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Effects of variable viscosity and temperature modulation on linear Rayleigh-Bénard convection in Newtonian dielectric liquid*

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Abstract The linear Rayleigh-Bénard electro-convective stability of the Newtonian dielectric liquid is determined theoretically subject to the temperature modulation with time. A perturbation method is used to compute the critical Rayleigh number and the wave number. The critical Rayleigh number is calculated as a function of the frequency of modulation, the temperature-dependent variable viscosity, the electric field dependent variable viscosity, the Prandtl number, and the electric Rayleigh number. The effects of all three cases of modulations are established to delay or advance the onset of the convection process. In addition, how the effect of variable viscosity controls the onset of convection is studied.

Key words dielectric liquid, temperature dependent variable viscosity, electric field dependent variable viscosity, electric Rayleigh number, temperature modulation

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Nomenclature

a,	wave number;
D,	electric displacement;
E,	electric field;
E_0,	reference electric field;
g,	gravitational acceleration, $(0, 0, -g)$;
L,	electric Rayleigh number;
P,	dielectric polarization;
Pr,	Prandtl number;
Ra_{c},	critical Rayleigh number;
p,	effective pressure;
q,	velocity vector, (u, v, w);
T,	temperature;
T_0,	reference temperature;
∇,	vector differential operator;
V_T,	temperature dependent variable viscosity;
V_E,	electric field dependent variable viscosity;
e,	positive free charge.

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Greek symbols

α, coefficient of thermal expansion;
β, small amplitude of the temperature modulation;
κ_1, thermal diffusivity;
κ, thermal conductivity;
ϕ, electric potential;
φ, phase angle;
ω, modulation frequency;
μ, temperature and electric field strength dependent variable viscosity;
ρ, fluid density;
ρ_0, reference density at $T = T_0$;
ϵ_0, electric permittivity;
ϵ_{r}, relative permittivity or dielectric constant;
χ_{e}, electric susceptibility.

Subscripts

b, basic state;
c, critical quantity;
0, reference value.

Superscripts

$'$, dimensionless quantity;
T, transpose;
$*$, dimensionless quantity.

1 Introduction

The study of Rayleigh-Bénard electroconvection in Newtonian dielectric liquid has many applications in various fields such as geothermal engineering, chemical engineering, insulating oils, power operators (power cables, power capacitors, circuit breaks, transformers, and so on), and material science engineering. The potential functionalities of these liquids include acting as a coolant, serving as a diagnostic medium, and providing electrical insulation. The problem of temperature modulation with the effect of alternating current (AC) or direct current (DC) electric field on convective instability in a dielectric liquid layer has been well surveyed. The convection can occur in a dielectric liquid layer, if the temperature gradient is destabilizing which is similar to Rayleigh-Bénard instability. The preliminary experiments in a uniform dielectric liquid layer were not established theoretically, and furthermore the effect of non-uniform temperature gradient controlled by time and position within a slight change in an AC electric field was not considered. This non-uniform temperature gradient can be attained by solving the energy equation with appropriate time-dependent temperature boundary conditions which controls the convection externally. However, the non-uniform temperature gradient reveals its provenance in transient cooling or heating at the walls. The basic state temperature depends on time and position explicitly and was studied by Gross and Porter[1] and Gross[2]. This phenomenon is known as temperature modulation.

The principal examination here is of the effect of modulated temperature on the onset of thermal volatility in a Newtonian liquid. Venezian[3] determined the onset of convection by the perturbation method in terms of amplitude oscillation and recognized that the onset of convection can be postponed or progressed by in-phase or out-of-phase modulation of the boundary temperatures, respectively, when contrasted with the unmodulated system. If heat is applied rapidly, the fundamental temperature gradient is non-uniform, being a function of time and position. The effect of non-uniform temperature gradient on low-frequency modulation of thermal convection was studied by Rosenblat and Herbert[4]. They got the asymptotic arrangement with arbitrary amplitude and made this examination with the known trial results. Rosenblat and Tanaka[5] considered the effect of thermal modulation on the onset of Rayleigh-Bénard convection, when the temperature gradient is time-periodic. In general, they established that change in the critical Rayleigh number is seen. At the same time, Nield[6] examined that the non-uniform temperature gradient controls convection in the absence of an electric field. In particular, if heat is supplied gradually, the fundamental temperature gradient is uniform and convection as a rule shows as rolls, and a number of studies on this in the presence of electric field[7] were investigated. Later, Siddheshwar and Annamma[8], Singh and Bajaj[9], and Siddheshwar and Annamma[10] examined the onset of thermal convection in a horizontal layer under the effect of temperature modulation. The thermal instability in a dielectric liquid layer in the dielectric liquid layer was subject to synchronous/asynchronous time-periodic temperatures. The consequence of thermal modulation in a horizontal dielectric liquid under the temperature gradient and simultaneous action of an electric field was investigated by Siddheshwar and Annamma[11], Finucane and Kelly[12], and Malkus and Veronis[13].

The combined effect of a uniform electric field and a non-uniform temperature field in a dielectric liquid was studied by Siddheshwar and Radhakrishna[14] using both linear and non-linear stability analyses. A weak nonlinear stability in a Newtonian liquid, confined between two parallel surfaces, subject to imposed time-periodic boundary temperature or gravity modulation, was investigated by Siddheshwar et al.[15] in a dielectric liquid with an AC electric field. Further, Pranesh and Sangeetha[16] studied the effect of time-periodic wall temperature of infinitesimal amplitude under an AC electric field in dielectric couple stress fluids using the linear stability theory. The Venezian approach is apt for obtaining the critical Rayleigh number.

The temperature modulation in a flat liquid layer within two rigid parallel plates, heating in a time periodic manner with two frequencies forcing and using the Fourier-Floquet method was examined by Puneet et al.[17]. Kiran and Narasimhulu[18] studied heat transfer with the effect of modulation parameters in a dielectric liquid using the Ginzburg-Landau model. The flow behavior of the Newtonian liquid based on viscosity and heat transfer has been investigated by Sadaf[19]. In this survey, we observe that the variable viscosity effect is missing. Therefore, the present study concentrates on the effects of variable viscosity and temperature modulation on the onset of electro-convection in a Newtonian dielectric liquid, and this study focuses on external control of convection in dielectric liquids.

2 Mathematical formulation

We consider an unbounded flat layer of the Newtonian dielectric liquid restricted between two parallel plates at a separation d apart. The Cartesian coordinate system is taken with the bottom plate in the xy-plane and z-axis, vertically upwards. The bottom plate at $z = 0$ and the upper plate at $z = d$ are maintained with temperatures $T_0 + \frac{\Delta T}{2}(1 + \beta\cos(\omega t))$ and $T_0 - \frac{\Delta T}{2}(1 - \beta\cos(\omega t + \varphi))$ subsequently (see Fig. 1). A uniform electric field E_0 is applied in the vertical direction.

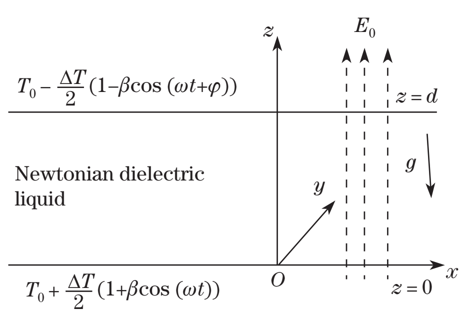

Fig. 1 Schematic of the flow configuration

The basic mathematical equations for the Rayleigh-Bénard convection in the Newtonian dielectric liquid are

$$\nabla \cdot q = 0, \tag{1}$$

$$\rho_0\Big(\frac{\partial q}{\partial t} + (q \cdot \nabla)q\Big) = -\nabla p - \rho g\hat{k} + \nabla \cdot \big(\mu\,(E,T)\,(\nabla q + \nabla q^{\mathrm{T}})\big) + (P \cdot \nabla)E, \tag{2}$$

$$\frac{\partial T}{\partial t} + (q \cdot \nabla)\,T = \kappa \nabla^2 T, \tag{3}$$

$$\rho = \rho_0\,(1 - \alpha\,(T - T_0))\,, \tag{4}$$

$$\nabla \cdot D = 0, \tag{5}$$

$$\nabla \times E = 0, \tag{6}$$

$$D = \epsilon_0 E + P, \quad P = \epsilon_0\,(\epsilon_{\mathrm{r}} - 1)\,E, \tag{7}$$

$$\epsilon_{\mathrm{r}} = \epsilon_{\mathrm{r}}^0 - e\,(T - T_0)\,, \tag{8}$$

$$\mu\,(E,T) = \frac{\mu_0}{1 + \delta_T\,(T - T_0) - \delta_E\,(E - E_0)}, \tag{9}$$

$$T\,(0,t) = T_0 + \frac{\Delta T}{2}\,(1 + \beta\cos(\omega t))\,, \tag{10}$$

$$T\,(d,t) = T_0 - \frac{\Delta T}{2}\,(1 - \beta\cos(\omega t + \varphi))\,. \tag{11}$$

Here, $\widehat{k}$ is the unit vector in the z-direction, α and e are positive, and $\epsilon_{\mathrm{r}}^0 = (1 + \chi_{\mathrm{e}})$, where χ_{e} is the electric susceptibility.

The Newtonian dielectric liquid in the basic state is characterized as follows:

$$\begin{cases} q_{\mathrm{b}} = 0, \quad \mu = \mu_{\mathrm{b}}\,(z), \quad T = T_{\mathrm{b}}\,(z), \\ \rho = \rho_{\mathrm{b}}\,(z,t)\,, \quad p = p_{\mathrm{b}}\,(z,t)\,, \quad P = P_{\mathrm{b}}\,(z)\,, \quad E = E_{\mathrm{b}}\,(z)\,. \end{cases} \tag{12}$$

Substituting Eq. (12) into fundamental governing equations (1)$-$(8), we obtain the solutions of quiescent state as follows:

$$\frac{\partial p_{\mathrm{b}}}{\partial z} + \rho_{\mathrm{b}} g + P_{\mathrm{b}}\frac{\partial E_{\mathrm{b}}}{\partial z} = 0, \tag{13}$$

$$\frac{\partial T_{\mathrm{b}}}{\partial t} = \kappa\frac{\partial^2 T_{\mathrm{b}}}{\partial z^2}. \tag{14}$$

We might just need the temperature field T_{b}, because Eq. (14) is linear and comprises the sum of a reference temperature field T_0 and the oscillating part $\beta F(z,t)$,

$$T_{\mathrm{b}} = T_0 + \frac{\Delta T}{2}\Big(1 - \frac{2z}{d}\Big) + \beta F(z,t), \tag{15}$$

$$F(z,t) = \mathrm{Re}\{H(\lambda)\mathrm{e}^{\frac{\lambda z}{d}} + H(-\lambda)\mathrm{e}^{\frac{-\lambda z}{d}}\}\mathrm{e}^{-\mathrm{i}\omega t}, \tag{16}$$

where Re stands for the real part, and the other parameters are

$$\begin{cases} \mu_{\mathrm{b}}(z) = \dfrac{\mu_1}{1 + V_T G_1(z) - V_E \frac{e G_1(z)}{(1+\chi_{\mathrm{e}})\left(1 - \frac{e G_1(z)}{1+\chi_{\mathrm{e}}}\right)}}, \\ \rho_{\mathrm{b}}(z) = \rho_0\,(1 - \alpha \Delta T G_1(z))\,, \\ \epsilon_{\mathrm{rb}}\,(z) = (1 + \chi_{\mathrm{e}})\,\Big(1 - \dfrac{e G_1\,(z)}{1 + \chi_{\mathrm{e}}}\Big), \\ E_{\mathrm{b}} = \dfrac{(1 + \chi_{\mathrm{e}})\,E_0}{(1 + \chi_{\mathrm{e}}) - e G_1\,(z)}, \\ P_{\mathrm{b}} = \epsilon_0\,(1 + \chi_{\mathrm{e}})\,E_0\Big(1 - \dfrac{1}{(1 + \chi_{\mathrm{e}}) - e G_1\,(z)}\Big)\hat{k}. \end{cases} \tag{17}$$

The solution to Eq. (14) that satisfies the temperature boundary conditions (10) and (11) is

$$G_1(z) = T_\mathrm{b} - T_0 = \frac{\Delta T}{2}\Big(1 - \frac{2z}{d}\Big) + \beta \mathrm{Re}\Big\{ \Big(H(\lambda)\mathrm{e}^{\frac{\lambda z}{d}} + H(-\lambda)\mathrm{e}^{\frac{-\lambda z}{d}}\Big)\,\mathrm{e}^{-\mathrm{i}\omega t}\Big\}.$$

Here,

$$\begin{cases} \lambda = (1-\mathrm{i})\Big(\dfrac{\omega d^2}{2\kappa_1}\Big)^{\frac{1}{2}}, \\ H(\lambda) = \dfrac{\Delta T}{2}\Big(\dfrac{\mathrm{e}^{-\mathrm{i}\varphi} - \mathrm{e}^{-\lambda}}{\mathrm{e}^{\lambda} - \mathrm{e}^{-\lambda}}\Big). \end{cases} \tag{18}$$

Presently, we superpose small disturbances to the system of this fundamental state to analyze the stability of the system.

2.1 Linear stability analysis

The fundamental state is disturbed by the small temperature disturbance so that

$$\begin{cases} q = q_\mathrm{b} + q', \quad T = T_\mathrm{b} + T', \quad \rho = \rho_\mathrm{b} + \rho', \quad p = p_\mathrm{b} + p', \quad \mu = \mu_\mathrm{b} + \mu', \\ E = E_\mathrm{b} + (E_1', E_3'), \quad P = P_\mathrm{b} + (P_1', P_3'). \end{cases} \tag{19}$$

Here, the prime denotes perturbation. Equation (5) on linearization leads to

$$P_1' = \epsilon_0\chi_\mathrm{e}E_1', \quad P_2' = \epsilon_0\chi_\mathrm{e}E_2', \quad P_3' = \epsilon_0\chi_\mathrm{e}E_3' - e\epsilon_0E_0T', \tag{20}$$

where it is implicit that $e\Delta T \ll (1+\chi_\mathrm{e})$. Equation (6) implies that it can be written as $E' = \nabla\phi'$. Introducing the electric potential ϕ', substituting Eq. (19) into Eq. (2) to eliminate the pressure p, and neglecting the primes, we get the vorticity transport equation in the following form:

$$\begin{aligned} \rho_0\frac{\partial}{\partial t}\left(\nabla^2 w\right) =& \epsilon_0 eE_0\frac{\partial T_\mathrm{b}}{\partial z}\nabla_1^2(D\phi) - \frac{\epsilon_0 e^2E_0^2}{1+\chi_\mathrm{e}}\frac{\partial T_\mathrm{b}}{\partial z}\nabla_1^2 T + \mu_\mathrm{b}(z)\nabla^4 w \\ &+ 2\frac{\partial \mu_\mathrm{b}}{\partial z}\frac{\partial}{\partial z}\left(\nabla^2 w\right) + \frac{\partial^2\mu_\mathrm{b}}{\partial z^2}\Big(\frac{\partial^2}{\partial z^2} - \nabla_1^2\Big)w + \alpha\rho_0 g\nabla_1^2 T. \end{aligned} \tag{21}$$

Substituting Eq. (19) into Eq. (3) and neglecting primes, we obtain

$$\frac{\partial T}{\partial t} - \nabla^2 T = -w\frac{\partial T_\mathrm{b}}{\partial z}. \tag{22}$$

Using Eq. (20) in Eq. (5), we get

$$\nabla_1^2\phi - \frac{eE_0}{1+\chi_\mathrm{e}}DT = 0. \tag{23}$$

We introduce the dimensionless variables such as

$$t^* = \frac{t\kappa}{d^2}, \quad w^* = \frac{wd}{\kappa}, \quad (x^*, y^*, z^*) = \Big(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\Big), \quad T^* = \frac{T}{\Delta T}, \quad \phi^* = \frac{\phi(1+\chi_\mathrm{e})}{eE_0(\Delta T)d}. \tag{24}$$

Substituting Eq. (24) into Eqs. (21)–(23) and neglecting * (asterisk) for simplicity, we obtain

$$\begin{aligned} \frac{1}{Pr}\frac{\partial}{\partial t}\left(\nabla^2 w\right) =& L\nabla_1^2 T - L\frac{\partial}{\partial z}\left(\nabla_1^2\phi\right) + g_1(z)\nabla^4 w + 2Dg_1(z)\nabla^2\Big(\frac{\partial w}{\partial z}\Big) \\ &+ D^2 g_1(z)\Big(\frac{\partial^2 w}{\partial z^2} - \nabla_1^2 w\Big) + Ra\nabla_1^2 T, \end{aligned} \tag{25}$$

where

$$g_1(z) = 1 - V_T G(z) + V_E\Big(\frac{eG(z)}{1+\chi_{\mathrm{e}}} + \Big(\frac{eG(z)}{1+\chi_{\mathrm{e}}}\Big)^2\Big). \tag{26}$$

Then,

$$\begin{cases} \dfrac{\partial T}{\partial t} + w\dfrac{\partial T_0}{\partial z} = \nabla^2 T, \\ \nabla^2\phi - \dfrac{\partial T}{\partial z} = 0. \end{cases} \tag{27}$$

Here, the dimensionless quantities (w, T, ϕ) are the velocity, the temperature, and the electric potential, respectively.

The dimensionless parameters are found in Eq. (25) as follows.

The Prandtl number is

$$Pr = \frac{\mu_0}{\rho_0\kappa}.$$

The Rayleigh number is

$$Ra = \frac{\alpha\rho_0 g d^3\Delta T}{\mu_0\kappa}.$$

The electric Rayleigh number is

$$L = \frac{\epsilon_0 E_0^2 e^2 d^2\Delta T}{\mu_0\kappa(1+\chi_{\mathrm{e}})}.$$

The temperature dependent variable viscosity is

$$V_T = \delta_T\Delta T.$$

The electric field dependent variable viscosity is

$$V_E = \delta_E E_0.$$

In Eq. (26), $\frac{\partial T_0}{\partial z}$ is the dimensionless form of $\frac{\partial T_{\mathrm{b}}}{\partial z}$, where

$$\frac{\partial T_0}{\partial z} = -1 + \beta F(z,t), \tag{28}$$

$$F(z,t) = \mathrm{Re}\left\{\left(H(\lambda)\,\mathrm{e}^{\lambda z} + H(-\lambda)\,\mathrm{e}^{-\lambda z}\right)\mathrm{e}^{-\mathrm{i}\omega t}\right\}, \tag{29}$$

$$H(\lambda) = \frac{\lambda}{2}\Big(\frac{\mathrm{e}^{-\mathrm{i}\varphi} - \mathrm{e}^{-\lambda}}{\mathrm{e}^{\lambda} - \mathrm{e}^{-\lambda}}\Big). \tag{30}$$

The free-free isothermal boundary conditions for solving Eqs. (25)–(27) are

$$w = \frac{\partial^2 w}{\partial z^2} = T = \frac{\partial\phi}{\partial z} = 0 \quad \text{at} \quad z = 0, 1. \tag{31}$$

Eliminating T and ϕ from Eqs. (25)–(27), we obtain the following differential equation for w of order 8:

$$\Big(\Big(\frac{\partial}{\partial t} - \nabla^2\Big)\Big(\frac{1}{Pr}\frac{\partial}{\partial t} - (1 + V_E - V_T)\nabla^2\Big)\nabla^4 + L\frac{\partial T_0}{\partial z}\nabla_1^2 + \nabla^2\frac{\partial T_0}{\partial z}Ra\nabla_1^2\Big)w = 0. \tag{32}$$

The boundary conditions for velocity in the dimensionless form for solving Eq. (32) are obtained from Eqs. (25)–(27), and Eq. (31) is represented as follows:

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = \frac{\partial^6 w}{\partial z^6} = 0 \quad \text{at} \quad z = 0, 1. \tag{33}$$

2.2 Method of solution

We are presently seeking the eigenfunctions w and eigenvalues Ra of Eqs. (32) and (33) for the temperature form so as to vanish from the linear form $\frac{\partial T_0}{\partial z} = -1$ with the quantities of order β. The resultant eigenfunctions and eigenvalues which are attained in the current problem to differ as of those related to the standard Bénard problem of quantities in order β. The expansion is given as

$$\begin{cases} w = w_0 + \beta w_1 + \beta^2 w_2 + \cdots , \\ Ra = Ra_0 + \beta Ra_1 + \beta^2 Ra_2 + \cdots . \end{cases} \tag{34}$$

This form was used in connection with convective problems to study the effects of finite amplitude convection by Venezian[3]. Substituting Eq. (34) into Eq. (32) and likening the coefficients of different powers of β on either side of the subsequent condition, we can obtain

$$L_1 w_0 = 0, \tag{35}$$

$$L_1 w_1 = -LF\nabla_1^4 w_0 + Ra_1 \nabla^2 \nabla_1^2 w_0 - Ra_0 F \nabla^2 \nabla_1^2 w_0, \tag{36}$$

$$L_1 w_2 = -LF\nabla_1^4 w_1 + Ra_1 \nabla^2 \nabla_1^2 w_1 + Ra_2 \nabla^2 \nabla_1^2 w_0 - Ra_0 F \nabla^2 \nabla_1^2 w_1 - Ra_1 F \nabla^2 \nabla_1^2 w_0. \tag{37}$$

The function w_0 is the solution to the classical Bénard problem with $\beta = 0$.

Here,

$$L_1 = \Big(\frac{\partial}{\partial t} - \nabla^2\Big)\Big(\frac{1}{Pr}\frac{\partial}{\partial t} - (1 + V_E - V_T)\,\nabla^2\Big)\nabla^4 - L\nabla_1^4 - Ra\nabla^2\nabla_1^2. \tag{38}$$

We accept the slightly stable solution to Eq. (35) in the form of

$$w_0^{(n)} = \sin(n\pi z)\exp(\mathrm{i}(lx + my)), \quad n = 1, 2, 3, \cdots , \tag{39}$$

where l and m are the wave numbers in the x- and y-directions, respectively, such that $l^2 + m^2 = a^2$. The corresponding eigenvalues are given by

$$Ra_0^{(n)} = \frac{\delta_n^6 (1 + V_E - V_T)}{a^2} - \frac{La^2}{\delta_n^2}, \tag{40}$$

where $\delta_n^2 = n^2\pi^2 + a^2$. For a fixed value of a, the least eigenvalue (see Ref. [3]) for $L = 0$ is

$$Ra_0 = \frac{(\pi^2 + a^2)^3}{a^2} \tag{41}$$

corresponding to $w_0 = \sin(\pi z)$, which is used as the starting-point of our solution. Equation (36) for w_1 then reads

$$L_1 w_1 = \left(-LF\nabla_1^4 + Ra_1\nabla^2\nabla_1^2 - Ra_0 F \nabla^2 \nabla_1^2\right) w_0. \tag{42}$$

If Eq. (42) is a solution, the right-hand side must be orthogonal to the operator L_1. This involves that the part independent of time in Eq. (42) should be perpendicular to $\sin(\pi z)$ of the right-hand side as F differs with respect to time resulting in the sine wave. The main substantial term in the right-hand side of Eq. (42) is $Ra_1 a^2 \delta_1^2 \sin(\pi z)$, in case $Ra_1 = 0$. Then, it has taken after that all the odd coefficients are equal to zero, i.e., $Ra_1 = Ra_3 = Ra_5 = \cdots = 0$ in Eq. (34).

To solve Eq. (42), we expand the right-hand side by using Fourier series expansion and inverting the operator L_1 to obtain w_1, and we require Fourier series for expansion of $\mathrm{e}^{\lambda z}$. For resulting steps in the problem, we need the expression of $\mathrm{e}^{\lambda z}\sin(m\pi z)$. It can be found by

$$Q_{nm}(\lambda) = 2\int_0^1 \mathrm{e}^{\lambda z}\sin(n\pi z)\sin(m\pi z)\mathrm{d}z = -\frac{4nm\pi^2\lambda\left(1 + (-1)^{n+m+1}\mathrm{e}^{\lambda}\right)}{\left(\lambda^2 + (n-m)^2\pi^2\right)\left(\lambda^2 + (n+m)^2\pi^2\right)} \tag{43}$$

so that

$$\mathrm{e}^{\lambda z}\sin(m\pi z)=\sum_{n=1}^{\infty}Q_{nm}\sin(m\pi z). \tag{44}$$

It is convenient to define

$$L_1(\omega,n)=-\frac{\omega^2}{Pr}\delta_n^4+\delta_n^8(1+V_E-V_T)-La^4-Ra_0a^2\delta_n^2-\mathrm{i}\omega\delta_n^6\Big(\frac{1}{Pr}+(1+V_E-V_T)\Big). \tag{45}$$

It follows

$$L_1\sin(n\pi z)\mathrm{e}^{-\mathrm{i}\omega t}=L_1(\omega,n)\sin(n\pi z)\mathrm{e}^{-\mathrm{i}\omega t}. \tag{46}$$

Now, Eq. (42) reads

$$L_1w_1=\left(-La^4-Ra_0a^2\delta_1^2\right)\mathrm{Re}\left\{\sum\left(H(\lambda)Q_{n1}(\lambda)+H(-\lambda)Q_{n1}(-\lambda)\right)\mathrm{e}^{-\mathrm{i}\omega t}\sin(n\pi z)\right\}$$

so that

$$w_1=\left(-La^4-Ra_0a^2\delta_1^2\right)\mathrm{Re}\left\{\sum\frac{B_n(\lambda)}{L_1(\omega,n)}\mathrm{e}^{-\mathrm{i}\omega t}\sin(n\pi z)\right\}, \tag{47}$$

where

$$B_n(\lambda)=H(\lambda)Q_{n1}(\lambda)+H(-\lambda)Q_{n1}(-\lambda). \tag{48}$$

For w_2, Eq. (37) becomes

$$L_1w_2=-a^2F\left(La^2+Ra_0\delta_n^2\right)w_1+Ra_2a^2\delta_1^2w_0. \tag{49}$$

Equation (49) is not required for solution, but can be simply used to determine Ra_2. The solvency condition requires that the constant part of the right-hand side should be orthogonal to $\sin(\pi z)$, and accordingly,

$$Ra_2=2\Big(\frac{La^2+Ra_0\delta_n^2}{\delta_1^2}\Big)\int_0^1\overline{Fw_1}\sin(\pi z)\,\mathrm{d}z. \tag{50}$$

Here, the bar denotes the time average.

From Eq. (42), we obtain

$$F\sin(\pi z)=\frac{1}{-La^4-Ra_0a^2\delta_1^2}L_1w_1$$

so that

$$\begin{aligned}\overline{Fw_1}\sin(\pi z)&=\frac{1}{-La^4-Ra_0a^2\delta_1^2}\overline{w_1L_1w_1}\\&=\frac{a^2\left(-La^2-Ra_0\delta_1^2\right)}{2}\mathrm{Re}\left\{\sum\frac{B_n(\lambda)}{L_1(\omega,n)}\sin(n\pi z)\sum B_n^*(\lambda)\right\},\end{aligned} \tag{51}$$

and

$$\begin{aligned}Ra_2&=\Big(\frac{-La^2-Ra_0\delta_1^2}{2\delta_1^2}\Big)\mathrm{Re}\left\{\sum\left(La^4+Ra_0a^2\delta_n^2\right)\frac{|B_n(\lambda)|^2}{L_1(\omega,n)}\right\}\\&=\Big(\frac{-La^2-Ra_0\delta_1^2}{2\delta_1^2}\Big)\sum_{n=1}^{\infty}\left(La^4+Ra_0a^2\delta_n^2\right)\frac{|B_n(\lambda)|^2}{|L_1(\omega,n)|^2}\Big(\frac{L_1(\omega,n)+L_1^{\mathrm{c}}(\omega,n)}{2}\Big),\end{aligned} \tag{52}$$

where $L_1^{\mathrm{c}}(\omega, n)$ is the conjugate of $L_1(\omega, n)$. To calculate the critical value of Ra_2, replace a by a_0 in Ra_2, where a_0 is the critical value obtained in getting from Ra_0 in Eq. (41). We have the following three cases to evaluate $Ra_{2\mathrm{c}}$.

Case 1 $\varphi = 0$ (in-phase modulation)

In this particular case,

$$B_n(\lambda) = \begin{cases} d_n & \text{if} \quad n \text{ is even}, \\ 0 & \text{if} \quad n \text{ is odd}. \end{cases}$$

Case 2 $\varphi = \pi$ (out-of-phase modulation)

Especially,

$$B_n(\lambda) = \begin{cases} 0 & \text{if} \quad n \text{ is even}, \\ d_n & \text{if} \quad n \text{ is odd}. \end{cases}$$

Case 3 $\varphi = -\mathrm{i}\infty$

The bottom wall is modulated, and the upper wall is fixed with a constant temperature. In this case $B_n(\lambda) = -d_n$ for all n, where

$$d_n = -\frac{4n\pi^2\lambda^2}{(\lambda^2 + (n+1)^2\pi^2)(\lambda^2 + (n-1)^2\pi^2)}.$$

The variable λ is represented in Eq. (18), as far as the frequency in the dimensionless form is

$$\lambda = (1 - \mathrm{i})\left(\frac{\omega}{2}\right)^{\frac{1}{2}},$$

and thus

$$|d_n|^2 = \frac{16n^2\pi^4\omega^2}{(\omega^2 + (n+1)^4\pi^4)(\omega^2 + (n-1)^4\pi^4)}.$$

Hence, from Eq. (52) and using $d_n(\lambda)$, we get the expression for $Ra_{2\mathrm{c}}$ in the following form:

$$Ra_{2\mathrm{c}} = \Big(\frac{-La^2 - Ra_0\delta_1^2}{2\delta_1^2}\Big)\sum_{n=1}^{\infty}\left(La^4 + Ra_0a^2\delta_n^2\right)\left|\frac{d_n(\lambda)}{L_1(\omega, n)}\right|^2\Big(\frac{L_1(\omega, n) + L_1^{\mathrm{c}}(\omega, n)}{2}\Big). \quad (53)$$

In Eq. (53), for Case 1, n is even, for Case 2, n is odd, and for Case 3, n is either even or odd. Equation (53) converges in all cases rapidly. The graphs of $Ra_{2\mathrm{c}}$ versus ω for various values of L, V_T, and V_E are interpreted in Figs. 2–13.

3 Results and discussion

The effects of the temperature-dependent variable viscosity, the electric field dependent variable viscosity, and the temperature modulation on linear Rayleigh-Bénard convection in the Newtonian dielectric liquid with free-free isothermal boundaries are studied. Our results in respect of linear Rayleigh-Bénard convection agree with those of Venezian[3] when $L = 0$, $V_T = 0$, and $V_E = 0$.

Figure 2 represents the variation of $Ra_{2\mathrm{c}}$ versus ω for various magnitudes of the electric Rayleigh number L with the Prandtl number $Pr = 10$ in the absence of V_T and V_E. It can be observed that the supercritical movements occur when L is greater than 2 973 which in turn means that $Ra_{2\mathrm{c}}$ increases with the increase in L with respect to ω. Hence, L stabilizes the system. From the figure, we can notice that when $L < 2$ 973, initially $Ra_{2\mathrm{c}}$ decreases with the increment in ω, it reaches a minimum, and then it increases with the increment in ω. When $L > 2\,973$, Ra_{2c} increases with ω and reaches a maximum. After that, it decreases with ω. Here, in a dielectric liquid, the system is destabilized for smaller values of the modulation frequency ω and is stabilized for larger ω. The temperature profiles consist of a steady part in addition to a parabolic part which oscillates with time. The parabolic part is significant when the amplitude of the modulation increases. It must be recognized that the nonlinear shape is conditional for the finite amplitude instability in case the convection occurs at lower Rayleigh numbers than that of the linear theory. There is likewise a significant range of ω in which the stabilizing influence is minimum and this minimum declines with an increase in L.

Figure 3 shows the plot of Ra_{2c} versus ω for various values of L with a fixed Prandtl number $Pr = 10$, $V_T = 0.1$, and in the absence of V_E. From this figure, we observe that the supercritical movements take place when L is greater than 2 700, and Ra_{2c} increases with an increment in L. Thus, L has stabilizing effects on the system. Figure 3 shows that when $L < 2\,700$, initially Ra_{2c} decreases with an increase in ω, attains a minimum, and then rises with ω. When $L > 2\,700$, initially, Ra_{2c} shows an increment with an increase in ω, attains a maximum, and then reduces with an increase in ω. This shows that for a dielectric liquid, the system has been destabilized for smaller values of ω, and it has been stabilized for larger values of ω. We conclude that the impact of V_T is felt all over the liquid when the modulated frequency is quite low.

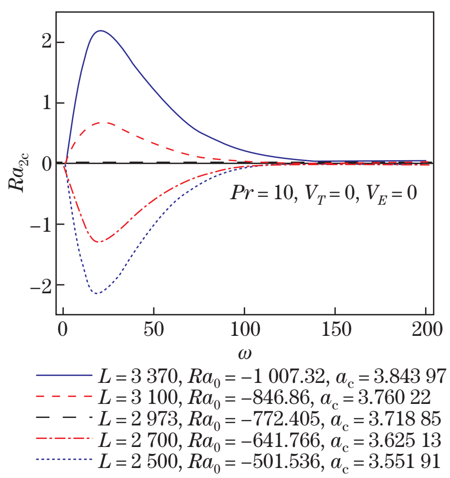

Fig. 2 Ra_{2c} versus ω for various values of L for in-phase temperature modulation, when $V_T = 0$ and $V_E = 0$ (color online)

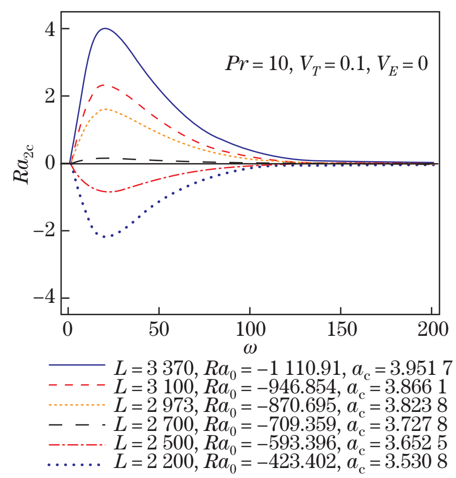

Fig. 3 Ra_{2c} versus ω for various values of L for in-phase temperature modulation, when $V_T = 0.1$ and $V_E = 0$ (color online)

The plot of Ra_{2c} versus ω for various values of L with a fixed Prandtl number $Pr = 10$ and $V_E = 0.1$ and in the absence of V_T is shown in Fig. 4. It can be noticed that when $L < 3\,120$, Ra_{2c} shows the reduction with an increase in ω, attains a minimum, and then increases with an increment in ω. When $L > 3\,120$, with an increment in ω, Ra_{2c} increases first and decreases after attaining a maximum with a further increment in ω. This shows that the system has been destabilized for smaller values of ω, and it is stabilized for larger values of ω in a dielectric liquid. We conclude that the impact of V_E is felt all over the liquid when the modulated frequency is quite low.

Figure 5 is the plot of Ra_{2c} versus ω for various values of L with a fixed Prandtl number $Pr = 10$, $V_T = 0.03$, and $V_E = 0.01$. When $L < 2\,700$, Ra_{2c} is reduced and reaches a minimum, and then it increases with an increase in ω. Furthermore, Ra_{2c} increases and reaches the peak followed by its decrement with an increment in ω as $L > 2\,700$. This shows that the system is stabilized for larger values of ω and destabilized for smaller values of ω in a dielectric liquid. The conclusion from this result is that the impacts of both V_T and V_E are felt all over the liquid when the frequency of modulation is low. From Figs. 4 and 5, we see that Ra_{2c} reduces with the increase in L.

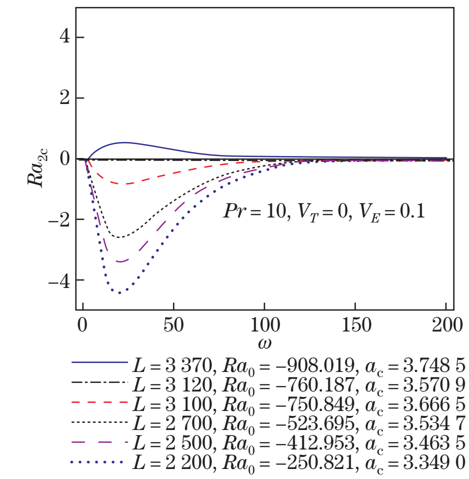

Fig. 4 Ra_{2c} versus ω for various values of L for in-phase temperature modulation when $V_E = 0.1$, $V_T = 0$, and $Pr = 10$ (color online)

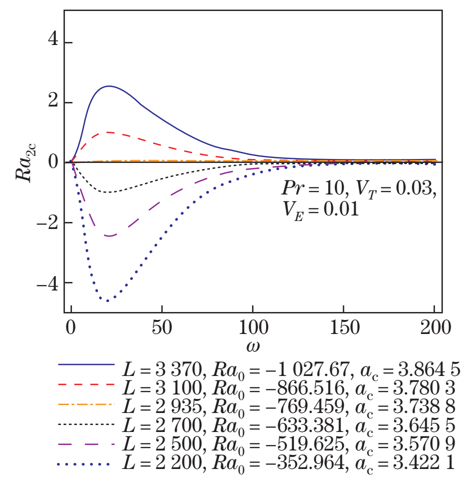

Fig. 5 Ra_{2c} versus ω for various values of L for in-phase temperature modulation when $V_T = 0.03$, $V_E = 0.01$, and $Pr = 10$ (color online)

We observe from Fig. 6 the variations of Ra_{2c} versus ω for out-of-phase temperature modulation for a fixed electric Rayleigh number L and in the absence of V_T and V_E. We find that Ra_{2c} decreases with an increment in ω. Therefore, in this case, the sub-critical motions are averted. We can also see from Fig. 7 the variation of Ra_{2c} versus ω for out-of-phase temperature modulation with a fixed electric Rayleigh number. Ra_{2c} decreases with an increase in ω for $V_T = 0.1$ and $V_E = 0$. Compaing Figs. 6 and 7, we also find that Ra_{2c} decreases with an increase in the variable viscosity parameter V_T for a fixed L.

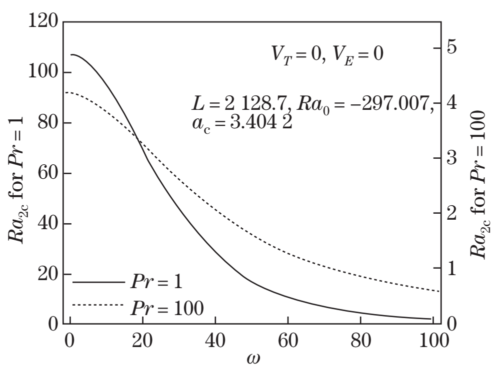

Fig. 6 Ra_{2c} versus ω for out-of-phase temperature modulation with fixed L. Note that the two cases use distinctive vertical scales

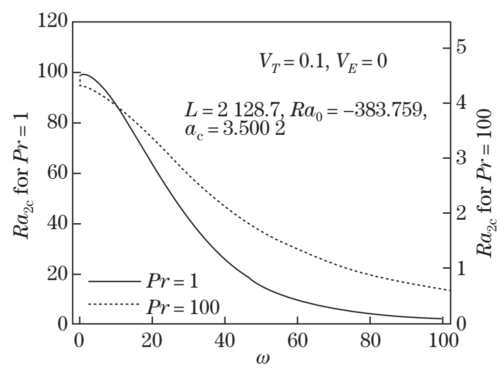

Fig. 7 Ra_{2c} versus ω for out-of-phase temperature modulation with fixed L and $V_T = 0.1$. Note that the two cases use distinctive vertical scales

We notice from Fig. 8 the variations of Ra_{2c} versus ω for out-of-phase temperature modulation with a fixed electric Rayleigh number L. We find that Ra_{2c} decreases with an increment in ω for fixed $V_E = 0.1$ and $V_T = 0$. Compaing Figs. 8 and 9, we find that $Ra_{2\rm c}$ increases with an increase in the variable viscosity parameter V_E for a fixed L.

From the results in Fig. 9, the plot of $Ra_{2\rm c}$ versus ω with a fixed electric Rayleigh number L reveals that $Ra_{2\rm c}$ decreases with an increase in ω for fixed $V_T = 0.03$ and $V_E = 0.01$. For Case 3 wherein only the bottom wall has been modulated with the upper wall being fixed steady temperature, Figs. 10–13 show that the effects of the temperature dependent variable viscosity, the electric field dependent variable viscosity, and the electric Rayleigh number on $Ra_{2\rm c}$ are qualitatively similar to the preceding two cases.

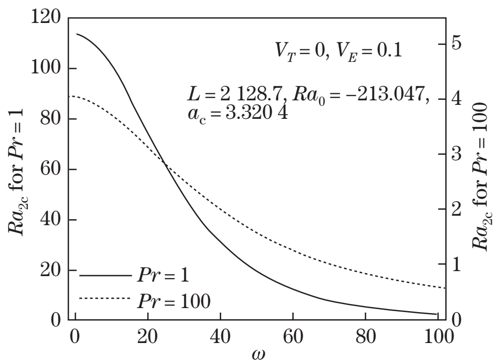

Fig. 8 $Ra_{2\rm c}$ versus ω for out-of-phase temperature modulation with fixed L, $V_T = 0$, and $V_E = 0.1$. Note that the two cases use distinctive vertical scales

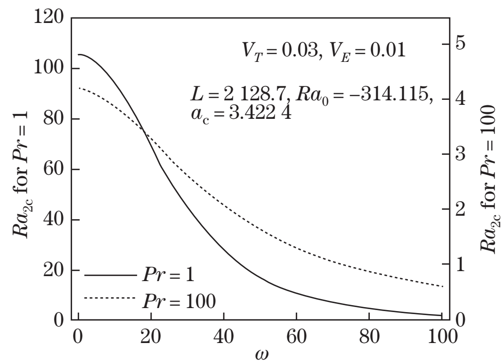

Fig. 9 $Ra_{2\rm c}$ versus ω for out-of-phase temperature modulation with fixed L, $V_T = 0.03$, and $V_E = 0.01$. Note that the two cases use distinctive vertical scales

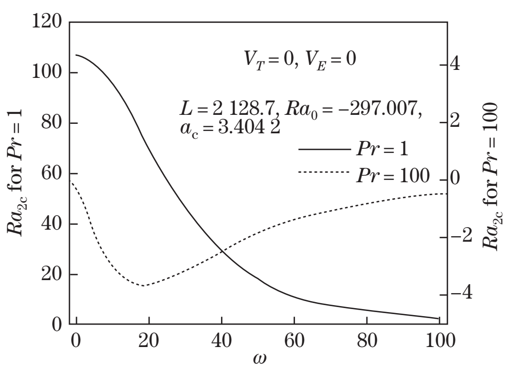

Fig. 10 $Ra_{2\rm c}$ versus ω at the point when just the temperature of the bottom wall has been modulated with fixed L and $V_T = V_E = 0$. Note that the two cases use distinctive vertical scales

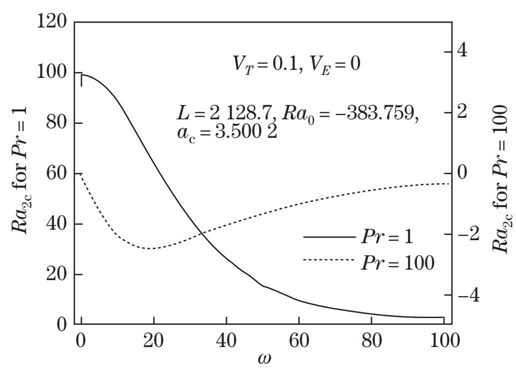

Fig. 11 $Ra_{2\rm c}$ versus ω at the point when just the temperature of the bottom wall has been modulated with fixed L, $V_E = 0$, and $V_T = 0.1$. Note that the two cases used distinctive vertical scales

4 Conclusions

The effects of variable viscosity and temperature modulation on linear Rayleigh-Bénard convection of the Newtonian dielectric liquid are examined by applying linear stability analysis. An expression for $Ra_{2\rm c}$ is obtained analytically from Eq. (53), and the numerical results are computed for different values of parameters such as Pr, V_T, V_E, and L for all the three cases of modulations. The conclusions may be drawn as follows.

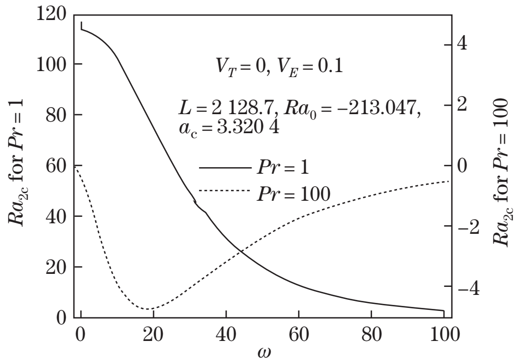

Fig. 12 $Ra_{2\mathrm{c}}$ versus ω at the point when just the temperature of the bottom wall has been modulated with fixed L, $V_T = 0$, and $V_E = 0.1$. Note that the two cases use distinctive vertical scales

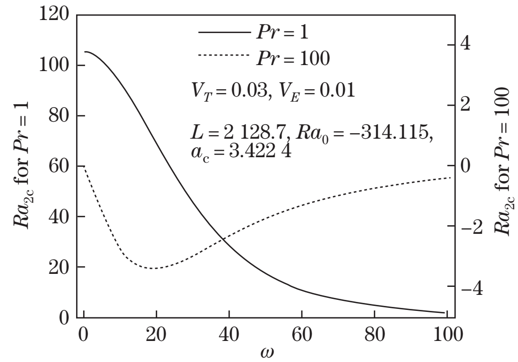

Fig. 13 $Ra_{2\mathrm{c}}$ versus ω at the point when just the temperature of the bottom wall has been modulated with fixed L, $V_T = 0.03$, and $V_E = 0.01$. Note that the two cases use distinctive vertical scales

(i) The consequence of all the three cases of modulations, i.e., in-phase, out-of-phase, and only the bottom wall temperature modulations has been found to be destabilizing compared with the un-modulated system.

(ii) At large frequencies, the impact of temperature modulation disappears in all three cases of modulation.

(iii) $Ra_{2\mathrm{c}}$ tends to zero with an increase in ω for larger values of V_T.

(iv) An increase in the values of L and V_T is to destabilize the system, while an increase in V_E is to stabilize the system in all the three cases.

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