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STRICTIFYING AND TAMING DIRECTED PATHS IN HIGHER DIMENSIONAL AUTOMATA

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Abstract. Directed paths have been used by several authors to describe concurrent executions of a program. Spaces of directed paths in an appropriate state space contain executions with all possible legal schedulings. It is interesting to investigate whether one obtains different topological properties of such a space of executions if one restricts attention to schedulings with "nice" properties, eg involving synchronizations. This note shows that this is not the case, ie that one may operate with nice schedulings without inflicting any harm.

Several of the results in this note had previously been obtained by Ziemiański in [17, 18]. We attempt to make them accessible for a wider audience by giving an easier proof for these findings by an application of quite elementary results from algebraic topology; notably the nerve lemma.

1. Introduction

1.1. **Schedules in Higher Dimensional Automata.** Higher Dimensional Automata (HDA) were introduced by V. Pratt [13] back in 1991 as a model for concurrent computation. Mathematically, HDA can be described as (labelled) pre-cubical or $\square$-sets (cf Definition 2.1). Those are obtained by glueing individual cubes of various dimensions together; directed paths corresponding to a $\square$-set respect the natural partial order in each cube of the model. These directed paths correspond to lawful schedules/executions of a concurrent computation; and paths that are homotopic in a directed sense (d-homotopic, cf [3]), will always lead to the same result.

Compared to other well-studied concurrency models like labelled transition systems, event structures, Petri nets etc., it has been shown by R.J. van Glabbeek [7] that Higher Dimensional Automata have the highest expressivity; on the other hand, they are certainly less studied and less often applied so far.

It is not evident which paths one should admit as directed paths: It is obvious that they should progress along each axis in each facet of the HDA; the time flow is not reversible. This is reflected in the notion of a *d-path* on such a complex. One may ask, moreover, that processes synchronize after a step (either a full step or an idle step) has been taken. This is what *tame d-paths* have to satisfy, on top. A natural question to ask is whether one can perform the same computations (and obtain the same results) according to whether synchronization is requested all along or not.

It has been shown by K. Ziemiański [17, 18] that the synchronization request has no essential significance: The spaces of directed paths and of tame d-paths between two states are always

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homotopy equivalent. This has two consequences: On the one hand, one may, without global effects, relax the computational model and allow quite general parallel compositions. On the other hand, in the analysis of the schedules on a HDA, one may restrict attention to tame d-paths, ie mandatory synchronization; these are combinatorially far easier to model and to analyze.

1.2. **Posets, poset categories, and algebraic topology.** Many (sets of) schedules can be formulated in the language of series-parallel pomsets (of events). Tame d-paths "live in" *serial* compositions of simple Higher Dimensional Automata consisting of a single cube each. General d-paths underpin more complicated schedules, for which *parallel* composition is involved in the description; cf eg [5] for a detailed description of finite step transition systems accepting pomset languages and [2] for newer developments.

In this paper, we are not interested in the analysis of individual paths/schedules, but in the analysis of the space of *all* schedules from a start state to an end state, equipped with a natural topology. It turns out that the way subspaces of schedules are glued together is essentially the same, whether synchronization is mandatory or not.

In that line of argument, posets enter the scene in a different manner: We divide the space of all executions (d-paths) into easy-to-analyze subspaces; for tame d-paths, for example, we simply fix a sequence of faces that they are kept in. Refinement is a partial order relation on these face sequences, and we will exploit the combinatorial/topological properties of the poset category of face sequences (called cube chains).

The use of methods from algebraic topology in the analysis of concurrency properties has been advocated in eg [4, 9, 3] to which we refer the reader for details. In this paper, we will (apart from the proof of Proposition 6.6) only apply one important result from algebraic topology, the so-called nerve lemma [1, 10], cf Theorem 6.2. At a first glance, one may say that it allows to apply a divide and conquer strategy: Cut a space into subspaces that are topologically trivial (contractible); also all non-empty intersections of such are assumed contractible. Then all essential information (up to homotopy equivalence) is contained in the way these subspaces are glued together. That glueing can be described by way of a simplicial complex, the *nerve* of the associated poset category. If the posets associated to different spaces are (naturally) isomorphic, then their nerves and hence the spaces they describe are homotopy equivalent.

2. Definitions and results

2.1. **Definitions.** We start with some notation: The unit interval $[0,1]$ is denoted by I. For two topological spaces X and Y, we let Y^X denote the space of all continuous maps from X to Y equipped with s compact open-topology. For an interval $J=[a,b]\subset \mathbf{R},\ a<b$, an element $p\in X^J$ is called a *path* in X. A path $\varphi\in J_1^{J_2}$ in an interval J_1 defines a *reparametrization map* $X^{J_1}\to X^{J_2}, p\mapsto p\circ\varphi$.

Let $p_0:[t_0,t_1]\to X$ and $p_1:[t_1,t_2]\to X$ denote two paths with $p_0(t_1)=p_1(t_1)$. Their *concatenation* at t_1 is denoted $p_0 *_{t_1} p_1 : [t_0,t_2]\to X$.

Definition 2.1. (1) [8, 9] A d-space consists of a topological space X together with a subspace $\vec{P}(X)\subset X^I$ of paths in X that contains the constant paths, is closed under concatenation and under *non-decreasing* reparametrizations $p: I\to I$. Elements of $\vec{P}(X)$ are called d-paths.

(2) For $x, y \in X$, we let $\vec{P}(X)_x^y = \{p \in \vec{P}(X) | \, p(0) = x, p(1) = y\}$ denote the subspace of all d-paths from x to y.
(3) A continuous map $f : X \to Y$ is called a *directed* map if $f(\vec{P}(X)) \subset \vec{P}(Y)$, ie if it maps d-paths in X into d-paths in Y.
(4) Let $J = [a, b] \subset \mathbf{R}$ denote an interval $(a < b)$ and let $\varphi : J \to I$ denote any *increasing homeomorphism*. Then $\vec{P}_J(X) := \{p \circ \varphi | \, p \in \vec{P}(X)\}$ – independent of the choice of φ.

In applications to concurrency, the d-spaces under consideration are usually directed $\square$-sets, or rather their geometric realizations [13, 6, 7, 3]:

Definition 2.2. (1) A $\square$*-set* X (also called a pre-cubical or semi-cubical set) is a sequence of disjoint sets X_n, $n \geq 0$; equipped, for $n > 0$, with *face maps* $d_i^\alpha : X_n \to X_{n-1}$, $\alpha \in \{0, 1\}, 1 \leq i \leq n$, satisfying the pre-cubical relations: $d_i^\alpha d_j^\beta = d_{j-1}^\beta d_i^\alpha$ for $i < j$.
Elements of X_n are called n-cubes, those of X_0 are called vertices.
(2) A $\square$-set X is called *proper* [17] if for every pair $x_0, x_1 \in X_0$ of vertices there exists at most one cube with bottom vertex x_0 and top vertex x_1 (cf also Section 4.1).
(3) A $\square$-set is called *non-self-linked* [3] if every cube $c \in X_n$ has $\binom{n}{k} 2^{n-k}$ different iterated faces in X^k (ie, iterated faces agree if and only they do so because of the pre-cubical relations).
(4) The *geometric realization* of a pre-cubical set X is the space

$$|X| = \bigcup_{n \geq 0} X_n \times I^n /_{[d_i^\alpha(c), x] \sim [c, \delta_i^\alpha(x)]}$$

with $\delta_i^\alpha(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{i-1}, \alpha, x_i, \ldots, x_{n-1})$.

It seems that typical Higher-Dimensional Automata as they occur in concurrency are proper and non-self-linked.

Speaking about a cube c in X (or rather in $|X|$; we will often suppress $||$ from the notation), we mean actually the image of the quotient map $\{c\} \times I^{\dim c} \hookrightarrow \bigcup_{n \geq 0} X_n \times I^n \downarrow |X|$. If X is non-self-linked, then this map is a homeomorphism onto its image in X; if not, then it may identify points on the boundary of $I^{\dim c}$.

What are the *directed* paths in the geometric realization of a $\square$-set?

Definition 2.3. (1) A path $p : J \to I$ on an interval J is called *strictly* increasing if it is increasing and moreover satisfies: $p(t) = p(t') \Rightarrow t = t'$ or $p(t) = 0$ or $p(t) = 1$.
(2) A path $p = (p_1, \ldots, p_n) : J \to I^n$ on an interval J is called *(strictly) increasing* if every component p_i is (strictly) increasing.
(3) Let X denote a $\square$-set. A path $p \in X^I$ is called a *d-path* if it admits a presentation [18] $[c_1; \beta_1] *_{t_1} [c_2; \beta_2] *_{t_2} \cdots *_{t_{l-1}} [c_l; \beta_l]$ consisting of a sequence (c_i) of cubes in X, a sequence $(\beta_i) \in \vec{P}_{[t_{i-1}, t_i]}(I^{\dim c_i})$, $1 \leq i \leq l$, of *increasing* paths β_i such that $0 = t_0 \leq t_1 \leq \ldots t_{i-1} \leq t_i \leq \cdots \leq t_l = 1$ and $p(t) = [c_i; \beta_i(t)], t_{i-1} \leq t \leq t_i$; ie, on this interval, $p = q_i \circ \beta_i$ with $q_i : I^{\dim c_i} \to c_i$ the resp. quotient map.
(4) A d-path $p : I \to X$ is called *strictly* directed if there exists a presentation $p = [c_1; \beta_1] *_{t_1} [c_2; \beta_2] *_{t_2} \cdots *_{t_{l-1}} [c_l; \beta_l]$ with *strictly* increasing paths $\beta_i : [t_{i-1}, t_i] \to I^{\dim c_i}$.
(5) A directed path $p : I \to X$ is called *tame* if the subdivision in (3) above can be chosen such that $p(t_i)$ is a *vertex* for every $0 \leq i \leq l$. A path that is strictly directed and tame is called strictly tame.

Observe that we allow d-paths that include non-trivial directed *loops*.

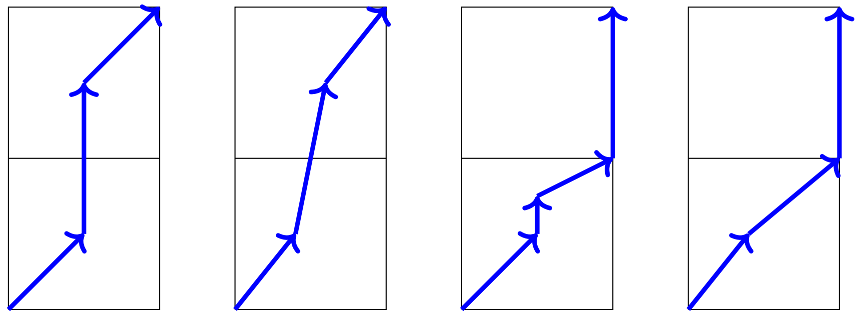

Figure 1. d-paths in a cubical complex consisting of two squares: directed, strict, tame, tame and strict.

Example 2.4. (1) A Euclidean cubical complex [16] K is a subset of Euclidean space $\mathbf{R}^n$ that is a union of elementary cubes $\prod[k_i, k_i + e_i] \subset \mathbf{R}^n$ with $k_i \in \mathbf{Z}$ and $e_i \in \{0, 1\}$. The maximal cubes in the cubical set that it realizes can be described by a pair of bottom and top vertices $(\mathbf{k}, \mathbf{l}) \in \mathbf{Z}^n \times \mathbf{Z}^n$ with $0 \leq l_i - k_i \leq 1$. A Euclidean cubical complex is obviously proper and non-self-linked. Euclidean cubical complexes arise as models for PV-programs (cf eg [3]).

(2) For an example of a non-proper cubical set, consider the cubical set X glued from two squares (2-cubes) along a common boundary (consisting of four oriented edges and of four vertices). Its geometric realization is homeomorphic to a 2-dimensional sphere; but remark that the directed paths on this sphere are quite peculiar. The space of all such directed paths is actually homotopy equivalent to a circle.

(3) [19] The $\square$-set Z_n with exactly one cube in every dimension $k \leq n$ is obviously self-linked. For a description of d-paths in the geometric realization of this space, cf Example 6.5.

(4) [19] The $\square$-set Q^n has $(n - k + 1)$ k-cubes $c_0^k, \dots, c_{n-k}^k$ and face maps $d_i^e c_j^k = c_{j+i+e}^{k-1}$. It arises from the cube I^n by identifying all faces spanned by two vertices with i, resp. $k + i$ coordinates 1 with each other ($0 \leq i \leq n - k$). This $\square$-set is proper, but also self-linked.

2.2. **Interpretation.**

2.2.1. *Different types of schedulings.*

D-(irected): paths correspond to executions of a concurrent program - without the possibility to let one or several processes run backwards in time.

Strict: d-paths correspond to programs where a particular process only may be idle at a vertex in the program (once a step is fully taken); between steps it needs to move forward in time at "positive speed".

Tame: d-paths correspond to programs where processes need to synchronize at every vertex (a number of processes may stay idle inbetween) before progressing; at synchronization events, a process has taken a full step or it has stayed idle.

Strict tame: d-paths correspond to programs combining both properties.

Our main result in Theorem 2.6 below states that the spaces of schedulings, regardless of the restrictions above, will have the same topological properties in all four cases.

In the final Section 6.3 we show that one may restrict (up to homotopy equivalence) the space of tame d-paths even further: It is enough to consider Pl d-paths that are piecewise linear with kink points at certain hyperplanes.

2.2.2. *A simple illustrative example.* We refer to Figure 2. Let $X = \partial I^3$ be the $\square$-set corresponding to the boundary of a 3-cube. It has twelve edges: four parallel to each of the axes and labelled x, y resp. z and six two-dimensional facets: two parallel to each of the three coordinate planes and labelled xy, xz resp. yz.

The image of every d-path from the bottom vertex $\mathbf{0}$ to the top vertex $\mathbf{1}$ is contained in two subsequent square facets; the image of every *tame* d-path from $\mathbf{0}$ to $\mathbf{1}$ is contained in a pair of an edge and a facet. Taking care of intersections, one arrives in both cases at a category with geometric realization in form of a hexagon; homotopy equivalent to the circle S^1. The space of all d-paths (whether tame or not) in ∂I^n is indeed homotopy equivalent to S^{n-2}.

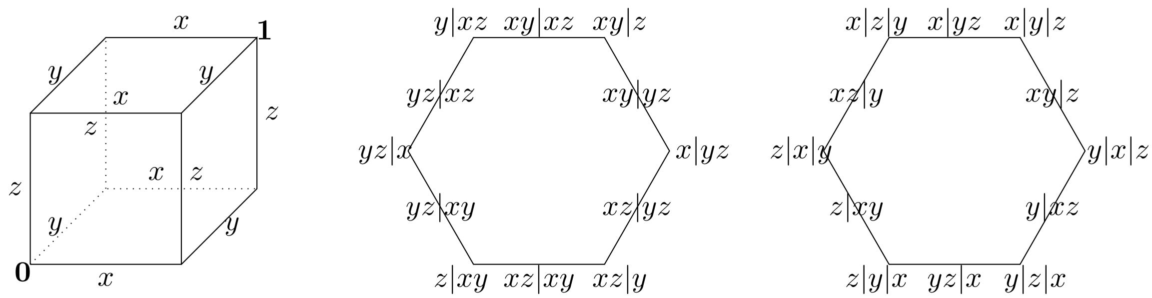

FIGURE 2. $X = \partial I^3$, and spaces of d-paths, resp. of tame d-paths

The $\square$-set $X = \partial I^3$ models the situation where a shared resource can serve two out of three processes but not all of them at the same time. Remark that a sequence like $xy|xz$ (on top of Figure 2) of two subsequent facets can be interpreted as $x \parallel (y|z)$, ie x and $y|z$ are executed concurrently. Allowing this may be very convenient and speed up a concurrent execution. For an analysis of the consequences, it is reassuring to realize that the space of schedules between two given states is qualitatively the same regardless whether one allows parallel execution over a series of steps (like in $x \parallel (y|z)$) or only over one step at a time (like in yz).

2.3. **Main result.** Let X denote a $\square$-set with finitely many cubes. For a given pair of vertices $x^-, x^+ \in X_0$, we let $\vec{P}(X)_{x^-}^{x^+}, \vec{S}(X)_{x^-}^{x^+}, \vec{T}(X)_{x^-}^{x^+}$, resp. $\vec{ST}(X)_{x^-}^{x^+}$ denote the spaces of directed, strictly directed, tame, and strictly tame dipaths from x^- to x^+ (considered as subspaces of X^I with the compact-open topology). Inclusion maps lead to the commutative diagram

(2.5)
$$\begin{tikzcd}
& {|\mathcal{C}(X)_{x^-}^{x^+}|} & \\
\vec{ST}(X)_{x^-}^{x^+} \arrow[ur, "5"] \arrow[rr, hook, "3"] \arrow[d, hook, "1"] & & \vec{S}(X)_{x^-}^{x^+} \arrow[ul, "6"'] \arrow[d, hook, "2"] \\
\vec{T}(X)_{x^-}^{x^+} \arrow[rr, hook, "4"] & & \vec{P}(X)_{x^-}^{x^+}.
\end{tikzcd}$$

that also contains (maps into) the nerve of a poset-category $\mathcal{C}(X)_{x^-}^{x^+}$ explained in the sketch of the proof of our result:

Theorem 2.6. (1) *All inclusion maps in (2.5) are homotopy equivalences.*

(2) *For a proper non-self-linked $\square$-set X (cf Definition 2.2(2)), all path spaces are homotopy equivalent to the nerve of the category $\mathcal{C}(X)_{x^-}^{x^+}$.*

Overview proof. It is shown in Proposition 3.4 by a cube-wise strictification construction that the maps with labels ① and ② are homotopy equivalences.

For a $\square$-set X and chosen end points x^-, x^+, we define a poset category $\mathcal{C}(X)_{x^-}^{x^+}$, cf Section 4.3: An object of that category is a *cube chain* (cf Definition 4.11(4)) in X connecting x^- with x^+; this a sequence of cubes such that the top vertex of each cube in that sequence agrees with the bottom vertex of the subsequent cube. Morphisms in $\mathcal{C}(X)_{x^-}^{x^+}$ correspond then to *refinements* of cube chains; for details consult Section 4.3.

We prove in Proposition 6.4 for paths in a *proper non-self-linked* $\square$-set X (cf Definition 2.2(2-3)) that both $\vec{S}(X)_{x^-}^{x^+}$ and $\vec{ST}(X)_{x^-}^{x^+}$ are homotopy equivalent to the nerve of $\mathcal{C}(X)_{x^-}^{x^+}$ (indicated by the maps ⑤ and ⑥) and can therefore deduce that also ③ is a homotopy equivalence: Both spaces have a common underlying combinatorial structure!

Our proof uses only the classical nerve lemma [1, 10], cf Theorem 6.2, and a transparent taming construction (Proposition 5.1) for strict d-paths subordinate to the *collar* of a cube chain (cf Definition 4.5). The remaining inclusion ④ is a homotopy equivalence as well by the 2-out-of-3 property for homotopy equivalences. In the more involved case of a *general* $\square$-set X, we show in Proposition 6.6 that ③ is a homotopy equivalence using the projection lemma and the homotopy lemma (cf eg [12]) underlying the proof of the nerve lemma. $\square$

Remark 2.7. Many of the results in this paper are not new. Ziemianski proved in [18], using elaborate homotopy theory tools, that the space of tame d-paths $\vec{T}(X)_{x^-}^{x^+}$ is always homotopy equivalent to the nerve of a more intricate category $Ch(X)$ of cube chains, even for a general $\square$-set X. This Reedy category (cf [11]) takes care of identifications on the boundary of cubes in a cube chain. Moreover, he shows by an ingenious global taming construction, that ④ is a homotopy equivalence. Apart from including spaces of strictly increasing paths (necessary in our proof for taming), this note presents a far more elementary argument that, for proper $\square$-sets, only uses the nerve lemma.

3. Strictification

3.1. Strictifying directed maps on $\square$-sets.

Lemma 3.1. *There exists a (continuous) directed map $F : I \times I \to I$ (cf Definition 2.1(3)) with the following properties:*

1. $t \in I \Rightarrow F(t,0) = 0, F(t,1) = 1.$
2. $0 < x_0 < 1,\ t \in I \Rightarrow 0 < F(t,x_0) < 1.$
3. $x_0 < y_0,\ t \in I \Rightarrow F(t,x_0) < F(t,y_0).$
4. $s,t \in I,\ s < t,\ 0 < x_0 < 1 \Rightarrow F(s,x_0) < F(t,x_0).$

Proof. One way to construct such a directed map is as the restriction of the flow of the differential equation $y' = g(y)$ corresponding to a smooth function $g : I \to \mathbf{R}$ with $g(0) = g(1) = 0$ and $g(t) > 0,\ 0 < t < 1$, e.g. $g(t) = t - t^2$. For this choice, the function given by $F(t,x) = \frac{xe^t}{1-x+xe^t}$ has the required properties. $\square$

The homotopy F on the interval I from Lemma 3.1 can be used to define a diagonal continuous directed homotopy $\mathbf{F} : I \times I^n \to I^n$ on the cube I^n by $\mathbf{F}(t; x_1, \dots, x_n) = (F(t, x_1), \dots, F(t, x_n))$. Remark that $\mathbf{F}$ respects all (sub)-faces of I^n because of Lemma 3.1(1). Applying this construction cube-wise (the same for every k-cube!), we define for every (geometric realization of a) semi-cubical set X, a continuous directed map $\mathbf{F} : I \times X \to X$ that lets all cubes – and in particular all vertices – invariant.

3.2. **Strictification is a homotopy equivalence.** Using such a directed map $\mathbf{F}$, we define a map $\vec{S} : \vec{P}(X)_{x^-}^{x^+} \to \vec{S}(X)_{x^-}^{x^+}$ by $\vec{S}(p)(t) := \mathbf{F}(t, p(t))$.

Lemma 3.2. *Let $p \in \vec{P}(X)_{x^-}^{x^+}$.*

1. *If $p(t) \in c$ for some cube c in X, then $\vec{S}(p)(t) \in c$ for all $t \in I$.*
2. *$\vec{S}(p) \in \vec{S}(X)_{x^-}^{x^+}$.*
3. *If p is tame, then $\vec{S}(p)$ is (strict and) tame.*

Proof. (1) follows from Lemma 3.1(1). In particular, if $p(t)$ is a vertex, then $\vec{S}(p)(t) = p(t)$ is the same vertex.

(2) Let $t < t'$ be such that $p[t, t']$ is contained in a cube c; on that interval, we may write $p(s) = \pi_c(\tilde{p}(s))$ for a d-path $\tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_n)$ in I^n. If $t < t'$ then $\tilde{p}_i(t) \le \tilde{p}_i(t')$. Assume $\tilde{p}_i(t) \neq 0, 1$. If $\tilde{p}_i(t') = 1$, then $F(t, \tilde{p}_i(t)) < 1 = F(t', \tilde{p}_i(t'))$. Otherwise, $F(t, \tilde{p}_i(t)) \le F(t, \tilde{p}_i(t')) < F(t', \tilde{p}_i(t'))$ by Lemma 3.1(3-4). Hence $\vec{S}(p)$ is strict on $[t, t']$.

(3) is a consequence of (1) and (2).

$\square$

Lemma 3.3. *For a finite $\square$-complex, the map $\vec{S} : \vec{P}(X)_{x^-}^{x^+} \to \vec{S}(X)_{x^-}^{x^+}$ is continuous (in the compact open topologies).*

Proof. The metric d_1 corresponding to the l^1-norm on individual cubes extends to a metric d_1 on X (the distance between two points being the infimum over the sum of distances on connecting chains) that induces the topology on X. The compact open topology on a function space with target X corresponds to the uniform convergence topology with respect to that metric d_1.

If X is a finite complex, then $I \times X$ is compact, and hence the restriction $\mathbf{F} : I \times X \to X$ is uniformly continuous with respect to the metrics induced by d_1. $\square$

Proposition 3.4. *Let X denote a finite pre-cubical set with vertices x^- and x^+. Then the inclusions $\iota : \vec{S}(X)_{x^-}^{x^+} \hookrightarrow \vec{P}(X)_{x^-}^{x^+}$ and its restriction $\iota_T : \vec{ST}(X)_{x^-}^{x^+} \hookrightarrow \vec{T}(X)_{x^-}^{x^+}$ are homotopy equivalences.*

Proof. The homotopy $\mathcal{S} : I \times \vec{P}(X)_{x^-}^{x^+} \to \vec{P}(X)_{x^-}^{x^+}$ given by $\mathcal{S}(s, p)(t) = \mathbf{F}(st, p(t))$ connects the identity map (for $s = 0$) with the map $\iota \circ S$ (for $s = 1$). Its restriction to $\vec{S}(X)_{x^-}^{x^+}$ connects the identity map on that space with $S \circ \iota$.

The restriction of $\mathcal{S}$ to a map from $\vec{T}(X)_{x^-}^{x^+}$ to $\vec{ST}(X)_{x^-}^{x^+}$ (well-defined because of Lemma 3.2(3)) is a homotopy inverse to the inclusion map $\vec{ST}(X)_{x^-}^{x^+} \hookrightarrow \vec{T}(X)_{x^-}^{x^+}$. $\square$

Remark 3.5. (1) Every d-path $p \in \vec{P}(X)_{x^-}^{x^+}$ can thus be arbitrarily well approximated by a strict d-path of the form $\mathcal{S}(s, p)$, $s > 0$. This shows that $\vec{S}(X)_{x^-}^{x^+}$ is *dense* in $\vec{P}(X)_{x^-}^{x^+}$.

(2) But $\vec{S}(X)_{x^-}^{x^+}$ is *not open* in $\vec{P}(X)_{x^-}^{x^+}$. Arbitrarily close to any strict d-path there is a d-path that "pauses" on a tiny interval.

4. Cube chains and collars

4.1. **The collar of a face in a cube.** We propose a user-friendly notation for repeated face maps in a cube – and then in a $\square$-set: Every partition $[1:n] = I_0 \sqcup I_* \sqcup I_1$ defines a face $d_{[I_0|I_*|I_1]}I^n = \{0\}^{I_0} \times I^{I_*} \times \{1\}^{I_1}$ of the cube I^n. Its (open) *collar* $C_{[I_0|I_*|I_1]}I^n$ is defined as $[0,0.5[^{I_0} \times I^{I_*} \times]0.5,1]^{I_1} \subset I^n$. In particular, the bottom vertex $\mathbf{0}$ is identified with $d_{[[1:n]|\emptyset|\emptyset]}I^n$ and the top vertex with $\mathbf{1} = d_{[\emptyset|\emptyset|[1:n]]}I^n$. Remark that the only vertices in a collar $C_{[I_0|I_*|I_1]}I^n$ are those that are already present in the face $d_{[I_0|I_*|I_1]}I^n$.

For a $\square$-set X, an n-cell c in X and a partition $I_0 \sqcup I_* \sqcup I_1$, the combinatorics of the quotient map $q : I^n \to c$ gives rise to a face $d_{[I_0|I_*|I_1]}c = q(d_{[I_0|I_*|I_1]}I^n)$ with collar $C_{[I_0|I_*|I_1]}c = q(C_{[I_0|I_*|I_1]}I^n)$. Remark that (for a self-linked $\square$-set), different partitions (of the same cardinality) can give rise to the same face.

If d and c are cubes in X, the collar of d in c is defined as $C(d,c) = \bigcup_{[I_0|I_*|I_1]} C_{[I_0|I_*|I_1]}c$; the union is taken over all $[I_0|I_*|I_1]$ such that $d_{[I_0|I_*|I_1]}c$ is a face of d, including d itself. The collar $C(d,X) = \bigcup_{c \in X_n, n \geq 0} C(d,c)$ of d in X agrees with a regular neighbourhood of d with respect to a barycentric subdivision of the $\square$-set X. The collar $C(x,X)$ of a vertex x is called the *star* $st(x)$ of x in X. For simple illustrations, cf Figure 3.

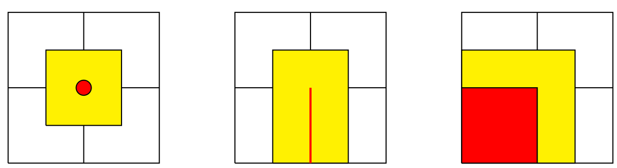

FIGURE 3. Star of a vertex, collar of an edge and of a 2-cube in a Euclidean cubical complex consisting of four 2-cubes

Remark 4.1. (1) The collar $C(d,X)$ of a face d in a $\square$-set X is open since it intersects every cube in X in an open set.

(2) If c is a face of d, then $C(c,X) \subseteq C(d,X)$.

4.2. **D-paths subordinate to a cube chain.**

Definition 4.2. Let X denote a $\square$-set with two vertices $x^-, x^+ \in X_0$ selected.

(1) A *cube chain* $\mathbf{c} = (c_1, \dots, c_n)$ in X from x^- to x^+ [17] is a sequence of cubes c_i, $0 \leq i \leq n$, $\dim c_i > 0$, such that $c_{1,\mathbf{0}} = x^-, c_{n,\mathbf{1}} = x^+, c_{i-1,\mathbf{1}} = c_{i,\mathbf{0}} = x_i$, $1 \leq i \leq n$.

(2) For a *cube chain with vertices*, we require only $\dim c_i \geq 0$, ie we allow vertices within such a cube chain.

(3) A cube d in X is a *coface* of a cube chain $\mathbf{c}$ with vertices in X if it is a coface of at least one cube c_i.

(4) The *length* of a cube chain $\mathbf{c}$ is defined as $|\mathbf{c}| = \sum_{i=1}^n \dim c_i$.

Remark 4.3. (1) To every cube chain with vertices one may associate a cube chain without vertices by simply leaving out every vertex.

(2) A cube chain in a $\square$-set X defines a vertex sequence $(x^- = x_0, x_1, \dots, x_n, x_{n+1} = x^+)$ but, if X is not proper, not the other way round: More than one cube may share the same bottom and top vertex.

Let us for a moment focus on a single n-cube c in a $\square$-set X (with quotient map $q : I^n \to c$) and on certain d-paths in its collar $C(c, X)$ connecting a point in $st(c_{\mathbf{0}})$ with a point in $st(c_{\mathbf{1}})$:

Definition 4.4. A d-path $p \in \vec{P}(C(c, X))$ is called *subordinate to the collar* of c if there exist

- a cube chain $(d_i)_1^l$ in X with quotient maps $q_i : I^{\dim d_i} \to d_i$,
- partitions $I_0^i \sqcup I_*^i \sqcup I_1^i$ of $\dim d_i$ such that $(d_{[I_0^i|I_*^i|I_1^i]} d_i)_1^l$ is a cube chain (with vertices) from $c_{\mathbf{0}}$ to $c_{\mathbf{1}}$ within c,
- a presentation (cf Definition 2.3) $p = [d_1; \beta_1] *_{s_1} [d_2; \beta_2] *_{s_2} \cdots *_{s_{l-1}} [d_l; \beta_l]$ with $\beta_i \in \vec{P}_{[s_{i-1}, s_i]}(C_{[I_0^i|I_*^i|I_1^i]} I^{\dim d_i})_{x_i^-}^{x_i^+}$ and $x_i^{\pm}$ is contained in the star of the vertices $d_{[I_0^i \cup I_*^i|\emptyset|I_1^i]} I^{\dim d_i}$, resp. $d_{[I_0^i|\emptyset|I_*^i \cup I_1^i]} I^{\dim d_i}$.

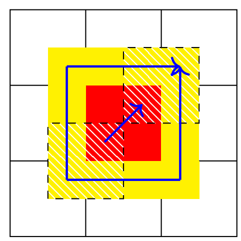

FIGURE 4. Cube c in red, collar $C(c)$ in yellow, stars $st(c_{\mathbf{0}})$ and $st(c_{\mathbf{1}})$ dashed. Several d-paths subordinate to c in blue.

Now let $\mathbf{c} = (c_1, \dots, c_n)$ denote a cube chain in a $\square$-set X with associated vertex sequence $(x^- = x_0, x_1, \dots, x_n, x_{n+1} = x^+)$.

Definition 4.5. Let $\mathbf{c} = (c_1, \dots, c_n)$ denote a cube chain in a $\square$-set from x^- to x^+ via $x_1, \dots, x_{n-1}$.

(1) The subspace $\vec{T}_{\mathbf{c}}(X) \subset \vec{T}(X)$ of d-paths subordinate to $\mathbf{c}$ consists of d-paths with presentation $p = [c_1; p_1] *_{t_1} \cdots *_{t_{n-1}} [c_n; p_n]$ with $p_i \in \vec{P}_{[t_{i-1}, t_i]}(I^{\dim c_i})_{\mathbf{0}}^{\mathbf{1}}$.
(2) The subspace $\vec{P}_{\mathbf{c}}^C(X) \subset \vec{P}(X)$ of d-paths subordinate to the *collar* $C(\mathbf{c})$ of $\mathbf{c}$ consists of d-paths $p = p_1 *_{t_1} \cdots *_{t_{n-1}} p_n$ with p_j subordinate to the collar of c_j in X.
(3) $\vec{ST}_{\mathbf{c}}(X)_{x^-}^{x^+} := \vec{T}_{\mathbf{c}}(X) \cap \vec{S}(X)$, $\vec{S}_{\mathbf{c}}^C(X)_{x^-}^{x^+} := \vec{P}_{\mathbf{c}}^C(X) \cap \vec{S}(X)_{x^-}^{x^+}$, $\vec{T}_{\mathbf{c}}^C(X)_{x^-}^{x^+} := \vec{P}_{\mathbf{c}}^C(X) \cap \vec{T}(X)_{x^-}^{x^+}$, $\vec{ST}_{\mathbf{c}}^C(X)_{x^-}^{x^+} := \vec{P}_{\mathbf{c}}^C(X) \cap \vec{ST}(X)_{x^-}^{x^+}$.

Remark 4.6.
For every d-path subordinate to the collar of $\mathbf{c}$, there exists a covering of the interval I by open intervals K_i such that $p_i(K_i) \subset C(c_i, X)$ and $p_i(K_i \cap K_{i+1}) \subset st(x_{i+1})$.

Example 4.7. Let Z_2 denote the 2-dimensional $\square$-set with one 0-cell v, one 1-cell e and one 2-cell c with quotient map $q : I^2 \to Z_2$ (cf Example 2.4(3)). The star of v has the form $st(v) = q(I^2 \setminus \{(x, y) |\ x \neq 0.5, y \neq 0.5\})$. The edge e has collar $C(e, X) = q(I^2 \setminus \{(0.5, 0.5)\})$. A d-path $p \in \vec{P}(Z_2)$ in X is subordinate to the collar of the cube chain (e) (consisting of a single

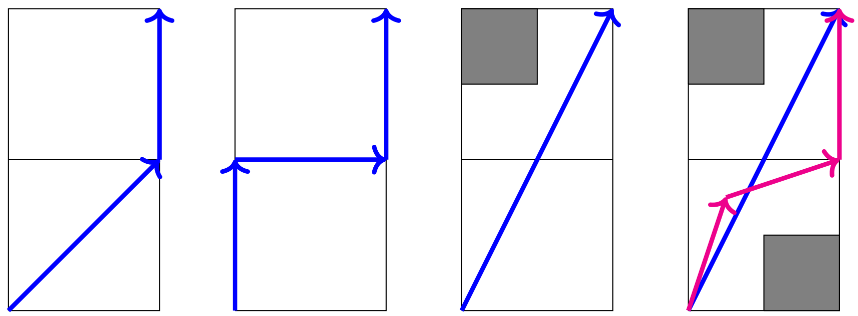

FIGURE 5. d-paths subordinate to a cube chain consisting of a 2-cube and a 1-cube, resp. of three 1-cubes; moreover subordinate to their respective collars. The path in magenta is contained in $\vec{T}^C_{\mathbf{c}}(X)$, but not in $\vec{T}_{\mathbf{c}}(X)$.

1-cube) if $p(0)$ and $p(1)$ are contained in the image of two subsequent open quadrants under the quotient map q. It is subordinate to the collar of the cube chain (e,e) if $p = q\circ\alpha$ with $\alpha\in\vec{P}(I^2\setminus\{(0.5,0.5)\})^{x^+}_{x^-}$ and $x^\pm$ contained in $st(\mathbf{0})$ resp. in $st(\mathbf{1})$, or if p is the concatenation of two paths subordinate to the collar of (e) at some point on the edge e. In either case, $q(0.5,0.5)$ is not contained in the image of p.

Proposition 4.8. *Let $\mathbf{c}$ denote a cube chain in X from x^- to x^+.*

(1) *The path space $\vec{P}^C_{\mathbf{c}}(X)^{x^+}_{x^-}\subset\vec{P}(X)^{x^+}_{x^-}$ subordinate to the collar of $\mathbf{c}$ is open.*
(2) *Likewise are $\vec{S}^C_{\mathbf{c}}(X)^{x^+}_{x^-}\subset\vec{S}(X)^{x^+}_{x^-}$, $\vec{T}^C_{\mathbf{c}}(X)^{x^+}_{x^-}\subset\vec{T}(X)^{x^+}_{x^-}$, and $\vec{ST}^C_{\mathbf{c}}(X)^{x^+}_{x^-}\subset\vec{ST}(X)^{x^+}_{x^-}$.*

Proof. (1) Let $p\in\vec{P}^C_{\mathbf{c}}(X)^{x^+}_{x^-}$ be such that $p[t_i,t_{i+1}]\in C(c_i,X)$ (open by Remark 4.1) and $p(t_i)\in st(x_i)$. The space of *all* d-paths q in X with $q[t_i,t_{i+1}]\in C(c_i,X)$ and $q(\{t_i\})\in st(x_i)$ is open in the compact-open topology and is an open neighbourhood of p contained in $\vec{P}^C_{\mathbf{c}}(X)^{x^+}_{x^-}$.
(2) by definition of the topology induced on subspaces.

□

Proposition 4.9. (1) *Every cube chain $\mathbf{c}$ – and hence also its collar $C(\mathbf{c})$ – contains a tame strict d-path.*
(2) *For every strict d-path $p\in\vec{S}(X)^{x^+}_{x^-}$, there exists a cube chain $\mathbf{c}(p)$ such that $p\in\vec{S}^C_{\mathbf{c}(p)}(X)^{x^+}_{x^-}$.*

In the following proof, we will make use of *coordinate hyperplanes* in a cube chain. In a cube I^n, consider middle hyperplanes given by the equations $x_i = 0.5,\ 1\le i\le n$. The elementary but crucial observation is that a *strict d-path* intersects any coordinate hyperplane in *at most* one point. Likewise, one defines coordinate middle hyperplanes (potentially with identifications on the boundary, one boundary middle hyperplane being identified with another such) in each cube in a □-set X.

Proof. (1) The diagonal path δ_n in $\vec{ST}(I^n)^{\mathbf{1}}_{\mathbf{0}}$ connects bottom and top vertex of the n-cube diagonally with constant speed. For an n-cell c in X, composition with the quotient map $I^n\downarrow c$ defines a strict tame path $\delta(c)$ in c from its bottom vertex to its top vertex. The concatenation $\delta(\mathbf{c}) := \delta(c_0)*\cdots*\delta(c_n)$ defines a strict tame path in the cube chain $\mathbf{c}$ connecting bottom and top vertex.

(2) For every strict d-path $p : J = [j^-, j^+] \to I^n$ in a single cube I^n defined on some interval $J \subseteq I$, there is a finite set $S \subset J$ (possibly empty) consisting of $s_j \in J$ at which p intersects one or several of the middle hyperplanes $x_i = 0.5,\ 1 \le i \le n$. Define I_0^j, I_*^j, I_1^j as the set of indices i for which $p_i(s_j)$ is less than, equal, resp. greater than 0.5. Since p is strict, we have that $I_1^{j+1} = I_1^j \cup I_*^j$ and $I_0^{j+1} = I_0^j \setminus I_*^{j+1}$. For $\max(j^-, s_{j-1}) < t < \min(j^+, s_{j+1})$, $p(t)$ is contained in the collar of the face $d_{[I_0^j|I_*^j|I_1^j]}I^n$ – the minimal face with this property.

For $\max(j^-, s_{j-1}) < t < \min(j^+, s_j)$, $p(t)$ is contained in the star of the *vertex* $d_{[I_0^j \cup I_*^j|\emptyset|I_1^j]}I^n$, and for $\max(j^-, s_j) < t < \min(j^+, s_{j+1})$ in the star of the *vertex* $d_{[I_0^j|\emptyset|I_*^j \cup I_1^j]}I^n$. The entire path is therefore contained in the collar of the cube chain defined by the cells $d_{[I_0^j|I_*^j|I_1^j]}I^n$.

Two special cases deserve particular attention:

(a) This cube chain degenerates to a single vertex if p does not intersect any of the hyperplanes $x_i = 0.5$.

(b) If $s_{min} = j^-$ and $p(j^-)$ is contained in a lower boundary face, resp. if $s_{max} = j^+$ and $p(j^+)$ is contained in an upper boundary face, then the first cube $d_{[I_0^{min}|I_*^{min}|I_1^{min}]}I^n$ of the cube chain is the minimal face containing $p(j^-)$; resp. the last cube $d_{[I_0^{max}|I_*^{max}|I_1^{max}]}I^n$ of the cube chain is minimal containing $p(j^+)$.

Now let $p : I \to X$ denote a strict d-path in a $\square$-set X from x^- to x^+ with presentation $p = [c_1; p^1] *_{t_1} [c_2, p^2] *_{t_2} \cdots *_{t_{l-1}} [c_l, p^l]$. Then the construction above can be performed for each individual cube c_i leading to a sequence $\mathbf{c}(p)$ of cubes the collar of which contains $p(I)$. One has to check that two subsequent cubes "match":

If $p^i(t_i) = p^{i+1}(t_i)$ is not contained in any middle hyperplane, then it is contained in the star of a vertex which is the top vertex in the cube chain corresponding to p_i and the bottom vertex in that corresponding to p^{i+1}. If $p^i(t_i) = p^{i+1}(t_i)$ is contained in a middle hyperplane, then the last cube in the cube chain corresponding to $[c_i, p^i]$ agrees with the first one in the cube chain corresponding to $[c_{i+1}, p_{i+1}]$, ie the minimal cube in the boundary of c_i and of c_{i+1} containing $p_i(t_i)$ – according to (b) above.

In a final step, one may erase cubes consisting of a single vertex; cf Remark 4.3(1). $\square$

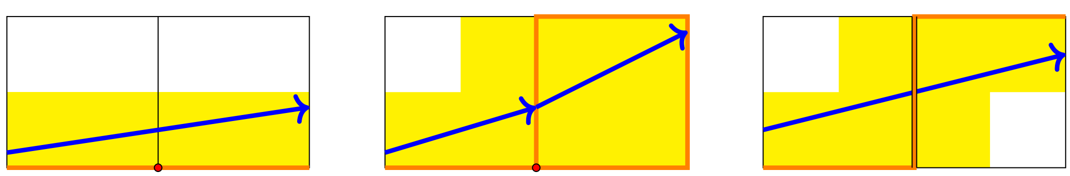

FIGURE 6. D-paths in two subsequent cubes in blue; the corresponding cube chains in orange, their collars in yellow. In the first two cases the two cube chains associated to each individual square consist of a single cube with common top, resp. bottom vertex; in the last case, the two cube chains share a common edge cube.

Remark 4.10. The analogue of Proposition 4.9(2) is wrong for (non-strict) d-paths. Let X be the cubical set (consisting of two 2-cubes) corresponding to $[0, 2] \times [0, 1]$. The d-path in

Figure 7 that linearly connects $(0,0),(0,0.5),(2,0,5)$ and $(2,1)$ is not contained in $\vec{P}^C_{\mathbf{c}}(X)^{(2,1)}_{(0,0)}$ for any cube chain connecting $(0,0)$ with $(2,1)$.

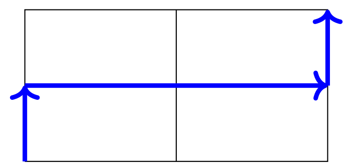

FIGURE 7. A d-path that is not contained in the collar of any cube chain from bottom to top

4.3. **A poset category of cube chains.**

Definition 4.11. (1) An *elementary refinement* of a cube c consists of two subsequent faces $d_{[I_0|[1:n]\setminus I_0|\emptyset]}c$ and $d_{[\emptyset|I_0|[1:n]\setminus I_0]}c$ with $\emptyset \neq I_0 \neq [1:n]$.
(2) An elementary refinement *of a cube chain* $\mathbf{c}$ arises by replacing a single cube c_i by one of its elementary refinements.
(3) A *refinement* of a cube chain arises by reflexive and transitive closure of elementary refinements.
(4) Refinement between cube chains in X from a vertex x^- to a vertex x^+ defines a *partial order* relation among cube chains that gives rise to a thin (poset) category $\mathcal{C}(X)^{x^+}_{x^-}$ with cube chains as objects and refinements as morphisms.

Remark that, for a general $\square$-set X, the category $Ch(X)$ in Ziemiański's [18] differs from this poset category. Ziemianski's cube chains are concatenations of *cubical maps* from a standard cube into a cube in X; moreover cubical symmetries of the standard cube that induce identities in the $\square$-set are an additional part of the structure.

Proposition 4.12. *All cube chains are supposed to be cube chains in X from x^- to x^+.*
(1) $\mathbf{c}'$ *refines* $\mathbf{c}$ *if and only if* $\vec{S}^C_{\mathbf{c}'}(X)^{x^+}_{x^-} \subseteq \vec{S}^C_{\mathbf{c}}(X)^{x^+}_{x^-}$.
(2) *If* $\mathbf{c}'$ *is a proper refinement of* $\mathbf{c}$, *then* $\vec{S}^C_{\mathbf{c}'}(X)^{x^+}_{x^-} \subset \vec{S}^C_{\mathbf{c}}(X)^{x^+}_{x^-}$ *is a proper subset.*
(3) *For every strict d-path* $p \in \vec{S}(X)$, *the chain* $\mathbf{c}(p)$ *(Proposition 4.9(2)) is* finest *among the cube chains* $\mathbf{c}$ *with* $p \in \vec{S}^C_{\mathbf{c}}(X)^{x^+}_{x^-}$.
(4) *Path spaces* $\vec{S}^C_{\mathbf{c}}(X)^{x^+}_{x^-}, \vec{S}^C_{\mathbf{c}'}(X)^{x^+}_{x^-}$ *intersect iff* $\mathbf{c}$ *and* $\mathbf{c}'$ *have a common refinement. Likewise for spaces of strict tame paths.*
(5) $\vec{S}^C_{\mathbf{c}''}(X)^{x^+}_{x^-} \subseteq \vec{S}^C_{\mathbf{c}}(X)^{x^+}_{x^-} \cap \vec{S}^C_{\mathbf{c}'}(X)^{x^+}_{x^-}$ *for a common refinement* $\mathbf{c}''$ *of* $\mathbf{c}$ *and* $\mathbf{c}'$. *If* $\mathbf{c}''$ *is a coarsest common refinement (if a such exists), then* $\vec{S}^C_{\mathbf{c}''}(X)^{x^+}_{x^-} = \vec{S}^C_{\mathbf{c}}(X)^{x^+}_{x^-} \cap \vec{S}^C_{\mathbf{c}'}(X)^{x^+}_{x^-}$.
(6) *If X is proper and non-self-linked (cf Definition 2.2(2-3)), then two cube chains either do not have a common refinement or they have a coarsest one (for which equality holds in (5)).*
(7) *Similar results hold for spaces of strict and tame d-paths.*

Proof. (1) $\Rightarrow$ from the definition of collars and paths in collars in Definition 4.5.
$\Leftarrow$: Assume $\mathbf{c}' = (c'_1, \dots, c_{l'})'$ does not refine $\mathbf{c} = (c_1, \dots c_l)$ and such that the prefix $(c'_1, \dots, c'_{k'})$ refines a minimal prefix of $(c_1, \dots, c_k)$ but $c'_{k'+1}$ does not refine neither c_k nor c_{k+1}. Then the diagonal path $\delta(c'_1) * \cdots * \delta(c'_{l'})$ is not contained in $\vec{S}^C_{\mathbf{c}}(X)^{x^+}_{x^-}$.

(2) The diagonal path in $\vec{ST}_{\mathbf{c}}(X)$ from the proof of Proposition 4.9(1) is not contained in $\vec{S}^C_{\mathbf{c}'}(X)^{x^+}_{x^-}$.
(3) Every elementary refinement (cf above) of the cube chain $\mathbf{c}(p)$ described above results in a collar that does not intersect at least one of the hyperplanes $x_j = 0.5$ with $j \in I_*$ within a cube c_i. In particular, it cannot contain $p(I)$ which has a non-empty intersection with this hyperplane.
(4) $\Leftarrow$ follows from (1) and Proposition 4.9.
$\Rightarrow$: By (3) above, the cube chain $\mathbf{c}(p)$ for a path p in the intersection refines both $\mathbf{c}$ and $\mathbf{c}'$.
(5) The first part follows from (1) above. Let $p \in \vec{S}^C_{\mathbf{c}}(X)^{x^+}_{x^-} \cap \vec{S}^C_{\mathbf{c}'}(X)^{x^+}_{x^-}$. By Proposition 4.9(2), $p \in \vec{S}^C_{\mathbf{c}(p)}(X)^{x^+}_{x^-}$, and by (3) above, $\mathbf{c}(p)$ refines both $\mathbf{c}$ and $\mathbf{c}'$ and thus $\mathbf{c}''$. Apply (1) above to $\mathbf{c}(p)$ and $\mathbf{c}''$.
(6) First note that the length of a cube chain (cf Definition 4.11(4)) is invariant under refinements. The proof is by induction on this length. The statement is obviously true for cube chains of length 0 and 1. Let $\mathbf{c} = (c_1, \dots, c_n), \mathbf{c}' = (c'_1, \dots, c'_n)$ denote two cube chains with a common refinement. There is a number of vertices (at least one) such that the edge from x^- to that vertex refines the first face of some common refinement of c_1 and of c'_1. All of these vertices span a (unique since X is proper) maximal common lower cube d of c_1 and c'_1 giving rise to elementary refinements (d, d_1) of c_1 and (d, d'_1) of c'_1. Then the cube chains arising by replacing c_1 by d_1 and c'_1 by d'_1 have a shorter length and hence a coarsest common refinement. Add d to the resulting cube chain at the beginning.

□

Example 4.13. Let X denote the non-proper □-set from Example 2.4(2) consisting of two 2-cubes c^1, c^2 glued along a common boundary consisting of four edges. Then the cube chains consisting solely of c^1, resp. of c^2 possess two common refinements (consisting of two consecutive boundary edges) but no coarsest such.

4.4. **Path spaces as colimits.** We define functors $\vec{S}, \vec{T}$ and $\vec{ST} : \mathcal{C}(X)^{x^+}_{x^-} \to \mathbf{Top}$ by $\vec{S}(\mathbf{c}) = \vec{S}^C_{\mathbf{c}}(X)^{x^+}_{x^-}$, $\vec{T}(\mathbf{c}) = \vec{T}^C_{\mathbf{c}}(X)^{x^+}_{x^-}$, and $\vec{ST}(\mathbf{c}) = \vec{ST}^C_{\mathbf{c}}(X)^{x^+}_{x^-}$. Refinement of cube sequences is reflected in inclusion of path spaces (Proposition 4.12(1)). As a result of Proposition 4.9 and 4.12, we conclude:

Corollary 4.14. Let X denote a □-set with vertices $x^-, x^+ \in X_0$. Then

$$\vec{S}(X)^{x^+}_{x^-} = \operatorname*{colim}_{\mathcal{C}(X)^{x^+}_{x^-}} \vec{S}, \vec{T}(X)^{x^+}_{x^-} = \operatorname*{colim}_{\mathcal{C}(X)^{x^+}_{x^-}} \vec{T} \text{ and } \vec{ST}(X)^{x^+}_{x^-} = \operatorname*{colim}_{\mathcal{C}(X)^{x^+}_{x^-}} \vec{ST}.$$

The colimit identifies path spaces in finer cube chains with subspaces of path spaces in coarser ones; the colimit is therefore just a union of topological spaces.

5. Comparing paths in a cube chain and in its collar

Let $\mathbf{c} = (c_1, \dots, c_n)$ denote a cube chain in a □- set X between vertices x^- and x^+. The purpose of this section is to establish

Proposition 5.1. (1) *$\vec{S}_{\mathbf{c}}(X)^{x^+}_{x^-} \hookrightarrow \vec{S}^C_{\mathbf{c}}(X)^{x^+}_{x^-}$ is a deformation retract.*
(2) *$\vec{ST}_{\mathbf{c}}(X)^{x^+}_{x^-} \hookrightarrow \vec{ST}^C_{\mathbf{c}}(X)^{x^+}_{x^-}$ is a deformation retract.*

Proof. (1) will be proved through a cubewise taming construction - first for individual d-paths and then for spaces of such. The proof of (2) follows the same pattern; you need to check that the d-paths and d-homotopies in the construction below *stay tame*.

To prove (1), we consider in a first step only strict d-paths subordinate to the collar of a cube chain $\mathbf{d} = (d_1, \dots, d_k)$ *within the standard cube* I^n; d_j is the face $d_{[I_0^j|I_*^j|I_1^j]}I^n$ with bottom vertex $v_{j-1} = d_{[I_0^j\cup I_*^j|\emptyset|I_1^j]}I^n$ and top vertex $v_j = d_{[I_0^j|\emptyset|I_*^j\cup I_1^j]}I^n$. Consider the sequence of piecewise linear hypersurfaces $H_j \subset I^n$, $1 \le j < k$, given by the equations $m_j(x) = 1, x \in I^n$, with $m_j(x_1, \dots, x_n) := \min_{i\in I_*^j} x_i + \max_{i\in I_*^{j+1}} x_i$. The intersection of H_j with the collar of the cube d_l in I^n is empty unless $l = j$ or $l = j+1$: In these collars, the coordinates x_i satisfy either all $x_i < 0.5$ or $x_i > 0.5$ for $i \in I_*^j \cup I_*^{j+1}$. H_j intersects the collars of d_j and d_{j+1} within the collar of the vertex v_j.

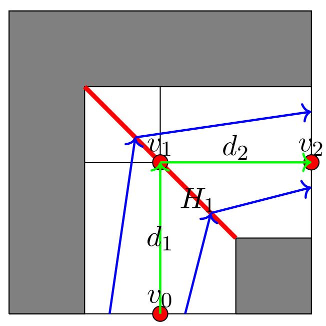

FIGURE 8. Cube chain (d_1, d_2), its collar, line hypersurface H_1, d-paths (blue) and their taming (green)

For a strict d-path, $p \in \vec{S}_{\mathbf{d}}^C(I^n)$, consider the functions given by the compositions $m_j \circ p : I \to \mathbf{R},\ 1 \le j \le k$; they are *strictly increasing*. When p enters $\overline{st(v_j)}$ at $t = t_-$, we have $\min_{i\in I_*^j} p_i(t_-) = 0.5$, whereas $\max_{i\in I_*^{j+1}} p_i(t_-) < 0.5$; their sum being less than 1; when p exits $\overline{st(v_j)}$ at $t = t_+$, we have $\max_{i\in I_*^{j+1}} p_i(t_+) = 0.5$ and $\min_{i\in I_*^j} p_i(t_+) > 0.5$; their sum being greater than 1. We conclude that there exists a unique ascending sequence $t_1 < t_2 < \dots < t_{k-1}$ such that $\min_{i\in I_*^j} p_i(t_j) + \max_{i\in I_*^{j+1}} p_i(t_j) = 1,\ 1 \le j < k$. Remark that $p_i(t_j) < 0.5$ for $i \in I_0^{j+1}$ and $p_i(t_j) > 0.5$ for $i \in I_1^j$.

We define a *tame* d-path $t(p) = q$ as the concatenation $q^1 *_{t_1} q^2 *_{t_2} \cdots *_{t_{k-1}} q^k$ of strict d-paths $q^j : [t_{j-1}, t_j] \to d_j$ (from v_{j-1} to v_j): the components $q_i^j, 1 \le i \le n$, of q^j are given by

$$q_i^j(t) = \begin{cases} 0 & i \in I_0^j \\ \frac{p_i(t)-p_i(t_{j-1})}{p_i(t_j)-p_i(t_{j-1})} & i \in I_*^j \\ 1 & i \in I_1^j \end{cases}. \tag{5.2}$$

Furthermore, we define a linear d-homotopy $H = H_s$ of strict d-paths $H_s \in \vec{S}_{\mathbf{d}}^C(I^n)$ connecting p and q as $H(t, s) = (1-s)p(t) + sq(t)$. □

Remark 5.3. (1) Formula (5.2) has the following interpretation: On the interval $[t_j, t_{j+1}]$, the components of p in I_0^j resp. in I_1^j are compressed to 0, resp. 1 (the respective processes in an HDA are idle), whereas its components in I_*^j are stretched to fill the entire interval (the respective processes take a full step). Remark that $q(t_j)$ is a vertex for every $0 \leq j \leq k$.

(2) If the cube chain $\mathbf{d}$ consists of a single cube only (a face, that may be just a vertex, but also the entire cube), then $t(p) = q$ is given by formula (5.2) for q on the entire domain – no concatenation involved.

(3) For a strict d-path p that is already contained in the cube chain $\mathbf{c}$ itself, $p(t)$ solves the equation $\min_{i\in I_*^j} x_i + \max_{i\in I_*^{j+1}} x_i = 1$ exactly at t_j with $p(t_j) = v_j$. Hence $t(p) = p$ and $H(t,s) = p(t)$ for $s \in I$.

Lemma 5.4. *For every cube chain* $\mathbf{d}$ *in* I^n*, the times* $t_j = t_j(p)$, $1 \leq j < k$, *define* continuous *functions* $t_j : \vec{S}_{\mathbf{d}}(I^n)_{st(v_0)}^{st(v_k)} \to I$.

Proof. Given a d-path $p \in \vec{S}_{\mathbf{d}}^C(I^n)$ and $\varepsilon > 0$ consider the *open* set of all strict d-paths q satisfying $m_j(q(t_j(p) - \varepsilon)) < 1$ and $m_j(q(t_j(p) + \varepsilon)) > 1$. Obviously, it contains the path p. For a strict d-path satisfying these two inequalities, the solution $t =: t_j(q)$ of $m_j(q(t)) = 1$ is contained in the interval $(t_j(p) - \varepsilon, t_j(p) + \varepsilon)$. □

Proof of Proposition 5.1 continued. In the sequel, we assume given a strict d-path $p \in \vec{S}_{\mathbf{c}}^C(X)_{x^-}^{x^+}$. Let us fix a presentation (cf Definition 2.3(3)) and perform the taming construction above cubewise. It is not difficult to check that, since the cube chain and its collar are unchanged, the resulting tamed path and d-homotopy does not depend on the chosen presentation. Moreover, the tamed d-paths in subsequent cubes (and the d-homotopies) "fit" at top, resp. bottom vertices of the cube chain.

Finally, Lemma 5.4 and formula (5.2) show that the construction yields a *continuous* taming map $T : \vec{S}_{\mathbf{c}}^C(X)_{x^-}^{x^+} \to \vec{S}_{\mathbf{c}}(X)_{x^-}^{x^+}$ and a continous deformation $H : \vec{S}_{\mathbf{c}}^C(X)_{x^-}^{x^+} \to (\vec{S}_{\mathbf{c}}^C(X)_{x^-}^{x^+})^I$ such that $H_0 = id$ and H_1 is given by T – composed with an inclusion map. Moreover, Remark 5.3(3) shows that H leaves $\vec{S}_{\mathbf{c}}(X)$ elementwise fixed. □

6. Taming is a homotopy equivalence

6.1. **Proper and non-self-linked □-sets.** In this section, we deal with a proper non-self-linked □-set X (cf Definition 2.2(2)). For a such, the taming result (③ in Theorem 2.6) is a consequence of the nerve lemma. This is essentially due to

Proposition 6.1. *Let* X *denote a proper non-self-linked* □*-set and let* $\mathbf{c}$ *denote a cube chain in* X *from* x^- *to* x^+*. Furthermore, let also* $\mathbf{c}_1, \dots \mathbf{c}_k$ *denote cube chains in* X *from* x^- *to* x^+.

(1) *The spaces* $\vec{S}_{\mathbf{c}}(X)_{x^-}^{x^+}$ *and* $\vec{P}_{\mathbf{c}}(X)_{x^-}^{x^+}$ *of (strict) d-paths subordinate to* $\mathbf{c}$ *are* contractible.

(2) *Likewise the spaces* $\vec{S}_{\mathbf{c}}^C(X)_{x^-}^{x^+} x^{-x^+}$ *and* $\vec{P}_{\mathbf{c}}^C(X)_{x^-}^{x^+}$ *of (strict) d-paths subordinate to the collar of* $\mathbf{c}$.

(3) *The intersections* $\bigcap_i \vec{S}_{\mathbf{c}_i}^C(X)_{x^-}^{x^+}$ *and* $\bigcap_i \vec{P}_{\mathbf{c}_i}^C(X)_{x^-}^{x^+}$ *are contractible if the cube chains* $\mathbf{c}_i$ *possess a common refinement and empty otherwise.*

Proof. (1) For $\vec{P}_{\mathbf{c}}(X)_{x^-}^{x^+}$, this has been observed in [17, Proposition 6.2(1)]. In brief, the space of d-paths in a *single* cube from the bottom to the top vertex is contractible (to the unit speed diagonal path joining them, say) by a linear d-homotopy. For paths in a general cube chain, perform first a reparametrization homotopy joining every d-path with its naturalization [15], ie, a unit speed path with respect to the l_1-norm along the same trajectory. The space of natural d-paths can then be contracted cube-wise.

For $\vec{S}_{\mathbf{c}}(X)_{x^-}^{x^+}$, one can either go through the same steps for strict d-paths or one can refer to Proposition 3.4 (or rather its proof) in the current paper.

(2) For $\vec{S}_{\mathbf{c}}^C(X)_{x^-}^{x^+}$, this follows from (1) and Proposition 5.1. For $\vec{P}_{\mathbf{c}}^C(X)_{x^-}^{x^+}$, apply then Proposition 3.4 (or rather its proof).

(3) follows from (2) and Proposition 4.12(6).

□

Theorem 6.2 (Nerve lemma [1, 10])**.** *Let Z denote a paracompact topological space and let $\mathcal{U}$ denote a good covering $Z = \bigcup U_i$ of Z by open sets $U_i \subset X$, ie all non-empty intersections $\bigcap_{i\in I} U_i$ are contractible. Then X is homotopy equivalent to the nerve of the poset of these non-empty intersections ordered by inclusion.*

This nerve is a simplicial complex with vertices corresponding to the U_i and $(k-1)$-dimensional simplices corresponding to non-empty intersections $\bigcap_{i\in I} U_i$.

Corollary 6.3. (1) The space $\vec{S}(X)_{x^-}^{x^+}$ of strict d-paths is homotopy equivalent to the nerve of a covering $\mathcal{U}(X)_{x^-}^{x^+}$; likewise, the space $\vec{ST}(X)_{x^-}^{x^+}$ of strict tame d-paths is homotopy equivalent to the nerve of a covering $\mathcal{V}(X)_{x^-}^{x^+}$.

(2) The two spaces are homotopy equivalent to each other.

Proof. The spaces $\vec{S}_{\mathbf{c}}^C(X)_{x^-}^{x^+}$ define a covering $\mathcal{U}(X)_{x^-}^{x^+}$ of $\vec{S}(X)_{x^-}^{x^+}$ (Proposition 4.9(2)) by open (Proposition 4.8(2)) and contractible (Proposition 6.1) sets; intersections of sets in the covering are empty or contractible by Proposition 6.1(3). Hence the covering $\mathcal{U}(X)_{x^-}^{x^+}$ is good. Similarly, the spaces $\vec{ST}_{\mathbf{c}}^C(X)_{x^-}^{x^+}$ define a good covering $\mathcal{V}(X)_{x^-}^{x^+}$ of $\vec{ST}_{\mathbf{c}}^C(X)$.

Moreover, the spaces $\vec{ST}(X)_{x^-}^{x^+} \subset \vec{S}_{\mathbf{c}}(X)_{x^-}^{x^+} \subset \vec{P}_{\mathbf{c}}(X)_{x^-}^{x^+}$ are metrizable and thus paracompact, cf [15]. Apply the nerve lemma, Theorem 6.2 to show (1) above.

The two coverings correspond to each other through inclusion maps over the *same poset* and giving rise to the *same nerve*: By Proposition 4.12(4), its objects can be enumerated by all sets $\{\mathbf{c}_i\}$ of cube chains in X from x^- to x^+ that possess a common refinement (the partial order is generated by refinement and superset). □

The remaining paragraphs in this section are not important for the main result, but they lead to a far simpler poset with a smaller nerve, better suited for calculations: The poset category corresponding to the covering $\mathcal{U}(X)_{x^-}^{x^+}$ is very redundant: By Proposition 4.12(4), its objects can be enumerated by all sets $\{\mathbf{c}_i\}$ of cube chains in X from x^- to x^+ that possess a common refinement (the partial order is generated by refinement and superset).

Proposition 6.4. *The nerves of the covering $\mathcal{U}(X)_{x^-}^{x^+}$ and of the poset category $\mathcal{C}(X)_{x^-}^{x^+}$ (cf Definition 4.11(4)) are homotopy equivalent. Hence, the spaces $\vec{ST}(X)_{x^-}^{x^+} \subset \vec{S}(X)_{x^-}^{x^+}$ are both homotopy equivalent to the nerve of $\mathcal{C}(X)_{x^-}^{x^+}$.*

Proof. Let $P\mathcal{U}(X)_{x^-}^{x^+}$ denote the poset correponding to the covering $\mathcal{U}(X)_{x^-}^{x^+}$ described in the proof of Corollary 6.3 Consider the poset map $P\mathcal{U}(X)_{x^-}^{x^+} \to \mathcal{C}(X)_{x^-}^{x^+}$ that associates to a set $\{\mathbf{c}_i\}$ of cube chains the *coarsest common refinement* of all the $\mathbf{c}_i$. The fiber (ie comma category) over any $\mathbf{c} \in \mathcal{C}(X)_{x^-}^{x^+}$ has the set $\{\mathbf{c}\}$ as an initial element, and is thus contractible. Apply Quillen's Theorem A! [14] □

6.2. **General $\square$-sets.** I am indebted to K. Ziemiański for pointing out to me that Proposition 6.1 is no longer true for non-proper $\square$-sets.

Example 6.5. Let Z_n denote the unique $\square$-set with exactly one cube $c_k,\ 0 \le k \le n$, from Example 2.4(3). For $n = 2$, consider the cube chain $\mathbf{c}$ consisting of the single cube c_2 from c_0 to c_0. Then $\vec{S}_{\mathbf{c}}(X)_{c_0}^{c_0}$ is homotopy equivalent to a circle S^1 – and hence not contractible! In fact, it deformation contracts to the subspace of piecewise linear d-paths in c_2 connecting c_0 with itself through a point on the anti-diagonal (connecting c_0 with itself).

For a general $\square$-set X, the proof of Theorem 2.6(1) requires a little more machinery from algebraic topology: Instead of the nerve lemma itself, we apply two more general results in homotopy theory that are used in proving it: the *projection lemma* comparing colimits with homotopy colimits and the *homotopy lemma* comparing homotopy colimits of spaces that can be glued together from pieces that are mutually homotopy equivalent. For a quite elementary presentation, cf eg [12]. Using these two results, we can prove that taming is a homotopy equivalence also for general $\square$-sets:

Proposition 6.6. *Let X denote a $\square$-set with selected vertices $x^-, x^+ \in X_0$. Then the inclusion map $\iota : \vec{ST}(X)_{x^-}^{x^+} \hookrightarrow \vec{S}(X)_{x^-}^{x^+}$ is a homotopy equivalence.*

Proof. The proof proceeds by a series of homotopy equivalences (denoted $\cong$) :
$\vec{S}(X)_{x^-}^{x^+} = \mathrm{colim}_{\mathcal{C}_{x^-}^{x^+}} \vec{S}_{\mathbf{c}}^C(X)_{x^-}^{x^+} \cong \mathrm{hocolim}_{\mathcal{C}_{x^-}^{x^+}} \vec{S}_{\mathbf{c}}^C(X)_{x^-}^{x^+} \cong \mathrm{hocolim}_{\mathcal{C}_{x^-}^{x^+}} \vec{S}_{\mathbf{c}}(X)_{x^-}^{x^+}$.
Likewise, $\vec{ST}(X) = \mathrm{colim}_{\mathcal{C}_{x^-}^{x^+}} \vec{ST}_{\mathbf{c}}^C(X)_{x^-}^{x^+} \cong \mathrm{hocolim}_{\mathcal{C}_{x^-}^{x^+}} \vec{ST}_{\mathbf{c}}^C(X)_{x^-}^{x^+} \cong \mathrm{hocolim}_{\mathcal{C}_{x^-}^{x^+}} \vec{ST}_{\mathbf{c}}(X)_{x^-}^{x^+}$.

In both cases, the first homotopy equivalence is due to the projection lemma, the second to the homotopy lemma and Propostion 5.1. Paths subordinate to a cube chain $\mathbf{c}$ are automatically tame, hence $\vec{S}_{\mathbf{c}}(X)_{x^-}^{x^+} = \vec{ST}_{\mathbf{c}}(X)_{x^-}^{x^+}$. $\square$

Remark 6.7. By far more sophisticated homotopy theoretical methods, Ziemiański proved in [18] that $\vec{T}(X)_{x^-}^{x^+}$ is homotopy equivalent to the nerve of a Reedy category $Ch(X)$ instead of our poset category $\mathcal{C}(X)_{x^-}^{x^+}$, also for general $\square$-sets. He uses a filtration of $\vec{T}(X)_{x^-}^{x^+}$ by differently defined contractible subsets using tame presentations. But these subspaces do *not* define an open covering, and therefore it is not possible to obtain the result by invoking the nerve lemma! Furthermore, Ziemiański shows that $\vec{T}(X)_{x^-}^{x^+} \hookrightarrow \vec{P}(X)_{x^-}^{x^+}$ is a deformation retract by a *global* taming construction that is far more tricky than our local one (which is supposed subordinate to the collar of a cube chain).

6.3. **PL d-paths and spaces of sequences.** In this final section, we *restrict attention to proper non-self linked* $\square$-sets (cf Definition 2.2(2-3)). At least for these, it is possible to find an even smaller model describing the homotopy type of the space of all d-paths between two vertices: To this end, consider the intersections H_k of an n-cube I^n (and hence of an n-cube in a $\square$-set X) with the hyperplanes given by the equations $x_1 + \cdots + x_n = k,\ k \in \mathbf{N}, 0 < k < n$; different from the hyperplanes previously considered. Every hyperplane section H_k is an $(n-1)$-dimensional polyhedron with the vertices with k entries 1 and $n-k$ entries 0 as extremal points. Requesting certain variables to take the value 0 or 1 yields the restriction of these hyperplane sections to faces; on which they again are hyperplane sections.

Observe that two elements $x_k \in H_k$ and $x_{k+1} \in H_{k+1}$ such that $x_k \le x_{k+1}$ (ie there exists a d-path from x_k to x_{k+1}) have l_1-distance (aka. Manhattan distance) $d_1(x_k, x_{k+1}) = 1$.

The hyperplane sections $H_k \subset I^n$ are *achronal*: Each d-path p in I^n intersects a hyperplane section H_k in at most one point $p_k \in H_k$; if defined, $d_1(p_k, p_{k+1}) = 1$. The union of all hyperplane sections H_k corresponding to all cells in the geometric realization of a $\square$-set X form a subspace $\mathcal{H}(X) \subset X$. Any d-path between two elements of $\mathcal{H}(X)$ has *integral* l_1-length (which is thus invariant under directed homotopy). The shortest length is called their l_1-distance d_1; compare [15] for this concept in greater generality.

A d-path $p : J \to X$ on an interval $J \subset \mathbf{R}$ is called *natural* if $d_1(p(t_1), p(t_2)) = t_2 - t_1$ for $t_1, t_2 \in J, t_1 \leq t_2$. The natural d-paths from a vertex x^- to a vertex x^+ in X with $p(0) = x^-$ form the space $\vec{N}(X)_{x^-}^{x^+}$. All paths in $\vec{N}(X)_{x^-}^{x^+}$ have integral d_1-length. A reparametrization linearly adjusting the domain to length one defines an inclusion map $\iota_N : \vec{N}(X)_{x^-}^{x^+} \hookrightarrow \vec{P}(X)_{x^-}^{x^+}$. Analogous to reparametrization by unit speed for curves, one defines a naturalization map in the opposite direction and proves it to be a homotopy inverse:

Proposition 6.8. [15, 18] *For a $\square$-set X with selected vertices $x^-, x^+ \in X_0$, the inclusion map $\iota_N : \vec{N}(X)_{x^-}^{x^+} \hookrightarrow \vec{P}(X)_{x^-}^{x^+}$ is a homotopy equivalence.*

The natural *tame* d-paths form a subspace with inclusion $\iota_{NT} : \vec{NT}(X)_{x^-}^{x^+} \hookrightarrow \vec{N}(X)_{x^-}^{x^+}$; naturalization preserves the trace of d-paths and hence tame d-paths and tame d-homotopies stay tame under naturalization:

Corollary 6.9. For a $\square$-set X with selected vertices $x^-, x^+ \in X_0$, the inclusion map $\iota_{NT} : \vec{NT}(X)_{x^-}^{x^+} \hookrightarrow \vec{T}(X)_{x^-}^{x^+}$ is a homotopy equivalence.

Remark 6.10. Let $p : J \to X$ on an interval with $\min J = 0$ denote a natural d-path in X.

(1) p intersects $\mathcal{H}(X)$ exactly at *integral* times: $p(t) \in \mathcal{H}(X) \Leftrightarrow t \in J \cap \mathbf{Z}$.
(2) If p is tame, then $p(i)$ and $p(i+1)$, i an integer, are contained in a common cube. A minimal such cube is uniquely determined since X is proper. Moreover, since X is non-self-linked, there is a unique unit speed line segment d-path (of length 1) in this (and any other) cube containing them.

A path $p \in \vec{NT}(X)_{x^-}^{x^+}$ with $p(0) = x^-$ is called PL (piecewise linear) if, for every integer i in its domain, the path between $p(i)$ and $p(i+1)$ is given by the unit speed line segment (of l_1-length 1) in the minimal cube that contains them both. These PL paths between x^- and x^+ form the subspace $\vec{PL}(X)_{x^-}^{x^+} \subset \vec{NT}(X)_{x^-}^{x^+}$.

Proposition 6.11. *Inclusion $\iota_{PL} : \vec{PL}(X)_{x^-}^{x^+} \hookrightarrow \vec{NT}(X)_{x^-}^{x^+} \hookrightarrow \vec{T}(X)_{x^-}^{x^+} \hookrightarrow \vec{P}(X)_{x^-}^{x^+}$ is a homotopy equivalence.*

Proof. In view of Theorem 2.6 and Corollary 6.9, it remains only to show that the first inclusion is a homotopy equivalence. In fact, $\vec{PL}(X)_{x^-}^{x^+}$ is a deformation retract in $\vec{NT}(X)_{x^-}^{x^+}$: To a natural tame d-path $p : J \to X$ associate the PL d-path $L(p)$ obtained by linearly connecting $p(i)$ with $p(i+1)$ for $i, i+1 \in \mathbf{Z} \cap J$ in the unique minimal cube containing both, cf Remark 6.10. Observe that $p = L(p)$ if p is PL. The (natural) convex combination homotopy joining $p \in \vec{NT}(X)_{x^-}^{x^+}$ with $L(p)$ shows that the linearization map L thus defined is a homotopy inverse to the first inclusion map. It restricts to the constant homotopy on $\vec{PL}(X)_{x^-}^{x^+}$. $\square$

The only data needed to describe PL d-paths are the *kink points* $p(i)$ in the cubes they traverse: The space $Seq(X)_{x^-}^{x^+}$ is defined as the space of all finite sequences $(x_0 = x^-, \dots, x_n = x^+) \in \bigcup_{n \geq 0} \mathcal{H}(X)^{n+1}$ with x_i, x_{i+1} in a common cube such that $x_i \leq x_{i+1}$ and $d_1(x_i, x_{i+1}) = 1$.

Example 6.12. Consider the boundary of $X = \partial I^3$ from Section 2.2.2. The hyperplane sections (diagonal lines) in the six boundary squares form two triangles, cf Figure 9. The associated pairs (= sequences) of kink points between the bottom and the top vertex form a hexagon, homotopy equivalent to a circle S^1:

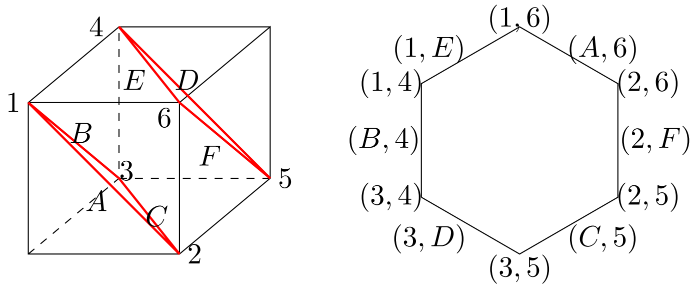

FIGURE 9. Red lines (= hyperplane sections) in the boundary ∂I^3 of a 3-cube and associated pairs of kink points. For example $(A, 6)$ indicates that from a point on the interior of A you can reach only 6 by a tame path; from its end points 1, resp. 2 you can reach E, resp. F.

With this definition, we obtain

Proposition 6.13. *Let X be a proper non-self-linked $\square$-set with selected vertices $x^-, x^+ \in X_0$. Then $\vec{P}(X)_{x^-}^{x^+}$ and $Seq(X)_{x^-}^{x^+}$ are homotopy equivalent.*

Proof. We may replace $\vec{P}(X)_{x^-}^{x^+}$ by $\vec{PL}(X)_{x^-}^{x^+}$ by Proposition 6.11. The forgetful map that associates to a path $p \in \vec{PL}(X)_{x^-}^{x^+}$ the sequence $(p(i)) \in Seq(X)_{x^-}^{x^+}$ – with i running through the integers in its domain – is a homeomorphism. Its inverse is the map that associates to a sequence in $Seq(X)_{x^-}^{x^+}$ the PL-path that connects any two subsequent elements in the sequence by the unit speed line segment in the unique minimal cube containing them both. $\square$

Remark 6.14. Extending the results of this section to a general $\square$-set X seems to be more intricate. The main reason is that, in some cases, two elements in successive hyperplane sections can be joined by more than one unit speed line segment paths – through different cubes if X is not proper and/or through the same cube if X is self-linked. For example, two vertices in subsequent hyperplane sections may be connected by various edges, after identification of vertices.

It seems to be necessary to replace the cube chains from this paper by the cube chains $Ch(X)$ in Ziemiański's paper [18]; those are generated by cubical maps from a wedge of cubes into X. Hyperplane sections in X are then replaced by hyperplane sections in a wedge of cubes. In such a wedge of cubes (which is obviously both proper and non-self-linked), there is a well-defined unit speed line segment between points on consecutive hyperplane sections.

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