Working Paper No. 334

An Endogenous Model of Heterogeneous Growth Ⅱ: A Generalization to the CES Utility Function

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July 7, 2020

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Abstract: Dohtani (2019) constructed an endogenous growth model with heterogeneous industries. In the growth model, the growth rate of each industry is determined by such fundamental parameters as the rate of technological progress of the industry. Through the growth model, the paper derived a fundamental equation that gives an analytical relation among the growth rates of relative price, consumption and capital stock of each industry. However, the fundamental equation is derived under the assumption that the household possesses an additive utility function. In the present paper, we generalize the model to a growth model with the CES utility function and introduce a new notion that gives a generalization of the growth rate in the usual sense. By using such a notion, we derive the same results derived by Dohtani (2019).

Keywords: Structural transformation; Heterogeneous growth; Change of relative prices;. Growth Rate of Trend

JEL classification: C61; L16; O40; O41

1. Introduction

Dohtani (2019) constructs an endogenous growth model. In the growth model, the growth rate of each industry is determined by such fundamental parameters as the rate of technological progress of the industry, the elasticity of marginal productivity of the industry and the elasticity of marginal utility of the goods produced by the industry. These fundamental parameters are different among the industries. Therefore, our model gives a theoretical explanation of the persistent transition of industrial structure accompanied by a change of relative prices. Through the growth model, Dohtani (2019) derives a fundamental equation that gives an analytical relation among the growth rates of relative price, consumption and capital stock of each industry. However, Dohtani (2019) assumes that the household possesses an additive utility function:

$$U_{AU}(C_{1t},\cdots,C_{nt}) = \sum_{k\in N} {C_{kt}}^{a_k} / a_k .$$

In this paper, we generalize the model to the growth model with the CES utility function and try to derive the same results as those derived by Dohtani (2019). However, the trouble is that the generalized growth model does not possess the growth rates in the usual sense. Therefore, we introduce a new notion of growth rate that gives a generalization of the usual growth rate. By using such a notion, we derive almost the same results (especially, the same fundamental equation) as those obtained by Dohtani (2019).

2. The Model and the Derivation of Main Results

The background of our model is the same as those of Dohtani (2019). We omit the explanation of the background. For the explanation, see Dohtani (2019). We now explain the extended version of the growth model constructed by Dohtani (2019). We consider the utility function that gives a generalization of the above-mentioned additive utility function:

$$U(C_{1t},\cdots,C_{nt}) = \left(\sum_{k\in N} b_k C_{kt}^{a_k} \right)^g .$$

We consider the following Hamiltonian of the intertemporal optimization problem of the representative household with the above CES utility function:

$$H = U(C_{1t},\cdots,C_{nt})e^{-st} + \eta\left(\sum_{k\in N} \Pi_{kt} + rK_{It} - \sum_{k\in N} P_{kt}C_{kt} \right).$$

Since we provided the optimal paths in the case where $g=1$, we consider the case where $g \neq 1$. Since the Hamiltonian is a concave function of the state and the control variables, the sufficient condition for optimization is given by

(1.1) $$\partial H / \partial C_{jt} = \left(\sum_{k \in N} b_k C_{kt}^{a_k} \right)^{g-1} \bullet a_j b_j g C_{jt}^{a_j - 1} e^{-st} - \eta_t P_{jt} = 0,$$

(1.2) $$\dot{\eta}_t = -\partial H / \partial K_{It} = -r \eta_t,$$

(1.3) $$\dot{K}_{It} = \sum_{k \in N} \Pi_{kt} + r K_{It} - \sum_{k \in N} P_{kt} C_{kt},$$

(1.4) $$\lim_{t \to \infty} K_{It} \eta_t = 0,$$

where $j \in N$. The equation (1.2) yields $\eta_t = \eta_0 e^{-rt}$, where the initial value η_0 is determined later. Then, from (1.1) we have the following dynamic inverse demand equation:

(2) $$P_{jt} = \frac{H_{C_{jt}}}{\eta_t} = \left(\sum_{k \in N} b_k C_{kt}^{a_k} \right)^{g-1} \bullet a_j b_j g C_{jt}^{a_j - 1} \bullet \frac{e^{(r-s)t}}{\eta_0}, \quad j \in N.$$

We have

(3) $$C_{jt} = Q_{jt} = e^{d_j t} K_{jt}{}^{m_j} \quad (j \in N).$$

Throughout this paper, we assume

Assumption 1: $1 > \max\{a_j m_j, g a_j m_j\}$.

We first consider the case of $g \neq 1$. Substituting (3) into the dynamic inverse demand equation (2), we have

(4) $$P_{jt} = \left(\sum_{k \in N} b_k D_{kt}^{a_k} K_{kt}^{a_k m_k} \right)^{g-1} \bullet a_j b_j g D_{jt}^{a_j - 1} K_{jt}^{(a_j - 1) m_j} \frac{e^{(r-s)t}}{\eta_0},.$$

for any $j \in N$. By assuming the price path P_{jt} $(j \in N)$ is given, the consumption-goods industry $j \in N.$ solves the following optimization problem[1]:

$$\max \Pi_{jt} = \max(P_{jt}Q_{jt} - K_{jt}) = \max(P_{jt}e^{d_j t}K_{jt}{}^{m_j} - K_{jt}), \quad j \in N.$$

The first condition of profit maximization entails

$$
\begin{aligned}
(5) \qquad 1 &= m_j P_{jt} D_{jt} K_{jt}^{m_j - 1} \\
&= m_j \left(\sum_{k \in N} b_k D_{kt}^{a_k} K_{kt}^{a_k m_k} \right)^{g-1} \bullet a_j b_j g D_{jt}^{a_j - 1} K_{jt}^{(a_j - 1)m_j} \frac{e^{(r-s)t}}{\eta_0} \bullet D_{jt} K_{jt}^{m_j - 1} \\
&= m_j \left(\sum_{k \in N} b_k D_{kt}^{a_k} K_{kt}^{a_k m_k} \right)^{g-1} \bullet a_j b_j D_{jt}^{a_j} K_{jt}^{a_j m_j - 1} \frac{e^{(r-s)t}}{\eta_0} \\
&= m_j \left(\sum_{k \in N} b_k e^{a_k d_k t} K_{kt}^{a_k m_k} \right)^{g-1} \bullet a_j b_j e^{a_j d_j t} K_{jt}^{a_j m_j - 1} \frac{e^{(r-s)t}}{\eta_0}.
\end{aligned}
$$

From now on, we will see that the equilibrium growth path is of the form:

$$K_{jt} = \xi_{jt} e^{G_j t}, \quad \xi_j(t) > 0 \text{ for any } t \geq 0 \text{ and any } j \in N,$$

where

$$G_j = \frac{r - s + a_j d_j}{1 - a_j m_j}.$$

In the following, we determine $\xi_j(t)$ in the case where $g \neq 1$. From (5), we have

$$
\begin{aligned}
1 &= \left(\sum_{k \in N} b_k e^{a_k d_k t} \xi_{kt}{}^{a_k m_k} e^{a_k m_k G_k t} \right)^{g-1} \\
&\qquad \bullet a_j b_j m_j e^{a_j d_j t} \xi_{jt}{}^{(a_j m_j - 1)} e^{(a_j m_j - 1) G_j t} \frac{e^{(r-s)t}}{\eta_0} \\
&= \left(\sum_{k \in N} b_k \xi_{kt}{}^{a_k m_k} e^{(a_k d_k + a_k m_k G_k)t} \right)^{g-1}
\end{aligned}
$$

[1] In Sections 3 and 4, we assume the firm that solves the static problem above. However, in Section 5, we will consider the firm that solves a dynamic optimization problem.

$$\bullet \frac{a_j b_j m_j}{\eta_0} \xi_{jt}{}^{(a_j m_j - 1)} e^{\{r - s + a_j d_j + (a_j m_j - 1) G_j\} t}.$$

From the definition, we have

$$(6) \qquad 1 = \left(\sum_{k \in N} b_k \xi_{kt}{}^{a_k m_k} e^{\psi_k t} \right)^{g-1} \bullet \frac{a_j b_j m_j}{\eta_0} \xi_{jt}{}^{(a_j m_j - 1)},$$

where

$$\psi_j \equiv a_j d_j + a_j m_j G_j .$$

In Eq. (6), $(\bullet)^{g-1}$ does not depends on j. Therefore, the following equation must be satisfied:

$$\frac{a_j b_j m_j}{\eta_0} \xi_{jt}{}^{(a_j m_j - 1)} = \frac{a_i b_i m_i}{\eta_0} \xi_{it}{}^{(a_i m_i - 1)} \equiv \Theta \quad \text{for any} \quad i, j \in N.$$

We have the following:

$$(7) \qquad \xi_{jt} = \left(\frac{a_j b_j m_j}{\eta_0 \Theta} \right)^{\frac{1}{1 - a_j m_j}}.$$

Therefore, we obtain

$$1 = \left\{ \sum_{k \in N} b_k \left(\frac{a_k b_k m_k}{\eta_0 \Theta} \right)^{\frac{a_k m_k}{1 - a_k m_k}} e^{\psi_k t} \right\}^{g-1} \Theta$$

$$= \left\{ \sum_{k \in N} b_k \left(\frac{a_k b_k m_k}{\eta_0} \right)^{\frac{a_k m_k}{1 - a_k m_k}} \Theta^{\frac{1}{g-1} - \frac{a_k m_k}{1 - a_k m_k}} e^{\psi_k t} \right\}^{g-1}.$$

Therefore, we have

$$(8) \qquad 1 = \sum_{k \in N} b_k \left(\frac{a_k b_k m_k}{\eta_0} \right)^{\frac{a_k m_k}{1 - a_k m_k}} \Theta^{\frac{1}{g-1} - \frac{a_k m_k}{1 - a_k m_k}} e^{\psi_k t}.$$

We here define

(9) $$0=\sum_{k\in N}\Omega_k\Theta^{\frac{1}{g-1}-\frac{a_k m_k}{1-a_k m_k}}e^{\psi_k t}-1\equiv F(\Theta,t),$$

where

$$\Omega_j\equiv b_j\left(\frac{a_j b_j m_j}{\eta_0}\right)^{\frac{a_k m_k}{1-a_k m_k}}\quad j\in N.$$

From the assumption, we see

(10) $$\Gamma_j\equiv\frac{1}{g-1}-\frac{a_j m_j}{1-a_j m_j}=\frac{1-g a_j m_j}{(g-1)(1-a_j m_j)}\begin{cases}>0 & \text{if } g>1,\\ <0 & \text{if } g<1,\end{cases}\quad\text{for any } j\in N.$$

Therefore, we have

(11.1) $$\frac{\partial F(\Theta,t)}{\partial\Theta}=\sum_{k\in N}\Gamma_k\Omega_k\Theta^{\Gamma_k-1}e^{\psi_k t}\begin{cases}>0 & \text{if } g>1,\\ <0 & \text{if } g<1,\end{cases}\quad\text{for any } j\in N,$$

(11.2) $$\frac{\partial F(\Theta,t)}{\partial t}=\sum_{k\in N}\psi_k\Omega_k\Theta^{\Gamma_k-1}e^{\psi_k t}>0.$$

Therefore, Equation (9) can be solved for Θ and we define $\Theta=\Theta_t$ Then, ξ_{jt} is determined by $\Theta=\Theta_t$ and Equation (7). Equation (8) shows

(12) $$1=\sum_{k\in N}\Omega_k\Theta_t^{\Gamma_k}e^{\psi_k t}.$$

We can prove the following:

Lemma 1: There are real numbers $\alpha>0$ and $\beta>0$ and $k^*\in N$ such that

$$\alpha\le\Theta_t^{\Gamma_{k^*}}e^{\psi_{k^*}t}\le\beta.\blacksquare$$

Proof: The following statement contradicts (12).

$$\inf\{\Theta_t^{\Gamma_j}e^{\psi_j t}:t\ge 0\}=0\qquad\text{for any } j\in N.$$

The following statement also contradicts (12).

$$\sup\{\Theta_t^{\Gamma_{\tilde{j}}}e^{\psi_{\tilde{j}}t}:t\ge 0\}=+\infty\quad\text{for some } \tilde{j}\in N.$$

These contradictions complete the proof.■

We here introduce the following definition:

Definition 1: If there are $A>0$ and $B>0$ such that $A<\frac{X_t}{e^{ht}}<B$ for any $t\geq 0$, then h is called the growth rate of trend (GRT) of X_t.■

The following lemma shows that Definition 1 is well-defined.

Lemma 2: If h and $\tilde{h}$ are the GRT of X_t, then $h=\tilde{h}$.■

Proof: From the assumption, we obtain that $A>0$, $B>0$, $\tilde{A}>0$, and $\tilde{B}>0$ such that $A<X_t/e^{ht}<B$ and $\tilde{A}<X_t/e^{\tilde{h}t}<\tilde{B}$ for any $t\geq 0$. Then, we have

$$0<A/\tilde{B}<\frac{X_t/e^{ht}}{X_t/e^{\tilde{h}t}}=e^{(\tilde{h}-h)t}<B/\tilde{A} \text{ for any } t\geq 0.$$

This implies $\tilde{h}=h$. Thus, we complete the proof.■

We here define the following.

Definition 2: We denote the GRT of X_t by $grt(X_t)$.■

We now obtain the following result.

Lemma 3: $grt(K_{jt})=\frac{\psi_{k^*}}{\Gamma_{k^*}(1-a_j m_j)}+G_j\equiv\Phi_j$ for any $j\in N$.■

Proof: Without loss of generality, we assume $\Gamma_{k^*}>0$. From Lemma 1, we have

$$\alpha^{-1}e^{(\psi_{k^*}/\Gamma_{k^*})t}\geq\Theta_t^{-1}\geq\beta^{-1}e^{(\psi_{k^*}/\Gamma_{k^*})t} \tag{13}$$

for any $j\in N$. Moreover, (7) yields

$$K_{jt}=\xi_{jt}e^{G_j t}=\left\{\frac{a_j b_j m_j}{\eta_0\Theta_t}\right\}^{\frac{1}{1-a_j m_j}}e^{G_j t}=\left(\frac{a_j b_j m_j}{\eta_0}\right)^{\frac{1}{1-a_j m_j}}\Theta_t^{\frac{-1}{1-a_j m_j}}e^{G_j t}.$$

We here define

$$\Delta_j \equiv \left(\frac{a_j b_j m_j}{\eta_0} \right)^{\frac{1}{1-a_j m_j}} .$$

Then, we have

$$\alpha^{-1} \Delta_j e^{\left\{ \frac{\psi_{k^*}}{\Gamma_{k^*}(1-a_j m_j)} + G_j \right\} t} \geq \Delta_j \Theta_t^{\frac{-1}{1-a_j m_j}} e^{G_j t} = K_{jt} \geq \beta^{-1} \Delta_j e^{\left\{ \frac{\psi_{k^*}}{\Gamma_{k^*}(1-a_j m_j)} + G_j \right\} t}$$

for any $j \in N$. This shows that

$$\text{(14)} \qquad \alpha^{-1} \Delta_j \geq \frac{K_{jt}}{e^{\Phi_j t}} \geq \beta^{-1} \Delta_j$$

for any $j \in N$. From the definition of GRT, this completes the proof.■

We here prove $grt(K_{jt}) = \Phi_j > 0$. Before proving it we prepare the following lemma.

Lemma 4: $\dfrac{\psi_j}{\Gamma_j(1-a_j m_j)} + G_j = \dfrac{r-s+ga_j d_j}{1-ga_j m_j}$ for any $j \in N$. ■

Proof: We have

$$\frac{\psi_j}{\Gamma_j(1-a_j m_j)} + G_j = \frac{1}{1-a_j m_j} \left\{ \frac{a_j d_j + a_j m_j \bullet \frac{r-s+a_j d_j}{1-a_j m_j}}{\frac{1-ga_j m_j}{(g-1)(1-a_j m_j)}} + r - s + a_j d_j \right\}$$

$$= \frac{1}{1-a_j m_j} \left\{ \frac{(g-1)(1-a_j m_j)\left(a_j d_j + a_j m_j \bullet \frac{r-s+a_j d_j}{1-a_j m_j} \right)}{1-ga_j m_j} + r - s + a_j d_j \right\}$$

$$= \frac{(g-1)(1-a_j m_j)\left(a_j d_j + a_j m_j \bullet \frac{r-s+a_j d_j}{1-a_j m_j} \right) + (1-ga_j m_j)(r-s+a_j d_j)}{(1-a_j m_j)(1-ga_j m_j)}$$

$$= \frac{(g-1)\{(1-a_j m_j)a_j d_j + a_j m_j (r-s+a_j d_j)\} + (1-ga_j m_j)(r-s+a_j d_j)}{(1-a_j m_j)(1-ga_j m_j)}$$

$$= \frac{(1-a_j m_j)ga_j d_j + (1-a_j m_j)(r-s)}{(1-a_j m_j)(1-ga_j m_j)} = \frac{r-s+ga_j d_j}{1-ga_j m_j}.$$

This completes the proof.■

With Lemma 4, we obtain the following.

Lemma 5: $\Phi_j > 0$ for any $j \in N$.■

Proof: Since $k^* \in \Xi$, from Assumption 1and Lemma 4, we have

$$\Phi_j = \frac{\psi_{k^*}}{\Gamma_{k^*}(1-a_j m_j)} + G_j \geq \frac{\psi_j}{\Gamma_j(1-a_j m_j)} + G_j = \frac{r-s+ga_j d_j}{1-ga_j m_j} > 0.$$

This completes the proof.■

We here define

$$\Xi \equiv Arg \max\{k \in N : \frac{\psi_k}{\Gamma_k}\}.$$

Then, we have the following result.

Lemma 6: $k^* \in \Xi$ and $\frac{\psi_k}{\Gamma_k} = \frac{\psi_j}{\Gamma_j}$ for any $k, j \in \Xi$.■

Proof: We start with the proof of the first half. We define

$$\rho_{jt} = \frac{\Theta_t}{e^{-(\psi_j / \Gamma_j)t}} \quad j \in N.$$

Then, we have

(15) $$1 = \sum_{k \in N} \Omega_k \Theta_t^{\Gamma_k} e^{\psi_k t} = \sum_{k \in N} \Omega_k \rho_{kt} \geq . \rho_{jt} \geq 0 \quad \text{for any} \quad j \in N.$$

We assume $k^* \in \Xi^C$ (the complement of Ξ). We here arbitrarily choose $k^\# \in \Xi$. Then, we have

$$\Theta_t = \rho_{k^*t} e^{-(\psi_{k^*} / \Gamma_{k^*})t} = \rho_{k^\# t} e^{-(\psi_{k^\#} / \Gamma_{k^\#})t}.$$

Then, Lemma 1 yields $\alpha \leq \Theta_t^{\Gamma_{k^*}} e^{\psi_{k^*} t} = \rho_{k^* t} \leq \beta$ and we have

(16) $$\alpha \leq \rho_{k^* t} = \rho_{k^\# t} e^{\{(\psi_{k^*} / \Gamma_{k^*}) - (\psi_{k^\#} / \Gamma_{k^\#})\}t} \leq \beta.$$

Since $k^{\#} \in \Xi$ and we assume $k^{*} \in \Xi^{C}$, the definition of Ξ yields

$$\psi_{k^*}/\Gamma_{k^*} - \psi_{k^\#}/\Gamma_{k^\#} < 0.$$

Therefore, we have

$$\lim_{t\to\infty} \rho_{k^*t} = \rho_{k^\# t} e^{\{(\psi_{k^*}/\Gamma_{k^*})-(\psi_{k^\#}/\Gamma_{k^\#})\}t} = 0.$$

This contradicts (16). This contradiction proves $k^{*} \in \Xi$. This completes the proof of the first half. The proof of the latter half proof follows directly from the definition of Ξ.■

We now obtain the following first main result:

Lemma 7: $grt(C_j) = m_j\Phi_j + d_j$ and $grt(P_j) = (1-m_j)\Phi_j - d_j$ for any $j \in N$.■

Proof: From Eqs. (2) and (3), we have

$$\frac{C_{jt}}{e^{(m_j\Phi_j + d_j)t}} = \frac{e^{d_j t}K_{jt}^{m_j}}{e^{(m_j\Phi_j + d_j)t}} = \left(\frac{K_{jt}}{e^{\Phi_j t}}\right)^{m_j},$$

$$\frac{P_{jt}}{e^{\{(1-m_j)\Phi_j - d_j\}t}} = \frac{e^{-d_j}K_{jt}^{1-m_j}/m}{e^{\{(1-m_j)\Phi_j - d_j\}t}} = \left(\frac{K_{jt}}{e^{\Phi_j t}}\right)^{m_j - 1} / m^{1/(m_j-1)},$$

for any $j \in N$. From the definition of GRT, this completes the proof.■

We now arrive at the GRT of relative prices:

Lemma 8: $grt(P_{jt}) = grt(K_{tt}) - grt(C_{jt})$ for any $j \in N$.■

Proof: The proof follows directly from Lemma 7.■

Before deriving the equilibrium growth path of K_I .We here prove the following lemma.

Lemma 9: Consider the differential equation:

(17) $\quad \dot{z}(t) = f(t)z(t) - v(t)e^{Bt}.$

where

(18.1) $v(t) > 0$ for any $t > 0$,

(18.2) $\sup\{v(t) : t \ge 0\} < +\infty$,

(18.3) $f(t)$ is convergent as $t \to \infty$ and $\lim_{t\to\infty} f(t) > B$.

Define

$$U(t) \equiv \int_{[0.t]} v(u)\, e^{\int_{[0,u]}\{B-f(w)\}dw} du$$

Then, $U(t)$ is convergent as $t \to \infty$. As a solution of equation (17) , we have

(19) $$z(t) = \{z_0 - U(t)\} e^{\int_{[0,t]} f(w)dw}.$$

where $z_0 = \lim_{t\to\infty} U(t)$. Then, we have $z(0) = z_0$ and for any $t > 0$.■

Proof: In the following, we prove that $U(t)$ is convergent as $t \to \infty$. From condition (18.3), we see that there exists a $T > 0$ such that

(20) $$-R = \sup\{B - f(t) : t > T\} < 0.$$

Now, define

$$H \equiv e^{\int_{[0,T]}\{B-f(w)\}dw} \quad \text{and} \quad \Lambda(t) \equiv \int_{[T.t]} v(u) e^{-R(u-T)} du.$$

Then, it follows from (20) that for any $t > T$

(21) $$\Xi(t) \equiv \int_{[T.t]} v(u) e^{\int_{[0,u]}\{B-f(w)\}dw} du = \int_{[T.t]} v(u) e^{\int_{[T,u]}\{B-f(w)\}dw} \bullet e^{\int_{[0,T]}\{B-f(w)\}dw} du$$
$$\le H \int_{[T.t]} v(u) e^{-R(u-T)} du \equiv H\Lambda(t).$$

We first prove that $\Lambda(t)$ is convergent as $t \to \infty$ Condition (18.1) shows that $\Lambda(t)$ is monotonously increasing. From condition (18.2), we have for any $t > T$

$$\Lambda(t)=\int_{[T.t]} v(u)e^{-R(u-T)}du<\sup\{v(t):t\geq T\}\bullet\frac{1-e^{R(T-t)}}{V}.$$

This implies that $\Lambda(t)$ is bounded from above. Since $\Lambda(t)$ is monotonously increasing, $\Lambda(t)$ is convergent as $t\to\infty$. On the other hand, Condition (18.1) shows that $\Xi(t)$ is monotonously increasing. Therefore, from (20) and convergence of $\Lambda(t)$ we see that $\Xi(t)$ converges as $t\to\infty$, so that as $t\to\infty$

$$U(t)=\int_{[T.t]} v(u)e^{\int_{[T,u]}\{B-f(w)\}dw}du+\int_{[0,T]} v(u)e^{\int_{[0,u]}\{B-f(w)\}dw}du$$

$$\to\ \lim_{t\to\infty}\Omega(t)+\int_{[0,T]} v(u)e^{\int_{[0,u]}\{B-f(w)\}dw}du.$$

Therefore, we see that $U(t)$ is bounded from above convergent as $t\to\infty$. On the other hand, it follows from condition (18.1) that $U(t)$ is monotonously increasing. Therefore we see that $U(t)$ is convergent as $t\to\infty$. We next prove that $z(t)>0$ for any $t\geq 0$. Condition (18.1) shows that $U(t)$ is a strictly monotone function. Therefore, from the definition of z_0, we see that $z_0>U(t)$ for any $t\geq 0$. This shows that $z(t)>0$ for any $t\geq 0$. Since $U(0)=0$, it follows directly from equation (19) that $z(0)=z_0$. Finally, we prove that Equation (19) is a solution of Equation (17). We have

$$\begin{aligned}\dot{z}(t)&=\{z_0-U(t)\}f(u)e^{\int_{[0,t]} f(u)du}-U'(t)e^{\int_{[0,t]} f(u)du}\\&=\{z_0-U(t)\}f(u)e^{\int_{[0,t]} f(u)du}-v(t)e^{\int_{[0,t]}\{B-f(u)\}du}e^{\int_{[0,t]} f(u)du}\\&=f(t)z(t)-v(t)e^{Bt}.\end{aligned}$$

This implies that Equation (19) is a solution of Equation (17). Thus we complete the proof.■

Using Lemma 9, we start to derive the equilibrium growth path of G_I. In the following, we assume the following:

Assumption 2: We assume $\Phi_{\max}\equiv\max\{\Phi_j : j\in N\}<r.$

For Assumption 2, it should be noted that Lemma 4 yields $\Phi_j = G_j$ for any $j \in \Xi$. We here define

$$\varphi_{jt} \equiv \frac{K_{jt}}{e^{\Phi_j t}}, \qquad j \in N.$$

Since we have $\Pi_{jt} = P_{jt}Q_{jt} - K_{jt} = P_{jt}C_{jt} - K_{jt}$ it follows from (1.3) that

$$\text{(22)} \qquad \dot{K}_{It} = \sum_{k \in N} \Pi_{kt} + rK_{It} - \sum_{k \in N} P_{kt}C_{kt} = rK_{It} - \sum_{k \in N} K_{kt}$$
$$= rK_{It} - \sum_{k \in N} \varphi_{kt} e^{\Phi_k t} = rK_{It} - \left\{ \sum_{k \in N} \varphi_{kt} e^{(\Phi_k - \Phi_{\max})t} \right\} e^{\Phi_{\max} t}.$$

We now obtain

Lemma 10: We define

$$F(t) \equiv \int_{[0,t]} \sum_{k \in N} \varphi_{ku} e^{(\Phi_k - r)u} du .$$

Then, $F(t)$ monotonously increases and converges as $t \to \infty$ and

$$K_{It} = \{\lim_{t \to \infty} F(t) - F(t)\} e^{rt}$$

is a solution of Equation (22).■

Proof: To use Lemma 9, we define

$$\theta(t) \equiv \sum_{k \in N} \varphi_{kt} e^{(\Phi_k - \Phi_{\max})t}.$$

Since $grt(K_{jt}) = \Phi_j > 0$, the $\varphi_{jt}-$function is bounded and therefore, the $\theta-$function is bounded. Therefore, under Assumption 2, it can be easily checked that Conditions (18) are all satisfied. We have

$$F(t) = \int_{[0,t]} \left\{ \sum_{k \in N} \varphi_{ku} e^{(\Phi_k - \Phi_{\max})u} \right\} e^{(\Phi_{\max} - r)u} du = \int_{[0,t]} \theta(u) e^{\int_{[0,u]} (\Phi_{\max} - r) dw} du.$$

We here define

$$f(t) \equiv r, \quad B \equiv \Phi_{\max}, \quad v(t) = \theta(t). \quad U(t) = F(t).$$

(22) yields

$$\dot{K}_{It} = f(t)K_{It} - v(t)e^{Bt}.$$

Thus, applying Lemma 9 to this equation, we can prove all the statements of Lemma 7. The proof follows directly from Lemma 9.■

In the following, we consider $K_{It} = \{\lim_{t\to\infty} F(t) - F(t)\}e^{rt}$ as a path of Equation 19. It follows directly from (1.3) that

$$\lim_{t\to\infty} K_{It}\eta_t = [\{\lim_{t\to\infty} F(t)\} - F(t)]\eta_0 = 0.$$

This implies that $K_{It} = [\{\lim_{t\to\infty} F(t)\} - F(t)]e^{rt}$ satisfies (1.4) and, therefore, is the equilibrium growth path of capital stock of the investment-goods industry. We now derive the GRT of K_{It}:

Lemma 11: $grt(K_{It}) = \Phi_{\max}$. ■

Proof: It follows directly from the definition that $F(0) = 0$. Therefore, Assumption 2 yields $K_{I0} = \lim_{t\to\infty} F(t)$. Thus, we have

$$\frac{K_{It}}{e^{\Phi_{\max}t}} = \{K_{I0} - F(t)\}e^{(r-\Phi_{\max})t} = \frac{K_{I0} - F(t)}{e^{(\Phi_{\max}-r)t}}.$$

Moreover, Assumption 2 yields $\Phi_j - r < \Phi_{\max} - r < 0$ for any $j \in N$, Therefore, we see from the l'Hopital's rule that

(23) $$\frac{K_{It}}{e^{\Phi_{\max}t}} = \frac{\dot{F}(t)}{(r - \Phi_{\max})e^{(\Phi_{\max}-r)t}} = \frac{\sum_{k\in N}\varphi_{kt}e^{(\Phi_k - r)t}}{(r - \Phi_{\max})e^{(\Phi_{\max}-r)t}} = \sum_{k\in N}\frac{\varphi_{kt}}{r - \Phi_{\max}}e^{(\Phi_k - \Phi_{\max})t}.$$

We have

$$\Phi_j \le \Phi_{\max}, \quad \alpha \Delta_j \le \varphi_{jt} \le \beta \Delta_j,$$

for any $j \in N$. Therefore, we see from (23) that

$$\sum_{k \in N} \frac{\alpha \Delta_k}{r - \Phi_{\max}} e^{(\Phi_k - \Phi_{\max})t} \le \frac{K_{It}}{e^{\Phi_{\max} t}} \le \sum_{k \in N} \frac{\beta \Delta_k}{r - \Phi_{\max}} e^{(\Phi_k - \Phi_{\max})t}. \tag{24}$$

We here define

$$M_\Phi \equiv \{ j \in N : \Phi_{\max} = \Phi_k \}.$$

We have

$$\sum_{k \in N} \frac{\alpha \Delta_k}{r - \Phi_{\max}} e^{(\Phi_k - \Phi_{\max})t} \ge \sum_{k \in M_\Phi} \frac{\alpha \Delta_k}{r - \Phi_{\max}},$$
$$\sum_{k \in N} \frac{\beta \Delta_k}{r - \Phi_{\max}} e^{(\Phi_k - \Phi_{\max})t} \le \sum_{k \in N} \frac{\beta \Delta_k}{r - \Phi_{\max}}.$$

Therefore, we obtain from (24) that

$$\sum_{k \in M_\Phi} \frac{\alpha \Delta_k}{r - \Phi_{\max}} \le \frac{K_{It}}{e^{\Phi_{\max} t}} \le \sum_{k \in N} \frac{\beta \Delta_k}{r - \Phi_{\max}}. \tag{25}$$

Since the light and left hands of (25) is constant, (25) implies that $grt(K_{It}) = \Phi_{\max}$. This completes the proof.■

We are now in the position to define η_0.

$$K_{j0} = \left(\frac{a_j b_j m_j}{\eta_0 \Theta_0} \right)^{\frac{1}{1 - a_j m_j}} \equiv W \bullet \left(\frac{1}{\eta_0} \right)^{\frac{-1}{1 - a_j m_j}},$$

where $j \in N$. From the definition, we have

$$K = K_{I0} + \sum_{k \in N} K_{k0} = K_{I0} + W \sum_{k \in N} \left(\frac{1}{\eta_0} \right)^{\frac{1}{1 - a_k m_k}}. \tag{26}$$

Before defining η_0, we prove the following result.

Lemma 12: $H_{jt} \equiv \int_{[0,t]} \left\{ \frac{a_j b_j m_j}{\Theta_u} \right\}^{\frac{1}{1-a_j m_j}} e^{(\Phi_j - r)u} du$ converges for any $j \in N$.■

Proof: It follows directly from the definition that $F(0) = 0$. Therefore, Assumption 2 yields $K_{I0} = \lim_{t\to\infty} F(t)$. Moreover, we obtain from the definition that

$$
\begin{aligned}
(27) \qquad F(t) &= \int_{[0,t]} \left\{ \sum_{k\in N} \varphi_{ku} e^{(\Phi_k - \Phi_{\max})u} \right\} e^{(\Phi_{\max} - r)u} du \\
&= \int_{[0,t]} \sum_{k\in N} \varphi_{ku} e^{(G_k - r)u} du \\
&= \int_{[0,t]} \sum\nolimits_{k\in N} \left\{ \frac{a_k b_k m_k}{\eta_0 \Theta_u} \right\}^{\frac{1}{1-a_k m_k}} e^{(G_k - r)u} du \\
&\equiv \sum_{k\in N} \left[\left(\frac{a_k b_k m_k}{\eta_0} \right)^{\frac{1}{1-a_k m_k}} \int_{[0,t]} \left\{ \frac{a_k b_k m_k}{\Theta_u} \right\}^{\frac{1}{1-a_k m_k}} e^{(\Phi_k - r)u} du \right] \ge Z \sum_{k\in N} H_{kt},
\end{aligned}
$$

where

$$
Z \equiv \min \left\{ \left(\frac{a_j b_j m_j}{\eta_0} \right)^{\frac{1}{1-a_j m_j}} : j \in N \right\}.
$$

$F(t)$ is an increasing function. Since $F(t)$ converges, we have

$$
K_{I0} = \lim_{t\to\infty} F(t) = \sup\{F(t) : t \ge 0\} < +\infty.
$$

Then, we obtain from (27) that

$$
(28) \qquad K_{I0} / Z \ge F(t) / Z \ge \sum_{k\in N} H_{kt}.
$$

Since $\Theta(t) > 0$, it follows directly from that $H_j(t)$ is an increasing function for any $j \in N$, Therefore, we see from (28) that $H_j(t)$ converges. This completes the proof.■

Noting Lemma 12, we define

$$\lim_{t\to\infty} H_{jt} = \sup\{H_{jt} : t \geq 0\} \equiv H_j^{\sup},$$

where $j \in N$. Then, we obtain

$$K_{I0} = \lim_{t\to\infty} \int_{[0,t]} \sum_{k\in N} \left\{\frac{a_k b_k m_k}{\eta_0 \Theta_u}\right\}^{\frac{1}{1-a_k m_k}} e^{(G_k - r)u} du$$

$$= \sum_{k\in N}\left[\left(\frac{a_k b_k m_k}{\eta_0}\right)^{\frac{1}{1-a_k m_k}} \bullet \{\lim_{t\to\infty} H_{kt}\}\right] = \sum_{k\in N} H_k^{\sup}\left(\frac{a_k b_k m_k}{\eta_0}\right)^{\frac{1}{1-a_k m_k}}.$$

Thus, from (26), we have the following:

$$\text{(29)} \qquad K = \sum_{k\in N} H_k^{\sup}\left(\frac{a_k b_k m_k}{\eta_0}\right)^{\frac{1}{1-a_k m_k}} + W \sum_{k\in N}\left(\frac{1}{\eta_0}\right)^{\frac{1}{1-a_k m_k}}.$$

Therefore, we have $\partial K / \partial \eta_0 < 0$ and (29) is uniquely solved for η_0. Thus, the value of η_0 is determined by (29).

Finally, we consider the case of $g = 1$. We prove that the GRT in the case is given as the limitation of the case where $g \neq 1$:

Lemma 13: $\lim_{g\to 1} grt(K_{jt}) = G_j$ for any $j \in N$ and $\lim_{g\to 1} grt(K_{It}) = G_{\max}$. ■

Proof: It should be noted here that ψ_{k^*} and G_j do not depend on the parameter g. We have

$$\lim_{g\to 1}\left|\Gamma_{k^*}\right| = \lim_{g\to 1}\left|\frac{1 - g a_{k^*} m_{k^*}}{(1-g)(1-a_{k^*} m_{k^*})}\right| = \infty.$$

Since ψ_{k^*} and G_j do not depend on the parameter g, we have

$$\text{(30)} \qquad \lim_{g\to 1} grt(K_{jt}) = \lim_{g\to 1} \Phi_j = \lim_{g\to 1}\left(\frac{\psi_{k^*}}{\Gamma_{k^*}(1 - a_j m_j)} + G_j\right) = G_j$$

for any $j \in N$

$$\lim_{g\to 1} grt(K_{It}) = \lim_{g\to 1} \Phi_{\max} = \lim_{g\to 1} \max\{\Phi_j : j \in N\}$$
$$= \max\{\lim_{g\to 1} \Phi_j : j \in N\} = \max\{G_j : j \in N\} = G_{\max}.$$

This completes the proof.■

Before providing the results on equilibrium growth paths, we here make one remark. From (30), we see that Assumptions 1 and 2 are the same as of Assumptions 1 and 2 of Dohtani (2019):

Lemma 14: In the case of $g = 1$, Assumption 1 and 2 becomes the followings, respectively.

(31.1) $1 > a_j m_j$;

(31.2) $r > G_j = \dfrac{r - s + a_j d_j}{1 - a_j m_j}$ for any $j \in N$. ■

Proof of Lemma 14: Inequality (31.1) follows directly from Assumption 1 and $g = 1$. On the other hand, Assumption 2 yields $r > \max\{\Phi_j : j \in N\}$ for any $j \in N$. We here consider the case of $g > 1$. In this case, from Assumption 1, the definition of ψ_k, and (10), we have $\Gamma_j > 0$ and $\psi_j > 0$ for any $j \in N$. Moreover, (10) shows that Γ_j is an increasing function. Therefore, Assumption 2 and (30) yield

$$r > \max\{\Phi_j = \frac{\psi_{k^*}}{\Gamma_{k^*}(1 - a_j m_j)} + G_j : j \in N\}$$
$$\geq \lim_{g \to 1} \max\{\Phi_j = \frac{\psi_{k^*}}{\Gamma_{k^*}(1 - a_j m_j)} + G_j : j \in N\} = \max\{G_j : j \in N\}.$$

This completes the proof.■

From Lemmas 3, 7, 8, 11 and 13, for any $g \in R^1$, we now arrive at the following result that gives the generalizations of Theorems 1 and 2 of Dohtani (2019):

Theorem 1: Suppose Assumptions 1 and 2 are satisfied. Then there exist equilibrium growth paths which satisfy that for any $j \in N$

$$grt(K_{jt}) = grt(I_{jt}) = \Phi_j, \quad grt(C_{jt}) = grt(Q_{jt}) = m_j \Phi_j + d_j,$$
$$grt(P_{jt}) = (1 - m_j)\Phi_j - d_j, \quad grt(K_{It}) = \Phi_{\max}.$$

Moreover, for the GRT of relative price, we obtain that

$$grt(P_{jt}) = grt(K_{jt}) - grt(C_{jt})$$

for any $j \in N$.■

Moreover, we obtain the following results.

Theorem 2: Suppose Assumptions 1 and 2 are satisfied. Then, for the GRTs, we have

the rate of technological progress $d_j \uparrow$

$\Rightarrow \; grt(C_{jt})\uparrow, \; grt(P_{jt})\downarrow, \; grt(K_{jt})\uparrow, \; grt(K_{It})\uparrow$ or $\rightarrow$

the elasticity of marginal productivity $(1-m_j)\downarrow \; (\Leftrightarrow \; m_j \uparrow)$

$\Rightarrow \; gr(C_{jt})\uparrow, gr(P_{jt})\downarrow, \; gr(K_{jt})\uparrow, \; agr(K_{It})\uparrow$ or $\rightarrow$,

the elasticity of marginal utility $(1-a_j)\downarrow \; (\Leftrightarrow \; a_j \uparrow)$

$\Rightarrow \; gr(C_{jt})\uparrow, gr(P_{jt})\uparrow, \; gr(K_{jt})\uparrow, \; agr(K_{It})\uparrow$ or $\rightarrow$,

the rate of time preference $s\uparrow$

$\Rightarrow \; grt(C_{jt})\downarrow, \; grt(P_{jt})\downarrow, \; grt(K_{jt})\downarrow, \; grt(K_{It})\downarrow$,

for any $j \in N$, where $\uparrow$ implies "increases" and $\rightarrow$ implies "be invariant".■

Proof of Theorem 2: Firstly, excepting for the proof of "$grt(K_{It})\downarrow$ or $\rightarrow$", we prove Theorem 2. We start with the case where $k^* \neq j$. We have

$$\Phi_j = \frac{\psi_{k^*}}{\Gamma_{k^*}(1-a_j m_j)} + G_j = \frac{\psi_{k^*}}{\Gamma_{k^*}(1-a_j m_j)} + \frac{r-s+a_j d_j}{1-a_j m_j}$$

$$= \frac{1}{1-a_j m_j}\left(\frac{a_{k^*}d_{k^*} + a_{k^*}m_{k^*} \bullet \dfrac{r-s+a_{k^*}d_{k^*}}{1-a_{k^*}m_{k^*}}}{\Gamma_{k^*}} + r - s + a_j d_j \right) \equiv \frac{Q_j}{1-a_j m_j},$$

for any $j \in N$. Therefore, we see

(31.1) $$\frac{\partial \Phi_j}{\partial d_j} = \frac{a_j}{1-a_j m_j} > 0.$$

(31.2) $$\frac{\partial \Phi_j}{\partial m_j} = \frac{a_j \Xi_j}{(1-a_j m_j)^2} > 0,$$

(31.3) $$\frac{\partial \Phi_j}{\partial a_j} = \frac{d_j(1-a_j m_j) + m_j \Xi_j}{(1-a_j m_j)^2} > 0,$$

for any $j \in N$. Thus, we have

(32.1) $$\frac{\partial grt(K_{jt})}{\partial d_j} = \frac{\partial grt(I_{jt})}{\partial d_j} = \frac{\partial \Phi_j}{\partial d_j} > 0,$$

(32.2) $$\frac{\partial grt(C_{jt})}{\partial d_j}=\frac{\partial(m_j\Phi_j+d_j)}{\partial d_j}=m_j\frac{\partial\Phi_j}{\partial d_j}+1>0,$$

(32.3) $$\frac{\partial grt(P_{jt})}{\partial d_j}=\frac{\partial\{(1-m_j)\Phi_j-d_j)}{\partial d_j}=\frac{(1-m_j)a_j}{1-a_jm_j}-1=-\frac{1-a_j}{1-a_jm_j}<0,$$

(32.4) $$\frac{\partial gr(K_{jt})}{\partial m_j}=\frac{\partial gr(I_{jt})}{\partial m_j}=\frac{\partial\Phi_j}{\partial m_j}>0,$$

(32.5) $$\frac{\partial gr(C_{jt})}{\partial m_j}=\Phi_j+m_j\frac{\partial\Phi_j}{\partial m_j}>0\,,$$

(32.6) $$\frac{\partial gr(P_{jt})}{\partial m_j}=-\Phi_j+(1-m_j)\frac{\partial\Phi_j}{\partial m_j}=\frac{-Q_j}{1-a_jm_j}+\frac{(1-m_j)a_jQ_j}{(1-a_jm_j)^2}=\frac{-(1-a_j)Q_j}{(1-a_jm_j)^2}<0\,,$$

(32.7) $$\frac{\partial gr(K_{jt})}{\partial a_j}=\frac{\partial gr(I_{jt})}{\partial a_j}=\frac{\partial\Phi_j}{\partial a_j}>0,$$

(32.8) $$\frac{\partial gr(C_{jt})}{\partial a_j}=m_j\frac{\partial\Phi_j}{\partial a_j}>0\,,$$

(32.9) $$\frac{\partial gr(P_{jt})}{\partial a_j}=(1-m_j)\frac{\partial\Phi_j}{\partial a_j}>0\,,$$

for any $j\in N$. Moreover, we have

(33) $$\begin{aligned}\frac{\partial\Phi_j}{\partial s}&=\frac{1}{1-a_jm_j}\left\{\frac{-a_{k^*}m_{k^*}}{\Gamma_{k^*}(1-a_{k^*}m_{k^*})}-1\right\}\\&=\frac{-1}{(1-a_jm_j)\Gamma_{k^*}}\left\{\frac{a_{k^*}m_{k^*}}{1-a_{k^*}m_{k^*}}+\frac{1-ga_{k^*}m_{k^*}}{(g-1)(1-a_{k^*}m_{k^*})}\right\}\\&=\frac{-\{(g-1)a_{k^*}m_{k^*}+1-ga_{k^*}m_{k^*}\}}{(1-a_jm_j)\Gamma_{k^*}(g-1)(1-a_{k^*}m_{k^*})}=\frac{-(1-a_{k^*}m_{k^*})}{(1-a_jm_j)(1-ga_{k^*}m_{k^*})}<0,\end{aligned}$$

for any $j\in N$. Therefore, we have

(34.1) $$\frac{\partial grt(K_{jt})}{\partial s}=\frac{\partial\Phi_j}{\partial s}<0,$$

(34.2) $$\frac{\partial grt(C_{jt})}{\partial s}=\frac{\partial(m_j\Phi_j+d_j)}{\partial a_j}=m_j\frac{\partial\Phi_j}{\partial s}<0,$$

(34.3) $$\frac{\partial grt(P_{jt})}{\partial s}=\frac{\partial\{(1-m_j)\Phi_j-d_j)}{\partial s}=(1-m_j)\frac{\partial\Phi_j}{\partial s}<0,$$

for any $j\in N$. Thus, we complete the proof in the case where $k^*\neq j$. We next consider the case where $j=k^*\in\Pi$. In this case, Lemma 4 yields $\Phi_j=(r-s+ga_jd_j)/(1-ga_jm_j)$.

Then, the proof of the case follows from that of Corollary 1 of Dohtani (2019). Excepting for the proof of "$grt(K_{It})\downarrow$ or $\rightarrow$", we complete the proof in the case of $k^* = j$. Finally, since $\Phi_{\max} = \max\{\Phi_j : j \in N\}$, the proof of "$grt(K_{It})\downarrow$ or $\rightarrow$" follows directly from the proof above. Thus, we complete the proof of Theorem 2.■

Theorem 3: Suppose Assumptions 1 and 2 are satisfied. Then, as a_j (resp. m_j) $(j \in N)$ becomes large[2], the effect of the rate of technological progress d_j on the GRT of C_{jt} and K_{jt} become large (resp. small). That is

$$\frac{\partial^2 grt(C_{jt})}{\partial a_j \partial d_j} > 0, \quad \frac{\partial^2 grt(K_{jt})}{\partial a_j \partial d_j} = \frac{\partial^2 grt(I_{jt})}{\partial a_j \partial d_j} > 0, \quad \frac{\partial^2 grt(C_{jt})}{\partial m_j \partial d_j} > 0, \text{ and}$$
$$\frac{\partial^2 grt(K_{jt})}{\partial m_j \partial d_j} = \frac{\partial^2 grt(I_{jt})}{\partial m_j \partial d_j} > 0,$$

for any $j \in N$, As a_j (resp. m_j) $(j \in N)$ becomes large, the effect of the rate of technological progress d_j on the GRT of P_{jt} becomes large (resp. small):

$$\frac{\partial^2 grt(P_{jt})}{\partial a_j \partial d_j} > 0 \text{ and } \frac{\partial^2 grt(P_{jt})}{\partial m_j \partial d_j} < 0,$$

for any $j \in N$.■

Proof of Theorem 3: We first consider the case where $k^* \neq j$. (31) yields

(35.1) $$\frac{\partial^2 \Phi_j}{\partial a_j \partial d_j} = \frac{\partial}{\partial a_j}\left(\frac{a_j}{1-a_j m_j}\right) = \frac{1}{(1-a_j m_j)^2} > 0,$$

(35.2) $$\frac{\partial^2 \Phi_j}{\partial m_j \partial d_j} = \frac{\partial}{\partial m_j}\left(\frac{a_j}{1-a_j m_j}\right) = \frac{a_j^2}{(1-a_j m_j)^2} > 0,$$

for any $j \in N$. Therefore, (31), (32.1), (32.2) and (35) yield

(36.1) $$\frac{\partial^2 grt(K_{jt})}{\partial a_j \partial d_j} = \frac{\partial^2 \Phi_j}{\partial a_j \partial d_j} > 0,$$

(36.2) $$\frac{\partial^2 grt(K_{jt})}{\partial m_j \partial d_j} = \frac{\partial^2 \Phi_j}{\partial m_j \partial d_j} > 0,$$

(36.1) $$\frac{\partial^2 grt(C_{jt})}{\partial a_j \partial d_j} = m_j \frac{\partial^2 \Phi_j}{\partial a_j \partial d_j} > 0,$$

---

[2] Although the CES utility function is homogeneous of degree one, we consider a more general CES utility function. It should be noted here that a change of a_j implies that the degree of homogeneity of CES utility function increases.

(36.3) $$\frac{\partial^2 grt(C_{jt})}{\partial m_j \partial d_j} = \frac{\partial \Phi_j}{\partial d_j} + m_j \frac{\partial^2 \Phi_j}{\partial m_j \partial d_j} > 0,$$

for any $j \in N$. Moreover, (32.3) and (35.2) yield

(37.2) $$\frac{\partial^2 grt(P_{jt})}{\partial a_j \partial d_j} = (1-m_j)\frac{\partial^2 \Phi_j}{\partial a_j \partial d_j} > 0,$$

(37.1) $$\frac{\partial^2 grt(P_{jt})}{\partial m_j \partial d_j} = \frac{\partial}{\partial m_j}\left(-\frac{1-a_j}{1-a_j m_j}\right) = -\frac{(1-a_j)a_j}{(1-a_j m_j)^2} < 0,$$

for any $j \in N$. Thus, (36) and (37) complete the proof in the case where $k^* \neq j$. We next consider the case where $k^* = j$. In this case, as stated in the proof of Theorem 2, we have $\Phi_j = (r - s + g a_j d_j)/(1 - g a_j m_j)$. Therefore, we see

(38) $$\frac{\partial \Phi_j}{\partial d_j} = \frac{g a_j}{1 - g a_j m_j}.$$

for any $j \in N$. Therefore, we obtain

(39.1) $$\frac{\partial^2 grt(K_{jt})}{\partial a_j \partial d_j} = \frac{\partial^2 grt(I_{jt})}{\partial a_j \partial d_j} = \frac{\partial^2 \Phi_j}{\partial a_j \partial d_j} = \frac{\partial}{\partial a_j}\left(\frac{g a_j}{1 - g a_j m_j}\right) = \frac{g}{(1 - g a_j m_j)^2} > 0,$$

(39.2) $$\frac{\partial^2 grt(K_{jt})}{\partial m_j \partial d_j} = \frac{\partial^2 grt(I_{jt})}{\partial m_j \partial d_j} = \frac{\partial^2 \Phi_j}{\partial m_j \partial d_j} = \frac{\partial}{\partial m_j}\left(\frac{g a_j}{1 - g a_j m_j}\right) = \frac{g^2 a_j{}^2}{(1 - g a_j m_j)^2} > 0.$$

(39.3) $$\frac{\partial^2 grt(C_{jt})}{\partial a_j \partial d_j} = m_j \frac{\partial^2 \Phi_j}{\partial a_j \partial d_j} > 0,$$

(39.4) $$\frac{\partial^2 grt(C_{jt})}{\partial m_j \partial d_j} = m_j \frac{\partial^2 \Phi_j}{\partial m_j \partial d_j} + \frac{\partial \Phi_j}{\partial d_j} > 0,$$

(39.5) $$\frac{\partial^2 grt(P_{jt})}{\partial a_j \partial d_j} = (1-m_j)\frac{\partial^2 \Phi_j}{\partial a_j \partial d_j} > 0,$$

for any $j \in N$. Moreover, (37) and (39) yield

(40) $$\frac{\partial^2 grt(P_{jt})}{\partial m_j \partial d_j} = -\frac{\partial \Phi_j}{\partial d_j} + (1-m_j)\frac{\partial^2 \Phi_j}{\partial m_j \partial d_j} = -\frac{g a_j}{1 - g a_j m_j} + (1-m_j)\frac{g^2 a_j^2}{(1 - g a_j m_j)^2}$$
$$= -\frac{(1 - g a_j) g a_j}{(1 - g a_j m_j)^2} < 0,$$

for any $j \in N$. (39) and (40) complete the proof in the case where $k^* = j$. Thus, we complete all the proof of Theorem 3.■

3. Conclusions and Final Remark

In this paper, we try to generalize the results on the growth model with the additive utility function, which were obtained by Dohtani (2019). Firstly, to generalize such a growth model, we constructed the growth model with the CES utility function. Secondly, we defined the growth rate of trend that gives a generalization of the growth rate in the usual sense. Thirdly, by using such a notion, we derived almost the same results as those obtained by Dohtani (2019). Especially, owing to the notion, we succeeded to derive a fundamental equation that gives an analytical relation among the growth rates of relative price, consumption and capital stock of each industry.

We expect that the notion introduced in this paper or a further modified version of the notion will be effectively utilized to discuss growth rates in more extensive growth models. However, we must leave it for future researches to construct extensive versions.

Reference

Dohtani, A. (2019): "An Endogenous Model of Heterogeneous Growth Ⅰ: Additive Utility Function," Working Paper No. ???, Faculty of Economics, University of Toyama.