Characterization of the lengths of binary circular words containing no squares other than 00, 11, and 0101

James D. Currie
Department of Mathematics & Statistics
The University of Winnipeg*
currie@uwinnipeg.ca

Jesse T. Johnson
Department of Mathematics & Statistics
University of Victoria
jessejoho@gmail.com

May 19, 2020

Abstract

We characterize exactly the lengths of binary circular words containing no squares other than 00, 11, and 0101. Key words: combinatorics on words, circular words, necklaces, square-free words, non-repetitive sequences

1 Introduction

Combinatorics on words started with the work of Thue [17], who showed the existence of arbitrarily long square-free words over a three letter alphabet. Thue studied circular words also [18], and completely characterized the circular overlap-free words on two letters. Circular words have been relatively unexplored until recently. In 2002, the first author characterized the lengths for which ternary circular square-free words exist:

Theorem 1. *For every positive integer n other than 5, 7, 9, 10, 14, or 17, there is a ternary circular square-free word of length n.*

Several other proofs [16, 1, 11] of this theorem have now been given, signaling increasing interest in circular words. The general question of when there is a circular word avoiding some pattern has also begun to receive attention [3, 12]. Circular words avoiding patterns seem harder to understand than linear words; while the set of linear words avoiding

*The author is supported by an NSERC Discovery Grant

some pattern is closed under taking factors, this is not true for the circular version.

Square-free words are objects of continuing interest in combinatorics on words. As proven by Thue, there exist arbitrarily long words over $\{a, b, c\}$ that contain no square factors. On the other hand, one quickly checks that every word over $\{0, 1\}$ of length 4 or greater contains a square. A binary word with as few square factors as possible was found by Fraenkel and Simpson [5].

Theorem 2. *There exist arbitrarily long words over* $\{0, 1\}$ *avoiding all factors of the form* xx, $x \neq 0, 1, 01$.

Simpler proofs of this result have been found [6, 14]. Call a binary word containing no squares other than 00, 11, and 0101 an **FS word**. Harju and Nowotka have shown [7] that there are arbitrarily long circular FS words. It is natural to ask: For exactly which lengths are there circular FS words? We answer this question completely:

Main Theorem. *There is a circular FS word of length* m *exactly when* m *is a non-negative integer other than 9, 10, 11, 13, 15, 16, 17, 18, 21, 22, 23, 25, 26, 27, 29, 31, 32, 33, 34, 35, 37, 40, 41, 42, 45, 47, 49, 53, 56, 59, 61, 64, and 73.*

It would be interesting to probe the structure of circular FS words more deeply.

Problem. *For each positive integer* n*, how many circular FS words are there of length* n*?*

2 Preliminaries

For general background on combinatorics on words, see the works of Lothaire [9, 10]. Let Σ be a finite set. We refer to Σ as an **alphabet**, and its elements as **letters**. We denote by Σ^* the free monoid over Σ, with identity ϵ, the **empty word**. We call the elements of Σ^* **words**. Informally, we think of the elements of Σ^* as finite strings of letters, and of its binary operation as concatenation. Thus, if $u = u_1u_2 \cdots u_n$, $u_i \in \Sigma$ and $v = v_1v_2 \cdots v_m$, $v_j \in \Sigma$, then $uv = u_1u_2 \cdots u_nv_1v_2 \cdots v_m$. In this case, we say that u is a **prefix** of uv and v is a **suffix**. More generally, if $w = uvz$, then v is a **factor** of w. We say that v appears in w at **index** i in the case where $|u| = i - 1$. We will work in particular with the alphabets $B = \{0, 1\}$, $S = \{a, b, c\}$, and $T = \{a, b, c, d\}$. Words over S are called **ternary words**.

A word of the form $s = uu$, $u \neq \epsilon$ is called a **square**. Thus a square uu has period $|u|$. We write z^2 for zz. A word w which doesn't contain a square factor is said to be **square-free**. We call a word over B containing no square factors other than 00, 11, and 0101 an **FS word** (for Fraenkel-Simpson word).

If $u = u_1u_2 \cdots u_n$, $u_i \in \Sigma$, then the **length** of u is defined to be n, the number of letters in u, and we write $|u| = n$. The set of words of length m over Σ is denoted by Σ^m. We use $\Sigma^{\geq n}$ to denote the set of words over Σ of length at least n. For $a \in \Sigma$, $u \in \Sigma^*$, we denote by $|u|_a$ the number of occurrences of a in u.

If $w = uv$, then define $wv^{-1} = u$. Thus $vu = v(uv)v^{-1}$, and we refer to vu as a **conjugate** of uv. The relation 'a is a conjugate of b' is an equivalence relation on Σ^*, and we refer to the equivalence classes of Σ^* under this equivalence relation as **circular words**. If $w \in \Sigma^*$, we denote the circular word containing w by $[w]$.We may consider the indices i of the letters of a circular word $[u] = [u_1 u_2 \cdots u_n]$ to belong to $\mathbb{Z}_n$, the integers modulo n. Thus $u_{n+1} = u_1$, for example. If $[w]$ is a circular word and $v \in \Sigma^*$, we say that v is a **factor** of $[w]$ if v is a factor of an element of $[w]$, i.e., if v is a factor of a conjugate of w. A circular word $[w]$ is **square-free** if no factor of $[w]$ is a square.

A circular word $[w]$ is called an **FS circular word**, if every conjugate of w is an FS word.

Let Σ and T be alphabets. A map $\mu : \Sigma^* \to T^*$ is called a **morphism** if it is a monoid homomorphism, that is, if $\mu(uv) = \mu(u)\mu(v)$, for $u, v \in \Sigma^*$. A morphism $f : \Sigma^* \to B^*$ such that $f(w)$ is an FS word whenever w is square-free is called an **FS morphism**.

A ternary w such that for any letters $x, y \in \{a, b, c\}$,

$$|w|_x - 1 \leq |w|_y \leq |w_x| + 1$$

is called **level**. The authors recently proved the following:

Theorem 3. *[4, 8] There is a level ternary circular square-free word of length n, exactly when n is a positive integer, $n \neq 5, 7, 9, 10, 14, 17$.*

3 Constructing circular FS words

We begin by proving a generalization of an approach used by Harju and Nowotka [6]. They used it to demonstrate that a particular morphism applied to ternary square-free words gave FS words. Here, we use it to find FS morphisms on alphabets of any size:

Lemma 1. *Fix $n \geq 3$. Suppose $f : \Sigma_n^* \to \Sigma_2^*$ is a morphism satisfying these conditions:*

1. *For any square-free $v \in \Sigma_n^3$, $f(v)$ is an FS word.*
2. *There is a word $p \in \Sigma_2^*$, $|p| \geq 3$, such that:*
   - *(a) For each $a \in \Sigma_n$, p is a prefix of $f(a)$.*
   - *(b) If $a_i \in \Sigma_n$, $1 \leq i \leq \ell$, and $f(a_1 a_2 \cdots a_\ell) = qpr$ for some words $q, r \in \Sigma_2^*$, then $q = \epsilon$ or $q = f(a_1 a_2 \cdots a_j)$, some $j \leq \ell$.*

Then f is an FS morphism.

Proof. To begin with, note that the conditions imply that if $a, b \in \Sigma_n$ and $f(a)$ is a prefix of $f(b)$, then $a = b$. Otherwise, aba is a square-free word of length 3, with square prefix $f(a)f(a)$. However, $|f(a)| \geq |p| \geq 3$, so $f(a)f(a) \neq 00, 11$, or 0101. This contradicts Condition 1.

For the sake of getting a contradiction, consider a square-free word $w = w_1 w_2 \cdots w_m$, with the $w_i \in \Sigma_n$, such that $f(w_1 w_2 \cdots w_m)$ contains a square xx, $x \neq \epsilon, 0, 1, 01$. Let m be as small as possible. By Condition 1, $m \geq 4$. Since m is minimal, write

$$xx = W_1'' W_2 \cdots W_m',$$

$$\begin{aligned} \text{where } f(w_1) &= W_1'W_1'', W_1'' \neq \epsilon \\ f(w_i) &= W_i, 2 \leq i \leq m-1 \\ f(w_m) &= W_m'W_m'', W_m' \neq \epsilon. \end{aligned}$$

As per Condition 2a, write $W_2 = pW_2''$.

Case A: $|x| < |W_1''|$ **or** $|x| < |W_m'|$

If $|x| < |W_1''|$, write $W_1'' = xW_1'''$, $W_1''' \neq \epsilon$. Then we find the second copy of x in xx can be written

$$x = W_1'''W_2 \cdots W_m' = W_1'''pW_2'' \cdots W_m'.$$

However, then

$$f(w_1) = W_1'W_1'' = W_1'xW_1''' = W_1'W_1'''pW_2'' \cdots W_m'W_1'''$$

contains an instance of p at an index which contradicts Condition 2b.

Similarly, if $|x| < |W_m'|$, write $W_m' = W_m'''x$, $W_m''' \neq \epsilon$. Then we find the first copy of x in xx can be written

$$x = W_1''W_2 \cdots W_m''' = W_1'''pW_2'' \cdots W_m'''.$$

However, then

$$f(w_m) = W_m'W_m'' = W_m'''xW_m'' = W_m'''W_1''pW_2'' \cdots W_m'''W_m''$$

contains an instance of p at an index which contradicts Condition 2b.

Case B: $|x| \geq |W_1''|, |W_m'|$

In this case we can write

$$\begin{aligned} x &= W_1'' \cdots W_j' \\ &= W_j'' \cdots W_m', \end{aligned}$$

for some j, $1 < j < m$, with $W_j = W_j'W_j''$.

If $j > 2$, then there is at least one instance of p in $x = W_1''W_2 \cdots W_j'$, appearing as a prefix of W_2. On the other hand, if $j = 2$, then an instance of p appears as a prefix of W_{j+1} in $x = W_j''W_{j+1} \cdots W_m'$. In either case, there is at least one instance of p in x. For the sake of definiteness, adjusting notation if necessary, choose j so that $W_j' = \epsilon$ if x starts with p; that is, assume in all cases that $W_j'' \neq \epsilon$.

Case B(i): Word x starts with p.

If x starts with p, then Condition 2b forces $W_1'' = W_1$. Our choice of notation gives $W_j'' = W_j$. Since W_1 and W_j are prefixes of x, one must be a prefix of the other, and, as noted at the beginning of this proof, this forces $w_1 = w_j$. Therefore $W_1 = W_j$.

We prove by induction that for $1 \leq i \leq j-2$, $w_1 \cdots w_i = w_j \cdots w_{j+i-1}$, and $W_{i+1} \cdots W_{j-1} = W_{j+i} \cdots W_m'$. We have just established the base case of this induction, when $i = 1$.

Suppose that for some k, $1 \le k < j-2$, we have $w_1 \cdots w_k = w_j \cdots w_{j+k-1}$, and $W_{k+1} \cdots W_{j-1} = W_{j+k} \cdots W'_m$. Then one of W_{k+1} and W_{j+k} is a prefix of the other, giving $w_{k+1} = w_{j+k}$, yielding the induction step.

Setting $i = j-1$, we see that $w_1 \cdots w_{j-1} = w_j \cdots w_{2j-2}$. However, now w contains the square $(w_1 \cdots w_{j-1})^2$. Since $|w_1 \cdots w_{j-1}| \ge |p| = 3$, this is a contradiction.

Case B(ii): Word x doesn't start with p.

The first p in x is at the beginning of W_2: If $x = W''_1 W_2 \cdots W'_j$ has an instance of p of index i, $1 < i < |W'_1| + 1$, then $f(w_1 w_2)$ contains an instance of p of index properly between 1 and $|f(w_1)| + 1$, violating property 2(b). Thus the least index of p in x is $|W'_1| + 1$. However, an analogous argument observing that $x = W''_j W_{j+1} \cdots W'_m$ yields least index of $p = |W''_j| + 1$. Thus W''_1 and W''_j are prefixes of x with the same length, forcing $W''_1 = W''_j$. Now, $W_2 \cdots W'_j = W_{j+1} \cdots W'_m$, so that one of W_2 and W_{j+1} is a prefix of the other, forcing $w_2 = w_{j+1}$.

We prove by induction that for $2 \le i \le j-2$, $w_2 \cdots w_i = w_{j+1} \cdots w_{j+i-1}$, and $W_{i+1} \cdots W_{j-1} W'_j = W_{j+i} \cdots W_{m-1} W'_m$. We have just established the base case of this induction, when $i = 2$.

Suppose that for some k, $1 \le k < j-1$, we have $w_2 \cdots w_k = w_{j+1} \cdots w_{j+k-1}$, and $W_{k+1} \cdots W_{j-1} W'_j = W_{j+k} \cdots W_{m-1} W'_m$. Then one of W_{k+1} and W_{j+k} is a prefix of the other, giving $w_{k+1} = w_{j+k}$, yielding the induction step.

When $i = j-1$, we find $W'_j = W'_m$. Since one of W_j and W_m must be a prefix of the other, $w_j = w_m$. Then w contains the square $w_2 \cdots w_j w_{j+1} \cdots w_m = (w_2 \cdots w_j)^2$. Since $|w_2 \cdots w_j| \ge |p| = 3$, this is a contradiction. □

One can find morphisms on T satisfying the conditions of Lemma 1 by computer search.

It is possible to build circular FS words from circular square-free words and FS morphisms, using the following Lemma and Corollary due to Rampersad[13].

Lemma 2. *If f is a square-free morphism from Σ_n to Σ_m, and $[w]$ is a square-free circular word with $|w| \ge 2$, then $[f(w)]$ is a square-free circular word.*

Proof. Write $w = w_1 w_2 \cdots w_\ell$, $w_\ell \in \Sigma_n$. Let $f(w_i) = W_i$, $1 \le i \le \ell$. Replacing w with one of its conjugates if necessary, we can assume that $W''_1 W_2 \cdots W_\ell W'_1$ is a representative of $[f(w)]$ containing a square, where $W_1 = W'_1 W''_1$. Then $W_1 W_2 \cdots W_\ell W_1 = f(w_1 w_2 \cdots w_\ell w_1)$ also contains this square. Since f is square-free, this implies that $w_1 w_2 \cdots w_\ell w_1$ contains some square xx. Both $w_1 w_2 \cdots w_\ell$ and $w_2 \cdots w_\ell w_1$ are representatives of w, and are thus square-free. It follows that $xx = w_1 w_2 \cdots w_\ell w_1$. However, x then begins and ends with letter w_1, so that $w_1 w_1$ appears at the center of xx, whence w contains the square $w_1 w_1$.This is a contradiction. □

Corollary 1. *If f is a FS morphism from Σ_n to Σ_2, and $[w]$ is a square-free circular word with $|w| \ge 2$, then $[f(w)]$ is an FS circular word.*

Proof. The previous proof goes through, replacing 'containing a square' by 'containing a square other than 00, 11, 0101', and 'square-free' by 'an FS morphism'. □

To produce circular FS words with specific lengths, we make use of the following recent result by the authors [4, 8]:

Theorem 4. *There is a level ternary circular square-free word of length n, for each positive integer n, $n \neq 5, 7, 9, 10, 14, 17$.*

Here is how we produce circular FS words with desired lengths: Given $n \geq 2$, $n \neq 5, 7, 9, 10, 14, 17$, write $n = 3i + j$, integers i and j such that $-1 \leq j \leq 1$. Note that $i \geq 1$. Let w be a level circular square-free word over S with $|w| = n$. Permuting a, b, c if necessary, assume that , $|w|_a = i + j$, $|w|_b = |w|_c = i$. For $k \leq i + j$, replacing k of the a's in w by d's gives a circular square-free word u over T with $|u|_a = i + j - k$, $|u|_b = |u|_c = i$, $|u|_d = k$.

Suppose that $f : T^* \to B^*$ is an FS morphism, with $|f(a)| = \alpha$, $|f()| = \beta$, $|f(c)| = \gamma$, $|f(d)| = \delta$. By Corollary 1, $[f(u)]$ is a circular FS word, with length

$$\begin{aligned} \sum_{t \in T} |f(t)||u|_t &= \alpha(i + j - k) + \beta i + \gamma i + \delta(k) \\ &= (\alpha + \beta + \gamma)i + \alpha j + k(\delta - \alpha) \end{aligned}$$

.

In a similar way, for $k \leq i$, replacing k of the b's in w by d's gives a circular square-free word v over T with $|v|_a = i + j$, $|v|_b = i - k$,$|v|_c = i$, $|v|_d = k$, and $[f(v)]$ is a circular FS word with length

$$(\alpha + \beta + \gamma)i + \alpha j + k(\delta - \beta)$$

We have proved the following:

Theorem 5. *Suppose there exists a FS morphism $f : T^* \to B^*$, with $|f(a)| = \alpha$, $|f(b)| = \beta$, $|f(c)| = \gamma$, $|f(d)| = \delta$. Then there exists a circular FS word of length m for every positive integer m of the form*

$$m = (\alpha + \beta + \gamma)i + \alpha j + k(\delta - \alpha) \tag{1}$$

with integers i, j, k such that

- $i \geq 1$
- $-1 \leq j \leq 1$
- $3i + j \neq 5, 7, 9, 10, 14, 17$
- $k \leq i + j$,

and of every length m of the form

$$m = (\alpha + \beta + \gamma)i + \alpha j + k(\delta - \beta) \tag{2}$$

with integers i, j, k such that

- $i \geq i$
- $-1 \leq j \leq 1$

- $3i + j \neq 5, 7, 9, 10, 14, 17$
- $k \leq i$.

As an example of the application of this theorem, we prove the following:

Lemma 3. *Suppose that m is an integer, $m \geq 7400$. There is a circular FS word of length m.*

Proof. One checks that the morphism $f : T^* \to S^*$ given by

$$\begin{aligned}
f(a) &= 01100111000101110010110001011100011001011000111001\\
f(b) &= 01100111000110010110001011100101100011100101110001\\
f(c) &= 01100111000110010111000101100011100101100010111001\\
f(d) &= 011001110001011100101100011100101110001011000111001
\end{aligned}$$

satisfies the conditions of Theorem 1. We have $|f(a)| = |f(b)| = |f(c)| = 50$, $|f(d)| = 51$. Write $m = 50\ell + k$, $0 \leq k \leq 49$. Write $\ell = 3i + j$, $-1 \leq j \leq 1$. Then $\ell = \lfloor m/50 \rfloor \geq 148$, so that $i = (\ell - j)/3 \geq 147/3 = 49 \geq k$. Then the conditions giving a length of form (2) hold, with $\alpha = \beta = \gamma = 50$, $\delta = 51$, and there is a circular FS word of length

$$\begin{aligned}
(\alpha + \beta + \gamma)i + \alpha j + k(\delta - \beta) &= 150i + 50j + k\\
&= 150\left(\frac{\ell - j}{3}\right) + 50j + m - 50\ell\\
&= 50\ell - 50j + 50j + m - 50\ell\\
&= m.
\end{aligned}$$

□

Several other morphisms satisfying the conditions of Theorem 1 are given in Tables 1, 2, and 3.

Remark 1. *Choose r, $0 \leq r \leq 32$, and consider the morphism f_r in Tables 1, 2, or 3. Let $[\alpha, \beta, \gamma, \delta]$ be a permutation of $[|f_r(a)|, |f_r(b)|, |f_r(c)|, |f_r(d)|]$. Letting i, j, k take on values allowable in Theorem 5, one produces FS words of various lengths. A computer search thus shows that all lengths less than 7400 are obtainable in this way except for*

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 55, 56, 57, 58, 59, 61, 63, 64, 65, 69, 70, 71, 73, 77, 116, 127, 232, 241, and 253 .

A further computer search finds circular FS words in each of these cases, or shows that no such word exists. Where circular FS words exist, not obtainable via Theorem 5 and the morphisms in Tables 1–3, they are listed in Table 4. The lengths for which no circular FS word exists are found to be

9, 10, 11, 13, 15, 16, 17, 18, 21, 22, 23, 25, 26, 27, 29, 31, 32, 33, 34, 35, 37, 40, 41, 42, 45, 47, 49, 53, 56, 59, 61, 64, and 73.

Main Theorem. *There is a circular FS word of length m exactly when m is a non-negative integer other than 9, 10, 11, 13, 15, 16, 17, 18, 21, 22, 23, 25, 26, 27, 29, 31, 32, 33, 34, 35, 37, 40, 41, 42, 45, 47, 49, 53, 56, 59, 61, 64, and 73.*

r	x	$f_r(x)$	$\lvert f_r(x)\rvert$
0	a	011001110001100101110001	24
	b	011001110001100101100010111001	30
	c	01100111000101110010110001011100011001011000111001	50
	d	011001110001011100101100011100101110001011000111001	51
1	a	110001100101110001011001	24
	b	11000110010110001110010110011100010111001011000101	50
	c	110001100101100010111001011001110001011000111001011001	54
	d	1100011001011000111001011100010110011100010111001011001	55
2	a	110001100101100010111001011001	30
	b	11000110010111000101100011100101110001011001	44
	c	11000110010110001110010110011100010111001011000101	50
	d	110001100101100011100101110001011000111001011000101	51
3	a	0110011100011001011000111001	28
	b	0110011100010111001011000101110001100101100010111001	52
	c	01100111000110010111000101100011100101110001100101100010111001	62
	d	011001110001011000111001011000101110001100101100011100101110001	63
4	a	0101100011100101100111000101110	31
	b	010110001110010111000101100111000110	36
	c	01011000111001011100011001011000101110010110011100	50
	d	010110001110010111000110010110001011100011001011100	51
5	a	010110001110010110011100011001011100	36
	b	01011000111001011001110001011100101100010111000110	50
	c	010110001110010111000110010110001011100101100111000110	54
	d	0101100011100101110001011001110001011100101100111000110	55
6	a	01011000111001011001110001011100101100010111000110	50
	b	0101100011100101110001011001110001011100101100111000110	55
	c	01011000111001011100011001011000101110010110011100011001011100	62
	d	010110001110010111000101100111000110010110001011100011001011100	63
7	a	011001110001011100101100011100101110001	39
	b	01100111000110010111000101100011100101110001	44
	c	011001110001100101100011100101110001100101100010111001	54
	d	0110011100011001011000101110001100101110001011000111001	55
8	a	0101100011100101100111000110	28
	b	010110001110010111000101100111000101110010110011100	51
	c	01011000111001011100011001011000101110010110011100011001011100	62
	d	010110001110010111000101100111000110010110001011100011001011100	63
9	a	110001100101100010111001011001	30
	b	110001100101100011100101110001011001	36
	c	1100011001011000111001011001110001011000111001011000101	55
	d	11000110010111000101100011100101100111000101100011100101	56
10	a	011001110001011100101100011100101110001	39
	b	01100111000101110010110001011100011001011000111001	50
	c	011001110001100101100011100101110001100101100010111001	54
	d	0110011100011001011000101110001100101110001011000111001	55

Table 1: Various FS morphisms with lengths

r	x	$f_r(x)$	$\lvert f_r(x)\rvert$
11	a	110001100101100010111001011001	30
	b	11000110010111000101100011100101	32
	c	11000110010110001110010110011100010111001011000101	50
	d	110001100101100011100101110001011000111001011000101	51
12	a	11000110010111000101100011100101110001011001	44
	b	110001100101100011100101110001011000111001011000101	51
	c	110001100101100010111001011001110001011000111001011000101	57
	d	1100011001011100010110001110010110011100010111001011000101	58
13	a	000111001011001110001100101110001011	36
	b	000111001011000101110010110011100011001011	42
	c	00011100101110001100101100010111001011001110001011	50
	d	000111001011100010110011100010111001011001110001011	51
14	a	000111001011001110001011	24
	b	00011100101110001011001110001100101110001011	44
	c	0001110010111000101100111000101110010110011100011001011	55
	d	0001110010111000110010110001011100011001011100010110011100011001011	67
15	a	110011100011001011100010110001110010111000110010110001110010	60
	b	11001110001100101100010111001011000111001011100010110001110010	62
	c	11001110001100101100011100101110001011000111001011000101110010	62
	d	110011100011001011000101110001100101110001011000111001011100010	63
16	a	01100111000101110010110001011100011001011000111001	50
	b	01100111000110010110001011100101100011100101110001	50
	c	01100111000110010111000101100011100101100010111001	50
	d	011001110001011100101100011100101110001011000111001	51
17	a	110011100011001011100010	24
	b	110011100011001011000111001011100010110001110010	48
	c	110011100010111001011000111001011100011001011000101110010	57
	d	1100111000101110010110001011100011001011100010110001110010	58
18	a	110011100011001011100010	24
	b	110011100010111001011000111001011100010110001110010	51
	c	110011100011001011000111001011100011001011000101110010	54
	d	1100111000110010110001011100011001011100010110001110010	55
19	a	110011100011001011100010	24
	b	1100111000101110010110001011100011001011000101110010	52
	c	110011100011001011000111001011100011001011000101110010	54
	d	1100111000110010110001011100011001011100010110001110010	55
20	a	1100111000110010110001110010	28
	b	11001110001011000111001011100010	32
	c	110011100010111001011000111001011100010110001110010	51
	d	1100111000101110010110001011100011001011000101110010	52
21	a	1100111000110010110001110010	28
	b	11001110001100101110001011000111001011100010	44
	c	110011100010111001011000111001011100011001011000101110010	57
	d	1100111000101110010110001011100011001011000111001011100010	58

Table 2: Various FS morphisms with lengths

r	x	$f_r(x)$	$\|f_r(x)\|$
22	a	1100111000110010110001110010	28
	b	1100111000101110010110001011100011001011100010	46
	c	110011100010111001011000111001011100010110001110010	51
	d	1100111000101110010110001011100011001011000101110010	52
23	a	1100111000110010110001110010	28
	b	11001110001100101110001011000111001011000101110010	50
	c	11001110001011000111001011100011001011000111001011100010	56
	d	110011100010111001011000111001011100011001011000101110010	57
24	a	1100111000110010110001110010	28
	b	11001110001100101110001011000111001011000101110010	50
	c	110011100010111001011000111001011100011001011000101110010	57
	d	1100111000101110010110001011100011001011000111001011100010	58
25	a	1100111000110010110001110010	28
	b	11001110001011000111001011100011001011000111001011100010	56
	c	110011100010111001011000111001011100011001011000101110010	57
	d	1100111000101110010110001011100011001011100010110001110010	58
26	a	1100111000110010110001110010	28
	b	110011100010111001011000111001011100011001011000101110010	57
	c	11001110001100101110001011000111001011100011001011000101110010	62
	d	110011100010110001110010110001011100011001011000111001011100010	63
27	a	11001110001011000111001011100010	32
	b	110011100011001011100010110001110010	36
	c	110011100011001011000111001011100011001011000101110010	54
	d	1100111000101110010110001110010111000110010110001110010	55
28	a	11001110001011000111001011100010	32
	b	11001110001011100101100010111000110010110001110010	50
	c	110011100011001011000111001011100011001011000101110010	54
	d	1100111000110010110001011100011001011100010110001110010	55
29	a	110011100011001011000111001011100010	36
	b	11001110001100101110001011000111001011000101110010	50
	c	110011100010111001011000111001011100010110001110010	51
	d	1100111000101110010110001011100011001011000101110010	52
30	a	110011100011001011100010110001110010	36
	b	11001110001011100101100010111000110010110001110010	50
	c	11001110001100101100011100101110001011000111001011000101110010	62
	d	110011100011001011000101110001100101110001011000111001011100010	63
31	a	110011100010111001011000111001011100010	39
	b	110011100011001011000111001011100010110001110010	48
	c	11001110001100101110001011000111001011100011001011000101110010	62
	d	110011100011001011000101110001100101110001011000111001011100010	63
32	a	11001110001100101110001011000111001011100010	44
	b	1100111000101110010110001011100011001011000101110010	52
	c	110011100011001011000111001011100011001011000101110010	54
	d	1100111000110010110001011100011001011100010110001110010	55

Table 3: Various FS morphisms with lengths

$\lvert w \rvert$	w
1	0
2	00
3	000
4	0001
5	00011
6	000111
7	0001011
8	00010111
12	000101100111
14	00010111001011
19	0001011100011001011
20	00010110001110010111
24	000101100011100101100111
28	0001100101100011100101100111
30	000101110010110011100011001011
36	000101100011100101100111000110010111
38	00010110001110010110001011100101100111
39	000101100011100101100010111000110010111
43	0001011001110001011100101100111000110010111
44	00010110001110010111000101100111000110010111
46	0001011001110001011100101100010111000110010111
48	000101100011100101100111000110010110001110010111
50	00010110001110010110001011100101100111000110010111
51	000101100011100101100010111000110010110001110010111
52	0001011001110001100101100011100101100111000110010111
55	0001011000111001011000101110001100101100011100101100111
57	000101100011100101100010111000110010110001011100101100111
58	0001011000111001011001110001011100101100010111000110010111
63	000101100011100101110001011001110001011100101100111000110010111
65	00010110001110010110001011100011001011000101110010110001110010111
69	000101100011100101100010111000110010110001011100101100111000110010111
70	0001011000111001011000101110001100101110001011001110001011100101100111
71	00010110011100010111001011001110001100101100011100101100111000110010111
77	00010110001110010110001011100011001011100010110001110010110001011100101100111
116	00010110001110010110001011100011001011000101110010110001110010111000101100011 1001011001110001011100101100011110010111
127	00011100101110001011001110001100101100010111000110010111000101100011100101100 11100010111001011000101110001100101100010111001011
232	$f_{27}(abdcd)$
241	$f_{29}(abdcd)$
253	$f_{10}(abdcd)$

Table 4: Circular SF words of various lengths

References

[1] T. Clokie, D. Gabric, J. Shallit, Circularly squarefree words and unbordered conjugates: a new approach. In Mercas R., Reidenbach D. (eds.) *Combinatorics on Words. WORDS 2019.* Lecture Notes in Computer Science, **11682**, Springer, 2019, pp. 264–274.

[2] J. D. Currie, There are circular square-free words of length n for $n \geq 18$, *Electron. J. Comb.* **9** (2002) #N10.

[3] J. D. Currie, D. S. Fitzpatrick, Circular words avoiding patterns. In: Ito M., Toyama M. (eds) *Developments in Language Theory. DLT 2002.* Lecture Notes in Computer Science, **2450**. Springer, Berlin, Heidelberg, 319–325.

[4] J. D. Currie, J. T. Johnson, There are level ternary circular square-free words of length n for $n \neq 5, 7, 9, 10, 14, 17$, doi: http://arXiv:2005.06235

[5] A.S. Fraenkel, R.J. Simpson, How many squares must a binary sequence contain? *Electron. J. Comb.* **2** (1995) #R2.

[6] T. Harju, D. Nowotka, Binary words with few squares *Bull. Eur. Assoc. Theor. Comput. Sci.*, **89** (2006), 164–166.

[7] T. Harju, D. Nowotka, Cyclically repetition-free words on small alphabets *Information Processing Letters*, **110** (2010), 591–595 ·

[8] J. T. Johnson, New and existing results on circular words, M.Sc. thesis, University of Victoria, doi: http://hdl.handle.net/1828/11743

[9] M. Lothaire, *Combinatorics on Words*, Encyclopedia of Mathematics and its Applications **17**, Addison-Wesley, Reading (1983).

[10] M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, Cambridge (2002).

[11] L. Mol, N. Rampersad, Personal communication.

[12] H. Mousavi, J. Shallit, Repetition avoidance in circular factors. In: Béal M.-P., Carton O. (eds) *Developments in Language Theory. DLT 2013.* Lecture Notes in Computer Science, **7907**. Springer, Berlin, Heidelberg, 384–395.

[13] N. Rampersad, Personal communication.

[14] N. Rampersad, J. Shallit, and M.-w. Wang. Avoiding large squares in infinite binary words. *Theoret. Comput. Sci.*, **339** (2005) 19–34.

[15] R. O. Shelton, On the structure and extendibility of square-free words, *Combinatorics on Words: Progress and Perspectives*, Academic Press, New York, 1983, 101–188.

[16] A. M. Shur, On Ternary Square-free Circular Words, *Electr. J. Comb.* **17(1)** (2010) #R140

[17] A. Thue, Über unendliche Zeichenreihen, *Norske Vid. Selsk. Skr. I. Mat. Nat. Kl. Christiana* No. 7 (1906).

[18] A. Thue, Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen, *Norske Vid. Selsk. Skr. I. Mat. Nat. Kl. Christiana* **1** (1912) 1–67.