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Equilibrium Pigou Cycle

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Abstract

We construct a simple endogenous model that describes business cycles resulting from self-fulfilling prophecies. In our model, the goods market is assumed to be continuously cleared. Being guided by Pigou's 1926 insight, we try to show that the expectation about future economy plays an important source of business cycles. We assume that firms expect the occurrence of business fluctuations. Then, the firms will try to calculate expected income. Under such an assumption, we demonstrate the occurrence of business cycles. Since the expectation of business fluctuations yields business fluctuations, the results of our model are based on self-fulfilling prophecies. Using the Hopf bifurcation theorem, we detect a limit cycle in our model. Moreover, by executing numerical simulations, we describe the possible occurrence of stable limit cycles and show that the width of such a limit cycle depends on the degree of intensity of animal spirits.

Keywords: Equilibrium Pigou Cycle; Expectation; Expected Income; Self-fulfilling Prophesy; Animal Spirits; Hopf Bifurcation

1. Introduction

We construct a simple endogenous model[1] that describes business cycles resulting from self-fulfilling prophecies. Our model is of Keynesian type. An important feature that is common to endogenous business cycles models of Keynesian type is market instability. However, we prove that for such a type of business cycles models, market instability is not always necessary for business cycles[2]. Pigou (1926) stressed that the expectation about future economy plays an important source of business cycles. To prove the Pigou's insight, we incorporate the expectation about future income (i.e. expected income) into a business cycle model of Keynesian type. Moreover, to stress a role of expectation, we assume that the goods market is continuously cleared.

By using the Hopf bifurcation theorem, we detect a limit cycle in our model. Moreover, by executing numerical simulations, we demonstrate the possible occurrence of stable limit cycles. Noting that goods market is continuously cleared in our model, we call such a cycle ***equilibrium Pigou cycle***. On the other hand, a cycle emerging in Keynesian disequilibrium models is called **disequilibrium Keynesian cycle**.

In this paper, as the intensity of animal spirits, we consider the degree of *the propensity to invest for expected income* (***PIEI***). We demonstrate that the intensity of animal spirits plays an important role in the occurrence of equilibrium Pigou cycles. It has often been well-known that *the propensity to invest for income* (***PII***) plays an important role in the occurrence of disequilibrium Keynesian cycles. In this sense, we stress the role of expectation. However, the source of occurrence of cycles is similar to that obtained in many Keynesian models. See Owase (1991) and Lorenz (1993).

2. The Model

We consider the Keynesian consumption function:

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[1] It has been well-known that there exist periodic paths in optimal growth models. See for example Grandmont (1987) and Benhabib (1992). Unlike these neoclassical-type endogenous models, our model is a business-cycle model of the Keynesian type in the sense that the investment function of our model is of a Keynesian type. For the original models of Keynesian type, see Kaldor (1940), and Goodwin (1951).

[2] Unlike the business cycles models of Keynesian type, the neoclassical-type of endogenous business cycles models of Footnote 1 are equilibrium models. However, the expectation about future economy is included in the neoclassical-type models. We argue business cycles from a different point of view from the neoclassical-type models.

$C = \alpha Y + C_0,$

where $\alpha \in (0,1)$, $C_0 > 0$, C is consumption and Y is income. The investment function is given by

$I = \beta Y + \beta_e Y_e - \eta K$,

where K is capital stock, Y_e is expected income, $\beta \geq 0$, $\beta_e \geq 0$, $\eta \in [0,1)$, and $i > 0$. Moreover, we assume that the expected income is adaptively adjusted by

(1) $\dot{Y}_e = f(Y - Y_e)$,

where the $f-$function (called the adjustment function) is assumed to be of class C^1 and, for simplicity, to be $f'(0 = 1$. Moreover, we assume:

Assumption 1: $f(0) = 0$ and $f'(u) > 0$ for any $u \in R^1$.

We stress the role of expectation in investment. Before discussing it, we first consider the case of $\beta_e = 0$ and the following dynamic model of Keynesian type:

$$\Lambda: \begin{cases} \dot{Y} = \theta(I + C - Y) = \theta\{(\alpha + \beta)Y - \eta K - Y + C_0, \\ \dot{K} = I - \delta K = \beta Y - (\eta + \delta)K, \end{cases}$$

where δ is the depreciation rate. For simplicity, we assume $\delta = 0$. It should be noted here that expectation about future economy is not incorporated into System Λ. System Λ has been used in the studies that demonstrate the endogenous occurrence of business cycles. See Gabisch and Lorenz (!987), Owase (1991), and Gandolfo (1996). A sigmoid nonlinearity of investment function yields a disequilibrium Keynesian cycle (in other words, a persistent and cyclic market disequilibrium). See Chang and Smyth (1971), Owase (1991), Lorenz (1993), Gandolfo (1996), and Puu (2000). Needless to say, such studies have assumed the instability of the equilibrium. However, to stress the important role of expectation in explaining business cycles, we consider the case where the equilibrium of System Λ is stable. Throughout this paper we work under

Assumption 1: $\alpha + \beta - 1 < \eta$.

We have the following:

Lemma 1: System Λ has a unique equilibrium in $R^2_{++} = \{(x, y) \in R^2 : x > 0, y > 0\}$. The equilibrium point of System Λ is asymptotically stable.■

Proof: See Appendix.■

With (1) and Lemma 1, it is not unnatural to assume that the market economy is in equilibrium. To stress the importance of role of expectation, we introduce such an assumption and we consider the case of $\beta_e > 0$. Then, we have

(2) $$Y = C + I = \beta_e Y_e + (\alpha + \beta)Y - \eta K + C_0.$$

From now on, we derive a two-dimensional differential equations system concerning K and Y_e. Eq. (2) yields

(3) $$Y = \frac{1}{\Omega}(\beta_e Y_e - \eta K + C_0), \quad \Omega \equiv 1 - \alpha - \beta > 0.$$

For simplicity, without loss of generality, we assume that the depreciation rate of capital stock is zero. Then, Eq. (3) yields

(4.1) $$\dot{K} = I = \beta Y + \beta_e Y_e - \eta K = \frac{(1-\alpha)\beta_e}{\Omega} Y_e - \frac{(1-\alpha)\eta}{\Omega} K + \frac{\beta C_0}{\Omega},$$

(4.2) $$\dot{Y}_e = f(Y - Y_e) = f\left(-\frac{\eta}{\Omega} K + \frac{\alpha + \beta + \beta_e - 1}{\Omega} Y_e + \frac{C_0}{\Omega}\right).$$

Thus, we have the following two-dimensional continuous-time system with an adaptive expectation formation for expected income:

$$\Lambda_e : \begin{cases} \dot{K} = -\frac{1-\alpha}{\Omega}\eta K + \frac{\beta_e(1-\alpha)}{\Omega} Y_e + \frac{\beta C_0}{\Omega}, \\ \dot{Y}_e = f\left(-\frac{\eta}{\Omega} K + \frac{\alpha + \beta + \beta_e - 1}{\Omega} Y_e + \frac{C_0}{\Omega}\right). \end{cases}$$

Since the goods market is continuously cleared on any path of System Λ, the notion of equilibrium point in the economic sense has no meaning in our model. Hence, it not suitable to call the point satisfying $(\dot{K}, \dot{Y}_e) = (0,0)$ the equilibrium point. We call it the stationary state. We work under the following assumptions.

Assumption 2: $\alpha + \beta - 1 > 1 - \alpha - \beta_e$.

This assumption yields the following lemma:

Lemma 2: Suppose Assumptions 1 to 3 are satisfied. Then the stationary state of System Λ_e is uniquely determined in R^2_{++}.■

Proof: See Appendix.■

We denote the stationary state by (K^*, Y_e^*) and define

$$(x, y) \equiv (K, Y_e) - (K^*, Y_e^*).$$

Then, system Λ_e becomes

$$\tilde{\Lambda}_e : \begin{cases} \dot{x} = -\dfrac{\eta(1-\alpha)}{\Omega} x + \dfrac{\beta_e(1-\alpha)}{\Omega} y, \\ \dot{y} = f(-\dfrac{\eta}{\Omega} x + \dfrac{\alpha + \beta + \beta_e - 1}{\Omega} y). \end{cases}$$

In this paper, we consider the dynamic behavior of System $\tilde{\Lambda}_e$. In the next section, we prove the occurrence of a Hopf cycle (i.e. an equilibrium Pigou cycle).

We here remark upon an important feature of our model. In our model, since both the goods market is assumed to be continuously cleared, business cycles result merely from the adjustment of expected income. As stated in the Introduction, the households expect there to be short-run fluctuations in consumption; therefore, the households estimate the expected income to smooth or stabilize fluctuations in consumption. Thus, the expectation of business cycles yields an estimation of expected income and, consequently, business cycles occur. In this sense, our model describes business cycles that result from self-fulfilling prophecies.

3. Local Analysis of Nonlinear Dynamics

We consider the emergence of a Hopf cycle. As a bifurcation parameter, we consider the PIEI β_e. We rewrite System $\tilde{\Lambda}_e$ as $\tilde{\Lambda}_e(\beta_e)$. Then, we obtain the following result:

Theorem 1: Suppose that Assumption 1 is satisfied. Then, System $\tilde{\Lambda}_e(\beta_e)$ undergoes a supercritical Hopf bifurcation at

$$\beta_e = \beta_e^{\#} \equiv 2(1-\alpha) + \beta > 0.$$

If $\beta_e < \beta_e^{\#}$ (resp. $\beta_e > \beta_e^{\#}$), the fixed point of System $\tilde{\Lambda}_e(\beta_e)$ is asymptotically stable (resp. completely unstable).■

Proof: See Appendix.■

We provide a numerical example of the occurrence of a Hopf bifurcation.

Example 1: We consider the case:

$$f(u) = 0.3\mathrm{Arctan}(u) + 0.7u, \quad \alpha_e = 0.85, \quad \eta = 0.1, \text{ and } il = 0.1.$$

Figure 1 describes the graph of the $f-$function. Then, we have

$$f'(0) = 1, \quad \Omega = 0.25, \quad \beta_e^{\#} = 0.3.$$

We here use Lemma 1. System $\widetilde{\Lambda}_e(\beta_e)$ with these parameters undergoes a Hopf bifurcation at $\beta_e = \beta_e^{\#}$. We numerically see the stability of the Hopf cycle. Paths of System $\widetilde{\Lambda}_e(\beta_e)$ rotate clockwise. The blue curve of Figure 2.1 describes a typical path of System $\widetilde{\Lambda}_e(0.2999)$. Since $0.2999 < \beta_e^{\#}$, Theorem 1 shows that the fixed point (0,0) of System $\widetilde{\Lambda}_e(0.2999)$ is asymptotically stable. Figure 2.1 demonstrates that the blue path converges to the fixed point. The blue curve of Figure 2.2 (resp. Figure 2.3) describes a typical path of Systems $\widetilde{\Lambda}_e(0.302)$ (resp. $\widetilde{\Lambda}_e(0.306)$). On the other hand, the red path of Figure 2.2 (resp. Figure 2.3) describes a limit cycle of Systems $\widetilde{\Lambda}_e(0.302)$ (resp. $\widetilde{\Lambda}_e(0.306)$). Since $0.306 > 0.302 > \beta_e^{\#}$, Theorem 1 shows that the fixed point (0,0) of Systems $\widetilde{\Lambda}_e(0.302)$ and $\widetilde{\Lambda}_e(0.306)$ is completely unstable. Through Figures 2.2 and 2.3、we numerically see that System $\widetilde{\Lambda}_e(\beta_e)$ undergoes a supercritical Hopf bifurcation at $\beta_e = \beta_e^{\#}$ and a stable Hopf cycle emerges. This observation also suggests that System $\widetilde{\Lambda}_e(\beta_e)$ has a stable limit cycle.■

We next provide a numerical example of typical stable limit cycle and demonstrate that the PIEI increases is closely related to the intensity of instability of System $\widetilde{\Lambda}_e(\beta_e)$.

Example 2: The blue paths of Figure 3.1 describe typical paths of System $\widetilde{\Lambda}_e(0.36)$. The red path of Figure 3.1 describes a limit cycle of System $\widetilde{\Lambda}_e(0.36)$. On the other hand, the black, blue and red curves of Figure 3.2 describe typical stable limit cycles of Systems $\widetilde{\Lambda}_e(0.33)$, $\widetilde{\Lambda}_e(0.35)$ and $\widetilde{\Lambda}_e(0.37)$, respectively. Figure 3.2 shows that as the PIEI increases, the intensity of instability of System $\widetilde{\Lambda}_e(\beta_e)$ becomes high.■

Figures 1, 2 and 3 about here.

4. Conclusions and Final Remark

We constructed an endogenous business-cycle model. In our model, both the goods market is continuously cleared. We assumed that the consumption plan is given by the Keynesian consumption function. On the other hand, we incorporated the expected income into the investment function, for simplicity we assumed a Keynesian investment plan that depends on capital stock, income, expected income and interest rate. Under these settings, we derived a dynamic model in the Keynesian type and considered local dynamics of the model. We stressed the importance of expectation in explaining business cycles. By using the Hopf bifurcation theorem, we demonstrated in our model the occurrence of a Hopf cycle. We also executed numerical simulations and demonstrated the occurrence of stable limit cycles. We numerically demonstrated that as the PIEI increases, the intensity of instability of our system becomes high.

An important feature of our model is as follows. In our model, business cycles result merely from the adjustment of expected income. Our model shows that if the firms expect business fluctuations, then the firms try to calculate expected income. Then, as we proved, there emerges a possibility that business cycles are realized. In this sense, business cycles in our model result from self-fulfilling prophecies in the sense that expectation of business cycles yields business cycles.

Finally, one remark should be made. Although we considered the possibility that expectation yields business cycles. We do not mean that the disequilibrium models[3] of Keynesian type are not important. Needless to say, they play important roles in explaining business cycles. The purpose of our paper is merely to provide another source of business cycle, which is different from those discussed so far in the endogenous business cycle models of Keynesian type.

5. Appendix

In this appendix, we prove Lemmas 2 and Theorems 1.

Proof of Lemma 2: Since we assume that $f(0)=0$ and for any $u \in R^1$, an equilibrium point of System Λ is given by the solution of the following two equations.

$$K = \frac{\beta_e}{\eta} Y_e + \frac{\beta}{1-\alpha} \bullet \frac{C_0}{\eta}, \quad K = \frac{\alpha + \beta + \beta_e - 1}{\eta} Y_e + \frac{C_0}{\eta}.$$

From assumption 3 and $\alpha + \beta - 1 < 0$, we have

$$\beta / (1-\alpha) < 1, \quad 0 < (\beta_e + \alpha + \beta - 1) / \eta < \beta_e / \eta.$$

This completes the proof. See Figure 4.■

Proof of Theorem 1: The Jacobian matrix of System $\Theta(\beta_e)$ is given by

[3] The word “disequilibrium” implies that an equilibrium point is unstable.

$$J_{\Theta}(\beta_e) \equiv \begin{bmatrix} -\frac{(1-\alpha)\eta}{\Omega} & \frac{\beta_e(1-\alpha)}{\Omega} \\ -\frac{\eta}{\Omega} & \frac{\alpha+\beta+\beta_e-1}{\Omega} \end{bmatrix}.$$

From the definition of $\beta_e^{\#}$, the trace of $J_{\Theta}(\beta_e^{\#})$ is given by

(6) $\quad \mathrm{tr}J_{\Theta}(\beta_e^{\#}) = \{-(1-\alpha)\eta + (\alpha+\beta+\beta_e^{\#}-1)\}/\Omega = 0.$

Moreover, Assumption 3 yields

(7) $\quad \det J_{\Theta}(\beta_e^{\#}) = (1-\alpha)\eta(1-\alpha-\beta_e^{\#})/\Omega^2 > 0.$

On the other hand, from Assumption 3, we have

(8) $\quad d\,\mathrm{Re}\,\lambda(\beta_e^{\#})/d\beta = 1/\Omega > 0.$

From the Hopf bifurcation Theorem[4], Eqs. (6) to (8) complete the proof.■

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[4] See Guckenheimer and Holmes (1983, Theorem 3.4.2)

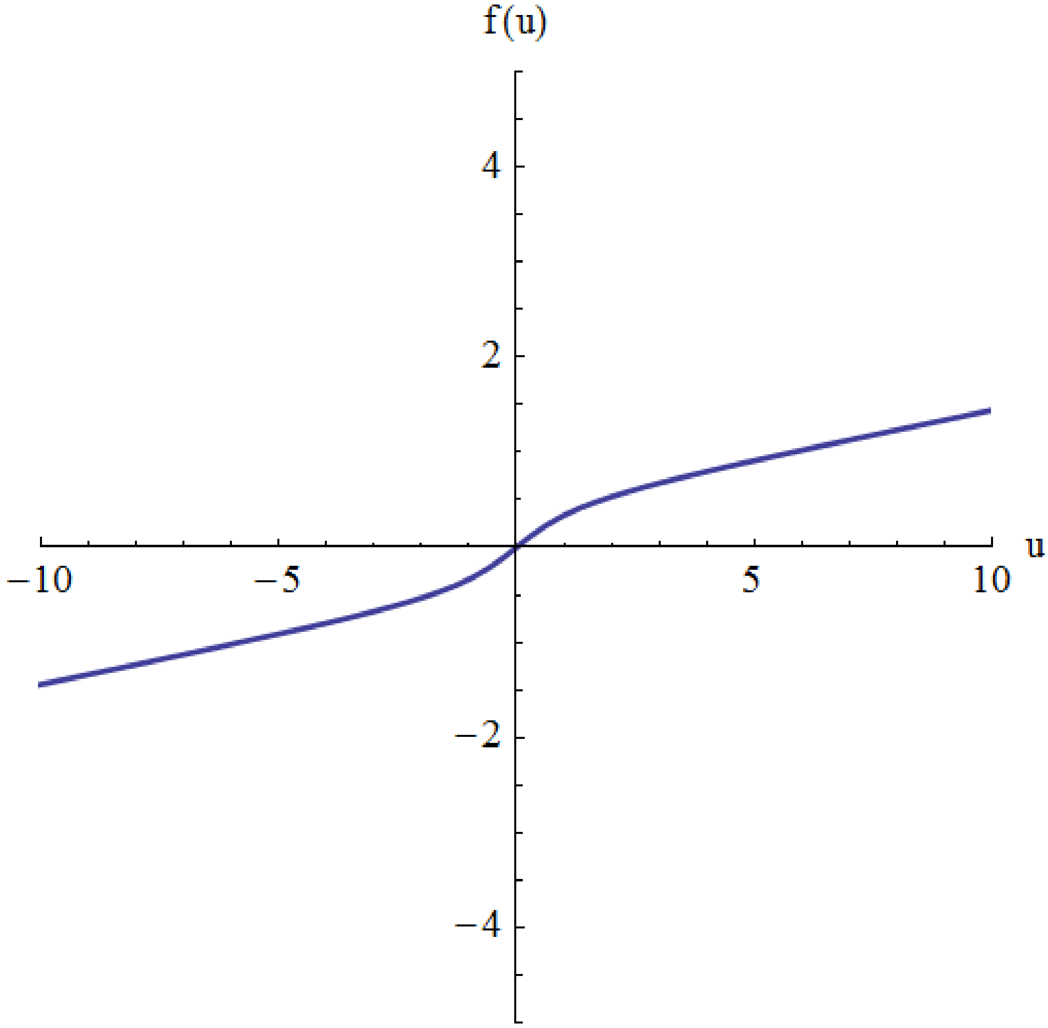

Figure 1

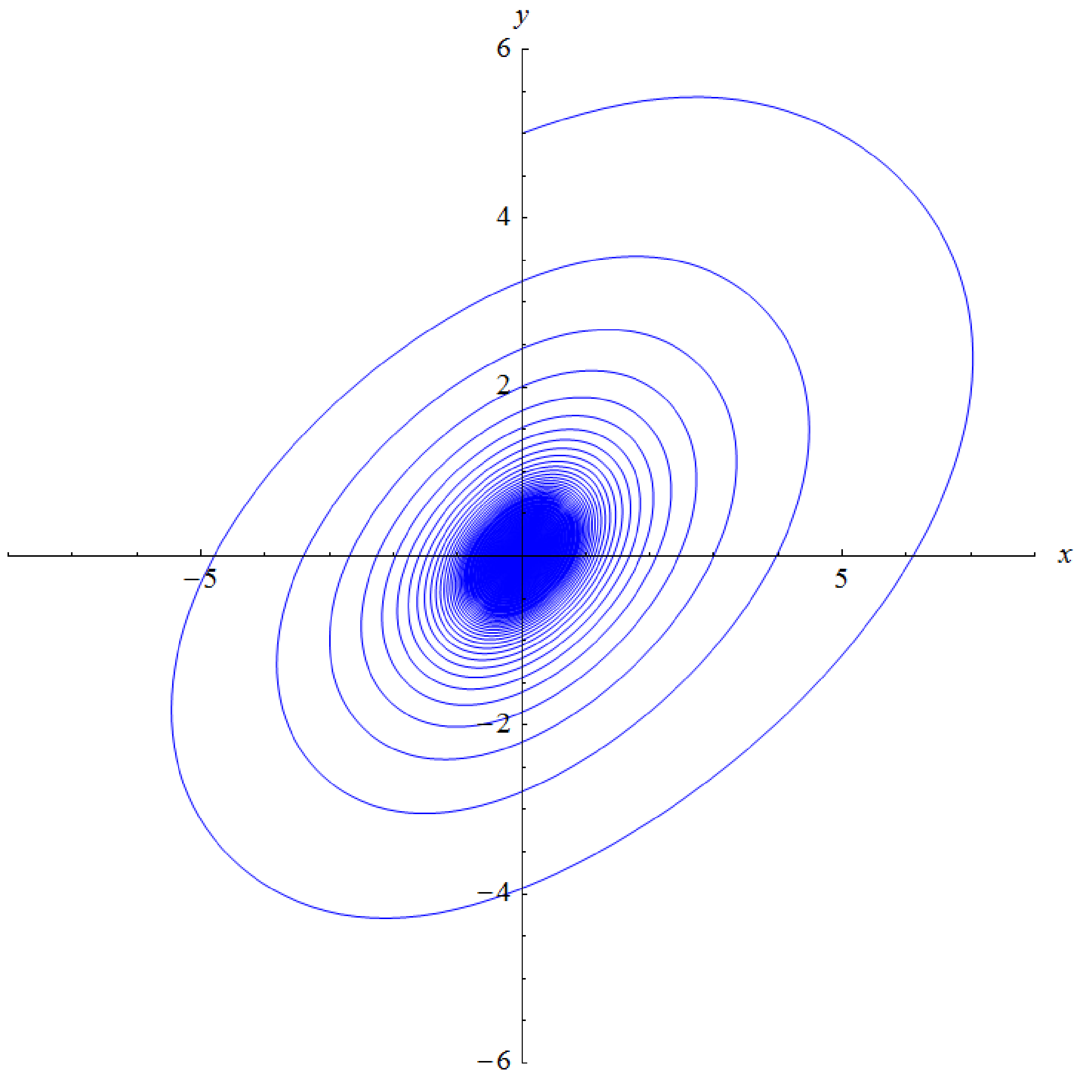

Figure 2.1

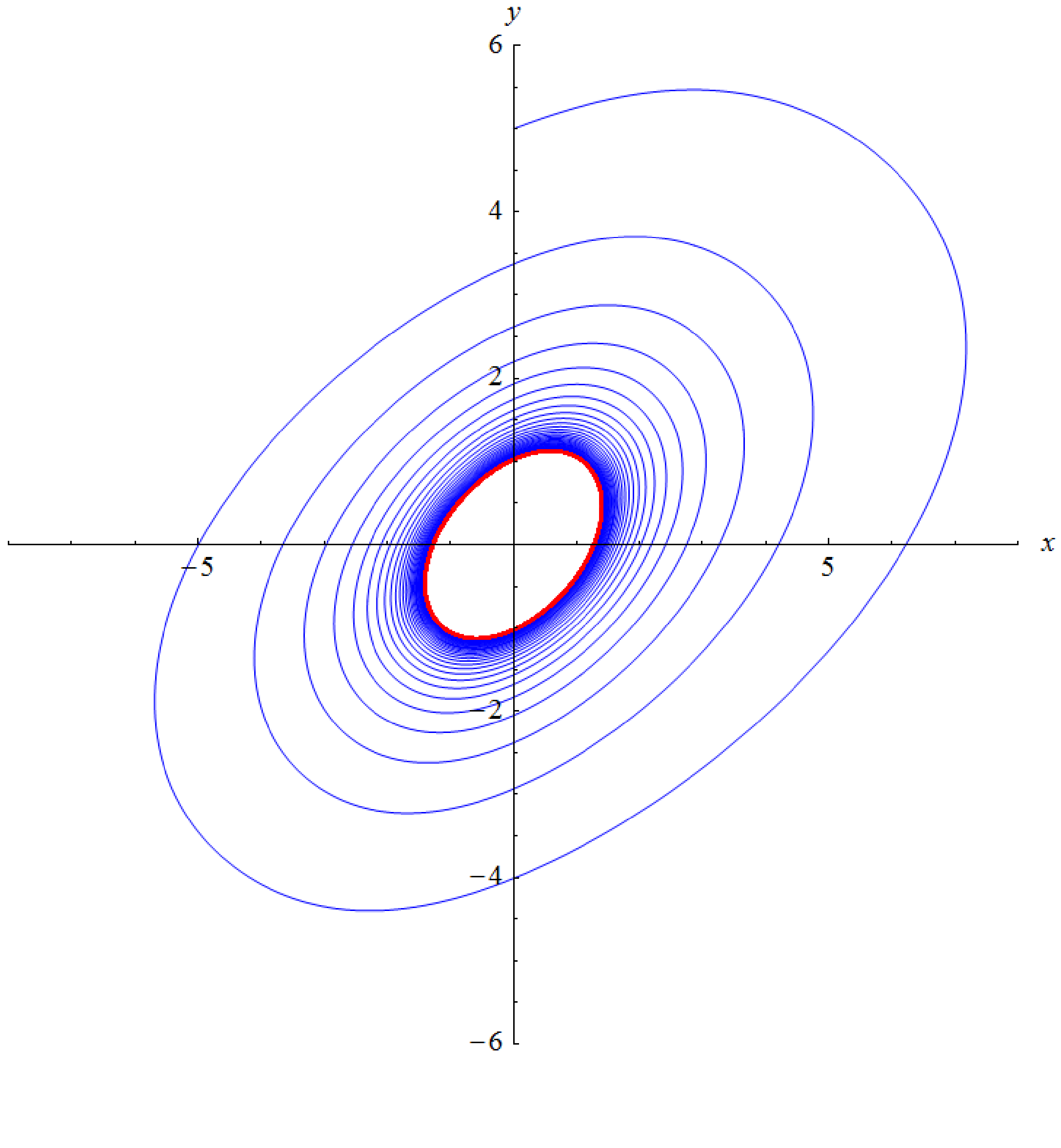

Figure 2.2

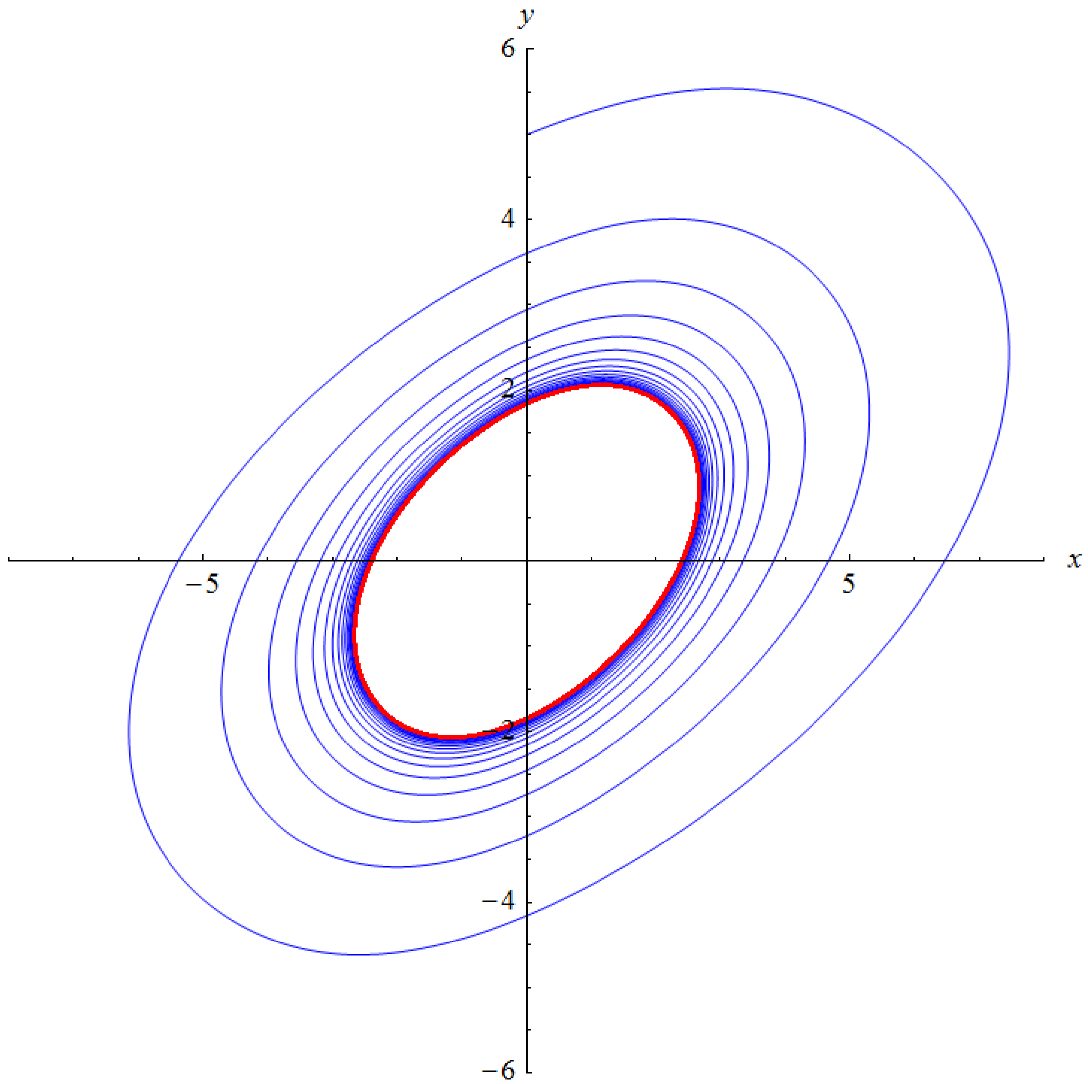

Figure 2.3

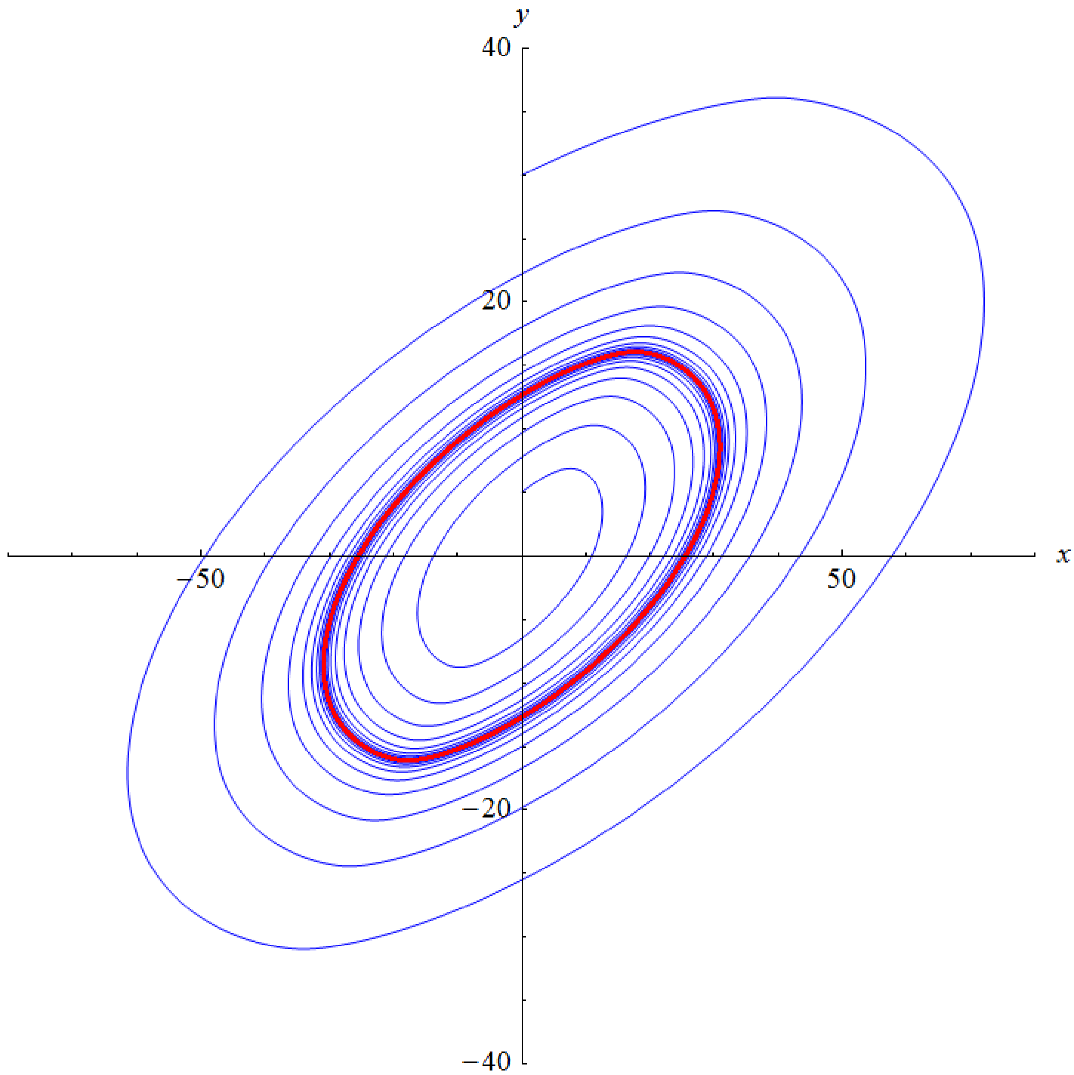

Figure 3.1

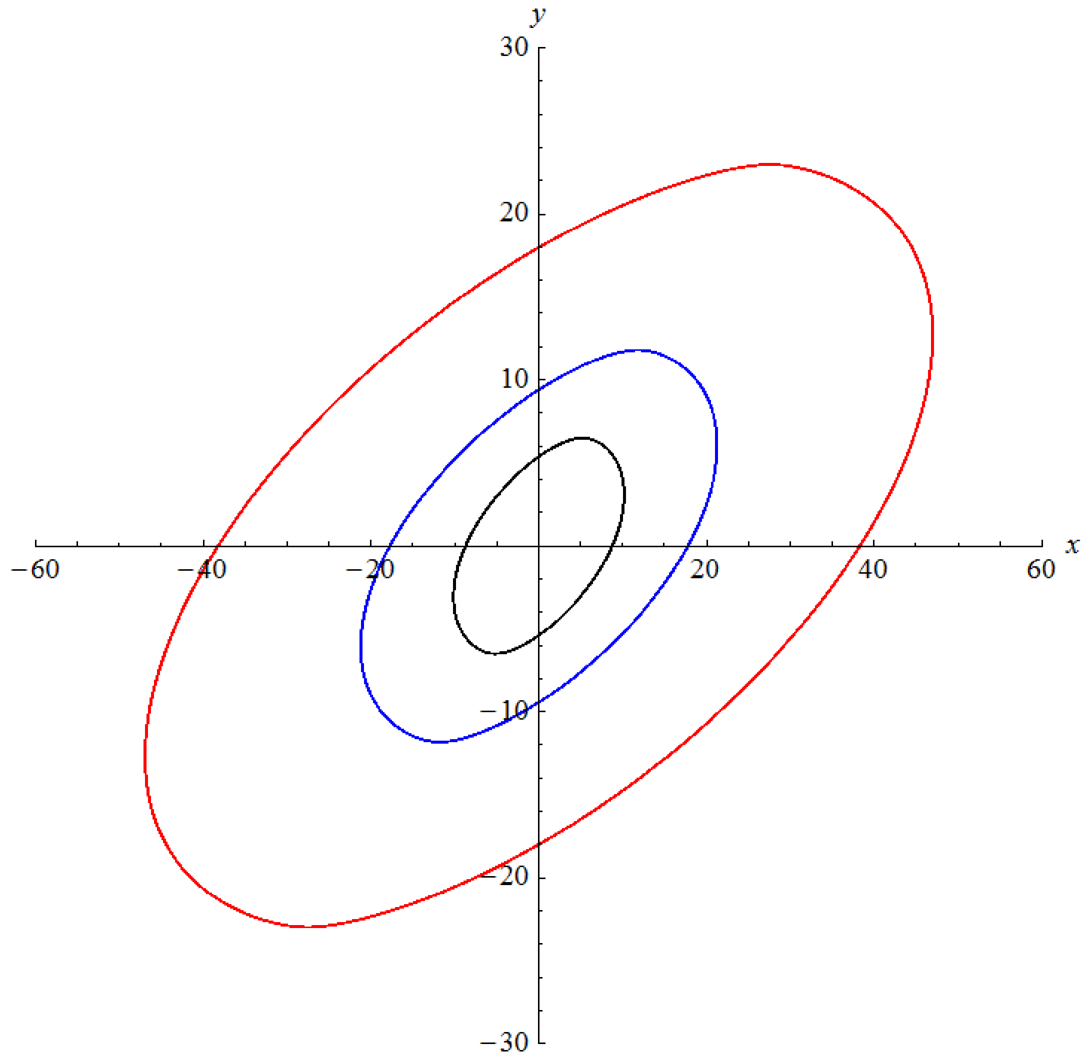

Figure 3.2