Entanglement and methods for its quantification

B.Sc Thesis
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Contents

Abstract

The aim of this thesis is to introduce the quantum phenomena called entanglement and theoretical models for quantifying it in the context of mainly bipartite systems. The approach taken is in terms of quantum information and the thesis discusses the uses of entanglement in this field.

1 Introduction

Entanglement as a resource is one of the cornerstones on which we are currently attempting to build information-processing units of non-classical nature. Although it is not currently very clear if quantum entanglement is an absolute requirement for a general quantum computer, entanglement is still the most popular and best known candidate for utilizing the non-classical effects of the quantum world.

Foundations for the principle of entanglement go back to the 1930s and the famous Einstein-Polsky-Rosen(EPR) - article [1]. After this initial discovery, entanglement remained somewhat poorly understood phenomena till the late 1990s with the relative explosion of new research with real contributions to new understanding. A major push to the research interest was the emerging field of quantum computing and the research that clarified the nature of quantum information entropy by Schumacher [2].

The emergence of entanglement in quantum mechanics is a direct result of the first postulate of quantum mechanics, which states that the quantum system is completely described by a wave funtion dependant on location in space and time. When defining a general quantum state, we require a few mathematical rules to hold true, namely continuity, finiteness and linear mapping. These properties alone imply that the linear system superposition principle holds. This principle also generalises to multi-particle states easily. Thus entanglement is fundamentally the superposition principle generalized to a composite system. Mathematical proof, beyond what was discussed, is omitted, being fundamentally simple in nature but somewhat laborious. Concrete experimental proof exists in multitudes and some can be found in Refs. [3; 4; 5].

Quantum entanglement can be discussed in relation to many types of systems. The path taken in this thesis approaches the problem mainly from the point of view of quantum information and specifically the unit of qubit. There exists theoretical models of entanglement for different kinds of quantum mechanical units like harmonic oscillators and other such systems with more than two states. These different kinds of systems would require a whole different framework to be developed and in many cases does not exist to a meaningful degree.

The work starts with a brief introduction into quantum information, from which we proceed to introduce two important applications of entanglement, super-dense coding and quantum teleportation. From here we continue to the actual theoretical framework of two-particle entanglement and define the existence and entanglement criteria for both pure and mixed states of these systems. The treatment of bipartite systems continues by examining mixed state entanglement measures a bit closer and the two-particle chapter closes with a brief but important introduction to true quantifiers of entanglement, entanglement cost and entanglement distillation. Naturally from here one continues to the larger systems of three or more qubits, which we are able to introduce briefly, and we also discuss the problems and details related to these systems. The thesis ends with a discussion portion that touches on most of the topics discussed previously.

2 Quantum information theory

Quantum information theory is partly a sub-discipline of the general quantum theory, with mixed in principles from traditional information theory. It focuses in the possibilities of information processing by the use of quantum phenomena. These new computational methods and processes are of great interest for their promise of additional computing power for certain traditionally very demanding problems. The new quantum mechanical unit of processing, the qubit, might seem trivial on the surface, but as even the relatively simple examples will show, the qubit allows us operations that are beyond the scope of traditional bits. This alone might imply that quantum information brings with it a second, more powerful age of information.

The field of quantum information is still very young in terms of science. The impetus for developing a theory of quantum information processing came from the realisation that a quantum information processing system might be more powerful than a classical one, credited to Richard Feynman [6], dating to 1981. Some short time later it was shown by Paul Benioff that quantum systems can model classical computers [7], important step towards quantum advantage over classical information processing. Finally in the mid to late eighties, David Deutsch concluded the question by demonstrating that quantum advantage existed [8; 9]. These are some of the works that started the interest in quantum information, and the real development of the field in part picks up from the articles; Schumacher in 1995 [2], Hill & Wootters in 1997 [10] and Wootters in 1998 [11]. Seminal textbook for quantum computation and quantum information was published in 2000, incidentally carrying this exact name [12].

The fundamental questions that quantum information seeks to answer are ultimately ones that have been and continue to be of interest to traditional information processing, including topics like data searching, path optimisation and cryptography. Some famous examples of so called quantum advantage in information processing are the prime factoring problem (Shor's algorithm) and data searching (Grover's algorithm). The quantum advantage continues to be an active research topic that relies on the discoveries of new algorithms of quantum mechanical nature.

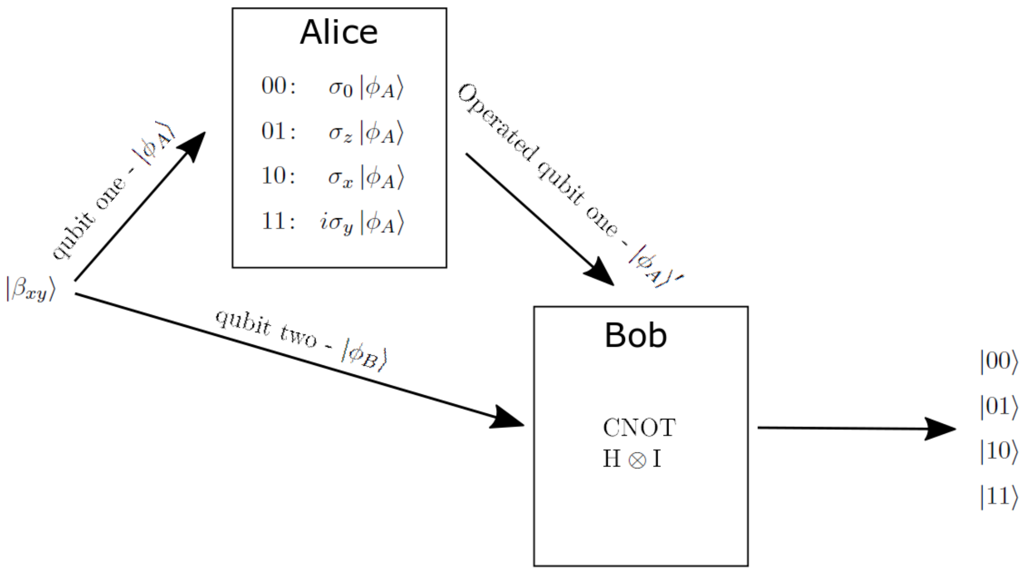

Figure 1: Schematic of the superdense coding protocol. First, an entangled Bell state is shared between Alice and Bob, where each receive their pair. After that Alice performs one of the four local operations on her qubit and sends the qubit to Bob. Then Bob measures the two-qubit state and is able to infer which of the four operations Alice did, this way transmitting two bits of classical information just by sending one quantum bit.

2.1 Entanglement in quantum information

Quantum information faces a host of problems, often titled no-go theorems [13], that restrict the behaviour and operations related to the fundamental unit of quantum processing, the qubit. Luckily there are ways to work around these problems. One of the most notable features of entanglement is its fundamental property as a carrier of both classical and quantum information, and the latter fact allows us to circumvent some of the difficulty with transporting said quantum information. This will become apparent from the forthcoming example of superdense coding and can also be realised from the fact that a measurement on an entangled state reduces the entropy of the given state.

2.1.1 Superdense coding

One example revealing some of the power of entanglement is the process titled superdense coding [12]. This way of making use of the entanglement of a pair of qubits allows two classical bits to be transferred by sending only one qubit of information. Briefly introducing the bra-ket qubit notation we can state their relation to matrices as

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix},\ |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}. \tag{1}$$

Multiplying the qubits together obey the Kronecker product

$$|0\rangle \otimes |0\rangle = |00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix},\ |1\rangle \otimes |0\rangle = |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}. \tag{2}$$

Beginning with superdense coding, we first need to introduce an important concept related to entanglement, the so called Bell basis states or EPR-pairs. These states are defined as maximally entangled states of two qubits and a way of presenting such states is as follows

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\Big(\,|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\Big), \tag{3}$$

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}\Big(\,|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B\Big), \tag{4}$$

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}\Big(\,|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B\Big), \tag{5}$$

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}\Big(\,|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\Big), \tag{6}$$

where the subscripts A and B denote the qubit that introduces the basis state.

Superdense coding is a task of transmitting information from a party A, traditionally called Alice, to the recipient, Bob, also denoted as B. These parties are defined to be an arbitrary distance apart and thus in principle the process of transferring information in this way works over any amount of distance.

Now let us say Alice has two classical bits of information that are to be sent to Bob. They go about this task first by sharing an entangled pair of qubits, for example the Bell state

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\Big(\,|0_A 0_B\rangle + |1_A 1_B\rangle\Big). \tag{7}$$

We can now call these qubits of the individual parties $|\phi_A\rangle$ for A and $|\phi_B\rangle$ for B.

After either sharing these qubits, or receiving them from a third party, they can separate. Now when Alice has decided on what classical bits she want to send, some local operations on their half of the qubit pair are performed. These local operations are denoted by the important Pauli spin matrices

$$\sigma_0 = I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix},\ \sigma_x = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix},\ \sigma_y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix},\ \sigma_z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}. \tag{8}$$

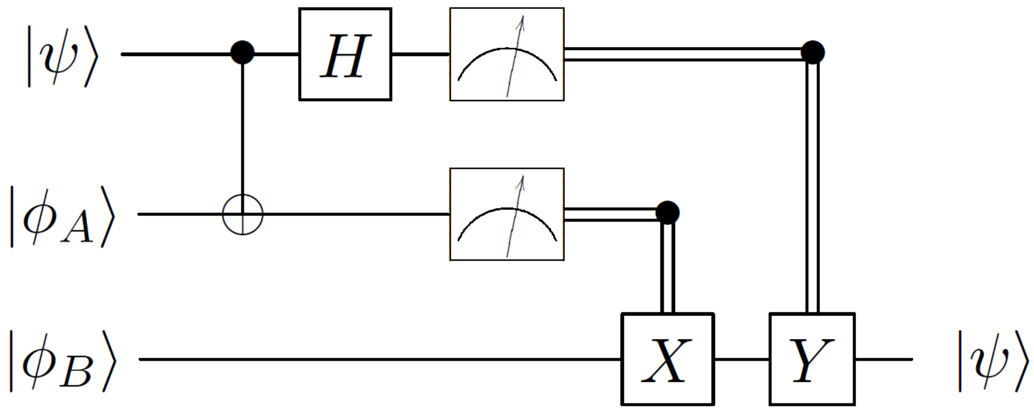

Figure 2: Quantum teleportation circuit. The protocol starts by sharing of an entangled qubit, after which the parties can separate. Now Alice, having a quantum state she wants to send, performs the required CNOT and Hadamard gates with her quantum state and the qubit pair. Then she performs two measurements and sends the results forward to Bob, classically, after which Bob performs the required operations depending on Alice's measurement outcomes, denoted by X and Y. The result is that Bob receives the Alice's quantum state.

These operations depend on the classical bits that are to be sent, illustrated in Figure 1.

Then after the local operation has been performed, that is, when the entangled quantum state has been appropriately left either intact (case 00) or operated on with the corresponding Pauli spin matrix, Alice sends her qubit to the receiver. After receiving the qubit, Bob makes a Bell basis measurement with the four possible outcomes, that is, equivalent to two bits of information. More specifically, the operations Bob performs after receiving the qubit are: first CNOT-gate with A being the control and B as the target and then $H \otimes I$ to qubit A. Hadamard-gate H is a gate that creates a superposition out of a single basis state. CNOT-gate flips the target qubit if the control qubit reads $|1\rangle$.

This is then how we are able to send two bits of classical information with the one quantum bit, in this case $|\phi_A\rangle$. We also see that this involves some trickery in that we always still had two qubits to work with, but the Alice never had classically defined access to the qubit $|\phi_B\rangle$, thus it is fair to say that two bits have indeed been transferred with one qubit.

2.1.2 Quantum teleportation

The famous phenomena of quantum teleportation utilizes the entanglement effect to carry units of quantum information across in theory arbitarily large distances. This example illustrates the power of entanglement in terms of manipulating quantum states without the loss of their quantum mechanical properties[12].

The quantum teleportation utilises the aforementioned maximally entangled pairs, the Bell states, to transport the state of a qubit across any distance, instantaneously. To an untrained eye this would seem like a violation of known physical laws, but we see this is discrepancy is rectified by the details of the phenomena.

We start again with two parties, Alice and Bob, that separate after sharing a pair of entangled qubits. We then require that Alice, the sender, has an additional qubit in a state $|\psi\rangle$ that she has not measured, i.e. the quantum state is unknown and undisturbed. The steps of transferring this qubit $|\psi\rangle$ to Bob are then the following; Alice takes her additional qubit, $|\psi\rangle$, and performs an interaction, CNOT-gate with her half of the entangled pair $|\phi_A\rangle$. The result of the CNOT-operation is then sent through a Hadamard gate. Output of the Hadamard gate and the original qubit $|\phi_A\rangle$ are measured by Alice, and these results are transferred to Bob via two classical information channels. This procedure is then finished at the Bobs end by performing operations, that depend on the information received from Alice, to the entangled qubit in his possession. This results in Bob receiving the original state $|\psi\rangle$ that was to be transferred and consuming the entanglement between the original pair. We can also illustrate this phenomena with the aid of a quantum circuit diagram, see Figure 2.

In conclusion, quantum teleportation allows the transfer of quantum mechanical information units, which in classical terms seems almost trivial. Classically we are able to send units of information freely, them being easily readable, describable, copyable and thus transferrable. The qubit on the other hand is governed by the no-cloning theorem that prohibits classical solutions to qubit transfer [14]. We can see that it can be circumvented to a degree by teleporting the qubit: the teleportation protocol allowing us to at least change the location of our information.

3 Two-particle entanglement

The phenomena of entanglement is naturally introduced with a system of two qubits, bipartite system being the most elementary system that can exhibit entanglement. Systems of qubits can be divided in two sub-categories, pure and mixed. We call a system pure when it consists of one state or a superposition between two states, while a mixed state is a statistical ensemble of pure states. This division already distinguishes some of the methods of which we can use to measure the entanglement present, even when treating only two particles.

The introduction begins by going over some mathematical tools at the core of the measures. We start by defining the density matrix. This is done mainly to get a handle on the mixed state quantum systems, but can be used in the pure state case as well. The density matrix describes the statistical state of a given quantum system and can be used to calculate outcomes of measurements on this system. The formal definition can be stated as

Definition 1. Density matrix;

$$\rho \equiv \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \tag{9}$$

where p_i denotes the probability of the system to be in the state $|\psi_i\rangle$, their sum is defined to be one $\sum_i p_i = 1$ and they obey $0 \leq p_i \leq 1$.

The density matrix, or often the density operator, has many interesting properties, one of which is relevant to entanglement, that allows us to quickly check if the given density matrix is of a pure state. For a pure state, it holds that

$$\rho = \rho^2. \tag{10}$$

By tracing the density matrix we are also able to distinguish pure and mixed states with the information that for pure states $\mathrm{Tr}(\rho^2) = 1$, and for mixed states $\mathrm{Tr}(\rho^2) < 1$. Tracing over any valid density matrix also gives us the result $\mathrm{Tr}(\rho) = 1$, which means that the eigenvalues of valid density matrices always sum to one.

Moving on to the second and very related mathematical concept, the reduced density matrix, that we define as

Definition 2. Reduced density matrix;

$$\rho^A \equiv \mathrm{Tr}_{\mathrm{B}}(\rho^{AB}), \tag{11}$$

where Tr_{B} is a partial trace over the system B, defined as

$$\mathrm{Tr}_{\mathrm{B}}(\left|a_1\right\rangle \left\langle a_2\right| \otimes \left|b_1\right\rangle \left\langle b_2\right|) \equiv \left|a_1\right\rangle \left\langle a_2\right| \mathrm{Tr}_{\mathrm{B}}(\left|b_1\right\rangle \left\langle b_2\right|). \tag{12}$$

The reduced density matrix allows us to quantify correlations within a multipartite quantum system, and to ignore unnecessary or undetected parts of a given system. This is also relevant in open quantum system time evolution, which alone signifies its importance.

Third and the final mathematical concept to be introduced is the von Neumann entropy. Moving from classical worlds entropy to the quantum equivalent, we need a new definition of entropy. This formulation is called the von Neumann entropy and it defines entropy with the aid of the density matrix formulation, which can be stated as

Definition 3. Von Neumann entropy;

$$\mathrm{S}(\rho) \equiv -\mathrm{Tr}(\rho \ln \rho), \tag{13}$$

or alternatively with the eigenvalues α_x of the density matrix

$$\mathrm{S}(\rho) = -\sum_x \alpha_x \ln \alpha_x. \tag{14}$$

Von Neumann entropy can be used to measure the purity of a given state by noting that for a given pure state, we find it to equal zero. The more physical interpretation to von Neumann entropy could be stated to mean the number of qubits required to transfer a quantum state by a source of statistical nature [2].

3.1 Pure state

Pure states are the most basic representations of objects of quantum mechanical nature that involve multiple particles. These are mostly unnatural states in the sense that virtually everything that exists is in some form of contact with the surrounding environment. Still, they are surprisingly useful and widely used. We start by defining the pure state

Definition 4. Pure state [15];

Bipartite quantum system consisting of Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, with dimensions d_A and d_B, can be defined as vectors in the tensor-product of the two Hilbert spaces $\mathcal{H}_A \otimes \mathcal{H}_B$. Then for any vector in this composite system, we have

$$|\psi\rangle = \sum_{i,j}^{d_A,d_B} c_{ij} |a_i\rangle \otimes |b_j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \tag{15}$$

where c_{ij} is a complex vector of length $d_a \times d_b$.

For pure states the probability co-efficient of equation (9) is equal to one, since we only have one possible state. In other words,

$$\rho = |\psi\rangle \langle\psi| . \tag{16}$$

This allows us to treat pure states in the bra-ket representation.

Quantifying entanglement of these states is considerably simpler than the mixed counterparts, since pure states have some simplifying mathematical properties like only one non-zero eigenvalue in their diagonal basis and the non-existence of a probabilistic nature of the mixed state. Many of the entanglement measures also reduce to the entanglement entropy because of the mathematical properties.

We are now ready to begin the definition for the entanglement of pure states, which can be formally expressed for two qubits as

Definition 5. Bipartite pure state entanglement[15];

For a pure state $|\psi\rangle \in \mathcal{H}$, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, we call it entangled if we can not find states $|\phi^A\rangle \in \mathcal{H}_\mathcal{A}$ and $|\phi^B\rangle \in \mathcal{H}_\mathcal{B}$ such that they satisfy

$$|\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle , \tag{17}$$

where $\mathcal{H}$ denotes the total Hilbert space and the indices their respective qubit. Composite states satisfying this equation are called separable or product states.

The meaning of separable or product states in physical terms is that the states exist completely void of any quantum mechanical correlations between them. Now working with the bra-ket representation, we are able to quantify entanglement with a tool called Schmidt decomposition.

Theorem 1. *Schmidt decomposition [12];*

Defining $|\psi\rangle$ as a pure state of $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, there exist orthonormal states; $|i_A\rangle$ for system A, and $|i_B\rangle$ for system B such that

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle , \tag{18}$$

where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$. These are called Schmidt co-efficients. The proof is essentially a restated form of the singular value decomposition.

Proof. Let $|a\rangle$ be any fixed orthonormal basis for system A and $|b\rangle$ similarly for system B. We can then write $|\psi\rangle$ as

$$|\psi\rangle = \sum_{ab} k_{ab} |a\rangle |b\rangle \tag{19}$$

for a matrix k with complex numbers k_{ab}. Then with the singular value decomposition [16] we decompose $k = udv$ as

$$|\psi\rangle = \sum_{iab} u_{ai} d_{ii} v_{ib} |a\rangle |b\rangle , \tag{20}$$

where u and v are unitary matrices and d is a diagonal matrix containing positive elements. Now we can define the elements of the singular value decomposition as

$$|i_A\rangle \equiv \sum_a u_{ai} |a\rangle , \; |i_B\rangle \equiv \sum_b v_{ib} |b\rangle \text{ and } d_{ii} \equiv \lambda_i. \tag{21}$$

Now combining equations (20) and (21) we end up with

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle , \tag{22}$$

which is the equation (16). □

The first notable mathematical result from Schmidt decomposition is that for a given pure state, the reduced density matrices of the subsystems have equal eigenvalues. The second useful result is that for a product state, we find that if the Schmidt rank (rank of the matrix d) equals to one, the state is a product state.

Next we define an important entanglement measure, the entropy of entanglement. All of the entropy measures introduced for bipartite states in this work reduce to the entropy of entanglement for pure states and this is the case for most but not all known measures at present overall. The formal definition of the entropy of entanglement can be stated as

Definition 6. Entropy of entanglement [17];

For a given pure bipartite state, entropy of entanglement E can be defined with the reduced density matrix of either of the subsystems

$$E(\psi) = -\mathrm{Tr}(\rho^A \ln \rho^A) = -\mathrm{Tr}(\rho^B \ln \rho^B), \tag{23}$$

or by the eigenvalues α_x of either of the reduced density matrices or by the Schmidt coefficients λ_x

$$E(\psi) = -\sum_x \alpha_x \ln \alpha_x = -\sum_x \lambda_x^2 \ln \lambda_x^2, \tag{24}$$

which can be derived easily with the Schmidt decomposition.

Notable property to this definition is that while an entangled pure state has a total entropy of zero, the reduced density matrices of the subsystems are fundamentally correlated with each other and, thus, they are in a mixed state with non-zero entropy. So a subsystem of a pure state can have a non-zero entropy while the entropy of the whole system is zero. This idea is expanded upon in the following example.

Example 1. *Bell state entanglement*

Let us examine the entropy definition of entanglement by calculating this measure for one Bell state

$$|B_{10}\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}. \tag{25}$$

The density matrix of this state can be expressed as

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| = |B_{10}\rangle \langle B_{10}| = \left(\frac{|10\rangle + |01\rangle}{\sqrt{2}}\right)\left(\frac{\langle 10| + \langle 01|}{\sqrt{2}}\right)$$

$$= \frac{|10\rangle\langle 10| + |10\rangle\langle 01| + |01\rangle\langle 10| + |01\rangle\langle 01|}{2} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{26}$$

For this matrix we calculate the eigenvalues

$$\det(\rho - \lambda I) = 0 \implies \lambda_1 = 0, \lambda_2 = 1. \tag{27}$$

Now using the alternate form of von Neumann entropy (14) we find

$$S(\rho) = -\sum_x \alpha_x \ln \alpha_x = -(3 * 0 \ln 0 + 1 \ln(1)) = 0. \tag{28}$$

This is the result that a pure state has zero total entropy. It is a general result for all pure bipartite states.

Moving on to the reduced density matrix, we can calculate it as follows

$$\rho^A = \mathrm{Tr_B}(\rho) = \frac{\mathrm{Tr_B}(|10\rangle\langle 10|) + \mathrm{Tr_B}(|10\rangle\langle 01|) + \mathrm{Tr_B}(|01\rangle\langle 10|) + \mathrm{Tr_B}(|01\rangle\langle 01|)}{2}$$

$$= \frac{|1\rangle\langle 1|\langle 0|0\rangle + |1\rangle\langle 0|\langle 1|0\rangle + |0\rangle\langle 1|\langle 1|0\rangle + |0\rangle\langle 0|\langle 1|1\rangle}{2}$$

$$= \frac{|1\rangle\langle 1| + |0\rangle\langle 0|}{2} = \frac{I}{2} \tag{29}$$

Now we can use this reduced density matrix to calculate the entropy of the subsystem A

$$E(\rho^A) = -\mathrm{Tr}(\rho^A \ln \rho^A) = -\mathrm{Tr}\Big(\frac{I}{2}\ln\Big(\frac{I}{2}\Big)\Big) = \ln 2 \tag{30}$$

In conclusion we find that the entanglement entropy of the partial system A is equal to $\ln 2$*, which indicates maximal bipartite entanglement. This holds equally for the subsystem B, as per Schmidt decomposition. This does not mean that the state as a whole has entropy, just that the two partials of this system have entropy. These partials are thus mixed in nature and as per Schmidt decomposition, we actually require these subsystems to be mixed for entanglement to exist. It also turns out that if we compute any of the Bell states in this manner, we find this exact result for all of them.*

3.2 Mixed state

Mixed states represent the case where a state of a given quantum system is probabilistic beyond superposition. This can be expressed as the familiar density matrix, equation (9), now with the coefficients $p_i \neq 1$. The properties governing the density matrix lead to a mathematical construct called the convex set. This means that if we have the states ρ_1 and ρ_2, the convex combination

$$\rho = \mathrm{p}\rho_1 + (1-\mathrm{p})\rho_2, \text{ where p} \in [0;1] \tag{31}$$

is also a state. It is also possible to generalise this to more than two states ρ_i, with more than two probabilities or convex weights p_i. Now we are able to define mixed state entanglement

Definition 7. Bipartite mixed state entanglement;

Let ρ be the density matrix representing the composite system. Then for a given state ρ we say that it is a product state if there exist states ρ^A and ρ^B satisfying

$$\rho = \rho^A \otimes \rho^B. \tag{32}$$

The state is said to be separable if there exist convex weights p_i and product states $\rho_i^A \otimes \rho_i^B$ such that we can write the state in the form

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B. \tag{33}$$

If this definition does not hold, we call the state entangled.

When compared to the pure state entanglement definition (17) we see that the only change that has been made is the addition of the probabilities for a given state. Next we are able to define a measure of entanglement for mixed state systems called entanglement of formation. The measure for two particle systems follows the procedure of Refs. [10; 11] and can be stated as follows

Definition 8. Entanglement of formation[10; 11];

Given a mixed state density matrix ρ, entanglement of formation is defined as the average entanglement of the decomposition of pure states that makes up the mixed state, minimized over all possible decompositions of the density matrix

$$E(\rho) = \min \sum_i p_i E(\psi_i). \tag{34}$$

We can luckily transform this to a more computable form, first by noting that equation (23) can be written as

$$E(\psi) = \mathbf{E}(C(\psi)), \tag{35}$$

and defining the concurrence C for pure states, an entanglement measure in itself as

$$C(\psi) = \langle \psi | \tilde{\psi} \rangle \,, \tag{36}$$

where the spin flip transformation for pure states of two qubits we define as

$$|\tilde{\psi}\rangle = \sigma_y \, |\psi^*\rangle \,. \tag{37}$$

Concurrence is a measure increasing from zero to one dependent on the entanglement present and as seen above, quite easy to calculate for a given pure state. The function $\mathbf{E}$ can now be explicitly stated as a function of concurrence

$$\mathbf{E}(C) = h\Big(\frac{1+\sqrt{1-C^2}}{2}\Big), \tag{38}$$

where the entropy function $h(x)$, in binary form is written as

$$h(x) = -x \log_2 x - (1-x)\log_2(1-x). \tag{39}$$

This concludes the entanglement of formation for a system of pure states and by some modifications, namely using the density matrix, we can proceed to more general case of the mixed states. Substituting to the pure state representation a density matrix we get

$$E(\rho) = \mathbf{E}(C(\rho)). \tag{40}$$

While at a glance not much has changed, the concurrence as a function of the density matrix behaves differently. To calculate concurrence for a mixed state, we try to find a minimum of all possible decompositions of the density matrix

$$C(\rho) = \inf \sum_i p_i C_i(|\psi_i\rangle). \tag{41}$$

In Ref. [24] the solution to two qubit system was found to be,

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{42}$$

where the eigenvalues λ_i are found with the aid of the spin flip procedure

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \tag{43}$$

and note that the eigenvalues are found to be the square roots of the matrix $\rho\tilde{\rho}$. Alternatively we can also find the eigenvalues with the equation

$$R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}, \tag{44}$$

where we take the eigenvalues in decreasing order.

The procedure of quantifying entanglement of formation shown above only works for density matrices of two qubits, that have two or less non-zero eigenvalues. The process of generalising this to any bipartite state[10] is more involved and does not have an effect for our purposes. The aim of this measure is to quantify the information transferring resource required to create a quantum state.

We can now take a closer look on entanglement of formation by calculating it for a mixed state of two Bell states

Example 2. *Mixed state entanglement*

Let us say we have two Bell states, the state B_{01} with a probability of $p_1 = \frac{3}{4}$ and the other state B_{10} with a probability of $p_2 = \frac{1}{4}$. We write the density matrix of this mixed state as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \frac{3}{4}\Big(|B_{01}\rangle\langle B_{01}|\Big) + \frac{1}{4}\Big(|B_{10}\rangle\langle B_{10}|\Big)$$

$$= \frac{3}{4}\frac{1}{2}\Big(|00\rangle - |11\rangle \Big)\Big(\langle 00| - \langle 11| \Big) + \frac{1}{4}\frac{1}{2}\Big(|10\rangle + |01\rangle \Big)\Big(\langle 10| + \langle 01| \Big) \tag{45}$$

We calculated the latter part of this expression in the first example, so by using that result and computing the other we find

$$\rho = \frac{3}{8}\Big(|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11| \Big)$$

$$+\frac{1}{8}\Big(|10\rangle\langle 10| + |10\rangle\langle 01| + |01\rangle\langle 10| + |01\rangle\langle 01| \Big) \tag{46}$$

and in the matrix form we have

$$\rho = \frac{1}{8}\begin{bmatrix} 3 & 0 & 0 & -3 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 0 & 3 \end{bmatrix}. \tag{47}$$

Now we can start quantifying the entanglement present by using the entanglement of formation measure. First we note that

$$\sigma_y \otimes \sigma_y = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \tag{48}$$

and we can proceed to write the spin flip transform for the density matrix

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$$

$$= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{8} & 0 & 0 & -\frac{3}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ -\frac{3}{8} & 0 & 0 & \frac{3}{8} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{8} & 0 & 0 & -\frac{3}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ -\frac{3}{8} & 0 & 0 & \frac{3}{8} \end{bmatrix}. \tag{49}$$

Now we use the spin flip matrix as

$$\rho\tilde{\rho} = \begin{bmatrix} \frac{3}{8} & 0 & 0 & -\frac{3}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ -\frac{3}{8} & 0 & 0 & \frac{3}{8} \end{bmatrix} \begin{bmatrix} \frac{3}{8} & 0 & 0 & -\frac{3}{8} \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ -\frac{3}{8} & 0 & 0 & \frac{3}{8} \end{bmatrix} = \begin{bmatrix} \frac{9}{32} & 0 & 0 & -\frac{9}{32} \\ 0 & \frac{1}{32} & \frac{1}{32} & 0 \\ 0 & \frac{1}{32} & \frac{1}{32} & 0 \\ -\frac{9}{32} & 0 & 0 & \frac{9}{32} \end{bmatrix}, \tag{50}$$

and the eigenvalues of this matrix are

$$\lambda_{1,2} = 0,\ \lambda_3 = \frac{1}{16},\ \lambda_4 = \frac{9}{16}. \tag{51}$$

Concurrence measure can be calculated with the eigenvalues

$$C(\rho) = \max\left\{0, \sqrt{\frac{9}{16}} - \sqrt{\frac{1}{16}} - 0 - 0\right\} = \max\left\{0, \frac{1}{2}\right\} = \frac{1}{2}. \tag{52}$$

While concurrence in it self is widely used as a measure of entanglement, we can still calculate the entanglement of formation

$$E(\rho) = \mathbf{E}(C(\rho)) = \mathrm{E}(\frac{1}{2}) = h\Big(\frac{1+\sqrt{1-(1/2)^2}}{2}\Big) = h(\frac{2+\sqrt{3}}{4})$$

$$= -\frac{2+\sqrt{3}}{4}\log_2(\frac{2+\sqrt{3}}{4}) - (1-\frac{2+\sqrt{3}}{4})\log_2(1-\frac{2+\sqrt{3}}{4})$$

$$\approx 0.355 \tag{53}$$

The concurrence measure is an entanglement monotone i.e a function quantifying the amount of entanglement in numerical interval between 0 and 1, unity meaning maximal entanglement and zero a separable or a product state. Entanglement monotone also implies that the quantity does not increase under local operations and classical communications. On the other hand, the numerical result of entanglement of formation is more physically meaningful since it quantifies the amount of entanglement needed to create the state in question. Lastly we might note that the quantities of concurrence and entanglement of formation are numerically close, and in the next section we examine this idea further.

3.2.1 Bell states entangled

A brief examination into what different probabilities of mixed state systems do to the entanglement measures of concurrence and entanglement of formation is now in order. The calculation is familiar from Example 2 and here we present only the results. Now we present a mixed state system with two possible states and define the probabilities of the two states, in our case Bell states, to have the form

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| = (1-p)\Big(\left|B_{nm}\right\rangle \left\langle B_{nm}\right|\Big) + p\Big(\left|B_{n'm'}\right\rangle \left\langle B_{n'm'}\right|\Big), \tag{54}$$

where nm and $n'm'$ represents the usual Bell basis state notation. From expressing the probabilities of the states in this manner we gain the ability to reduce the mixed state of two Bell states to one variable of probability. When we choose an

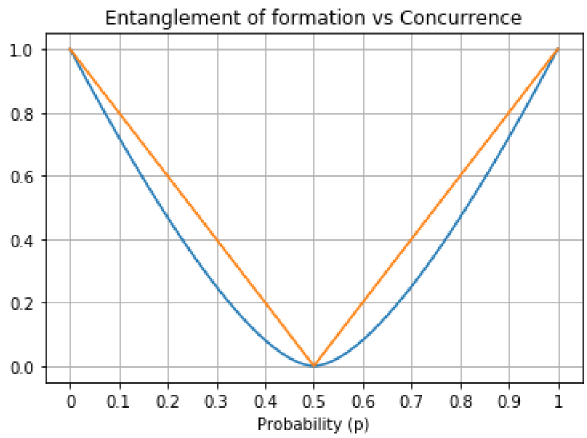

Figure 3: Concurrence C (orange) and entanglement of formation E (blue) as a function of the mixing probability of p in Eq. (54).

appropriate numerical value to our probability, we continue the exact procedure shown in Example 2, and calculate concurrence and entanglement of formation. If we repeat this procedure in sufficiently small steps, from 0 to 1, we can plot the measures of entanglement as a function of the probability, see Figure 3.

The plotted concurrence and entanglement of formation illustrate why concurrence in itself has been widely adopted as a measure of entanglement, the two quantities being very similar in value.

3.3 Positive partial transpose criterion

We will now consider a measure of entanglement that is quite different from the previous ones in a crucial way, that is, it does not concern itself with entropy of the system. This criteria we call the positive partial transpose(PPT) criterion [15] and the use case is for situations when the Schmidt decomposition fails, mainly in the mixed state systems. To introduce PPT criterion we need to introduce a partial transposition which relies on the fact that we are able to expand a density matrix in a selected product basis as

$$\rho = \sum_{i,j,k,l} p_{kl}^{ij} \left| i \right\rangle \left\langle j \right| \otimes \left| k \right\rangle \left\langle l \right|, \tag{55}$$

where ρ acts on the familiar $\mathcal{H}_A \otimes \mathcal{H}_B$ product space. From this decomposition we have the definition of the partial transposition of ρ with respect to subsystem B as

$$\rho^{T_B} = (I \otimes T)\rho = \sum_{i,j,k,l} p_{lk}^{ij} \left| i \right\rangle \left\langle j \right| \otimes \left| k \right\rangle \left\langle l \right|, \tag{56}$$

and similarly for ρ^{T_A} by exchanging i and j indices. We can also find that the partial transpositions are related by $\rho^T = (\rho^{T_A})^{T_B}$ and thus $\rho^{T_A} = (\rho^{T_B})^T$. Now we say that the density matrix ρ is PPT if the partial transposition has no negative eigenvalues or in other words it is positive semidefinite. After this, stating the formal PPT criterion is simple

Theorem 2. *PPT criterion*

Let ρ be a bipartite separable state. Then ρ is PPT.

PPT criterion is a simple yet powerful way to detect and quantify entanglement, with a caveat that it works only for the low dimensional cases of 2×2 and 2×3 systems [15]. It is also worth noting that the PPT criterion is stronger than all of the criteria relying on entropy and for the purposes of two-qubit entanglement, this criterion is sufficient for detecting all entanglement present [18].

3.4 Entanglement cost and entanglement distillation

Spending the last sub-chapter of bipartite entanglement on justification that even the most simple forms of entanglement carry immense value is in order. The usefulness of bipartite entanglement and the interest in developing a good theoretical base for it is intimately tied with the idea that entanglement can be used as a communication resource [19; 20]. That leads us to briefly introduce two related concepts, entanglement cost and entanglement distillation. First we introduce entanglement distillation, which quantifies the rate of which we are able to obtain the state $|\Phi^+\rangle$ i.e B_{00} from a given starting state ρ, in asymptotic terms. The formal definition of entanglement of distillation is quite involved[18], but verbally we can say that the equation states that entanglement of distillation is equal to the supremum of the rates of all possible distillation protocols. We can write this in mathematical terms [15]

$$E_D(\rho) = \sup_{LOCC} \left\{ \lim_{n_{in} \to \infty} \frac{n_{out}}{n_{in}} \right\}, \tag{57}$$

where we can see the process of local operations and classical communications (LOCC) mapping n_{in} input copies ρ onto n_{out} output singlet states. Entanglement distillation is the underlying quantity allowing us to perform quantum teleportation and superdense coding, and thus quite central to the importance of bipartite entanglement.

Entanglement cost is the reverse process and a measure dual to entanglement of distillation. The measure quantifies the process of transforming a maximally entangled state asymptotically to some non-maximally entangled state ρ. Again the formal definition[18] stating that entanglement cost equals the minimum rate of Bell states used to create ρ using LOCC. In mathematical terms we can write[15]

$$E_C(\rho) = \inf_{LOCC} \left\{ \lim_{n_{out} \to \infty} \frac{n_{in}}{n_{out}} \right\}, \tag{58}$$

which states that entanglement cost is the minimization over all LOCC that map n_{in} input singlet states onto n_{out} output copies of state ρ. Entanglement cost might sound familiar to entanglement of formation, and it turns out it equals regularised entanglement of formation[21].

The entanglement measures presented have some relation beyond the opposite nature of the two, and from the definitions we are able to state a relationship between the them[15]

$$E_C(\rho) \geq E_D(\rho). \tag{59}$$

The nature of bound entangled states is such that there exist states which require entanglement for their generation, but entanglement is not distillable from them. This generates the inequation between out two measures. For pure states however, entanglement cost, entanglement distillation and entropy of entanglement are equal[19]

$$E_C(\psi) = E_D(\psi) = E(\psi) = -\mathrm{Tr}(\rho^A \ln \rho^A), \tag{60}$$

which means that pure states are reversibly transformable into singlet states.

4 Multi-particle entanglement

Entanglement of a system of more than two qubits, even when only adding a single qubit more, can introduce a large amount of difficulty in analytic solvability of the system. In this chapter we will introduce some concepts of multi-partite entanglement that will hopefully illustrate some of the increasing complexity of these systems.

4.1 Three qubits

Separable pure states of three qubits can be divided into two distinct categories, the fully separable and bi-separable states. We can define these similarly to the bipartite case. This introduction can be found in Ref. [15]

Definition 9. Bi-separable three-qubit state

$$|\psi\rangle_{A|BC} = |\alpha\rangle_A \otimes |\delta\rangle_{BC} \tag{61}$$

Definition 10. Fully separable three-qubit state

$$|\psi\rangle_{A|B|C} = |\alpha\rangle_A \otimes |\beta\rangle_B \otimes |\gamma\rangle_C \tag{62}$$

These definitions allow us to conclude that a pure state tripartite system is genuinely entangled when it is neither bi-separable nor fully separable. We can also note that only the genuine entangled state requires a physical interaction between all the parties and we see that when we compare this to the bipartite case, there is an additional degree of separability present. From the definitions above we can now present actual states in the familiar form. First we have the Greenberger-Horne-Zeilinger state

$$|GHZ\rangle_3 = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{63}$$

The second kind of entangled form of pure three qubit states, is called the W state

$$|W\rangle_3 = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{64}$$

The states presented here represent the two definitions when we call forth the classical communications and stochastic local operations, i.e probabilistic local operations, shortened to SLOCC. The SLOCC operations, in our case of three qubits, can be used to transform any entangled three qubit pure state to either of the defined W or GHZ state. These states are distinct and inequivalent, as shown in Ref. [22]. Mathematically these local unitary operations take the form

$$|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\theta} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle , \tag{65}$$

where we see that there are six parameters required to describe the state, that are defined to satisfy $\lambda_i \geq 0, \sum_i \lambda_i^2 = 1$ and $\theta \in [0;\pi]$. Additionally when the state is in W class, we have $\theta = \lambda_4 = 0$, which states that in the set of all pure states, W class is a set of measure zero. This is also a clear indication that the W states are a smaller group of states than the more general GHZ states. This mathematical construct is a generalization of the Schmidt decomposition to three qubits [23].

When considering bipartite mixed state entanglement, one has to refer back to the possible pure states of the bipartite system. Increasing the qubit count of the system to three or more parts is maybe somewhat obviously no different. Thus we are able to define thee qubit mixed state full separation[15] as

Definition 11. Fully separable mixed state of three qubits

Mixed state ρ is fully separable if ρ can be written as a convex combination of fully separable pure states. In other words we say that a mixed state is fully separable if we can write the state with convex weights p_i and fully separable states $|\phi_i^{fs}\rangle$ as

$$\rho^{fs} = \sum_i p_i |\phi_i^{fs}\rangle \langle\phi_i^{fs}| . \tag{66}$$

The full separation of mixed state of three qubits is then quite similar to the bipartite case, and increasing the qubit count of the system would introduce new terms and computational complexity but at its core the principle would remain the same. Bi-separability follows the same basic principle and can be defined, again in a three qubit system [15] as

Definition 12. Bi-separable mixed state of three qubits

Mixed state ρ is fully bi-separable if we are able to write it as a convex combination of bi-separable pure states

$$\rho^{bs} = \sum_i p_i \left|\phi_i^{bs}\right\rangle \left\langle\phi_i^{bs}\right| . \tag{67}$$

Bi-separable states bring out the diverging properties of multipartite states in comparison to the bipartite systems, i.e the separability within the system can now be in respect to different partitions. While the bi-separable states contain entanglement, we might quite justifiably classify them as a different group to the genuinely entangled states, since the measures and properties of these states differ.

Lastly full entanglement of the three qubit mixed state follows the two pure state classes of GHZ and W states, and the W state can be defined as

Definition 13. Fully entangled mixed state of three qubits

Mixed state ρ is fully entangled if we can write it as a convex combination of W-pure states

$$\rho^{W} = \sum_i p_i \left|\phi_i^{W}\right\rangle \left\langle\phi_i^{W}\right| . \tag{68}$$

If the state does not belong to the W class, it belongs to the GHZ class. The W class in itself has been shown to belong inside the GHZ class. Unlike the pure case, W class in mixed state systems is not a set of measure zero in comparison to the GHZ class [24]. The problem of assigning a state to the appropriate class has to be treated as separate problem for the mixed and pure states. The pure state system has been solved [25], but the general solution to a mixed state ensemble is yet to be found.

4.2 Generalizing to n-part systems

Defining the quantum system of an arbitrary amount of particles mostly follows the familiar formula introduced in two and three qubit sections. In the cases of full separability we could transform the three qubit equation (62) to N-partite system by adding N-amount of Kronecker products with the corresponding qubits. The mixed system full separability is similar given we add the state probabilities to the equation. A notable departure to the three qubit definitions comes when we consider partial separation. Considering that the correlations can be between any of the parts of the system we have the notion of m-separability[15]

Definition 14. Partial separation of N-partite pure state

We call N-partite pure state $|\psi\rangle$ m-separable, for $1 < m < N$, if we can separate the N parties to m parts of $p_1, ..., p_m$ such that

$$|\psi\rangle = \bigotimes_{i=1}^{m} |\phi_i\rangle_{p_i} . \tag{69}$$

The N-partite system has $(\mathrm{m}^N)/\mathrm{m}!$ possible ways of partitioning N parties to m parts. In other words a mixed state is m-separable if we can write it as a convex combination of m-separable states, that might belong to different partitions.

From here we would start defining the mixed state partial separability, but we run in to the uncomfortable realisation that it is not yet understood well enough to briefly glance over. As for the specific criteria, we can note that the CCNR (computable cross-norm or realignment, omitted because of length) [15] and PPT criterion are roughly generalisable. The two criteria however can only rule out full separability. Rest of the known criteria are too involved or rely on the consciously omitted entanglement witness part of entanglement to go over here. The often cited article titled entanglement detection illustrates this point well [15].

5 Discussion and Conclusions

Let us go over the main points of the thesis. First we went over the quantum information theory and introduced some examples that hopefully convinces the reader of the importance of two-qubit entanglement. It is a point worth repeating that the bipartite entanglement in itself is exceedingly important in terms of quantum information, and is not just an introductory tool. The main portion of the work focused on the details of two-qubit entanglement, where we introduced a working theory of the phenomena, albeit omitting the entanglement witness formulation completely. Some noteworthy parts there were the convex roof construction, the PPT criterion and the entanglement distillation and cost sections. Lastly we went over some of the easier parts of multipartite entanglement, mainly in the three qubit realm.

Entanglement as a topic of a roughly twenty page thesis, that is geared towards a reader that might not be familiar with the subject, is in some ways a difficult task. The balancing act of trying to fit enough information to build up the readers knowledge of the details that build the definitions, while simultaneously trying to introduce a meaningful amount of phenomena and mathematical tools pertaining to entanglement involves some hard decisions. One of these hard decisions was made to cut entanglement witness formulation completely, which means the side of entanglement from direct observables is absent here. It also means that some of the multipartite definitions and phenomena are not able to be discussed, and that some proofs have to be omitted. Bipartite entanglement section underwent some cuts as well, namely the CCNR criterion is something that would have been worth it to discuss in addition to the entanglement witness.

As an important side-note regarding the future of quantum mechanical processing and to note that the entanglement of a system of particles is not necessarily the only non-classical correlation; there exist different types of correlations under the umbrella term quantum discord. The entanglement introduced here is only one of these non-classical correlations that a quantum system can exhibit. Entanglement as a phenomena is the most studied of these correlations, and the earliest discovered.

Finally I would like to point the reader towards Refs. [15; 18] for a more detailed overview of the phenomena of entanglement, and as an illustration of the depths of this phenomena. It is truly remarkable how vast this arguably very small portion of quantum mechanics can be.

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