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Stability of buoyancy-driven convection in an Oldroyd-B fluid-saturated anisotropic porous layer*

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Abstract The nonlinear stability of thermal convection in a layer of an Oldroyd-B fluid-saturated Darcy porous medium with anisotropic permeability and thermal diffusivity is investigated with the perturbation method. A modified Darcy-Oldroyd model is used to describe the flow in a layer of an anisotropic porous medium. The results of the linear instability theory are delineated. The thresholds for the stationary and oscillatory convection boundaries are established, and the crossover boundary between them is demarcated by identifying a codimension-two point in the viscoelastic parameter plane. The stability of the stationary and oscillatory bifurcating solutions is analyzed by deriving the cubic Landau equations. It shows that these solutions always bifurcate supercritically. The heat transfer is estimated in terms of the Nusselt number for the stationary and oscillatory modes. The result shows that, when the ratio of the thermal to mechanical anisotropy parameters increases, the heat transfer decreases.

Key words convection, porous medium, Oldroyd-B fluid, cubic Landau equation

Chinese Library Classification O175.8, O357.3
2010 Mathematics Subject Classification 76A10, 76E06, 76S05

Nomenclature

d,	depth of the porous layer;
$\boldsymbol{g}$,	gravitational acceleration;
$\hat{k}$,	unit vector in the vertical direction;
$\widetilde{K}^{-1}$,	inverse permeability tensor;
p,	pressure;
$\boldsymbol{q}$,	velocity vector;
R_{D},	Darcy-Rayleigh number;
t,	time;
x, y, z,	space coordinates;
α,	effective thermal diffusivity tensor;
β,	thermal expansion coefficient;
η,	thermal anisotropy parameter;
λ_1,	stress relaxation time;
λ_2,	strain retardation time;
Λ_1,	stress relaxation parameter;
Λ_2,	strain retardation parameter;
μ,	dynamic viscosity;
ν,	kinematic viscosity;
ρ,	fluid density;
ω,	growth rate;
ψ,	stream function;
ξ,	mechanical anisotropy parameter.

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Subscripts/superscripts

b,	basic state;	$'$,	perturbed variable.

1 Introduction

The buoyancy-driven flow instability in a layer of fluid-saturated porous media has received increasing attention over the past fifty years because of its numerous applications in geophysics and energy-related systems[1–5]. The porous media involved in many engineering and industrial applications are usually anisotropic in their mechanical and thermal properties, and several investigators have addressed these effects on the stability characteristics of the system. Castinel and Combarnous[6] were the first to investigate the effect of the anisotropy in the permeability of a porous medium on the natural convection in the porous medium. Epherre[7] extended this work to the case of anisotropy in both thermal diffusivity and permeability. Subsequently, several studies were undertaken, covering various effects on this problem, and the results have been reported in the open literature[8–11].

Most of the investigations on the thermal convection in anisotropic porous media are for Newtonian fluids, and seldom for non-Newtonian fluids. However, the thermal convection encountered in many engineering applications, e.g., geophysics, material processing, petroleum, chemical and nuclear industries, reservoir engineering, and bioengineering, exhibits non-Newtonian characteristics. This has encouraged researchers to consider non-Newtonian fluids in their investigations. Among different kinds of non-Newtonian fluids, viscoelastic fluids are found to be of considerable importance in various engineering applications[12]. Alishaev and Mirzadjanzade[13] investigated the viscoelastic flows in porous media for the calculations of delay phenomenon in the filtration theory. Rudraiah and Kaloni[14] provided a review on some of the constitutive equations of non-Newtonian fluids flowing through porous media. By applying the linear stability theory, Rudraiah et al.[15] examined the onset of convection in an Oldroyd-B fluid-saturated horizontal porous layer heated from below. Shenoy[16] gave a comprehensive review on the non-Newtonian fluids and heat transfer in porous media. Kim et al.[17] studied the thermal instability in a porous layer saturated with a viscoelastic fluid. Malashetty et al.[18] and Shivakumara et al.[19] analyzed the effects of local thermal non-equilibrium on the convection onset in a viscoelastic fluid-saturated porous layer. Sheu et al.[20] investigated the buoyancy-induced convection in a viscoelastic fluid-saturated porous medium. Wang and Tan[21] used both linear and nonlinear stability theories to investigate double diffusive convection with the modified Darcy-Maxwell model with the Soret effect. Recently, various types of flow problems for the Oldroyd-B and Maxwell viscoelastic fluids have been analyzed[22–24]. Considering another non-Newtonian fluid known as the Casson fluid, Makinde and Eegunjobi[25] discussed the thermally radiating magnetohydrodynamics slip flow in a micro channel filled with Casson fluid-saturated porous media, and Makinde and Rundora[26] analyzed the unsteady mixed convection flow in a permeable wall channel with a reactive Casson fluid-saturated porous medium.

The investigations on the thermal convection in a viscoelastic fluid-saturated porous medium are mainly dispensed with isotropic porous media except the study of Malashetty and Swamy[12], where the convection onset in a layer of viscoelastic liquid-saturated anisotropic Darcy porous media was investigated. Since the stability of the viscoelastic fluid-saturated anisotropic porous layer takes the form of overstable motion only, it is of interest to perform the nonlinear stability analysis and quantify the role of anisotropy and viscoelastic parameters on the same layer. The goal of the present paper is to investigate the nonlinear stability of thermal convection in an Oldroyd-B fluid-saturated anisotropic porous layer with the perturbation method. The stability of bifurcating equilibrium solutions is discussed by deriving cubic Landau equations. The results of the linear instability analysis are delineated. Besides, the consequence of the viscoelastic and anisotropy parameters on the variation of the Nusselt number with respect to the Darcy-Rayleigh number is examined.

2 Mathematical formulation

The physical configuration is as shown in Fig. 1. We consider an infinite horizontal anisotropic porous layer heated from below and saturated with a viscoelastic fluid of an Oldroyd-B type confined between the impermeable planes $z = 0$ and $z = d$ in the presence of gravity. The anisotropy in both thermal diffusivity and permeability is considered. A Cartesian coordinate system (x, y, z) is selected such that the origin is located at the lower boundary and the z-axis is measured vertically upward. The lower and upper boundaries are maintained at $T_0 + \Delta T$ $(\Delta T > 0)$ and T_0, respectively.

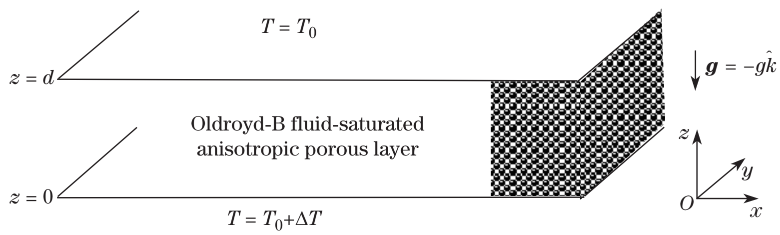


Fig. 1 Physical configuration

The basic governing equations under the Boussinesq approximation are[13–17]

$$\nabla \cdot \boldsymbol{q} = \boldsymbol{0}, \tag{1}$$

$$\Big(1 + \lambda_1 \frac{\partial}{\partial t}\Big)(-\nabla p + \rho \boldsymbol{g}) = \mu\Big(1 + \lambda_2 \frac{\partial}{\partial t}\Big)\widetilde{K}^{-1} \cdot \boldsymbol{q}, \tag{2}$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{q} \cdot \nabla)T = \nabla \cdot (\widetilde{\alpha} \cdot \nabla T), \tag{3}$$

$$\rho = \rho_0(1 - \beta(T - T_0)), \tag{4}$$

where $\boldsymbol{q} = (u, v, w)$ is the velocity vector, p is the pressure, μ is the fluid viscosity, λ_1 is the stress relaxation time, λ_2 is the strain retardation time, $\boldsymbol{g}$ is the gravitational acceleration, ρ is the fluid density, T is the temperature, β is the thermal expansion coefficient, ρ_0 is the reference density at $T = T_0$, and

$$\widetilde{\alpha} = \alpha_x \hat{i}\hat{i} + \alpha_y \hat{j}\,\hat{j} + \alpha_z \hat{k}\hat{k}, \quad \widetilde{K}^{-1} = k_x^{-1}\hat{i}\hat{i} + k_y^{-1}\hat{j}\,\hat{j} + k_z^{-1}\hat{k}\hat{k} \tag{5}$$

are the effective thermal diffusivity and the inverse permeability tensors, respectively, whose principal axes are associated with the coordinate system. A horizontal isotropy in the permeability and thermal diffusivity is assumed and considered, i.e.,

$$k_x = k_y (= k_{\rm h}), \quad k_z = k_{\rm v}; \quad \alpha_x = \alpha_y (= \alpha_{\rm h}), \quad \alpha_z = \alpha_{\rm v},$$

where $k_{\rm h}$ and $\alpha_{\rm h}$ are the permeability and the thermal diffusivity in the horizontal $\hat{i}$ and $\hat{j}$ directions, respectively, while $k_{\rm v}$ and $\alpha_{\rm v}$ are the corresponding values in the vertical $\widehat{k}$ direction, respectively.

The steady basic state is quiescent, and is given by

$$\boldsymbol{q}_{\rm b} = \boldsymbol{0}, \quad T_{\rm b} = T_0 + \Delta T\Big(1 - \frac{z}{d}\Big), \quad p_{\rm b} = p_0 - \rho_0 g\Big(z - \beta \Delta T\Big(z - \frac{z^2}{2d}\Big)\Big). \tag{6}$$

To study the stability of the basic state, the perturbations on the basic state are superimposed as follows:

$$\boldsymbol{q} = \boldsymbol{0} + \boldsymbol{q}', \quad p = p_{\rm b}(z) + p', \quad \rho = \rho_{\rm b} + \rho', \quad T = T_{\rm b}(z) + T', \tag{7}$$

where primes designate the perturbed quantities. Substituting Eq. (7) into Eqs. (1)–(4), eliminating the pressure term from Eq. (2) by operating curl, introducing the stream function $\psi(x,z,t)$ through

$$u=\frac{\partial\psi}{\partial z},\quad w=-\frac{\partial\psi}{\partial x},\tag{8}$$

and rendering the resulting equations to dimensionless form by using d, $d^2/\alpha_{\rm v}$, $\alpha_{\rm v}$, and ΔT as the units of the length, time, stream function, and temperature, respectively, we have

$$\boldsymbol{L}\begin{pmatrix}\psi\\ T\end{pmatrix}=\begin{pmatrix}0\\ J(\psi,T)\end{pmatrix},\tag{9}$$

where

$$\boldsymbol{L}=\begin{pmatrix}\left(1+\Lambda_2\dfrac{\partial}{\partial t}\right)\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{1}{\xi}\dfrac{\partial^2}{\partial z^2}\right) & R_{\rm D}\left(1+\Lambda_1\dfrac{\partial}{\partial t}\right)\dfrac{\partial}{\partial x}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial t}-\eta\dfrac{\partial^2}{\partial x^2}-\dfrac{\partial^2}{\partial z^2}\end{pmatrix},\tag{10}$$

and $J(\cdot,\cdot)$ stands for the Jacobian with respect to x and z. Here, $R_{\rm D}$ is the Darcy-Rayleigh number defined by

$$R_{\rm D}=\frac{\beta g\Delta T k_{\rm v}d}{\nu\alpha_{\rm v}},$$

Λ_1 is the relaxation parameter defined by $\Lambda_1=\lambda_1\alpha_{\rm v}/d^2$, Λ_2 is the retardation parameter defined by $\Lambda_2=\lambda_2\alpha_{\rm v}/d^2$, η is the thermal anisotropy parameter defined by $\eta=\alpha_{\rm h}/\alpha_{\rm v}$, and ξ is the mechanical anisotropy parameter defined by $\xi=k_{\rm h}/k_{\rm v}$.

Since the boundaries are impermeable and isothermal, the appropriate boundary conditions are

$$\psi=T=0\quad\text{at}\quad z=0,1.\tag{11}$$

3 Nonlinear stability analysis

The nonlinear stability analysis near the convection threshold is performed with the perturbation method. The cubic Landau equations are derived for stationary and oscillatory convection modes. Such a study helps in analyzing the stability of the bifurcating equilibrium solutions (subcritical/supercritical) and also in estimating the convective rate of heat transfer. Accordingly, the dependent variables ψ, T, and $R_{\rm D}$ are expanded in the power series of a small perturbation parameter $\chi\,(\ll 1)$ as follows[27–28]:

$$R_{\rm D}=R_{\rm Dc}+\chi^2R_{\rm D2}+\cdots,\quad \psi=\sum_{n=1}^{\infty}\chi^n\psi_n,\quad T=\sum_{n=1}^{\infty}\chi^nT_n.\tag{12}$$

The other parameters Λ_1, Λ_2, ξ, and η are taken as given, and $R_{\rm Dc}$ is the critical value at the threshold as the case may be. A slow time scale s is introduced, i.e., $s=\chi^2t$. The operator $\frac{\partial}{\partial t}$ is replaced, depending on the nature of the bifurcating solutions.

3.1 Bifurcation of the stationary solution

In this case, $R_{\rm D}=R_{\rm Dc}^{\rm s}$. Substituting Eq. (12) and $\frac{\partial}{\partial t}=\chi^2\frac{\partial}{\partial s}$ into Eq. (9) and equating the coefficients of different powers of χ lead to a sequence of equations.

For the first-order power of χ, the resulting stability equations are homogeneous, and are

$$\boldsymbol{L}_1\begin{pmatrix}\psi_1\\ T_1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix},\tag{13}$$

where

$$\boldsymbol{L}_1 = \begin{pmatrix} \dfrac{\partial^2}{\partial x^2} + \dfrac{1}{\xi}\dfrac{\partial^2}{\partial z^2} & R_{\mathrm{D}}\dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & -\eta\dfrac{\partial^2}{\partial x^2} - \dfrac{\partial^2}{\partial z^2} \end{pmatrix}.$$

The eigenvalue and eigenfunctions of this problem are

$$R_{\mathrm{D}}^{\mathrm{s}} = \frac{\pi^4 + (\xi+\eta)\pi^2 a^2 + \xi\eta a^4}{\xi a^2}, \tag{14}$$

$$\psi_1 = A_1 \sin(ax)\sin(\pi z), \quad T_1 = B_1 \cos(ax)\sin(\pi z). \tag{15}$$

The undetermined amplitudes A_1 and B_1 are related by

$$A_1 = -\frac{c}{a}B_1, \tag{16}$$

where $c = \eta a^2 + \pi^2$. The eigenvalue $R_{\mathrm{D}}^{\mathrm{s}}$ attains the critical value at $a_{\mathrm{c}} = (\xi\eta)^{-\frac{1}{4}}\pi$, and the corresponding critical value is

$$R_{\mathrm{Dc}}^{\mathrm{s}} = \pi^2\Big(1 + \sqrt{\frac{\eta}{\xi}}\Big)^2, \tag{17}$$

which is free from the viscoelastic parameters and coincides with the Newtonian case[9]. For the isotropic case ($\xi = \eta$), $a_{\mathrm{c}} = \pi$, and $R_{\mathrm{Dc}}^{\mathrm{s}} = 4\pi^2$, which are the known exact values[2].

For the second-order power of χ, the stability equations are inhomogeneous, and are

$$\boldsymbol{L}_1 \begin{pmatrix} \psi_2 \\ T_2 \end{pmatrix} = \begin{pmatrix} 0 \\ J(\psi_1, T_1) \end{pmatrix}, \tag{18}$$

where $J(\psi_1, T_1) = \dfrac{1}{2}\pi a A_1 B_1 \sin(2\pi z)$. The solution of the above system of equations is

$$T_2 = \frac{a}{8\pi}A_1 B_1 \sin(2\pi z), \quad \psi_2 = 0. \tag{19}$$

For the third-order power of χ, the stability equations become

$$\boldsymbol{L}_1 \begin{pmatrix} \psi_3 \\ T_3 \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ J(\psi_1, T_2) \end{pmatrix}, \tag{20}$$

where

$$\Delta_1 = \Big(\Big(a\Lambda_1 R_{\mathrm{D}} - \frac{c}{a}\Big(a^2 + \frac{1}{\xi}\pi^2\Big)\Lambda_2\Big)\frac{\mathrm{d}B_1}{\mathrm{d}s} + aR_{\mathrm{D2}}B_1\Big)\sin(ax)\sin(\pi z),$$

$$J(\psi_1, T_2) = -\Big(\frac{\mathrm{d}B_1}{\mathrm{d}s} + \frac{c^2}{8}B_1^3\Big)\cos(ax)\sin(\pi z) + \frac{c^2}{8}B_1^3\cos(ax)\sin(3\pi z).$$

The solution of the above equations is

$$\psi_3 = A_3\sin(ax)\sin(\pi z) + \cdots, \quad T_3 = B_3\cos(ax)\sin(\pi z) + \cdots. \tag{21}$$

The solvability condition has been derived for Eq. (20), which is in the form of the first-order nonlinear ordinary differential equation (the cubic Landau equation) for the unknown amplitude B_1 as follows:

$$\Gamma\frac{\partial B_1}{\partial s}=\frac{a^2}{c}R_{\mathrm{D}2}B_1-\varpi B_1^3, \tag{22}$$

where

$$\Gamma=\Big(\frac{1}{c}+\Lambda_2\Big)\Big(a^2+\frac{1}{\xi}\pi^2\Big)-\frac{a^2\Lambda_1}{c}R_{\mathrm{Dc}}^{\mathrm{s}}, \tag{23}$$

$$\varpi=\frac{c}{8}\Big(a^2+\frac{1}{\xi}\pi^2\Big). \tag{24}$$

For the steady state, the amplitude is found to be

$$B_1^2=\frac{8a^2}{c^2}\Big(a^2+\frac{1}{\xi}\pi^2\Big)^{-1}R_{\mathrm{D}2}. \tag{25}$$

Equation (25) is independent of viscoelastic parameters while depends on anisotropy parameters. Note that $R_{\mathrm{D}2}>0$, which indicates that the stationary bifurcation is always supercritical (stable).

The convective heat transfer is determined in terms of the area-averaged thermal Nusselt number. The Nusselt number is defined by

$$Nu=1-\chi^2\left.\frac{\mathrm{d}T_2}{\mathrm{d}z}\right|_{z=0}=1+\frac{2a^2}{c}\frac{R_{\mathrm{D}}^{\mathrm{s}}-R_{\mathrm{Dc}}^{\mathrm{s}}}{a^2+\frac{1}{\xi}\pi^2}. \tag{26}$$

By substituting the critical values into the above equation, we have

$$Nu=1+\frac{2}{\pi^2}\Big(1+\sqrt{\frac{\eta}{\xi}}\Big)^{-2}\Big(R_{\mathrm{D}}-\pi^2\Big(1+\sqrt{\frac{\eta}{\xi}}\Big)^2\Big). \tag{27}$$

If $\eta=\xi$, Eq. (27) coincides with the Newtonian case[29].

3.2 Bifurcation of the oscillatory solution

In this case, $R_{\mathrm{D}}=R_{\mathrm{Dc}}^{\mathrm{o}}$. A minor alteration of the method applied in the earlier section is used to find out the bifurcation of the oscillatory convection. The time derivative is not zero in the present case, and $\frac{\partial}{\partial t}$ is replaced by $\frac{\partial}{\partial t}+\chi^2\frac{\partial}{\partial s}$.

For the first-order power of χ, the equations reduce to the linear instability problem for overstability as follows:

$$\boldsymbol{L}\begin{pmatrix}\psi_1\\ T_1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}. \tag{28}$$

The eigenfunctions are

$$\begin{cases}\psi_1=(A_1\mathrm{e}^{\mathrm{i}\omega t}+\overline{A}_1\mathrm{e}^{-\mathrm{i}\omega t})\sin(ax)\sin(\pi z),\\ T_1=(B_1\mathrm{e}^{\mathrm{i}\omega t}+\overline{B}_1\mathrm{e}^{-\mathrm{i}\omega t})\cos(ax)\sin(\pi z),\end{cases} \tag{29}$$

where the overline denotes the complex conjugate, ω and a are taken to be the critical conditions associated with the oscillatory onset. The amplitudes A_1 and B_1 are functions of the slow time scale, and are related by

$$A_1=-\frac{c+\mathrm{i}\omega}{a}B_1. \tag{30}$$

The eigenvalue is found to be

$$R_{\mathrm{D}}^{\mathrm{o}} = \Big(a^2 + \frac{1}{\xi}\pi^2\Big)\frac{c + (\Lambda_1 - \Lambda_2)\omega^2 + c\Lambda_1\Lambda_2\omega^2}{a^2(1 + \Lambda_1^2\,\omega^2)}, \tag{31}$$

where

$$\omega^2 = \frac{c(\Lambda_1 - \Lambda_2) - 1}{\Lambda_1\Lambda_2} > 0. \tag{32}$$

Equation (32) shows that the oscillatory convection is not possible if $\Lambda_1 < \Lambda_2$. It is seen that $R_{\mathrm{D}}^{\mathrm{o}}$ attains its critical value $R_{\mathrm{Dc}}^{\mathrm{o}}$ at $a^2 = a_{\mathrm{c}}^2$, where

$$a_{\mathrm{c}}^2 = \frac{\sqrt{\pi^4 + \pi^2/\Lambda_2}}{\sqrt{\xi\eta}}, \tag{33}$$

which is independent of Λ_1, and the corresponding expression for $R_{\mathrm{Dc}}^{\mathrm{o}}$ is

$$R_{\mathrm{Dc}}^{\mathrm{o}} = \frac{1}{\Lambda_1}\Big(\sqrt{1 + \pi^2\,\Lambda_2} + \pi\sqrt{\Lambda_2}\sqrt{\frac{\eta}{\xi}}\Big)^2. \tag{34}$$

For the second-order power of χ, the equations are inhomogeneous and found to be

$$\boldsymbol{L}\begin{pmatrix}\psi_2\\ T_2\end{pmatrix} = \begin{pmatrix}0\\ J(\psi_1, T_1)\end{pmatrix}, \tag{35}$$

where

$$J(\psi_1, T_1) = \frac{1}{2}\pi a(A_1 B_1 \mathrm{e}^{2\mathrm{i}\omega t} + \overline{A}_1\overline{B}_1\mathrm{e}^{-2\mathrm{i}\omega t} + A_1\overline{B}_1 + B_1\overline{A}_1)\sin(2\pi z). \tag{36}$$

Equation (36) suggests that the stream function and temperature should contain the terms involving the frequency 2ω. Based on this fact, the second-order stream function and temperature can be expressed as follows:

$$\begin{cases}\psi_2 = (\psi_{20} + \psi_{22}\mathrm{e}^{2\mathrm{i}\omega t} + \overline{\psi}_{22}\mathrm{e}^{-2\mathrm{i}\omega t})\sin(2\pi z),\\ T_2 = (T_{20} + T_{22}\mathrm{e}^{2\mathrm{i}\omega t} + \overline{T}_{22}\mathrm{e}^{-2\mathrm{i}\omega t})\sin(2\pi z).\end{cases} \tag{37}$$

The solution of the second-order problem is now found to be

$$T_{20} = \frac{a}{8\pi}(A_1\overline{B}_1 + B_1\overline{A}_1), \quad \psi_{20} = 0, \quad T_{22} = \frac{\pi a A_1 B_1}{8\pi^2 + 4\mathrm{i}\omega}, \quad \psi_{22} = 0. \tag{38}$$

For the third-order power of χ, the stability equations are

$$\boldsymbol{L}\begin{pmatrix}\psi_3\\ T_3\end{pmatrix} = \begin{pmatrix}\Delta_2\\ J(\psi_1, T_2)\end{pmatrix}, \tag{39}$$

where

$$\Delta_2 = \Big(a(1 + \mathrm{i}\omega\Lambda_1)R_{\mathrm{D}2}B_1 + \Big(aR_{\mathrm{D}}\Lambda_1 - \frac{\Lambda_2}{a}\Big(a^2 + \frac{1}{\xi}\pi^2\Big)$$

$$\cdot (c + \mathrm{i}\omega)\Big)\frac{\mathrm{d}B_1}{\mathrm{d}s}\Big)\mathrm{e}^{\mathrm{i}\omega t}\sin(ax)\sin(\pi z) + \cdots,$$

$$J(\psi_1, T_2) = -\Big(\frac{\mathrm{d}B_1}{\mathrm{d}s} + \pi a(A_1T_{20} + \overline{A}_1T_{22})\Big)\mathrm{e}^{\mathrm{i}\omega t}\cos(ax)\sin(\pi z) + \cdots.$$

The third-order problem has the solution as follows:

$$\psi_3 = A_3 \mathrm{e}^{\mathrm{i}\omega t}\sin(ax)\sin(\pi z) + \cdots, \quad T_3 = B_3\mathrm{e}^{\mathrm{i}\omega t}\cos(ax)\sin(\pi z) + \cdots. \tag{40}$$

Equation (39) gives the following cubic Landau equation that explains the temporal variation of B_1 of the convection cell:

$$\gamma\frac{\mathrm{d}B_1}{\mathrm{d}s} = \frac{a^2(1+\mathrm{i}\omega\Lambda_1)}{(1+\mathrm{i}\omega\Lambda_2)\big(a^2+\frac{1}{\xi}\pi^2\big)} Ra_{\mathrm{D2}}B_1 - \kappa|B_1|^2B_1, \tag{41}$$

where

$$\gamma = 1 + \frac{1}{1+\mathrm{i}\Lambda_2\omega}\Big(\Lambda_2(c+\mathrm{i}\omega) - a^2\Lambda_1 Ra^0_{\mathrm{Dc}}\Big(a^2+\frac{1}{\xi}\pi^2\Big)^{-1}\Big), \tag{42}$$

$$\kappa = \pi^2\Big(\frac{c^2+\omega^2}{8\pi^2} + \frac{(c+\mathrm{i}\omega)^2}{8\pi^2} + \frac{c^2+\omega^2}{(8\pi^2+4\mathrm{i}\omega)}\Big). \tag{43}$$

From Eq. (42), the following relation can be obtained:

$$\frac{\mathrm{d}\,|B_1|^2}{\mathrm{d}s} = 2p_r\,|B_1|^2 - 2l_r\,|B_1|^4, \tag{44}$$

$$\frac{\mathrm{d}\,(ph\,(B_1))}{\mathrm{d}s} = p_i - l_i\,|B_1|^2, \tag{45}$$

where

$$\frac{a^2(1+\mathrm{i}\omega\Lambda_1)}{(1+\mathrm{i}\omega\Lambda_2)(a^2+\frac{1}{\xi}\pi^2)}R_{\mathrm{D2}}\gamma^{-1} = p_r + \mathrm{i}p_i, \quad \kappa\gamma^{-1} = l_r + \mathrm{i}l_i,$$

and $ph(\cdot)$ represents the phase shift. The temporal evolution of $|B_1|$ can be expressed as a function of the initial amplitude B_0 as follows:

$$|B_1|^2 = \frac{B_0^2}{(l_r/p_r)B_0^2 + (1-(l_r/p_r)B_0^2)\exp(-2p_r s)}. \tag{46}$$

From the above equation, it is seen that $|B_1| \sim B_0\exp(p_r s)$ as $s \to -\infty$ and $|B_1| \to 0$, just as the linear theory, but $|B_1| \to \sqrt{p_r/l_r}$ as $s \to \infty$, which is independent of the value of B_0. For the post-transient state, Eq. (45) yields an expression for the amplitude as follows:

$$|B_1|^2 = \frac{p_r}{l_r} = \frac{R_{\mathrm{D2}}}{\Omega}. \tag{47}$$

If $\Omega > 0$, the bifurcation is supercritical. If $\Omega < 0$, the bifurcation is subcritical. This can be achieved by evaluating the expression for Ω for various values of the physical parameters since there is no simple way to analyze this expression. For this case, the area and time-averaged thermal Nusselt number can be represented by using Eq. (11) as follows:

$$Nu = 1 + \frac{\Big(\frac{\alpha_{\mathrm{c}}}{2\pi}\int_0^{2\pi/\alpha_{\mathrm{c}}}\frac{\mathrm{d}T}{\mathrm{d}z}\mathrm{d}x\Big)_{z=0}}{\Big(\frac{\alpha_{\mathrm{c}}}{2\pi}\int_0^{2\pi/\alpha_{\mathrm{c}}}\frac{\mathrm{d}T_{\mathrm{b}}}{\mathrm{d}z}\mathrm{d}x\Big)_{z=0}} = 1 - \chi^2\Big(\frac{\mathrm{d}T_2}{\mathrm{d}z}\Big)_{z=0}. \tag{48}$$

With Eqs. (37) and (38), we can rewrite Eq. (48) as follows:

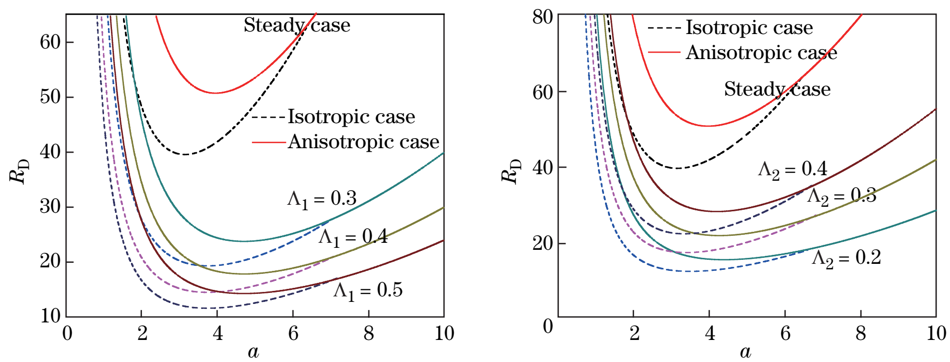
$$Nu = 1 + \frac{1}{2}\frac{c}{\Omega}(R_{\mathrm{D}} - R^{\mathrm{o}}_{\mathrm{Dc}}). \tag{49}$$

4 Results and discussion

The effects of the anisotropy in permeability and thermal diffusivity on the nonlinear stability of thermal convection in a horizontal porous layer saturated by an Oldroyd-B fluid are investigated. Since the considered nonlinear stability analysis is based on the linear instability analysis, the results of the linear instability theory are also discussed. Although the stationary convection boundary depends on anisotropy parameters while is independent of viscoelastic parameters, it concurs with the Newtonian fluid-saturated anisotropic porous layer when the base flow is quiescent. The oscillatory convection boundary, however, depends on viscoelastic parameters, e.g., mechanical and thermal anisotropy parameters.

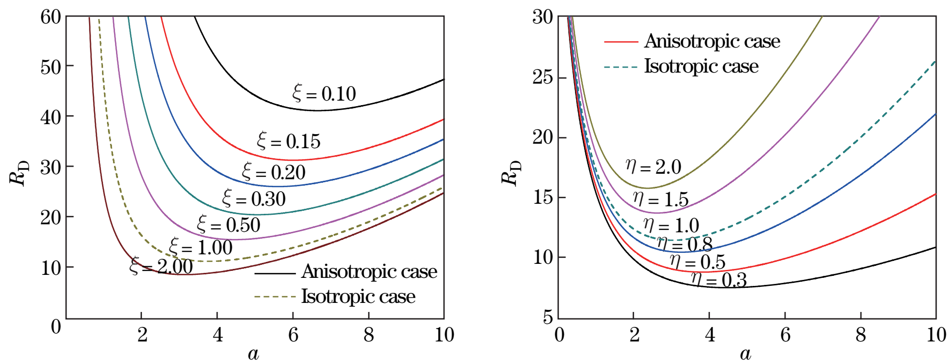
The neutral stability curves on the (a, R_{D})-plane for different values of the stress relaxation parameter Λ_1, strain retardation parameter Λ_2, mechanical anisotropy parameter ξ, and thermal anisotropy parameter η are presented in Figs. 2 and 3. The region underneath the neutral curve corresponds to the stability region, above which it is unstable. It is observed that the effects of increasing Λ_1 (see Fig. 2(a)) and ξ (see Fig. 3(a)) as well as decreasing Λ_2 (see Fig. 2(b)) are to decrease the stability region, while the effect of increasing η (see Fig. 3(b)) is to increase the stability region. Besides, the oscillatory neutral stability curves shift towards lower values of the wavenumber when Λ_1 and ξ increase, which indicates that the cell width at the critical state increases while Λ_2 and η decrease.

The critical Darcy-Rayleigh number R_{Dc} and the corresponding critical oscillation frequency ω_{c} are obtained for various physical parameters. The results are presented in Figs. 4–6.



(a) $\Lambda_2 = 0.1$ (b) $\Lambda_1 = 0.8$

Fig. 2 Neutral stability curves for different values of Λ_1 and Λ_2 for isotropic ($\eta/\xi = 1$) and anisotropic cases ($\eta/\xi = 1.6$)



(a) $\eta = 1.0$ (b) $\xi = 1.0$

Fig. 3 Oscillatory neutral stability curves for different values of ξ and η when $\Lambda_1 = 0.5$ and $\Lambda_2 = 0.1$

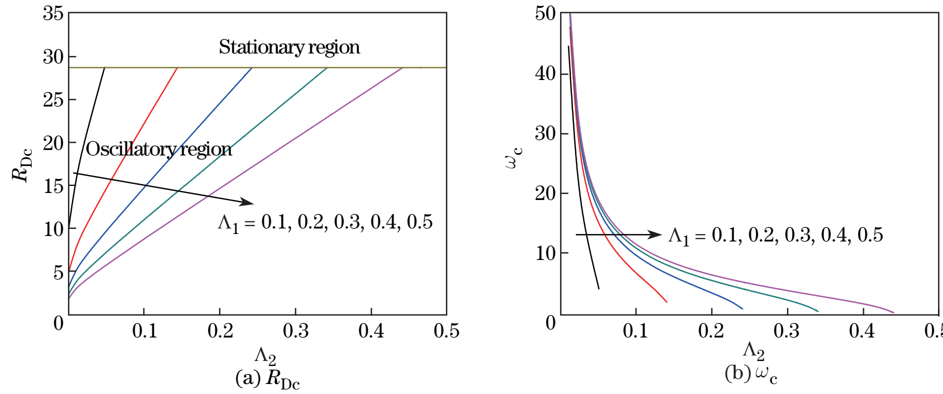


Fig. 4 Variations of R_{Dc} and ω_{c} with respect to Λ_2 for different values of Λ_1, where $\eta/\xi = 0.5$

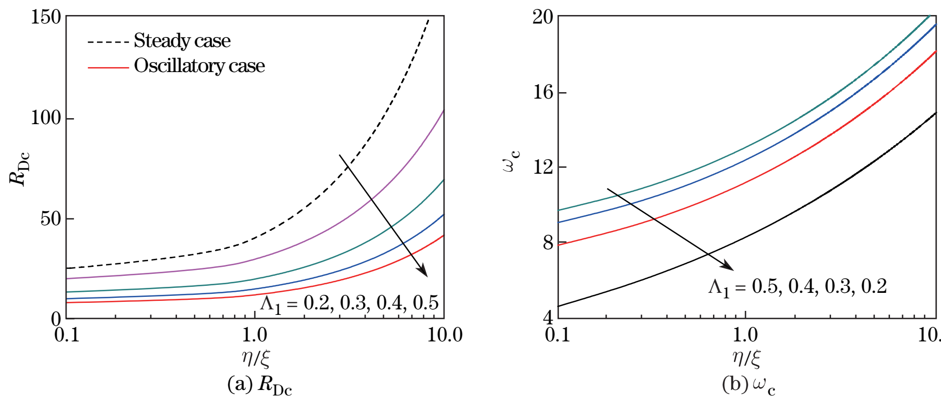


Fig. 5 Variations of R_{Dc} and ω_{c} with respect to η/ξ for different values of Λ_1, where $\Lambda_2 = 0.1$

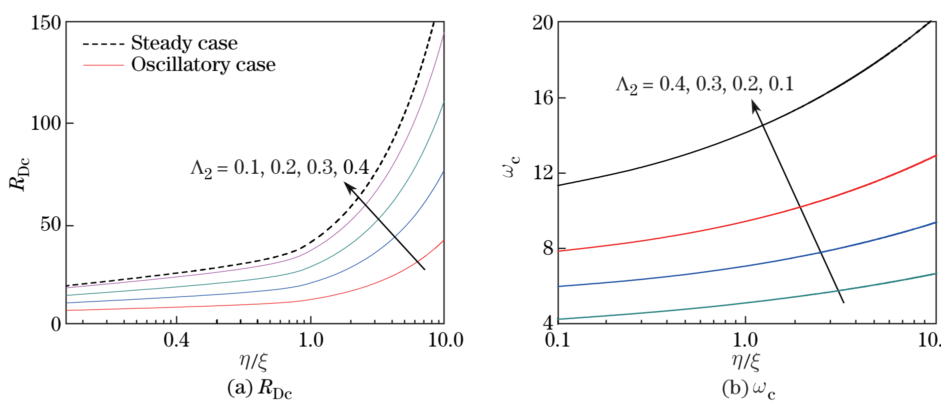


Fig. 6 Variations of R_{Dc} and ω_{c} with respect to η/ξ for different values of Λ_2 when $\Lambda_1 = 0.5$

Figures 4(a) and 4(b), respectively, show the variations of R_{Dc} and ω_{c} as functions of Λ_2 for different values of Λ_1, where $\eta/\xi = 0.5$. It is noted that increasing Λ_1 is to advance the onset of oscillatory convection for any fixed value of Λ_2. This may be attributed to the fact that increasing the relaxation parameter ceases the stickiness of the viscoelastic fluid and hence the effect of friction will be reduced so that the convection sets in at lower values of R_{Dc}. On the contrary, increasing Λ_2 delays the onset of oscillatory convection for a fixed value of Λ_1 because increasing Λ_2 amounts to increasing the time taken by the fluid element to respond to the applied stress. Further inspection of Fig. 4(a) reveals that the range of the values of Λ_2, within which the oscillatory convection possibly increases with increasing Λ_1. In other words, for a fixed value of Λ_1, there exists a threshold value Λ_2^* which divides the boundary of regimes between the oscillatory and stationary convection. Initially, convection begins in the form of an oscillatory mode. As the value of Λ_2 reaches Λ_2^*, convection ceases to be oscillatory and stationary convection becomes the preferred mode of instability. The value of Λ_2^* depends on other physical parameters as well. The critical frequency ω_{c} shown in Fig. 4(b) exhibits that it decreases with increasing Λ_2 and increases with increasing Λ_1 due to the increase in the elasticity of the fluid.

The variations of (R_{Dc} and ω_{c}) as functions of η/ξ are shown in Figs. 5 and 6 for different values of Λ_1 (with $\Lambda_2 = 0.1$) and Λ_2 (with $\Lambda_1 = 0.5$), respectively. These figures clearly indicate that the effects of increasing η/ξ is to delay the onset of convection and to increase the frequency of oscillations. The increase in η/ξ amounts to either decreasing ξ or increasing η. We note that the decrease in ξ amounts to decreasing the horizontal permeability, which impedes the motion of the fluid in the parallel direction. As a result, the transfer process in the porous medium gets suppressed, and hence higher values of R_{Dc} are needed for the onset of instability. Moreover, the increase in η amounts to increasing the horizontal thermal diffusivity. Thus, heat can be transported with ease in the porous layer, the horizontal temperature differences in the fluid, which are necessary to maintain convection, are more competently dissipated with increasing η, and higher values of R_{Dc} are required for the onset of convection. It is intriguing to note that by altering the anisotropy in the permeability and thermal diffusivity, it is possible to control the (augment/suppress) convective instability. From the figures, it is further evident that the increase in Λ_1 and decrease in Λ_2 are to hasten the onset of oscillatory convection. Moreover, the critical frequency increases with increasing Λ_1 while decreases with increasing Λ_2.

The parameters for the boundary separating stationary and oscillatory solutions are estimated. Figures 7(a) and 7(b) show the bifurcation of the stationary and oscillatory solutions in the viscoelastic parameter plane for different values of η and ξ, respectively. The region above each curve corresponds to the system which is unstable under oscillatory convection, and the region below the curve corresponds to the system which is unstable under stationary convection. From Figs. 7(a) and 7(b), for a fixed value of Λ_2/Λ_1, it is seen that the value of Λ_1, at which codimension-two bifurcation occurs, decreases when η increases (see Fig. 7(a)), while an opposite trend is observed when ξ increases (see Fig. 7(b)). As the value of Λ_2/Λ_1 advances towards 0.9, there is a steep rise in the value of Λ_1.

The stability of the stationary and oscillatory bifurcating solutions is analyzed by deriving the cubic Landau equations for these cases. It is an observable fact that the stationary solution always bifurcates supercritically (see Eq. (25)), while the stability of the oscillatory solution

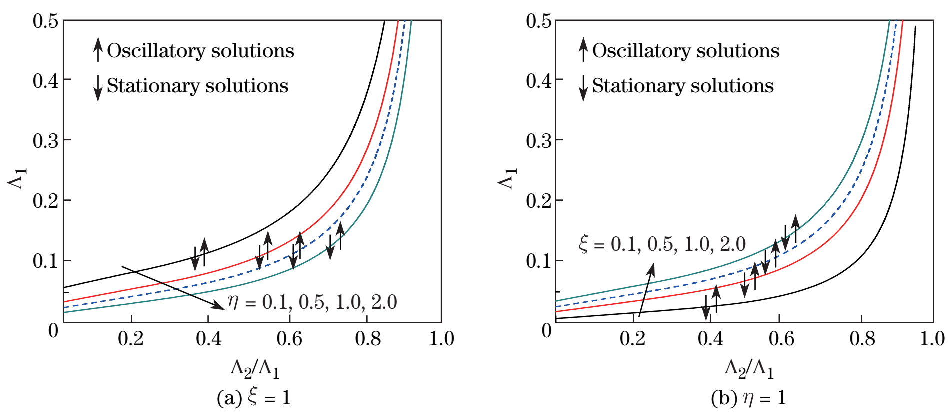


Fig. 7 Bifurcations of stationary and oscillatory solutions in the viscoelastic parameter plane for different values of η and ξ

can be understood from the sign of Ω. When $\Omega > 0$, the bifurcation is supercritical. When $\Omega < 0$, the bifurcation is subcritical (see Eq. (47)). Hence, the expression Ω is evaluated for a wide range of parametric values at the critical values of oscillatory convection, and is denoted by Ω_c. Figures 8(a) and 8(b) represent the computed values of Ω_c as a function of η/ξ for different values of Λ_1 and Λ_2, respectively. These figures indicate that the oscillatory solution always bifurcates supercritically. Thus, the linear instability analysis provides the necessary and sufficient conditions for instability.

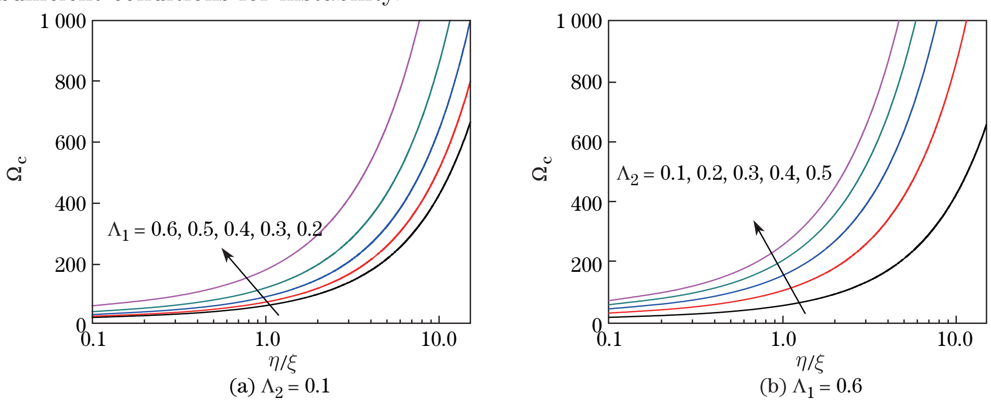


Fig. 8 Variations of Ω_c with respect to η/ξ for different values of Λ_1 and Λ_2

The heat transfer is estimated in terms of the Nusselt number for both stationary and oscillatory cases. For the stationary case, the area-averaged Nusselt number is calculated as a function of R_D for different values of η/ξ. It is seen that the Nusselt number originates from higher values of R_D with increasing η/ξ. The value of the Nusselt number increases with increasing R_D for any fixed value of η/ξ, and the heat transfer decreases with increasing η/ξ (see Fig. 9).

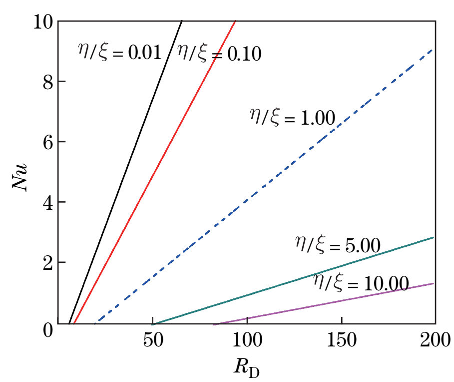
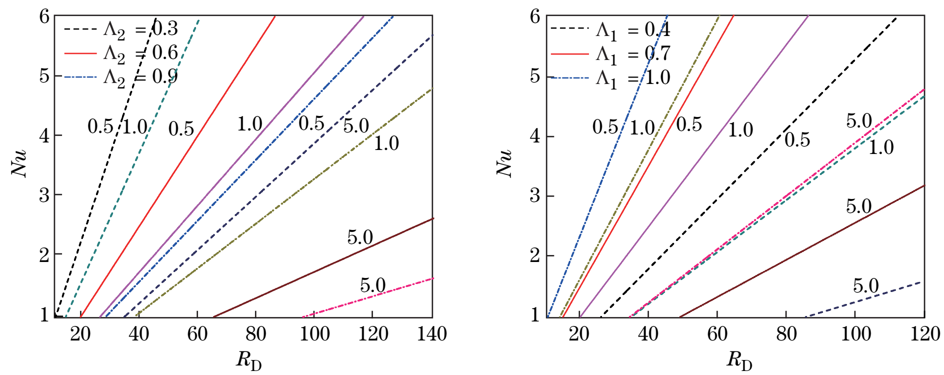


Fig. 9 Effects of η/ξ on the area-averaged Nusselt number Nu

For the oscillatory case, the area and time-averaged Nusselt number Nu is calculated as a function of R_D for various values of the physical parameters. The variations of Nu as a function of R_D for different values of η/ξ as well as Λ_2 (with $\Lambda_1 = 1$) and Λ_1 (with $\Lambda_2 = 0.3$) are illustrated in Figs. 10(a) and 10(b), respectively. It is noted that the value of Nu increases when R_D increases for a fixed value of η/ξ. Moreover, the heat transfer increases with increasing Λ_1 while decreases with increasing η/ξ and Λ_2.

5 Conclusions

The nonlinear stability of thermal convection in an Oldroyd-B fluid-saturated anisotropic porous layer is investigated with the perturbation method. The onset of stationary and



(a) $\Lambda_1 = 1.0$ (b) $\Lambda_2 = 0.3$

Fig. 10 Variations of the area- and time-averaged Nusselt numbers with respect to R_{D} for different values of η/ξ (0.5, 1.0, and 1.5), Λ_2, and Λ_1

oscillatory convection is delineated since the nonlinear stability analysis is based on the results of the linear instability analysis. The instability sets in via the oscillatory mode under certain conditions, and the effect of increasing the mechanical and thermal anisotropy parameters is to advance and delay the onset of the oscillatory convection, respectively. A codimension-two bifurcation occurs at well-defined parametric conditions, and the value of the relaxation parameter, at which it occurs, decreases with increasing the thermal anisotropy parameter and decreasing the mechanical anisotropy parameter in the viscoelastic parameter plane. The stability of the stationary and oscillatory cases is discussed by deriving the cubic Landau equations. It is observed that these solutions always bifurcate supercritically. The increases in the value of the relaxation and retardation parameters are to enhance and suppress the time and area-averaged heat transfer, respectively. Besides, the increase in the ratio of the thermal anisotropy parameter to the mechanical anisotropy parameter is to decrease the heat transfer. By tuning the anisotropy of the porous medium, it is possible to control the convective instability of the system.

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References

[1] McKibbin, R. Convection and heat transfer in layered and anisotropic porous media. *ASME* 2008 *Heat Transfer Summer*, American Society of Mechanical Engineers, New York, 327–336 (1992)

[2] Nield, D. A. and Bejan, A. *Convection in Porous Media*, Springer, New York (2013)

[3] Storesletten, L. Effects of anisotropy on convective flow through porous media. *Transport Phenomena in Porous Media*, Elsevier, the Netherlands, 261–283 (1998)

[4] Straughan, B. *The Energy Method*, *Stability and Nonlinear Convection*, Springer, New York (2004)

[5] Storesletten, L. Effects of anisotropy on convection in horizontal and inclined porous layers. *Emerging Technologies and Techniques in Porous Media*, Springer, Dordrecht, 285–306 (2004)

[6] Castinel, G. and Combarnous, M. Natural convection in an anisotropic porous layer. *International Journal of Chemical Engineering*, **17**, 605–613 (1977)

[7] Epherre, J. F. Criterion for the appearance of natural convection in an anisotropic porous layer. *International Journal of Chemical Engineering*, **17**, 615–616 (1977)

[8] Capone, F., Gentile, M., and Hill, A. A. Anisotropy and symmetry in porous media convection. *Acta Mechanica*, **208**, 205–214 (2009)

[9] Kvernvold, O. and Tyvand, P. A. Nonlinear thermal convection in anisotropic porous media. *Journal of Fluid Mechanics*, **90**, 609–624 (1979)

[10] Govender, S. On the effect of anisotropy on the stability of convection in rotating porous media. *Transport in Porous Media*, **64**, 413–422 (2006)

[11] Tyvand, P. A. and Storesletten, L. Onset of convection in an anisotropic porous layer with vertical principal axes. *Transport in Porous Media*, **108**, 581–593 (2015)

[12] Malashetty, M. S. and Swamy, M. The onset of convection in a viscoelastic liquid saturated anisotropic porous layer. *Transport in Porous Media*, **67**, 203–218(2007)

[13] Alishaev, M. G. and Mirzadjanzade, A. K. For the calculation of delay phenomenon in filtration theory. *Izvestiya Vuzov Neft i Gaz*, **6**, 71–77 (1975)

[14] Rudraiah, N. and Kaloni, P. N. Flow of non-Newtonian fluids. *Encyclopedia of Fluid Mechanics*, **9**, 1–69 (1990)

[15] Rudraiah, N., Kaloni, P. N., and Radhadevi, P. V. Oscillatory convection in a viscoelastic fluid through a porous layer heated from below. *Rheologica Acta*, **28**, 48–53 (1989)

[16] Shenoy, A. V. Non-Newtonian fluid heat transfer in porous media. *Advances in Heat Transfer*, **24**, 101–190 (1994)

[17] Kim, M. C., Lee, S. B., Kim, S., and Chung, B. J. Thermal instability of viscoelastic fluids in porous media. *International Journal of Heat and Mass Transfer*, **46**, 5065–5072 (2003)

[18] Malashetty, M. S., Shivakumara, I. S., Sridharkulkarni, and Swamy, M. Convective instability of Oldroyd-B fluid saturated porous layer heated from below using a thermal non-equilibrium model. *Transport in Porous Media*, **64**, 123–39 (2006)

[19] Shivakumara, I. S., Malashetty, M. S., and Chavaraddi, K. B. Onset of convection in a viscoelastic fluid saturated sparsely packed porous layer using a thermal non-equilibrium model. *Canadian Journal of Physics*, **84**, 973–90 (2006)

[20] Sheu, L. J., Tam, L. M., Chen, J. H., Chen, H. K., Lin, K. T., and Kang, Y. Chaotic convection of viscoelastic fluids in porous media. *Chaos*, *Solitons and Fractals*, **37**, 113–124 (2008)

[21] Wang, S. and Tan, W. Stability analysis of soret-driven double-diffusive convection of Maxwell fluid in a porous medium. *International Journal of Heat and Fluid Flow*, **32**, 88–94 (2011)

[22] Raghunatha, K. R., Shivakumara, I. S., and Shankar, B. M. Weakly nonlinear stability analysis of triple diffusive convection in a Maxwell fluid saturated porous layer. *Applied Mathematics and Mechanics* (*English Edition*), **39**(2), 153–168 (2017) https://doi.org/10.1007/s10483-018-2298-6

[23] Cao, L. M., Si, X. H., and Zheng, L. C. Convection of Maxwell fluid over stretching porous surface with heat source/sink in presence of nanoparticles: Lie group analysis. *Applied Mathematics and Mechanics* (*English Edition*), **37**(4), 433–442 (2016) https://doi.org/10.1007/s10483-016-2052-9

[24] Mahanthesh, B., Gireesha, B. J., Shehzad, S. A., Abbasi, F. M., and Gorla, R. S. R. Nonlinear three-dimensional stretched flow of an Oldroyd-B fluid with convective condition, thermal radiation, and mixed convection. *Applied Mathematics and Mechanics* (*English Edition*), **38**(7), 969–980 (2017) https://doi.org/10.1007/s10483-017-2219-6

[25] Makinde, O. D. and Eegunjobi, A. S. Entropy analysis of thermally radiating magnetohydrodynamics slip flow of Casson fluid in a microchannel filled with saturated porous media. *Journal of Porous Media*, **19**, 799–810 (2016)

[26] Makinde, O. D. and Rundora, L. Unsteady mixed convection flow of a reactive Casson fluid in a permeable wall channel filled with a porous medium. *Defect and Diffusion Forum*, **377**, 166–179 (2017)

[27] Malkus, W. V. R. and Veronis, G. Finite amplitude cellular convection. *Journal of Fluid Mechanics*, **4**, 225–260 (1985)

[28] Venezian, G. Effect of modulation on the onset of thermal convection. *Journal of Fluid Mechanics*, **35**, 243–254 (1969)

[29] Gupta, V. P. and Joseph, D. D. Bounds for heat transport in a porous layer. *Journal of Fluid Mechanics*, **57**, 491–514 (1973)