Afrika Matematika (2019) 30:1–21
https://doi.org/10.1007/s13370-018-0624-y

Congruences for overpartitions with restricted odd differences

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Received: 8 June 2017 / Accepted: 23 August 2018 / Published online: 30 August 2018

Abstract
In recent work, Bringmann et al. used q-difference equations to compute a two-variable q-hypergeometric generating function for the number of overpartitions where (i) the difference between two successive parts may be odd only if the larger of the two is overlined, and (ii) if the smallest part is odd then it is overlined, given by $\overline{t}(n)$. They also established the two-variable generating function for the same overpartitions where (i) consecutive parts differ by a multiple of $(k+1)$ unless the larger of the two is overlined, and (ii) the smallest part is overlined unless it is divisible by $k+1$, enumerated by $\overline{t}^{(k)}(n)$. As an application they proved that $\overline{t}(n) \equiv 0 \pmod 3$ if n is not a square. In this paper, we extend the study of congruence properties of $\overline{t}(n)$, and we prove congruences modulo 3 and 6 for $\overline{t}(n)$, congruences modulo 2 and 4 for $\overline{t}^{(3)}(n)$ and $\overline{t}^{(7)}(n)$, congruences modulo 4 and 5 for $\overline{t}^{(4)}(n)$, and congruences modulo 3, 6 and 12 for $\overline{t}^{(8)}(n)$.

Keywords Partitions · Overpartitions · Congruences

Mathematics Subject Classification 05A17 · 11P82

1 Introduction

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{1.1}$$

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Using Jacobi's famous triple product identity [5, Entry 19, p.35], (1.1) takes the form

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty,$$

where $(a;b)_\infty = (1-a)(1-ab)(1-ab^2)\cdots$.

Throughout this paper, we will use

$$f_k := (q^k;q^k)_\infty.$$

The most important special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n\geq 1} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1.2}$$

$$\psi(q) := f\left(q,q^3\right) = \sum_{n\geq 0} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1} \tag{1.3}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1. \tag{1.4}$$

Equation (1.4) is the famous pentagonal number theorem [2, pp. 9–12].

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n; the number of partitions of n is denoted by $p(n)$. It is well known that the generating function of $p(n)$ is

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Ramanujan's [14], [15, pp. 210–213], three famous congruences satisfied by $p(n)$ are

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.5}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.6}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.7}$$

Motivated by these congruences mathematicians are engaged in finding such congruences for different partition functions. One of the partition functions we discuss here is overpartitions. An overpartition of n is a partition of n in which the first occurrence (equivalently, the final occurrence) of a part may be overlined. Let $\overline{p}(n)$ denote the number of overpartitions of n. Corteel and Lovejoy [9] showed that the generating function of $\overline{p}(n)$ is

$$\sum_{n\geq 0} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{1}{\varphi(-q)}.$$

For example, the 24 overpartitions of 5 are

$$5, \overline{5}, 4+1, \overline{4}+1, 4+\overline{1}, \overline{4}+\overline{1}, 3+2, \overline{3}+2, 3+\overline{2}, \overline{3}+\overline{2}, 3+1+1, \overline{3}+1+1,$$
$$3+1+\overline{1}, \overline{3}+1+\overline{1}, 2+2+1, \overline{2}+2+1, 2+2+\overline{1}, \overline{2}+2+\overline{1}, 2+1+1+1,$$
$$\overline{2}+1+1+1, 2+1+1+\overline{1}, \overline{2}+1+1+\overline{1}, 1+1+1+1+1, 1+1+1+1+\overline{1}.$$

Andrews [3] defined combinatorial objects that he called singular overpartitions which are overpartitions in which no part is divisible by δ and only parts $\equiv \pm i \pmod{\delta}$ may be overlined. The number of singular overpartitions of n is denoted by $\overline{C}_{\delta,i}(n)$. The ten singular overpartitions counted $\overline{C}_{3,1}(4)$ are

$$4,\ \overline{4},\ 2+2,\ \overline{2}+2,\ 2+1+1,\ \overline{2}+1+1,\ 2+\overline{1}+1,\ \overline{2}+\overline{1}+1,\ 1+1+1+1,\ \overline{1}+1+1+1.$$

For $\delta \geq 3$ and $1 < i < \lfloor \frac{\delta}{2} \rfloor$, the generating function for $\overline{C}_{\delta,i}(n)$ is

$$\sum_{n\geq 0} \overline{C}_{\delta,i}(n)q^n = \frac{(q^\delta;q^\delta)_\infty(-q^i;q^\delta)_\infty(-q^{\delta-i};q^\delta)_\infty}{(q;q)_\infty}. \tag{1.8}$$

Andrews found that for each $n \geq 0$,

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}, \tag{1.9}$$

and also that, for all $n \geq 0$, $\overline{C}_{3,1}(n) = \overline{A}_3(n)$, where $\overline{A}_3(n)$ is the number of overpartitions of n into parts not divisible by 3. To know more about overpartitions one can see [1,7,13].

Let $\overline{t}(n)$ denote the number of overpartitions of n where (i) the difference between two successive parts may be odd only if the larger of two is overlined, and (ii) if the smallest part is odd then it is overlined. Let $\overline{s}(n)$ denote the number of overpartitions counted by $\overline{t}(n)$ but with odd smallest part. Let $\overline{t}(m,n)$ and $\overline{s}(m,n)$ denote the number of overpartitions counted by $\overline{t}(n)$ (resp. $\overline{s}(n)$) having m parts. For example, the nine overpartitions counted by $\overline{t}(5)$ are

$$\overline{5},\ \overline{4}+\overline{1},\ \overline{3}+2,\ \overline{3}+\overline{2},\ 3+1+\overline{1},\ \overline{3}+1+\overline{1},\ 2+\overline{2}+\overline{1},\ \overline{2}+1+1+\overline{1},\ 1+1+1+1+\overline{1}$$

and seven overpartitions counted by $\overline{s}(5)$ are

$$\overline{5},\ \overline{4}+\overline{1},\ 3+1+\overline{1},\ \overline{3}+1+\overline{1},\ 2+\overline{2}+\overline{1},\ \overline{2}+1+1+\overline{1},\ 1+1+1+1+\overline{1}.$$

Bringmann et al. [6] have proved the following identities:

$$\sum_{m,n\geq 0} \overline{t}(m,n)x^m q^n = \frac{(-xq;q)_\infty}{(xq;q)_\infty}\left(1+\sum_{n\geq 1}\frac{(-q^3;q^3)_{n-1}(-x)^n q^n}{(-q;q)_{n-1}(q^2;q^2)_n}\right),$$
$$\sum_{m,n\geq 1} \overline{s}(m,n)x^m q^n = \sum_{n\geq 1}\frac{(q^3;q^3)_{n-1}x^n q^n}{(q;q)_{n-1}(q^2;q^2)_n}.$$

For some particular values of x and using the mock theta functions $\overline{\gamma}(q)$ and $\overline{\chi}(q)$ defined by

$$\overline{\gamma}(q) := \sum_{n\geq 0}\frac{(-1;q)_n(q;q)_n q^{\binom{n+1}{2}}}{(q^3;q^3)_n} \tag{1.10}$$

and

$$\overline{\chi}(q) := \sum_{n\geq 0}\frac{(-1;q)_n(-q;q)_n q^{\binom{n+1}{2}}}{(-q^3;q^3)_n}. \tag{1.11}$$

Bringmann et al. [6] have also proved the following identities:

$$\sum_{n\geq 0}\overline{t}(n)q^n=\frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^2;q^2)_\infty}, \tag{1.12}$$

$$\sum_{n\geq 0}\left(\overline{t}_+(n)-\overline{t}_-(n)\right)q^n=\frac{(-q^3;q^3)_\infty}{(-q;q)_\infty^3}\overline{\chi}(q), \tag{1.13}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\overline{\overline{s}}(n)q^n &:= 1+3\sum_{n\geq 1}(\overline{s}_+(n)-\overline{s}_-(n))q^n\\ &=\frac{(-q^3;q^3)_\infty}{(-q;q)_\infty^3},\end{aligned} \tag{1.14}$$

$$1+3\sum_{n\geq 1}\overline{s}(n)q^n=\frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^2;q^2)_\infty}\overline{\chi}(q), \tag{1.15}$$

where $\overline{t}_+(n)$ (resp. $\overline{s}_+(n)$) denotes the number of overpartitions counted by $\overline{t}(n)$ (resp. $\overline{s}(n)$) with largest part even and $\overline{t}_-(n)$ (resp. $\overline{s}_-(n)$) denotes the number of overpartitions counted by $\overline{t}(n)$ (resp. $\overline{s}(n)$) with largest part odd. Bringmann et al. [6] have also proved the following congruence identities:
For a prime $\ell\neq 2, 3$, and $n\geq 0$,

$$\begin{aligned}&\overline{t}_+(\ell^2n)+\left(\left(\frac{-n}{\ell}\right)-\ell-1\right)\overline{t}_+(n)+\ell\overline{t}_+\left(\frac{n}{\ell^2}\right)\\ &\equiv\overline{t}_-(\ell^2n)+\left(\left(\frac{-n}{\ell}\right)-\ell-1\right)\overline{t}_-(n)+\ell\overline{t}_-\left(\frac{n}{\ell^2}\right) \pmod 3.\end{aligned} \tag{1.16}$$

For $n\geq 1$,

$$\overline{t}(n)\equiv\begin{cases}(-1)^{h+1} \pmod 3 & \text{if } n=h^2,\\ 0 \pmod 3 & \text{otherwise.}\end{cases} \tag{1.17}$$

Bringmann et al. [6] generalized (1.12) as

$$\sum_{n\geq 0}\overline{t}^{(k)}(n)q^n=\frac{\left(q^{k+1};q^{k+1}\right)_\infty}{(q^k;q^k)_\infty(q;q)_\infty}, \tag{1.18}$$

where $\overline{t}^{(k)}(n)$ denotes the number of overpartitions of n where (i) consecutive parts differ by a multiple of $(k+1)$ unless the larger of the two is overlined, and (ii) the smallest part is overlined unless it is divisible by $k+1$.

Chern et al. [8] studied Ramanujan-type congruences for the partition functions $\overline{t}(n)$ and $\overline{\overline{s}}(n)$.

By (1.11) and (1.15), we deduce that

$$1+3\sum_{n\geq 1}\overline{s}(n)q^n=\frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^2;q^2)_\infty}\left(1+\sum_{n\geq 1}\frac{(-1;q)_n(-q;q)_nq^{\binom{n+1}{2}}}{(-q^3;q^3)_n}\right), \tag{1.19}$$

which implies that

$$1+\sum_{n\geq 1}\overline{s}(n)q^n\equiv\frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^2;q^2)_\infty} \pmod 2. \tag{1.20}$$

Applying (1.12) in (1.20), we find that

$$\sum_{n\geq 0} \overline{s}(n)q^n \equiv \sum_{n\geq 0} \overline{t}(n)q^n \pmod{2}, \tag{1.21}$$

from which we can say that "For any nonnegative integer n, $\overline{t}(n)$ is even (or odd) iff $\overline{s}(n)$ is even (or odd)".

Motivated by the above work, in this paper, we extend the study of congruence properties of overpartitions with restricted odd differences. In Sect. 3, we prove congruences modulo 6 for $\overline{t}(n)$, while in Sects. 4–7, we prove congruences modulo 2, 4 for $\overline{t}^{(3)}(n)$ and $\overline{t}^{(7)}(n)$, congruences modulo 4 and 5 for $\overline{t}^{(4)}(n)$ and congruences modulo 3, 6, and 12 for $\overline{t}^{(8)}(n)$.

2 Preliminaries

In this section, we recall 2-dissection identities for certain quotients of theta functions and p-dissection identities for theta functions $f(-q)$ and $\psi(q)$ which plays a key role in proving our main results.

Lemma 2.1 *The following 2-dissections hold.*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.1}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{2.2}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.3}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}, \tag{2.4}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{2.5}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{2.6}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{2.7}$$

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}, \tag{2.8}$$

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}, \tag{2.9}$$

$$\frac{1}{f_1 f_7} = \frac{f_{16}^2 f_{56}^5}{f_2^2 f_8 f_{14}^2 f_{28}^2 f_{112}^2} + q\frac{f_4^2 f_{28}^2}{f_2^3 f_{14}^3} + q^6\frac{f_8^5 f_{112}^2}{f_2^2 f_4^2 f_{14}^2 f_{16}^2 f_{56}}, \tag{2.10}$$

$$f_1 f_7 = \frac{f_2 f_{14} f_{16}^2 f_{56}^5}{f_4 f_8 f_{28}^3 f_{112}^2} - q f_4 f_{28} + q^6\frac{f_2 f_8^5 f_{14} f_{112}^2}{f_4^3 f_{16}^2 f_{28} f_{56}}. \tag{2.11}$$

Equations (2.1)–(2.3) are consequences of dissection formulas of Ramanujan, collected in Berndt's book [5, Entry 25 p. 40]. Xia and Yao [17] proved (2.4) by employing Jacobi triple product identity. Equation (2.5) was proved by Baruah and Ojah [4] and (2.6) was proved by Hirschhorn, Garvan, and Borwein [11]. Replacing q by $-q$ in (2.6) and using the relation

$$(-q;-q)_\infty = \frac{f_2^3}{f_1 f_4}$$

we obtain (2.7). Equation (2.8) was proved by Xia and Yao [18]. (2.9) was proved by Hirschhorn and Sellers [12]. (2.10) and (2.11) was proved by Xia [16, Lemma 3.4]

Lemma 2.2 [10, Theorem 2.1] *For any odd prime p,*

$$\psi(q) = \sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{2.12}$$

Furthermore, $\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod p$ *for* $0 \le m \le \frac{p-3}{2}$.

Lemma 2.3 [10, Theorem 2.2] *For any prime $p \ge 5$,*

$$f_1 = \sum_{\substack{k=-\frac{p-1}{2} \\ k \ne (\pm p-1)/6}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}. \tag{2.13}$$

Furthermore, for $-(p-1)/2 \le k \le (p-1)/2$ *and* $k \ne \dfrac{\pm p-1}{6}$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p,$$

where $\dfrac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & p \equiv 1 \mod 6, \\ \frac{-p-1}{6}, & p \equiv -1 \mod 6. \end{cases}$

3 Congruences for $\bar{t}(n)$

In this section, we prove infinite family of congruences modulo 6 for $\bar{t}(n)$ by using dissections of theta function identities.

Theorem 3.1 *If p is any prime $p \ge 5$ such that $\left(\frac{-2}{p}\right) = -1$ and $1 \le j \le p-1$, then for any nonnegative integer α*

$$\sum_{n \ge 0} \bar{t}\left(8p^{2\alpha}n + 3p^{2\alpha}\right) q^n \equiv 3\psi(q) f_6 \pmod 6 \tag{3.1}$$

and for $n \ge 0$,

$$\bar{t}\left(8p^{2\alpha+1}(pn+j) + 3p^{2\alpha+2}\right) \equiv 0 \pmod 6. \tag{3.2}$$

Proof From (1.12) and (2.4), we find that

$$\sum_{n\geq 0}\overline{t}(n)q^n=\frac{f_4f_6f_{16}f_{24}^2}{f_2^3f_8f_{12}f_{48}}+q\frac{f_6f_8^2f_{48}}{f_2^3f_{16}f_{24}},\tag{3.3}$$

which yields

$$\sum_{n\geq 0}\overline{t}(2n+1)q^n=\frac{f_3f_4^2f_{24}}{f_1^3f_8f_{12}}.\tag{3.4}$$

Invoking (2.7) and (3.4), we see that

$$\sum_{n\geq 0}\overline{t}(2n+1)q^n=\frac{f_4^8f_6^3f_{24}}{f_2^9f_8f_{12}^3}+3q\frac{f_4^4f_6f_{12}f_{24}}{f_2^7f_8}.\tag{3.5}$$

Equating the coefficients of q^{2n+1}, dividing throughout by q and then replacing q^2 by q in (3.5), we find that

$$\sum_{n\geq 0}\overline{t}(4n+3)q^n=3\frac{f_2^4f_3f_6f_{12}}{f_1^7f_4}.\tag{3.6}$$

Substituting (2.3) and (2.7) into (3.6), we find that

$$\sum_{n\geq 0}\overline{t}(4n+3)q^n=3\frac{f_2^4f_6f_{12}}{f_4}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2}+3q\frac{f_4^2f_6f_{12}^2}{f_2^7}\right)\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right),\tag{3.7}$$

from which we get

$$\sum_{n\geq 0}\overline{t}(8n+3)q^n=3\frac{f_2^{19}f_3^4}{f_1^{19}f_4^4f_6}+36q\frac{f_2^3f_3^2f_4^4f_6^3}{f_1^{13}}.\tag{3.8}$$

By the binomial theorem, it is easy to see that for all positive integers r and m

$$f_r^{2m}\equiv f_{2r}^m \pmod 2,\tag{3.9}$$
$$f_r^{4m}\equiv f_{2r}^{2m} \pmod 4.\tag{3.10}$$

From (3.9), it follows that

$$\frac{f_2^{19}f_3^4}{f_1^{19}f_4^4f_6}\equiv\frac{f_2^2f_6}{f_1}=\psi(q)f_6 \pmod 2.\tag{3.11}$$

In view of (3.11), we can rewrite (3.8) as

$$\sum_{n\geq 0}\overline{t}(8n+3)q^n\equiv 3\psi(q)f_6 \pmod 6.\tag{3.12}$$

From here our proof relies on mathematical induction. The congruence (3.12) is the case $\alpha=0$ of (3.1). Now assume that (3.1) holds for some $\alpha\geq 0$. Substituting (2.12) and (2.13) into (3.12), we deduce that

$$\sum_{n\geq 0}\bar{t}\left(8p^{2\alpha}n+3p^{2\alpha}\right)q^n \equiv 3\left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi\left(q^{p^2}\right)\right) \times\left(\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} q^{6\times\frac{3k^2+k}{2}} f\left(-q^{6\times\frac{3p^2+(6k+1)p}{2}}, -q^{6\times\frac{3p^2-(6k+1)p}{2}}\right) +q^{6\times\frac{p^2-1}{24}} f_{6p^2}\right) \pmod 6. \tag{3.13}$$

For a prime $p\geq 5$, $-(p-1)/2\leq k\leq (p-1)/2$ and $0\leq m\leq (p-3)/2$, consider the congruence

$$\frac{m^2+m}{2}+6\times\frac{3k^2+k}{2}\equiv\frac{9p^2-9}{24} \pmod p,$$

which is equivalent to

$$(2m+1)^2+2(6k+1)^2\equiv 0 \pmod p.$$

Since $\left(\frac{-2}{p}\right)=-1$, the only solution of the above congruence is $m=(p-1)/2$ and $k=(\pm p-1)/6$. Therefore, extracting the terms containing $q^{pn+\frac{9p^2-9}{24}}$ from both sides of (3.13), dividing throughout by $q^{\frac{9p^2-9}{24}}$ and then replacing q^p by q, we find that

$$\sum_{n\geq 0}\bar{t}\left(8p^{2\alpha+1}n+3p^{2\alpha+2}\right)q^n\equiv 3\psi(q^p)f_{6p} \pmod 6, \tag{3.14}$$

which yields

$$\sum_{n\geq 0}\bar{t}\left(8p^{2\alpha+2}n+3p^{2\alpha+2}\right)q^n\equiv 3\psi(q)f_{6} \pmod 6, \tag{3.15}$$

which is the (3.1) with $\alpha+1$ for α. Comparing the coefficients of q^{pn+j}, for $1\leq j\leq p-1$, from both sides of (3.14), we arrive at (3.2). □

4 Congruences for $\bar{t}^{(3)}(n)$

In this section, we prove congruences and infinite family of congruences modulo 2 and 4 for $\bar{t}^{(3)}(n)$.

Theorem 4.1 *If n and α are any nonnegative integers, then*

$$\overline{t}^{(3)}(4^{\alpha}n)+\overline{t}^{(3)}(n)\equiv 0 \pmod 2, \tag{4.1}$$

$$\overline{t}^{(3)}(4^{\alpha}(4n+2))\equiv 0 \pmod 2, \tag{4.2}$$

$$\overline{t}^{(3)}(4^{\alpha}(4n+3))\equiv 0 \pmod 2, \tag{4.3}$$

$$\overline{t}^{(3)}(4^{\alpha}(8n+5))\equiv 0 \pmod 2, \tag{4.4}$$

$$\overline{t}^{(3)}(24n+1)\equiv \begin{cases} 1 \pmod 2 & \textit{if n is a pentagonal number,}\\ 0 \pmod 2 & \textit{otherwise,}\end{cases} \tag{4.5}$$

$$\overline{t}^{(3)}(6n+3)\equiv 0 \pmod 4, \tag{4.6}$$

$$\overline{t}^{(3)}(6n+5)\equiv 0 \pmod 4, \tag{4.7}$$

$$\overline{t}^{(3)}(16n+10)\equiv 0 \pmod 4, \tag{4.8}$$

$$\overline{t}^{(3)}(16n+14)\equiv 0 \pmod 4, \tag{4.9}$$

$$\overline{t}^{(3)}(24n+19)\equiv 0 \pmod 4. \tag{4.10}$$

Proof Setting $k=3$ in (1.18) and using (2.5), we find that

$$\sum_{n\geq 0}\overline{t}^{(3)}(n)q^n=\frac{f_4}{f_1f_3}=f_4\left(\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2}+q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}\right). \tag{4.11}$$

Extracting the coefficients of even and odd powers of q on both sides of the above equation, we obtain

$$\sum_{n\geq 0}\overline{t}^{(3)}(2n)q^n=\frac{f_4^2f_6^5}{f_1^2f_3^4f_{12}^2}, \tag{4.12}$$

$$\sum_{n\geq 0}\overline{t}^{(3)}(2n+1)q^n=\frac{f_2^6f_{12}^2}{f_1^4f_3^2f_4^2f_6}. \tag{4.13}$$

Using (3.9), (4.12) can be rewritten

$$\sum_{n\geq 0}\overline{t}^{(3)}(2n)q^n\equiv\frac{f_2^3}{f_6} \pmod 2, \tag{4.14}$$

which yields

$$\sum_{n\geq 0}\overline{t}^{(3)}(4n)q^n\equiv\frac{f_1^3}{f_3}\equiv\sum_{n\geq 0}\overline{t}^{(3)}(n)q^n \pmod 2, \tag{4.15}$$

$$\overline{t}^{(3)}(4n+2)\equiv 0 \pmod 2. \tag{4.16}$$

Equating the coefficients of q^n in (4.15) and by mathematical induction, we arrive at (4.1). Using (4.16) in (4.1), we obtain (4.2).

In view of (3.9), we have

$$\frac{f_2^6f_{12}^2}{f_1^4f_3^2f_4^2f_6}\equiv f_{12} \pmod 2. \tag{4.17}$$

By (4.13) and (4.17), we find that

$$\sum_{n\geq 0} \overline{t}^{(3)}(2n+1)q^n \equiv f_{12} \pmod{2}, \tag{4.18}$$

from which we obtain

$$\overline{t}^{(3)}(4n+3) \equiv 0 \pmod{2}, \tag{4.19}$$

$$\overline{t}^{(3)}(8n+5) \equiv 0 \pmod{2}, \tag{4.20}$$

$$\sum_{n\geq 0} \overline{t}^{(3)}(24n+1)q^n \equiv f_1 \pmod{2}. \tag{4.21}$$

Equations (4.3) and (4.4) follows from (4.1), (4.19) and (4.20).

The result (4.5) is obtained from (4.21) and (1.4).

Thanks to (3.10),

$$\frac{f_2^6 f_{12}^2}{f_1^4 f_3^2 f_4^2 f_6} \equiv \frac{f_6^3}{f_3^2} \pmod{4}. \tag{4.22}$$

In view of (4.13) and (4.22),

$$\sum_{n\geq 0} \overline{t}^{(3)}(2n+1)q^n \equiv \frac{f_6^3}{f_3^2} \pmod{4}, \tag{4.23}$$

which yields the desired results (4.6) and (4.7).

By (3.10), we have

$$\frac{f_4^2 f_6^5}{f_1^2 f_3^4 f_{12}^2} \equiv \frac{f_4^2 f_6^3}{f_1^2 f_{12}^2} \pmod{4}. \tag{4.24}$$

Using (2.1) in the right hand side of (4.24) and then applying the resulting equation in (4.12), we deduce that

$$\sum_{n\geq 0} \overline{t}^{(3)}(2n)q^n \equiv \frac{f_4^2 f_6^3}{f_{12}^2} \left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8} \right) \pmod{4}. \tag{4.25}$$

Extracting the terms involving odd powers of q on both sides of (4.25), we obtain

$$\begin{aligned} \sum_{n\geq 0} \overline{t}^{(3)}(4n+2)q^n &\equiv 2 \frac{f_2^4 f_3^3 f_8^2}{f_1^5 f_4 f_6^2} \pmod{4} \\ &\equiv 2 \frac{f_8^2}{f_1 f_3} \pmod{4}. \end{aligned} \tag{4.26}$$

But in view of (3.9), we can rewrite (2.5) as

$$\frac{1}{f_1 f_3} \equiv \frac{f_8}{f_{12}} + q \frac{f_{24}}{f_4} \pmod{2}. \tag{4.27}$$

Substituting (4.27) into (4.26), we find that

$$\sum_{n\geq 0} \overline{t}^{(3)}(4n+2)q^n \equiv 2 \frac{f_8^3}{f_{12}} + 2q \frac{f_8^2 f_{24}}{f_4} \pmod{4}. \tag{4.28}$$

Equating the coefficients of q^{4n+2} and q^{4n+3} on both sides of (4.28), we arrive at (4.8) and (4.9).

Extracting the coefficients of q^{3n} on both sides of (4.23) and then replacing q^3 by q, we find that

$$\sum_{n\geq 0}\overline{t}^{(3)}(6n+1)q^n \equiv \frac{f_2^3}{f_1^2} \pmod{4}. \tag{4.29}$$

Applying (2.1) in (4.29), we obtain

$$\sum_{n\geq 0}\overline{t}^{(3)}(6n+1)q^n \equiv \frac{f_8^5}{f_2^2 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^2 f_8} \pmod{4}. \tag{4.30}$$

In view of (3.9) and (3.10), we can rewrite the above equations as

$$\sum_{n\geq 0}\overline{t}^{(3)}(6n+1)q^n \equiv \frac{f_8}{f_2^2} + 2q\frac{f_8^2 f_{16}}{f_4} \pmod{4}, \tag{4.31}$$

which yields

$$\sum_{n\geq 0}\overline{t}^{(3)}(12n+7)q^n \equiv 2\frac{f_4^2 f_8}{f_2} \pmod{4}. \tag{4.32}$$

If we equate the coefficients of q^{2n+1} on both sides of the above equation, we obtain (4.10). □

Theorem 4.2 *Let $p \geq 5$ be any prime such that $\left(\frac{-3}{p}\right) = -1$ and $1 \leq j \leq p-1$. Then for any nonnegative integers α,*

$$\sum_{n\geq 0}\overline{t}^{(3)}\left(24p^{2\alpha}n + 7p^{2\alpha}\right)q^n \equiv 2\psi(q)f_4 \pmod{4} \tag{4.33}$$

and for each $n \geq 0$,

$$\overline{t}^{(3)}\left(24p^{2\alpha+1}(pn+j) + 7p^{2\alpha+2}\right) \equiv 0 \pmod{4}. \tag{4.34}$$

Proof We prove (4.33) by mathematical induction. Extracting the terms involving even powers of q on both sides of (4.32), we find that

$$\sum_{n\geq 0}\overline{t}^{(3)}(24n+7)q^n \equiv 2\frac{f_2^2 f_4}{f_1} = 2\psi(q)f_4 \pmod{4}, \tag{4.35}$$

which is the $\alpha = 0$ case of (4.33). Now assume that (4.33) holds for some $\alpha \geq 0$. Substituting (2.12) and (2.13) into (4.33), we deduce that

$$\sum_{n\geq 0} \bar{t}^{(3)}\left(24p^{2\alpha}n + 7p^{2\alpha}\right)q^n$$
$$\equiv 2\left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi\left(q^{p^2}\right)\right)$$
$$\times\left(\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} q^{4\times\frac{3k^2+k}{2}} f\left(-q^{4\times\frac{3p^2+(6k+1)p}{2}}, -q^{4\times\frac{3p^2-(6k+1)p}{2}}\right)\right.$$
$$\left.+q^{4\times\frac{p^2-1}{24}} f_{4p^2}\right) \pmod 4. \tag{4.36}$$

For a prime $p \geq 5$, $-(p-1)/2 \leq k \leq (p-1)/2$ and $0 \leq m \leq (p-3)/2$, consider the congruence

$$4\times\frac{3k^2+k}{2} + \frac{m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod p,$$

that is

$$(12k+2)^2 + 3(2m+1)^2 \equiv 0 \pmod p.$$

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the above congruence is $k = (\pm p - 1)/6$ and $m = (p-1)/2$. Therefore, extracting the terms containing $q^{pn+\frac{7p^2-7}{24}}$ from both sides of (4.36), dividing throughout by $q^{\frac{7p^2-7}{24}}$ and then replacing q^p by q, we find that

$$\sum_{n\geq 0} \bar{t}\left(24p^{2\alpha+1}n + 7p^{2\alpha+2}\right)q^n \equiv 2\psi(q^p)f_{4p} \pmod 4, \tag{4.37}$$

which yields

$$\sum_{n\geq 0} \bar{t}\left(24p^{2\alpha+2}n + 7p^{2\alpha+2}\right)q^n \equiv 2\psi(q)f_4 \pmod 4, \tag{4.38}$$

which is the (4.33) with $\alpha+1$ for α . Comparing the coefficients of q^{pn+j}, for $1 \leq j \leq p-1$, from both sides of (4.37), we arrive at (4.34). □

Theorem 4.3 *Let $p \geq 5$ be any prime with $\left(\frac{-2}{p}\right) = -1$ and $1 \leq j \leq p-1$. Then for any nonnegative integers α,*

$$\sum_{n\geq 0} \bar{t}^{(3)}\left(16p^{2\alpha}n + 6p^{2\alpha}\right)q^n \equiv 2\psi(q)f_6 \pmod 4, \tag{4.39}$$

and for each $n \geq 0$,

$$\bar{t}^{(3)}\left(16p^{2\alpha+1}(pn+j) + 6p^{2\alpha+2}\right) \equiv 0 \pmod 4. \tag{4.40}$$

Proof Equating the coefficients of q^{4n+1}, dividing throughout by q and then replacing q^4 by q in (4.28), we obtain

$$\sum_{n\geq 0}\bar{t}^{(3)}(16n+6)q^n \equiv 2\frac{f_2^2 f_6}{f_1} \equiv 2\psi(q)f_6 \pmod 4. \tag{4.41}$$

Rest of the proof is similar to that of Eqs. (3.1) and (3.2) in the Theorem 3.1, so we omit the proof here. □

5 Congruences for $\bar{t}^{(4)}(n)$

In this section, we obtain congruences modulo 4 and 5 for $\bar{t}^{(4)}(n)$.

Theorem 5.1 *If n is any nonnegative integer, then*

$$\bar{t}^{(4)}(8n+6) \equiv 0 \pmod 4, \tag{5.1}$$

$$\bar{t}^{(4)}(16n+10) \equiv 0 \pmod 4, \tag{5.2}$$

$$\bar{t}^{(4)}(16n+2) \equiv \begin{cases} 2 \pmod 4 & \text{if } n \text{ is a triangular number,} \\ 0 \pmod 4 & \text{otherwise.} \end{cases} \tag{5.3}$$

Proof Setting $k=4$ in (1.18) and then using (2.9), we obtain

$$\sum_{n\geq 0}\bar{t}^{(4)}(n)q^n = \frac{f_8 f_{20}^2}{f_2^2 f_4 f_{40}} + q\frac{f_4^2 f_{10} f_{40}}{f_2^3 f_8 f_{20}}, \tag{5.4}$$

which yields

$$\sum_{n\geq 0}\bar{t}^{(4)}(2n)q^n = \frac{f_4 f_{10}^2}{f_1^2 f_2 f_{20}}. \tag{5.5}$$

Combining (2.1) and (5.5), we find that

$$\sum_{n\geq 0}\bar{t}^{(4)}(2n)q^n = \frac{f_4 f_{10}^2}{f_2 f_{20}}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right). \tag{5.6}$$

Extracting the coefficients of odd powers of q on both sides of the above equation, we obtain

$$\sum_{n\geq 0}\bar{t}^{(4)}(4n+2)q^n = 2\frac{f_2^3 f_5^2 f_8^2}{f_1^6 f_4 f_{10}}. \tag{5.7}$$

It follows from (3.9) that

$$\frac{f_2^3 f_5^2 f_8^2}{f_1^6 f_4 f_{10}} \equiv f_4 f_8 \equiv \frac{f_8^2}{f_4} \equiv \psi(q^4) \pmod 2. \tag{5.8}$$

Using (5.8), (5.7) can be reduced to

$$\sum_{n\geq 0}\bar{t}^{(4)}(4n+2)q^n \equiv 2f_4 f_8 \equiv 2\psi(q^4) \pmod 4. \tag{5.9}$$

Equating the coefficients of q^{2n+1} and q^{4n+2} on both sides of the above equation, we arrive at (5.1) and (5.2).

Extracting the terms of (5.9) in which powers of q is congruent to 0 modulo 4, we obtain

$$\sum_{n\geq 0}\bar{t}^{(4)}(16n+2)q^n \equiv 2\psi(q) \pmod 4. \tag{5.10}$$

A positive integer x is said to be triangular number, if it is of the form $\frac{x(x+1)}{2}$. The result (5.3) follows from (1.3) and (5.10). □

Theorem 5.2 *For any prime $p \geq 5$ with $\left(\frac{-2}{p}\right) = -1$, $1 \leq j \leq p-1$, and $\alpha \geq 0$, we have*

$$\sum_{n\geq 0}\bar{t}^{(4)}\left(16p^{2\alpha}n+2p^{2\alpha}\right)q^n \equiv 2f_1f_2 \pmod 4, \tag{5.11}$$

and for each $n \geq 0$,

$$\bar{t}^{(4)}\left(16p^{2\alpha+1}(pn+j)+2p^{2\alpha+2}\right) \equiv 0 \pmod 4. \tag{5.12}$$

Proof From (5.9), we have

$$\sum_{n\geq 0}\bar{t}^{(4)}(16n+2)q^n \equiv 2f_1f_2 \pmod 4. \tag{5.13}$$

For a prime $p \geq 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$\frac{3k^2+k}{2} + 2\times\frac{3m^2+m}{2} \equiv \frac{3p^2-3}{24} \pmod p,$$

which implies that

$$(6k+1)^2 + 2(6m+1)^2 \equiv 0 \pmod p.$$

Since $\left(\frac{-2}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p-1)/6$. Therefore, using Lemma 2.3, we have

$$\sum_{n\geq 0}\bar{t}^{(4)}\left(16\left(p^2n+\frac{3p^2-3}{24}\right)+2\right)q^n \equiv 2f_1f_2 \pmod 4. \tag{5.14}$$

Invoking (5.13) and (5.14), we arrive at

$$\sum_{n\geq 0}\bar{t}^{(4)}\left(16p^2n+2p^2\right)q^n \equiv \sum_{n\geq 0}\bar{t}^{(4)}(16n+2)q^n \equiv 2f_1f_2 \pmod 4. \tag{5.15}$$

The result (5.11) follows from the above equation and by induction on α.

Substituting (2.13) into (5.11), we deduce that

$$\begin{aligned}\sum_{n\geq 0}\bar{t}^{(4)}\left(16p^{2\alpha}n+2p^{2\alpha}\right)q^n &\equiv 2\left(\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}}\right)+q^{\frac{p^2-1}{24}}f_{p^2}\right)\\ &\times\left(\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} q^{2\times\frac{3k^2+k}{2}} f\left(-q^{2\times\frac{3p^2+(6k+1)p}{2}},-q^{2\times\frac{3p^2-(6k+1)p}{2}}\right)\right.\\ &\left.+q^{2\times\frac{p^2-1}{24}}f_{2p^2}\right) \pmod 4, \end{aligned} \tag{5.16}$$

which yields

$$\sum_{n\geq 0}\bar{t}^{(4)}\left(16p^{2\alpha+1}n+2p^{2\alpha+2}\right)q^n\equiv f_p f_{2p} \pmod 4. \tag{5.17}$$

Equating the coefficients of q^{pn+j} for $j=1,2,\ldots,p-1$ in (5.17), we obtain (5.12). □

Theorem 5.3 *For $n\geq 0$, we have*

$$\bar{t}^{(4)}(16n+6)\equiv 2\triangle_3(n) \pmod 5 \tag{5.18}$$

and

$$\bar{t}^{(4)}(16n+14)\equiv 0 \pmod 5. \tag{5.19}$$

Proof Setting $k=4$ in (1.18), we obtain

$$\sum_{n\geq 0}\bar{t}^{(4)}(n)q^n=\frac{f_5}{f_1f_4}. \tag{5.20}$$

In view of (3.10), we can rewrite (5.20) as

$$\sum_{n\geq 0}\bar{t}^{(4)}(n)q^n\equiv\frac{f_1^4}{f_4} \pmod 5. \tag{5.21}$$

Substituting (2.2) into (5.21) and then extracting even powers of q on both sides, we obtain

$$\begin{aligned}\sum_{n\geq 0}\bar{t}^{(4)}(2n)q^n &\equiv\frac{f_2^9}{f_1^2f_4^4} \pmod 5\\ &\equiv\frac{f_2^9}{f_4^4}\left(\frac{f_8^5}{f_2^5f_{16}^2}+2q\frac{f_4^2f_{16}^2}{f_2^5f_8}\right) \pmod 5,\end{aligned} \tag{5.22}$$

which implies that

$$\sum_{n\geq 0} \overline{t}^{(4)}(4n+2)q^n \equiv 2\frac{f_1^4 f_8^2}{f_2^2 f_4} \pmod 5$$

$$\equiv 2\frac{f_8^2}{f_2^2 f_4}\left(\frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}\right) \pmod 5. \tag{5.23}$$

Extracting odd powers of q on both sides, we obtain

$$\sum_{n\geq 0} \overline{t}^{(4)}(8n+6)q^n \equiv 2\frac{f_4^6}{f_2^3} \pmod 5$$

$$\equiv 2\psi^3(q^2) \pmod 5. \tag{5.24}$$

If we extract even and odd powers of q on both sides of the above equation, we arrive at (5.18) and (5.19), respectively. □

6 Congruences for $\overline{t}^{(7)}(n)$

In this section, we prove congruences and infinite family of congruences modulo 2 and 4 for $\overline{t}^{(7)}(n)$.

Theorem 6.1 *If n is any nonnegative integer, then*

$$\overline{t}^{(7)}(8n+7) \equiv 0 \pmod 4. \tag{6.1}$$

Proof Setting $k = 7$ in (1.18), we find that

$$\sum_{n\geq 0} \overline{t}^{(7)}(n)q^n = \frac{f_8}{f_1 f_7}. \tag{6.2}$$

Substituting (2.10) into (6.2) and then extracting the terms involving odd powers of q on both sides of the resulting equation, we obtain

$$\sum_{n\geq 0} \overline{t}^{(7)}(2n+1)q^n = \frac{f_2^2 f_4 f_{14}^2}{f_1^3 f_7^3}. \tag{6.3}$$

In view of (3.10), we can rewrite (6.3) as

$$\sum_{n\geq 0} \overline{t}^{(7)}(2n+1)q^n \equiv f_1 f_4 f_7 \pmod 4. \tag{6.4}$$

Invoking (2.11) and (6.4), we deduce that

$$\sum_{n\geq 0} \overline{t}^{(7)}(2n+1)q^n \equiv \frac{f_2 f_{14} f_{16}^2 f_{56}^5}{f_8 f_{28}^3 f_{112}^2} - q f_4^2 f_{28} + q^6 \frac{f_2 f_8^5 f_{14} f_{112}^2}{f_4^2 f_{16}^2 f_{28} f_{56}} \pmod 4. \tag{6.5}$$

Equating the coefficients of odd powers of q on both sides of the above equation, we find that

$$\sum_{n\geq 0} \overline{t}^{(7)}(4n+3)q^n \equiv 3 f_2^2 f_{14} \pmod 4, \tag{6.6}$$

which yields the desired result (6.1). □

Theorem 6.2 *Let $p \geq 5$ be any prime with $\left(\frac{-14}{p}\right) = -1$ and $1 \leq j \leq p-1$. Then for any nonnegative integers α,*

$$\sum_{n\geq 0} \overline{t}^{(7)}\left(8p^{2\alpha}n + 3p^{2\alpha}\right) q^n \equiv f_2 f_7 \pmod 2, \tag{6.7}$$

and for each $n \geq 0$,

$$\overline{t}^{(7)}\left(8p^{2\alpha+1}(pn+j) + 3p^{2\alpha+2}\right) \equiv 0 \pmod 2. \tag{6.8}$$

Proof We prove (6.7) by mathematical induction. In view of (3.9), we can rewrite (6.6) as

$$\sum_{n\geq 0} \overline{t}^{(7)}(4n+3)q^n \equiv f_4 f_{14} \pmod 2, \tag{6.9}$$

which yields

$$\sum_{n\geq 0} \overline{t}^{(7)}(8n+3)q^n \equiv f_2 f_7 \pmod 2. \tag{6.10}$$

which is the $\alpha = 0$ case of (6.7). Now assume that (6.7) holds for some $\alpha \geq 0$. Substituting (2.13) into (6.7), we deduce that

$$\begin{aligned}
&\sum_{n\geq 0} \overline{t}^{(7)}\left(8p^{2\alpha}n + 3p^{2\alpha}\right) q^n \\
&\equiv \left(\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} q^{3k^2+k} f\left(-q^{3p^2+(6k+1)p}, -q^{3p^2-(6k+1)p}\right) + q^{\frac{p^2-1}{12}} f_{2p^2} \right) \\
&\quad \times \left(\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} q^{7\times\frac{3k^2+k}{2}} f\left(-q^{7\times\frac{3p^2+(6k+1)p}{2}}, -q^{7\times\frac{3p^2-(6k+1)p}{2}}\right) \right. \\
&\quad \left. + q^{7\times\frac{p^2-1}{24}} f_{7p^2} \right) \pmod 2.
\end{aligned} \tag{6.11}$$

For a prime $p \geq 5$, $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider the congruence

$$2 \times \frac{3k^2+k}{2} + 7 \times \frac{3m^2+m}{2} \equiv \frac{9p^2-9}{24} \pmod p,$$

that is

$$(12k+2)^2 + 14(6m+1)^2 \equiv 0 \pmod p.$$

Since $\left(\frac{-14}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p - 1)/6$. Therefore, extracting the terms containing $q^{pn+\frac{9p^2-9}{24}}$ from both sides of (6.11), dividing throughout by $q^{\frac{9p^2-9}{24}}$ and then replacing q^p by q, we find that

$$\sum_{n\geq 0} \bar{t}^{(7)}\left(8p^{2\alpha+1}n+3p^{2\alpha+2}\right)q^n \equiv f_{2p}f_{7p} \pmod 2, \tag{6.12}$$

which yields

$$\sum_{n\geq 0} \bar{t}^{(7)}\left(8p^{2\alpha+2}n+3p^{2\alpha+2}\right)q^n \equiv f_2f_7 \pmod 2, \tag{6.13}$$

which is the (6.7) with $\alpha+1$ for α. Comparing the coefficients of q^{pn+j}, for $1\leq j\leq p-1$, from both sides of (6.13), we arrive at (6.8). □

7 Congruences for $\bar{t}^{(8)}(n)$

In this section, we prove congruences and infinite family of congruences modulo 3, 6, and 12 for $\bar{t}^{(8)}(n)$.

Theorem 7.1 *For each nonnegative integer n,*

$$\bar{t}^{(8)}(4n+3) \equiv 0 \pmod 3, \tag{7.1}$$

$$\bar{t}^{(8)}(24n+15) \equiv 0 \pmod{12}, \tag{7.2}$$

$$\bar{t}^{(8)}(24n+23) \equiv 0 \pmod{12}. \tag{7.3}$$

Proof Setting $k=8$ in (1.18), we obtain

$$\sum_{n\geq 0} \bar{t}^{(8)}(n)q^n = \frac{f_9}{f_8f_1}. \tag{7.4}$$

Invoking (2.8) and (7.4), we find that

$$\sum_{n\geq 0} \bar{t}^{(8)}(n)q^n = \frac{f_{12}^3f_{18}}{f_2^2f_6f_8f_{36}} + q\frac{f_4^2f_6f_{36}}{f_2^3f_8f_{12}}. \tag{7.5}$$

Extracting the coefficients of odd powers of q on both sides of the above equation, we obtain

$$\sum_{n\geq 0} \bar{t}^{(8)}(2n+1)q^n = \frac{f_2^2f_3f_{18}}{f_1^3f_4f_6}. \tag{7.6}$$

Substituting (2.7) into (7.6) and then extracting the coefficients of odd powers of q on both sides of the resulting equation, we deduce that

$$\sum_{n\geq 0} \bar{t}^{(8)}(4n+3)q^n = 3\frac{f_2f_6^2f_9}{f_1^5}, \tag{7.7}$$

from which, we obtain the result (7.1).

In view of (3.10), we have

$$\frac{f_2f_6^2f_9}{f_1^5} \equiv \frac{f_6^2f_9}{f_1f_2} \pmod 4. \tag{7.8}$$

Using (7.8), (7.7) can be reduces to

$$\sum_{n\geq 0}\bar{t}^{(8)}(4n+3)q^n \equiv 3\frac{f_6^2 f_9}{f_1 f_2} \pmod{12}. \tag{7.9}$$

Applying (2.8) in (7.9), we find that

$$\sum_{n\geq 0}\bar{t}^{(8)}(4n+3)q^n \equiv 3\frac{f_6^2}{f_2}\left(\frac{f_{18}f_{12}^3}{f_2^2 f_6 f_{36}}+q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}\right) \pmod{12}. \tag{7.10}$$

Equating the coefficients of q^{2n+1}, dividing throughout by q and then replacing q^2 by q in (7.10), we obtain

$$\sum_{n\geq 0}\bar{t}^{(8)}(8n+7)q^n \equiv 3\frac{f_2^2 f_3^3 f_{18}}{f_1^4 f_6} \pmod{12}. \tag{7.11}$$

But, from (3.10)

$$\frac{f_2^2 f_3^3 f_{18}}{f_1^4 f_6} \equiv \frac{f_3^3 f_{18}}{f_6} \pmod{4}. \tag{7.12}$$

Invoking (7.11) and (7.12), we find that

$$\sum_{n\geq 0}\bar{t}^{(8)}(8n+7)q^n \equiv 3\frac{f_3^3 f_{18}}{f_6} \pmod{12}, \tag{7.13}$$

which yields the desired results (7.2) and (7.3). □

Theorem 7.2 *For any prime* $p\geq 5$ *with* $\left(\frac{-6}{p}\right)=-1$, $1\leq j\leq p-1$, $\alpha\geq 0$, *we have*

$$\sum_{n\geq 0}\bar{t}^{(8)}\left(24p^{2\alpha}n+7p^{2\alpha}\right)q^n \equiv 3f_1 f_6 \pmod{6}, \tag{7.14}$$

and for each $n\geq 0$,

$$\bar{t}^{(8)}\left(24p^{2\alpha+1}(pn+j)+7p^{2\alpha+2}\right) \equiv 0 \pmod{6}. \tag{7.15}$$

Proof From (3.9), we have

$$\frac{f_3^3 f_{18}}{f_6} \equiv f_3 f_{18} \pmod{2}. \tag{7.16}$$

Using (7.16), we can rewrite (7.13) as

$$\sum_{n\geq 0}\bar{t}^{(8)}(8n+7)q^n \equiv 3f_3 f_{18} \pmod{6}, \tag{7.17}$$

which yields

$$\sum_{n\geq 0}\bar{t}^{(8)}(24n+7)q^n \equiv 3f_1 f_6 \pmod{6}. \tag{7.18}$$

For a prime $p\geq 5$ and $-(p-1)/2\leq k, m\leq (p-1)/2$, consider

$$\frac{3k^2+k}{2}+6\times\frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod{p},$$

which is equivalent to

$$(6k+1)^2 + 6(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-6}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p - 1)/6$. Therefore, using Lemma 2.3, we have

$$\sum_{n\geq 0} \overline{t}^{(8)} \left(24\left(p^2 n + \frac{7p^2-7}{24}\right)+7\right) q^n \equiv 3 f_1 f_6 \pmod{6}. \tag{7.19}$$

Invoking (7.18) and (7.19), we arrive at

$$\sum_{n\geq 0} \overline{t}^{(8)} \left(24p^2 n + 7p^2\right) q^n \equiv \sum_{n=0}^{\infty} \overline{t}^{(8)} (24n+7) q^n \equiv 3 f_1 f_6 \pmod{6}. \tag{7.20}$$

The result (7.14) follows from the above equation and by induction on α.

Substituting (2.13) into (7.14), we deduce that

$$\begin{aligned}\sum_{n\geq 0} & \overline{t}^{(8)} \left(24p^{2\alpha} n + 7p^{2\alpha}\right) q^n \\ &\equiv 3 \left(\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + q^{\frac{p^2-1}{24}} f_{p^2} \right) \\ &\times \left(\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} q^{6\times\frac{3k^2+k}{2}} f\left(-q^{6\times\frac{3p^2+(6k+1)p}{2}}, -q^{6\times\frac{3p^2-(6k+1)p}{2}}\right) \right. \\ &\left. + q^{6\times\frac{p^2-1}{24}} f_{6p^2} \right) \pmod{6}, \end{aligned} \tag{7.21}$$

which yields

$$\sum_{n\geq 0} \overline{t}^{(8)} \left(24p^{2\alpha+1} n + 7p^{2\alpha+2}\right) q^n \equiv 3 f_p f_{6p} \pmod{6}. \tag{7.22}$$

Equating the coefficients of q^{pn+j} for $j = 1, 2, \ldots, p-1$ in (7.22), we obtain (7.15). □

Acknowledgements The authors would like to thank the referees for helpfull suggestions and comments.

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