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On 5-regular bipartitions with even parts distinct

M. S. Mahadeva Naika[1] · **T. Harishkumar**[1]

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Abstract In 2010, Andrews, Michael D. Hirschhorn and James A. Sellers considered the function $ped(n)$, the number of partition of an integer n with even parts distinct (the odd parts are unrestricted). They obtained infinite families of congruences in the spirit of Ramanujan's congruences for the unrestricted partition function $p(n)$. Let $b(n)$ denote the number of 5-regular bipartitions of a positive integer n with even parts distinct (odd parts are unrestricted). In this paper, we establish many infinite families of congruences modulo powers of 2 for $b(n)$. For example,

$$\sum_{n=0}^{\infty} b\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} n + 14 \cdot 3^{2\alpha} \cdot 5^{2\beta} + 1\right) q^n \equiv 8 f_2^3 f_5^3 \pmod{16},$$

where $\alpha, \beta \geq 0$.

Keywords Partition identities · Theta-functions · Partition congruences · Regular bipartition

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1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. An ℓ-regular partition is a partition in which none of the parts is

✉ M. S. Mahadeva Naika
msmnaika@rediffmail.com

T. Harishkumar
harishhaf@gmail.com

[1] Department of Mathematics, Central College Campus, Bangalore University, Bengaluru, Karnataka, India

divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n with $b_\ell(0) = 1$. The generating function for $b_\ell(n)$ is

$$\sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{f_\ell}{f_1},$$

where $f_\ell := (q^\ell; q^\ell)_\infty = \prod_{n=1}^{\infty} (1 - q^{n\ell})$.

Arithmetic properties of ℓ-regular partition functions have been studied by a number of mathematicians. Calkin et al. [3] established congruences for 5-regular partitions modulo 2 and for 13-regular partitions modulo 2 and 3 using the theory of modular forms. For more details, one can see [5,7,9].

In 2010, Andrews et al. [1] obtained infinite families of congruences for $ped(n)$, the number of partitions of n with even parts distinct. For more details one can see [4,11].

Recently, Lin [8] proved two infinite families of congruences modulo 3 for $Bped(n)$, the number of bipartitions of n with even parts distinct.

Let $b(n)$ denote the number of 5-regular bipartitions of n with even parts distinct with $b(0) = 1$. The generating function is given by

$$\sum_{n=0}^{\infty} b(n) q^n = \frac{f_4^2 f_5^2}{f_1^2 f_{20}^2}. \tag{1.1}$$

We prove many congruences of the following form. For $\alpha, \beta \geq 0$,

$$\sum_{n=0}^{\infty} b\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} n + 14 \cdot 3^{2\alpha} \cdot 5^{2\beta} + 1\right) q^n \equiv 8 f_2^3 f_5^3 \pmod{16}.$$

2 Preliminary results

In this section, we record several identities which are useful in proving our main results.

Lemma 2.1 *The following 2-dissections hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8} \tag{2.1}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.2}$$

For proofs, see [2, p. 40].

Lemma 2.2 *We have*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} - 3q f_9^3 + 4q^3 \frac{f_3^2 f_{18}^6}{f_6^2 f_9^3}. \tag{2.3}$$

For a proof, see [2, p. 345].

Lemma 2.3 *The following 2-dissections hold:*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \tag{2.4}$$

and

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q \frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \tag{2.5}$$

For proofs, see [7].

Lemma 2.4 *We have*

$$\frac{1}{f_1^3 f_5} = \frac{f_4^4}{f_2^7 f_{10}} - 2q \frac{f_4^6 f_{20}^2}{f_2^9 f_{10}^3} + 5q \frac{f_4^3 f_{20}}{f_2^8} + 2q^2 \frac{f_4^9 f_{40}^2}{f_2^{10} f_8^2 f_{10}^2 f_{20}}, \tag{2.6}$$

$$f_1^3 f_5 = \frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3 + 2q^2 \frac{f_4^6 f_{10} f_{40}^2}{f_2 f_8^2 f_{20}^2} \tag{2.7}$$

and

$$f_1 f_5^3 = f_2^3 f_{10} - q \frac{f_2^2 f_{10}^2 f_{20}}{f_4} + 2q^2 f_4 f_{20}^3 - 2q^3 \frac{f_4^4 f_{10} f_{40}^2}{f_2 f_8^2}. \tag{2.8}$$

For proofs, see [10].

Lemma 2.5 [2, p. 303, Entry 17(v)] *We have*

$$(q;q)_\infty = (q^{49};q^{49})_\infty \left(\frac{B(q^7)}{C(q^7)} - q \frac{A(q^7)}{B(q^7)} - q^2 + q^5 \frac{C(q^7)}{A(q^7)} \right), \tag{2.9}$$

where $A(q) = f(-q^3, -q^4)$, $B(q) = f(-q^2, -q^5)$ *and* $C(q) = f(-q, -q^6)$.

We prove the following theorem,

Theorem 2.1 *Let* $r_1 \in \{62, 78\}$, $r_2 \in \{62, 158\}$, $r_3 \in \{166, 214\}$, $r_4 \in \{142, 238\}$, $r_5 \in \{86, 134\}$, $r_6 \in \{10, 26\}$, $r_7 \in \{28, 92, 124, 156\}$, $r_8 \in \{124, 156\}$, $r_9 \in \{22, 38\}$, $r_{10} \in \{34, 66\}$, $r_{11} \in \{26, 42, 58, 74\}$, $r_{12} \in \{44, 76\}$, $r_{13} \in \{68, 132\}$ *and* $r_{14} \in \{52, 84, 116, 148\}$. *Then for* $\alpha, \beta, \gamma \geq 0$, *we have, modulo 16,*

$$\sum_{n=0}^{\infty} b\left(16 \cdot 5^{2\beta} n + 6 \cdot 5^{2\beta} + 1\right) q^n \equiv 8 f_1^9 + 8 f_4 f_5, \tag{2.10}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 5^{2\beta+1}n+14\cdot 5^{2\beta+1}+1\right)q^n \equiv f_1f_{20}+8qf_5^9, \tag{2.11}$$

$$b\left(16\cdot 5^{2\beta+2}n+r_1\cdot 5^{2\beta+1}+1\right)\equiv 0, \tag{2.12}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}n+14\cdot 3^{2\alpha}\cdot 5^{2\beta}+1\right)q^n \equiv 8f_2^3f_5^3, \tag{2.13}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+6\cdot 3^{2\alpha}\cdot 5^{2\beta+1}+1\right)q^n \equiv 8qf_1^3f_{10}^3, \tag{2.14}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}n+14\cdot 3^{2\alpha}\cdot 5^{2\beta}+1\right)q^n \equiv 8f_2f_5, \tag{2.15}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+22\cdot 3^{2\alpha}\cdot 5^{2\beta+1}+1\right)q^n \equiv 8f_1f_{10}, \tag{2.16}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}+1\right)q^n \equiv 8q^2f_6^3f_{15}^3, \tag{2.17}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}n+46\cdot 3^{2\alpha}\cdot 5^{2\beta}+1\right)q^n \equiv 8f_5f_6^3+8qf_2f_{15}^3, \tag{2.18}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+38\cdot 3^{2\alpha}\cdot 5^{2\beta+1}+1\right)q^n \equiv 8f_3^3f_{10}+8q^3f_1f_{30}^3, \tag{2.19}$$

$$b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+r_2\cdot 3^{2\alpha}\cdot 5^{2\beta}+1\right)\equiv 0, \tag{2.20}$$

$$b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+r_3\cdot 3^{2\alpha}\cdot 5^{2\beta+1}+1\right)\equiv 0, \tag{2.21}$$

$$b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+r_4\cdot 3^{2\alpha}\cdot 5^{2\beta}+1\right)\equiv 0, \tag{2.22}$$

$$b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+r_5\cdot 3^{2\alpha}\cdot 5^{2\beta+1}+1\right)\equiv 0, \tag{2.23}$$

$$b\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}n+r_6\cdot 3^{2\alpha+1}\cdot 5^{2\beta}+1\right)\equiv 0, \tag{2.24}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+12\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n \equiv 8f_1^9, \tag{2.25}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+4\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}+1\right)q^n \equiv 8q^2f_7^9, \tag{2.26}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}+1\right)q^n \equiv 8qf_5^9, \tag{2.27}$$

$$b\left(32\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_7\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.28}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+44\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n\equiv 8f_2f_3^3, \tag{2.29}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+76\cdot 3^{4\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n\equiv 8f_1f_6^3, \tag{2.30}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}+1\right)q^n \equiv 8q^2f_{10}f_{15}^3, \tag{2.31}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+92\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)q^n \equiv 8q^3f_5f_{30}^3, \tag{2.32}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 5^{2\beta}n+28\cdot 5^{2\beta}+1\right)q^n\equiv 8f_1f_{20}+8f_2^3f_5^3, \tag{2.33}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 5^{2\beta+1}n+12\cdot 5^{2\beta+1}+1\right)q^n\equiv 8f_4f_5+8qf_1^3f_{10}^3, \tag{2.34}$$

$$b\left(32\cdot 5^{2\beta+1}n+r_8\cdot 5^{2\beta}+1\right)\equiv 0 \tag{2.35}$$

and modulo 4,

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_1^3, \tag{2.36}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+1\right)q^n\equiv 2f_7^3, \tag{2.37}$$

$$b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right) \equiv \begin{cases} 2 & \text{if } n=k(3k+1)/2 \text{ for some } k\in\mathbb{Z},\\ 0 & \text{otherwise.}\end{cases} \tag{2.38}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+6\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_3^3, \tag{2.39}$$

$$b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+34\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.40}$$

$$b\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+r_9\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.41}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_5^3, \tag{2.42}$$

$$b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_{10}\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.43}$$

$$b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_{11}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.44}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_1^3, \tag{2.45}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}+1\right)q^n\equiv 2f_7^3, \tag{2.46}$$

$$b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv\begin{cases}2 & \textit{if } n=k(3k+1)/2 \textit{ for some } k\in\mathbb{Z},\\ 0 & \textit{otherwise.}\end{cases} \tag{2.47}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+6\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_3^3, \tag{2.48}$$

$$b\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+34\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.49}$$

$$b\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_9\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.50}$$

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_5^3, \tag{2.51}$$

$$b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_{10}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.52}$$

$$b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_{11}\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.53}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_1^3, \tag{2.54}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+1\right)q^n\equiv 2f_7^3, \tag{2.55}$$

$$b\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)\equiv\begin{cases}2 & \textit{if } n=k(3k+1)/2 \textit{ for some } k\in\mathbb{Z},\\ 0 & \textit{otherwise.}\end{cases} \tag{2.56}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+12\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_3^3, \tag{2.57}$$

$$b\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+68\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.58}$$

$$b\left(32\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+r_{12}\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.59}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_5^3, \tag{2.60}$$

$$b\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_{13}\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.61}$$

$$b\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_{14}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.62}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_1^3, \tag{2.63}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}+1\right)q^n\equiv 2f_7^3, \tag{2.64}$$

$$b\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)$$
$$\equiv \begin{cases} 2 & \textit{if } n=k(3k+1)/2 \textit{ for some } k\in\mathbb{Z},\\ 0 & \textit{otherwise}.\end{cases} \tag{2.65}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+12\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_3^3, \tag{2.66}$$

$$b\left(32\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+68\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.67}$$

$$b\left(32\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_{12}\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.68}$$

$$\sum_{n=0}^{\infty} b\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+4\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}+1\right)q^n\equiv 2f_5^3, \tag{2.69}$$

$$b\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_{13}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}+1\right)\equiv 0, \tag{2.70}$$

$$b\left(32\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_{14}\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}+1\right)\equiv 0. \tag{2.71}$$

3 Proof of the Theorem (2.1)

Using (2.4) in (1.1) and comparing the terms involving q^{2n+1} from both sides, we arrive at

$$\sum_{n=0}^{\infty} b(2n+1)q^n = 2\frac{f_2^5 f_5}{f_1^5 f_{10}}. \tag{3.1}$$

From the binomial theorem, it is easy to see that for any positive integers k and m,

$$f_k^{2m}\equiv f_{2k}^m \pmod 2, \tag{3.2}$$

$$f_k^{4m}\equiv f_{2k}^{2m} \pmod 4, \tag{3.3}$$

$$f_k^{8m}\equiv f_{2k}^{4m} \pmod 8. \tag{3.4}$$

Employing (2.2) and (2.4) along with (3.2) and (3.4) in (3.1), we get, modulo 16,

$$\sum_{n=0}^{\infty} b(4n+1)q^n\equiv 2\frac{f_2^2 f_4 f_{10}^2}{f_1^3 f_5 f_{20}}+8qf_2^7 f_{10}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} b\,(4n+3)\,q^n \equiv 2\frac{f_2^5 f_{20}}{f_1^4 f_4 f_{10}} + 8\frac{f_4^5}{f_1^3 f_5}. \tag{3.6}$$

Utilizing (2.2) and (2.6) in (3.6), we arrive at

$$\sum_{n=0}^{\infty} b\,(8n+3)\,q^n \equiv 2\frac{f_2 f_{10}}{f_1 f_5} + 8\frac{f_2^7}{f_1^3 f_5} \tag{3.7}$$

and

$$\sum_{n=0}^{\infty} b\,(8n+7)\,q^n \equiv 8 f_8 f_{10} + 8 f_1^3 f_2^5 f_5. \tag{3.8}$$

Using (2.7) in (3.8), we get

$$\sum_{n=0}^{\infty} b\,(16n+7)\,q^n \equiv 8 f_1^9 + 8 f_4 f_5 \tag{3.9}$$

and

$$\sum_{n=0}^{\infty} b\,(16n+15)\,q^n \equiv 8 f_2^3 f_5^3. \tag{3.10}$$

The congruence (3.9) is $\beta = 0$ case of (2.10). Suppose that the congruence (2.10) is true for some integer $\beta \geq 0$.

Ramanujan recorded the following identity in his notebooks without proof:

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2 R(q^5)), \tag{3.11}$$

where $R(q) = \dfrac{f(-q, -q^4)}{f(-q^2, -q^3)}$.

For a proof of (3.11), one can see [6, 12].

Using (3.11) in (2.10) and then comparing the coefficients of q^{5n+4}, we get

$$\sum_{n=0}^{\infty} b\left(16\cdot 5^{2\beta+1} n + 14\cdot 5^{2\beta+1} + 1\right) q^n \equiv 8 f_1 f_{20} + 8q f_5^9, \tag{3.12}$$

which proves (2.11). Again using (3.11) in (3.12) and then comparing the coefficients of q^{5n+1} in the resultant equation, we obtain

$$\sum_{n=0}^{\infty} b\left(16\cdot 5^{2\beta+2} n + 6\cdot 5^{2\beta+2} + 1\right) q^n \equiv 8 f_1^9 + 8 f_4 f_5, \tag{3.13}$$

which implies that the congruence (2.10) is true for $\beta+1$. By mathematical induction, the congruence (2.10) is true for all integers β.

Comparing the coefficients of q^{5n+i} for $i = 3, 4$ in (3.12), we obtain (2.12).

The congruence (3.10) is the $\alpha = \beta = 0$ case of (2.13). Suppose that the congruence (2.13) holds for $\alpha > 0$ with $\beta = 0$. From (2.3), the congruence (2.13) with $\beta = 0$ becomes

$$\sum_{n=0}^{\infty} b\left(16 \cdot 3^{2\alpha} n + 14 \cdot 3^{2\alpha} + 1\right) q^n \equiv 8 f_6 f_{15} + 8q^2 f_{15} f_{18}^3 + 8q^5 f_6 f_{45}^3 + 8q^7 f_{18}^3 f_{45}^3, \tag{3.14}$$

which implies

$$\sum_{n=0}^{\infty} b\left(16 \cdot 3^{2\alpha+1} n + 14 \cdot 3^{2\alpha} + 1\right) q^n \equiv 8 f_2 f_5, \tag{3.15}$$

$$\sum_{n=0}^{\infty} b\left(16 \cdot 3^{2\alpha+1} n + 10 \cdot 3^{2\alpha+1} + 1\right) q^n \equiv 8q^2 f_6^3 f_{15}^3 \tag{3.16}$$

and

$$\sum_{n=0}^{\infty} b\left(16 \cdot 3^{2\alpha+1} n + 46 \cdot 3^{2\alpha} + 1\right) q^n \equiv 8 f_5 f_6^3 + 8q f_2 f_{15}^3. \tag{3.17}$$

Comparing the coefficients of q^{3n+2} on both sides of the equation (3.16), we get

$$\sum_{n=0}^{\infty} b\left(16 \cdot 3^{2\alpha+2} n + 14 \cdot 3^{2\alpha+2} + 1\right) q^n \equiv 8 f_2^3 f_5^3, \tag{3.18}$$

which proves the congruence (2.13) is true for $\alpha + 1$ with $\beta = 0$. Hence, by induction, the congruence (2.13) is true for any integer α with $\beta = 0$. Suppose that the congruence (2.13) holds for some integers $\alpha, \beta > 0$.

Employing the equation (3.11) in (2.13) and then comparing the coefficients of q^{5n+1}, we find that

$$\sum_{n=0}^{\infty} b\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} n + 6 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} + 1\right) q^n \equiv 8q f_1^3 f_{10}^3, \tag{3.19}$$

which proves (2.14). Employing the equation (3.11) in (3.19) and extracting the coefficients of q^{5n+4}, we arrive at

$$\sum_{n=0}^{\infty} b\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} n + 14 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} + 1\right) q^n \equiv 8 f_2^3 f_5^3, \tag{3.20}$$

which implies that (2.13) is true for $\beta + 1$. Hence, by induction, the congruence (2.13) is true for any non-negative integers $\alpha, \beta > 0$.

Employing (2.3) in (2.13) and then comparing the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we obtain respectively (2.15), (2.17) and (2.18).

Employing the equation (3.11) in the equation (2.15) and (2.18), we get respectively (2.16) and (2.19).

Using the equations (2.15) and (2.16) along with the equation (3.11), we obtain (2.20) and (2.21) respectively. Using the equations (2.18) and (2.19) along with the equation (3.11), we obtain (2.22) and (2.23) respectively.

Comparing the coefficients of q^{3n} and q^{3n+1} in Eq. (2.17), we get (2.24).

Using (2.2) and (2.5) in (3.5), we get

$$\sum_{n=0}^{\infty} b(8n+1)\,q^n \equiv 2\frac{f_2^{14} f_{10}^2}{f_1^{11} f_4^3 f_5 f_{20}} + 8q\frac{f_2^5 f_4^3 f_{20}}{f_1^8 f_{10}} \tag{3.21}$$

and

$$\sum_{n=0}^{\infty} b(8n+5)\,q^n \equiv 8\frac{f_4^5}{f_1^3 f_5} - 2\frac{f_2^5 f_{20}}{f_1^4 f_4 f_{10}} + 8 f_1^3 f_2^2 f_5. \tag{3.22}$$

Utilizing (3.2) and (3.4) in (3.21), we get

$$b\left(2^{\alpha+2}n+1\right) \equiv b(4n+1)\,.$$

Using (2.2), (2.6) and (2.7) in (3.22), we obtain

$$\sum_{n=0}^{\infty} b(16n+5)\,q^n \equiv 8\frac{f_2^6}{f_1 f_5} + 8 f_2^3 - 2\frac{f_2 f_{10}}{f_1 f_5} \tag{3.23}$$

and

$$\sum_{n=0}^{\infty} b(16n+13)\,q^n \equiv 8 f_8 f_{10} + 8 f_1 f_2 f_5^3 + 8 f_1^3 f_2^5 f_5. \tag{3.24}$$

Employing (2.7) and (2.8) in (3.24), we get

$$\sum_{n=0}^{\infty} b(32n+13)\,q^n \equiv 8 f_1^9 \tag{3.25}$$

and

$$\sum_{n=0}^{\infty} b(32n+29)\,q^n \equiv 8 f_1 f_{20} + 8 f_2^3 f_5^3. \tag{3.26}$$

The congruence (3.25) is $\alpha = \beta = \gamma = 0$ case of (2.25). Suppose that the congruence (2.25) is true for $\alpha \geq 0$ with $\beta = \gamma = 0$. From (2.25) with $\beta = \gamma = 0$, we arrive at

$$\sum_{n=0}^{\infty} b\left(32 \cdot 3^{4\alpha} + 12 \cdot 3^{4\alpha} + 1\right) q^n \equiv 8 f_1^9. \tag{3.27}$$

Using (2.3) in (3.27) and then comparing the coefficients of q^{3n}, we get

$$\sum_{n=0}^{\infty} b\left(32 \cdot 3^{4\alpha+1} + 12 \cdot 3^{4\alpha} + 1\right) q^n \equiv 8f_1^3 + 8qf_3^9 \equiv 8f_3 + 8qf_3^9 + 8qf_9^3, \tag{3.28}$$

which implies

$$\sum_{n=0}^{\infty} b\left(32 \cdot 3^{4\alpha+2} + 12 \cdot 3^{4\alpha+2} + 1\right) q^n \equiv 8f_1^9 + 8f_3^3 \equiv 8qf_6f_9^3 + 8q^2f_3f_9^6 + 8q^3f_9^9, \tag{3.29}$$

which yields

$$\sum_{n=0}^{\infty} b\left(32 \cdot 3^{4\alpha+3} + 12 \cdot 3^{4\alpha+2} + 1\right) q^n \equiv 8qf_3^9, \tag{3.30}$$

which implies

$$\sum_{n=0}^{\infty} b\left(32 \cdot 3^{4\alpha+4} + 12 \cdot 3^{4\alpha+4} + 1\right) q^n \equiv 8f_1^9, \tag{3.31}$$

which proves the congruence (2.25) is true for $\alpha + 1$. By mathematical induction, the congruence (2.25) is true for all integers $\alpha \geq 0$ with $\beta = \gamma = 0$. Suppose the congruence (2.25) is true for $\alpha, \beta \geq 0$ with $\gamma = 0$. From the equation (2.25) with $\gamma = 0$ and then employing (3.11), we get

$$\sum_{n=0}^{\infty} b\left(32 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} + 28 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} + 1\right) q^n \equiv 8qf_5^9, \tag{3.32}$$

which implies

$$\sum_{n=0}^{\infty} b\left(32 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} + 12 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} + 1\right) q^n \equiv 8f_1^9, \tag{3.33}$$

which proves that the congruence (2.25) with $\gamma = 0$ is true for $\beta + 1$. So, by induction, the congruence (2.25) with $\gamma = 0$ is true for all integers $\alpha, \beta \geq 0$. Suppose that the congruence (2.25) is true for $\alpha, \beta, \gamma \geq 0$ and then utilizing (2.9), we arrive at

$$\sum_{n=0}^{\infty} b\left(32 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1}n + 4 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} + 1\right) q^n \equiv 8q^2f_7^9, \tag{3.34}$$

which proves (2.26). Comparing the coefficients of q^{7n+2} in (3.34), we obtain

$$\sum_{n=0}^{\infty} b\left(32 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2}n + 12 \cdot 3^{4\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} + 1\right) q^n \equiv 8 f_1^9, \tag{3.35}$$

which implies that the congruence (2.25) is true for $\gamma+1$. By induction, the congruence (2.25) is true for all integers $\alpha, \beta, \gamma \geq 0$. Using (3.11) in (2.25), we obtain (2.27). The congruence (2.27) implies (2.28). Using (2.3) in (2.25) and then comparing the coefficients of q^{3n+1} and q^{3n+2} in the resultant equation, we obtain (2.29) and (2.30) respectively.

Utilizing (3.11) in (2.29) and (2.30), we obtain respectively (2.31) and (2.32).

The congruence (3.26) is $\beta = 0$ case of (2.33). Suppose that the congruence (2.33) is true for some integer $\beta \geq 0$. Using (3.11) in (2.33) and then comparing the coefficients of q^{5n+1}, we get

$$\sum_{n=0}^{\infty} b\left(32 \cdot 5^{2\beta+1}n + 12 \cdot 5^{2\beta+1} + 1\right) q^n \equiv 8 f_4 f_5 + 8q f_1^3 f_{10}^3, \tag{3.36}$$

which is (2.34). Again using (3.11) in (3.36) and then comparing the coefficients of q^{5n+4} in the resultant equation, we obtain

$$\sum_{n=0}^{\infty} b\left(32 \cdot 5^{2\beta+2}n + 28 \cdot 5^{2\beta+2} + 1\right) q^n \equiv 8 f_1 f_{20} + 8 f_2^3 f_5^3, \tag{3.37}$$

which implies that the congruence (2.33) is true for $\beta+1$. By mathematical induction, the congruence (2.33) is true for all integers $\beta \geq 0$.

Comparing the coefficients of q^{5n+i} for $i = 3, 4$ in (2.33) along with (3.11), we obtain (2.35).

From (3.7), we have, modulo 4,

$$\begin{aligned}\sum_{n=0}^{\infty} b(8n+3) q^n &\equiv 2\frac{f_1 f_{10}}{f_5} \\ &\equiv 2 f_2^3 + 2q f_{10}^3, \end{aligned}\tag{3.38}$$

which reduces to

$$\sum_{n=0}^{\infty} b(16n+3) q^n \equiv 2 f_1^3 \tag{3.39}$$

and

$$\sum_{n=0}^{\infty} b(16n+11) q^n \equiv 2 f_5^3. \tag{3.40}$$

Equation (3.39) is the $\alpha = \beta = \gamma = 0$ case of (2.36). Suppose that the congruence (2.36) is true for $\alpha \geq 0$ with $\beta = \gamma = 0$. From (2.36) with $\beta = \gamma = 0$,

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}n+2\cdot 3^{2\alpha}+1\right)q^n \equiv 2f_1^3. \tag{3.41}$$

Utilizing (2.3), the equation (3.41) reduces to

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha+1}n+2\cdot 3^{2\alpha+2}+1\right)q^n \equiv 2f_3^3, \tag{3.42}$$

which yields

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha+2}n+2\cdot 3^{2\alpha+2}+1\right)q^n \equiv 2f_1^3, \tag{3.43}$$

which implies that the congruence (2.36) is true for $\alpha+1$ with $\beta=\gamma=0$. By mathematical induction, the congruence (2.36) is true for all $\alpha\geq 0$. Suppose that the congruence (2.36) holds for $\alpha, \beta\geq 0$ with $\gamma=0$. Employing (3.11) in (2.36) with $\gamma=0$, we get

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}+1\right)q^n \equiv 2f_5^3, \tag{3.44}$$

which implies

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}+1\right)q^n \equiv 2f_1^3, \tag{3.45}$$

which implies that the congruence (2.36) is true for $\beta+1$ with $\gamma=0$. By mathematical induction, the congruence (2.36) is true for all non-negative integers α, β with $\gamma=0$.

Suppose that the congruence (2.36) holds for $\alpha, \beta, \gamma\geq 0$. Employing (2.9) in (2.36), we get

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+1\right)q^n \equiv 2f_7^3, \tag{3.46}$$

which proves (2.37). The congruence (3.46) reduces to

$$\sum_{n=0}^{\infty} b\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}+1\right)q^n \equiv 2f_1^3, \tag{3.47}$$

which implies that the congruence (2.36) is true for $\gamma+1$. By mathematical induction, the congruence (2.36) is true for all integers α, β and γ.

Using (2.3) in (2.36) and comparing the coefficients of q^{3n}, q^{3n+1} and q^{3n+2} respectively, we obtain (2.38), (2.39) and (2.40) respectively.

Comparing the coefficients of q^{3n+1} and q^{3n+2} in (2.39), we get (2.41).

Employing (3.11) in (2.36) and comparing the coefficients of q^{5n+3}, we obtain (2.42).

Comparing the coefficients of q^{5n+2} and q^{5n+4} from (2.36) along with (3.11), we obtain (2.43).

Comparing the coefficients of q^{5n+i} for $i = 1, 2, 3, 4$ from (2.42), we arrive at (2.44).

From (3.40), we deduce

$$\sum_{n=0}^{\infty} b\,(80n + 11)\, q^n \equiv 2 f_1^3. \tag{3.48}$$

The congruence (3.48) is the $\alpha = \beta = \gamma = 0$ case of (2.45). The rest of the proofs of the identities (2.45)–(2.53) are similar to the proofs of the identities (2.36)–(2.44). So, we omit the details.

Employing (2.5) in (3.23), we obtain

$$\sum_{n=0}^{\infty} b\,(32n + 5)\, q^n \equiv 2 f_1^3 \tag{3.49}$$

and

$$\sum_{n=0}^{\infty} b\,(32n + 21)\, q^n \equiv 2 f_5^3. \tag{3.50}$$

The rest of the proofs of the identities (2.54)–(2.71) are similar to the proofs of the identities (2.36)–(2.44). So, we omit the details.

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