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Firefighter Problem Played on Infinite Graphs

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1 Abstract

The Firefighter Problem was introduced over 30 years ago and continues to be studied by researchers today. The problem consists of a graph of interest where a fire breaks out at time $t = 0$ on any given vertex of the graph G. The player, then, gets to place a firefighter to "protect" a vertex from the fire. Each consecutive turn, the fire spreads to adjacent vertices. These vertices are then referred to as "burned". The firefighter also gets to move to protect an additional, unburned vertex, completing the first round. Each vertex that the firefighter "defends" stays protected for the remainder of the game. The game ends when the fire cannot move to any adjacent vertices. This game can be used to solve real world problems. For example, we can model the spread of diseases in a community or the spread of a wildfire using the Firefighter Problem. There are many known results for the Firefighter Problem on finite graphs. In this project, we study the Firefighter Problem on infinite graphs, with the goal of expanding on known results. We are exploring various infinite grids by imposing the additional requirement that firefighters can only move to adjacent, unburned vertices.

2 Introduction

The Firefighter Problem incorporates ideas and techniques introduced in Graph Theory, the study of graphical structures and their characteristics. Using graphs allows us to illustrate the overall game at each time step. Using terminology from Graph Theory, we will define a vertex as a location where the fire or firefighter can be placed. The edges of the graph define the path that the fire can take to further spread. Two vertices are adjacent if they share an edge.

Starting with a graph, the game is played as followed. To begin, a fire is placed on any vertex of a given graph, in this case, a finite graph (Figure 1). Then the player gets to place a firefighter on any unburned vertex, see Figure 2. This vertex will remain "protected" for the rest of the game so that the fire cannot spread to this vertex. Each consecutive turn, the fire will move to any unprotected, adjacent vertices as shown in Figure 3. The player will then get to place another firefighter on any unburnt vertex, completing the first round, see Figure 4. Notice that the original game rules allows the firefighter to move with only one restriction: a firefighter cannot move to a burned vertex. The game will continue until the fire can no longer spread to an unprotected, adjacent vertex, illustrated in Figure 4. In

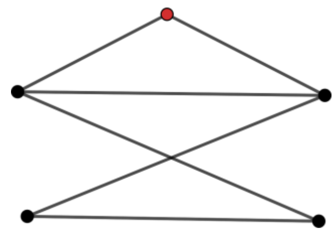

Figure 1: Fire is placed at the start of game.

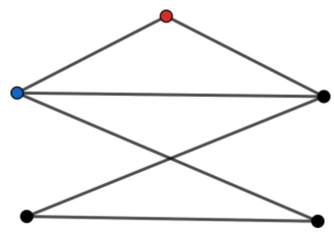

Figure 2: One firefighter is placed anywhere on the finite graph.

other words, the game will end when the fire is surrounded by "protected" vertices or all the vertices of a graph are "protected" or burned.

Now, consider playing the game on an infinite graph. We ask if it is possible to contain the fire and if so, how many time intervals are needed to contain the fire and is one firefighter enough to contain the fire?

Suppose we are looking at the integer number line. A fire breaks out at a vertex along the line, see Figure 5. Then a firefighter can be placed on any unburned vertex. We will place the firefighter on an adjacent vertex of the fire, blocking the spread of fire to the left shown in Figure 6. At the start of the first round, the fire spreads to an adjacent, unburned vertex (Figure 7). Notice that on the player's turn, the firefighter is placed, blocking the fire from spreading further to the right (Figure 8), ending the game. The fire is surrounded by "protected" vertices. Thus the fire cannot spread to any adjacent, unprotected vertices.

On the other hand, there are cases where a game on an infinite graph will never end, i.e.

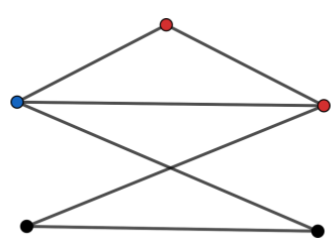

Figure 3: The fire spreads to the only burned, adjacent vertex.

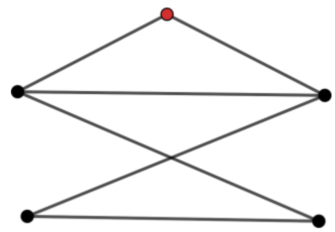

Figure 4: A firefighter is placed on an unburned vertex and ends the game.

Figure 5: Fire is placed at the start of game.

for every $n \in \mathbb{Z}$ the fire can continue to spread after n rounds. Suppose a fire breaks out on a vertex that has infinite degree, see Figure 9. By adding one firefighter at each time interval, the fire always will be able to spread to an adjacent vertex after n rounds. Then in this case, it is impossible to contain a fire with one firefighter.

Considering infinite graphs, specifically the square grid and the strong grid, we want to expand on known results and add additional rules to the game. The square grid is denoted as $\mathbb{Z}\square\mathbb{Z}$ and the strong grid is denoted as $\mathbb{Z}\boxtimes\mathbb{Z}$. The $\mathbb{Z}\boxtimes\mathbb{Z}$ grid is similar to the $\mathbb{Z}\square\mathbb{Z}$ grid with added edges such that the points (x, y) are adjacent to $(x+1, y+1)$, $(x-1, y+1)$, $(x-1, y-1)$, and $(x+1, y-1)$ for all $x, y \in \mathbb{Z}$. For the infinite setting, the Firefighter Problem has been studied on the square grid. A study considered the effects of adding multiple firefighters per turn as opposed to a single firefighter [1]. This is due to the fact that a fire that breaks out at the origin on a $\mathbb{Z}\square\mathbb{Z}$ grid cannot be contained with a single firefighter. Notice that at the first time interval, the firefighter can "protect" a vertex adjacent to the fire. Then on the next round, the fire will spread to the remaining three unprotected, adjacent vertices. Again, the player can place a single firefighter anywhere on the graph. However, since at each time interval there will always be at least one unprotected, adjacent vertex where the fire can spread to, one firefighter is unable to protect all the surrounding adjacent vertices by the point to which the fire has spread. Notice, that if we use more than a single firefighter to contain the fire, it is now possible. Consider using three firefighters. At the start of the game, place all three firefighters adjacent to the fire, so that only one unprotected, adjacent vertex

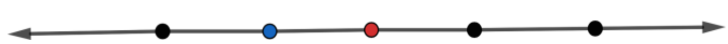

Figure 6: One firefighter is added.

Figure 7: Fire spreads to an adjacent, unprotected vertex.

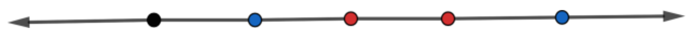

Figure 8: One more firefighter is added and the game ends since the fire can no longer spread.

to the fire exists. Then for the next round, the fire will spread to the one unprotected, adjacent vertex. The firefighters can then be placed on all the surrounding unprotected, adjacent vertices, ending the game.

It is also proven that on the $\mathbb{Z}\square\mathbb{Z}$ grid, a fire cannot be contained with one firefighter, but it can be with two firefighters [2]. Figure 10 shows the movements that the two firefighters should make to contain the fire in the least number of rounds.

Another known result for infinite graphs is, for an r-regular graph, a fire can be contained by r 1 firefighters in two time intervals, and the minimum number of vertices burned is two [3]. Note that $\mathbb{Z}\square\mathbb{Z}$ is a 4-regular graph and Z � Z is an 8-regular graph. Then a fire on a $\mathbb{Z}\square\mathbb{Z}$ grid can be contained with three firefighters and a fire on a Z � Z grid can be contained with seven firefighters. Notice that it does not necessarily give the minimum number of firefighters needed.

In this paper, when looking at the $\mathbb{Z}\square\mathbb{Z}$ grid and Z�Z grid, we want to consider restricting the movement of the firefighters such that they can only move to adjacent, unburned vertices, unlike in Figure 8 where the firefighter was able to move to any unburned vertex. With this additional condition, we are considering how many firefighters are needed to prevent the infinite spread of fire on these graphs. To do this, we have constructed lemmas to call upon to determine if the firefighters will be able to contain the fire. We plan to use these lemmas to investigate the number of firefighters needed to contain the fire under this condition. In

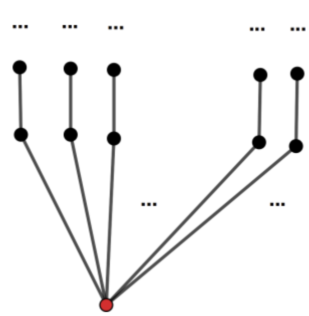

Figure 9: Fire breaks out on a vertex of infinite degree.

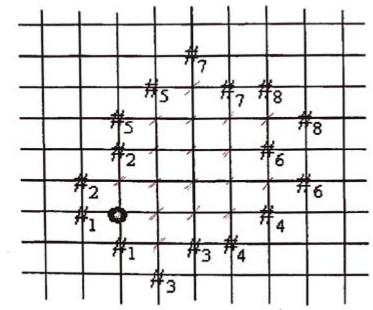

Figure 10: Fire breaks out on a vertex on a Z□Z grid and two firefighters are placed on the graph. The movements of the firefighters for each time interval are shown [2].

the square grid, using four firefighters to contain the fire is trivial (Figure 11). Thus we want to explore our hypothesis that using three firefighters to contain the fire is impossible. Similarly, with the strong grid, using eight firefighters is trivial (Figure 12). Therefore we want to investigate our hypothesis that seven firefighters is not sufficient to contain the fire.

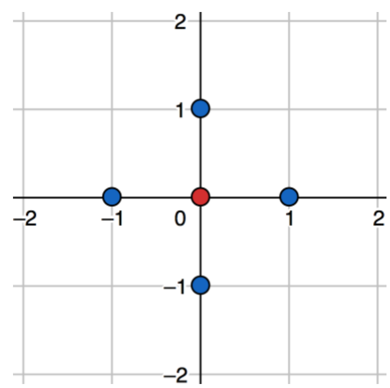

Figure 11: Trivial case on Z□Z grid using four firefighters to contain the fire.

Using graphs will allow us to illustrate many real-world examples. One application that comes out of the Firefighter Problem is the spread of wildfires. We will be able to outline the forest graphically where the vertices symbolize trees. Exploring this problem, we will be able to gather a plan for an effective stop to the wildfire. Another application for this problem includes modeling the spread of diseases. With graphical modeling, we will be able to predict how long it might take for the disease to stop spreading, as well as analyzing how far it has spread. With our additional requirement on the way that the firefighters can move, it is a more accurate simulation of the movement of an actual firefighter during the spread of a wildfire or the movement of a disease within a given area.

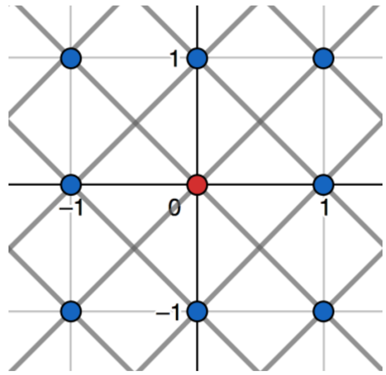

Figure 12: Trivial case on Z � Z grid using eight firefighters to contain the fire.

3 Z□Z

Researchers have studied that in the typical game setting, one firefighter is unable to contain the fire on Z□Z, but it can be with two firefighters [2]. We look to investigate how the number of firefighters will change with the additional restriction that firefighters can only move to adjacent, unburnt vertices. Trivially, four firefighters can contain a fire on a Z□Z grid, see Figure 11.

Lemma 3.1. *Suppose the firefighters can only move to adjacent, unburned vertices. If the fire starts at* (0, 0) *on* Z□Z *and after placing the firefighters, there are no protected vertices in one of the four directions (North, South, East, and West) of the fire, then the fire cannot be contained.*

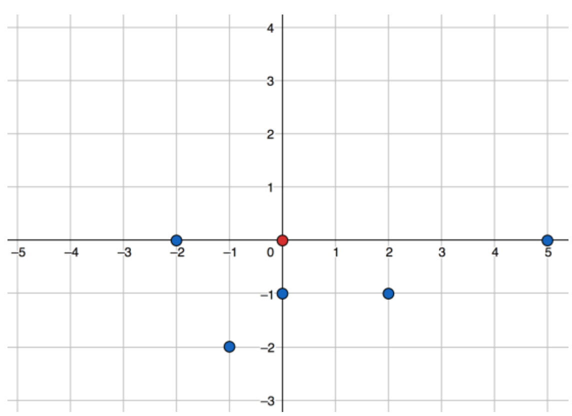

Figure 13: The fire starts at (0, 0) on a Z□Z grid with no firefighters North of the fire.

Proof. Without loss of generality, suppose a fire starts at (0, 0) with no protected vertices North of the fire, see Figure 14. Then, after we place the firefighters, there are no firefighters

North of the fire such that their y-coordinate is greater than 0. Then the firefighters' y-coordinates are less than or equal to 0. Since at each time interval, the fire spreads to adjacent vertices, then the fire at least will move North such that the fire's y-coordinate increases by 1. The firefighters are also allowed to move one unit each time interval after the fire spreads. At best there is a firefighter that is level with the fire such that its y-value is 0. If at any time interval t the firefighter travels East or West, its y-value will remain the same as its y-value at the previous $t-1$ time interval. At the same time, the fire will increase its y-value by 1 unit. Thus the firefighter's y-coordinate is less than the fire's. On the other hand, if the firefighter moves South, its y-value will decrease by 1 while the y-value of the fire will increase by 1. Thus the firefighter's y-coordinate is less than the fire's y-coordinate by 2 units. Then if the firefighter moves North, both its y-value and the y-value of the fire will increase by 1. Thus, at best, the firefighter can remain level with the fire by moving strictly North. However, it can never increase its y-value in such a way that it is greater than the fire's y-value if the firefighter's y-value was not larger than the fire's y-value at the start of the game. Thus it will never be able to block the path of the fire in the North direction and the fire cannot be contained. Thus if there are no protected vertices North of the fire, then the fire cannot be contained.

By rotational symmetry, the argument holds when there are no protected vertices East, South, or West of the fire. □

Knowing that the firefighters should be placed in such a way that there is at least one firefighter in each direction of the fire, we want to consider the ways in which the firefighters must move to protect a vertex along an axis.

Lemma 3.2. *Suppose a fire starts at* $(0,0)$ *on* $\mathbb{Z}\square\mathbb{Z}$*. Let* (x,y) *be the coordinates of a firefighter. If* $|y| < |x|$*, then the firefighter must move North/ South to reach a vertex along the x-axis before the fire spreads to that vertex.*

Proof. Suppose a fire starts at $(0,0)$ on $\mathbb{Z}\square\mathbb{Z}$. Let the firefighter start at (x,y). Suppose $|y| < |x|$. We show that the firefighter can reach the x-axis by moving North/South depending if $y > 0$ or $y < 0$. Without loss of generality, assume $y > 0$. Then the firefighter is on the North side of the graph. Then it will take the firefighter $|y|$ time intervals to reach the x-axis by moving strictly South. At the same time, the fire is traveling along the x-axis, so that it will take the fire $|x|$ time intervals to reach the x-value of the firefighter along the x-axis. Then since $|y| < |x|$, we have that the firefighter can reach the x-axis before the fire.

□

Corollary 1. *Suppose a fire starts at* $(0,0)$ *on* $\mathbb{Z}\square\mathbb{Z}$*. Let* (x,y) *be the coordinates of a firefighter. If* $|x| < |y|$*, then the firefighter must move East/ West to reach the y-axis before the fire spreads there.*

Proof. Argument is symmetric to proof for Lemma 3.2.

□

Knowing that the firefighter must move a specific direction to reach a vertex along an axis, we can also determine which firefighters will never be able to reach a vertex along a specific axis.

Lemma 3.3. *Suppose a fire starts at* $(0,0)$ *on* $\mathbb{Z}\square\mathbb{Z}$*. Let* (x,y) *be the coordinates of a firefighter. If* $|x| < |y|$*, then it is impossible for the firefighter to protect any vertices on the x-axis. (Assuming there are no other firefighters on the graph that can protect the x-axis).*

Proof. Suppose a fire starts at $(0,0)$ on $\mathbb{Z}\square\mathbb{Z}$. Let (x,y) be the coordinates of a firefighter. Assume $|x| < |y|$. Then to reach the x-axis it would take the firefighter y time intervals. At the same time, the fire is spreading along the x-axis. After x time intervals, the fire would reach the height of the firefighter. Since $|x| < |y|$, then it would take the fire less time to reach that height along the x-axis. Thus the firefighter cannot protect any vertices on the x-axis. □

Corollary 2. *Suppose a fire starts at* $(0,0)$ *on* $\mathbb{Z}\square\mathbb{Z}$*. Let* (x,y) *be the coordinates of a firefighter. If* $|y| < |x|$*, then it is impossible for the firefighter to protect any vertices on the y-axis. (Assuming there are no other firefighters on the graph that can protect the y-axis).*

Proof. Argument is symmetric to proof for Lemma 3.3.

□

Consider a firefighter whose $|x| = |y|$. We will show that a firefighter placed in this way will be unable to reach an axis.

Lemma 3.4. *Suppose a fire starts at* $(0,0)$*. Then a firefighter at point* (n,m) *for all* $n,m \in \mathbb{Z}$ *where* $|n| = |m|$ *cannot protect a vertex on an axis, assuming there are no other protected vertices on the axes.*

Proof. Suppose a fire starts at $(0, 0)$. Assume the firefighter starts at (n, m) where $n, m \in \mathbb{Z}$ and $|n| = |m|$. Without loss of generality, assume $n > 0$ and $m < 0$, see Figure 14. Then the firefighter is located in the Southeast quadrant. Thus it can travel either to the positive x-axis or negative y-axis.

Case 1: The firefighter travels to the positive x-axis. For the firefighter to reach the x-axis, the firefighter has to increase its y-value at every time interval. At the same time, the fire is increasing its x-value along the x-axis. Since the fire moves first, at n time intervals, fire will reach $(n,0)$ on the x-axis before the fire.

Case 2: The firefighter travels to the negative y-axis. For the firefighter to reach the y-axis, the firefighter has to increase its x-value at every time interval. At the same time, the fire is increasing its y-value along the y-axis. Since the fire moves first, at n time intervals, fire will reach $(0,m)$ on the y-axis before the fire.

Therefore the firefighter cannot reach an axis. □

We know that a firefighter with position (n, m) where $n, m \in \mathbb{Z}$ where $|n| = |m|$ is unable to protect any axis. Notice that once a firefighter tries to cross the diagonal lines ($y = x$ and $y = (-x)$), it is no longer able to protect any axes. Therefore, once a firefighter protects one axis, it is unable to protect another.

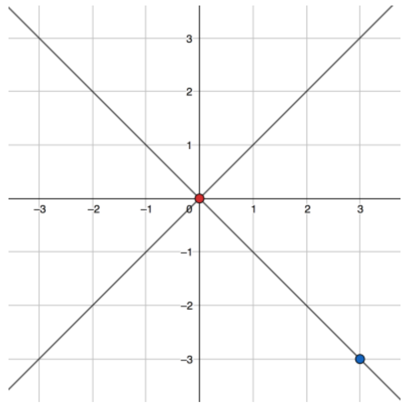

Figure 14: Firefighter located on the (n, m) diagonal where $|n| = |m|$ and $n > 0$ and $m < 0$ on the Z□Z grid.

Theorem 3.5. *There is no way to contain a fire in* Z□Z *with 3 firefighters, where the firefighters are restricted to moving to an unburned, adjacent vertex.*

Proof. Let the fire starts at point (0, 0) on Z□Z. Suppose the 3 firefighters are placed anywhere on the graph. We will show that at least one axis will not be blocked by a firefighter.

Case 1: Suppose the firefighters all start in the same quadrant. Without loss of generality, assume they start in the Northwest quadrant. Then after 1 time interval, there will be no protected vertices to the East. Thus the fire cannot be contained, by Lemma 3.1.

Case 2: Suppose two firefighters start in the same quadrant. There are two subcases:

Consider two firefighters starting in adjacent quadrants Without loss of generality, suppose the two firefighters are in the Northwest quadrant and the other firefighter is in the Northeast quadrant, see Figure 15. Then after 1 time interval, there are no protected vertices to the South of the fire. Thus the fire cannot be contained, Lemma 3.1.

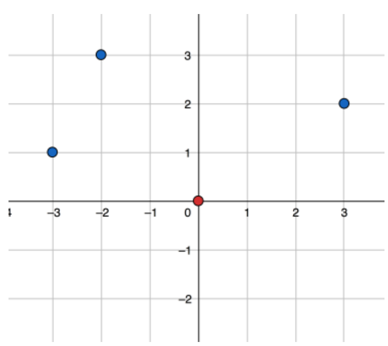

Figure 15: Two firefighters are placed in the Northwest quadrant and one firefighter is placed in the Northeast quadrant of a Z□Z grid.

Consider two firefighters starting in opposite quadrants. Without loss of generality, suppose the two firefighters are in the Northwest quadrant and the other firefighter is in the

Southeast quadrant, see Figure 16. Then at best the two firefighters in the Northwest quadrant can protected the positive y-axis and the negative x-axis. Then the third firefighter has to protected both the positive x-axis and negative y-axis. Notice that after protecting an axis, to protect the other, at some point the firefighter's $|x| = |y|$. By Lemma 3.4, the firefighter will be unable to protect a vertex on an axis. Then, at best, the firefighter can only protect one axis, not both. Thus the fire cannot be contained.

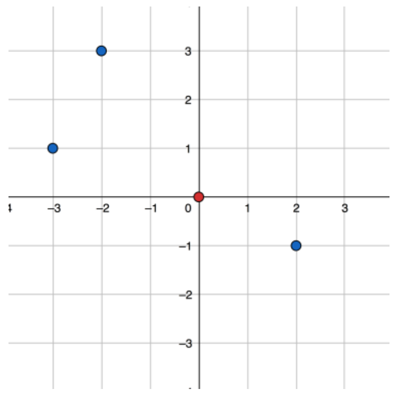

Figure 16: Two firefighters are placed in the Northwest quadrant and one firefighter is placed in the Southeast quadrant of a $\mathbb{Z}\square\mathbb{Z}$ grid.

Assume the firefighters start in different quadrants.

Case 3: Suppose none of the firefighters start on an axis and the firefighters start in different quadrants (Figure 17. Without loss of generality, assume that the Northeast quadrant does not contain a protected vertex. Then there is a firefighter in the Northwest, Southeast, and Southwest quadrants. According to the lemma, if any of the firefighter's $|x|$ value is the same as its $|y|$ - value, then the firefighter cannot reach an axis. Thus, if $|x| = |y|$ for any firefighter, then the other firefighters can contain, at best, two of the four half-axes, leaving two half-axes unprotected. Without loss of generality, assume none of the firefighters' $|x|$- value is the same as its $|y|$ - value. There are two subcases:

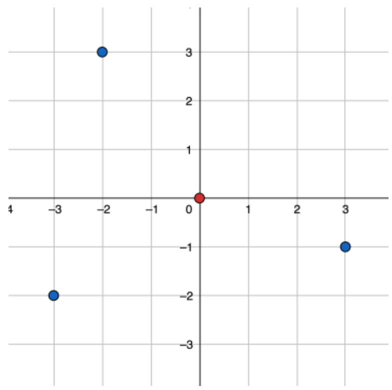

Figure 17: Three firefighters are placed in different quadrants.

Consider, for all firefighters, their $|x|$-value is greater than their $|y|$-value or $|y|$-value is greater than their $|x|$-value. Without loss of generality, assume $|x|$-value is greater than their $|y|$-value, see Figure 18. Then according to the lemma above, the firefighters must move North/South to reach an axis. Then the firefighters to the South will both move North, while the firefighter in the Northwest will move South. Notice that both the firefighters to the Northwest and Southwest will both be reaching the negative x-axis. Therefore both the positive and negative y-axis will be unprotected. Therefore the fire cannot be contained.

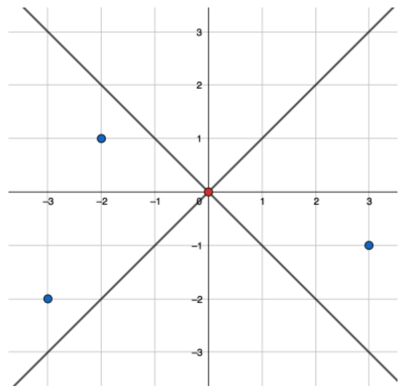

Figure 18: The diagonal lines represent where $|x| = |y|$ for all $x, y \in \mathbb{Z}$. Here all firefighters' $|x| > |y|$.

Consider one of the firefighter's $|x|$-value is less than its $|y|$-value, while for the other two firefighters, their $|x|$-value is greater than their $|y|$-value. Suppose the Northwest firefighter has $|x|$-value less than $|y|$. Then the Northwest firefighter must move to the East to reach an axis, namely the positive y-axis. The other two firefighter will both be moving North to reach the negative and positive x-axis. Notice that the negative y-axis will be unprotected. Therefore the fire cannot be contained.

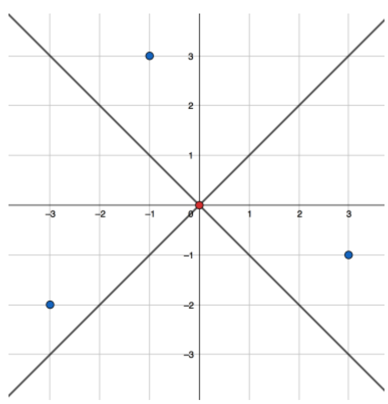

Figure 19: The diagonal lines represent where $|x| = |y|$ for all $x, y \in \mathbb{Z}$. Here two firefighters' $|x| > |y|$ and one firefighter's $|x| < |y|$.

Case 4: Suppose one of the firefighter's starting location is on an axis. Without loss of generality assume it is starting on the positive x-axis. Notice that this firefighter will be unable to protect any other axis since at some point in time their x-value will be equal to its y. There are four subcases:

Consider the other two firefighters are in the Northeast and Southeast, see Figure 20. Then after 1 time interval, there will be no protected vertices to the West of the fire. Thus the fire cannot be contained.

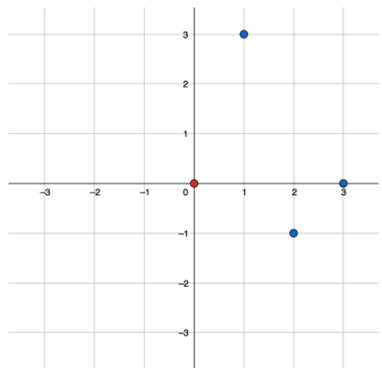

Figure 20: One firefighter starts along the positive x-axis. The other two firefighters are placed in the Northeast and Southeast quadrants.

Consider the other two firefighters are in the Northwest and Southwest, see Figure 21. Then the firefighters must move such that they can reach an axis. At best, they each are able to reach different axes. Without loss of generality, assume they are able to protect the positive y-axis and negative x-axis. Then the negative y-axis is unprotected. Thus the fire cannot be contained.

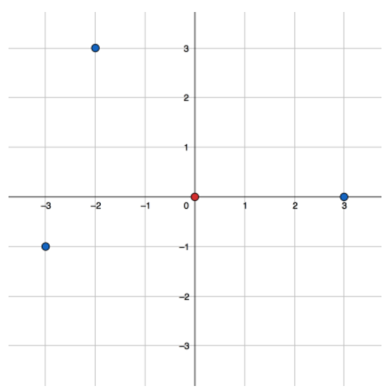

Figure 21: One firefighter starts along the positive x-axis. The other two firefighters are placed in the Northwest and Southwest quadrants.

Consider the other two firefighters are in the Northeast and Northwest, see Figure 22. Then after 1 time interval, there will be no protected vertices to the South of the fire. Thus the fire cannot be contained.

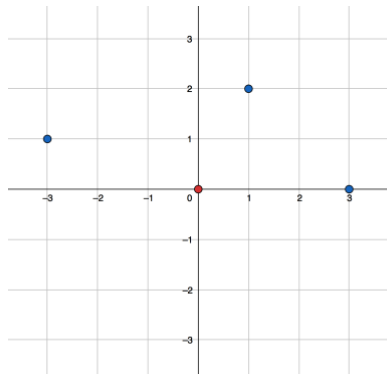

Figure 22: One firefighter starts along the positive x-axis. The other two firefighters are placed in the Northeast and Northwest quadrants.

Consider the other two firefighters are in the Northeast and Southwest, see Figure 23. Then at best, both firefighters will be able to protect an axis depending on its location. Without loss of generality assume the firefighters are able to protect the positive y-axis and the negative x-axis. Then the negative y-axis will be unprotected. Thus the fire cannot be contained.

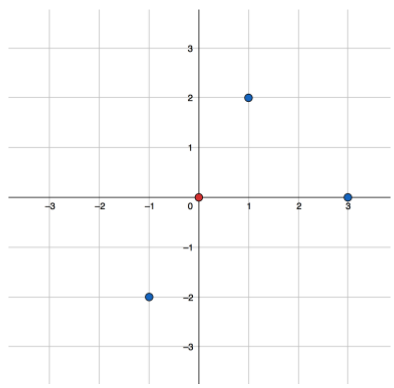

Figure 23: One firefighter starts along the positive x-axis. The other two firefighters are placed in the Northeast and Southwest quadrants.

Case 5: Suppose two firefighters started on adjacent axes. Without loss of generality, assume they started on the positive x-axis and positive y-axis. There are two subcases:

Consider the third firefighter is located in the Northeast quadrant, see Figure 24. Then the firefighter at best can either reach the positive x-axis or positive y-axis. Then both the negative x-axis and negative y-axis will be unprotected. Thus the fire cannot be contained.

Consider the third firefighter is located in either the Northwest, Southeast, and Southwest quadrants. Without loss of generality, assume the firefighter is in the Southwest quadrant, see Figure 25. Then depending on its x- and y-values, at best the firefighter will be able to protect either the negative x-axis or negative y-axis, but not both. Thus the fire cannot be contained.

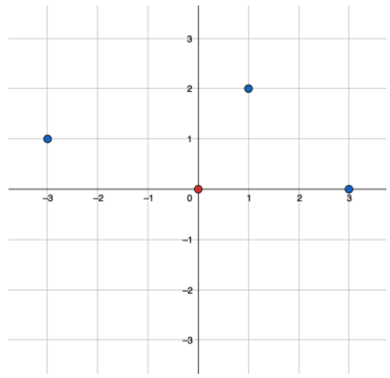

Figure 24: Two firefighters are along the positive x-axis and positive y-axis. The third firefighter is placed in the Northeast quadrant.

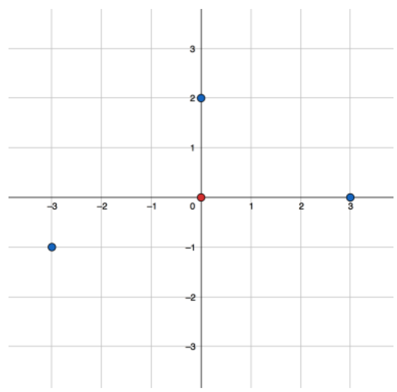

Figure 25: Two firefighters are along the positive x-axis and positive y-axis. The third firefighter is placed in the Southwest quadrant.

Case 6: Suppose two firefighters started on opposite axes. Without loss of generality, assume they start on the negative x-axis and positive x-axis, see Figure 26. Then the third firefighter can either start in the Northern half of the graph or Southern half of the graph. Without loss of generality, assume the firefighter starts in the Northern half of the graph. Then after 1 time interval, the fire will have no protected vertices to the South. Thus the fire cannot be contained.

Case 7: Suppose all three firefighters started on an axis. Without loss of generality, assume there are no protected vertices on the negative y-axis, see Figure 27. Then after 1 time interval, there will be no protected vertices to the South of the fire. Thus the fire will not be contained.

Therefore the fire cannot be contained. □

The argument would be symmetric for a fire starting at (x_0, y_0) on $\mathbb{Z}\Box\mathbb{Z}$ for all $x_0, y_0 \in \mathbb{Z}$.

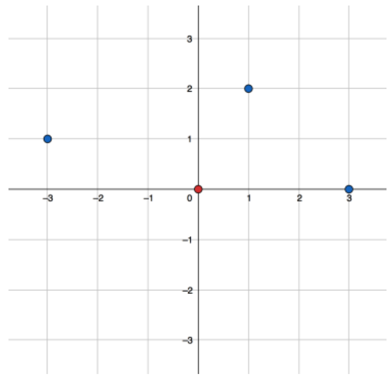

Figure 26: Two firefighters are along the positive x-axis and negative x-axis. The third firefighter is placed in the Northern (positive y-values) of the graph.

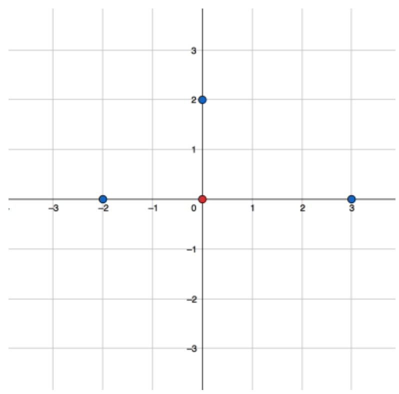

Figure 27: The three firefighters are along the positive x-axis, positive y-axis, and negative x-axis.

4 N �N

With our goal to explore the Z�Z grid, we start with looking strictly at the positive quadrant of that graph. Therefore, we want to investigate the number of firefighters needed to contain the N � N grid. Trivially, three firefighters can contain a fire starting at $(0, 0)$ on a N � N grid.

Lemma 4.1. *If a fire starts at* $(0, 0)$ *on* N�N, *then two firefighters are insufficient to contain the graph including the axes, if firefighters can only move to adjacent, unburned vertices.*

Proof. Let a fire start at $(0, 0)$ on N � N. Assume two firefighters are placed anywhere on the graph.

Case 1: Suppose both firefighters start on the axes. Then their locations are $(x, 0)$ and $(0, y)$. At each time interval, the fire spreads to adjacent, unprotected vertices. Since the firefighters are starting on the axes, they will be able to stop the fire from spreading infinitely to the Northern and Eastern direction. Then the firefighters must protect some vertex on the (n, n) diagonal in order to prevent the fire from spreading to (n, n) for all $n \in$ N. Notice that for a firefighter to reach the (n,n) diagonal for a specific n value, it will take each firefighter at least n time intervals and the fire exactly n time intervals to reach that point. Since the fire moves first, the fire will burn that vertex before the firefighter can protect it. Thus the fire cannot be contained.

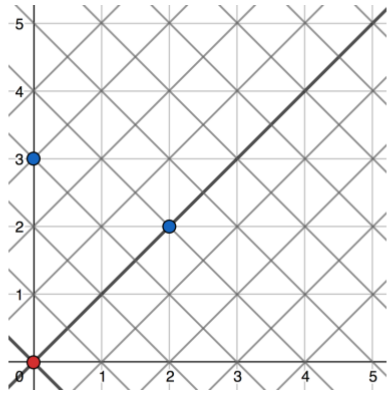

Figure 28: One firefighter starts along the y-axis and the second firefighter starts along the (n, n) diagonal on a N � N grid.

Case 2: Without loss of generality, suppose one firefighter is located on the y-axis at $(0, y)$ and the other firefighter is located along the diagonal (n,n), see Figure 28. We know from Case 1 that the firefighter along the y-axis is unable to reach the (n, n) diagonal. At best, it will end up spreading to be adjacent with the firefighter located along the diagonal, preventing the fire from spreading North (Figures 29 and 30). We also know that this firefighter will be unable to protect the x-axis. It will take the firefighter y time intervals to reach the x-axis and the fire will reach the point $(y, 0)$ on the x-axis before the firefighter does unless the other firefighter has protected the x-axis. Then the firefighter along the diagonal must protect the x-axis. Notice that by strictly moving in any of the four directions (North, South, East, or West), the firefighter will be unable to reach the x-axis before the fire. Then it must move diagonally Southeast at some point in time. Notice that if the firefighter decides to move Southwest, then they will reduce the amount of time they have to reach the axis. Going strictly South does not give the firefighter enough time intervals to reach the axis, thus moving Southwest reduces the amount of time intervals for the firefighter. Therefore they must move Southeast at least once to reach the axis. Suppose the firefighter moves diagonally from (x_0, y_0) to $(x_0 + 1, y_0 - 1)$, see Figure 30. Then once the fire becomes adjacent with (x_0, y_0) and $(x_0 + 1, y_0 - 1)$, the fire will be able to spread diagonally Northeast to $(x_0 + 1, y_0)$. Consider this new location of the fire as the new origin for the graph, see Figure 31. Then at best the firefighter starting along the y-axis is located along the y-axis along the new origin. The other firefighter would be South of the new x-axis if it continues to move to protect the original x-axis or by moving Northeast, along the x-axis. Thus at this point, we are in a case similar to Case 1, where the firefighters are unable to have both an x- and y-value greater than the fire's. Thus the fire cannot be contained.

Case 3: Without loss of generality, suppose the two firefighters are located (n, n) and

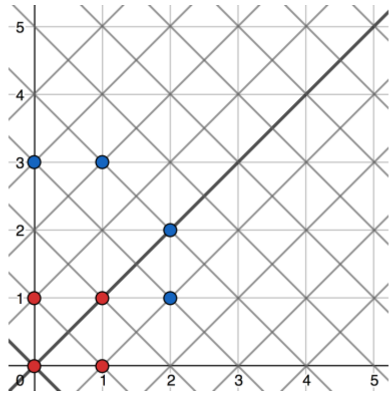

Figure 29: The fire spreads to the adjacent, unprotected vertices. The firefighters also move such that the firefighter on the y-axis travels towards the diagonal and the firefighter on the diagonal moves towards the x-axis.

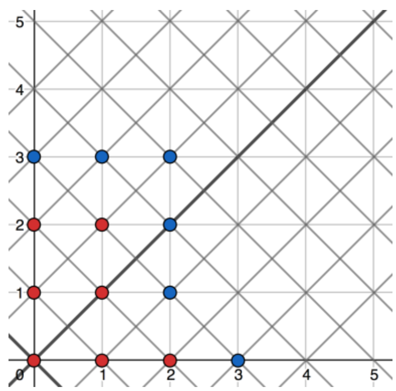

Figure 30: Another time interval has passed and the fire has continued to spread. The firefighter from the y-axis is now adjacent to the diagonal and the firefighter from the diagonal had to travel diagonally Southeast to reach the axis.

(x, y), such that $x > y > 0$, see Figure 32. Then the firefighter located at (x, y) will be able to reach the x-axis before the fire to block it from spreading East. Once this firefighter reaches the x-axis, it will be unable to protect the y-axis. Then the firefighter located at (n, n) must protected the y-axis. Notice if the firefighter travels strictly in any of the four directions (North, South, East, or West), the fire will reach the y-axis first and the firefighter will be unable to block it. Then the firefighter must move diagonally in the Northwest direction to reach the axis before the fire. Suppose the firefighter moves diagonally at (x_0, y_0) to $(x_0 - 1, y_0 + 1)$. Then once the fire spreads to be adjacent to (x_0, y_0) and $(x_0 - 1, y_0 + 1)$, the fire will be able to spread diagonally Northeast to $(x_0 + 1, y_0 + 1)$. Consider this point of the fire as the new origin. At this point, with the firefighters moving to protect the original axes, there is no firefighter to protect the new axes and quadrant. Thus the fire cannot be contained.

Case 4: Suppose none of the firefighters start on an axis or along the (n, n) diagonal. Let

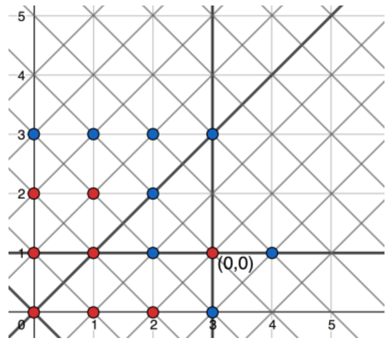

Figure 31: The fire travels diagonally, adjacent to where the firefighter travelled diagonally. Consider the new point of the fire as the new origin. At best, both firefighters are located on the new axes.

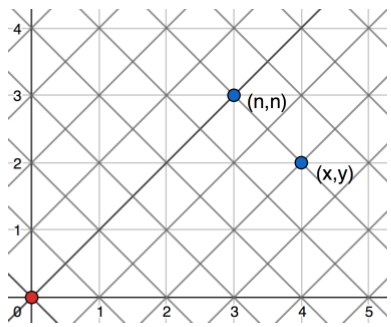

Figure 32: Firefighters are placed on the (n, n) diagonal and $(n + 1, n)$.

their positions be (x_1, y_1) and (x_2, y_2), see Figure 33. Then the firefighters must protect the x- and y-axes as well as the (n, n) diagonal. There are two subcases:

Consider both firefighters choose to protect the axes first such that the firefighter at (x_1, y_1) protects the y-axis and the firefighter at (x_2, y_2) protects the x-axis. Then after $max\{x_1, y_2\}$ time intervals, the axes are both protected. Now both firefighters are trying to reach the (n, n) diagonal. Without loss of generality suppose x_1 y_2. Then the the y-axis was protected first and that firefighter will take less time to reach the (n, n) diagonal. However, we've shown above, that a firefighter located at an axis is unable to reach the (n, n) diagonal before the fire. Then for this firefighter it would take $n + x_1$ time intervals to reach the diagonal, but the fire would reach it at n time intervals. Thus the fire cannot be contained.

Consider one of the firefighters choose to protect an axis and the other firefighter tries to protect the diagonal. We will show that with this strategy, one of the axes will remain unprotected. Without loss of generality, suppose the firefighter at (x_1, y_1) protects the y-axis. Notice that it will take x_1 time intervals to reach the y-axis and then n time intervals to reach the (n, n) diagonal such that the firefighter does not move diagonally at any point (preventing the fire from spread diagonally adjacent to where the firefighter moved diagonally. Then it will take the firefighter a total of $n + x_1$ time intervals to reach the diagonal, while the fire

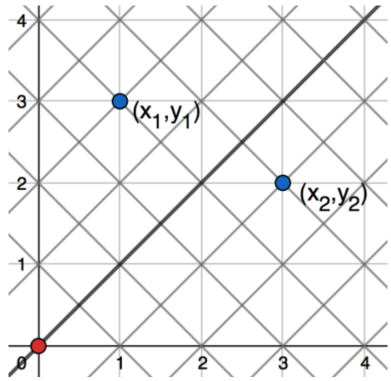

Figure 33: Firefighters are placed such that $x_1 \leq y_2$.

takes n time intervals. Therefore, that firefighter will be unable to protect the diagonal. At best, if $x_1 = 1$, the firefighter will be able to be adjacent to the (n, n) diagonal after $n + x_1$ time intervals, preventing the fire from spreading directly North (the fire will never be able to escape off the diagonal). If $x_1 > 1$, the firefighter will be unable to block the fire from spreading North. In the case where $x_1 > 1$, the fire cannot be contained. Then suppose $x_1 = 1$. Then the second firefighter at (x_2, y_2) must protect the (n, n) as well as the x-axis. Then after $\frac{|x_2-y_2|}{2}$ time intervals, the firefighters will reach the (n, n) diagonal. Then the firefighter must protect the x-axis. Notice that it would take n time intervals to reach the x-axis for any given n-value. Then it would take the firefighter $n + \frac{|x_2-y_2|}{2}$ time intervals to protect both the diagonal and the x-axis. However the fire would reach any point x_0 along the x *axis* before the firefighter. Then the firefighter must move diagonally Southeast to reach a point along the x-axis before the fire. However once the firefighter is adjacent to the point where the firefighter moved diagonally, it will be able to escape through the diagonal. Then considering that point as the new origin, similar to Figure **??** for Case 2, we have that at best, the two firefighters are located along the new axes. Then by Case 1, the fire cannot be contained.

□

Using these results from the N � N, we can construct ideas that will help us prove the number of firefighters needed to contain a fire on Z � Z.

5 Z �N

Now we look to only the all possible x-values and the positive y-values of Z � Z, namely the Z�N grid. We plan to investigate the number of firefighters needed to contain the fire. Trivially, five firefighters can contain a fire starting at $(0, 0)$ on a Z � N grid.

Lemma 5.1. *If a fire starts at* $(0,0)$ *on* Z�N*, then four firefighters are insufficient to contain the fire including the positive and negative* x*-axes and positive* y*-axis.*

Proof. Let a fire start at (0, 0) on $\mathbb{Z} \times \mathbb{N}$. Assume four firefighters are placed anywhere on the graph.

Case 1: Suppose three firefighters are located on the three axes (positive and negative x-axes and the positive y-axis). Without loss of generality, assume the fourth firefighter is anywhere in Q1 (the positive quadrant), see Figure 34. Then the firefighters must be able to protect the (n, n) and $(-n, n)$ diagonals for all $n \in \mathbb{N}$. Notice that it would take n time intervals for the firefighters along the axes to reach the either of the diagonals. At the same time, the fire will take n time intervals to reach any point along the diagonals. Since the fire moves first, the fire will always reach any point along the diagonal before the firefighters. Then the fourth firefighter must protect both, but since it is located in Q1, it can only protect the (n, n) diagonal. Thus the fire cannot be contained.

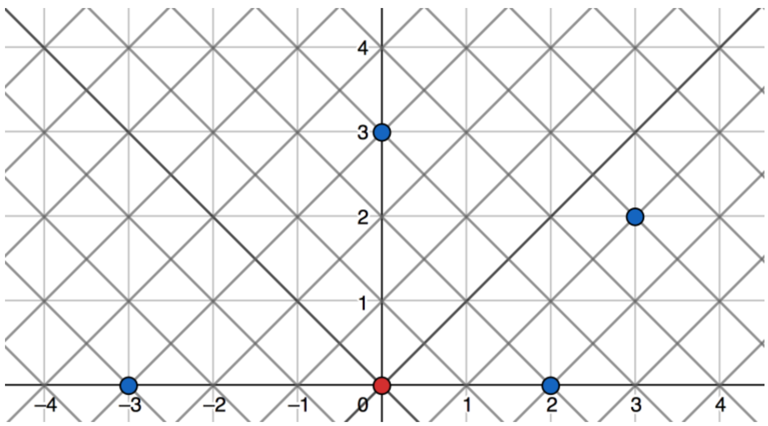

Figure 34: Three firefighters are placed on the axes and one firefighter is placed in Q1.

Case 2: Suppose two of the firefighters are along the positive and negative x-axis. Suppose the other two firefighters are along the (n, n) and $(-n, n)$ diagonals, see Figure 35. Then the firefighters must protect the positive y-axis. Notice if the the firefighter move strictly in the four directions (North,South,East,West), then the firefighters will not reach the y-axis before the fire. Then the firefighters must move diagonally at some point in time. The firefighters along the axes will never be able to travel above the diagonals. The other firefighter traveling to the y-axis will also never be able to protect the vertices adjacent to where the other firefighter moves diagonal. Then the fire will be able to spread diagonally, adjacent to where the firefighters moved diagonally. Thus the fire cannot be contained.

Case 3: Suppose two of the firefighters are along the negative x-axis and the positive y-axis. Suppose the other firefighters are along the (n,n) and $(-n,n)$ diagonals, see Figure 36. Then the firefighters have to protect the positive x-axis. Notice that only the firefighter on the (n,n) diagonal can reach the axis. The firefighter must move diagonal at some point to reach the axis before the fire. Then we only have two firefighters trying to protect Q1 (positive x and y values). According to Lemma 3.1, two firefighters are insufficient to contain a fire on $\mathbb{N} \times \mathbb{N}$. Thus the fire cannot be contained.

Case 4: Suppose one firefighter is on the x-axis. Without loss of generality, suppose it is on the negative x-axis. Suppose two firefighters are along the $(-n, n)$ and the (n, n)

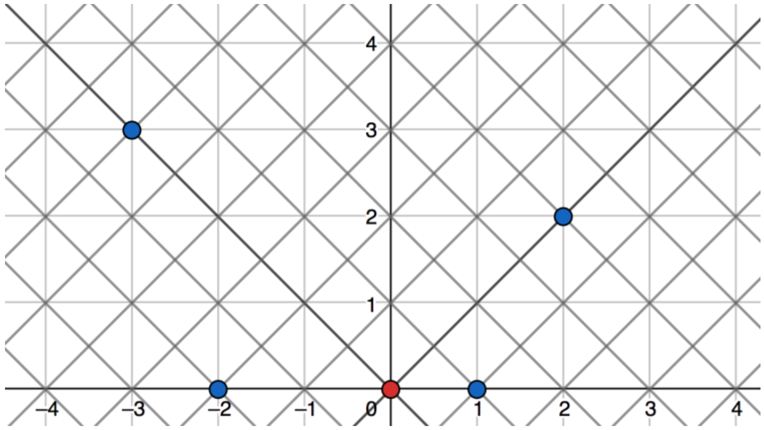

Figure 35: Four firefighters are placed on the positive and negative x-axis and the $(-n, n)$ and (n, n) diagonals.

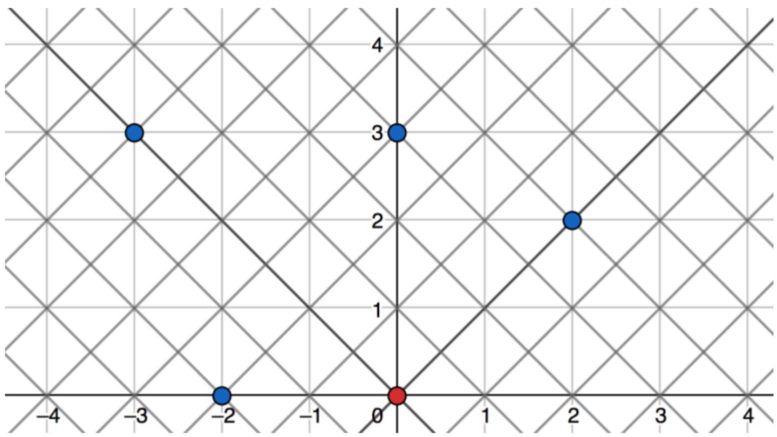

Figure 36: Four firefighters are placed on the negative x-axis, positive y-axis, and the $(-n, n)$ and (n, n) diagonals.

diagonals and the fourth firefighter is anywhere on the graph. Assume it is located in Q1, see Figure 37. Then we know that the firefighter along the negative x-axis will be unable to travel to the other side of the $(-n, n)$ diagonal. We also know that the firefighters on the $(-n, n)$ and (n, n) diagonals will be unable to reach the positive y-axis in such a way that the fire will not be able to spread through a diagonal move. Notice that the firefighter on the (n, n) diagonal will also be unable to travel to the positive x-axis without allowing the fire to spread through a diagonal. Then the fourth firefighter must protect both the positive x- and y-axes. However, once the firefighter chooses to protect one of these axes, it is unable to protect the other. Thus the fire cannot be contained.

Case 5: Suppose none of the firefighters start along an axis or a diagonal. Then the four firefighters are placed anywhere on the graph with coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, and (x_4, y_4), see Figure 38. The firefighters have to protect the three half axes (negative and positive x-axes and the positive y-axis) as well as the $(-n, n)$ and (n, n) diagonals.

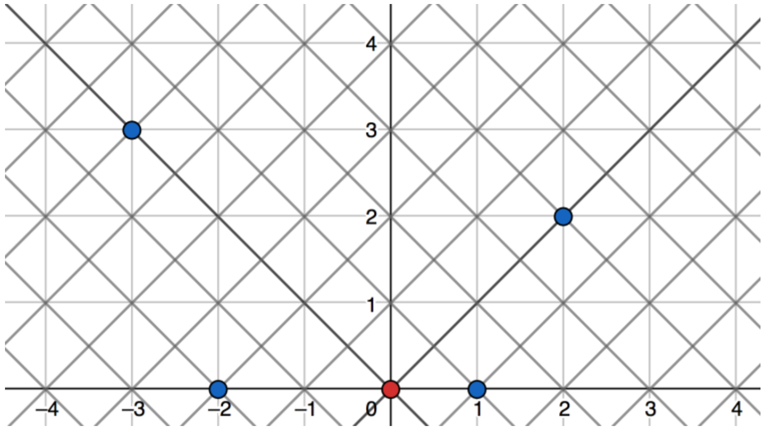

Figure 37: Three firefighters are placed on the negative x-axis and the $(-n, n)$ and (n, n) diagonals. The fourth firefighter is placed anywhere in Q1.

Notice that once a firefighter protects an axis, then they are unable to protect either of the diagonals. Similarly as soon as a firefighter protects either of the diagonals, they can only protect an adjacent axis by moving diagonally. Then as the fire spreads, it will be able to escape through the diagonal. Thus the fire cannot be contained.

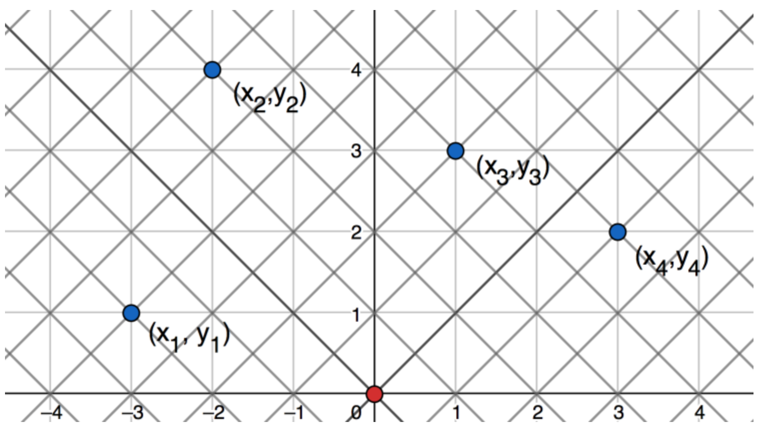

Figure 38: Four firefighters are placed anywhere on the graph.

□

With both the $\mathbb{N} \times \mathbb{N}$ and the $\mathbb{Z} \times \mathbb{N}$ grids explored, we are ready to use the results we found to determine the number of firefighters needed to contain a fire on $\mathbb{Z} \times \mathbb{Z}$.

6 $\mathbb{Z} \times \mathbb{Z}$

Trivially, eight firefighters can contain a fire that starts at $(0, 0)$ on a $\mathbb{Z} \times \mathbb{Z}$ grid. It is known that seven firefighters can contain a fire on a $\mathbb{Z} \times \mathbb{Z}$ grid when the firefighters are not restricted to move to adjacent, unburned vertices [3]. We will explore to see whether this result holds true when we do restrict the movement of the firefighters such that they can only move to adjacent, unburned vertices.

Lemma 6.1. *Suppose the firefighters can only move to adjacent, unburned vertices. If fire starts at* $(0, 0)$ *on* $\mathbb{Z} \times \mathbb{Z}$ *with no protected vertices in at least one of the four directions of the fire (North, Northeast, and Northwest; South, Southwest, and Southeast; East, Northeast, and Southeast; West, Northwest, and Southwest), then the fire cannot be contained.*

Proof. Suppose the fire starts at $(0, 0)$ on $\mathbb{Z} \times \mathbb{Z}$. Without loss of generality, assume there are no protected vertices North (North, Northeast, and Northwest) of the fire. Then there are no firefighters whose y-coordinate is greater than 0. At most the firefighter's y-value is equal to the 0. Then at each time interval, the fire will increase its y-value while also spreading one unit over in both the positive and negative direction parallel to the x-axis. The firefighter then must travel diagonally Northwest or Northeast (if adjacent to the fire) depending on which side of the fire it started on or North to also increase its y-value. Continuing this process, the firefighter will only be able to stay level (have the same y-value as the fire) with the fire and will never be able to block the fire from spreading to the North. Thus the fire cannot be contained.

By rotational symmetry, the argument holds when there are no protects vertices to the South, East, or West of the fire.

□

Notice, in the lemma above that the firefighter must move diagonally to stay level with the fire. However, when the firefighter travels diagonally, the fire would then be able to travel diagonally adjacent to the diagonal movement of the firefighter.

Lemma 6.2. *Let a fire start at* $(0, 0)$ *on* $\mathbb{Z} \times \mathbb{Z}$*, then a single firefighter trying to contain a fire in a quadrant, including the adjacent axes, must move either North, East, South, or West (it cannot move diagonally).*

Proof. Let a fire start at $(0, 0)$ on $\mathbb{Z} \times \mathbb{Z}$. Without loss of generality, assume a firefighter starts in the Northeast quadrant with coordinates (x, y). Suppose by means of contradiction that the firefighter can contain a fire by moving diagonally. Then the firefighter can move diagonally such that at each time interval its x- and y-values both increase or decrease by 1 unit.

Case 1: Assume the firefighter can reach the (n, n) diagonal for any $n \geq 1$. Then notice that after 1 time interval, the fire will increase its y-value by 1 and form a new diagonal that needs to be protected, the $(n, n + 1)$ diagonal. Since the firefighter is moving diagonally, it will always increase or decrease both its x- and y-values. Thus it will never touch that diagonal allowing the fire to continuously spread.

Case 2: Assume the firefighter cannot reach the (n, n) diagonal for all $n \geq 1$. Then the fire will continuously spread in that direction.

Thus we have a contradiction that the fire can be contained with a firefighter moving diagonally. Therefore a firefighter must move North, East, South, or West to contain a fire.

□

We know that we need at least three firefighters to contain a fire on $\mathbb{N} \times \mathbb{N}$ and at least five firefighters to contain a fire on $\mathbb{Z} \times \mathbb{N}$. The lemmas above further explain how the firefighters should be placed as well as how they should travel.

Theorem 6.3. *There is no way to contain a fire on $\mathbb{Z} \times \mathbb{Z}$ with 7 firefighters, where the firefighters are restricted to moving to an unburnt adjacent vertex.*

Proof. Suppose a fire starts at $(0, 0)$ on $\mathbb{Z} \times \mathbb{Z}$. Let seven firefighters be placed anywhere on the graph.

Case 1: Suppose three firefighter are located on the positive, northern half of the graph and four firefighters are located on the negative, southern half of the graph, see Figure 39. Then according to Lemma 7, three firefighters are clearly insufficient to protect the northern half of the graph, and four firefighters are insufficient to protect the southern half of the graph.

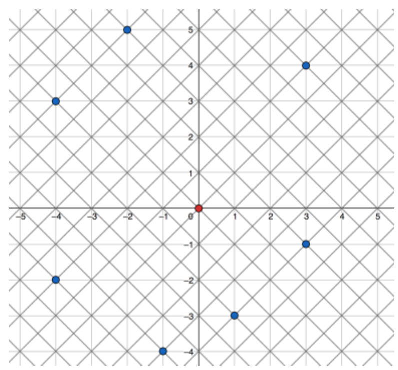

Figure 39: Three firefighters located in the northern half of the graph and four firefighters located in the southern half of the graph.

Case 2: Suppose two firefighters are located on the northern half of the graph and five firefighters are located on the southern half of the graph, see Figure 40. The five fighters are able to protect the southern half of the graph. Notice that once the firefighters to the south reach the x-axis (they might even start on the x-axis), then they can help protect the northern half of the graph. At best, there are any number of firefighters along the x-axis. Notice that if we have two or more firefighters along the x-axis, it would be just as effective as having two firefighters such that we have four firefighters protecting the northern half of the graph, since these firefighters will still be unable to protect the $(-n, n)$ and (n, n) diagonals. Then the two firefighters that started in the northern half of the graph have to protect both diagonals and the positive y-axis. Then once each firefighter decided to protect one of the diagonals or the y-axis, they will be unable to protect the others. Thus one of the diagonals or the axis will remain unprotected. Thus the fire cannot be contained.

Case 3: Suppose one firefighter is located on the northern half of the graph and six firefighters are located on the southern half of the graph, see Figure 41. The six firefighters are able to protect the southern half of the graph. Then at best, there are any number of firefighters along the x-axis. Notice that if we have two or more firefighters along the x-axis,

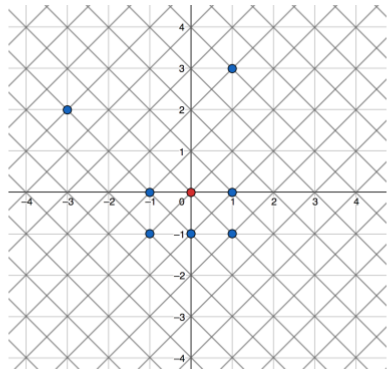

Figure 40: Two firefighters located in the northern half of the graph. Five firefighters protecting the southern half of the graph.

it would be just as effective as having two firefighters such that there are three firefighters that can help protect the northern half of the graph, since they will be unable to protect either the $(-n, n)$ or (n, n) diagonals. According to Lemma 7, three firefighters is insufficient to contain the fire. Thus the fire cannot be contained.

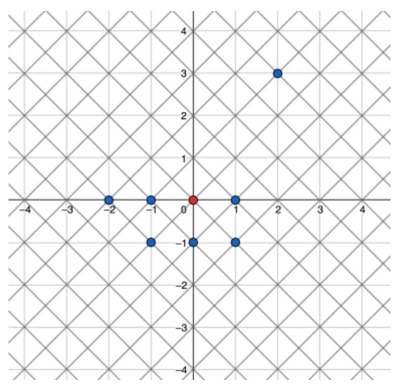

Figure 41: One firefighter located in the northern half of the graph. Six firefighters protecting the southern half of the graph.

Case 4: Suppose no firefighters are located on the northern half of the graph and seven firefighters are located on the southern half of the graph, see Figure 42. Then at best, there are firefighters located on the x-axis such that they are level with the height of the fire. Notice that there are no firefighters North of the fire. Thus the fire cannot be contained. □

7 Conclusions and Future Research

Our goal was to show the number of firefighters needed to contain a fire on the Z□Z grid and the Z � Z grid. Trivially, four firefighters are sufficient on the Z□Z grid and eight

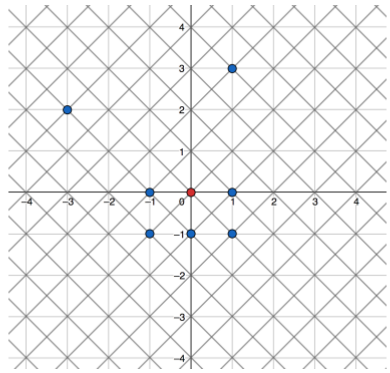

Figure 42: No firefighters are located in the northern half of the graph. Seven firefighters protecting the southern half of the graph.

firefighters on the Z � Z grid. Based on the results of the Lemmas and Corollaries, we are able to conclude that three firefighters are insufficient to contain a fire on the Z□Z grid and seven firefighters are insufficient to contain a fire on the Z � Z grid.

For future work, we want explore other infinite graphs that are not grids. We want to consider the infinite random graph to see if the fire can be contained in a finite number of time intervals. Additionally, we are interested in finding a graph where the trivial amount of firefighters is not the minimum number of firefighters needed to contain the fire with our restriction to the movement of the firefighters, if possible.

8 Bibliography

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