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A Novel Mathematical Model of the Trojan Y-Chromosome Strategy with Optimal Control

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A Novel Mathematical Model of the Trojan Y-Chromosome Strategy with
Optimal Control
by

Christopher Turner, B.S.

Presented to the Faculty of the Graduate School of

Stephen F. Austin State University

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A Novel Trojan Y-Chromosome Strategy with Optimal Control
by

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ABSTRACT

Invasive species are a prevalent problem all over the world. Controlling and eradicating an invasive species is an even more difficult problem. The Trojan Y Chromosome (TYC) eradication strategy is one control method. This method alters the female to male sex ratio by introducing sex reversed males called supermales. These sex reversed males can only produce male progeny. Mathematical models of this strategy have shown that a population can be driven to extinction with a continuous supply of these sex reversed males. There are many different mathematical models of this strategy, but most have serious flaws, such as negative solutions or finite time blow up. In this paper, a new model for the TYC strategy is introduced and an optimal control is established for the introduction rate of the supermales for this new model.

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1 INTRODUCTION TO TROJAN Y CHROMOSOME MODEL AND INVASIVE SPECIES CONTROL

1.1 Motivation

The United States Department of the Interior defines an invasive species as “an alien species whose introduction does or is likely to cause economic or environmental harm to human health [13].” The Invasive Species Advisory Committee further defines an invasive species as “a species that is non-native to the ecosystem under consideration and whose introduction causes or is likely to cause economic or environmental harm or harm to human health [1].” In other words, an invasive species is a foreign species that has tremendous impact on the ecosystem that it is invading. Invasive species can be difficult to manage and even more difficult to eradicate. For these reasons, modeling the spread of invasive species is an important problem and much has been devoted to this issue. Here, we analyze a strategy to control an invasive species.

1.2 Trojan Y-Chromosome Strategy

The Trojan Y Chromosome (TYC) model is an eradication technique developed by Gutierrez and Teem [6]. This was further developed and analyzed by Gutierrez, Juan B., et al in “Analysis of the Trojan Y chromosome model for eradication of invasive species in a dendritic riverine system” [5]. The optimal control of the classical and certain modifications to the model was developed by Beauregard et al in “Optimal control and analysis of a modified trojan Y-Chromosome strategy” [2]. It uses the idea of supermales to alter the sex ratio of the species to be predominately male and thus drive the species towards extinction. A supermale is a male with two Y chromosomes and thus when it mates with a wild female, the offspring will be male. A traditional Punnett square is given in Fig. 1.1 for a normal male and female species would be, where, for a large enough population, the sex ratio for the species is 50% male and 50% female.

	X	Y
X	XX	XY
X	XX	XY

Figure 1.1: Punnet Squares of Normal Male and Female Reproduction

The Punnett squares when we add the supermale component are given in Fig. 1.2.

	Y	Y
X	XY	XY
X	XY	XY

Figure 1.2: Punnet Squares of Supermale and Female Reproduction

As the population becomes predominantly male, new females are no longer produced. As older females die, reproduction becomes increasingly rare. This in turn will lead to the extinction of the invasive species.

In addition, certain environmental pressures, such as resource availability, affect the long term dynamics of this system relative to no environmental pressures. A preliminary study conducted by the United States Geological Survey (USGS) in Florida has shown that the females of this species are more likely to stress their young, resulting in death of the young, regardless of the amount of food available to them. Mesocosm experiments were conducted by placing juvenile fish into tanks with either adult female or adult male fish with varying food supplies. Effectively, they had a tank with males and juveniles with low food, males with juveniles with high food, females and juveniles with low food, females and juveniles with high food, and a control tank of just juveniles. [12] This study demonstrated that the presence

of adult fish, regardless of the available food supply, applied detrimental pressure to the juvenile population. This effect is predominant in the female/juvenile tanks, especially if the available food supply is low.

This study motivates the idea to specifically look at the effects of environmental pressures of wild females on supermales.

2 VARIOUS MODELS OF THE TROJAN Y-CHROMOSOME STRATEGY

2.1 Modified TYC Model

The traditional mating model for a species may be modeled by

$$\begin{aligned}\dot{f} &= \frac{1}{2}\beta fmL - \delta f, \\ \dot{m} &= \frac{1}{2}\beta fmL - \delta m,\end{aligned}$$

where $L = 1 - \frac{f+m}{K}$ and K, β , and δ are non-negative parameters. Specifically K is the carrying capacity, β is the birth rate, and δ is the death rate for the species. The derivatives \dot{f} and \dot{m} document the change in the female and male population in time, where f is the number of females and m is the number of males. The traditional TYC model has these same elements, and adds an equation for the supermales. The traditional TYC model with supermales was first given by Gutierrez and Teem [6]:

$$\begin{aligned}\dot{f} &= \frac{1}{2}\beta fmL - \delta f, \\ \dot{m} &= \frac{1}{2}\beta fmL + \beta fsL - \delta m, \\ \dot{s} &= \mu - \delta s,\end{aligned}$$

where $L = 1 - \frac{f+m+s}{K}$ and K, β, δ , and μ are non-negative parameters. The derivative \dot{s} is the change in the supermale population over a certain time interval, s is the number of supermales at that time, and μ is the time dependent introduction rate for the supermales. A modification of this model was proposed in [2], which included a factor for intraspecies competition for mates:

$$\dot{f} = \frac{1}{2}\beta fmL \left(\frac{m}{m+s} \right) - \delta f, \tag{2.1}$$

$$\dot{m} = \frac{1}{2}\beta fmL \left(\frac{m}{m+s} \right) + \beta fsL \left(\frac{s}{m+s} \right) - \delta m, \tag{2.2}$$

$$\dot{s} = \mu - \delta s, \tag{2.3}$$

where $L = 1 - \frac{f + m + s}{K}$ and $K, \beta, \delta,$ and μ are non-negative parameters. The intraspecies competition factors, $\frac{m}{m + s}$ and $\frac{s}{m + s}$, are really a nonnegative saturation term that would provide the percentage of the male population that comes from either the normal male population or the supermale population. Let us now look at the equilibrium of the modified TYC model.

2.1.1 Equilibria of Modified TYC Model

We now solve Equations (2.1), (2.2), and (2.3) to find the equilibrium solutions to this nonlinear system of equations. Since μ is a continuous rate of introduction for supermales, which does not seem physically possible, we set $\mu = 0$. Hence at equilibrium $s = 0$.

Substituting $s = 0$ into (2.1) we have

$$0 = \frac{1}{2}\beta f m L - \delta f. \quad (2.4)$$

Similarly, substituting $s = 0$ into (2.2) we have

$$0 = \frac{1}{2}\beta f m L - \delta m. \quad (2.5)$$

Solving (2.4) and (2.5), we have

$$\begin{aligned} 0 &= f \left(\frac{1}{2}\beta m L - \delta \right) \\ 0 &= m \left(\frac{1}{2}\beta f L - \delta \right). \end{aligned}$$

Then $f = 0$ or $m = \frac{2\delta}{\beta L}$. Similarly, $m = 0$ or $\frac{2\delta}{\beta L} = f$. This shows us that $f = m$. This also gives us an equilibrium solution when $s = m = f = 0$. The equilibrium solution of the form $(f, m, s) \equiv (0, 0, 0)$ is called the extinction state.

Assume $f \neq 0$, then $\frac{1}{2}\beta m L - \delta = 0$, and $L = 1 - \frac{f + m + s}{K}$ combine to

$$\frac{2\delta K}{\beta(K - f - m)} = m.$$

Since $m = f$, then

$$\begin{aligned}\frac{2\delta K}{\beta} &= m(K - m - m) \\ 2m^2 - Km + \frac{2\delta K}{\beta} &= 0.\end{aligned}$$

Solving this using the quadratic formula we have

$$m = \frac{K}{4} \pm \frac{K}{4} \sqrt{1 - \frac{16\delta}{\beta K}}.$$

Let $K^+ = \frac{K}{4} + \frac{K}{4} \sqrt{1 - \frac{16\delta}{\beta K}}$ and $K^- = \frac{K}{4} - \frac{K}{4} \sqrt{1 - \frac{16\delta}{\beta K}}$. Looking at the discriminant, if $1 - \frac{16\delta}{\beta K} = 0$, then we have two equilibrium solution at $(\frac{K}{4}, \frac{K}{4}, 0)$ and $(0, 0, 0)$. If $1 - \frac{16\delta}{\beta K} < 1$ then we have no real solutions. If $1 - \frac{16\delta}{\beta K} > 1$, then we have three equilibrium solutions, $(K^+, K^+, 0)$, $(K^-, K^-, 0)$, and $(0, 0, 0)$. We determine the stability of these equilibrium solutions in the next section.

2.1.2 Linear Stability Analysis of Modified TYC Model

Since this is a nonlinear system, evaluating the stability of this system can be difficult. A reliable approach would be to linearize the system. Let

$$\begin{aligned}g_1(f, m, s) &= \dot{f}, \\ g_2(f, m, s) &= \dot{m}, \\ g_3(f, m, s) &= \dot{s}.\end{aligned}$$

Let f^* , m^* , s^* be the equilibrium solutions to g_1, g_2, g_3 . In other words,

$$\begin{aligned}g_1(f^*, m^*, s^*) &= 0, \\ g_2(f^*, m^*, s^*) &= 0, \\ g_3(f^*, m^*, s^*) &= 0.\end{aligned}$$

Let $u = f - f^*$, $v = m - m^*$, and $w = s - s^*$ denote the small perturbations about the equilibrium points. Note that f^* , m^* , s^* are all constants. Substituting $f = f^* + u$,

$g = g^* + v$, and $s = s^* + w$ into the the equations above and Taylor expanding on u , we have

$$\begin{aligned}\dot{u} &= g_1(u + f^*, v + m^*, w + s^*) \\ &= g_1(f^*, m^*, s^*) + u \frac{\partial g_1}{\partial f}(f^*, m^*, s^*) + v \frac{\partial g_1}{\partial m}(f^*, m^*, s^*) \\ &\quad + w \frac{\partial g_1}{\partial s}(f^*, m^*, s^*) + \text{H.O.T}\end{aligned}$$

where H.O.T are the Higher Order Terms, that is, $u^2, v^2, w^2, uv, uw, vw, \dots$. Since our perturbations are small, these H.O.T. are negligible. A similar argument can be made for v and w . Since we have neglected the H.O.T., this system is now linearized about (f^*, m^*, s^*) . Now we can rewrite this system as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial f}(f^*, m^*, s^*) & \frac{\partial g_1}{\partial m}(f^*, m^*, s^*) & \frac{\partial g_1}{\partial s}(f^*, m^*, s^*) \\ \frac{\partial g_2}{\partial f}(f^*, m^*, s^*) & \frac{\partial g_2}{\partial m}(f^*, m^*, s^*) & \frac{\partial g_2}{\partial s}(f^*, m^*, s^*) \\ \frac{\partial g_3}{\partial f}(f^*, m^*, s^*) & \frac{\partial g_3}{\partial m}(f^*, m^*, s^*) & \frac{\partial g_3}{\partial s}(f^*, m^*, s^*) \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

where the matrix

$$J = \begin{pmatrix} \frac{\partial g_1}{\partial f}(f^*, m^*, s^*) & \frac{\partial g_1}{\partial m}(f^*, m^*, s^*) & \frac{\partial g_1}{\partial s}(f^*, m^*, s^*) \\ \frac{\partial g_2}{\partial f}(f^*, m^*, s^*) & \frac{\partial g_2}{\partial m}(f^*, m^*, s^*) & \frac{\partial g_2}{\partial s}(f^*, m^*, s^*) \\ \frac{\partial g_3}{\partial f}(f^*, m^*, s^*) & \frac{\partial g_3}{\partial m}(f^*, m^*, s^*) & \frac{\partial g_3}{\partial s}(f^*, m^*, s^*) \end{pmatrix}$$

is the Jacobian matrix evaluated at (f^*, m^*, s^*) . Now we can evaluate the eigenvalues of the Jacobian to determine whether the equilibria are stable or not. If the real

components of eigenvalues are all negative, then this is a stable equilibrium. If one or more of the real components are positive, then we have an unstable equilibrium.

Evaluating Jacobian at the extinction state yields,

$$J(0, 0, 0) = \begin{pmatrix} -\delta & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & -\delta \end{pmatrix}.$$

The eigenvalues for this matrix are $-\delta$ with a multiplicity of 3. Since the death term, δ , is nonnegative then all the eigenvalues are negative. Therefore the extinction state is a stable equilibrium. This means that the extinction state is stable, which indicates that if we can get the population arbitrarily close to the extinction state, extinction will occur. For eradication of an invasive species, this is ideal.

The eigenvalues for the Jacobian evaluated at $(K^-, K^-, 0)$ are

$$\frac{\beta K \sqrt{1 - \frac{16\delta}{\beta K}}}{8} - \frac{\beta K}{8} + 2\delta, \quad -\delta, \quad -\delta$$

It is clear that we have two negative eigenvalues. We are interested in $\frac{16\delta}{\beta K} < 1$. In [2] it was shown that utilizing available population data that $\frac{16\delta}{\beta K} < 1$ is reasonable and biologically feasible. Since $\frac{16\delta}{\beta K} < 1$, then $\frac{\beta K \sqrt{1 - \frac{16\delta}{\beta K}}}{8} > \frac{\beta K}{8}$, this leads us to the fact that this eigenvalue is positive. Thus $(K^-, K^-, 0)$ is an unstable equilibrium. In particular, it is a saddle point equilibrium in the female/male phase space.

The eigenvalues for the Jacobian evaluated at $(K^+, K^+, 0)$ are

$$\frac{-\beta K \sqrt{1 - \frac{16\delta}{\beta K}}}{8} - \frac{\beta K}{8} + 2\delta, \quad -\delta, \quad \text{and} \quad -\delta.$$

Factoring out δ , the first eigenvalue becomes

$$\delta \left(2 - \frac{\beta K}{8\delta} - \frac{\beta K}{8\delta} \sqrt{1 - \frac{16\delta}{\beta K}} \right).$$

Let $r = \frac{\beta K}{8\delta}$. Then this eigenvalue is

$$\delta \left(2 - r - r \sqrt{1 - \frac{2}{r}} \right).$$

Let

$$\phi(r) = 2 - r - r\sqrt{1 - \frac{2}{r}} = \dots = -\sqrt{r-2}(\sqrt{r-2} + \sqrt{r}).$$

Since these eigenvalues correspond to the equilibrium solution when $\frac{16\delta}{\beta K} < 1$, then $\frac{2}{r} = \frac{16\delta}{\beta K} < 1$ implies that $\frac{2}{r} < 1$. Hence, this the eigenvalue is only real valued when $r > 2$. Note that $\phi(r)$ has negative values for $r > 2$. Thus we have only negative eigenvalues and this is therefore a stable equilibrium.

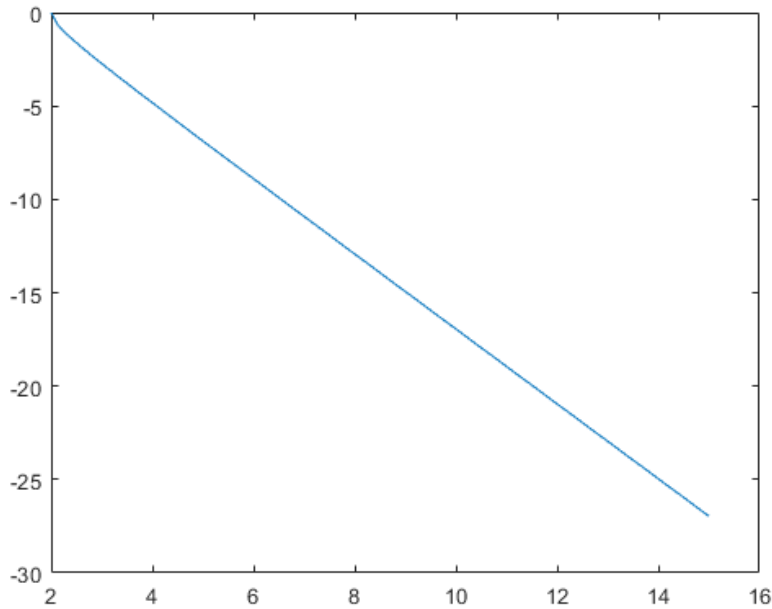


Figure 2.1: $\phi(r)$ along the domain $r \in [2, 15]$

From this analysis, there are two stable equilibria at $(K^+, K^+, 0)$ and $(0, 0, 0)$ and an unstable equilibrium at $(K^-, K^-, 0)$ for the modified TYC model. The long term dynamics indicate that either the species heads toward $(0, 0, 0)$, the extinction state, or the species heads toward $(K^+, K^+, 0)$, the recovery state. Figure 2.1 illustrates that we can get to either the extinction state or the recovery state depending on both the percentage of males and females and the percentage of initial supermales. For clarity, the females and males in Figure 2.1 are both at 25% of the carrying capacity and the graph illustrates the trajectories for the female and male population together over time. Figure 2.2 illustrates that we can get to the extinction state with a large enough initial super male population, no matter the percentage of females and males. Now let us look at the modified TYC model with cannibalism effects.

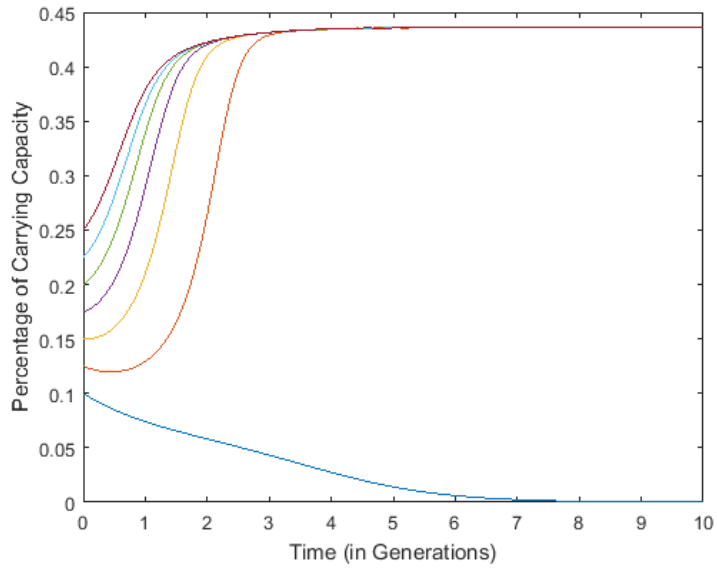


Figure 2.2: Simulation with Initial Supermales at 10% of Carrying capacity. $r = 18$

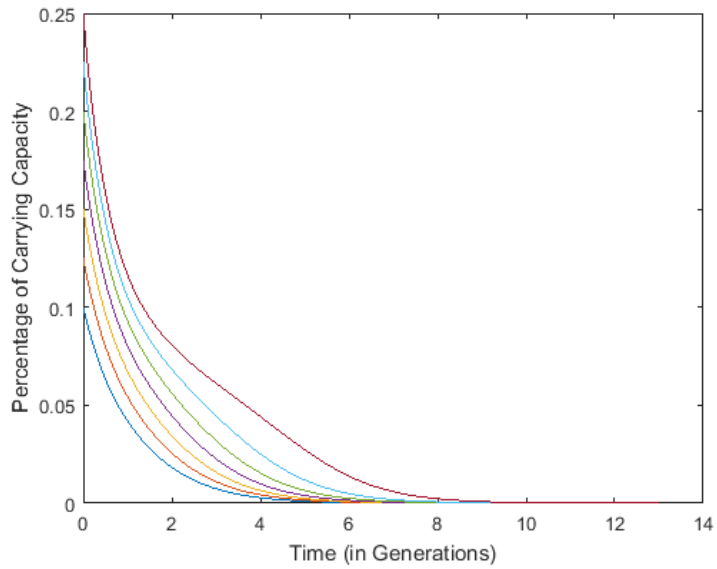


Figure 2.3: Simulation with Initial Supermales at 65% Carrying capacity. $r = 18$

2.2 Modified TYC Model with Cannibalism

Cannibalism is found in many ecological systems. If food is scarce, fish and other animals may eat their young or put pressure on the young so that the overall survival

for the young is low. Fish especially will eat smaller fish of the same species to survive. The TYC eradication strategy develops a supermale, but the name is a misnomer. The supermales are much smaller than their wild male counterparts and are at risk of being eaten or stressed when introduced to the ecosystem when the food supply is much too low. Preliminary results from the USGS, studying resource pressures in guppies have shown that the females will place added pressures on the young, regardless of the amount of food available to them. Keeping this in mind, we add an effect due to environmental pressures/cannibalism to (2.1)-(2.3).

$$\dot{f} = \frac{1}{2}\beta f m L \left(\frac{m}{m+s} \right) + (-\delta + \epsilon s)f, \quad (2.6)$$

$$\dot{m} = \frac{1}{2}\beta f m L \left(\frac{m}{m+s} \right) + \beta f s L \left(\frac{s}{m+s} \right) - \delta m, \quad (2.7)$$

$$\dot{s} = \mu + (-\delta - \epsilon_1 f)s, \quad (2.8)$$

where $L = 1 - \frac{f+m+s}{K}$ and K , β , δ , μ , ϵ , and ϵ_1 are non-negative parameters. The parameter ϵ is the benefit of the environmental pressure/cannibalism for the female population and ϵ_1 is the detrimental effect that the environmental pressures/cannibalism has on the supermale population. We have added the term ϵ into the death rate for the females to mimic the beneficial aspect and we added ϵ_1 to the death rate for the supermales to mimic the added removal from the ecosystem due to environmental pressures/cannibalism along with the natural death rate δ . If the female population cannibalizes the supermales or if the environmental pressures aid the females, then the death rate of the females should decrease, hence an additive effect on the death rate term. If the supermales are eaten or if the environmental pressures harm the supermales, then the death rate for the supermale population should increase, hence the subtraction. Note that $\epsilon \leq \epsilon_1 \ll \delta$. We made this distinction since the added benefit for the cannibalism/environmental pressures placed on the supermales should be minuscule in comparison to the detrimental effect that cannibalism/environmental pressures would place on the supermales, along with the fact that the benefit of cannibalism/environmental pressures should not lead to no death in the female population.

2.2.1 Scaling the Modified TYC with Cannibalism Effects

There are six parameters: carrying capacity (K), birth rate (β), death rate (δ), supermale introduction rate (μ), effect of cannibalism for the females (ϵ), effects of cannibalism on the supermales (ϵ_1). There are four variables: time (t , which is embedded in the equations), females (f), males (m), supermales (s) Hence, scaling can

help us reduce the number of parameters and simplify the equations we are working with. Let $\tau = \delta t$, $\hat{f} = \frac{f}{K}$, $\hat{m} = \frac{m}{K}$, and $\hat{s} = \frac{s}{K}$. Using the chain rule, then $\frac{df}{dt} = \frac{d\hat{f}K}{d\tau} \frac{d\tau}{dt}$. Note that $\frac{d\tau}{dt} = \delta$. Then $\frac{df}{dt} = K\delta \frac{d\hat{f}}{d\tau}$. This is similar for \hat{m} and \hat{s} .

Rewriting (2.6) we get

$$\frac{d\hat{f}}{d\tau} = \frac{\beta K}{2\delta} \hat{f} \hat{m} (1 - \hat{f} - \hat{m} - \hat{s}) \left(\frac{\hat{m}}{\hat{m} + \hat{s}} \right) - \hat{f} + \frac{\epsilon K}{\delta} \hat{s} \hat{f}.$$

Rewriting (2.7) we get

$$\frac{d\hat{m}}{d\tau} = \frac{\beta K}{2\delta} \hat{f} \hat{m} (1 - \hat{f} - \hat{m} - \hat{s}) \left(\frac{\hat{m}}{\hat{m} + \hat{s}} \right) + \frac{\beta K}{\delta} \hat{f} \hat{s} \left(1 - \hat{f} - \hat{m} - \hat{s} \right) \left(\frac{\hat{s}}{\hat{m} + \hat{s}} \right) - \hat{m}.$$

Finally, rewriting (2.8) and setting $\mu = 0$ we get

$$\frac{d\hat{s}}{d\tau} = -\hat{s} - \frac{\epsilon_1 K}{\delta} \hat{s} \hat{f}.$$

Let $r = \frac{\beta K}{2\delta}$, $\epsilon \mapsto \frac{\epsilon K}{\delta}$, $\epsilon_1 \mapsto \frac{\epsilon_1 K}{\delta}$, $\hat{f} \mapsto f$, $\hat{m} \mapsto m$, and $\hat{s} \mapsto s$. Our dimensionless equations are

$$\dot{f} = r f m (1 - f - m - s) \left(\frac{m}{m + s} \right) - f + \epsilon s f, \quad (2.9)$$

$$\dot{m} = r f m (1 - f - m - s) \left(\frac{m}{m + s} \right) + 2r f s (1 - f - m - s) \left(\frac{s}{m + s} \right) - m, \quad (2.10)$$

$$\dot{s} = -s - \epsilon_1 s f. \quad (2.11)$$

Now our equations are simplified and the only parameters we have to adjust are r , ϵ , and ϵ_1 . These parameters are the most important according to our scaling method. Also of note, all of our results will be in percentages of carrying capacity and our time will be measured in generations, instead of actual days, months, or years.

2.2.2 Equilibrium for the Modified TYC with Cannibalism Effects

Once again, since μ is a continuous rate of introduction for the supermales, we set $\mu = 0$. The goal of setting $\mu = 0$ is to ultimately stop the introduction of the supermales at some point and then want the population to go to the extinction

state. Solving for $\dot{s} = 0$, we have

$$\begin{aligned}\dot{s} &= -s - \epsilon_1 s f \\ 0 &= s(-1 - \epsilon_1 f).\end{aligned}$$

Then $s = 0$ or $-\frac{1}{\epsilon_1}$. Since $-\frac{1}{\epsilon_1}$ does not have a physical representation, we have not included this in our further calculations.

Substituting $s = 0$ into (2.9) we have

$$0 = f(rm(1 - f - m) - 1).$$

Similarly, substituting $s = 0$ into (2.10) we have

$$0 = m(rf(1 - f - m) - 1).$$

Solving (2.12) we have that $f = 0$ or $f = \frac{1}{r(1-f-m)}$. Solving (2.13) we have $m = 0$ or $m = \frac{1}{r(1-f-m)}$. This implies that $m = f$ when $s = 0$. This is the same as the Modified TYC model as seen in Section 2.1. Since $m = f$, then

$$2rm^2 - rm + 1 = 0.$$

Solving this quadratic, we have

$$m = \frac{1}{4} \pm \frac{1}{4} \sqrt{1 - \frac{8}{r}}.$$

Let $f^+ = \frac{1}{4} + \frac{1}{4} \sqrt{1 - \frac{8}{r}}$ and let $f^- = \frac{1}{4} - \frac{1}{4} \sqrt{1 - \frac{8}{r}}$. If $1 - \frac{8}{r} < 0$, then we have no real solutions and only one equilibrium at $(0, 0, 0)$. If $1 - \frac{8}{r} = 0$ then we have two equilibrium solutions at $(0, 0, 0)$ and $(\frac{1}{4}, \frac{1}{4}, 0)$. Finally, if $1 - \frac{8}{r} > 0$, then we have equilibrium solutions at $(0, 0, 0)$, $(f^+, f^+, 0)$, and $(f^-, f^-, 0)$. This is the same equilibria as the modified TYC without cannibalism as seen in Section 2.1.

2.2.3 Linear Stability Analysis of the Modified TYC Model with Cannibalism Effects

The modified TYC model is a nonlinear system of equations. Looking at the stability of the equilibrium is important. Only looking at the cases when $1 - \frac{8}{r} > 0$ and using the same linearization techniques as described in Section 2.1.2, the eigenvalues of the linearized system at $J(0, 0, 0)$ are -1 with a multiplicity of three. Since the

eigenvalues are all negative, this is a stable equilibrium. Analysis shows us that $(f^+, f^+, 0)$ is stable (all eigenvalues are negative) and $(f^-, f^-, 0)$ is unstable (one eigenvalue is always positive). This also shows us that our modified TYC model with cannibalism effects have the same equilibria and stability as the Modified TYC without cannibalism effects.

2.3 Unrealistic Behavior in Current TYC Models

The problems with the current models of TYC is that they can end up having a negative male population when we add a significant number of supermales initially, which is not biologically possible. In fact, the TYC model with cannibalism effects, if we introduce the critical number of supermales into the system that can guarantee extinction, then if any cannibalism occurs, for any given cannibalism rate, the population of the invasive species can rebound into the recovery state. Figure 2.3 illustrates one case of rebounding to the recovery state when we vary the cannibalism rate.

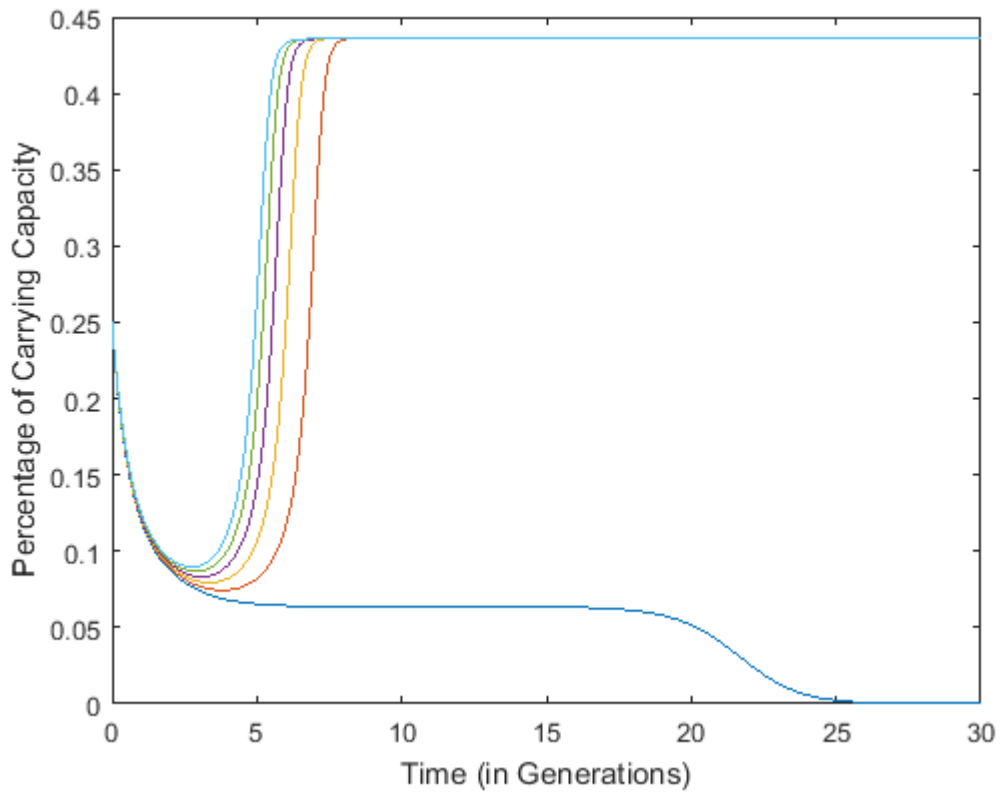


Figure 2.4: Varying Cannibalism Rates from 0 to 0.1, Critical Number of Supermales

This indicates that even if we can introduce the critical number of supermales, if there are any sort of environmental pressures/cannibalism in the system, we might still not eradicate the invasive species. With these issues in mind, a new model is proposed in the forthcoming section.

3 A PROPOSED CORRECTION TO CLASSICAL TYC MODEL

Building off of models proposed by Kendall [7] & Keyfitz [8] and Dietz & Hadelar [4] and Chavez & Huang [3], a proposed mating model that uses a pairing function for mating is given below.

$$\begin{aligned}\frac{dP}{dt} &= \phi(f, m) - \sigma P - \delta_m P - \delta_f P, \\ \frac{dm}{dt} &= -\phi(f, m) + \beta_m P + \sigma P + \delta_f P - \delta_m m, \\ \frac{df}{dt} &= -\phi(f, m) + \beta_f P + \sigma P + \delta_m P - \delta_f f,\end{aligned}$$

where P is the number of female-male pairings, f is the number of females, and m is the number of males, $\frac{dP}{dt}$ is the change in female-male pairing population over time, $\frac{dm}{dt}$ is the change in the male population who are not in the pairing population over time, $\frac{df}{dt}$ is the change in the female population who are not in the pairing population over time, ϕ is a function representing how the females and males leave their respective populations and enter the pairing population, σ is the proportion of the of the pairing population returning to their male and female counterparts, δ_m is the death rate for the males, δ_f is the death rate for the females, β_m is the birth rate for the males, and β_f is the birth rate for the females . One should note that some of the key components of ϕ are that it is differentiable and satisfies the following properties:

- 1) $\phi(f, m) \geq 0$
- 2) $\phi(f + c, m + d) \geq \phi(f, m)$ where $c, d \geq 0$
- 3) $\phi(af, am) = a\phi(f, m)$
- 4) $\phi(f, 0) = \phi(0, m) = 0$
- 5) $\phi(f, m) = \phi(m, f)$

Using this model and adding supermale introduction leads us to a new system of

equations

$$\begin{aligned}
\frac{dP}{dt} &= \phi(f, m) - \sigma P - \delta_m P - \delta_f P, \\
\frac{dm}{dt} &= -\phi(f, m) + \beta_m P + \beta_s P_s + \sigma P + \delta_f P - \delta_m m, \\
\frac{df}{dt} &= -\phi(f, m) - \phi(f, s) + \beta_f P + \sigma P + \sigma P_s + \delta_m P + \delta_s P_s - \delta_f f, \\
\frac{dP_s}{dt} &= \phi(f, s) - \sigma P_s - \delta_f P_s - \delta_s P_s, \\
\frac{ds}{dt} &= -\phi(f, s) + \mu + \sigma P_s + \delta_f P_s - \delta_s s,
\end{aligned}$$

where $\frac{dP}{dt}$, $\frac{dm}{dt}$, $\frac{df}{dt}$, $\phi(f, m)$, σ , δ_m , δ_f , β_m , β_f , P , f , m are the same as above, $\frac{dP_s}{dt}$ is the change in the female-supermale pairing population over time, $\frac{ds}{dt}$ is the change in the supermale population over time, $\phi(f, s)$ is a function representing how the females and supermales leave their respective populations and enter the pairing population, δ_s is the death rate for the supermales, and μ is the introduction rate for the supermales.

4 OPTIMAL CONTROL

4.1 Preliminaries

Optimal control is a branch of applied mathematics that tries to find a control for a dynamical system in which some objective function is optimized. Before we start the discussion of optimal control and its uses, we need to establish various theorems and definitions of optimal control theory. For a thorough development consider the book *Optimal Control Applied to Biological Models* by Lenhart and Workman [9].

Definition 4.1. Let $I \subseteq \mathbb{R}$ be an interval, which can be finite or infinite. The function $u : I \rightarrow \mathbb{R}$ is continuous at $x = c$ if $f(c)$ is defined and

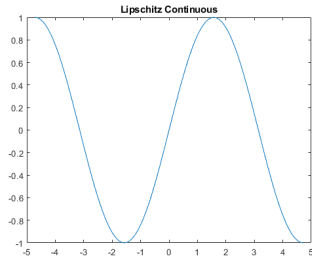
$$\lim_{x \rightarrow c} u(x) = f(c)$$

A function is said to be **continuous** if the limit exists at every point along the domain, I .

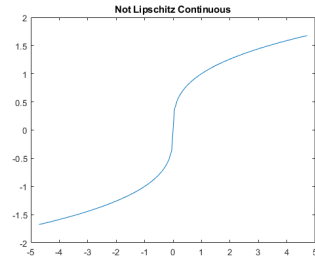
Definition 4.2. Let $I \subseteq \mathbb{R}$ be an interval. We say that a finite-valued function $u : I \subseteq \mathbb{R}$ is **piece-wise continuous** if it is continuous at each $x \in I$, with the possible exception of a finite number of x and u is equal to either the left limit or the right limit at every $x \in I$.

Definition 4.3. A function, u , is **Lipschitz continuous** if there exists a constant, c , such that $|u(x_1) - u(x_2)| \leq c|x_1 - x_2|$ for all points in the domain of k . Here c is called the **Lipschitz constant** of the function k .

Below are examples of a function that is Lipschitz continuous and a function that is not Lipschitz continuous. One should note that if a function is Lipschitz continuous, then it is continuous in the traditional sense, and if a function is continuous, then it is piece-wise continuous. However, if a function is piece-wise continuous, this does not mean that is continuous nor Lipschitz continuous.



(a) $\sin(x)$ is Lipschitz



(b) $\sqrt[3]{x}$ is not Lipschitz

Figure 4.1: Examples illustrating Lipschitz continuous and not Lipschitz continuous functions

Definition 4.4. A function $u(x)$ is said to be **convex** on $[a, b]$ if

$$\alpha u(x_1) + (1 - \alpha)u(x_2) \geq u(\alpha x_1 + (1 - \alpha)x_2)$$

for all $0 \leq \alpha \leq 1$ and for any $a \leq x_1, x_2 \leq b$.

4.2 Basic Optimal Control Problems and Necessary Conditions

In the basic optimal control schemes, we usually have some controlling function that acts on some state function. For our purposes, let $u(t)$ denote the control function and let $x(t)$ be the state function. The state variable satisfies a differential equation which depends on the control, $u(t)$. Define the given state variable as

$$x'(t) = g(t, x(t), u(t)).$$

We define a mapping of the control, $u(t)$, to the state function, $x(t)$, as $u(t) \mapsto x = x(u(t))$, which we will write as $x(u)$. We are trying to find a piece-wise continuous control function, $u(t)$, and the associated state variable, $x(t)$ to maximize or minimize a given objective function typically given as some integral, such as:

$$J(u) = \max_u \int_{t_0}^{t_1} f(t, x(t), u(t)) dt$$

subject to $x'(t) = g(t, x(t), u(t))$

$$x(t_0) = x_0 \quad \text{and} \quad x(t_1) \text{ free.} \tag{4.1}$$

The meaning of " $x(t_1)$ free" is that $x(t_1)$ is unrestrained. Now, assume an optimal control exists. Let u^* be that optimal control. Then we also have an optimal state, x^* . Since we are looking for a maximum u , then $J(u) \leq J(u^*) < \infty$. Now let us perturb u^* , denoted as

$$u^\epsilon(t) = u^*(t) + \epsilon h(t) \quad (4.2)$$

where $h(t)$ is a variation function and $\epsilon \in \mathbb{R}$. Then $u^\epsilon(t)$ is another control function. Also $x^\epsilon(t)$ is the state function corresponding to the control. Notice $x^\epsilon(t)$ satisfies

$$\frac{d}{dt}[x^\epsilon(t)] = g(t, x^\epsilon(t), u^\epsilon(t)). \quad (4.3)$$

Since all trajectories start at the same position, then $x^\epsilon(t_0) = x_0$. As $\epsilon \rightarrow 0$, then $u^\epsilon(t) \rightarrow u^*(t)$. Also, for all t ,

$$\left. \frac{\partial u^\epsilon(t)}{\partial \epsilon} \right|_{\epsilon=0} = h(t).$$

We have that $x^\epsilon(t) \rightarrow x^*(t)$. Since we know that $\frac{d}{dt}x^\epsilon(t)$ exists and it is equivalent to $g(t, x^\epsilon(t), u^\epsilon(t))$, then $\frac{\partial}{\partial \epsilon}x^\epsilon(t)$ must also exist. Now our new objective function, after the perturbations is

$$J(u^\epsilon) = \int_{t_0}^{t_1} f(t, x^\epsilon(t), u^\epsilon(t))dt. \quad (4.4)$$

Now, let λ be a piecewise differentiable function on the interval $[t_0, t_1]$ to be determined. We call λ the adjoint function and it serves as a kind of Lagrange multiplier. By the Fundamental Theorem of Calculus,

$$\begin{aligned} \int_{t_0}^{t_1} \frac{d}{dt}[\lambda(t)x^\epsilon(t)]dt &= \lambda(t_1)x^\epsilon(t_1) - \lambda(t_0)x^\epsilon(t_0), \text{ which implies} \\ \int_{t_0}^{t_1} \frac{d}{dt}[\lambda(t)x^\epsilon(t)]dt - \lambda(t_1)x^\epsilon(t_1) + \lambda(t_0)x^\epsilon(t_0) &= 0. \end{aligned} \quad (4.5)$$

Adding (4.4) to (4.5), we have

$$\begin{aligned}
J(u^\epsilon) &= \int_{t_0}^{t_1} f(t, x^\epsilon(t), u^\epsilon(t)) dt + \int_{t_0}^{t_1} \frac{d}{dt} [\lambda(t)x^\epsilon(t)] dt - \lambda(t_1)x^\epsilon(t_1) + \lambda(t_0)x^\epsilon(t_0) \\
&= \int_{t_0}^{t_1} \left(f(t, x^\epsilon(t), u^\epsilon(t)) + \frac{d}{dt} [\lambda(t)x^\epsilon(t)] \right) dt - \lambda(t_1)x^\epsilon(t_1) + \lambda(t_0)x^\epsilon(t_0) \\
&= \int_{t_0}^{t_1} \left[f(t, x^\epsilon(t), u^\epsilon(t)) + \lambda'(t)x^\epsilon(t) + \lambda(t) \frac{d}{dt} [x^\epsilon(t)] \right] dt - \lambda(t_1)x^\epsilon(t_1) + \lambda(t_0)x^\epsilon(t_0).
\end{aligned} \tag{4.6}$$

Using the fact that $x(t_0) = x_0$ and $\frac{d}{dt}[x^\epsilon(t)] = g(t, x^\epsilon(t), u^\epsilon(t))$, (4.6) becomes

$$J(u^\epsilon) = \int_{t_0}^{t_1} [f(t, x^\epsilon(t), u^\epsilon(t)) + \lambda'(t)x^\epsilon(t) + \lambda(t)g(t, x^\epsilon(t), u^\epsilon(t))] dt - \lambda(t_1)x^\epsilon(t_1) + \lambda(t_0)x_0.$$

Since u^* is where the maximum occurs by definition, then

$$0 = \left. \frac{d}{d\epsilon} J(u^\epsilon) \right|_{\epsilon=0} = \lim_{\epsilon \rightarrow 0} \frac{J(u^\epsilon) - J(u^*)}{\epsilon}.$$

Using a version of the Lebesgue Dominated Convergence Theorem as stated in Optimal Control Applied to Biological Models (pg. 10) [9] and the fact that the integrand is differentiable, we have

$$\begin{aligned}
0 &= \left. \frac{d}{d\epsilon} J(u^\epsilon) \right|_{\epsilon=0} \\
0 &= \int_{t_0}^{t_1} \left. \frac{\partial}{\partial \epsilon} [f(t, x^\epsilon(t), u^\epsilon(t)) + \lambda'(t)x^\epsilon(t) + \lambda(t)g(t, x^\epsilon(t), u^\epsilon(t))] \right|_{\epsilon=0} dt \\
&\quad - \left. \frac{\partial}{\partial \epsilon} \lambda(t_1)x^\epsilon(t_1) \right|_{\epsilon=0} + \left. \frac{\partial}{\partial \epsilon} \lambda(t_0)x_0 \right|_{\epsilon=0} \\
0 &= \int_{t_0}^{t_1} \left. \frac{\partial}{\partial \epsilon} [f(t, x^\epsilon(t), u^\epsilon(t)) + \lambda'(t)x^\epsilon(t) + \lambda(t)g(t, x^\epsilon(t), u^\epsilon(t))] \right|_{\epsilon=0} dt \\
&\quad - \left. \frac{\partial}{\partial \epsilon} \lambda(t_1)x^\epsilon(t_1) \right|_{\epsilon=0}.
\end{aligned}$$

Applying the chain rule to f and g , we have

$$\begin{aligned}
0 &= \int_{t_0}^{t_1} \left[f_x \frac{\partial x^\epsilon}{\partial \epsilon} + f_u \frac{\partial u^\epsilon}{\partial \epsilon} + \lambda'(t) \frac{\partial x^\epsilon}{\partial \epsilon} + \lambda(t) \left(g_x \frac{\partial x^\epsilon}{\partial \epsilon} + g_u \frac{\partial u^\epsilon}{\partial \epsilon} \right) \right] \Bigg|_{\epsilon=0} dt \\
&\quad - \lambda(t_1) \frac{\partial x^\epsilon}{\partial \epsilon} \Bigg|_{\epsilon=0}. \tag{4.7}
\end{aligned}$$

Note that as $\epsilon \rightarrow 0$, $u^\epsilon(t) \rightarrow u^*(t)$ and $x^\epsilon(t) \rightarrow x^*(t)$, then the partials f_x, f_u, g_x, g_u have inputs $(t, x^*(t), u^*(t))$. Rearranging the terms in (4.7), we now have

$$\begin{aligned}
0 &= \int_{t_0}^{t_1} \left[\left(f_x(t, x^*(t), u^*(t)) + \lambda(t)g_x(t, x^*(t), u^*(t)) + \lambda'(t) \right) \frac{\partial x^\epsilon}{\partial \epsilon} \Bigg|_{\epsilon=0} \right. \\
&\quad \left. + \left(f_u(t, x^*(t), u^*(t)) + \lambda(t)g_u(t, x^*(t), u^*(t)) \right) \frac{\partial u^\epsilon}{\partial \epsilon} \Bigg|_{\epsilon=0} \right] dt \\
&\quad - \lambda(t_1) \frac{\partial x^\epsilon}{\partial \epsilon} \Bigg|_{\epsilon=0}. \tag{4.8}
\end{aligned}$$

Note that $\frac{\partial u^\epsilon}{\partial \epsilon} = h(t)$ from (4.2). Substituting this back into (4.8) we have

$$\begin{aligned}
0 &= \int_{t_0}^{t_1} \left[\left(f_x(t, x^*(t), u^*(t)) + \lambda(t)g_x(t, x^*(t), u^*(t)) + \lambda'(t) \right) \frac{\partial x^\epsilon}{\partial \epsilon} \Bigg|_{\epsilon=0} \right. \\
&\quad \left. + \left(f_u(t, x^*(t), u^*(t)) + \lambda(t)g_u(t, x^*(t), u^*(t)) \right) h(t) \right] dt \\
&\quad - \lambda(t_1) \frac{\partial x^\epsilon}{\partial \epsilon} \Bigg|_{\epsilon=0}. \tag{4.9}
\end{aligned}$$

Now, we choose the adjoint function, $\lambda(t)$, that minimizes the coefficients of $f_x(t, x^*(t), u^*(t)) + \lambda(t)g_x(t, x^*(t), u^*(t)) + \lambda'(t)$. Let us choose $\lambda(t)$ to satisfy

$$\lambda'(t) = f_x(t, x^*(t), u^*(t)) - \lambda(t)g_x(t, x^*(t), u^*(t)) \tag{4.10}$$

and the boundary condition

$$\lambda(t_1) = 0. \tag{4.11}$$

We denote (4.10) as the **adjoint equation** and (4.11) as the **transversality condition**. Substituting (4.10) and (4.11) back into (4.9) we have

$$0 = \int_{t_0}^{t_1} (f_u(t, x^*(t), u^*(t)) + \lambda(t)g_u(t, x^*(t), u^*(t))h(t))dt.$$

Since this holds for any piece-wise function, then let

$$h(t) = f_u(t, x^*(t), u^*(t)) + \lambda(t)g_u(t, x^*(t), u^*(t)).$$

Then

$$0 = \int_{t_0}^{t_1} \left(f_u(t, x^*(t), u^*(t)) + \lambda(t)g_u(t, x^*(t), u^*(t)) \right)^2 dt.$$

This implies that the optimal condition holds when

$$f_u(t, x^*(t), u^*(t)) + \lambda(t)g_u(t, x^*(t), u^*(t)) = 0 \quad \forall t_0 \leq t \leq t_1.$$

Thankfully, we do not have to derive this equation for every problem. We can generate the necessary conditions for the **Hamiltonian**, H , which is defined as

$$\begin{aligned} H(t, x, u, \lambda) &= f(t, x, u) + \lambda(t)g(t, x, u) \\ &= \text{integrand} + \text{adjoint} * \text{RHS of DE.} \end{aligned}$$

We are maximizing H with respect to u at the critical value u^* and the conditions above can be written in terms of the Hamiltonian as follows:

$$\begin{aligned} \frac{\partial H}{\partial u} = 0 \text{ at } u^* &\implies f_u + \lambda g_u = 0 && (\text{optimality condition}) \\ \lambda' = -\frac{\partial H}{\partial x} &\implies \lambda' = -f_x - \lambda g_x && (\text{adjoint equation}) \\ \lambda(t_1) = 0 &&& (\text{transversality condition}) \end{aligned}$$

Note that we are given the dynamics of the state equation:

$$x' = g(t, x, u) = \frac{\partial H}{\partial \lambda}, \quad x(t_0) = x_0.$$

We have converted the problem of finding a control that maximizes or minimizes the objective function subject to the differential equation and given initial condition to one of maximizing the Hamiltonian point-wise with respect to the control parameter. We can view our optimal control problem as having two unknowns, u^* and x^* . We add in the adjoint variable, λ , which essentially attaches the differential equation information onto the maximization of the objective function. Generally, to solve optimization problems, we use the following steps:

1. Form the Hamiltonian for the given problem.
2. Write the adjoint, the transversality condition and the optimality conditions. Doing so gives you three unknowns, u^* , x^* , and λ .
3. Solve u^* in terms of x^* and λ .
4. Solve the two differential equations for x^* and λ with two boundary conditions, substituting u^* in the differential equations with the expression for the optimal control from step 3.
5. Once the optimal state and adjoint are found, solve for the optimal control.

4.3 Optimal Control Examples

Below is a simple example of how to calculate the Hamiltonian and solve the Hamiltonian for the optimal control and optimal state.

4.3.1 Example

Solve

$$\min_u \int_1^2 tu(t)^2 + t^2x(t) dt$$

subject to $x'(t) = -u(t); x(1) = 1$

Step One: Form the Hamiltonian.

$$H = f(t, x^*, u^*) + \lambda g(t, x^*, u^*)$$

$$H = tu^{*2} + t^2x^* + \lambda(-u^*)$$

$$H = tu^{*2} + t^2x^* - \lambda u^*.$$

Step Two: Write the adjoint, transversality condition, and the optimality condition.

$$\frac{\partial H}{\partial u} = 2tu^* - \lambda \tag{4.12}$$

$$\lambda' = -\frac{\partial H}{\partial x} = -2t \tag{4.13}$$

$$\lambda(2) = 0 \tag{4.14}$$

As we can see, (4.12) is the optimality condition for this example, (4.13) is the adjoint equation for this example, and (4.14) is the transversality condition for this example. Step Three: Solve u^* in terms of x^* and λ .

$$2tu^* - \lambda = 0 \implies u^* = \frac{\lambda}{2t}.$$

Step Four: Solve the differential equations.

$$\begin{aligned}\lambda' &= -2t \\ \lambda &= -t^2 + C.\end{aligned}$$

Using the transversality condition, we have

$$\lambda(2) = 0 \implies C = 4.$$

Thus we have that our adjoint equation at the optimal control is $\lambda(t) = -t^2 + 4$. Substituting this value back into the u^* equation from step three, we have

$$u^* = \frac{\lambda}{2t} = \frac{-t^2 + 4}{2t} = \frac{-t}{2} - \frac{2}{t}.$$

Step Five: Solve for the optimal state.

$$x'(t) = -u(t) = \frac{t}{2} + \frac{2}{t}.$$

Integrating,

$$\begin{aligned}\int x'(t) dt &= \int \left(\frac{t}{2} + \frac{2}{t} \right) dt \\ x(t) &= \frac{1}{2} \frac{t^2}{2} + 2 \ln(t) + C.\end{aligned}$$

Solving $x(t)$ at the given initial value condition, we have

$$\begin{aligned}x(t) &= \frac{t^2}{4} + 2 \ln(t) + C \\ x(1) = 1 &= \frac{1}{4} + 2 \ln(1) + C \implies C = \frac{3}{4}.\end{aligned}$$

Thus, for our objective function, we have an optimal control

$$u^*(t) = -\frac{t}{2} - \frac{2}{t}$$

with a corresponding optimal state at

$$x^*(t) = \frac{t^2}{4} + 2 \ln(t) + \frac{3}{4}.$$

This example shows the steps to solve an optimal control problem, but the steps don't necessarily have to follow the general guidelines as stated above. Our objective function happened to work out that the optimal control was only in terms of the adjoint and the adjoint equation had a relatively simple solution.

4.3.2 Example

Solve

$$\begin{aligned} & \max_u \int_0^1 x(t) + u(t) dt \\ & \text{subject to } x'(t) = 1 - u(t)^2; x(0) = 1 \end{aligned}$$

Step One: Form the Hamiltonian.

$$\begin{aligned} H &= f(t, x^*, u^*) + \lambda g(t, x^*, u^*) \\ H &= x^* + u^* + \lambda(1 - u^{*2}) \\ H &= x^* + u^* + \lambda + \lambda u^{*2}. \end{aligned}$$

Step Two: Write the adjoint, transversality condition, and the optimality condition.

$$\begin{aligned} \frac{\partial H}{\partial u} &= 1 - 2\lambda u^* \\ \lambda' &= -\frac{\partial H}{\partial x} = -1 \\ \lambda(1) &= 0 \end{aligned}$$

Step Three: Solve u^* in terms of x^* and λ .

$$\begin{aligned} 1 - 2\lambda u^* &= 0 \\ u^* &= \frac{1}{2\lambda}. \end{aligned}$$

Step Four: Solve the differential equations.

$$\begin{aligned}\lambda' &= -1 \\ \int \lambda' dt &= \int -1 dt \\ \lambda &= -t + C.\end{aligned}$$

Using the transversality condition, we have

$$\lambda(1) = 0 \implies C = 1.$$

Thus we have our adjoint equation at the optimal control to be $\lambda(t) = -t + 1$. Substituting this value into the optimal control, u^* , we have

$$u^* = \frac{1}{2\lambda} = \frac{1}{2(1-t)}.$$

Step Five: Solve for the optimal state.

$$x'(t) = 1 - u(t)^2 = 1 - \frac{1}{4(1-t)^2}.$$

Integrating,

$$\begin{aligned}\int x'(t) dt &= \int 1 - \frac{1}{4(1-t)^2} dt \\ x(t) &= t - \frac{1}{4(1-t)} + C.\end{aligned}$$

Solving $x(t)$ at the given initial value condition leads to

$$x(0) = 1 = 0 - \frac{1}{4(1-0)} + C \implies C = \frac{5}{4}.$$

Thus for our objective function, we have an optimal control when

$$u^*(t) = \frac{1}{2(1-t)}$$

with a corresponding optimal state at

$$x^*(t) = t - \frac{1}{4(1-t)} + \frac{5}{4}.$$

4.4 Existence of Optimal Control and the Principal of Optimality

Looking at Example 4.3.2 in section 4.3, when we evaluate the objective function at the optimal state and the optimal control we have

$$\begin{aligned} \int_0^1 x^*(t) + u^*(t) dt &= \int_0^1 t - \frac{1}{4(1-t)} + \frac{5}{4} + \frac{1}{2(1-t)} dt \\ &= \int_0^1 t + \frac{5}{4} + \frac{3}{4(1-t)} dt \\ &= \frac{t^2}{2} + \frac{5}{4}t - \frac{3}{4} \ln(|1-t|) \Big|_0^1. \end{aligned}$$

Notice at $t = 1$, we have $\ln(0)$ which does not exist. Evaluating the limit as $t \rightarrow 1^-$

$$\lim_{t \rightarrow 1^-} \left(\frac{t^2}{2} + \frac{5}{4}t - \frac{3}{4} \ln(|1-t|) \right) \rightarrow \infty.$$

In other words, the objective function becomes unbounded at $t = 1$. One explanation for the unbounded nature of the objective function is the nonlinearity of the optimal control, so one way to guarantee existence of an optimal control that is bounded is to place some restrictions on f and/or g . This is problem dependent and difficult and is an open area of research. One way to guarantee existence is given in the following theorem [9].

Theorem 4.5. *Let the set of controls for problem (4.1) be **Lebesgue integrable** [11] functions on $[t_0, t_1]$ with values in \mathbb{R} . Suppose that $f(t, x, u)$ is **convex** in u , and there exists constants C_4 and $C_1, C_2, C_3 > 0$ and $\beta > 1$ such that*

$$g(t, x, u) = \alpha(t, x) + \beta(t, x) * 0u$$

$$|g(t, x, u)| \leq C_1(1 + |x| + |u|)$$

$$|g(t, x_1, u) - g(t, x, u)| \leq C_2|x_1 - x|(1 + |u|)$$

$$f(t, x, u) \geq C_3|u|^\beta - C_4$$

for all t with $t_0 \leq t \leq t_1$, and all $x, x_1, u \in \mathbb{R}$. Then there exists an optimal control u^* maximizing $J(u)$, with $J(u^*)$ finite.

4.5 Forward and Backward Sweep

Most optimal control problems do not have an analytical solution, so our only recourse is to numerically approximate these solutions. One way to approximate a solution to an optimal control problem is the Forward-Backward Sweep Method proposed in [9]. Let \mathbf{u} be the vector solution to the optimal control problem, \mathbf{x} be the vector approximation to the state function, and $\boldsymbol{\lambda}$ be the vector approximation to the adjoint equation. The general outline for the Forward-Backward Sweep algorithm is as follows:

1. Make an initial guess for \mathbf{u} over the given interval. (Generally, we let \mathbf{u} be a vector of zeros).
2. Use the initial condition and the values for \mathbf{u} , solve \mathbf{x} forward in time according to its differential equations in the optimality system.
3. Using the transversality condition and the values for \mathbf{u} and \mathbf{x} from step 2, solve $\boldsymbol{\lambda}$ backward in time according to its differential equations in the optimality system.
4. Update \mathbf{u} with our newly found \mathbf{x} and $\boldsymbol{\lambda}$.
5. Check convergence. If the values found are marginally close, output the results, if not marginally close, return to step 2.

If problems arise with convergence, adjusting the initial guess for \mathbf{u} can solve this problem. Using this method, we now look at the optimal control for both the modified TYC model and the proposed correction to the TYC model.

5 OPTIMAL CONTROL ANALYSIS FOR VARIOUS TYC MODELS

In their paper, Lyu et al have analyzed the optimal control for the classic TYC model, including the calculations of the Hamiltonian and the various adjoint equations [10].

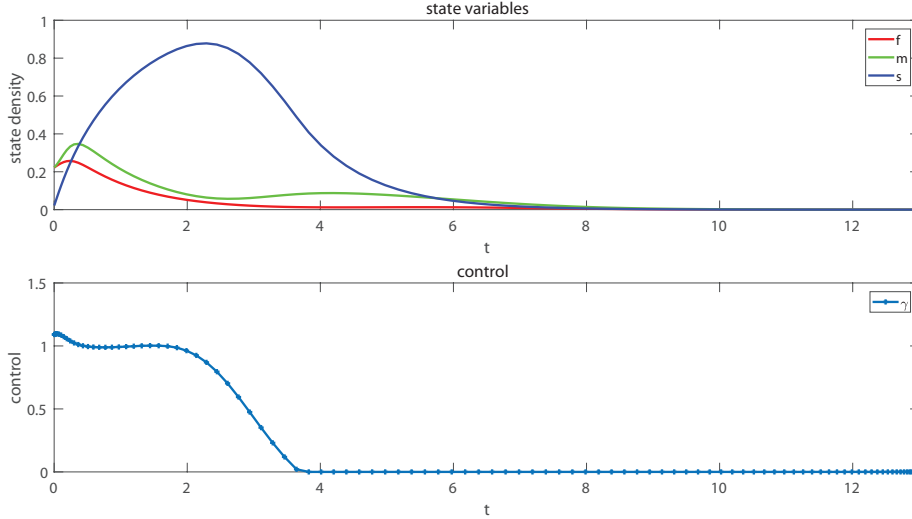


Figure 5.1: Optimal Control for Classic TYC [10]

Let us now look at the Modified TYC model and investigate an optimal μ , the introduction rate of the supermales, that will drive the female and male population to zero. Assuming that the introduction rate is not known *a priori*, consider the following objective function:

$$J_m(\mu) = \int_0^T -(f + m) - \frac{1}{2}\mu^2 dt.$$

We want to find an optimal μ , μ^* , such that

$$J_m(\mu^*) = \max_u \int_0^T -(f + m) - \frac{1}{2}\mu^2 dt.$$

The Hamiltonian for J_m is given by

$$H_m = -(f + m) - \frac{1}{2}\mu^2 + \lambda_1 \dot{f} + \lambda_2 \dot{m} + \lambda_3 \dot{s}.$$

We use the Hamiltonian to derive differential equations for the adjoint equations.

These are

$$\begin{aligned}
\lambda'_1(t) &= 1 - \lambda_1 \left\{ \frac{\beta m^2}{2K(m+s)} [K - 2f - m - s] - \delta \right\} \\
&\quad - \lambda_2 \left\{ \frac{\beta m^2}{2K(m+s)} [K - 2f - m - s] + \frac{\beta s^2}{K(m+s)} [sK - 2fs - m - s] \right\} \\
\lambda'_2(t) &= 1 - \lambda_1 \left\{ \frac{\beta fm}{m+s} \left[1 - \frac{1}{2K(m+s)} (mK + 2f + fmK + 3(m+s) + m^2K + 2s + msK) \right] \right\} \\
&\quad - \lambda_2 \left\{ \frac{\beta fm}{m+s} \left[1 - \frac{m}{2(m+s)} - \frac{f}{K} + \frac{fm}{2K(m+s)} - \frac{3m}{K} + \frac{m^2}{2K(m+s)} - \frac{s}{K} + \frac{ms}{2K(m+s)} \right] \right. \\
&\quad \left. + \frac{\beta fs^2}{m+s} \left[-\frac{s}{(m+s)} + \frac{fs}{(m+s)} - \frac{1}{K} + \frac{m}{K(m+s)} + \frac{s}{K(m+s)} \right] - \delta \right\} \\
\lambda'_3(t) &= -\lambda_1 \left\{ \frac{\beta fm^2}{2(m+s)} \left[-\frac{1}{m+s} + \frac{f}{K(m+s)} + \frac{m}{K(m+s)} - \frac{1}{K} + \frac{s}{K(m+s)} \right] \right\} \\
&\quad - \lambda_2 \left\{ \frac{\beta fm^2}{2(m+s)} \left[-\frac{1}{m+s} + \frac{f}{K(m+s)} + \frac{1}{K(m+s)} - \frac{m}{K} + \frac{s}{K} \right] \right. \\
&\quad \left. + \frac{\beta fs}{m+s} \left[3s - \frac{s^2}{m+s} - \frac{3fs}{K} + \frac{fs^2}{K(m+s)} - \frac{2m}{K} + \frac{ms}{K(m+s)} - \frac{3s}{K} + \frac{s^2}{K(m+s)} \right] \right\} \\
&\quad + \lambda_3 \delta.
\end{aligned}$$

Now, looking at the Modified TYC model with cannibalism and investigating an optimal μ , let us consider the following objective function:

$$J_c(\mu) = \int_0^T -(f + m) - \frac{1}{2}\mu^2 dt.$$

We want to find an optimal μ , μ^* , such that

$$J_c(\mu^*) = \max_u \int_0^T -(f + m) - \frac{1}{2}\mu^2 dt.$$

The Hamiltonian for J_c is given by

$$H_c = -(f + m) - \frac{1}{2}\mu^2 + \lambda_1 \dot{f} + \lambda_2 \dot{m} + \lambda_3 \dot{s}.$$

We use the Hamiltonian to derive differential equations for the adjoint equations.

These are

$$\begin{aligned}
\lambda'_1(t) &= 1 - \lambda_1 \left\{ \frac{\beta m^2}{2K(m+s)} [K - 2f - m - s] - \delta + \epsilon s \right\} \\
&\quad - \lambda_2 \left\{ \frac{\beta m^2}{2K(m+s)} [K - 2f - m - s] + \frac{\beta s^2}{K(m+s)} [sK - 2fs - m - s] \right\} \\
&\quad + \lambda_3 \{ \epsilon_1 s \} \\
\lambda'_2(t) &= 1 - \lambda_1 \left\{ \frac{\beta f m}{m+s} \left[1 - \frac{1}{2K(m+s)} (mK + 2f + fmK + 3(m+s) + m^2K + 2s + msK) \right] \right\} \\
&\quad - \lambda_2 \left\{ \frac{\beta f m}{m+s} \left[1 - \frac{m}{2(m+s)} - \frac{f}{K} + \frac{fm}{2K(m+s)} - \frac{3m}{K} + \frac{m^2}{2K(m+s)} - \frac{s}{K} + \frac{ms}{2K(m+s)} \right] \right. \\
&\quad \quad \left. + \frac{\beta f s^2}{m+s} \left[-\frac{s}{(m+s)} + \frac{fs}{(m+s)} - \frac{1}{K} + \frac{m}{K(m+s)} + \frac{s}{K(m+s)} \right] - \delta \right\} \\
\lambda'_3(t) &= -\lambda_1 \left\{ \frac{\beta f m^2}{2(m+s)} \left[-\frac{1}{m+s} + \frac{f}{K(m+s)} + \frac{m}{K(m+s)} - \frac{1}{K} + \frac{s}{K(m+s)} \right] + \epsilon f \right\} \\
&\quad - \lambda_2 \left\{ \frac{\beta f m^2}{2(m+s)} \left[-\frac{1}{m+s} + \frac{f}{K(m+s)} + \frac{1}{K(m+s)} - \frac{m}{K} + \frac{s}{K} \right] \right. \\
&\quad \quad \left. + \frac{\beta f s}{m+s} \left[3s - \frac{s^2}{m+s} - \frac{3fs}{K} + \frac{fs^2}{K(m+s)} - \frac{2m}{K} + \frac{ms}{K(m+s)} - \frac{3s}{K} + \frac{s^2}{K(m+s)} \right] \right\} \\
&\quad + \lambda_3 \{ \delta + \epsilon_1 f \}.
\end{aligned}$$

The optimal control for the modified TYC and the modified TYC with cannibalism are being investigated using the forward-backward sweep method at the time of publication.

6 SUMMARY AND FUTURE WORK

In summary, the Trojan Y Chromosome eradication strategy can lead to a species extinction, but the mathematical model can lead to finite time blow up as well as having negative solutions. Modifying the TYC strategy can help alleviate some of these issues, although they are still there. Adding environmental pressures/cannibalism can lead to a population thought to head towards extinction to bounce back into recovery. The proposed pairing model seems to alleviate the problem with finite time blow up and negative solutions, but further linear stability analysis must be done. Finally, we should further investigate the proposed pairing model and the optimal μ . With the latter in mind, let us consider the following objective function:

$$J_p(\mu) = \int_0^T -(f + m) - \frac{1}{2}\mu^2 dt.$$

We want to find an optimal μ , μ^* , such that

$$J_p(\mu^*) = \max_u \int_0^T -(f + m) - \frac{1}{2}\mu^2 dt.$$

The Hamiltonian for J_p is given by

$$H_p = -(f + m) - \frac{1}{2}\mu^2 + \lambda_1 \dot{P} + \lambda_2 \dot{m} + \lambda_3 \dot{f} + \lambda_4 \dot{P}_s + \lambda_5 \dot{s}.$$

We use the Hamiltonian to derive differential equations for the adjoint equations. These are

$$\begin{aligned} \lambda_1'(t) &= -\lambda_1(-\sigma - \delta_m - \delta_f) - \lambda_2(\beta_m + \sigma + \delta_f) - \lambda_3(\beta_f + \sigma + \delta_m) \\ \lambda_2'(t) &= 1 - \lambda_1(\phi_m(f, m)) - \lambda_2(-\phi_m(f, m) - \delta_m) - \lambda_3(-\phi_m(f, m)) \\ \lambda_3'(t) &= 1 - \lambda_1(\phi_f(f, m)) - \lambda_2(-\phi_f(f, m)) - \lambda_3(-\phi_f(f, m) - \phi_f(f, s) - \delta_f) \\ &\quad - \lambda_4(\phi_f(f, s)) - \lambda_5(-\phi_f(f, s)) \\ \lambda_4'(t) &= -\lambda_2(\beta_s) - \lambda_3(\sigma + \delta_s) - \lambda_4(-\sigma - \delta_f - \delta_s) - \lambda_5(\sigma - \delta_f) \\ \lambda_5'(t) &= -\lambda_3(-\phi_s(f, s)) - \lambda_4(\phi_s(f, s)) - \lambda_5(-\phi_s(f, s) - \delta_s). \end{aligned}$$

Using the forward and backward sweep method, attempt to find an optimal introduction rate for the modified TYC, the modified TYC with cannibalism, and the pairing model.

BIBLIOGRAPHY

- [1] What are invasive species? — national invasive species information center — usda.
- [2] E. A. BEAUREGARD, MATTHEW A., Optimal control and analysis of a modified trojan y-chromosome strategy., *Ecological Modelling*, (2020).
- [3] W. H. CASTILLO-CHAVEZ, CARLOS AND J. LI., On the existence of stable pairing distributions., *Journal of Mathematical Biology*, 34 (1996), pp. 413–441.
- [4] K. DIETZ AND K. P. HADELER., Epidemiological models for sexually transmitted diseases., *Journal of Mathematical Biology*, 26 (1988), pp. 1–25.
- [5] E. A. GUTIERREZ, JUAN B., Analysis of the trojan y chromosome model for eradication of invasive species in a dendritic riverine system., *Journal of Mathematical Biology*, 64 (2012), pp. 319–340.
- [6] J. B. GUTIERREZ AND J. L. TEEM., A model describing the effect of sex-reversed yy fish in an established wild population: the use of a trojan y chromosome to cause extinction of an introduced exotic species., *Journal of Theoretical Biology*, 241 (2006), pp. 333–341.
- [7] D. G. KENDALL, Stochastic processes and population growth., *Journal of the Royal Statistical Society. Series B (Methodological)*, 11 (1949), pp. 230–282.
- [8] N. KEYFITZ, The mathematics of sex and marriage., *Proceedings of the sixth Berkeley symposium on mathematical statistics and probability.*, 4 (1972).
- [9] S. LENHART AND J. T. WORKMAN., Optimal control applied to biological models., CRC press, 2007.
- [10] E. A. LYU, JINGJING, A comparison of the trojan y chromosome strategy to harvesting models for eradication of nonnative species., *Natural Resource Modeling*, (2019).
- [11] W. RUDIN, Principles of Mathematical Analysis., vol. 3., McGraw-hill., 1964.
- [12] P. SCHOFIELD, Personal communication.
- [13] C. STREET, Invasive species definition clarification and guidance., (2006).

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