Petrov D vacuum spaces revisited: complete tables and invariant classification by GHP analysis

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Abstract. The efficiency of the GHP formalism combined with the power of the computer system $xAcT$ enables an exhaustive systematic analysis to be made of Petrov D vacuum spaces. This gives a complete involutive set of tables for the four GHP derivatives on each of the four GHP spin coefficients and the one Weyl tensor component. It follows directly from these results that the theoretical upper bound on the order of covariant differentiation of the Riemann tensor required for a Karlhede invariant classification of these spaces is reduced to two.

1. Introduction

Kinnersley's work [16] is not just the definitive work on the Type D vacuum spacetimes, but it was also the calculation which established the power of the NP formalism [18]. So it would appear rather presumptuous, 40 years later, to draw attention to some related results which had not been presented by Kinnersley, and also to suggest that the GHP formalism [12] (the lesser-known descendant of the NP formalism) is an even more powerful formalism for such calculations. However, the difference today — compared to 40 years ago — is the availability of powerful computer support: in particular, we use the system $xAct$ [17]; this permits calculations which would not have been possible 40 years ago.

So although the results in this talk are important in their own right — for Petrov type D vacuum spaces — they also illustrate how methods and techniques in the GHP formalism [7] [8], [10], [13], [14] can be exploited even more fully when powerful computer support is available.

2. Calculations

Nonvanishing weighted GHP spin coefficients and curvature components

$$\mathcal{S} \equiv \{\rho, \quad \tau, \quad \rho', \quad \tau', \quad \Psi_2\}.$$

Beginning with the familiar Ricci and Bianchi equations in the GHP formalism [12], we use the GHP commutator equations and the introduction of *extension variables* in a systematic manner

to obtain complete and involutive tables for all non-zero spin coefficients and Ψ_2. In particular, all information is extracted by checking integrability conditions and consistency at all stages.
Ricci equations

$$
\begin{aligned}
\text{Þ}\rho &= \rho^2, & \partial\rho &= \tau(\rho - \bar{\rho}), \\
\text{Þ}\tau &= \rho(\tau - \bar{\tau}'), & \partial\tau &= \tau^2, \\
\text{Þ}'\rho &= \partial'\tau + \rho\bar{\rho}' - \tau\bar{\tau} - \Psi_2,
\end{aligned}
$$

and their $'$ counterparts;
Bianchi equations

$$
\text{Þ}\Psi_2 = 3\rho\Psi_2, \qquad \partial\Psi_2 = 3\tau\Psi_2,
$$

and their $'$ counterparts;
Commutator equations

$$
\begin{aligned}
[\text{Þ}\text{Þ}' - \text{Þ}'\text{Þ}]\eta_{pq} &= \Big((\bar{\tau} - \tau')\partial + (\tau - \bar{\tau}')\partial' - p(\Psi_2 - \tau\tau') - q(\bar{\Psi}_2 - \bar{\tau}\bar{\tau}')\Big)\eta_{pq}, \\
[\text{Þ}\partial - \partial\text{Þ}]\eta_{pq} &= \Big(\bar{\rho}\partial - \bar{\tau}'\text{Þ} + q\bar{\rho}\bar{\tau}'\Big)\eta_{pq}, \\
[\text{Þ}\partial' - \partial'\text{Þ}]\eta_{pq} &= \Big(\rho\partial' - \tau'\text{Þ} + p\rho\tau'\Big)\eta_{pq}, \\
[\partial\partial' - \partial'\partial]\eta_{pq} &= \Big((\bar{\rho}' - \rho')\text{Þ} + (\rho - \bar{\rho})\text{Þ}' + p(\Psi_2 + \rho\rho') - q(\bar{\Psi}_2 + \bar{\rho}\bar{\rho}')\Big)\eta_{pq},
\end{aligned}
$$

together with their $'$ counterparts, acting on a scalar η_{pq} of spin and boost weight $(p-q)/2$, $(p+q)/2$ respectively.

When we substitute the Bianchi equations into the commutators for Ψ_2 and also make use of the Ricci equations, we obtain
Post-Bianchi equations

$$
\begin{aligned}
\Psi_2(-\text{Þ}'\rho + \text{Þ}\rho' - \tau\bar{\tau} + \tau'\bar{\tau}') = 0 & \quad \Leftrightarrow \quad & -\text{Þ}'\rho + \text{Þ}\rho' - \tau\bar{\tau} + \tau'\bar{\tau}' = 0, \\
\Psi_2(\partial'\rho - \text{Þ}\tau') = 0 & \quad \Leftrightarrow \quad & \partial'\rho - \text{Þ}\tau' = 0
\end{aligned}
$$

and their $'$ counterparts.
To continue the analysis, it will be convenient to introduce the zero-weighted Z_1, the $\{0,2\}$-weighted Z_2 and the $\{2,0\}$-weighted Z_2' (using the notation of Carminati and Vu [1]),
Extension variables

$$
Z_1 = \partial'\tau \;(= Z_1'), \qquad Z_2 = \partial'\rho, \qquad Z_2' = \partial\rho'.
$$

These variables enable us to write down
Complete tables for all spin coefficients

$$
\text{Þ}\rho = \rho^2, \quad \partial\rho = \tau(\rho - \bar{\rho}), \quad \partial'\rho = Z_2, \quad \text{Þ}'\rho = Z_1 - \tau\bar{\tau} + \tau'\bar{\tau}'
$$

$$
\text{Þ}\tau = \rho(\tau - \bar{\tau}'), \quad \partial\tau = \tau^2, \quad \partial'\tau = Z_1, \quad \text{Þ}'\tau = Z_2',
$$

and their $'$ counterparts.
Next, by applying commutators to all spin coefficients, we obtain
Complete tables for Z_1, Z_2, Z_2'

$$
\text{Þ}Z_1 = \rho(2Z_1 - \bar{Z}_1 + \tau'\bar{\tau}' - \bar{\rho}\rho' + \bar{\rho}'\rho) + (\tau - \bar{\tau}')Z_2,
$$

$$\partial Z_1 = \tau(2Z_1 + \Psi_2 + \bar{\Psi}_2 + \rho\bar{\rho}' + \rho'\bar{\rho}) - (\bar{\rho} - \rho)Z_2' - \bar{\tau}'(\rho\bar{\rho}' - \rho\rho')$$

$$\text{P}Z_2 = 3\rho Z_2,$$

$$\partial' Z_2 = 3\tau' Z_2,$$

$$\partial Z_2 = \tau Z_2 + 2(\rho - \bar{\rho})Z_1 + \bar{\rho}\Psi_2 - \rho\bar{\Psi}_2 + 2\rho\bar{\rho}'(\rho - \bar{\rho}),$$

$$\text{P}'Z_2 = \rho' Z_2 + \bar{\Psi}_2\tau' + \Psi_2(\bar{\tau} - 2\tau') - 2(\bar{\tau} - \tau')(Z_1 + \rho\bar{\rho}')$$

and their $'$ counterparts.

At this stage we have a complete and involutive set of tables: for the eight complex quantities $\{\rho, \tau, \rho', \tau', \Psi_2, Z_1, Z_2, Z_2'\}$.

Next, by applying commutators to Z_1, Z_2, Z_2', we obtain
Complex identity

$$\Psi_2(\tau\bar{\tau} - \tau'\bar{\tau}' + \rho\bar{\rho}' - \bar{\rho}\rho') = 0 \quad \Leftrightarrow \quad \tau\bar{\tau} - \tau'\bar{\tau}' + \rho\bar{\rho}' - \bar{\rho}\rho' = 0;$$

and
Explicit expressions for Z_2, Z_2' in terms of spin coefficients

$$\Psi_2(Z_2 + \rho\bar{\tau} - 2\rho\tau' + \tau'\bar{\rho}) = 0 \quad \Leftrightarrow \quad Z_2 = -\rho\bar{\tau} + 2\rho\tau' - \tau'\bar{\rho},$$

together with the $'$ counterpart.

Substituting these expressions for Z_2, Z_2' into the above tables for Z_2, Z_2' respectively, we obtain
Consistency conditions for Z_2, Z_2'

$$\bar{\tau}'Z_1 - \tau\bar{Z}_1 = \tau'\bar{\tau}'(\tau - -\bar{\tau}'),$$

$$\bar{\rho}'Z_1 - \rho'\bar{Z}_1 = 2\rho'\bar{\rho}'(\rho - \bar{\rho}) + \tau'\bar{\tau}'(\bar{\rho}' - \rho') + \bar{\rho}'\Psi_2 - \rho'\bar{\Psi}_2.$$

In this talk, we restrict attention to the *generic case*: $\tau\tau'\rho'\rho \neq 0 \neq \bar{\rho}'\tau - \rho'\bar{\tau}'$; we then obtain
Explicit expression for Z_1

$$Z_1 = \tau\left(2\bar{\rho}\rho'^2 - \bar{\Psi}_2\rho' + (\bar{\tau}' - 2\tau)\bar{\tau}\rho' - 2\bar{\rho}'\bar{\rho}\rho' + \Psi_2\bar{\rho}' + \bar{\rho}'\tau\bar{\tau}\right)\Big/\bar{\rho}'\tau - \rho'\bar{\tau}'.$$

Substitution into the above table for Z_1 of this value for Z_1 — together with the values for Z_2, Z_2' — gives only trivial identities.
So finally we have obtained
Complete and involutive tables for all spin coefficients

$$\text{P}\rho = \rho^2, \quad \partial\rho = \tau(\rho - \bar{\rho}), \quad \partial'\rho = -\rho\bar{\tau} + 2\rho\tau' - \tau'\bar{\rho},$$

$$\text{P}'\rho = \tau\left(2\bar{\rho}\rho'^2 - \bar{\Psi}_2\rho' + (\bar{\tau}' - 2\tau)\bar{\tau}\rho' - 2\bar{\rho}'\bar{\rho}\rho' + \Psi_2\bar{\rho}' + \bar{\rho}'\tau\bar{\tau}\right)\Big/\bar{\rho}'\tau - \rho'\bar{\tau}';$$

$$\text{P}\tau = \rho(\tau - \bar{\tau}'), \quad \partial\tau = \tau^2, \quad \text{P}'\tau = -\rho'\bar{\tau}' + 2\rho'\tau - \tau\bar{\rho}',$$

$$\partial'\tau = \tau\left(2\bar{\rho}\rho'^2 - \bar{\Psi}_2\rho' + (\bar{\tau}' - 2\tau)\bar{\tau}\rho' - 2\bar{\rho}'\bar{\rho}\rho' + \Psi_2\bar{\rho}' + \bar{\rho}'\tau\bar{\tau}\right)\Big/\bar{\rho}'\tau - \rho'\bar{\tau}',$$

together with the $'$ counterparts.

The analogous calculations for the non-generic cases are easily carried out. Full details of all calculations are given in [9].

The system *xAct* was used to cross-check all the computations derived by hand and to derive others which due to their complication were difficult to obtain without the aid of the computer. A *Mathematica* notebook in which all the equations of the GHP formalism are obtained from scratch is available in [17] at [11].

Earlier relevant work on these spacetimes is given in [4], [5], [6], [1]. In particular, in [5] Czapor and Mclenaghan state that they have obtained the complete tables for all of the NP spin coefficients, but these tables, together with the intermediate calculations, were too long to print out in the NP formalism; in [1], Carminati and Vu presented *partial* tables (for *three* GHP operators acting on each GHP spin coefficient) in the GHP formalism.

3. Conclusions

The calculations in the previous section supply a *complete and involutive set of tables* for

$$\mathcal{S} \equiv \{\rho, \quad \tau, \quad \rho', \quad \tau', \quad \Psi_2\}.$$

There is an immediate application to the invariant classification scheme [15], [20]. All elements of $\mathcal{S}$ occur at the order of first derivative of the Weyl tensor, and no new quantities appear at second derivative order; moreover, the invariance group remains unchanged from first to second derivative order: therefore *the Karlhede algorithm for the invariant classification of these spacetimes terminates at second order.* This result strengthens earlier results on the invariant classification of Petrov type D vacuum spaces [2], [3], [21].

These results enable the classification scheme for vacuum Petrov type D spacetimes in [4] to be strengthened, and the full details are given in [9]; recently, a similar classification has been given in [19].

Finally we point out, that when a non-zero cosmological constant is introduced, it is very straightforward to repeat the calculations using *xAct*, and once again a complete and involutive set of tables is obtained, with the same result for the invariant classification.

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