenarios of job 1, there are four corresponding scenarios for job 2 and 8 corresponding scenarios for job 3.
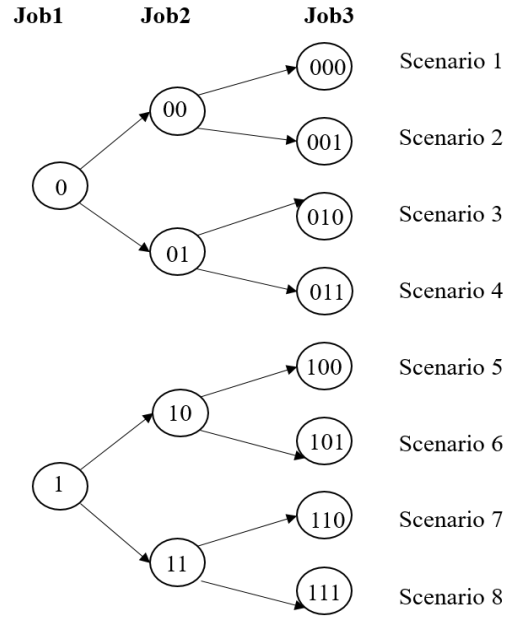
Figure 3: Tree representation of scenarios for 3 jobs

4.4 Illustrative Example

In this section, two illustrative examples are provided to explain the implementation procedure, the scenario generation and the performance evaluation of the proposed robust model for appointment scheduling.

Example 1: In Table 2, an illustrative example of Robust Appointment Scheduling problem for two jobs is presented. It contains the data of underage time, overage time, lower limit and upper time of the uncertain processing times for two jobs which need to be scheduled sequentially. The processing times of the two jobs can have any value within its lower and upper limit given in the dataset. The objective is to provide an appointment schedule that will minimize the total underage and overage cost for the worst case scenario for any realization of the processing time instance.

Job, i	Overage Cost, o_i	Underage Cost, u_i	Lower processing time, l_i	Upper processing time, k_i
1	1	2	5	7
2	3	4	6	8

Since there are 2 jobs to schedule, according to the scenario planning approach there is a total of 2^n or 2^2 or 4 scenarios (considering extreme scenario case) as shown in Table 3.

Table 3: Total scenarios for 2 jobs in Example 1

Scenarios	Scenario label	Job 1	Job 2
1	00	$p_1^1 = l_1 = 5$	$p_2^1 = l_2 = 6$
2	01	$p_1^2 = l_1 = 5$	$p_2^2 = k_2 = 8$
3	10	$p_1^3 = k_1 = 7$	$p_2^3 = l_2 = 6$
4	11	$p_1^4 = k_1 = 7$	$p_2^4 = k_2 = 8$

Xpress Optimizer 25.01.05, algebraic model language and optimizer ('mmnl' module) has been used to code the mathematical Model 2 (Mixed Integer Linear Programming) for solving the problem. The results found are presented in Table 4.

Table 4: Results obtained for 2 jobs in Example 1

Scenarios	Job 1, $d_i^{-\omega}$	$d_i^{+\omega}$	c_i^ω	Job 2, $d_i^{-\omega}$	$d_i^{+\omega}$	c_i^ω	Cost	Appointments, A_i	Objective, Z
1	1.33	0.00	5.00	0.00	0.67	12.33	6.10	$A_1 = 0$,	$Z = 6.10$
2	0.86	0.00	5.00	0.19	0.00	14.33	6.10	$A_2 = 6.33$,	
3	1.33	0.00	7.00	0.00	0.67	13.00	1.43	$A_3 = 13.19$,	
4	0.00	1.14	7.00	0.00	1.81	15.00	6.10		

The optimal appointment dates are found to be $A_2 = 6.33$, $A_3 = 13.19$. The objective function value, $Z = 6.1$ means that the worst cost that may occur for any realization of the

processing time is 6.1. Figure 4 shows the cost for different scenarios. It can be seen that among the 4 scenarios, the worst cost is 6.1 as found from the objective function Z.

Figure 4: Total cost of different scenarios for 2 jobs

Figure 5 and Figure 6 show the underage time and the overage time of job1 and job 2 respectively for all 4 scenarios. It can be seen that both the underage time and the overage time do not occur at the same time for a job in a particular scenarios.

Figure 5: Underage time and overage time of job 1
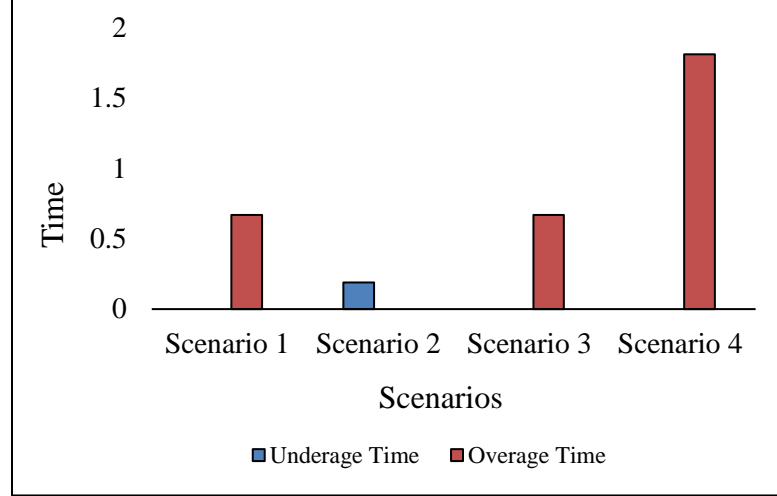
Figure 6: Underage time and overage time of job 2

If the nonlinear constraint 11 is deleted from Model 1 or constraints 19 and 20 are deleted from Model 2, then the rest of the constraints do not ensure that when $d_i^{-\omega} > 0$, $d_i^{+\omega} = 0$ and when $d_i^{+\omega} > 0$, $d_i^{-\omega} = 0$. For proving that such deletion of constraints results in an incorrect solution with both $d_i^{+\omega} > 0$, and $d_i^{-\omega} > 0$ at the same time, Example 1 is solved again using Model 2 but this time deleting the constraints 19 and 20. The results obtained is shown in Table 5.

Table 5: Results obtained for 2 jobs in Example 1 [without constraint (19) and (20)]

Scenarios	Job 1			Job 2			Cost	Appointments A_i	Objective Z
	$d_i^{-\omega}$	$d_i^{+\omega}$	c_i^{ω}	$d_i^{-\omega}$	$d_i^{+\omega}$	c_i^{ω}			
1	2.33	1.00	5	0.00	0.00	12	5.67	$A_1 = 0$,	$Z = 5.67$
2	1.33	0.00	5	0.00	1.00	14	5.67	$A_2 = 6.33$,	
3	0.00	0.67	7	0.33	0.00	13	2.00	$A_3 = 13.33$	
4	0.00	0.67	7	0.00	1.67	15	5.67		

Table 5 shows that when constraints 19 and 20 are deleted from Model 2, the rest of the model yield an incorrect optimal solution, $A_2 = 6.33$, $A_3 = 13.33$, and $Z = 5.67$ with both $d_1^{+\omega} = 1 > 0$, and $d_1^{-\omega} = 2.33 > 0$ for scenario 1, i.e. $\omega: \{p_1 = 5, p_2 = 6\}$. This means that for job 1, both the underage time and the overage time happens at the same time in scenario 1, which cannot happen. Figure 7 explains this issue.
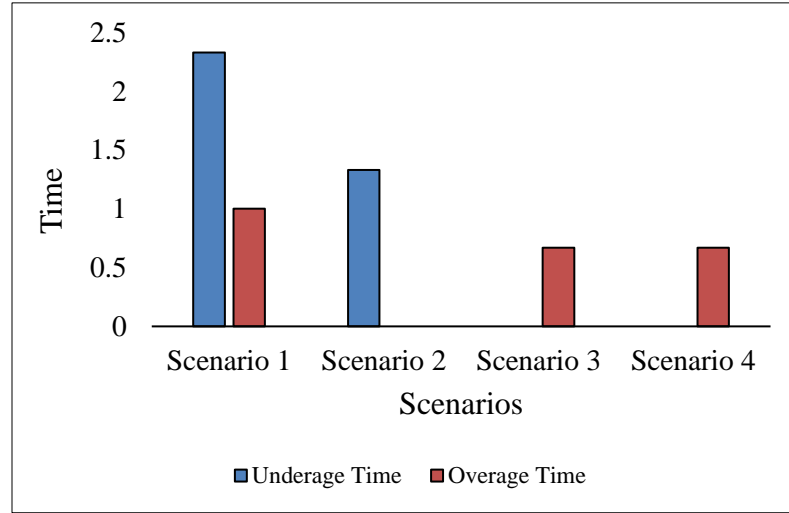
Figure 7: Underage time and overage time of job 1 in different scenarios

However, as it has been shown previously in Table 4 and Figure 5 that, when constraints 19 and 20 are used to ensure that when $d_i^{-\omega} > 0$, $d_i^{+\omega} = 0$ and when $d_i^{+\omega} > 0$, $d_i^{-\omega} = 0$, Models 2 yield a correct optimal solution to Example 1, which is $A_2 = 6.33$, $A_3 = 13.19$, $Z = 6.10$, and which gives $d_1^{+\omega} = 0$, and $d_1^{-\omega} = 1.33$ for scenario 1, i.e. ω: $\{p_1 = 5, p_2 = 6\}$. This proves that the underage time and the overage time do not occur at the same time for job 1 in scenario 1.

Hence, it can be conclude that the nonlinear constraint 11 in Model 1 or the linear constraints 19 and 20 in Model 2 are required to be included to ensure correct solution.

Example 2: In Table 6, we present another illustrative example of the Robust Appointment Scheduling problem. It contains the data of underage time, overage time, lower limit and upper time of the uncertain processing times for two jobs, which need to be scheduled sequentially. The processing times of the two jobs can have any values within its lower and upper limit given in the dataset. The objective is to provide an appointment schedule that will minimize the total underage and overage cost for the worst case scenario for any realization of the processing time instance.

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Table 6: Data for Illustrative Example 2

Job, i	Overage cost o_i	Underage cost u_i	Lower processing time l_i	Upper processing time k_i
1	2	1	5	7
2	3	4	6	8

Since there are 2 jobs to schedule, according to the scenario planning approach there is a total of 2^n or 2^2 or 4 scenarios (considering extreme scenario case) as shown in Table 7.

Table 7: Total scenarios for 2 jobs in Example 2

Scenarios	Scenario label	Job 1	Job 2
1	00	$p_1^1 = l_1 = 5$	$p_2^1 = l_2 = 6$
2	01	$p_1^2 = l_1 = 5$	$p_2^2 = k_2 = 8$
3	10	$p_1^3 = k_1 = 7$	$p_2^3 = l_2 = 6$
4	11	$p_1^4 = k = 7$	$p_2^4 = k_2 = 8$

Xpress Optimizer 25.01.05, algebraic model language and optimizer (mmnl module) has been used to code the mathematical Model 2 (Mixed Integer Linear Programming) for solving the problem. The results found is presented in Table 8.

Table 8: Results obtained for 2 jobs in Example 2

Scenarios	Job 1			Job 2			Cost	Appointments A_i	Objective Z
	$d_i^{-\omega}$	$d_i^{+\omega}$	c_i^ω	$d_i^{-\omega}$	$d_i^{+\omega}$	c_i^ω			
1	1.67	0.00	5	0.00	0.33	12.67	5.10	$A_1 = 0,$	$Z = 5.09$
2	0.86	0.00	5	0.52	0.00	14.67	5.10	$A_2 = 6.67,$	
3	1.67	0.00	7	0.00	0.33	13.00	2.76	$A_3 = 13.52,$	
4	0.00	1.14	7	0.00	1.48	15.00	5.10		

The optimal appointment dates are found to be $A_2 = 6.67,\ A_3 = 13.52$. The objective function value, $Z = 5.09$ means that the worst cost that may occur for any realization of

the processing time is 5.09. Figure 4.2 shows the cost for different scenarios. It is seen that among the 4 scenarios, the worst cost is 5.09 as found from the objective function Z.
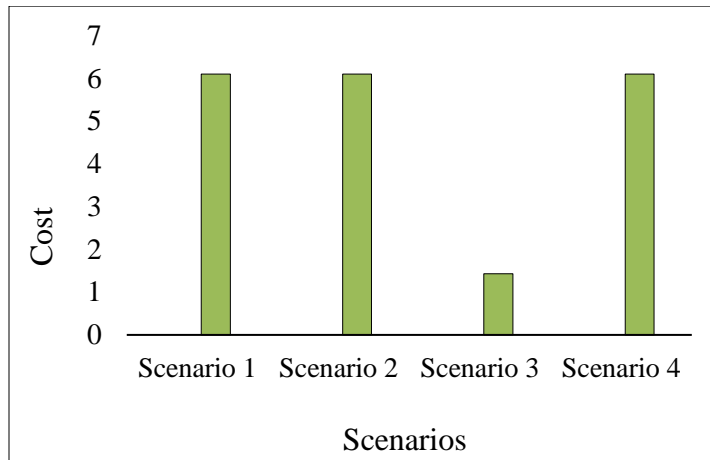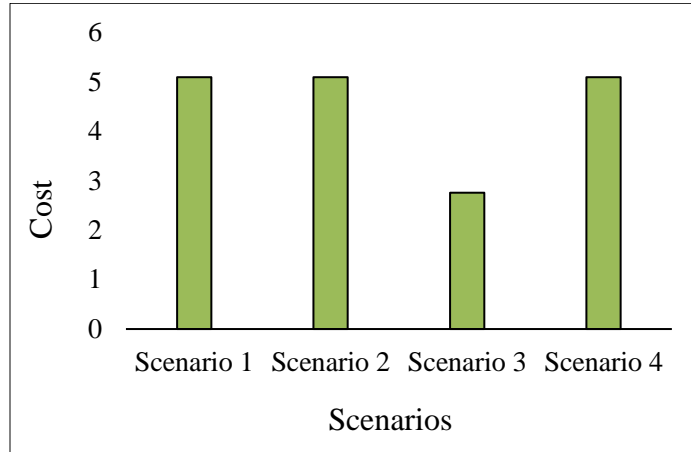

Figure 8: Total cost of different scenarios for 2 jobs

Figure 9 and Figure 10 show the underage time and the overage time of job1 and job 2 respectively for all 4 scenarios. It can be seen that the underage time and the overage time do not occur at the same time for a job in a particular scenarios.
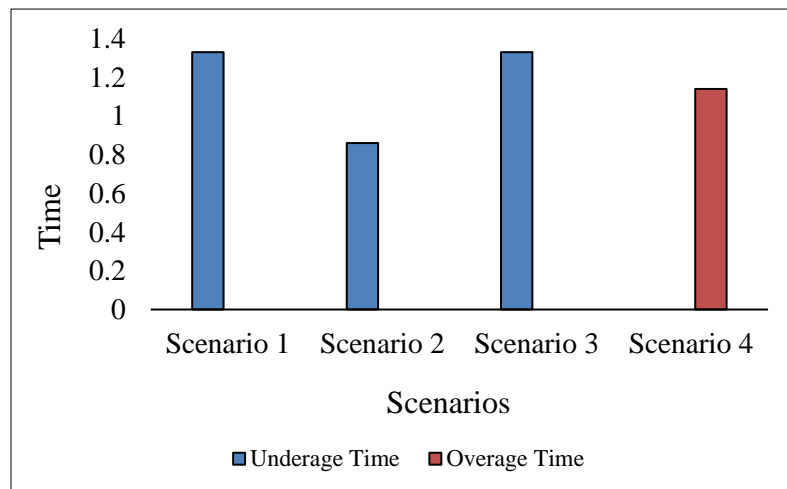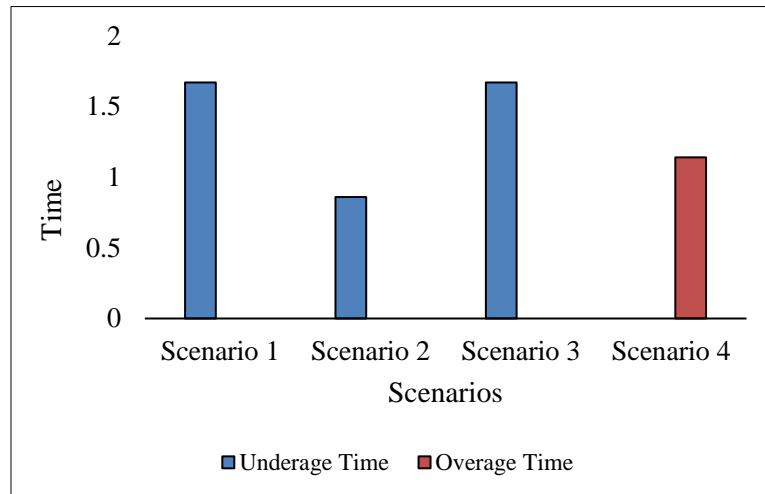

Figure 9: Underage time and overage time of job1 in different scenarios
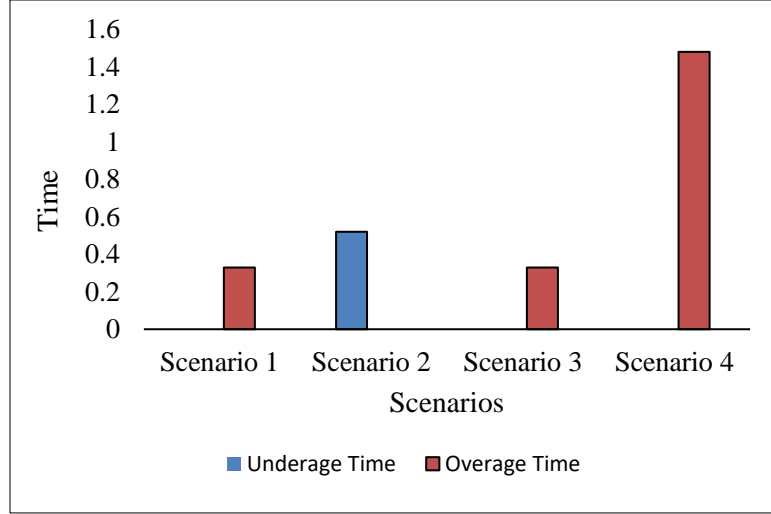
Figure 10: Underage time and overage time of job2 in different scenarios

Now, for this example, it is shown that if the nonlinear constraint 11 is deleted from Model 1 or constraints 19 and 20 are deleted from Model 2, then the rest of the constraints do not ensure that when $d_i^{-\omega} > 0$, $d_i^{+\omega} = 0$ and when $d_i^{+\omega} > 0$, $d_i^{-\omega} = 0$. For proving that such deletion of constraints results in an incorrect solution with both $d_i^{+\omega} > 0$, and $d_i^{-\omega} > 0$ at the same time, Example 2 is solved again using Model 2 (Mixed Integer Linear Programming) but for this time, deleting the constraints 19 and 20. The results obtained is shown in Table 9.

Table 9: Results obtained for 2 jobs in Example 2 [without constraint (19) and (20)]

Scenarios	Job 1			Job 2			Cost	Appointments	Objective
	$d_i^{-\omega}$	$d_i^{+\omega}$	c_i^{ω}	$d_i^{-\omega}$	$d_i^{+\omega}$	c_i^{ω}		A_i	Z
1	2.67	1.00	5.00	0.00	0.33	13.67	4.67	$A_1 = 0$,	$Z = 4.67$
2	0.00	0.00	5.00	0.67	0.00	14.67	4.67	$A_2 = 6.67$,	
3	1.67	0.00	7.00	0.00	0.33	13.00	3.33	$A_3 = 13.67$	
4	0.00	1.00	7.00	0.00	1.33	15.00	4.67		

Table 9 shows that when constraints 19 and 20 are deleted from Model 2, the rest of the model yield an incorrect optimal solution, $A_2 = 6.67$, $A_3 = 13.67$, and $Z = 4.67$ with both $d_1^{+\omega} = 1 > 0$, and $d_1^{-\omega} = 2.67 > 0$ for scenario 1, i.e. ω: $\{p_1 = 5, p_2 = 6\}$. Meaning

that for job 1, both the underage time and the overage time happens at the same time in scenario 1. Figure 11 shows this result.
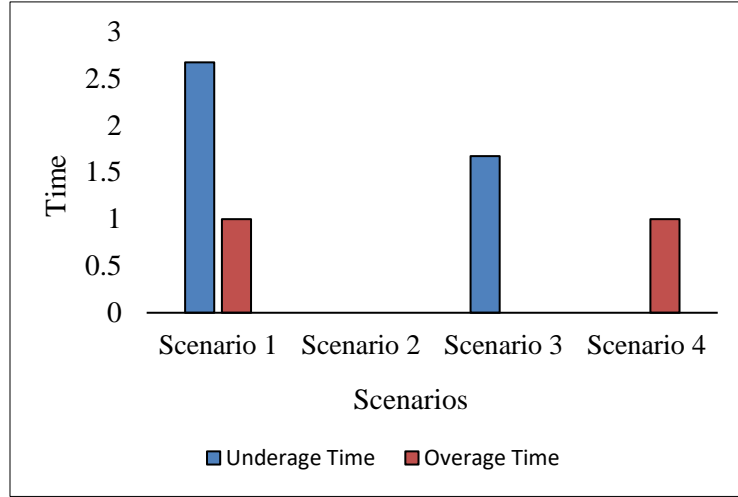
Figure 11: Underage time and overage time of job 1 for different scenarios

However, as shown in Table 8 and Figure 9, when constraints (19) and (20) are used to ensure that when $d_i^{-\omega} > 0$, $d_i^{+\omega} = 0$ and when $d_i^{+\omega} > 0$, $d_i^{-\omega} = 0$, Models 2 yields a correct optimal solution to Example 2, which is $A_2 = 6.67$, $A_3 = 13.52$, $Z = 5.09$, and which gives $d_1^{+\omega} = 0$, and $d_1^{-\omega} = 1.67$ for scenario 1, i.e. ω: $\{p_1 = 5, p_2 = 6\}$. This proves that the underage time and the overage time do not occur at the same time for job 1 in scenario 1.

Hence it can be concluded that the nonlinear constraint 11 in Model 1 or the linear constraints 19 and 20 in Model 2 are required to be included to ensure correct solution.

The illustrative example 1 and 2 contain different values of processing times for job 1 and jobs, however they reach to similar conclusion that the nonlinear constraint 11 in Model 1 or the linear constraints 19 and 20 in Model 2 are required to be included to ensure correct solution. Through these illustrative examples, the implementation procedure of the proposed robust model have been explained extensively.

4.5 Propositions

In this section we provide propositions that support the Robust Appointment Scheduling model.

Proposition 1: There exists an optimal solution $\{A_j\}$, such that,

$$A_j + l_j \leq A_{j+1} \leq A_j + k_j \tag{23}$$

Proof: Given an optimal solution $\{A_j\}$, let j^* be the first job that violates the condition.

Case1: At first we will prove,

$$A_j + l_j \leq A_{j+1} \tag{24}$$

Let, $A_{j^*-1} + l_{j^*-1} > A_{j^*}$ and $A'_j = A_j \forall j \neq j^*$ and $A'_{j^*} = A_{j^*-1} + l_{j^*-1}$. Also $C'_j = C_j \forall j < j^*$

Now,

$$C'_{j^*} = max\left(A'_{j^*}, C'_{j^*-1}\right) + P_{j^*} = C'_{j^*-1} + P_{j^*} \ (\because C'_{j^*-1} > A'_{j^*-1} + l_{j^*-1} = A'_{j^*}) \tag{25}$$

And,

$$C_{j^*} = max\left(A_{j^*}, C_{j^*-1}\right) + P_{j^*} = C_{j^*-1} + P_{j^*}$$
$$(\because C_{j^*-1} = max\left(A_{j^*-1}, C_{j^*-2}\right) + P_{j^*-1} \geq A_{j^*-1} + P_{j^*-1} \geq A_{j^*-1} + l_{j^*-1} \geq A_{j^*}) \tag{26}$$

Now from (25) and (26), we can write,

$$\therefore C'_{j^*} = C_{j^*} \tag{27}$$

All underage and overage costs are same in both the schedules except for job $j^* - 1$.

Overage cost of job $j^* - 1$ in Schedule A is $(C_{j^*-1} - A_{j^*})o_{j^*-1}$

Overage cost of job $j^* - 1$ in Schedule A' is $(C'_{j^*-1} - A'_{j^*})o_{j^*-1}$

$$(C_{j^*-1} - A_{j^*})o_{j^*-1} > (C'_{j^*-1} - A'_{j^*})o_{j^*-1} \ (\because A_{j^*} < A'_{j^*} \ \& \ C_{j^*-1} = C'_{j^*-1}) \tag{28}$$

Overage cost of job $j^* - 1$ is more in Schedule A than in schedule A' so the given schedule is not optimal.

Case 2: Here we will prove,

$$A_j + k_j \geq A_{j+1} \tag{29}$$

Let, $A_{j^*-1} + k_{j^*-1} < A_{j^*}$ and $A'_j = A_j \ \forall j < j^*$, $A'_{j^*} = A_{j^*-1} + k_{j^*}$ and $A'_{j^*+1} = A'_{j^*} + (A_{j^*+1} - A_{j^*}) \forall j > j^*$. Also, $C'_j = C_j \ \forall j < j^*$. So, we can write,

$$C'_{j^*} - A'_{j^*+1} = C_{j^*} - A_{j^*+1} \forall j \geq j^* \tag{30}$$

$$A'_{j^*} = A_{j^*-1} + k_{j^*} < A_{j^*} \tag{31}$$

All underage and overage costs are same in both the schedules except for job $j^* - 1$.

Underage cost of job $j^* - 1$ in Schedule A is $(A_{j^*} - C_{j^*-1})u_{j^*-1}$

Underage cost of job $j^* - 1$ in Schedule A' is $(A'_{j^*} - C'_{j^*-1})u_{j^*-1}$

Hence,

$$(A_{j^*} - C_{j^*-1})u_{j^*-1} > (A'_{j^*} - C'_{j^*-1})u_{j^*-1} \ (\because C_{j^*-1} = C'_{j^*-1} \ \& \ A_{j^*} > A'_{j^*}) \tag{32}$$

Underage cost of job $j^* - 1$ is more in Schedule A than in schedule A' so the given schedule is not optimal.

Proposition 2: If $n = 1$, an optimal solution is given by,

$$A_2^* = \frac{u_1 l_1 + o_1 k_1}{u_1 + o_1} \tag{33}$$

with,

$$Z^* = \frac{u_1 o_1}{u_1 + o_1}(k_1 - l_1) \tag{34}$$

Proof: The underage and overage cost is given by the functions $(A_2 - l_1)u_1$ and $(k_1 - A_2)o_1$ respectively. Total cost is minimized if underage cost is equal to the overage cost. So,

$$(A_2 - l_1)u_1 = (k_1 - A_2)o_1 \tag{35}$$

$$\therefore A_2^* = \frac{u_1 l_1 + o_1 k_1}{u_1 + o_1} \tag{36}$$

Substituting optimal appointment time in underage or overage cost gives the total cost,

$$Z^* = \left(\frac{u_1 l_1 + o_1 k_1}{u_1 + o_1} - l_1\right)u_1 = \left(\frac{u_1 l_1 + o_1 k_1 - l_1 u_1 - l_1 o_1}{u_1 + o_1}\right)u_1 = \frac{u_1 o_1}{u_1 + o_1}(k_1 - l_1) \tag{37}$$

Proposition 3: There exists an optimal solution $\{A_j\}$, and a scenario $\omega^* = argmax_\omega f(\omega)$ such that $p_j^{\omega^*} \in \{l_j, k_j\} \forall j \leq n$.

Proof: The case of $n = 1$ is proven by Proposition 2.

Let $n \geq 2$. We shall show that there exists an optimal solution $\{A_j\}$ such that for any scenario ω such that $l_j \leq p_j^\omega \leq k_j$,

$$f_{\{A_j\}}(\omega) \leq max\left\{f_{\{A_j\}}(\omega^* | p_j^{\omega^*} \in \{l_j, k_j\})\right\} \tag{38}$$

If this is not true, consider a scenario ω such that $l_j < p_j^\omega < k_j$ for at least one job j and for this scenario ω,

$$f_{\{A_j\}}(\omega) \geq max\left\{f_{\{A_j\}}(\omega' | l_j < p_j^\omega < k_j \text{ for at least one job } j)\right\} >$$
$$max\left\{f_{\{A_j\}}(\omega^* | p_j^{\omega^*} \in \{l_j, k_j\})\right\} \tag{39}$$

Let job i be the least indexed job with processing time neither minimum nor maximum. That is, $i = min\{j \mid l_j < p_j^\omega < k_j\}$. If $d_i^{-\omega} > 0$, then for ω' with $p_j^{\omega'} = p_j^\omega \forall j \neq i$ and $p_i^{\omega'} = l_i$, we get $f_{\{A_j\}}(\omega) < f_{\{A_j\}}(\omega')$. Therefore, $d_i^{-\omega} = 0$ and $d_i^{+\omega} \geq 0$.

If $d_j^{-\omega} = 0$ and $d_j^{+\omega} \geq 0 \ \forall j > i$, then for ω' with $p_j^{\omega'} = p_j^{\omega} \forall j \neq i$ and $p_i^{\omega'} = k_i$, we get $f_{\{A_j\}}(\omega) < f_{\{A_j\}}(\omega')$.

Therefore, there exists a job $i' > i$ such that $d_{i'}^{-\omega} > 0, d_{i'}^{+\omega} = 0$ and $\forall i \leq j < i', d_j^{-\omega} = 0$ and $d_j^{+\omega} \geq 0$.

If $\sum_{j=i}^{i'-1} o_j > u_{i'}$, the ε −perturbation gives a scenario ω' with $p_j^{\omega'} = p_j^{\omega} \forall j \neq i$ and $p_i^{\omega'} = p_i^{\omega} + \varepsilon$ for some small $\varepsilon > 0$ such that $f_{\{A_j\}}(\omega) < f_{\{A_j\}}(\omega')$. Therefore, $\sum_{j=i}^{i'-1} o_j < u_i$.

Now consider scenario ω' with $p_j^{\omega'} = p_j^{\omega} \forall j < i$, $p_j^{\omega'} = l_j \ \forall i \leq j \leq i'$, and $p_j^{\omega'} = p_j^{\omega} \forall j > i'$. We have $f_{\{A_j\}}(\omega) \leq f_{\{A_j\}}(\omega')$. Which is a contradiction of (39).

4.6 Stochastic Programming Approach of the Robust Model

It is possible to formulate the appointment scheduling problem using the stochastic programming approach. For that, it is assumed that the processing durations are discrete and stochastically independent for each job, which is followed similarly by Begen and Queyranne, (2011). This leads to a discrete time version of the appointment scheduling problem. In this study, they incorporate the joint discrete distribution of the processing time. They also assume that this joint distribution is known to the decision maker. The objective of the stochastic programming approach is to minimize the total expected cost of all the scenarios for a given processing sequence.

4.6.1 Problem Formulation

In this section, the appointment scheduling problem is formulated using the stochastic programming approach. There are n jobs that need to be scheduled on a single processor. Let, $\{1, 2, 3, \ldots, n + 1\}$ denote the set of jobs. The $n + 1^{st}$ job is a dummy job with processing duration equal to 0. The random processing duration of job i is denoted by p_{ij} where j indicates the uncertain instance of job i. Hence, the random vector of processing

duration is, $\boldsymbol{p} = (p_{11}, p_{12}, \ldots, p_{1j}, p_{21}, p_{22}, \ldots, p_{2j}, \ldots, p_{nj}, 0)$. Let $\overline{p_{ij}}$ and $\underline{p_{ij}}$ denote the maximum and minimum possible value of processing duration of job i respectively, over all the instances of job i. For stochastic programming, the total scenario set Ω will have $m^{\,n}$ number of scenarios, where m denotes the total number of the uncertain instances for a job and n denotes the total number of jobs to schedule.

Now, the objective is to minimize the total expected cost of all the scenarios. Therefore, the objective function is expressed as follows:

$$\sum_{\omega \in \Omega} \sum_{i=1}^{n} E_p \left[u_i d_i^{-\omega} + o_i d_i^{+\omega} \right] \tag{40}$$

Let, E_p is the expected probability with respect to the random processing duration vector $\boldsymbol{p}$. The rest of the constraints will be same as the Nonlinear model (Model 1) or the Mixed Integer Linear Programming model (Model 2).

4.6.2 Comparison of Stochastic Programming model and Robust Model

In Table 10, an illustrative example of the stochastic programming approach for the appointment scheduling problem is presented. It contains the data of underage time, overage time and the uncertain processing time instances for two jobs which need to be scheduled sequentially. There are three processing time instances for each of the two jobs. Since there are two jobs to schedule so, $n = 2$, and each job has three uncertain instances so, $m = 3$. The random processing vector is, $\mathbf{p} = (p_{11}, p_{12}, p_{13}, p_{21}, p_{22}, p_{23})$. Since each job has three uncertain instances, there is a total of m^n scenarios, i.e. for this example, $3^2 = 9$ scenarios as presented in Table 11. Processing time of a job can have only the values given in the dataset. The objective is to provide an appointment schedule that will minimize the total expected underage and overage cost of all the scenarios for the given processing time instances.

Table 10: Dataset for an Illustrative example of Stochastic Programming approach

Job, i	Overage cost, o_i	Underage cost, u_i	Processing time instances,		
			$j = 1$	$j = 2$	$j = 3$
1	2	1	5	6	9
2	3	4	7	8	11

Now, by assuming that the three instances of each job are equally likely to happen, the joint probability distribution can be obtained as shown in Table 11. Some studies assume that this joint probability distribution is known to the decision makers.

Table 11: Joint probability distribution of processing times for two jobs

Scenarios	Job 1	Job 2	Joint Probability Distribution
1	$p_{11} = 5$	$p_{21} = 7$	$\frac{1}{3} * \frac{1}{3} = \frac{1}{9}$
2	$p_{11} = 5$	$p_{22} = 8$	$\frac{1}{3} * \frac{1}{3} = \frac{1}{9}$
3	$p_{11} = 5$	$p_{23} = 11$	$\frac{1}{3} * \frac{1}{3} = \frac{1}{9}$
4	$p_{12} = 6$	$p_{21} = 7$	$\frac{1}{3} * \frac{1}{3} = \frac{1}{9}$
5	$p_{12} = 6$	$p_{22} = 8$	$\frac{1}{3} * \frac{1}{3} = \frac{1}{9}$
6	$p_{12} = 6$	$p_{23} = 11$	$\frac{1}{3} * \frac{1}{3} = \frac{1}{9}$
7	$p_{13} = 9$	$p_{21} = 7$	$\frac{1}{3} * \frac{1}{3} = \frac{1}{9}$
8	$p_{13} = 9$	$p_{22} = 8$	$\frac{1}{3} * \frac{1}{3} = \frac{1}{9}$
9	$p_{13} = 9$	$p_{23} = 11$	$\frac{1}{3} * \frac{1}{3} = \frac{1}{9}$
		Total	1

Xpress Optimizer 25.01.05, algebraic model language and optimizer (mmnl module) has been used to code the stochastic programming approach of the appointment scheduling problem. The result found is presented in Table 12.

Table 12: Results obtained for Stochastic Programming approach

Scenarios	Job 1 $d_i^{-\omega}$	$d_i^{+\omega}$	Job 2 $d_i^{-\omega}$	$d_i^{+\omega}$	Expected Cost	Scenario Cost	Appointments A_i	Objective Z
1	1.00	0.00	1.00	0.00	0.66	6	$A_1 = 0$,	$Z = 6.6$
2	1.00	0.00	0.00	0.00	0.22	2	$A_2 = 6$,	
3	1.00	0.00	1.00	0.00	0.66	6	$A_3 = 14$,	
4	0.00	0.00	0.00	0.00	0	0		
5	0.00	0.00	1.00	0.00	0.44	4		
6	0.00	0.00	0.00	0.00	0	0		
7	0.00	3.00	0.00	2.00	0.99	9		
8	0.00	3.00	0.00	3.00	1.32	12		
9	0.00	3.00	0.00	6.00	2.31	21		

It is shown in Table 12 that the total scenario set Ω now have 9 scenarios. The objective value, $Z = 6.6$ indicates the minimum expected cost of all 9 scenarios.

Next the proposed Mixed Integer Linear Programming model (Model 2) for performing Robust Appointment Scheduling is applied on the same dataset and then the obtained result is compared with the results found from the Stochastic Programming approach. Since for the proposed robust model only the extreme point instances of the processing times are considered for each job, the total scenario set Ω will contain a total of 2^n scenarios or 2^2 or 4 scenarios as presented in Table 14.

Table 13: Dataset for applying Robust Appointment Scheduling model

Job, i	Overage cost o_i	Underage cost u_i	Processing time instances $j = 1 = l_i$	$j = 3 = k_i$
1	2	1	5	9
2	3	4	7	11

Xpress Optimizer 25.01.05, algebraic model language and optimizer (mmnl module) has been used to code the robust model of the appointment scheduling problem. The result found is presented in Table 14.

Table 14: Results obtained for Robust Appointment Scheduling

Scenarios	Job 1		Job 2		Scenario Cost	Appointments A_i	Objective Z
	$d_i^{-\omega}$	$d_i^{+\omega}$	$d_i^{-\omega}$	$d_i^{+\omega}$			
1	2.67	0.00	0.00	0.00	12.1905	$A_1 = 0$,	$Z = 12.1905$
2	2.67	0.00	0.00	0.00	12.1905	$A_2 = 7.67$,	
3	0.00	0.00	1.33	0.00	2.85714	$A_3 = 16.38$,	
4	0.00	0.00	1.33	0.00	12.1905		

By comparing the results, it can be seen that for the stochastic model, the worst cost is for scenario 9 i.e. $\omega : \{p_1 = 9, \ p_2 = 11\}$ with $A_2 = 6$, $A_3 = 14$, $Z = 6.6$, and which, gives underage time for Job 1, $d_1^{+\omega} = 3$, and overage time for Job 2, $d_2^{+\omega} = 6$ and the worst cost is 21. Whereas for the robust model, the same scenario 9 i.e. $\omega : \{p_1 = 9, p_2 = 11\}$ constitutes the worst cost which is only 12.1905 with $A_2 = 7.67$, $A_3 = 16.38$, $Z = 12.1905$, and underage time for Job 1, $d_1^{-\omega} = 0$, overage time for Job 2, $d_2^{-\omega} = 1.33$. Although for the stochastic model the scenarios are assumed to be equally likely to happen, the worst cost is 21 whereas for robust model it is only 12.1905. Figure 4.9 shows the comparison of total costs in each scenario between robust model and stochastic programming model.

From the comparison between the robust model and the stochastic programming model, conclusions can be drawn saying that for the robust model, the worst cost is much less than that of the stochastic programming model, although the expected cost can be less for stochastic programming model compared to the robust model. However, considering the better computations efforts, eliminating the negative impacts of the lack of information regarding uncertainty and ability to provide better results for low probable scenarios with high impacts, the proposed robust model provides greater merits than the stochastic programming model.
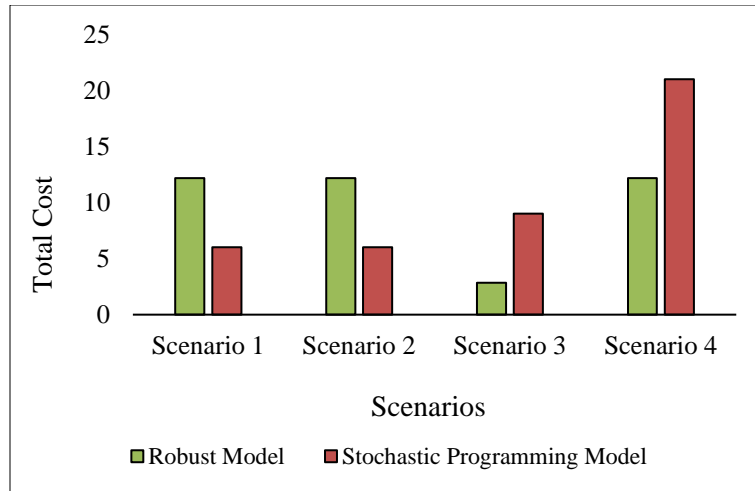
Figure 12: Comparison of cost between Robust Model and Stochastic Programming Model

4.6.3 Summary of the Comparison

From the above analysis, it can be summarized that for the stochastic programming approach, the greater the number of uncertain instances, the more is the number of total scenarios, i.e. Ω will have a total of m^n number of scenarios. Computational complexity also increases as the number of scenarios increase. On the other hand, for the robust model, the total number of scenario is 2^n, considering that the optimal solution corresponds to the extreme case scenarios for appointment scheduling of jobs with a single processor. As a result computational complexity is much less for the robust model than the stochastic model.

The worst cost for robust model is significantly lower than the worst cost found for stochastic model. Because the objective of robust model is to minimize the worst cost whereas the objective of stochastic model is to minimize the total expected cost of all the scenarios.

It is more difficult to find the joint probability distribution for applying stochastic programming than to find the extreme two bounds of the uncertain processing duration for applying robust model. Moreover, some studies assume that this joint probability

distribution is known to the decision makers (Begen and Queyranne, 2011) which may not hold true in real practice.

4.7 Special Cases

In this section, special cases of the appointment scheduling problem that might arise while considering real world application are discussed.

4.7.1 Underage Cost is Infinitely Greater than the Overage Cost

There can be cases where the underage cost of each jobs tend to be infinity greater compared to the overage cost of the jobs. This situation can arise when the processor and the resources are more important than the customers. As a result the cost of the system is much higher than the customers' time. In such a case, the optimal appointment dates will be equal to the sum of the lower bounds of the processing times. The cost of such a schedule will be zero for any realization of the processing times. This is because, since appointment dates are set to the lower limit of the processing times, there will be no job subjected to underage time and so no underage cost will incur. Also since the overage cost tends to zero, the total cost of the schedule will be zero. If $o_i \approx 0$ and $u_i \approx \infty$, then $A_i = \sum_{i=1}^{n} l_i$.

Table 15 shows an illustrative example of this case. Here, the overage cost of all the jobs are equal to 0 and the underage costs of all the jobs tend to infinity compared to the overage costs. Xpress Optimizer 25.01.05, algebraic model language and optimizer (mmnl module) is used to code the mathematical Model 2 (Mixed Integer Linear Programming) for solving the problem.

Table 15: Data for an illustrative example of the special case (underage cost tend to infinity)

Job, i	Overage Cost, o_i	Underage Cost, u_i	Lower processing time, l_i	Upper processing time, k_i
1	0	999999	5	6
2	0	999999	7	8
3	0	999999	9	11

The result found is presented in Table16. The optimal appointment dates are found to be $A_2 = 5, A_3 = 12, A_4 = 21$. These appointment dates are actually the sum of the lower bounds of the processing times. The objective function value, $Z = 0$, means that the worst cost that may occur for this schedule for any realization of the processing time is 0. It can be seen that for all the extreme scenarios, the cost is equal to 0 as found from the objective function Z.

Table 16: Results obtained for the special case (underage cost tend to infinity)

Scenarios	Job 1, p_1^ω	Job 2, p_2^ω	Job 3, p_3^ω	Cost	Appointment, A_i	Objective, Z
1	5	7	9	0	$A_1 = 0,$	$Z = 0$
2	5	7	11	0	$A_2 = 5,$	
3	5	8	9	0	$A_3 = 12,$	
4	5	8	11	0	$A_4 = 21$	
5	6	7	9	0		
6	6	7	11	0		
7	6	8	9	0		
8	6	8	11	0		

4.7.2 Overage Cost is Infinitely Greater than the Underage Cost

There can be cases where the overage cost of each jobs tend to be infinitely higher compared to the underage cost of the jobs. This situation can arise when the processor and the resources are less important than the customers. As a result, the cost of the customers' time is much higher than the cost of the system. In such a case, the optimal appointment dates will be equal to the sum of the upper bounds of the processing times. The cost of such a schedule will be zero for any realization of the processing times. This is because, since appointment dates are set at the upper limits of the processing times, there will not be any job subjected to overage time and so no overage cost will incur. Also, since the underage

cost tend to zero, the total cost of the schedule will be zero for all scenarios. If $u_i \approx$ 0 and $o_i \approx \infty$, then $A_i = \sum_{i=1}^{n} k_i$.

Table 17: Data for an illustrative example of the special case (overage cost tend to infinity)

Job, i	Overage Cost, o_i	Underage Cost, u_i	Lower processing time, l_i	Upper processing time, k_i
1	999999	0	5	6
2	999999	0	7	8
3	999999	0	9	11

Table 17 shows an illustrative example of this case. Here, the underage cost of all the jobs are equal to 0 and the overage costs of all the jobs tend to infinity compared to the underage costs. Xpress Optimizer 25.01.05, algebraic model language and optimizer (mmnl module) is used to code the mathematical Model 2 (Mixed Integer Linear Programming) for solving the problem.

Table 18: Results obtained for the special case (overage cost tend to infinity)

Scenarios	Job 1, p_1^ω	Job 2, p_2^ω	Job 3, p_3^ω	Cost	Appointment, A_i	Objective, Z
1	5	7	9	0	$A_1 = 0$,	$Z = 0$
2	5	7	11	0	$A_2 = 6$,	
3	5	8	9	0	$A_3 = 14$,	
4	5	8	11	0	$A_4 = 25$	
5	6	7	9	0		
6	6	7	11	0		
7	6	8	9	0		
8	6	8	11	0		

The result found is presented in Table 18. The optimal appointment dates are found to be $A_2 = 6, A_3 = 14, A_4 = 25$. These appointment dates are equal to the sum of the upper bounds of the processing times. The objective function value, $Z = 0$, means that the worst cost that may occur for any realization of the processing time is 0. It is seen that for all the extreme scenarios, the cost is equal to 0 as found from the objective function Z.

CHAPTER 5

SOLUTION PROCEDURE

Scheduling problem is known to be NP-complete (Ullman, 1975). Since Model 1 (Nonlinear Programming) and Model 2 (Mixed Integer Linear Programming) for the proposed Robust Appointment Scheduling increases in an order of 2^n (considering the extreme scenario cases), it results in an NP-hard problem. Therefore, a polynomial time solution procedure for solving the larger instances of the problem is proposed. An Iterative Search Procedure and a Dynamic Programming model are developed to solve the larger instances of the Robust Appointment Scheduling model.

5.1 Iterative Search Procedure

The objective of Min-max optimization is to minimize the maximum cost among all the potential realizable scenarios. Considering that, in Iterative Search Procedure, initially any random scenario is taken and it is added to the initial scenario set. Then the scenarios that constitute the worst case scenarios (the scenarios for which the cost is found to be maximum) are iteratively added to the total scenario set. Then the robust model is applied to optimize those scenarios in the scenario set. This iterative procedure is continued until an appointment schedule is obtained for which there are no more worst case scenario left to be added in the total scenario set. The steps for this Iterative Search Procedure are explained below:

1. Consider any random scenario $\omega \in \Omega$. This can be done by generating n random numbers 0 or 1; $p_i^\omega = l_i$, if the i −th random number is 0 and $p_i^\omega = k_i$, if the i −th random number is 1. Let $\Omega^0 = \{\omega\}$. This is the initial scenario set.

2. Let $A_1 = 0, A_{j+1} = \sum_{i=1}^{j} p_i^\omega \ \forall j \geq 1$

3. Find worst scenario $\omega' \in \Omega$ (using Dynamic Programming model explained in section 5.2). Replace Ω^0 by $\Omega^0 \cup \{\omega\}$ by adding this worst scenario to the scenario set.

4. Revise appointment schedule $\{A_j\}$ using Model 2 $\forall\ \omega \in \Omega^0$.

5. Go to step 3 if termination condition is not met, else stop.

Termination condition may be considered as a pre-specified maximum number of scenarios to be added in the scenario set. Termination can also be done considering convergence to an appointment schedule. For instance, the iterative search process can be terminated at an appointment schedule for which there are no worst case scenarios left to add to the scenario set. Which would mean that a solution is achieved that minimizes the maximum cost for all the scenarios. This indicates that for all the other scenarios, even the scenarios which have not been added in the scenario list, this current solution (found in this last iteration where the search process is terminated) gives lower cost and there is no worst case scenarios to add further to the scenario list.

Figure 13 exhibits the flowchart for the Iterative Search Procedure. In the next section, a Dynamic Programming model is proposed to calculate the worst cost among all the scenarios given an appointment schedule.
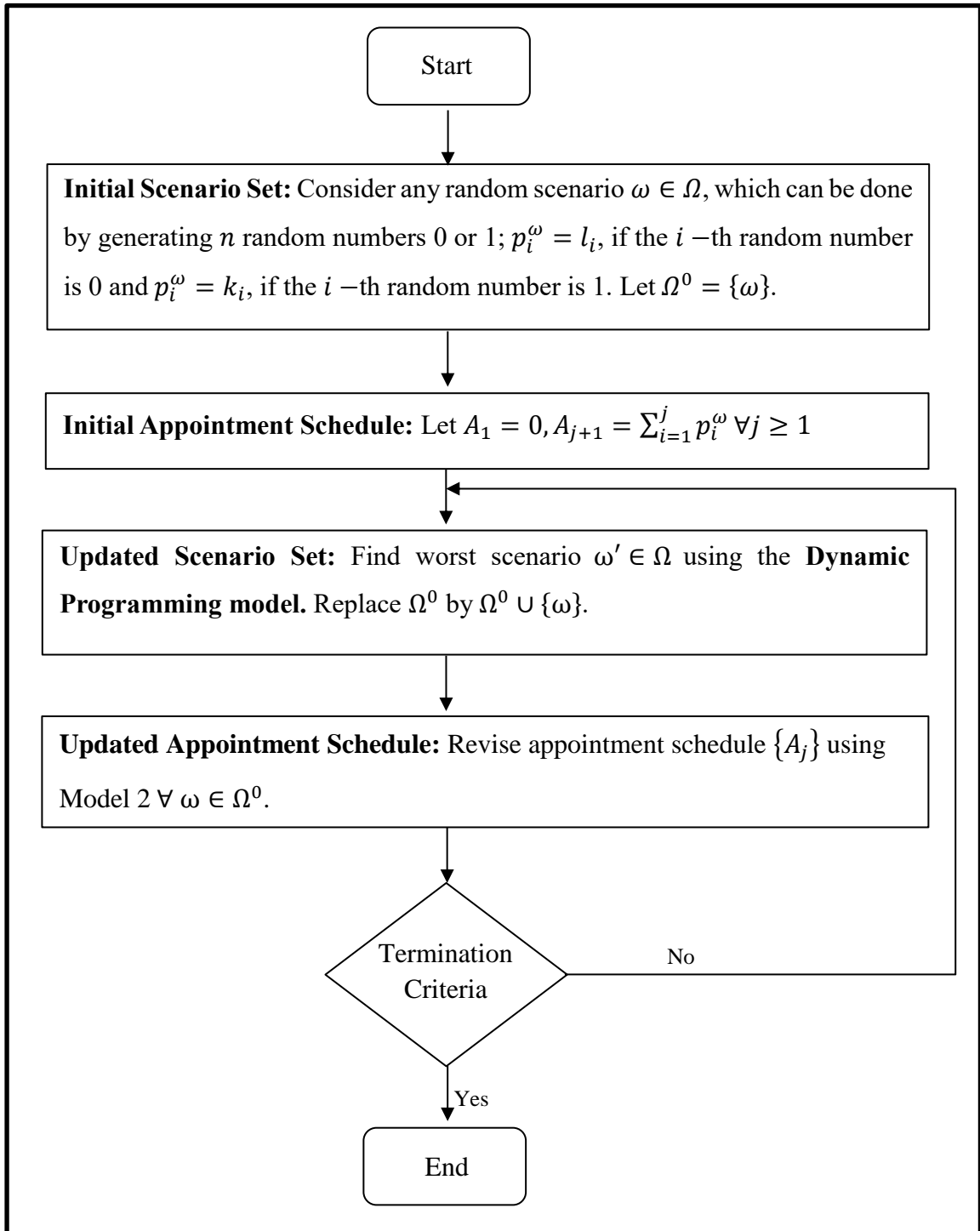
Figure 13: Flowchart of the Iterative Search Procedure

The flowchart contains the following boxes:

Start

Initial Scenario Set: Consider any random scenario $\omega \in \Omega$, which can be done by generating n random numbers 0 or 1; $p_i^\omega = l_i$, if the $i-$th random number is 0 and $p_i^\omega = k_i$, if the $i-$th random number is 1. Let $\Omega^0 = \{\omega\}$.

Initial Appointment Schedule: Let $A_1 = 0, A_{j+1} = \sum_{i=1}^{j} p_i^\omega \; \forall j \geq 1$

Updated Scenario Set: Find worst scenario $\omega' \in \Omega$ using the **Dynamic Programming model.** Replace Ω^0 by $\Omega^0 \cup \{\omega\}$.

Updated Appointment Schedule: Revise appointment schedule $\{A_j\}$ using Model 2 $\forall \; \omega \in \Omega^0$.

Termination Criteria — No / Yes

End

5.2 Finding the Worst Case Scenario Given an Appointment Schedule

For applying Iterative Search Procedure, in step 3, it is required to find the worst case scenario for a particular appointment schedule among all the scenarios in the total scenario set. A worst case scenario is the scenario for which the cost is maximum among all the other potentially realizable scenarios. Given an appointment schedule, finding the worst case scenario follows an exponential time computation process. Therefore, a Dynamic programming model is proposed which would allow to find the worst case scenario in $O(n^2)$ time.

5.2.1 Finding Worst Case Scenario in Exponential Time

Given an appointment schedule $\{A_j \mid j = 1,2, \dots, n + 1, A_1 = 0\}$ and given parameters o_j, u_j, l_j and $k_j \, \forall j = 1,2,..n$, it is possible to find the worst case scenario among the total scenarios. For that, at first the completion time of all the jobs in all the scenarios is calculated using the following equation,

$$C_i^\omega = p_i^\omega \; for \; i = 1 \; and \; C_i^\omega = \; max(C_{i-1}^\omega, A_i) + p_i^\omega \; \forall \omega \in \Omega, 2 \le i \le n + 1 \tag{41}$$

Then for a given appointment schedule $\{A_i \mid i = 1,2, \dots, n + 1, A_1 = 0\}$, the cost incurred for job i is calculated using Equation 42 or Equation 43,

$$cost_i^\omega = (A_{i+1} - C_i^\omega) * u_i \; \forall \omega \in \Omega, \forall \, 1 \le i \le n \mid A_{i+1} > C_i^\omega \tag{42}$$

$$cost_i^\omega = (C_i^\omega - A_{i+1}) * o_i \; \forall \omega \in \Omega, \forall \, 1 \le i \le n, \mid C_i^\omega > A_{i+1} \tag{43}$$

Next, the total cost of a scenario is found by adding the costs incurred for all jobs in that scenario as shown in Equation 44,

$$total \; cost^\omega = \sum_{i=1}^{n} cost_i^\omega \; \forall \omega \in \Omega, 1 \le i \le n \tag{44}$$

In this approach, it is required to calculate the cost of all the scenarios and then to choose the maximum cost among all the scenarios for finding the worst case scenario for a

particular schedule. Now, since there is a total number of 2^n scenarios, cost is calculated for 2^n scenarios resulting in an exponential time computation process.

5.2.2 Dynamic Programming Model for Finding Worst Case Scenario

The idea of the Dynamic programming is to solve a recursive problem where at each step the solution of the subproblem is computed and stored for future decision making process. By the time the last level is solved, the result of the total problem is achieved.

The objective of the proposed Dynamic Programming model is to find the scenario that maximizes the total overage and underage cost for a given appointment schedule $\{A_j \mid j = 1,2, \dots, n + 1, A_1 = 0\}$ and given parameters $o_j, u_j, l_j,$ and $k_j \; \forall j = 1,2, \dots n.$

In this approach, the costs incurred for one job in all the different scenarios are calculated one at a time before moving on to the next job to repeat the process. Here, each level corresponds to each job and the calculation of a level only depends on its immediate previous level. For each job, the costs occurring for different scenarios are calculated along with the total cost incurred up to that level as the process continues to move further. By the time the calculation for the last job is completed, the solution of the total problem is also achieved.

The dynamic program contains nodes and arcs to represent feasible scenarios. For each job i, there are nodes $(i, 1), (i, 2), \dots, (i, i)$. Node $(i, 1)$ represents the start time of job i at A_i. Node $(i, 2)$ represents the start time of job i at $A_{i-1} + k_{i-1}$. Node $(i, 3)$ represents the start time of job i at $A_{i-2} + k_{i-2} + k_{i-1}$. In general, node (i, r), where $1 < r \leq i$, represents the start time of job i at $A_{i-r+1} + k_{i-r+1} + k_{i-r+2} + \dots + k_{i-1}.$

Each node $(i, r) \; \forall \; 1 \leq i \leq n, 1 \leq r \leq i$ is connected to two nodes $(i + 1,1)$ and $(i + 1, r + 1)$ via two arcs $[i, r, i + 1,1]$ and $[i, r, i + 1, r + 1].$

The cost of arc $[i, r, i + 1,1]$ is given by the cost of l_i starting at time represented by node (i, r) and it is computed as Equation 45,

$$c[i,r,i + 1,1] = d_i^- u_i = \begin{cases} (A_{i+1} - A_i - l_i)u_i & \text{if } r = 1 \\ (A_{i+1} - A_{i-r+1} - k_{i-r+1} - \dots - k_{i-1} - l_i)u_i & \text{if } r > 1 \end{cases} \tag{45}$$

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The cost of arc $[i, r, i + 1, r + 1]$ is given by the cost of k_i starting at time represented by node (i, r) and it is computed as Equation 46,

$$c[i, r, i + 1, r + 1] = d_i^+ o_i = \begin{cases} (A_i + k_i - A_{i+1}) o_i & if\ r = 1 \\ (A_{i-r+1} + k_{i-r+1} + \cdots + k_i - A_{i+1}) o_i\ if\ r > 1 \end{cases} \tag{46}$$

Initialize $G(1,1) = 0$. For each, $i = 2, 3, \ldots, (n + 1)$ we compute Equation 47 and 48,

$$G(i, 1) = max\{G(i - 1, r) + c[i - 1, r, i, 1] | 1 \leq r \leq i - 1\} \tag{47}$$

$$G(i, r) = G(i - 1, r - 1) + c[i - 1, r - 1, i, r] \forall 2 \leq r \leq i \tag{48}$$

The cost of the worst scenario is calculated using Equation 49,

$$ma\,x\{G(n + 1, r) | 1 \leq r \leq n + 1\} \tag{49}$$

It is to be noted that if $\{A_j\}$ is optimal then,

$$ma\,x\{G(n + 1, r) | 1 \leq r \leq n + 1\} = G\ (n + 1, 1) = ma\,x\{G(n + 1, r) | 2 \leq r \leq n + 1\}. \tag{50}$$

Once the cost of the worst scenario is obtained, the worst scenario can be found by backtracking.

5.3 Illustrative Example of Iterative Search Procedure

In this section, an illustrative example of the proposed Iterative Solution Procedure and Dynamic Programing model is provided. Table 19 contains the data of underage time, overage time, lower limit and upper time of the uncertain processing times for three jobs which need to be scheduled sequentially.

Table 19: Data for an Illustrative Example of the Iterative Solution Procedure

Job, i	Overage Cost, o_i	Underage Cost, u_i	Lower processing time, l_i	Upper processing time, k_i
1	1	2	5	7
2	3	4	6	8
3	1	2	5	7

For 3 jobs, there is a total number of 2^3 or 8 scenarios. Table 20 shows the scenarios for 3 jobs.

Table 20: Binary representation of scenarios for 3 jobs

Scenarios	Scenario label	Job 1	Job 2	Job 3
1	000	$p_1^1 = l_1 = 5$	$p_2^1 = l_2 = 6$	$p_3^1 = l_3 = 5$
2	001	$p_1^2 = l_1 = 5$	$p_2^2 = l_2 = 6$	$p_3^2 = k_3 = 7$
3	010	$p_1^3 = l_1 = 5$	$p_2^3 = l_2 = 8$	$p_3^3 = l_3 = 5$
4	011	$p_1^4 = l_1 = 5$	$p_2^4 = l_2 = 8$	$p_3^4 = k_3 = 7$
5	100	$p_1^5 = k_1 = 7$	$p_2^5 = l_2 = 6$	$p_3^5 = l_3 = 5$
6	101	$p_1^6 = k_1 = 7$	$p_2^6 = l_2 = 6$	$p_3^6 = k_3 = 7$
7	110	$p_1^7 = k_1 = 7$	$p_2^7 = l_2 = 8$	$p_3^7 = l_3 = 5$
8	111	$p_1^8 = k_1 = 7$	$p_2^8 = l_2 = 8$	$p_3^8 = k_3 = 7$

At first, the problem is solved using the proposed Robust Appointment Scheduling model (Mixed Integer Linear Programming). Xpress Optimizer 25.01.05, algebraic model language and optimizer (mmnl module) has been used to code the mathematical Model 2 (Mixed Integer Linear Programming) for solving the problem. The result found is presented in Table 21.

Table 21: Results obtained from Xpress for Mixed Integer Linear Programming model

Scenarios	Scenario Cost	Appointment, A_i	Objective, Z
1	8.1905	$A_1 = 0,$	$Z = 8.1905$
2	8.1905	$A_2 = 6.4286,$	
3	6.1905	$A_3 = 13.4286,$	
4	8.1905	$A_4 = 19.0952,$	
5	3.6191		
6	3.6191		
7	6.1905		
8	8.1905		

The optimal appointment dates are found to be $A_2 = 6.4286$, $A_3 = 13.4286$, $A_4 = 19.0952$. The objective function value, $Z = 8.1905$ means that the worst cost that may occur for any realization of the processing time is 8.1905.

Then the same dataset is solved using the proposed Iterative Search Procedure. The results obtained from the Iterative Search Procedure for solving the dataset is presented in Table 22. The iterative procedure converges quickly; it requires only 3 iterations and proceeds as Table 22 when started with scenario 2 or "001".

Table 22: Results obtained for the Iterative Solution Procedure

Iteration	Scenario added	LP solution	LP cost	DP cost	DP worst scenario
1	2 or "001" $p_1^2 = 5, p_2^2 = 6, p_3^2 = 7$	$A_2 = 5,$ $A_3 = 11,$ $A_4 = 18$	$Z = 0$	$G(4,4)$ $= 18$	8
2	8 or "111" $p_1^8 = 7, p_2^8 = 8, p_3^8 = 7$	$A_2 = 7,$ $A_3 = 13.5,$ $A_4 = 20.5$	$Z = 6$	$G(4,1)$ $= G(4,3)$ $= 10$	1 and 4
3	1 or "000" and 4 or "011" $p_1^1 = 5, p_2^1 = 6, p_3^1 = 5$ $p_1^4 = 5, p_2^4 = 8, p_3^4 = 7$	$A_2 = 6.4286,$ $A_3 = 13.4286,$ $A_4 = 19.0952$	$Z = 8.1905$	$G(4,1)$ $= G(4,2)$ $= G(4,3)$ $= G(4,4)$ $= 8.1905$ (*optimal*)	1, 4, 2, and 8

Although for 3 jobs, there are 8 scenarios in the total scenario set, the Iterative Search Procedure converges within only three iterations and reaches to the optimal solution which is found from Xpress using Mixed Integer Linear Programming model (Model2) as presented in Table 21. Figure 14 shows the convergence of the solution method.
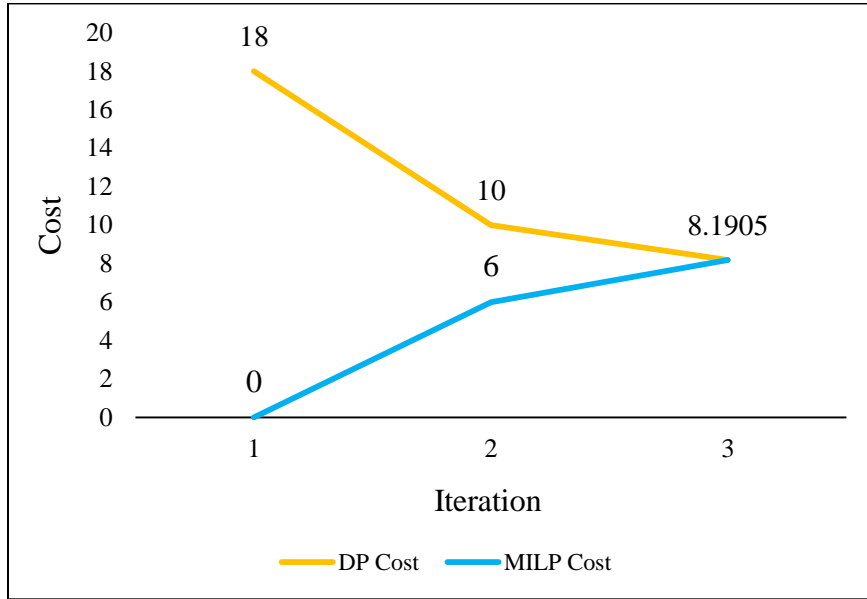

Figure 14: Convergence of the Iterative Search Procedure

Figure 15 shows the result obtained from the Dynamic Programming model for iteration 3 of the Iterative Solution Procedure.
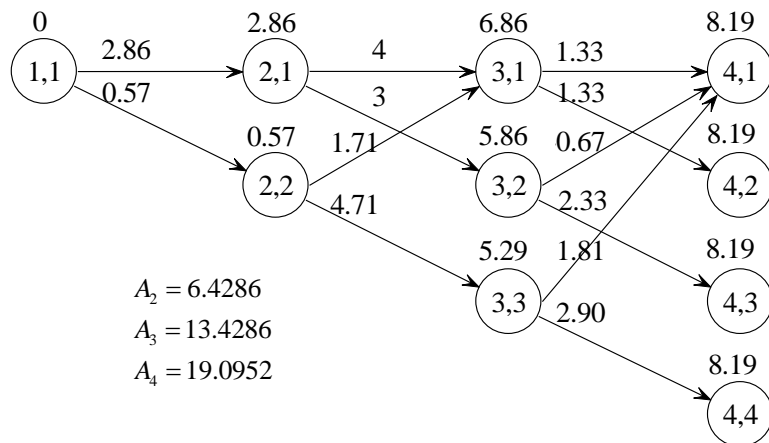

Figure 15: Result obtained from the Dynamic Programming model

CHAPTER 6
COMPUTATIONAL EXPERIENCE

The proposed methodologies have been applied to conduct two case studies; one is for scheduling VIA Rail Canada stations and another is for scheduling appointment dates in a Dentist's clinic.

6.1 Case Study 1: VIA Rail Canada

VIA Rail is Canada's national rail service providing ways to travel across Canada for 4.74 million passengers covering 12,500 kilometers of rail network. In this case study, the Robust Appointment Scheduling model is implemented for scheduling VIA Rail Canada stations to evaluate the performance of the robust model when applied to real world scheduling problem. The data for this case study is collected from VIA Rail Canada website (https://www.viarail.ca/en) where they provide the information regarding travelling routes, connecting stations, scheduled time table, actual train arrival time and actual train departure time for each stations.

6.1.1 Data Collection

In this case study, the travelling route of Train 60 which travels from Toronto (Union Station), Ontario to Montreal (Central Station), Quebec is considered. This route has nine connecting stations and so there are 8 travelling routes as mentioned in Table 23. Processing time here means the travelling time for a train from one station to next station which is uncertain and varies from time to time. Robust Appointment Scheduling model is applied to provide appointment dates for each of these stations. The objective is to minimize the total underage and overage cost for the worst case scenario under any realization of the travelling time of the train.

Table 23: Travelling route for Train 60 of VIA Rail Canada

Train No	60
Starting Station	Toronto Union Station, ON
End Station	Montreal Central Station, QC
No of total stations	9
No of travelling routes	8
Scheduled start time for the starting station	6:40 AM

The scheduled start time for Toronto (Union Station) is always 6:40 AM. The different actual travelling times of Train 60 for the month of January 2020 is collected from VIA Rail Canada website, from which the lower limit and upper limit of the travelling time for each station are calculated as shown in Table 24. To apply the robust model only these extreme point values are considered.

In this problem, underage cost is incurred when the train arrives a station before its scheduled appointment time. In this case, the system experiences idle time and the associated losses. The cost parameters are set from the perspective and knowledge of the decision maker. However, for this case study, the underage cost is considered to be equal to the opportunity cost that the system could have achieved in the absence of the idle time. The underage cost for a station is calculated using Equation 51,

$$underage\ cost = underage\ time * ticket\ cost\ per\ passenger\ per\ unit\ time *$$ $$total\ number\ of\ passengers \tag{51}$$

On the other hand, overage cost is incurred when the train arrives a station later than its appointment time. In that case, the passengers at that station have to wait until the train arrives. As a result, the passengers face idle time and the associated losses. The overage cost is considered to be equal to the opportunity cost for the passengers in the absence of the idle time incurred . The overage cost is calculated using Equation 52,

$$overage\ cost = overage\ time * income\ of\ per\ passenger\ per\ unit\ time *$$ $$total\ number\ of\ passengers \tag{52}$$

6.1.2 Application of Mixed Integer Linear Programming model (Model 2)

Mixed Integer Linear Programming model (Model 2) is applied for scheduling the stations. Since there are eight stations to schedule, according to the scenario planning approach, there is a total of 2^n or 2^8 or 256 scenarios (considering extreme scenario case).

Xpress Optimizer 25.01.05, algebraic model language and optimizer (mmnl module) is used to code the mathematical Model 2 (Mixed Integer Linear Programming) for solving the problem. The result found is presented in Table 24.

Table 24: Appointment dates for VIA Rail from Mixed Integer Linear Programming model

Stations	Lower processing time (sec), l_i	Upper processing time (sec), u_i	Appointments, A_i	Objective, Z
Toronto	900	1380	A_1= 6.40 AM,	Z = 716.315
Guildwood	1080	1860	A_2 = 7.03 AM,	
Oshawa	1980	2820	A_3 = 7.33 AM,	
Cobourg	1980	3660	A_4 = 8.20 AM,	
Belleville	2220	5700	A_5 = 9.20 AM,	
Kingston	5340	7680	A_6 = 10.51 AM,	
Cornwall	2700	3480	A_7 = 12.56 PM,	
Dorval	1320	1920	A_8 = 1.52 PM,	
Montreal	-	-	A_9 = 2.22 PM,	

The optimal appointment dates are found to be $A_1 = 6.40\ AM$, $A_2 = 7.03\ AM$, $A_3 = 7.33\ AM$, $A_4 = 8.20\ AM$, $A_5 = 9.20\ AM$, $A_6 = 10.51\ AM$, $A_7 = 12.56\ AM$, $A_8 = 1.52\ PM$, $A_9 = 2.22\ PM$, . The objective function value, $Z = 716.315$ means that the worst cost that may occur for any realization of the processing time is 716.315 CAD.

6.1.3 Application of the Iterative Search Procedure

In this section, the proposed Iterative Search Procedure is applied for solving the VIA Rail scheduling problem and the obtained result is presented in Table 25. The objective value found from Iterative Search Procedure is 716.373.

Table 25: Appointment dates for VIA Rail from the Iterative Search Procedure

Stations	Appointments, A_i	Objective, Z
Toronto	$A_1 = 6.40$ AM,	$Z = 716.373$
Guildwood	$A_2 = 7.03$ AM,	
Oshawa	$A_3 = 7.33$ AM,	
Cobourg	$A_4 = 8.20$ AM,	
Belleville	$A_5 = 9.20$ AM,	
Kingston	$A_6 = 10.51$ AM,	
Cornwall	$A_7 = 12.56$ PM,	
Dorval	$A_8 = 1.52$ PM,	
Montreal	$A_9 = 2.22$ PM,	

The Iterative Search Procedure converges very quickly, after 9 iterations as shown in Figure 16. This means that by selectively considering only 9 scenarios using the Dynamic Programming model, Iterative Search Procedure achieves the optimal solution instead of considering all the 256 scenarios used while implementing Model 2 (Mixed Integer Linear Programming) in Xpress Optimizer.
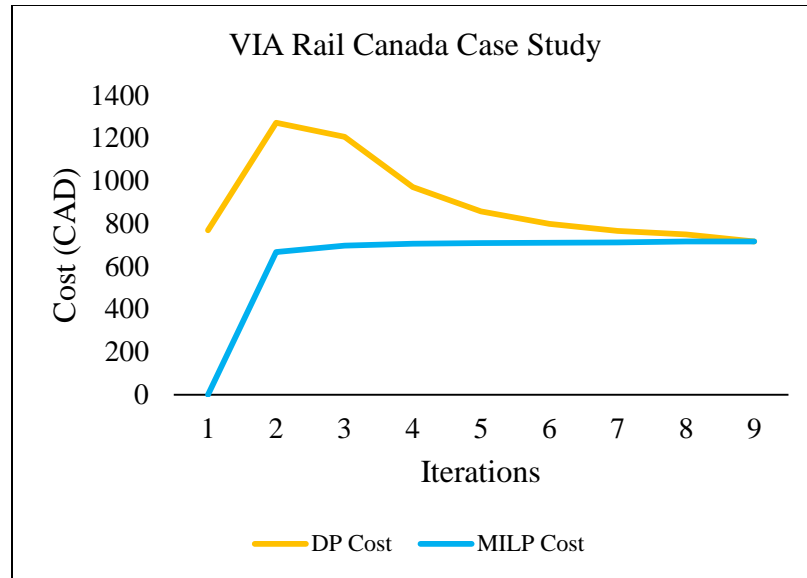
VIA Rail Canada Case Study

Figure 16: Convergence of the Iterative Solution Procedure

Table 26 shows the comparison of the Mixed Integer Linear Programming model and the Iterative Solution Procedure for VIA Rail case study.

Table 26: Performance Evaluation of the Iterative Solution Procedure

		Mixed Integer Linear Programming (Xpress)	Iterative Solution Procedure
VIA Rail Canada	Objective function value	716.315	716.373
	Optimality gap (%)	-	.008
	No of scenarios	256	9
	CPU time (seconds)	14862	3.1

6.1.4 Comparison of Cost Incurred for Robust Appointment Schedule and Via Rail Schedule Considering Actual Scenarios

This section provides a comparison of the cost incurred for Robust Appointment Schedule to that of VIA Rail Schedule for actual case scenarios for the month of January and February, 2020. The actual arriving time for Train 60 for different days of January and

February, 2020 has been collected from VIA Rail Canada website (https://www.viarail.ca/en).

Figure 17 shows that the cost incurred for VIA Rail schedule is much higher than that of Robust Appointment Schedule for almost all the days. This means that idle time is more for VIA Rail schedule. On the other hand, Robust Appointment schedule not only reduces cost, it also reduces the variation of cost among different days. The worst cost among these days incurred for VIA Rail schedule is 5913.6 CAD for 1.31.2020 while for the same day Robust Appointment Schedule would incur only 523.1184 CAD. Among these days, the worst cost found for Robust Appointment Schedule is 632.352 CAD for 1.1.2020 which is less than the worst cost found from the objective function value of the robust model which 716.315 CAD (both for the Mixed Integer Linear Programming model and the Iterative Solution Procedure) as shown in Table 26. This shows that the worst cost that is incurred for applying Robust Appointment Schedule is 716.315 CAD for any realization of the travelling time scenarios.
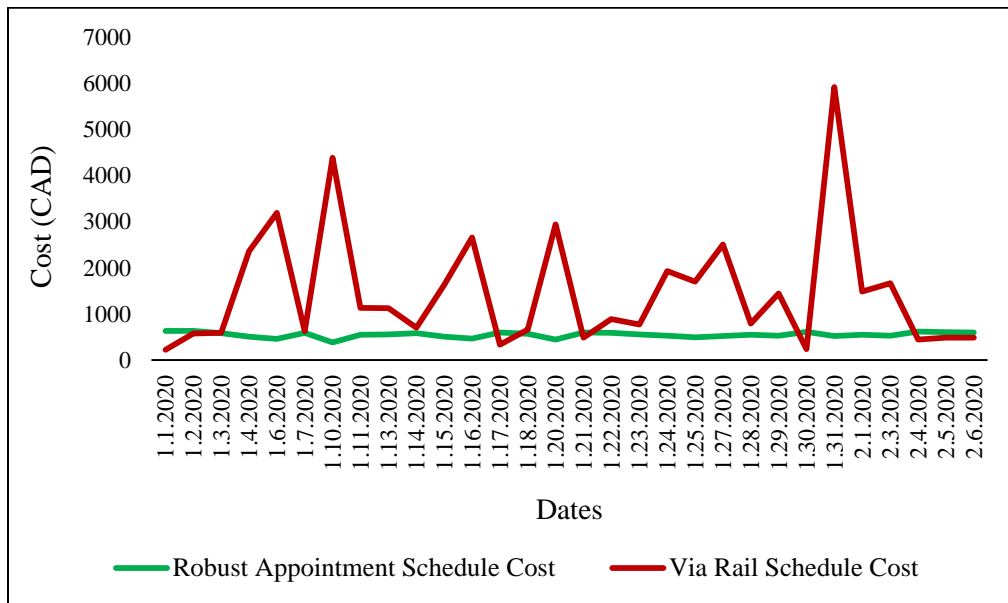

Figure 17: Comparison of VIA Rail schedule cost and Robust Appointment Schedule cost

6.2 Case Study 2: A Dentist's Clinic

In this case study, the Robust Appointment Scheduling model is implemented to provide appointment dates for a local Dentist's clinic in Montreal, Canada. The clinic provides different oral health services for the patients and the processing times of these services are uncertain, varying from one patient to another. The objective is to provide appointment dates for each patients that minimize the total underage and overage cost of both the processor and the patients for the worst case scenario under any realization of the processing time of the services.

6.2.1 Data Collection

Three months of data containing different processing durations for each of the oral treatments is collected from the clinic from November, 2019 to January 2020. From these data, the lower limit and upper limit of the processing durations for each oral treatments is calculated as shown in Table 27. These extreme point values is used to apply in the robust model.

In this problem, underage cost is incurred when the dentist finishes providing service to a patient before the appointment date of the next patient. As a result the dentist and the associated resources have to remain idle and face the associated cost until the next patients' appointment date. The underage cost is taken as equivalent to unit time worth of the system.

On the other hand, overage cost is incurred when completion time of a patient is more than the appointment date of the next patient. In this case, the next patients have to wait until the completion time of the previous patient and are subjected to idle time and associated costs. The overage cost is taken as equivalent to unit time worth of a patient.

6.2.2 Application of the Iterative Solution Procedure

The proposed Iterative Solution Procedure is applied for scheduling the patients of the clinic for 5 different days. For each day, there are 15 patients to schedule. Table 27 shows the name of the 15 oral health treatments provided for Day 1. Since there are 15 patients to schedule, according to the scenario planning approach, there are total 2^n or 2^{15} or 32,728 scenarios (considering extreme scenario case). The result obtained from Iterative Search Procedure is presented in Table 27.

The optimal appointment dates for Day 1 are found to be $A_1 = 10{:}00\ AM$, $A_2 = 10.30\ AM$, $A_3 = 10.55\ AM$, $A_4 = 11.17\ AM$, $A_5 = 11.39\ AM$, $A_6 = 11.58\ AM$, $A_7 = 12.27\ AM$, $A_8 = 1.20\ PM$, $A_9 = 2.12\ PM$, $A_{10} = 2.59\ PM$, $A_{11} = 3.15\ PM$, $A_{12} = 3.37\ PM$, $A_{13} = 4.51\ PM$, $A_{14} = 5.40\ PM$, $A_{15} = 6.48\ PM$, $A_{16} = 7.13$.

The objective function value, $Z = 1039.84$ means that the worst cost that may occur for this schedule for any realization of the processing time is 1039.84 CAD.

Table 27: Results obtained for Day 1 from the Iterative Search Procedure

Serial no.	Process Name	Lower processing time (sec), l_i	Upper processing time (sec), u_i	Robust Appointment Schedule (Iterative Solution Procedure)	
				Appointment, A_i	Objective, Z
1	Braces	1500.00	2100.00	$A_1 = 10.00$ AM	1039.84
2	Crowns and Caps	1200.00	1800.00	$A_2 = 10.30$ AM	
3	Extractions	900.00	1800.00	$A_3 = 10.55$ AM	
4	Oral Cancer examination	900.00	1800.00	$A_4 = 11.17$ AM	
5	Sealants	600.00	1800.00	$A_5 = 11.39$ AM	
6	Braces	1500.00	2100.00	$A_6 = 11.58$ AM	
7	Teeth bonding	1800.00	5400.00	$A_7 = 12.27$ PM	
8	Bridges and Implants	1800.00	5400.00	$A_8 = 1.20$ PM	
9	Fillings, Repairs/ Canals	2400.00	3600.00	$A_9 = 2.12$ PM	
10	Sealants	600.00	1800.00	$A_{10} = 2.59$ PM	
11	Crowns and Caps	1200.00	1800.00	$A_{11} = 3.15$ PM	
12	Teeth Whitening	3600.00	7200.00	$A_{12} = 3.37$ PM	
13	Teeth Veneers	2700.00	4200.00	$A_{13} = 4.51$ PM	
14	Root Canals	3600.00	7200.00	$A_{14} = 5.40$ PM	
15	Braces	1500.00	2100.00	$A_{15} = 6.48$ PM	
-	-	-	-	$A_{16} = 7.13$ PM	

The Iterative Search Procedure converges very quickly, after 16 iterations as shown in figure 18.
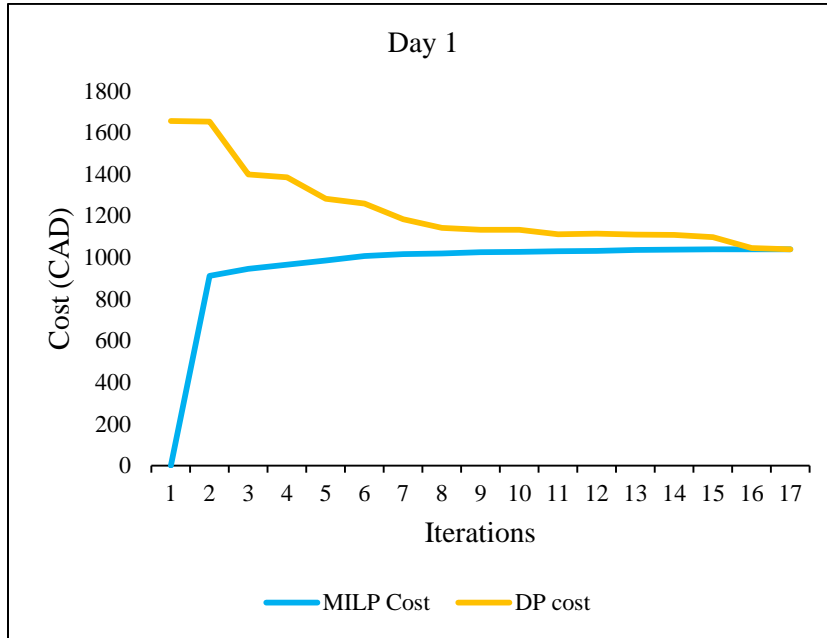

Figure 18: Convergence of the Iterative Solution Procedure (Day 1)

Similar to Day 1, the Robust Appointment Schedule model is implemented to provide appointment dates for another 4 days for the dentist's clinic. Table 28 shows the performance of Robust Appointment Schedule compared to the dentist's appointment schedule considering actual scenarios of the 5 days considered. Figure 19 shows the comparison of costs between the Robust Appointment Schedule and the dentist's appointment schedule considering actual processing time scenarios of 5 days. It is to be noted that for each instance of Table 28, the worst cost found for the Robust Appointment Schedule model (objective value) is less than the actual scenario cost of that day.

Table 28: Performance of Robust Appointment Schedule for the Dentist's clinic

|	Robust Appointment Schedule (Iterative Solution Procedure)			Dentist's Appointment Schedule	Savings by Robust Model (%)
	Objective Value, Z	CPU time (sec)	Cost incurred for actual scenario (CAD)	Cost incurred for actual scenario (CAD)	
Day 1	1039.84	5.1	232.2552	3752.8	93.81
Day 2	1094.988	7.1	323.9167	1741.7	81.40
Day 3	964.37	4.4	478.7	697.40	31.36
Day 4	1351.48	6.3	637.4756	1447.8	55.97
Day 5	1288.5	6.9	392.5581	508.22	22.76
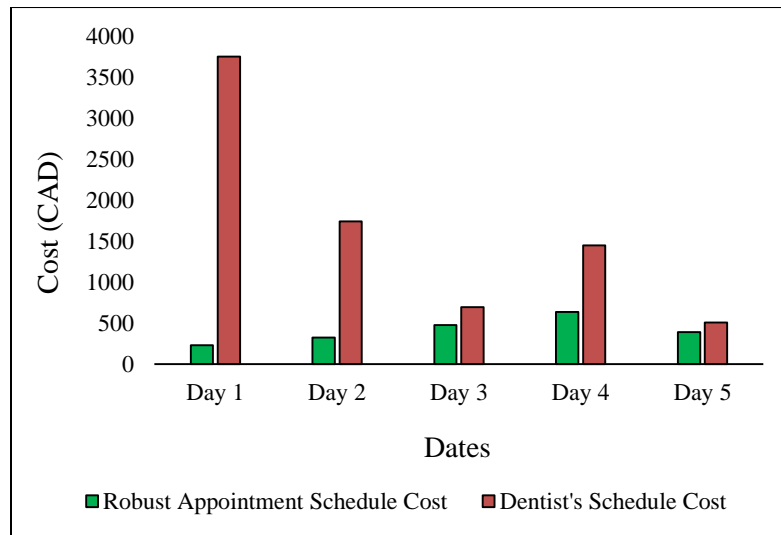
Figure 19: Comparison of the Robust Appointment Schedule cost and the Dentist's schedule cost

In chapter 6, the Robust Appointment Scheduling model is implemented for two case studies; one for VIA Rail Canada and the other for a Dentist's clinic. The case of VIA Rail Canada represents a smaller instance of the appointment scheduling problem having a total of eight jobs, i.e. stations to schedule. Therefore, the problem is solved implementing both the Mixed Integer Linear Programming model and the proposed Iterative Search Procedure. A comparison of results obtained from both the approaches indicates that the Iterative Solution Procedure not only converges very quickly but also provides a solution that is equal to the optimal solution obtained from the Mixed Integer Linear Programming

model. On the other hand, the case of the Dentist's clinic represents a larger instance of the appointment scheduling problem having a total of fifteen jobs, i.e. patients to schedule. As a result, this larger instance of the problem is solved implementing the proposed Iterative Search Procedure.

CHAPTER 7
CONCLUSION AND FUTURE RESEARCH SCOPES

In this research, uncertainty of processing time in appointment scheduling is considered. The existing modelling approaches for appointment systems do not provide protection against low probable yet high impact events. While applying stochastic programming approach, some studies assume identical service time distribution, which might lead to a faulty assumption for heterogeneous service types. On the other hand, some studies assume that the service time distribution is known to the decision maker, which may not be the real case. There is a lack of studies that help handling uncertainty in decision making without assigning probabilities to the future uncertain parameters. The proposed Robust Appointment Scheduling model helps dealing with uncertainty without assigning probabilities to uncertain parameters and achieves a solution that perform well for all possible realization of the scenarios and hedges against the worst-case scenario. In addition, the proposed robust model is applicable for any distribution of the uncertain processing time.

To begin with, a nonlinear robust appointment scheduling model is developed that minimizes the total underage and overage cost of the worst-case scenario for any realization of the processing time. To remove the nonlinearity, a Mixed Integer Linear Programming model is proposed. Proposition that states that the worst case scenario for robust scheduling belongs to the set of extreme point instances of the processing times of each job is provided along with theoretical proof (see Proposition 3). As a result, scenario realization and computational effort for the robust appointment scheduling model have reduced to a great extent without loss of optimality compared to other modelling approaches. Furthermore, some illustrative examples are provided, where it is shown that worst case scenario belongs to the extreme point scenarios of the processing times. A Stochastic Programming version of the Robust Model is provided to compare both the approaches. Robust Appointment Scheduling model not only reduces the computational effort, it also reduces the worst cost compared to the stochastic programming approach.

Since the proposed Robust Appointment Scheduling model is NP- hard, an Iterative Search Procedure is provided for solving the lager instances of the problem in polynomial time. The objective of the Iterative Search Procedure is to selectively choose the scenarios that incur worst cost and then optimizing among those scenarios to provide an appointment schedule that will work well for all other scenarios. For finding the worst case scenario, i.e. the scenario that incurs the worst cost, a Dynamic Programming model is proposed which allows to find the worst case scenario among all the scenarios in $O(n^2)$ time. Through an illustrative example it is shown that the Iterative Search Procedure converges very quickly.

Two case studies are conducted using the proposed methodologies; one is for scheduling the VIA Rail Canada stations and another is for a Dentist's clinic. For both the case studies, Robust Appointment Schedule exhibits high performance in terms of computational efforts and cost reduction.

This study will contribute both to the literature related to uncertainty handling in decision making and to the industries which aim to achieve an efficient service system.

The robust model has some limitations. Since the objective of the robust model is to minimize the worst case performance, it addresses the risk aversion nature of the decision maker with certainty, however, the robust model does not consider the expected scenario like the Stochastic Programming Approach. As a result, robust model can protect against very high impact event with low probability unlike the stochastic programming approach. Although the appointment schedule that minimizing the worst case performance might not always be profitable considering the expected case scenarios for the long term. For future work, research can be carried out for Robust Appointment Scheduling considering few of the assumption taken for appointment scheduling such as sequencing problem, no shows, fairness, overbooking, emergency arrival, service interruption, processor lateness, uncertainty of demand and capacity, etc. In addition, research can be carried out regarding uncertainty handling using Robust Optimization for decision support systems.

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VITA AUCTORIS

NAME: Tasmia Jannat Tumpa

PLACE OF BIRTH: Dhaka, Bangladesh

YEAR OF BIRTH: 1995

EDUCATION: Bangladesh University of Engineering and
                         Technology, B.Sc., Dhaka, Bangladesh, 2017

                         University of Windsor, M.Sc., Windsor, ON, 2020